UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM
Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

UNIVERSITY OF L'AQUILA

PHD IN MATHEMATICS AND MODELLING
Cycle XXXVII

PHD THESIS

Legendrian cycles and Alexandrov sphere theorems for $W^{2,n}$-hypersurfaces

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Math-02/B

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a.y. 2023/2024

“Quello che non so è quasi tutto. Quello che so è qualcosa che, per quanto limitato, è però importante.”

Ennio De Giorgi

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UNIVERSITY OF L'AQUILA

Abstract

Department of Information Engineering, Computer Science and Mathematics

Doctor of Philosophy

Legendrian cycles and Alexandrov sphere theorems for $W^{2,n}$-hypersurfaces

by Paolo VALENTINI

In this thesis, we prove that the proximal unit normal bundle of the graph of a $W^{2,n}$-function in n-variables carries a natural structure of Legendrian cycle. We then generalize Alexandrov's sphere theorems for higher-order mean curvature functions to hypersurfaces in $\mathbb{R}^{n+1}$ which are locally graphs of arbitrary $W^{2,n}$-functions, under a general degenerate ellipticity condition. The proof relies on extending the Montiel-Ros argument to this class of hypersurfaces and on the existence of the aforementioned Legendrian cycles. We also prove the existence of n-dimensional Legendrian cycles with $2n$-dimensional support, thus answering a question posed by Rataj and Zähle. Furthermore, we extend some of these results to Sobolev-type manifolds, representable as finite unions of $W^{2,n}$-regular graphs, and generalize Reilly's variational formulas in this context. Finally, we provide a very general version of the umbilicality theorem for Sobolev-type hypersurfaces.

Acknowledgements

The first person I would like to thank is my advisor, Professor Mario Santilli. Thanks to him, I was able to deepen my understanding of Geometric Measure Theory, a topic I love in every aspect. I am also grateful for his overwhelming passion for research.

I would like to sincerely thank Professor Debora Amadori for always believing in me and for always being there for me.

A special thanks to Professor Davide Gabrielli for his support and his unconditional passion for mathematics.

A deep thank you to Professor Marta Nolasco, for all the teachings, valuable advice, stimulating discussions, and laughter she has shared with me.

I am deeply grateful to Professor Luigi Ambrosio for giving me the opportunity to attend his inspiring lectures during my PhD years and for his invaluable support whenever I needed it.

I thank Professor Norberto Gavioli for his help and availability.

Thanks to Professor Francesco Leonetti for his sensitivity and his sincere interest in my thesis work.

I would like to thank Professor Diego Pallara for the nice conversations and for all the advice.

I sincerely thank my dear friend Roberta Marziani for her invaluable help.

I would also like to thank Elisa Continelli, Filomena De Filippis, Federica Iacovissi, Alessio Barbieri, Stefano Di Giovacchino and Lorenzo Pescatore for their deep and dear friendship and for giving each other strength throughout the delicate PhD journey.

But the biggest thanks go to Mathematics! For its ability to continually inspire wonder in those who tirelessly chase after it.

List of Symbols

Topological and metric space notations:

$\mathbb{R}^n$, $\mathbb{S}^{n-1}$	euclidean n-dimensional space and its unit sphere.
$\bullet$, $\lvert\cdot\rvert$	euclidean inner product and the induced norm.
$\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$	canonical basis of $\mathbb{R}^n$.
$\Omega_n := dX_1 \wedge \ldots \wedge dX_n$	standard volume form of $\mathbb{R}^n$.
$\mathbb{S}^{n-1}_+$, $\mathbb{S}^{n-1}_-$	subsets of points of $\mathbb{S}^{n-1}$ whose last coordinate is, respectively, positive and negative.
$v \perp \omega$	$v \bullet \omega = 0$, for $v, \omega \in \mathbb{R}^n$.
$\overline{E}$	topological closure of E.
$\mathrm{Int}(E)$	interior of E.
$E \subset\subset F$	$\overline{E} \subseteq F$, $\overline{E}$ compact.
∂E	topological boundary of E.
$B^n_\rho(x) = B_\rho(x)$	open ball in $\mathbb{R}^n$ with centre in x and radius $\rho > 0$.
$\boldsymbol{\delta}_C$	distance function from a set $C \subseteq \mathbb{R}^n$, i.e. $\boldsymbol{\delta}_C(\cdot) := \mathrm{dist}(\cdot, C)$.
$Lip(f)$	Lipschitz constant of a function f beetwen metric spaces.

Geometric Measure Theory:

$\mathscr{P}(S)$	power set of a set S.
$\mathcal{B}(X)$	Borel σ-algebra on a topological space X.
$\mathcal{L}^n$	Lebesgue outer measure in $\mathbb{R}^n$.
$\boldsymbol{\alpha}(n)$	Lebesgue measure of the unit ball of $\mathbb{R}^n$.
$\mathcal{H}^k$	k-dimensional Hausdorff outer measure in $\mathbb{R}^n$.
$\Theta^k(\mathcal{H}^k \llcorner E, x)$	k-dimensional density of E at x.
$\Theta^{*k}(\mathcal{H}^k \llcorner E, x)$, $\Theta^k_*(\mathcal{H}^k \llcorner E, x)$	upper and lower k-dimensional density of E at x.
$\mathrm{Tan}^k(\mathcal{H}^k \llcorner E, x)$	$(\mathcal{H}^k \llcorner E)$-approximate tangent cone of E at x.
$J^E_k f(x)$	$(\mathcal{H}^k \llcorner E)$-approximate Jabobian of f at x.
$\mathcal{E}^k(U)$	space of smooth k-forms on U.
$\mathcal{D}^k(U)$	space of smooth k-forms on U, with compact support.
$\mathcal{D}_k(U)$	space of k-currents on U.
α	contact 1-form.
$\omega := d\alpha$	symplectic 2-form.

Functions and function spaces: let $f : X \to Y$.

id	identity map.
$f \vert E$	restriction of f to $E \subseteq X$.
$\mathrm{Im}(f)$	range of f.
$\mathrm{dmn}(f)$	domain of f.
$\overline{f}$	$\overline{f}(x) := \big(x, f(x)\big)$ for every $x \in X$.
$\mathrm{graph}(f)$, E_f, C_f	assume that $f : \mathbb{R}^n \to \mathbb{R}$, then: $\mathrm{graph}(f) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y = f(x)\}$ is the graph of f, $E_f := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y \geq f(x)\}$ is the epi-graph of f, $C_f := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y \leq f(x)\}$ is the cato-graph of f.
$W^{k,p}(U)$	Sobolev space of functions whose weak partial derivatives, up to order k, lie in $L^p(U)$ for some open set $U \subseteq \mathbb{R}^n$.
$W^{k,p}_{loc}(U)$	local Sobolev space of functions $f \in L^1_{loc}(U)$, such that $f \in W^{k,p}(V)$ for every open set $V \subset\subset U$.
$Lip(X; Y)$	space of Lipschitz functions beetwen metric spaces X and Y.

$\pi_0(x,y) := x$, for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$	canonical projection on the first factor.
$\pi_1(x,y) := y$, for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$	canonical projection on the second factor.
$\bigwedge(n,k)$, for $k \leq n$	set of all increasing functions from $\{1,\dots,k\}$ to $\{1,\dots,n\}$.

Vector spaces:

$\mathbf{G}(n,k)$	Grassmann manifold of k-planes in $\mathbb{R}^n$.
$\nu^\perp \in \mathbf{G}(n,n-1)$	$\nu^\perp := \{z \in \mathbb{R}^n : z \bullet \nu = 0\}$, for $\nu \in \mathbb{R}^n$.

Differentiation notations:

∇, D	classical gradient and differential.
D_i, D^2_{ij}	classical partial derivatives.
$\boldsymbol{\nabla}, \mathbf{D}^k$	distributional gradient and k-differential of Sobolev functions.
$\mathbf{D}_i, \mathbf{D}^2_{ij}$	distributional partial derivatives of Sobolev functions.
d	exterior derivative of differential forms.

Other notations:

$c(p_1,\dots,p_n)$	positive constant depending only on the parameters $p_1,\dots,p_n$.

Introduction

Background and motivation

One of the fundamental and most well-known theorems in geometric analysis is the following Alexandrov's sphere theorem (cf. [1]).

Theorem (Alexandrov)**.** *A bounded and connected C^2-domain $\Omega \subset \mathbb{R}^{n+1}$ must be a round ball, provided there exist a C^1-function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that*

$$\varphi\big(\chi_{\Omega,1}(p), \ldots, \chi_{\Omega,n}(p)\big) = \lambda$$

and

$$\partial_i \varphi(\chi_{\Omega,1}(p), \ldots, \chi_{\Omega,n}(p)) > 0 \quad \text{for } i \in \{1, \ldots, n\}\,, \tag{0.0.1}$$

for every $p \in \partial\Omega$. Here $\chi_{\Omega,1} \leq \ldots \leq \chi_{\Omega,n}$ are the principal curvatures of $\partial\Omega$.

This result was proved by Alexandrov using the *moving plane method*, based on the classical maximum principle for linear elliptic operators and the Hopf lemma. Its simplest case is given by the famous rigidity result for hypersurfaces with constant mean curvature. More generally, choosing $\varphi = \sigma_k$ (where σ_k is the k-th elementary symmetric function; cf. Definition 3.1.11), one can deduce the following claim:

if $\Omega \subset \mathbb{R}^{n+1}$ is a bounded and connected C^2-domain such that $H_{\Omega,k}$ is constant, for some $k \in \{1, \ldots, n\}$, then Ω is a round ball.

In fact, if $\varphi = \sigma_k$, then (0.0.1) is automatically satisfied. To prove it, we consider the sets

$$\Gamma_i := \{\lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0\} \quad \text{for } i \in \{1, \ldots, n\}$$

and we define C_i to be the connected component of Γ_i, with $(1, \ldots, 1) \in C_i$. Since Ω is bounded there exists $x_0 \in \partial\Omega$ such that $\chi_{\Omega,i}(x_0) > 0$ for each $i \in \{1, \ldots, n\}$, whence we infer that $H_{\Omega,k}(x) > 0$ for any $x \in \partial\Omega$. Now we consider the continuous function $\boldsymbol{\chi}_\Omega : \partial\Omega \to \mathbb{R}^n$, defined as $\boldsymbol{\chi}_\Omega := (\chi_{\Omega,1}, \ldots, \chi_{\Omega,n})$, and since $\boldsymbol{\chi}_\Omega(\partial\Omega)$ is connected we deduce that

$$\boldsymbol{\chi}_\Omega(x_0) \in \boldsymbol{\chi}_\Omega(\partial\Omega) \subseteq C_k\,.$$

Now, employing Garding's theory of hyperbolic polynomials (cf. [18]), we infer that (cf. [50, Proposition 1.3.3 (3)])

$$\partial_n \sigma_k\big(\boldsymbol{\chi}_\Omega(x)\big) \geq \ldots \geq \partial_1 \sigma_k\big(\boldsymbol{\chi}_\Omega(x)\big) > 0 \quad \text{for every } x \in \partial\Omega\,,$$

so (0.0.1) is satisfied.

Subsequently, using the maximum principle for $W^{2,n}$-solutions of uniformly elliptic linear PDE's (cf. [20, Chapter 9]), Alexandrov observed in [2, p. 305] that the moving plane method can still be applied to certain Sobolev-type hypersurfaces. In particular, the sphere theorem can be generalized as follows.

Theorem (Alexandrov)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded and connected domain, such that $\partial\Omega$ can be locally represented by n-dimensional graphs of C^1-functions with second-order distributional derivatives belong to L^n. Assume that there exist a C^1-function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and constants $\lambda, \mu_1, \mu_2 \in \mathbb{R}$ such that*

$$\varphi\big(\chi_{\Omega,1}(p), \ldots, \chi_{\Omega,n}(p)\big) = \lambda$$

and

$$0 < \mu_1 \le \partial_i \varphi(\chi_{\Omega,1}(p), \dots, \chi_{\Omega,n}(p)) \le \mu_2 < \infty \quad \text{for } i \in \{1, \dots, n\} \tag{0.0.2}$$

for $\mathcal{H}^n$-a.e. every $p \in \partial\Omega$, where $\chi_{\Omega,1} \le \ldots \le \chi_{\Omega,n}$ are the generalized principal curvatures of $\partial\Omega$. Then Ω must be a round ball.

The proof of the previous theorem is based on the generalization of the moving plane method through the maximum principle for $W^{2,n}$-solutions of uniformly elliptic PDE's. The *uniform ellipticity condition* (0.0.2) guarantees the uniform ellipticity of the underlying linear equations, see [2, Statement A, p. 304]. Furthermore, C^1-regularity is used in connection with the Hopf boundary lemma [2, Statement B]. These two hypotheses are certainly useful conditions to continue addressing the problem using the moving plane method, but are they truly necessary to prove Alexandrov's theorem? Furthermore, since $\boldsymbol{\chi}_\Omega$ is no longer continuous, we note that the assumption (0.0.2) in the previous theorem, unlike in the C^2-case, is not automatically satisfied if $\varphi = \sigma_k$.

In order to analyze Alexandrov's sphere theorem from a different perspective, let us first recall two results. In the first of these, Hsiung [23, Theorem 1] extends the Minkowski integral formulas, for convex hypersurfaces, to compact C^2-hypersurfaces. Such *Minkowski-Hsiung identities* stating that:

$$\text{if } \Omega \subset \mathbb{R}^{n+1} \text{ is a bounded } C^2\text{-domain, then}$$
$$\int_{\partial\Omega} H_{\Omega,k-1}(x)\, d\mathcal{H}^n(x) = \int_{\partial\Omega} \big(x \bullet \nu_\Omega(x)\big)\, H_{\Omega,k}(x)\, d\mathcal{H}^n(x) \quad \text{for } k \in \{1, \dots, n\}.$$

The second result is given by the *Heintze-Karcher inequality* (cf. [22]), that is to say:

if $\Omega \subset \mathbb{R}^{n+1}$ is a bounded and connected C^2-domain,
with mean curvature $H_\Omega(x) \ge 0$ for every $x \in \partial\Omega$, then

$$\mathcal{L}^{n+1}(\Omega) \le \frac{n}{n+1} \int_{\partial\Omega} \frac{1}{H_\Omega(x)}\, d\mathcal{H}^n(x)$$

and the equality holds if and only if Ω is a ball.

In the 1980s, Ros in [47], and later jointly with Montiel in [39], combined the Minkowski-Hsiung identities with the Heintze-Karcher inequality to reprove Alexandrov's sphere theorem for C^2-domains, with a different approach and when φ is the k-th symmetric function σ_k. We notice that this approach is based on the area formula for the C^1-map $\overline{\nu}_\Omega(z) := (z, \nu_\Omega(z))$, where ν_Ω is the outer unit-normal vectorfield to Ω.

It is interesting to mention that Minkowski-Hsiung formulae can be derived as special cases of variational formulae of certain curvature integrals. Namely, if we define the *k-th total curvature measure* of a bounded smooth domain $\Omega \subset \mathbb{R}^{n+1}$ as

$$\mathcal{A}_k(\Omega) := \int_{\partial\Omega} H_{\Omega,k}\, d\mathcal{H}^n \quad \text{for } k \in \{0, \dots, n\}$$

then Reilly proved in [45] that

$$\frac{d}{dt}\mathcal{A}_{k-1}\big(F_t(\Omega)\big)\Big|_{t=0} = (n-k+1)\int_{\partial\Omega} (V \bullet \nu_\Omega)\, H_{\Omega,k}\, d\mathcal{H}^n \quad \text{for } k \in \{1, \dots, n\}, \tag{0.0.3}$$

whenever $\{F_t\}_{t \in (-\epsilon,\epsilon)}$ is a local variation of $\mathbb{R}^{n+1}$ with $F_0 = \mathrm{id}|\mathbb{R}^{n+1}$ and initial velocity V. Variational formulae for more general integrands, as well as in space forms and in higher codimension, were proved in [45] and [44] (cf. also [41]).

One of our aims in this thesis is to investigate the Montiel-Ros approach to prove Alexandrov's sphere theorem, on domains whose topological boundary can be locally represented as graph of functions with Sobolev $W^{2,n}$-regularity. Primarily, with the intention of weakening the assumptions originally required by Alexandrov. Moreover, in this approach, it is necessary to apply the area formula in the Sobolev $W^{1,n}$-case, which makes our goal interesting from another perspective. In fact, as explained in the following subsection, the validity of the area formula is critical in the Sobolev $W^{1,n}$-setting.

We prove Alexandrov's theorem in the fundamental case where φ is the k-th elementary symmetric function σ_k, and for $W^{2,n}$-domains, namely open sets whose topological boundary is locally representable as graph of $(C^0 \cap W^{2,n})$-regular functions, where neither C^1 nor Lipschitz regularity is required.

As a consequence of classical pointwise differentiability results for Sobolev functions (cf. [9]), it is not difficult to show that the boundary $\partial\Omega$ of a $W^{2,n}$-domain can be $\mathcal{H}^n$-almost covered by the union of countably many C^2-hypersurfaces (cf. Lemma 3.1.5 *(i)*). Moreover, it can be touched from both the inside and the outside by mutually tangent balls at $\mathcal{H}^n$-a.e. points (cf. Lemma 3.1.5), namely the *viscosity boundary*[1] $\partial^v_+\Omega$ of Ω has full measure in $\partial\Omega$. Therefore, an exterior unit normal ν_Ω is well defined, and it is also approximately differentiable at $\mathcal{H}^n$-a.e. points, with a symmetric approximate differential (cf. Lemma 3.1.2). We also notice that every $W^{2,n}$-domain Ω is a set of locally finite perimeter, that $\partial^v_+\Omega$ is contained in the essential boundary of Ω and ν_Ω is the measure-theoretic exterior normal.

For a $W^{2,n}$-domain we denote by $\chi_{\Omega,1}(p) \leq \ldots \leq \chi_{\Omega,n}(p)$ the eigenvalues of the approximate differential $\operatorname{ap} D\nu_\Omega$ of ν_Ω at $\mathcal{H}^n$-a.e. $p \in \partial\Omega$, and we define the k-th mean curvature function $H_{\Omega,k}$ and the total k-th mean curvature $\mathcal{A}_k(\Omega)$ as in the smooth case.

Fine properties of gradients of $W^{2,n}$-functions

An important part of our work is based on the analysis of fine properties of $W^{2,n}$-functions. If we consider $U \subseteq \mathbb{R}^n$ an open set and $f \in C^0(U) \cap W^{2,n}(U)$, first we notice that the set $\mathcal{S}(f)$, of points where f is twice pointwise differentiable, has full $\mathcal{L}^n$-measure in U (cf. Theorem 2.1.7). This is a result obtained by Calderón-Zygmund (cf. [9] or [8, Proposition 2.2]), and we provide an alternative proof based on the methods used by Trudinger, in [59, Theorem 1], to treat the second-order differentiability of viscosity solutions of second-order elliptic PDE's. A first result in our work, which relies on oscillation estimates obtained by Ulrich Menne (cf. [37, Appendix B]) and is based on a Rado-Reichelderfer type argument (cf. [32] and references therein), is that *the graph of ∇f satisfies the Lusin (N)-property on $\mathcal{S}(f)$* (cf. Lemma 2.1.15 and Remark 2.1.16), that is to say

$$\mathcal{H}^n\big(\overline{\nabla f}(Z)\big) = 0 \quad \text{for every } Z \subset \mathcal{S}(f) \text{ such that } \mathcal{L}^n(Z) = 0 \tag{0.0.4}$$

where $\overline{\nabla f}(x) := \big(x, \nabla f(x)\big)$. Note that, since $\mathcal{S}(f)$ has full $\mathcal{L}^n$-measure in U, it follows that $\nabla f = \boldsymbol{\nabla} f$ $\mathcal{L}^n$-a.e. in U (cf. Remark 2.1.13), where ∇f and $\boldsymbol{\nabla} f$ represent, respectively, the classical gradient and the distributional gradient of f. From (0.0.4), a fundamental tool for our analysis follows, namely *the area formula for $\overline{\nabla f}$ on $\mathcal{S}(f)$* given in (0.0.5). Additionally, we observe that every $W^{2,n}$-function is Monge-Ampère (cf. Lemma 2.3.29), and therefore we can associate to f a unique n-dimensional current $[df] \in \mathcal{D}_n(U \times \mathbb{R}^n)$ that satisfies the conditions in Definition 2.3.27. In conclusion, we have the following.

Theorem A (cf. Theorem 2.3.30). *Given $U \subseteq \mathbb{R}^n$ an open set and $f \in C^0(U) \cap W^{2,n}_{loc}(U)$, then*

$$\mathcal{H}^n\big(\overline{\nabla f}(E)\big) = \int_E J_n \overline{\nabla f}\, d\mathcal{L}^n \tag{0.0.5}$$

for every $\mathcal{L}^n$-measurable set $E \subseteq \mathcal{S}(f)$. Moreover $\overline{\nabla f}\big(\mathcal{S}(f) \cap K\big)$ is $\mathcal{H}^n$-rectifiable for every $K \subset U$ compact,

$$[df] = \big[\mathcal{H}^n \llcorner \overline{\nabla f}\big(\mathcal{S}(f)\big)\big] \wedge (\vec{\eta}_f \circ \pi_0) \tag{0.0.6}$$

where the n-vectorfield $\vec{\eta}_f$ is defined as

$$\vec{\eta}_f(x) := \frac{\big(\boldsymbol{e}_1, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_1\big) \wedge \ldots \wedge \big(\boldsymbol{e}_n, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_n\big)}{\big|\big(\boldsymbol{e}_1, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_1\big) \wedge \ldots \wedge \big(\boldsymbol{e}_n, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_n\big)\big|} \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in U\,,$$

where $\mathbf{D}$ denotes the distributional differential for Sobolev functions.

[1]The *viscosity boundary* $\partial^v_+\Omega$ of Ω is the set of all points $p \in \partial\Omega$ for which there exists $\nu \in S^n$ and $r > 0$ such that $B_r(p + r\nu) \cap \Omega = \emptyset$ and $B_r(p - r\nu) \subseteq \Omega$.

From representation (0.0.6), it follows that:

$$[df] \text{ is carried by } \overline{\nabla f}(\mathcal{S}(f)),$$
$$\text{and it is not possible to replace } \overline{\nabla f}(\mathcal{S}(f)) \text{ with } \overline{\nabla f}(U),$$
$$\text{even if } f \text{ were } C^1\text{-regular.}$$

To prove the above, first we observe that Tomas Roskovec (cf. [48]), using a Cesari-type construction, provides an example of a function $f \in C^1([-1,1]^n)$ such that

$$\nabla f \in W^{1,n}((-1,1)^n;\mathbb{R}^n) \quad \text{and} \quad [-1,1]^n \subseteq \nabla f([-1,1] \times \{0\}^{n-1}).$$

In other words, ∇f is a $(C^0 \cap W^{1,n})$-regular vector field, but it does not satisfy the Lusin (N)-property since it maps a segment into an n-cube. Naturally, $\overline{\nabla f}([-1,1] \times \{0\}^{n-1})$ will also have positive $\mathcal{H}^n$-measure. Taking into account that $\overline{\nabla f}$ satisfies the Lusin (N)-property on $\mathcal{S}(f)$, we deduce that

$$\mathcal{H}^n\Big(\overline{\nabla f}((-1,1)^n) \setminus \overline{\nabla f}(\mathcal{S}(f))\Big) \geq \mathcal{H}^n\Big(\overline{\nabla f}([-1,1] \times \{0\}^{n-1})\Big) > 0,$$

from which we conclude.

Legendrian cycles and sphere theorem

In the context of describing our proof of Alexandrov's sphere theorem, we first provide a precise definition of $W^{2,n}$-domains.

Definition A. *An open set $\Omega \subseteq \mathbb{R}^{n+1}$ is a $W^{2,n}$-domain if and only there exists a pair (Ω', F), that satisfies the following properties:*

(i) *$\Omega' \subseteq \mathbb{R}^{n+1}$ is an open set such that for each $p \in \partial\Omega'$ there exist $\epsilon > 0$, $\nu \in \mathbb{S}^n$, a bounded set U open in $\nu^\perp$ with $0 \in U$ and $f \in C^0(U) \cap W^{2,n}(U)$ with $f(0) = 0$ such that*

$$\big\{p + b + \tau\nu : b \in U,\ -\epsilon < \tau \leq f(b)\big\} = \overline{\Omega'} \cap \big\{p + b + \tau\nu : b \in U,\ -\epsilon < \tau < \epsilon\big\};$$

(ii) *F is a C^2-diffeomorphism defined over an open set $V \subseteq \mathbb{R}^{n+1}$, where $\overline{\Omega'} \subseteq V$;*

(iii) *$F(\Omega') = \Omega$.*

The definition above incorporates two key conditions: first, that such domains be locally representable as cato-graph of $(C^0 \cap W^{2,n})$-functions, allowing us to study them through fine properties of local graphs; and second, that they be invariant under the images of diffeomorphisms, which is evidently a necessary condition for generalizing Reilly's variational formulas in (0.0.3). However, we are uncertain whether it is truly essential to introduce the diffeomorphism F in that definition. In other words, if Ω' belongs to the class $\mathcal{S}$ of domains that satisfy only condition *(i)* in Definition A, is it then true that $F(\Omega')$ also belongs to $\mathcal{S}$?

Such domains can be highly singular. To provide an example, consider the following construction by Tatiana Toro (cf. [58, Example 2]). Given a countable dense subset $\{x_i\}_{i\in\mathbb{N}}$ of $B^n_{1/4}(0)$ and the function $f^{\mathrm{T}} \in W^{2,n}(B^n_{1/2}(0))$, for $n \geq 2$, defined by

$$f^{\mathrm{T}}(x) := x \bullet \boldsymbol{e}_1 \ \ln\big|\ln|x|\big| \ \sin\big(\ln\big|\ln|x|\big|\big) \quad \text{if } x \in B^n_{1/2}(0) \setminus \{0\}.$$

Then, by the completeness of Sobolev spaces, the function

$$f(x) := \sum_{i=1}^{\infty} 2^{-i} f^{\mathrm{T}}(x - x_i) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in B^n_{1/4}(0)$$

belongs to $W^{2,n}(B^n_{1/4}(0))$ and has a countable dense set of singular points.

Now we discuss in detail the generalization, of the Montiel-Ros method (cf. [39]), to $W^{2,n}$-domains. This approach, as previously specified, is based on the Minkowski-Hsiung identities and the Heintze-Karcher inequality. The most delicate part is the extension of the

Minkowski-Hsiung formulas to $W^{2,n}$-domains, which can be obtained thanks to a new and fundamental result on the structure of the Legendrian cycle associated with a $W^{2,n}$-domain. We recall that an integer multiplicity locally rectifiable n-current T of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, with support in $\mathbb{R}^{n+1} \times \mathbb{S}^n$, is called *Legendrian cycle* of $\mathbb{R}^{n+1}$ if $\partial T = 0$ and $T \llcorner \alpha = 0$, where α is the contact 1-form of $\mathbb{R}^{n+1}$ (cf. Definition 1.3.4). It is easy to see that the (exterior) unit normal-bundle of a C^2-domain carries a Legendrian cycle. For readers familiar with the theory of Monge-Ampère functions, it would not be surprising to assert that Legendrian cycles can be associated with $W^{2,n}$-domains (cf. [16] and [26]). On the other hand, this information is not sufficient for our purpose. What we need is much more precise information, namely that the Legendrian cycle over a $W^{2,n}$-domain is carried by the *proximal unit normal bundle* $\mathrm{nor}(\Omega)$ of Ω, which is defined as

$$\mathrm{nor}(\Omega) := \big\{(x,u) \in \partial\Omega \times \mathbb{S}^n : \mathrm{dist}(x+su,\Omega) = s \text{ for some } s > 0\big\}. \tag{0.0.7}$$

It does not seem possible to deduce this fact from classical results for Monge-Ampère functions. Moreover, an analogous result holds for sets of positive reach (cf. [60]). However, while the proof for the latter is based on the regularity of the level sets of the distance function, our proof for $W^{2,n}$-domains, which is exactly the content of the next theorem, is completely different and makes use of certain fine properties of the gradients of $W^{2,n}$-regular functions (some of which have been mentioned previously).

Theorem B (cf. Theorem 3.1.7)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded $W^{2,n}$-domain, then $\mathrm{nor}(\Omega)$ has finite $\mathcal{H}^n$-measure and there exists an unique n-dimensional Legendrian cycle T such that*

$$T = \big(\mathcal{H}^n \llcorner \mathrm{nor}(\Omega)\big) \wedge \vec{\eta}\,,$$

where $\vec{\eta}$ is a $\big(\mathcal{H}^n \llcorner \mathrm{nor}(\Omega)\big)$-measurable n-vectorfield such that:

$$|\vec{\eta}(x,u)| = 1\,, \quad \vec{\eta}(x,u) \text{ is simple}\,,$$

$$\mathrm{Tan}^n\big(\mathcal{H}^n \llcorner \mathrm{nor}(\Omega), (x,u)\big) \text{ is associated with } \vec{\eta}(x,u)$$

and

$$\langle [\textstyle\bigwedge_n \pi_0]\big(\vec{\eta}(x,u)\big) \wedge u, dX_1 \wedge \ldots \wedge dX_{n+1}\rangle > 0\,,$$

for $\mathcal{H}^n$-a.e. $(x,u) \in \mathrm{nor}(\Omega)$.

If Ω were a domain whose topological boundary is locally representable as a graph of $W^{2,p}$-regular functions with $p > n$, then $\mathrm{nor}(\Omega)$ could be replaced by the classical unit-normal bundle (since their difference is $\mathcal{H}^n$-negligible[2]) and Theorem B could be proved relying on well-known properties of $W^{1,p}$-maps (which do not hold even for continuous $W^{1,n}$-maps!) cf. [33]. We particularly notice that, for the $(W^{2,n} \cap C^1)$-regular graph of Roskovec's function, the difference between the classical unit-normal bundle and the proximal unit-normal bundle is a set of positive $\mathcal{H}^n$-measure.

Combining Theorem B with the variational formulae for the curvature measures associated to general Legendrian cycles obtained by Fu in [17] (see also Appendix B), we can extend Reilly variational formulae (cf. [45]) to $W^{2,n}$-domains, hence we deduce the Minkowski-Hsiung formulae in our setting.

Theorem C (cf. Theorem 3.1.15 and Corollary 3.1.17)**.** *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a bounded $W^{2,n}$-domain and $\{F_t\}_{t\in(-\epsilon,\epsilon)}$ is a local variation of $\mathbb{R}^{n+1}$ with initial velocity vector field V. Then*

$$\frac{d}{dt}\mathcal{A}_{k-1}(F_t(\Omega))\Big|_{t=0} = (n-k+1)\int_{\partial\Omega} \nu_\Omega(x) \bullet V(x)\, H_{\Omega,k}(x)\, d\mathcal{H}^n \qquad \text{for } k \in \{1,\ldots,n\}$$

and

$$\frac{d}{dt}\mathcal{A}_n(F_t(\Omega))\Big|_{t=0} = 0\,.$$

In particular

$$\int_{\partial\Omega} H_{\Omega,r-1}(x)\, d\mathcal{H}^n(x) = \int_{\partial\Omega} x \bullet \nu_\Omega(x)\, H_{\Omega,r}(x)\, d\mathcal{H}^n(x)$$

[2] Note that the proximal unit-normal bundle is a subset of the classical unit-normal bundle, but their difference can have positive $\mathcal{H}^n$-measure even for C^1-regular domains.

for $k \in \{1, \dots, n\}$*.*

As for the Heintze-Karcher inequality for $W^{2,n}$-domains, it can be derived from the general inequality [24, Theorem 3.20] by exploiting certain fine properties of the normal bundle (cf. Theorem 3.1.7*(i)-(iii)*) already used to prove Theorem B.

Theorem D (cf. Theorem 3.2.19)**.** *Given* $\Omega \subset \mathbb{R}^{n+1}$ *a bounded and connected* $W^{2,n}$*-domain such that* $H_{\Omega,1}(z) \geq 0$ *for* $\mathcal{H}^n$*-a.e.* $z \in \partial\Omega$*, then*

$$(n+1)\,\mathcal{L}^{n+1}(\Omega) \leq \int_{\partial\Omega} \frac{1}{H_{\Omega,1}(x)}\, d\mathcal{L}^n(x)\,.$$

Moreover, if $H_{\Omega,1}(z) \geq \frac{\mathcal{H}^n(\partial\Omega)}{(n+1)\,\mathcal{L}^{n+1}(\Omega)}$ *for* $\mathcal{H}^n$*-a.e.* $z \in \partial\Omega$ *then* Ω *is a round ball.*

We have now all the ingredients to prove the following generalization of Alexandrov's sphere theorem.

Theorem E (cf. Theorem 3.2.20 and Remark 3.2.21)**.** *A bounded and connected* $W^{2,n}$*-domain* $\Omega \subset \mathbb{R}^{n+1}$ *must be a round ball, provided there exist* $k \in \{2, \dots, n\}$ *and* $\lambda \in \mathbb{R}$ *such that*

$$\sigma_k(\chi_{\Omega,1}(p), \dots, \chi_{\Omega,n}(p)) = \lambda$$

and

$$\partial_i\sigma_k(\chi_{\Omega,1}(p), \dots, \chi_{\Omega,n}(p)) \geq 0 \quad \text{for } i \in \{1, \dots, n\}\,, \tag{0.0.8}$$

for $\mathcal{H}^n$*-a.e.* $p \in \partial\Omega$*.*

Theorem E contains a new statement only for $k \geq 2$. Indeed, if $k = 1$, this result reduces to the smooth Alexandrov's sphere theorem for hypersurfaces with constant mean curvature, since the condition $H_{\Omega,1}(z) = \lambda$ for $\mathcal{H}^n$-a.e. $z \in \partial\Omega$ implies that $\partial\Omega$ is smooth by Allard's regularity theorem (notice that, according to Theorem 3.1.15, the function $H_{\Omega,1}$ is the generalized mean curvature of $\partial\Omega$ in the sense of varifolds; cf. [3]).

Regarding the extension of the Montiel-Ros method to $W^{2,n}$-domains, we note that there are already generalizations of this approach to sets of *positive reach* (cf. [24]). However, the singularities of Sobolev-type domains and sets of positive reach are of a very different nature (the latter being analogous to the singularities of convex bodies). Therefore, it was necessary to develop new techniques to extend the Montiel-Ros approach in the context of $W^{2,n}$-domains.

These results were presented in [55].

The support of Legendrian cycles

Theorem B finds a natural application in other problems, beyond the rigidity questions we have considered so far. In Section 2.2, we use it to answer a question implicit in [43]. In [43, Remark 2.3], the authors ask whether there exist n-dimensional Legendrian cycles in $\mathbb{R}^{n+1}$ whose support is not locally $\mathcal{H}^n$-rectifiable, or even has positive $\mathcal{H}^{n+1}$-measure. By combining Theorem B with an observation by J. Fu in [15] about the existence of $W^{2,n}$-functions whose differential has a graph dense in $\mathbb{R}^n \times \mathbb{R}^n$, we prove the following result.

Theorem F. *There exist n-dimensional Legendrian cycles of* $\mathbb{R}^{n+1}$ *whose support has positive* $\mathcal{H}^{2n}$*-measure.*

The proof of the previous result is based on two arguments. First, given a set $C \subseteq \mathbb{R}^{n+1}$ we note that $\big(\mathcal{H}^n \llcorner \operatorname{nor}(C)\big)\big(B^{2n+2}_r(z)\big) > 0$ for every $z \in \operatorname{nor}(C)$ and for every $r > 0$ (cf. Lemma 2.2.22 and Remark 2.2.23), where $B^{2n+2}_r(z)$ is the open ball in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ with centre in z and radius $r > 0$, and this implies that

$$\operatorname{nor}(C) \subseteq \operatorname{spt}\big(\mathcal{H}^n \llcorner \operatorname{nor}(C)\big)\,. \tag{0.0.9}$$

Then, we consider $f \in C^0(U) \cap W^{2,n}(U)$ such that $\overline{\nabla f}(\operatorname{Diff}(f))$ is a dense subset of $U \times \mathbb{R}^n$ (cf. [15, p. 2260]), where $U \subset \mathbb{R}^n$ is a bounded open set and $\operatorname{Diff}(f)$ is the set of points where

f is pointwise differentiable. We also consider the Legendrian cycle of $U \times \mathbb{R}$ associated with $\Gamma := \mathrm{graph}(f)$, which is given by (cf. (2.1.30) in Remark 2.1.21)

$$\mathcal{N}_\Gamma := \left(\mathcal{H}^n \llcorner N(\Gamma)\right) \wedge \vec{\eta}_\Gamma \,,$$

where $N(\Gamma) := \mathrm{nor}(\Gamma) \cap (U \times \mathbb{R} \times \mathbb{R}^{n+1})$ and $\vec{\eta}_\Gamma$ is a Borel n-vectorfield such that

$$|\vec{\eta}_\Gamma(z,\nu)| = 1 \,, \quad \vec{\eta}_\Gamma(z,\nu) \text{ is simple}\,,$$

$$\mathrm{Tan}^n\left(\mathcal{H}^n \llcorner N(\Gamma), (z,\nu)\right) \text{ is associated with } \vec{\eta}_\Gamma(z,\nu)$$

and

$$\langle [\Lambda_n \pi_0] \left(\vec{\eta}_\Gamma(z,\nu)\right) \wedge \nu \,, dX_1 \wedge \ldots \wedge dX_{n+1} \rangle > 0$$

for $\mathcal{H}^n$-a.e. $(z,\nu) \in N(\Gamma)$. From (0.0.9), we infer that

$$N(\Gamma) \subseteq \mathrm{spt}(\mathcal{N}_\Gamma) \subseteq \Gamma \times \mathbb{S}^n$$

where $N(\Gamma)$ is dense in $\Gamma \times \mathbb{S}^n$ (cf. Lemma 2.2.24), so $\mathrm{spt}(\mathcal{N}_\Gamma) = \Gamma \times \mathbb{S}^n$. Since $\mathcal{H}^{2n}(\Gamma \times \mathbb{S}^n) > 0$, we obtain the desidered result.

This result was presented in [55].

Legendrian cycles on $F_n W^{2,n}$-sets

Ambrosio, Gobbino and Pallara (cf. [4]), based on an idea of De Giorgi, introduce a notion of Sobolev-type manifold denoted by $F_d W^{2,p}$, whose members are (locally) finite union of graphs of $(C^0 \cap W^{2,p})$-functions of d variables. More precisely, $\mathcal{S} \subset \mathbb{R}^{n+1}$ is an *$F_d W^{2,p}$-set* if

$$\mathcal{S} = \{\boldsymbol{\iota} > 0\}\,,$$

where $\boldsymbol{\iota} : \mathbb{R}^{n+1} \to \mathbb{N}$ is a function such that, for every $z \in \mathbb{R}^{n+1}$ where $\boldsymbol{\iota}(z) > 0$, there exist a positive integer $q(z)$ and an open neighborhood U of z such that

$$\boldsymbol{\iota}(x) = \sum_{i=1}^{q(z)} \mathbf{1}_{\Gamma_i}(x) \quad \text{for any } x \in U\,,$$

where every $\Gamma_i \cap U$ coincides with the graph of a $(C^0 \cap W^{2,p})$-function in d variables. In particular, $F_d W^{2,p}$-sets form a class of *curvature varifolds* (cf. [4, Remark 1.7]) in the sense of Hutchinson (cf. [25]). This means that, given $\mathcal{S} \subset \mathbb{R}^{n+1}$, an $F_d W^{2,p}$-set with multiplicity $\boldsymbol{\iota}$, then $V = \boldsymbol{v}(\mathcal{S}, \boldsymbol{\iota})$ is a curvature varifold (cf. [25, Definition 5.2.1]).

Definition B. *We say that that a closed set $\mathcal{S} \subset \mathbb{R}^{n+1}$ is a $\mathscr{W}^{2,n}$-set if there exists a pair $(\mathcal{S}', F)$, that satisfies the following properties:*

(i) $\mathcal{S}'$ is a $F_n W^{2,n}$-set;

(ii) $F(\mathcal{S}') = \mathcal{S}$, where F is a C^2-diffeomorphism of $\mathbb{R}^{n+1}$.

As in the definition A, this class of domains is invariant under the images of C^2 diffeomorphisms, which is a necessary condition to generalize Reilly's variational formulas.

Given a $\mathscr{W}^{2,n}$-set $\mathcal{S}$, it is possible to consider an $\mathcal{H}^n$-measurable unit vector field $\nu_\mathcal{S}$ such that $\nu_\mathcal{S}(p) \in \mathrm{Nor}^n(\mathcal{H}^n \llcorner \mathcal{S}, p)$ for $\mathcal{H}^n$-a.e. $p \in \mathcal{S}$ (cf. section 4.2). Furthermore, $\nu_\mathcal{S}$ is approximately differentiable at $\mathcal{H}^n$-a.e. points of $\mathcal{S}$ with a symmetric approximate differential (cf. Lemma 3.1.2). As usual, we denote by $\chi_{\mathcal{S},1}(p) \leq \ldots \leq \chi_{\mathcal{S},n}(p)$ the eigenvalues of $\mathrm{ap}\, D\nu_\mathcal{S}(p)$ at $\mathcal{H}^n$-a.e. $p \in \mathcal{S}$, and we define the k-th mean curvature $H_{\mathcal{S},k}$ and the total k-th mean curvature $\mathcal{A}_k(\mathcal{S})$ as in the smooth case.

As for $W^{2,n}$-domains, the most delicate part of the results achieved on $\mathscr{W}^{2,n}$-sets is to obtain a structure of Legendrian cycles on them. First, given a $\mathscr{W}^{2,n}$-set $\mathcal{S}$ with associated pair $(\mathcal{S}', F)$, we introduce the C^1-diffeomorphism

$$\Psi_F(x,u) := \left(F(x), \frac{(DF(x)^{-1})^*(u)}{|(DF(x)^{-1})^*(u)|}\right) \quad \text{for every } (x,u) \in \mathbb{R}^{n+1} \times \mathbb{S}^n$$

for which we have $\Psi_F\big(\mathrm{nor}(\mathcal{S}')\big) = \mathrm{nor}(\mathcal{S})$ (cf. [54, Lemma 2.1]). Then, the following version of Theorem B about $\mathscr{W}^{2,n}$-sets holds true.

Theorem G (cf. Theorem 4.3.22)**.** *Given a compact $\mathscr{W}^{2,n}$-set $\mathcal{S}$ with associated pair $(\mathcal{S}', F)$, then the integer multiplicity rectifiable current $T \in \mathcal{D}_n(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ given by*

$$T := \big(\boldsymbol{\iota} \circ \pi_0 \circ (\Psi_F|\mathrm{nor}(\mathcal{S}'))^{-1}\big)\,\big(\mathcal{H}^n \llcorner \mathrm{nor}(\mathcal{S})\big) \wedge \vec{\xi}_{\mathcal{S}}$$

is a Legendrian cycle, where $\pi_0 : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the canonical projection on the first factor and $\vec{\xi}_{\mathcal{S}}$ is an $\big(\mathcal{H}^n \llcorner \mathrm{nor}(\mathcal{S})\big)$-measurable n-vectorfield such that:

$$|\vec{\xi}_{\mathcal{S}}(x,u)| = 1\,, \quad \vec{\xi}_{\mathcal{S}}(x,u) \text{ is simple}\,,$$

$$\mathrm{Tan}^n\big(\mathcal{H}^n \llcorner \mathrm{nor}(\mathcal{S}), (x,u)\big) \text{ is associated with } \vec{\xi}_{\mathcal{S}}(x,u)$$

and

$$\langle [\textstyle\bigwedge_n \pi_0]\big(\vec{\xi}_{\mathcal{S}}(x,u)\big) \wedge u, dX_1 \wedge \ldots \wedge dX_{n+1}\rangle > 0$$

for $\mathcal{H}^n$-a.e. $(x,u) \in \mathrm{nor}(\mathcal{S})$.

Combining Theorem G with the variational formulae for the curvature measures associated to general Legendrian cycles obtained by Fu in [17] (see also Appendix B), we can extend Reilly variational formulae (cf. [45]) to $\mathscr{W}^{2,n}$-sets.

Theorem H (cf. Theorem 4.3.24)**.** *Let $\mathcal{S}$ be a compact $\mathscr{W}^{2,n}$-set and let $\{F_t\}_{t\in(-\epsilon,\epsilon)}$ be a local variation of $\mathbb{R}^{n+1}$, with initial velocity vector field V. If $k \in \{1,\ldots,n\}$ is odd, we have*

$$\frac{d}{dt}\mathcal{A}_{k-1}\big(F_t(\mathcal{S})\big)\Big|_{t=0} = (n-k+1)\int_{\mathcal{S}} V(x) \bullet \nu_{\mathcal{S}}(x)\, H_{\mathcal{S},k}(x)\, \boldsymbol{\iota}\big(F^{-1}(x)\big)\, d\mathcal{H}^n(x)\,.$$

Moreover, if n is even

$$\frac{d}{dt}\mathcal{A}_n\big(F_t(\mathcal{S})\big)\Big|_{t=0} = 0\,.$$

The Nabelpunktsatz

In the final chapter, we study the problem of extending the *umbilicality theorem* (or Nabelpunktsatz) from smooth hypersurfaces to those with Sobolev regularity. The classical proof of this theorem works for hypersurfaces that are at least C^3-regular. A proof for C^2-hypersurfaces is given in [57] (see also [40] or [34]), and in the case of C^2-regular graphs, the theorem takes the following form.

> *Suppose $U \subseteq \mathbb{R}^n$ is a connected open set, $f \in C^2(U)$, $\overline{f}(x) := \big(x, f(x)\big)$ and $\Gamma := \mathrm{graph}(f)$. We also consider $\nu : \Gamma \to \mathbb{S}^n$, the unit normal vector field to Γ, given by*
>
> $$\nu\big(\overline{f}(x)\big) := \frac{(-\nabla f(x), 1)}{\sqrt{1+|\nabla f(x)|^2}} \quad \text{for any } x \in U\,.$$
>
> *If Γ satisfies the umbilicality condition, namely there exists a function $\mu : \Gamma \to \mathbb{R}$ such that*
>
> $$D\nu(z)(\tau) = \mu(z)\tau \quad \text{for every } \tau \in \mathrm{Tan}(\Gamma, z) \text{ and every } z \in \Gamma\,, \tag{0.0.10}$$
>
> *then Γ is contained, either in an n-dimensional plane or in an n-dimensional sphere.*

Considering more general hypersurfaces, with curvatures defined only almost everywhere, the question of the validity of the Nabelpunktsatz goes back to the classical paper by Busemann and Feller [7], where they also highlight the existence of convex C^1-regular hypersurfaces that are umbilical at almost every point (see also Remark 5.3.5). In [11], it was observed that the Nabelpunktsatz still holds for $C^{1,1}$-hypersurfaces.

In the context of generalizing the previous results, we first note that if $f \in C^2(U)$ where $U \subseteq \mathbb{R}^n$ is a connected open set, the umbilicality condition (0.0.10) on $\Gamma = \mathrm{graph}(f)$ can be written in the following form

$$\mu(\overline{f}(x))\left[\boldsymbol{e}_i \bullet \boldsymbol{e}_j + D_i f(x) D_j f(x)\right] = -\frac{D^2_{ij} f(x)}{\sqrt{1+|\nabla f(x)|^2}}\,,$$

for every $x \in U$ and for every $i, j \in \{1, \dots, n\}$, where $D_i f$ and $D^2_{ij} f$ denotes the classical partial derivatives of f. Considering this condition in a weak sense, we obtain the following broad generalization.

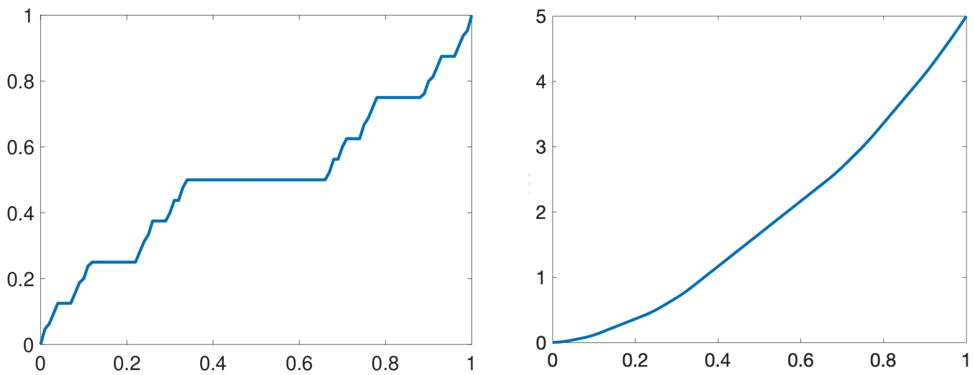
Theorem I (cf. Theorem 5.2.2)**.** *Suppose $U \subseteq \mathbb{R}^n$ is a connected open set, $f \in W^{2,1}_{loc}(U)$ and $\mu : U \to \mathbb{R}$ is a function such that for $\mathcal{L}^n$-a.e. $x \in U$ and for every $i, j \in \{1, \dots, n\}$, the following condition holds*

$$\mu(x)\left[\boldsymbol{e}_i \bullet \boldsymbol{e}_j + \mathbf{D}_i f(x) \mathbf{D}_j f(x)\right] = -\frac{\mathbf{D}^2_{ij} f(x)}{\sqrt{1+|\boldsymbol{\nabla} f(x)|^2}}\,, \tag{0.0.11}$$

where $\boldsymbol{\nabla} f$, $\mathbf{D}_i f$ and $\mathbf{D}^2_{ij} f$ denote the distributional gradient, the distributional partial derivatives and the second-order distributional partial derivatives of the Sobolev function f, respectively.

Then, either f is $\mathcal{L}^n$-a.e. equal to a linear function on U, or there exists a n-dimensional sphere S in $\mathbb{R}^{n+1}$ such that $\overline{f}(x) \in S$ for $\mathcal{L}^n$-a.e. $x \in U$.

We also note that the assumption of $W^{2,1}_{loc}$-regularity is essential. To justify this assertion, we consider the following example. Let $\mathcal{C} \subset [0,1]$ be the Cantor ternary set and $f : [0,1] \to [0,1]$ the Cantor-Vitali function. Recall that $f \in C^{0,\alpha}([0,1])$ with $\alpha = \log_3 2$ and $f(\mathcal{C}) = [0,1]$, in particular it is also increasing with $f(0) = 0$, $f(1) = 1$ and, finally, $f'(x) = 0$ for every x in the open set $[0,1] \setminus \mathcal{C}$. The function f provides an example of a BV-function that is not absolutely continuous, namely $f \in BV(0,1) \setminus W^{1,1}(0,1)$. In fact, since f is increasing, we deduce that the total variation of f is 1. Furthermore, since $\mathcal{L}^1(\mathcal{C}) = 0$ and $f(\mathcal{C}) = [0,1]$, we infer that f does not satisfy the Lusin (N)-property and hence is not absolutely continuous (cf. [31, Theorem 3.41]). If we now consider the primitive of f, denoted by F, we deduce that $F \in C^{1,\alpha}([0,1]) \setminus W^{2,1}(0,1)$ and since F is piecewise linear in $[0,1] \setminus \mathcal{C}$, we conclude that F satisfies condition (0.0.11) but not the statement of Theorem I.



Graph of the ternary Cantor function (left) and its primitive (right).

Notice that the umbilicality condition (0.0.11) acts $\mathcal{L}^n$-a.e. on f, in terms of a strong solution of a system of elliptic PDE's, and does not involve the concept of curvature on $\Gamma = \operatorname{graph}(f)$. To give a geometric interpretation of condition (0.0.11), that is, in terms of weak curvatures, the necessary assumption is that $\overline{f}$ satisfies the Lusin (N)-property, namely

$$\mathcal{H}^n\left(\overline{f}(E)\right) = 0 \quad \text{whenever } E \subset U \text{ is } \mathcal{L}^n\text{-negligible}\,.$$

Such condition is satisfied if $f \in W^{2,p}_{loc}(U)$, with $p > \frac{n}{2}$ (cf. Remark 5.3.4), and guarantees that Γ is locally $\mathcal{H}^n$-rectifiable of class 2 and that there exists an $(\mathcal{H}^n \llcorner \Gamma)$-measurable map ν such that, for $\mathcal{H}^n$-a.e. $x \in \Gamma$, we have $\nu(x) \in \operatorname{Nor}^n(\mathcal{H}^n \llcorner \Gamma, x) \cap \mathbb{S}^n$; moreover ν is $(\mathcal{H}^n \llcorner \Gamma)$-approximately differentiable at x and $\operatorname{ap} D\nu(x)$ is a symmetric endomorphism of $\operatorname{Tan}^n(\mathcal{H}^n \llcorner \Gamma, x)$, whose eigenvalues, as usual, represent the approximate principal curvatures of Γ. Overall, the following result holds.

Theorem L (cf. Theorem 5.3.3). *Let $U \subset \mathbb{R}^n$ be a bounded and connected open set, $f \in W^{2,1}_{loc}(U)$, $\Gamma := \mathrm{graph}(f)$ and suppose that $\overline{f}(x) := \big(x, f(x)\big)$ satisfies the Lusin's (N)-condition. Assume also that there exists an $(\mathcal{H}^n \llcorner \Gamma)$-measurable map ν such that $\nu(x) \in \mathrm{Nor}^n(\mathcal{H}^n \llcorner \Gamma, x) \cap \mathbb{S}^n$ and there exists a function $\mu : \Gamma \to \mathbb{R}$ such that*

$$\mathrm{ap}\, D\nu(x)(\tau) = \mu(x)\tau \quad \text{for every } \tau \in \mathrm{Tan}^n(\mathcal{H}^n \llcorner \Gamma, x)\,, \tag{0.0.12}$$

for $\mathcal{H}^n$-a.e. $x \in \Gamma$.

Then Γ is $\mathcal{H}^n$-rectifiable of class 2 and, up to a $\mathcal{H}^n$-negligible set, either Γ is a subset of a n-dimensional plane or a subset of a n-dimensional sphere.

These results were presented in [55].

Chapter 1

Notation and background

Given a set of parameters $\{p_1, p_2, \ldots, p_n\}$, we denote a *generic* positive constant depending only $p_1, \ldots, p_n$ by $c(p_1, \ldots, p_n)$.
If $f : S \to T$ is a function we define

$$\overline{f} : S \to S \times T\,, \qquad \overline{f}(x) := \big(x, f(x)\big)\,. \tag{1.0.1}$$

Moreover, we often use the following projection maps

$$\pi_0 : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \quad \pi_1 : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \tag{1.0.2}$$

defined as $\pi_0(x,u) := x$ and $\pi_1(x,u) := u$.
In this thesis we use the symbol $\bullet$ to denote scalar product. In particular we fix a scalar product $\bullet$ on $\mathbb{R}^{n+1}$ and an orthonormal basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1}\}$ of $\mathbb{R}^{n+1}$. For a subset E of an Euclidean space, $\overline{E}$ is the topological closure of E. We use the symbols $B_r(a) = B^k_r(a)$ for the *open* ball in $\mathbb{R}^k$ centered at a with radius r, and the symbols ∇ and D for the classical gradient and differential. On the other hand, we denote by $\boldsymbol{\nabla} f$ and $\mathbf{D}^k f$ for the distributional gradient and the distributional k-differential of a Sobolev map f. If $f : U \to \mathbb{R}$ is a continuous function defined on an open set U, we denote the set of $x \in U$ where f is pointwise differentiable by $\mathrm{Diff}(f)$. The characteristic function of a set X is $\mathbf{1}_X$. The Grassmannian of m-dimensional subspaces of $\mathbb{R}^k$ is $\mathbf{G}(k,m)$, and if $T \in \mathbf{G}(k,m)$ then $\pi_T : \mathbb{R}^k \to \mathbb{R}^k$ is the orthogonal projection onto T.

1.1 Basic notions from geometric measure theory

In this thesis we use standard notation from geometric measure theory, for which we refer to [14]. For reader's convenience we recall some basic notions here.

Given $X \subset \mathbb{R}^m$ and $a \in \mathbb{R}^m$, we define $\mathrm{Tan}(X,a)$, the *tangent cone* of X at a, as the set of all $v \in \mathbb{R}^m$ such that there exists a sequence $\{a_k\}_{k\in\mathbb{N}} \subset X \setminus \{a\}$ satisfying

$$\lim_{k\to\infty} a_k = a \quad \text{and} \quad \lim_{k\to\infty} \frac{a_k - a}{|a_k - a|} = \frac{v}{|v|}\,.$$

In other words, $\mathrm{Tan}(X,a)$ is the set of all $v \in \mathbb{R}^m$ such that for every $\epsilon > 0$, there exist $x \in X$ and $r > 0$ with $|x - a| < r$ such that $|r(x-a) - v| < \epsilon$. We also define the *normal cone* of X at a, with vertex at 0, as follows

$$\mathrm{Nor}(X,a) := \big\{u \in \mathbb{R}^m : u \bullet v \le 0 \text{ for } v \in \mathrm{Tan}(X,a)\big\}\,.$$

Given $X \subset \mathbb{R}^m$, $a \in \mathbb{R}^m$ and a positive integer μ, we define $\mathrm{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a)$, the $(\mathcal{H}^\mu \llcorner X)$-*approximate tangent cone* of X at a, as the set of all $v \in \mathbb{R}^m$ such that

$$\Theta^{*\mu}\big(\mathcal{H}^\mu \llcorner X \cap \{x : |r(x-a) - v| < \epsilon \text{ for some } r > 0\}, a\big) > 0 \tag{1.1.3}$$

for every $\epsilon > 0$. We also define the $(\mathcal{H}^\mu \llcorner X)$-*approximate normal cone* of X at a, with vertex at 0, as follows

$$\mathrm{Nor}^\mu(\mathcal{H}^\mu \llcorner X, a) := \big\{u \in \mathbb{R}^m : u \bullet v \le 0 \text{ for } v \in \mathrm{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a)\big\}\,.$$

Remark 1.1.1. *(i)* We can provide an equivalent definition of the $(\mathcal{H}^\mu \llcorner X)$-approximate tangent cone $\operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a)$ (cf. [14, 3.2.16]), namely

$$\operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a) = \bigcap \left\{ \operatorname{Tan}(E, a) : E \subseteq X \,,\, \Theta^\mu(\mathcal{H}^\mu \llcorner X \setminus E, a) = 0 \right\}.$$

Clearly, $\operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a)$ *is a subcone of* $\operatorname{Tan}(X, a)$.
(ii) (Locality property of approximate tangent spaces)**.** Let X and Y be two $\mathcal{H}^\mu$-measurable subsets of $\mathbb{R}^m$, each with finite $\mathcal{H}^\mu$-measure. Then, we have the following locality result

$$\operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a) = \operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner Y, a) \quad \text{for } \mathcal{H}^\mu\text{-a.e. } a \in X \cap Y\,. \tag{1.1.4}$$

To prove this, we show that $\operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a) = \operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X \cap Y, a)$ for $\mathcal{H}^\mu$-a.e. $a \in X \cap Y$. Clearly, the inclusion $\operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X \cap Y, a) \subseteq \operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a)$ holds for every $a \in \mathbb{R}^m$ (cf. (1.1.3)). We now prove the reverse inclusion

$$\operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a) \subseteq \operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X \cap Y, a) \quad \text{for } \mathcal{H}^\mu\text{-a.e. } a \in X \cap Y\,.$$

First, from [14, 2.10.19 (4)], we have that

$$\Theta^\mu(\mathcal{H}^\mu \llcorner X \setminus Y, a) = 0 \quad \text{for } \mathcal{H}^\mu\text{-a.e. } a \in X \cap Y\,. \tag{1.1.5}$$

Now, consider an arbitrary $v \in \operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a)$, and define

$$\mathbf{E}(a, v, \epsilon) := \{x \in \mathbb{R}^m : |r(x - a) - v| < \epsilon \text{ for some } r > 0\} \quad \text{for } \epsilon > 0\,.$$

Then, from (1.1.5), we infer that

$$\begin{aligned} &\Theta^{*\mu}\big(\mathcal{H}^\mu \llcorner (X \cap Y) \cap \mathbf{E}(a, v, \epsilon), a\big) \\ &\qquad = \Theta^{*\mu}\big(\mathcal{H}^\mu \llcorner X \cap \mathbf{E}(a, v, \epsilon), a\big) > 0 \quad \text{for every } \epsilon > 0 \end{aligned}$$

for $\mathcal{H}^\mu$-a.e. $a \in X \cap Y$, from which the desidered result follows.

Suppose $X \subset \mathbb{R}^m$ and f maps a subset of $\mathbb{R}^m$ into $\mathbb{R}^k$. Given a positive integer μ and $a \in \mathbb{R}^m$ we say that f is $(\mathcal{H}^\mu \llcorner X)$*-approximately differentiable* at a (cf. [14, 3.2.16]) if and only if there exists a mapping $g : \mathbb{R}^m \to \mathbb{R}^k$ pointwise differentiable at a such that $f(a) = g(a)$ if $a \in \operatorname{dmn}(f)$ and

$$\Theta^\mu\big(\mathcal{H}^\mu \llcorner X \cap \{b : f(b) \neq g(b)\}, a\big) = 0\,.$$

In this case (see [14, 3.2.16]) f determines the restriction of $Dg(a)$ on the approximate tangent cone $\operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a)$ and we define

$$\operatorname{ap} Df(a) := Dg(a) | \operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a)\,.$$

Suppose $X \subset \mathbb{R}^m$ and μ is a positive integer. We say that X is *countably $\mathcal{H}^\mu$-rectifiable* if there exist countably many μ-dimensional C^1-submanifolds Σ_i of $\mathbb{R}^m$ such that

$$\mathcal{H}^\mu\Big(X \setminus \bigcup_{i=1}^\infty \Sigma_i\Big) = 0\,. \tag{1.1.6}$$

In particular, X is said to be *locally $\mathcal{H}^\mu$-rectifiable* if condition (1.1.6) holds and $\mathcal{H}^\mu(X \cap K) < \infty$ for every compact set $K \subset \mathbb{R}^m$. Finally, if (1.1.6) holds and $\mathcal{H}^\mu(X) < \infty$, we simply say that X is *$\mathcal{H}^\mu$-rectifiable*. It is worth mentioning that X is said to be *countably $\mathcal{H}^\mu$-rectifiable of class* k if condition (1.1.6) holds, where the family $\{\Sigma_i\}_{i \in \mathbb{N}}$ consists specifically of μ-dimensional C^k-submanifolds of $\mathbb{R}^m$.

It is well known that if X is countably $\mathcal{H}^\mu$-rectifiable with $\mathcal{H}^\mu(X) < \infty$, then $\operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a)$ is a μ-dimensional plane at $\mathcal{H}^\mu$-a.e. $a \in X$, and every Lipschitz function $f : X \to \mathbb{R}^k$ has an $(\mathcal{H}^\mu \llcorner X)$-approximate differential $\operatorname{ap} Df(a) : \operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a) \to \mathbb{R}^k$ at $\mathcal{H}^\mu$-a.e. $a \in X$. For such points a we define, for each $h \in \{1, \ldots, k\}$, the $(\mathcal{H}^h \llcorner X)$*-approximate Jabobian* of f at a as

$$J_h^X f(a) := \sup \left\{ \big| [\textstyle\bigwedge_h \operatorname{ap} Df(a)](\xi) \big| : \xi \in \textstyle\bigwedge_h \operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a),\ |\xi| = 1 \right\}$$

(see (1.2.7) for the definition of $\bigwedge_h \operatorname{ap} Df(a)$). The approximate Jacobian naturally appears in area and coarea formula for f; cf. [14, 3.2.20, 3.2.22].

1.2 Differential forms and currents

Let V be a vector space. We denote by $v_1 \wedge \ldots \wedge v_m$ the simple m-vector obtained by the exterior multiplication of vectors $v_1, \ldots, v_m$ in V and $\bigwedge_m V$ is the vector space generated by all simple m vectors of V. Each linear map $f : V \to V'$ can be uniquely extended to a linear map

$$\textstyle\bigwedge_m f : \bigwedge_m V \to \bigwedge_m V' \tag{1.2.7}$$

such that $\bigwedge_m f(v_1 \wedge \ldots \wedge v_m) := f(v_1) \wedge \ldots \wedge f(v_m)$ for every $v_1, \ldots, v_m \in V$.

The vector space of all alternating m-linear functions $f : V^m \to \mathbb{R}$ (i.e. $f(v_1, \ldots, v_m) = 0$ whenever $v_1, \ldots, v_m \in V$ and $v_i = v_j$ for some $i \neq j$) is denoted by $\bigwedge^m V$. There is an obvious isomorphism between $\bigwedge^m V$ and the space of all linear $\mathbb{R}$-valued maps on $\bigwedge_m V$. It is often convenient to use the following customary notation (cf. [14])

$$\langle \xi, h \rangle := h(\xi) \quad \text{whenever } \xi \in \textstyle\bigwedge_m V \text{ and } h \in \bigwedge^m V.$$

If V is an inner product space, then both $\bigwedge_m V$ and $\bigwedge^m V$ can be endowed with natural scalar products, whose *associated norms* are denoted by $|\cdot|$ (cf. [14, 1.7.5]).

Suppose $U \subseteq \mathbb{R}^p$ is open and $k \in \mathbb{N}$. A k-form is a smooth map $\phi : U \to \bigwedge^k \mathbb{R}^p$ (if $k = 0$ we set $\bigwedge^0 \mathbb{R}^p = \mathbb{R}$). Following [14, 4.1.1, 4.1.7], we denote by $\mathcal{E}^k(U)$ the space of all smooth *k-forms on U* and we denote by $\mathcal{D}^k(U)$ the space of all smooth *k-forms with compact support in U*. If $\phi \in \mathcal{E}^k(U)$ we denote by $d\phi$ the *exterior derivative* of ϕ (cf. [14, 4.1.6]). If $X_1, \ldots, X_p$ are the coordinate functions of $\mathbb{R}^p$, then

$$\Omega_p := dX_1 \wedge \ldots \wedge dX_p$$

is the standard *volume form* of $\mathbb{R}^p$. Moreover, if f is a smooth function mapping U into $\mathbb{R}^q$ and ψ is a k-form defined on an open subset V of $\mathbb{R}^q$ with $f(U) \subseteq V$, then we define the k-form $f^\# \psi$ on U by the formula

$$\langle v_1 \wedge \ldots \wedge v_k, f^\# \psi(x) \rangle := \langle \textstyle\bigwedge_k Df(x)(v_1 \wedge \ldots \wedge v_k), \psi(f(x)) \rangle$$

for $x \in U$ and $v_1, \ldots, v_k \in \mathbb{R}^p$. We refer to [14, 4.1.6] for the basic properties of $f^\#$. Functions mapping a subset of U into $\bigwedge_k(\mathbb{R}^p)$ are called *k-vectorfields*.

Suppose $U \subseteq \mathbb{R}^p$ is open and $k \in \mathbb{N}$. A k-current is a continuous $\mathbb{R}$-valued linear map on $\mathcal{D}^k(U)$, with respect to the canonical LF-topology (*inductive limit of Fréchet topologies*) described in [14, 4.1.1] and we denote the space of all *k-currents* on U by $\mathcal{D}_k(U)$. We say that a sequence $\{T_\ell\}_{\ell \in \mathbb{N}} \subset \mathcal{D}_k(U)$ *weakly* converges* to $T \in \mathcal{D}_k(U)$, and we write $T_\ell \to T$, if

$$T_\ell(\phi) \to T(\phi) \quad \text{for every } \phi \in \mathcal{D}^k(U)\,.$$

If T is a k-current on U, then *the boundary of T* is the $(k-1)$-current $\partial T \in \mathcal{D}_{k-1}(U)$ given by

$$\partial T(\phi) := T(d\phi) \quad \text{for every } \phi \in \mathcal{D}^k(U)\,,$$

if $\partial T = 0$ we call T a *a cycle*. The *support of T* is defined as

$$\operatorname{spt}(T) := U \setminus \bigcup \{ V : V \subseteq U \text{ open and } T(\phi) = 0 \text{ for every } \phi \in \mathcal{D}^k(V) \}\,.$$

If $T \in \mathcal{D}_k(U)$ has compact support, then T can be naturally extended to a continuous linear map on $\mathcal{E}^k(U)$. If $\psi \in \mathcal{E}^h(U)$, $T \in \mathcal{D}_k(U)$ and $h \le k$ we set

$$(T \llcorner \psi)(\phi) := T(\psi \wedge \phi) \quad \text{for every } \phi \in \mathcal{D}^{k-h}(U)\,.$$

If $T \in \mathcal{D}_k(U)$, V is an open subset of $\mathbb{R}^q$ and $f : U \to V$ is a smooth map such that $f | \operatorname{spt}(T)$ is proper, then noting that $\operatorname{spt} f^\# \phi \subseteq f^{-1}(\operatorname{spt} \phi)$ and $f^{-1}(\operatorname{spt} \phi) \cap \operatorname{spt} T$ is a compact subset of U for each $\phi \in \mathcal{D}^k(V)$, we define $f_\# T \in \mathcal{D}_k(V)$ by the formula

$$f_\# T(\phi) := T\big[\gamma \wedge f^\# \phi\big] \tag{1.2.8}$$

whenever $\phi \in \mathcal{D}^k(V)$ and $\gamma \in \mathcal{D}^0(U)$ with $f^{-1}(\operatorname{spt}\phi) \cap \operatorname{spt} T \subseteq \operatorname{Int}\big(\gamma^{-1}(\{1\})\big)$. If $\operatorname{spt}(T)$ is a compact subset of U then $f_\# T(\phi) = T(f^\# \phi)$ whenever $\phi \in \mathcal{E}^k(V)$. We refer to [14, 4.1.7] for the basic properties of the map $f_\#$.

If $W \subseteq U$ is an open subset, we define the *mass* of a k-current $T \in \mathcal{D}_k(U)$ on W as

$$\mathbf{M}_W(T) := \sup\left\{T(\phi) : \phi \in \mathcal{D}^k(U)\,,\ \sup_{x \in U} |\phi(x)| \le 1\,,\ \operatorname{spt}(\phi) \subset W\right\}.$$

Let $T \in \mathcal{D}_k(U)$ be such that $\mathbf{M}_W(T) < \infty$ for every open set $W \subset\subset U$. Then, as a conseguence of the Riesz Representation Theorem, there exists a positive Radon measure $\|T\|$ on U and a $\|T\|$-measurable k-vectorfield $\vec{\tau} : U \to \bigwedge^k \mathbb{R}^n$ such that

$$|\vec{\tau}(x)| = 1 \quad \text{for } \|T\|\text{-a.e. } x \in U\,,$$

and the following *representation by integration* holds (cf. [56, 26.7])

$$T(\phi) = \int_U \langle \vec{\tau}(x), \phi(x) \rangle \, d\|T\|(x) \quad \text{for every } \phi \in \mathcal{D}^k(U)\,.$$

A particularly important class of currents, representable by integration, consists of those associated with rectifiable sets. We say that a current $T \in \mathcal{D}_k(U)$ is a *integer multiplicity locally rectifiable k-current of U* if

$$T(\phi) = \int_M \langle \vec{\eta}(x), \phi(x) \rangle \, d\mathcal{H}^k(x) \quad \text{for every } \phi \in \mathcal{D}^k(U)$$

where $M \subset U$ is an $\mathcal{H}^k$-measurable and countably $\mathcal{H}^k$-rectifiable set, while $\vec{\eta}$ is an $(\mathcal{H}^k \llcorner M)$-measurable k-vectorfield such that:

(i) $\int_{K \cap M} |\vec{\eta}| \, d\mathcal{H}^k < \infty$ for every compact subset K of U;

(ii) $\vec{\eta}(x)$ is a simple and $|\vec{\eta}(x)|$ is a positive integer for $\mathcal{H}^k$-a.e. $x \in M$;

(iii) $\operatorname{Tan}^k(\mathcal{H}^k \llcorner M, x)$ is associated with $\vec{\eta}(x)$ for $\mathcal{H}^k$-a.e. $x \in M$.

The set M is called *carrier* of T, it is $\mathcal{H}^k$-almost uniquely determined by T and we denote it by W_T. Moreover, if we define the *multiplicity* and *orientation* of T, respectively as

$$i_T := |\vec{\eta}| \quad \text{and} \quad \vec{\eta}_T := \frac{\vec{\eta}}{|\vec{\eta}|}\,,$$

then

$$T(\phi) = \int_{W_T} \langle \vec{\eta}_T(x), \phi(x) \rangle \, i_T(x) \, d\mathcal{H}^k(x) \quad \text{for every } \phi \in \mathcal{D}^k(U)$$

and it will be convenient to write $T = i_T(\mathcal{H}^k \llcorner W_T) \wedge \vec{\eta}_T$.

1.3 Legendrian currents and Minkowski-Hsiung identities

Here we introduce the notion of Legendrian cycles and we collect some fundamental facts.

Definition 1.3.2 (Contact 1-form)**.** *We say that $\alpha \in \mathcal{E}^1(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ is the contact 1-form of $\mathbb{R}^{n+1}$ if*

$$\langle (y,v), \alpha(x,u) \rangle := y \bullet u \quad \textit{for every } y, v, x, u \in \mathbb{R}^{n+1}\,.$$

Definition 1.3.3 (Legendrian current)**.** *Let $\Omega \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ be an open set and let $k \ge 1$ be an integer. A current $S \in \mathcal{D}_k(\Omega)$ is called Legendrian if $S \llcorner \alpha = 0$.*

Definition 1.3.4 (Legendrian cycle). *Let $W \subseteq \mathbb{R}^{n+1}$ be an open set. We say that an integer-multiplicity locally rectifiable n-current T of $W \times \mathbb{R}^{n+1}$ is a Legendrian cycle of W if the following three conditions are satisfied:*

(i) $\operatorname{spt}(T) \subseteq W \times \mathbb{S}^n$;

(ii) $\partial T = 0$;

(iii) $T \llcorner \alpha = 0$.

Lemma 1.3.5. *Suppose $W_1, \ldots, W_m \subset \mathbb{R}^{n+1}$ are bounded open sets and $T \in \mathcal{D}_n(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ such that $T \llcorner (W_i \times \mathbb{R}^{n+1})$ is a Legendrian cycle of W_i for every $i \in \{1, \ldots, m\}$ and $\operatorname{spt}(T)$ is a compact subset of $\bigcup_{i=1}^m W_i \times \mathbb{S}^n$. Then T is a Legendrian cycle of $\mathbb{R}^{n+1}$.*

Proof. For every $i \in \{1, \ldots, m\}$ choose an open set V_i with compact closure in W_i and a function $f_i \in C_c^\infty(\mathbb{R}^{n+1})$ such that $\operatorname{spt}(f_i)$ is a compact subset of W_i, $\sum_{i=1}^m f_i(x) = 1$ for every $x \in \bigcup_{i=1}^m V_i$ and $\operatorname{spt}(T) \subseteq \bigcup_{i=1}^m V_i \times \mathbb{S}^n$. Then $T = T \llcorner (\sum_{i=1}^m f_i \circ \pi_0)$, $\sum_{i=1}^m d(f_i \circ \pi_0) = 0$ on $\bigcup_{i=1}^m V_i \times \mathbb{S}^n$,

$$(T \llcorner \alpha)(\phi) = \sum_{i=1}^m (T \llcorner \alpha)\big((f_i \circ \pi_0)\phi\big) = 0$$

and

$$\partial T(\phi) = \sum_{i=1}^m \partial T\big((f_i \circ \pi_0)\phi\big) - T\Big(\big(\textstyle\sum_{i=1}^m d(f_i \circ \pi_0)\big) \wedge \phi\Big) = 0\,,$$

for every $\phi \in \mathcal{D}^{n-1}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$. The proof is complete. □

Definition 1.3.6 (Lipschitz-Killing forms). *For $k \in \{0, \ldots, n\}$ the k-th Lipschitz-Killing differential form $\varphi_k \in \mathcal{E}^n(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ is defined by*

$$\langle \xi_1 \wedge \ldots \wedge \xi_n, \varphi_k(x,u) \rangle := \sum_{\sigma \in \Sigma_{n,k}} \langle \pi_{\sigma(1)}(\xi_1) \wedge \ldots \wedge \pi_{\sigma(n)}(\xi_n) \wedge u, dX_1 \wedge \ldots \wedge dX_{n+1} \rangle\,,$$

for every $\xi_1, \ldots, \xi_n \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Where

$$\Sigma_{n,k} := \Big\{ \sigma : \{1, \ldots, n\} \to \{0,1\} \,\Big|\, \sum_{i=1}^n \sigma(i) = n - k \Big\}\,.$$

The Radon measures $T \llcorner \varphi_i$, where $i \in \{0, \ldots, n\}$ and T is a n-dimensional Legendrian cycle of $\mathbb{R}^{n+1}$, are called *curvature measures of T*. In [17], Joseph Fu derived a simple formula for the first variation of the measures $T \llcorner \varphi_i$, even in the more general setting of space forms, hence generalizing a well known result obtained by Reilly (cf. [45]) for smooth submanifolds. Fu's elegant proof of this formula employs the Maurer-Cartan forms of the Lie algebra associated with the space form to compute the exterior derivative of the Lipschitz-Killing forms (cf. [17, pp. 183-185]). On the other hand, it is also possible to compute such exterior derivatives in a somewhat more direct way, without relying on any Lie algebra theory. In the following lemma, we present such a computation. For completeness, we also provide a full proof of Lemma 1.3.11, following Fu's original approach.

We recall that, for $1 \leq k \leq m$, the set $\Lambda(m,k)$ denotes the collection of all increasing maps from $\{1, \ldots, k\}$ into $\{1, \ldots, m\}$.

Lemma 1.3.7. [17, Lemma 3.1]. *The exterior derivatives of the Lipschitz-Killing differential forms satisfy the following relations*

$$\langle \xi_1 \wedge \ldots \wedge \xi_{n+1}, d\varphi_k(x,u) \rangle = \langle \xi_1 \wedge \ldots \wedge \xi_{n+1}, \alpha(x,u) \wedge (n-k+1)\varphi_{k-1}(x,u) \rangle \qquad (1.3.9)$$

for $k \in \{1, \ldots, n\}$, and

$$\langle \xi_1 \wedge \ldots \wedge \xi_{n+1}, d\varphi_0(x,u) \rangle = 0\,, \qquad (1.3.10)$$

whenever $\xi_1, \ldots, \xi_{n+1} \in \operatorname{Tan}\big(\mathbb{R}^{n+1} \times \mathbb{S}^n, (x,u)\big)$ and $(x,u) \in \mathbb{R}^{n+1} \times \mathbb{S}^n$.

Proof. We fix $(x,u) \in \mathbb{R}^{n+1} \times \mathbb{S}^n$. Suppose $k \in \mathbb{N}_{\geq 1}$ and notice that

$$\varphi_k : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \textstyle\bigwedge^n(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$$

is a linear map. Therefore, we compute (cf. [14, p. 352])

$$\begin{aligned}
&\langle \xi_1 \wedge \ldots \wedge \xi_{n+1}, d\varphi_k(x,u)\rangle \qquad (1.3.11)\\
&= \sum_{j=1}^{n+1} (-1)^{j-1} \langle \xi_1 \wedge \ldots \wedge \xi_{j-1} \wedge \xi_{j+1} \wedge \ldots \wedge \xi_{n+1}, \langle \xi_j, D\varphi_k(x,u)\rangle\rangle\\
&= \sum_{j=1}^{n+1} (-1)^{j-1} \langle \xi_1 \wedge \ldots \wedge \xi_{j-1} \wedge \xi_{j+1} \wedge \ldots \wedge \xi_{n+1}, \varphi_k(\xi_j)\rangle\\
&= \sum_{j=1}^{n+1} (-1)^{j-1} \sum_{\sigma \in \Sigma_{n,k}} \langle \pi_{\sigma(1)}(\xi_1) \wedge \ldots \wedge \pi_{\sigma(j-1)}(\xi_{j-1}) \wedge \pi_{\sigma(j)}(\xi_{j+1}) \wedge \ldots\\
&\qquad \ldots \wedge \pi_{\sigma(n)}(\xi_{n+1}) \wedge \pi_1(\xi_j), \Omega_{n+1}\rangle\\
&= (-1)^n \sum_{j=1}^{n+1} \sum_{\sigma \in \Sigma_{n,k}} \langle \pi_{\sigma(1)}(\xi_1) \wedge \ldots \wedge \pi_{\sigma(j-1)}(\xi_{j-1}) \wedge \pi_1(\xi_j) \wedge \pi_{\sigma(j)}(\xi_{j+1}) \wedge \ldots\\
&\qquad \ldots \wedge \pi_{\sigma(n)}(\xi_{n+1}), \Omega_{n+1}\rangle
\end{aligned}$$

for $\xi_1, \ldots, \xi_{n+1} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Letting $p_u : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the orthogonal projection onto $\operatorname{span}\{u\}$, we use the shuffle formula (cf. [14, 1.4.2]) to compute

$$\begin{aligned}
&\langle \xi_1 \wedge \ldots \wedge \xi_{n+1}, \alpha(x,u) \wedge \varphi_{k-1}(x,u)\rangle \qquad (1.3.12)\\
&= \sum_{j=1}^{n+1} (-1)^{j-1} \langle \xi_j, \alpha(x,u)\rangle \langle \xi_1 \wedge \ldots \wedge \xi_{j-1} \wedge \xi_{j+1} \wedge \ldots \wedge \xi_{n+1}, \varphi_{k-1}(x,u)\rangle\\
&= \sum_{j=1}^{n+1} (-1)^{j-1} \sum_{\sigma \in \Sigma_{n,k-1}} \langle \pi_{\sigma(1)}(\xi_1) \wedge \ldots \wedge \pi_{\sigma(j-1)}(\xi_{j-1}) \wedge \pi_{\sigma(j)}(\xi_{j+1}) \wedge \ldots\\
&\qquad \ldots \wedge \pi_{\sigma(n)}(\xi_{n+1}) \wedge (\pi_0(\xi_j) \bullet u)\, u, \Omega_{n+1}\rangle\\
&= (-1)^n \sum_{j=1}^{n+1} \sum_{\sigma \in \Sigma_{n,k-1}} \langle \pi_{\sigma(1)}(\xi_1) \wedge \ldots \wedge \pi_{\sigma(j-1)}(\xi_{j-1}) \wedge p_u\big(\pi_0(\xi_j)\big) \wedge \pi_{\sigma(j)}(\xi_{j+1}) \wedge \ldots\\
&\qquad \ldots \wedge \pi_{\sigma(n)}(\xi_{n+1}), \Omega_{n+1}\rangle
\end{aligned}$$

whenever $\xi_1, \ldots, \xi_{n+1} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Suppose $\{\tau_1, \ldots, \tau_n\} \subset u^\perp$ is an orthonormal set, we define

$$v_i := (\tau_i, 0) \quad \text{for } i \in \{1, \ldots, n\}, \quad v_i := (0, \tau_{i-n}) \quad \text{for } i \in \{n+1, \ldots, 2n\}, \quad v_{2n+1} := (u, 0)$$

and we notice that $\{v_1, \ldots, v_{2n+1}\}$ is an orthonormal basis of

$$\operatorname{Tan}\big(\mathbb{R}^{n+1} \times \mathbb{S}^n, (x,u)\big) = \mathbb{R}^{n+1} \times \operatorname{Tan}(\mathbb{S}^n, u) = \mathbb{R}^{n+1} \times u^\perp .$$

Then we define

$$v_\lambda = v_{\lambda(1)} \wedge \ldots \wedge v_{\lambda(n+1)} \qquad \text{whenever } \lambda \in \Lambda(2n+1, n+1)$$

and, recalling that $\{v_\lambda : \lambda \in \Lambda(2n+1, n+1)\}$ is a basis of $\bigwedge_{n+1}(\mathbb{R}^{n+1} \times u^\perp)$ (cf. [14, 1.3.2]), we notice that (1.3.9) reduces to check

$$\langle v_\lambda, d\varphi_k(x,u)\rangle = (n-k+1)\, \langle v_\lambda, \alpha(x,u) \wedge \varphi_{k-1}(x,u)\rangle$$

whenever $\lambda \in \Lambda(2n+1, n+1)$.

First, if $\lambda \in \Lambda(2n+1, n+1)$ and $2n+1 \notin \mathrm{Im}(\lambda)$, it follows that $p_u(\pi_0(v_{\lambda(j)})) = 0$ for every $j \in \{1, \ldots, n+1\}$. Hence, from (1.3.12), we infer

$$\langle v_\lambda, \alpha(x,u) \wedge \varphi_{k-1}(x,u)\rangle = 0\,.$$

Furthermore, in (1.3.11), for each $j \in \{1, \ldots, n+1\}$ and each $\sigma \in \Sigma_{n,k}$, the vectors

$$\{\pi_{\sigma(1)}(v_{\lambda(1)}), \ldots, \pi_{\sigma(j-1)}(v_{\lambda(j-1)}), \pi_1(v_{\lambda(j)}), \pi_{\sigma(j)}(v_{\lambda(j+1)}), \pi_{\sigma(n)}(v_{\lambda(n+1)})\}$$

are linearly dependent. Therefore

$$\langle v_\lambda, d\varphi_k(x,u)\rangle = 0\,.$$

Now, assume that $\lambda \in \Lambda(2n+1, n+1)$ where $2n+1 \in \mathrm{Im}(\lambda)$, namely $\lambda(n+1) = 2n+1$. Since $\pi_1(v_{\lambda(j)}) = 0$ if $j \in \lambda^{-1}\{1, \ldots, n, 2n+1\}$, we apply (1.3.11) to deduce

$$\begin{aligned}&\langle v_\lambda, d\varphi_k(x,u)\rangle \\ &= (-1)^n \sum_{j\in\lambda^{-1}\{n+1,\ldots,2n\}} \sum_{\substack{\sigma\in\Sigma_{n,k}\\ \sigma(n)=0}} \langle \pi_{\sigma(1)}(v_{\lambda(1)}) \wedge \ldots \wedge \pi_{\sigma(j-1)}(v_{\lambda(j-1)}) \wedge \tau_{\lambda(j)-n} \wedge \pi_{\sigma(j)}(v_{\lambda(j+1)}) \wedge \ldots \\ &\qquad\qquad \ldots \wedge \pi_{\sigma(n-1)}(v_{\lambda(n)}) \wedge u, \Omega_{n+1}\rangle\end{aligned}$$

and, since $p_u(\pi_0(v_{\lambda(n+1)})) = u$ and $p_u(\pi_0(v_{\lambda(j)})) = 0$ if $j \in \{1, \ldots, n\}$, we infer from (1.3.12)

$$\begin{aligned}&\langle v_\lambda, \alpha(x,u) \wedge \varphi_{k-1}(x,u)\rangle \\ &\quad = (-1)^n \sum_{\sigma\in\Sigma_{n,k-1}} \langle \pi_{\sigma(1)}(v_{\lambda(1)}) \wedge \ldots \wedge \pi_{\sigma(n)}(v_{\lambda(n)}) \wedge u, \Omega_{n+1}\rangle\,.\end{aligned}$$

Obviously, if $\mathcal{H}^0(\lambda^{-1}\{1, \ldots, n\}) \neq k-1$, we obtain that

$$\langle v_\lambda, d\varphi_k(x,u)\rangle = \langle v_\lambda, \alpha(x,u) \wedge \varphi_{k-1}(x,u)\rangle = 0\,.$$

Whereas, if $\mathcal{H}^0(\lambda^{-1}\{1, \ldots, n\}) = k-1$ (i.e. $\mathcal{H}^0(\lambda^{-1}\{n+1, \ldots, 2n\}) = n-k+1$), we deduce that there exists an unique $\sigma \in \{\sigma \in \Sigma_{n,k} : \sigma(n) = 0\}$ such that

$$\begin{aligned}&\langle v_\lambda, d\varphi_k(x,u)\rangle \\ &= (-1)^n \sum_{j\in\lambda^{-1}\{n+1,\ldots,2n\}} \langle \tau_{\lambda(1)} \wedge \ldots \wedge \tau_{\lambda(k-1)} \wedge \tau_{\lambda(k)-n} \wedge \ldots \wedge \tau_{\lambda(n)-n} \wedge u, \Omega_{n+1}\rangle \\ &= (-1)^n\,(n-k+1)\,\langle \tau_{\lambda(1)} \wedge \ldots \wedge \tau_{\lambda(k-1)} \wedge \tau_{\lambda(k)-n} \wedge \ldots \wedge \tau_{\lambda(n)-n} \wedge u, \Omega_{n+1}\rangle\,.\end{aligned}$$

Similarly, there exists an unique $\sigma \in \Sigma_{n,k-1}$ such that

$$\begin{aligned}&\langle v_\lambda, \alpha(x,u) \wedge \varphi_{k-1}(x,u)\rangle \\ &\quad = (-1)^n\,\langle \tau_{\lambda(1)} \wedge \ldots \wedge \tau_{\lambda(k-1)} \wedge \tau_{\lambda(k)-n} \wedge \ldots \wedge \tau_{\lambda(n)-n} \wedge u, \Omega_{n+1}\rangle\,.\end{aligned}$$

Furthermore, from (1.3.11), and since $\mathcal{H}^0(\{i : \pi_1(v_i) \neq 0\}) = n$, we deduce that

$$\langle v_\lambda, d\varphi_0(x,u)\rangle = (-1)^n\,(n+1)\,\langle \bigwedge_{i=1}^{n+1} \pi_1(v_{\lambda(i)}), \Omega_{n+1}\rangle = 0$$

for every $\lambda \in \Lambda(2n+1, n+1)$. The proof is complete. □

Definition 1.3.8. (Local variation). *Given $\epsilon > 0$ and a smooth map*

$$F : \mathbb{R}^{n+1} \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}\,.$$

We say that $\{F_t\}_{t\in(-\epsilon,\epsilon)}$, where $F_t(x) := F(x,t)$, is a local variation of $\mathbb{R}^{n+1}$ if, for every fixed $t \in (-\epsilon, \epsilon)$, the map $F_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a diffeomorphism and F_0 is the identity of $\mathbb{R}^{n+1}$. For *such local variation we define the initial velocity field V by*

$$V(x) := \lim_{t\to 0} \frac{F_t(x) - x}{t} \quad \textit{for } x \in \mathbb{R}^{n+1}.$$

Given a C^2-diffeomorphism $F : U \to V$ between open subsets of $\mathbb{R}^{n+1}$, we define the C^1-diffeomorphism $\Psi_F : \mathbb{R}^{n+1} \times \mathbb{S}^n \to \mathbb{R}^{n+1} \times \mathbb{S}^n$ by

$$\Psi_F(x,u) := \left(F(x), \frac{(DF(x)^{-1})^*(u)}{|(DF(x)^{-1})^*(u)|} \right) \quad \text{for } (x,u) \in \mathbb{R}^{n+1} \times \mathbb{S}^n . \tag{1.3.13}$$

Example 1.3.9. The smooth map

$$F_t(x) := e^t x \quad \text{for } (x,t) \in \mathbb{R}^{n+1} \times \mathbb{R} ,$$

defines a local variation of $\mathbb{R}^{n+1}$. In this case, $V = \mathrm{id}|\mathbb{R}^{n+1}$ and

$$\Psi_{F_t}(x,y) = (e^t x, y) \quad \text{for } (x,y,t) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R} .$$

Definition 1.3.10. *We define $\mathbb{R}^{n+1}_0 := \mathbb{R}^{n+1} \setminus \{0\}$. Given a local variation $\{F_t\}_{t\in(-\epsilon,\epsilon)}$, we introduce the smooth map $h : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}_0 \times (-\epsilon,\epsilon) \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ by setting (cf. (1.3.13))*

$$h(x,u,t) := \Psi_{F_t}(x,u) \qquad \textit{for } (x,u,t) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}_0 \times (-\epsilon,\epsilon) ,$$

and we notice that $h(x,u,0) = (x,u)$ for $(x,u) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}_0$. Moreover, we define

$$p : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, \qquad p(x,u,t) := (x,u) ,$$

$$q : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R} , \qquad q(x,u,t) := t ,$$

$$P : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R} , \qquad P(x,u) := (x,u,0) .$$

Lemma 1.3.11. *Suppose T is a Legendrian cycle of $\mathbb{R}^{n+1}$ with compact support, $\{F_t\}_{t\in(-\epsilon,\epsilon)}$ is a local variation of $\mathbb{R}^{n+1}$ with initial velocity field V and*

$$\theta_V : (x,y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mapsto V(x) \bullet y \in \mathbb{R} ,$$

$$\theta : (x,y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mapsto x \bullet y \in \mathbb{R} .$$

Then

$$\frac{d}{dt} \left[(\Psi_{F_t})_\# T \right] (\varphi_k) \Big|_{t=0} = (n+1-k)\, T(\theta_V\, \varphi_{k-1}) \quad \textit{for } k \in \{1,\dots,n\}$$

and

$$\frac{d}{dt} \left[(\Psi_{F_t})_\# T \right] (\varphi_0) \Big|_{t=0} = 0 .$$

In particular, T satisfies the following Minkowski-Hsiung identities

$$k\, T(\varphi_k) = (n+1-k)\, T(\theta\, \varphi_{k-1}) , \tag{1.3.14}$$

for $k \in \{1,\dots,n\}$.

Proof. Firstly, we observe that

$$h^\# \alpha \circ P = \left(p^\# \alpha + \theta_V\, dq \right) \circ P , \tag{1.3.15}$$

$$\left(h^\# \varphi_k \wedge dq \right) \circ P = \left(p^\# \varphi_k \wedge dq \right) \circ P \tag{1.3.16}$$

and, if $\{F_t\}_{t\in\mathbb{R}}$ is the local variation in Example 1.3.9, then for a fixed $t \in \mathbb{R}$ we have

$$(\Psi_{F_t})^\# \varphi_k = e^{kt} \varphi_k \quad \text{for } k \in \{0,\dots,n\} . \tag{1.3.17}$$

To prove the relations (1.3.15) and (1.3.16), we choose an orthonormal basis $\{w_1,\dots,w_{2n+3}\}$ of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R} \simeq \mathbb{R}^{2n+3}$, whose dual basis is $\{w'_1,\dots,w'_{2n+3}\}$ and satisfies $w'_{2n+3} = dq$.

Moreover, we define

$$w_\mu := w_{\mu(1)} \wedge \ldots \wedge w_{\mu(n)} \quad , \quad w'_\mu := w'_{\mu(1)} \wedge \ldots \wedge w'_{\mu(n)}$$

whenever $\mu \in \Lambda(m,l)$ for $l \leq m \leq 2n+3$. Since $h \circ P = p$, we note that

$$Dh(x,u,0)(w_i) = p(w_i) \quad \text{whenever } i \in \{1,\ldots,2n+2\}$$

and

$$\langle Dh(x,u,0)(w_{2n+3}), \alpha(x,u)\rangle = V(x) \bullet u = \theta_V(x,u)\,,$$

for $(x,u) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Thus, recalling [14, 1.4.3], we compute

$$\begin{aligned} h^\# \alpha(x,u,0) &= \sum_{i=1}^{2n+3} \langle w_i, h^\# \alpha(x,u,0)\rangle\, w'_i \\ &= \sum_{i=1}^{2n+2} \langle p(w_i), \alpha(x,u)\rangle\, w'_i + \theta_V(x,u)\, w'_{2n+3} \end{aligned}$$

and

$$\begin{aligned} h^\# \varphi_k\,(x,u,0) &= \sum_{\lambda \in \Lambda(2n+3,n)} \langle w_\lambda, h^\# \varphi_k\,(x,u,0)\rangle\, w'_\lambda \\ &= \sum_{\lambda \in \Lambda(2n+2,n)} \langle \textstyle\bigwedge_n p(w_\lambda), \varphi_k\,(x,u)\rangle\, w'_\lambda \\ &\quad + \sum_{\lambda \in \Lambda(2n+2,n-1)} \langle \textstyle\bigwedge_n Dh(x,u,0)(w_\lambda \wedge w_{2n+3}), \varphi_k\,(x,u)\rangle\, w'_\lambda \wedge w'_{2n+3} \end{aligned}$$

for $(x,u) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, whence we readily infer (1.3.15) and (1.3.16).

Now, to prove (1.3.17), let $\{\boldsymbol{e}_1,\ldots,\boldsymbol{e}_{2n+2}\}$ be the canonical basis of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, whose dual basis is $\{\boldsymbol{e}'_1,\ldots,\boldsymbol{e}'_{2n+2}\}$. For any $\mu \in \Lambda(2n+2,n)$, we define

$$\boldsymbol{e}_\mu := \boldsymbol{e}_{\mu(1)} \wedge \ldots \wedge \boldsymbol{e}_{\mu(n)} \quad , \quad \boldsymbol{e}'_\mu := \boldsymbol{e}'_{\mu(1)} \wedge \ldots \wedge \boldsymbol{e}'_{\mu(n)}\,.$$

Since $\Psi_{F_t}(x,y) = (e^t x, y)$, we infer that

$$D(\Psi_{F_t})(x,y)(\boldsymbol{e}_j) = \begin{cases} e^t \boldsymbol{e}_j & \text{if } j \leq n+1 \\ \boldsymbol{e}_j & \text{if } j \geq n+2 \end{cases} \qquad \text{for } j \in \{1,\ldots,2n+2\}\,.$$

Then, for every fixed $\lambda \in \Lambda(2n+2,n)$ and for $(x,y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, from Definition 1.3.6 we deduce that

$$\langle D(\Psi_{F_t})(x,y)(\boldsymbol{e}_{\lambda(1)}) \wedge \ldots \wedge D(\Psi_{F_t})(x,y)(\boldsymbol{e}_{\lambda(n)}), \varphi_k(e^t x, y)\rangle \neq 0$$

if and only if $\mathcal{H}^0\big(\{m \in \{1,\ldots,n\} : \lambda(m) \leq n+1\}\big) = k$. In that case

$$\begin{aligned} &\langle D(\Psi_{F_t})(x,y)(\boldsymbol{e}_{\lambda(1)}) \wedge \ldots \wedge D(\Psi_{F_t})(x,y)(\boldsymbol{e}_{\lambda(n)}), \varphi_k(e^t x, y)\rangle \\ &\quad = e^{kt} \langle \bigwedge_{j=1}^{k} \pi_0(\boldsymbol{e}_{\lambda(j)}) \wedge \bigwedge_{j=k+1}^{n} \pi_1(\boldsymbol{e}_{\lambda(j)}) \wedge y, dX_1 \wedge \ldots \wedge dX_{n+1}\rangle\,. \end{aligned}$$

Overall, since φ_k depends only on the second coordinate, we conclude that (cf. [14, 1.4.3])

$$\begin{aligned} (\Psi_{F_t})^\# \varphi_k(x,y) &= \sum_{\lambda \in \Lambda(2n+2,n)} \langle \boldsymbol{e}_\lambda\,, (\Psi_{F_t})^\# \varphi_k(x,y)\rangle\, \boldsymbol{e}'_\lambda \\ &= \sum_{\lambda \in \Lambda(2n+2,n)} \langle \Big[\textstyle\bigwedge_n D(\Psi_{F_t})(x,y) \Big](\boldsymbol{e}_\lambda)\,, \varphi_k(e^t x, y)\rangle\, \boldsymbol{e}'_\lambda \end{aligned}$$

$$= e^{kt} \sum_{\lambda \in \Lambda(2n+2,n)} \langle \boldsymbol{e}_\lambda, \varphi_k(e^t x, y) \rangle \, \boldsymbol{e}'_\lambda = e^{kt} \varphi_k(x,y) \, .$$

Suppose that T has the form $T = i_T(\mathcal{H}^k \llcorner W_T) \wedge \vec{\eta}_T$ (cf. section 1.2, p. 14), that is

$$T(\phi) = \int_{W_T} \langle \vec{\eta}_T(x,u), \phi(x,u) \rangle \, i_T(x,u) \, d\mathcal{H}^n(x,u) \quad \text{for each } \phi \in \mathcal{D}^n(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \, .$$

For $t > 0$, we define $[\![0,t]\!] \in \mathcal{D}_1(\mathbb{R})$ by the formula

$$[\![0,t]\!](\alpha) := \int_0^1 \langle t, \alpha(st) \rangle \, d\mathcal{L}^1(s) = \int_0^t \langle 1, \alpha(s) \rangle \, d\mathcal{L}^1(s) \quad \text{for } \alpha \in \mathcal{E}^1(\mathbb{R}) \, .$$

Denoting by $T \times [\![0,t]\!] \in \mathcal{D}_{n+1}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R})$ the cartesian product of T and $[\![0,t]\!]$, we use [14, 4.1.8] to compute

$$(T \times [\![0,t]\!])(\phi) = \int_{W_T} \int_0^t \big\langle \vec{\zeta}_T(x,u), \phi(x,u,s) \big\rangle \, d\mathcal{H}^1(s) \, d\mathcal{H}^n(x,u)$$

whenever $\phi \in \mathcal{E}^{n+1}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R})$, where

$$\vec{\zeta}_T(x,u) := \big(\textstyle\bigwedge_n P\big)\big(i_T(x,u)\vec{\eta}_T(x,u)\big) \wedge w_{2n+3} \quad \text{for } \mathcal{H}^n\text{-a.e. } (x,u) \in W_T \, .$$

Employing [14, 4.1.8, 4.1.9] and taking into account that $\partial T = 0$, we derive the homotopy formula

$$(\Psi_{F_t})_\# T - T = (-1)^n \, \partial \big[h_\#(T \times [\![0,t]\!]) \big] \, .$$

Since $\operatorname{spt}\big(h_\#(T \times [\![0,t]\!])\big) \subseteq \mathbb{R}^{n+1} \times \mathbb{S}^n$ we use Lemma 1.3.7 to compute

$$\begin{aligned} \big[(\Psi_{F_t})_\# T - T\big](\varphi_k) &= (-1)^n \, (n-k+1) \, h_\#(T \times [\![0,t]\!])(\alpha \wedge \varphi_{k-1}) \\ &= (-1)^n \, (n-k+1) \int_{W_T} \int_0^t \big\langle \vec{\zeta}_T(x,u), (h^\# \alpha \wedge h^\# \varphi_{k-1})(x,u,s) \big\rangle \, d\mathcal{H}^1(s) \, d\mathcal{H}^n(x,u) \end{aligned}$$

whence we infer that

$$\begin{aligned} \lim_{t \to 0} & \frac{\big[(\Psi_{F_t})_\# T - T\big](\varphi_k)}{t} \\ &= (-1)^n \, (n-k+1) \int_{W_T} \big\langle \vec{\zeta}_T(x,u), (h^\# \alpha \wedge h^\# \varphi_{k-1})(x,u,0) \big\rangle \, d\mathcal{H}^n(x,u) \, . \end{aligned}$$

Using (1.3.15) and (1.3.16) we deduce that

$$\begin{aligned} &\big\langle \vec{\zeta}_T(x,u), (h^\# \alpha \wedge h^\# \varphi_{k-1})(x,u,0) \big\rangle \\ &\quad = \langle \vec{\zeta}_T(x,u), (p^\# \alpha \wedge h^\# \varphi_{k-1})(x,u,0) \rangle + \langle \vec{\zeta}_T(x,u), \theta_V(x,u) \, (dq \wedge p^\# \varphi_{k-1})(x,u,0) \rangle \end{aligned}$$

for $\mathcal{H}^n$-a.e. $(x,u) \in W_T$. Moreover, noting that $p(w_{2n+3}) = 0$ and $\langle \tau, \alpha(x,u) \rangle = 0$ whenever $\tau \in \operatorname{Tan}^n(\mathcal{H}^n \llcorner W_T, (x,u))$ for $\mathcal{H}^n$-a.e. $(x,u) \in W_T$ by [42, Theorem 9.2], we obtain that

$$\langle \vec{\zeta}_T(x,u), (p^\# \alpha \wedge h^\# \varphi_{k-1})(x,u,0) \rangle = 0 \qquad \text{for } \mathcal{H}^n\text{-a.e. } (x,u) \in W_T$$

and employing the shuffle formula (cf. [14, 1.4.2]) we compute

$$\begin{aligned} &\langle \vec{\zeta}_T(x,u), \theta_V(x,u) \, (dq \wedge p^\# \varphi_{k-1})(x,u,0) \rangle \\ &\qquad = (-1)^n \, \theta_V(x,u) \, \langle \big(\textstyle\bigwedge_n P\big)\big(i_T(x,u)\vec{\eta}_T(x,u)\big), p^\# \varphi_{k-1}(x,u,0) \rangle \\ &\qquad = (-1)^n \, \theta_V(x,u) \, i_T(x,u) \, \langle \vec{\eta}_T(x,u), \varphi_{k-1}(x,u) \rangle \end{aligned}$$

for $\mathcal{H}^n$-a.e. $(x,u) \in W_T$. Moreover, from (1.3.10), we obtain

$$\frac{d}{dt}\big[(\Psi_{F_t})_\# T\big](\varphi_0)\Big|_{t=0} = (-1)^n \lim_{t\to 0} \frac{\big[h_\#(T \times [\![0,t]\!])\big](d\varphi_0)}{t} = 0\,.$$

If $\{F_t\}_{t\in\mathbb{R}}$ is the one-parameter family in Example 1.3.9 we can directly compute the first variation. Indeed, for every $k \in \{1,\ldots,n\}$, by applying (1.3.17), we have

$$\begin{aligned}(n+1-k)\,T(\theta\varphi_{k-1}) &= \lim_{t\to 0} \frac{\big[(\Psi_{F_t})_\# T - T\big](\varphi_k)}{t} \\ &= \lim_{t\to 0} \frac{T\big((\Psi_{F_t})^\# \varphi_k\big) - T(\varphi_k)}{t} \\ &= \lim_{t\to 0} \frac{T(e^{kt}\varphi_k) - T(\varphi_k)}{t} \\ &= k\,T(\varphi_k)\,\lim_{t\to 0}\frac{e^{kt}-1}{kt} = k\,T(\varphi_k)\,.\end{aligned}$$

The proof is complete. □

The following result describes the approximate tangent space of the carrier of a Legendrian cycle.

Theorem 1.3.12 (cf. [42, Theorem 9.2])**.** *Let T be a Legendrian cycle of $\mathbb{R}^{n+1}$, with carrier W_T. For $\mathcal{H}^n$-a.e. $(x,u) \in W_T$ there exist numbers*

$$-\infty < \kappa_1(x,u) \le \ldots \le \kappa_n(x,u) \le +\infty$$

and vectors $\tau_1(x,u),\ldots,\tau_n(x,u)$ such that $\{\tau_1(x,u),\ldots,\tau_n(x,u),u\}$ is a positively oriented orthonormal basis of $\mathbb{R}^{n+1}$ (i.e. $\langle \bigwedge_{i=1}^n \tau_i(x,u)\wedge u, dX_1\wedge\ldots\wedge dX_{n+1}\rangle = 1$) and the vectors

$$\xi_i(x,u) := \begin{cases} (\tau_i(x,u), \kappa_i(x,u)\tau_i(x,u)) & \text{if } \kappa_i(x,u) < +\infty \\ (0,\tau_i(x,u)) & \text{if } \kappa_i(x,u) = +\infty \end{cases} \qquad i \in \{1,\ldots,n\}$$

form an orthogonal basis of $\mathrm{Tan}^n\big(\mathcal{H}^n \llcorner Q,(x,u)\big)$ for every $\mathcal{H}^n$-measurable set $Q \subseteq W_T$ with $\mathcal{H}^n(Q) < \infty$ and for $\mathcal{H}^n$-a.e. $(x,u)\in Q$.

The maps $\kappa_1,\ldots,\kappa_n$ are $(\mathcal{H}^n \llcorner W_T)$-almost uniquely determined, as well as the substaces of $\mathbb{R}^{n+1}$ spanned by vectors $\tau_j(x,u)$ belonging to a fixed value among the $\kappa_i(x,u)$ ($i\in\{1,\ldots,n\}$) is uniquely determinated.

Definition 1.3.13. *Given T, $\{\tau_1,\ldots,\tau_n\}$ and $\{\xi_1,\ldots,\xi_n\}$ as in Theorem 1.3.12, we define*

$$\vec{\xi}_T(x,u) := \frac{\xi_1(x,u)\wedge\ldots\wedge\xi_n(x,u)}{|\xi_1(x,u)\wedge\ldots\wedge\xi_n(x,u)|} \in \textstyle\bigwedge_n(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})\,,$$

$$\zeta_T(x,u) := \frac{1}{|\xi_1(x,u)\wedge\ldots\wedge\xi_n(x,u)|} \in (0,+\infty)\,,$$

for $\mathcal{H}^n$-a.e. $(x,u)\in W_T$.

Remark 1.3.14. This definition is well-posed, in the sense that $\vec{\xi}_T$ and ζ_T do not depend, up to a set of $\mathcal{H}^n$-measure zero, on the choice of $\{\tau_1,\ldots,\tau_n\}$ (cf. [52, Remark 3.1]).

Definition 1.3.15 (Principal curvatures of Legendrian cycles)**.** *Let T be a Legendrian cycle of $\mathbb{R}^{n+1}$, with carrier W_T. We define the principal curvatures of T as $K_{T,i} := \kappa_i$, where $\kappa_1 \le \ldots \le \kappa_n$ are the functions defined $\mathcal{H}^n$-a.e. on W_T given by Theorem 1.3.12.*

Definition 1.3.16 (k-th mean curvature of Legendrian cycles)**.** *Let T be a Legendrian cycle of $\mathbb{R}^{n+1}$, with carrier Σ_T. We define the sets*

$$W_T^{(i)} := \{(x,u)\in W_T : K_{T,i}(x,u) < +\infty\,,\ K_{T,i+1}(x,u) = +\infty\} \quad \textit{for } i\in\{1,\ldots,n-1\}\,,$$

$$W_T^{(0)} := \{(x,u) \in W_T : K_{T,1}(x,u) = +\infty\},$$
$$W_T^{(n)} := \{(x,u) \in \Sigma_T : K_{T,n}(x,u) < +\infty\}.$$

Then we define the k-th mean curvature function of T as

$$H_{T,k} := \mathbf{1}_{W_T^{(n-k)}} + \sum_{i=1}^{k} \sum_{\lambda \in \Lambda(n-k+i,i)} K_{T,\lambda(1)} \dots K_{T,\lambda(i)} \mathbf{1}_{W_T^{(n-k+i)}} \quad \text{for } k \in \{1,\dots,n\},$$
$$H_{T,0} := \mathbf{1}_{W_T^{(n)}}.$$

The definition of mean curvature functions is motivated by the following result.

Lemma 1.3.17 (cf. [52, Lemma 3.2]). *If $T = i_T(\mathcal{H}^n \llcorner W_T) \wedge \vec{\eta}_T$ is a Legendrian cycle of $\mathbb{R}^{n+1}$ and $k \in \{0,\dots,n\}$, then*

$$(T \llcorner \varphi_k)(\phi) = \int_{W_T} \phi(x,u) i_T(x,u) \zeta_T(x,u) H_{T,n-k}(x,u)\, d\mathcal{H}^n(x,u)$$

for every $\phi \in \mathcal{D}^0(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$.

Corollary 1.3.18 (Minkowski-Hsiung identities for Legendrian cycles)**.**
Let $T = i_T(\mathcal{H}^n \llcorner W_T) \wedge \vec{\eta}_T$ be a Legendrian cycle of $\mathbb{R}^{n+1}$, with compact support. Then, for any $k \in \{1,\dots,n\}$, we have

$$\int_{W_T} \left(k\, H_{T,k}(x,u)\, \theta(x,u) - (n+1-k)\, H_{T,k-1}(x,u)\right) i_T(x,u)\, \zeta_T(x,u)\, d\mathcal{H}^n(x,u) = 0,$$

where $\theta(x,y) := x \bullet y$ for $x,y \in \mathbb{R}^{n+1}$.

Proof. Since T has compact support, from Lemma 1.3.17, we infer

$$T(\varphi_j) = \int_{W_T} i_T(x,u) \zeta_T(x,u) H_{T,n-j}(x,u)\, d\mathcal{H}^n(x,u),$$

$$T(\theta\varphi_{j-1}) = \int_{W_T} \theta(x,u) i_T(x,u) \zeta_T(x,u) H_{T,n+1-j}(x,u)\, d\mathcal{H}^n(x,u),$$

for $j \in \{1,\dots,n\}$. Applying the Minkowski-Hsiung identities (1.3.14), we deduce

$$j \int_{W_T} i_T(x,u) \zeta_T(x,u) H_{T,n-j}(x,u)\, d\mathcal{H}^n(x,u)$$
$$= (n+1-j) \int_{W_T} \theta(x,u) i_T(x,u) \zeta_T(x,u) H_{T,n+1-j}(x,u)\, d\mathcal{H}^n(x,u).$$

Setting $n+1-j = k$, we obtain the desidered result. □

1.4 The proximal unit normal bundle

For an arbitrary nonempty subset $C \subseteq \mathbb{R}^{n+1}$ we define the *distance function* $\boldsymbol{\delta}_C$ from C as

$$\boldsymbol{\delta}_C : x \in \mathbb{R}^{n+1} \mapsto \inf\{|x-a| : a \in C\} \in [0,+\infty).$$

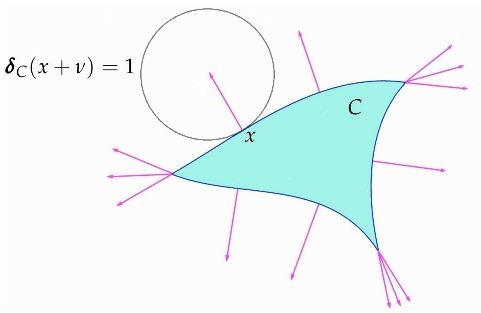
Definition 1.4.19. (Proximal unit normal bundle; cf. [46, p. 212])**.** *Given $C \subseteq \mathbb{R}^{n+1}$, we define the proximal unit normal bundle of C as the set*

$$\operatorname{nor}(C) := \{(x,\nu) \in \overline{C} \times \mathbb{S}^n : \boldsymbol{\delta}_C(x+s\nu) = s \text{ for some } s > 0\}.$$

Moreover, we define

$$\operatorname{nor}(C,x) := \{\nu \in \mathbb{S}^n : (x,\nu) \in \operatorname{nor}(C)\} \quad \text{for } x \in \overline{C}$$
$$\operatorname{nor}(C) \llcorner E := \{(x,\nu) \in \operatorname{nor}(C) : x \in E\} \quad \text{for } E \subseteq \mathbb{R}^{n+1}.$$

Notice that $\mathrm{nor}(C) = \mathrm{nor}(\overline{C})$. We recall that $\mathrm{nor}(C)$ is a Borel set and it is always countably $\mathcal{H}^n$-rectifiable (cf. [51, Remark 4.3][1]). Moreover, we say that $\mathrm{nor}(C)$ satisfies the *Lusin* (N)*-property* on $E \subset \mathbb{R}^{n+1}$ if $\mathcal{H}^n(\mathrm{nor}(C) \llcorner E) = 0$, provided that $\mathcal{H}^n(E) = 0$.



It is proved in [51] that there exists a subset $\widetilde{\mathrm{nor}}(C)$ of $\mathrm{nor}(C)$ (cf. [51, Definition 4.4]), where $\mathcal{H}^n\big(\mathrm{nor}(C) \setminus \widetilde{\mathrm{nor}}(C)\big) = 0$ (cf. [51, Remark 4.5]), such that for every $(x,u) \in \widetilde{\mathrm{nor}}(C)$ there exists a linear subspace $T_C(x,u)$ of $u^\perp$ and a symmetric bilinear form

$$Q_C(x,u) : T_C(x,u) \times T_C(x,u) \to \mathbb{R},$$

whose eigenvalues can be used to provide an explicit representation of the approximate tangent space of $\mathrm{nor}(C)$ at $\mathcal{H}^n$-a.e. points. To this end, for every $(x,u) \in \widetilde{\mathrm{nor}}(C)$, we define

$$-\infty < \kappa_1(x,u) \leq \ldots \leq \kappa_n(x,u) \leq +\infty$$

in the following way:

$$\begin{gathered}\kappa_1(x,u), \ldots, \kappa_m(x,u) \text{ are the eigenvalues of } Q_C(x,u), \\ \text{where } m = \dim T_C(x,u), \text{ and } \kappa_i(x,u) = +\infty \text{ for any } i \in \{m+1, \ldots, n\}.\end{gathered}$$

The following lemma is a simple extension of well known results for sets of positive reach (cf. [42, Proposition 4.23 and Lemma 4.24]).

Lemma 1.4.20. *Suppose $C \subseteq \mathbb{R}^{n+1}$. Then, for $\mathcal{H}^n$-a.e. $(x,u) \in \mathrm{nor}(C)$, there exist the vectors $\tau_1(x,u), \ldots, \tau_n(x,u) \in \mathbb{R}^{n+1}$ such that: $\{\tau_1(x,u), \ldots, \tau_n(x,u), u\}$ is a positively oriented orthonormal basis of $\mathbb{R}^{n+1}$ (i.e. $\langle \bigwedge_{i=1}^n \tau_i(x,u) \wedge u, dX_1 \wedge \ldots \wedge dX_{n+1} \rangle = 1$) and the vectors*

$$\xi_i(x,u) := \left(\frac{1}{\sqrt{1+\kappa_i(x,u)^2}}\, \tau_i(x,u), \frac{\kappa_i(x,u)}{\sqrt{1+\kappa_i(x,u)^2}}\, \tau_i(x,u) \right) \qquad i \in \{1, \ldots, n\}$$

form an orthonormal basis of $\mathrm{Tan}^n(\mathcal{H}^n \llcorner Q, (x,u))$ for every $\mathcal{H}^n$-measurable set $Q \subseteq \mathrm{nor}(C)$ with $\mathcal{H}^n(Q) < \infty$ and for $\mathcal{H}^n$-a.e. $(x,u) \in Q$ (we set $\frac{1}{\infty} = 0$ and $\frac{\infty}{\infty} = 1$).

The maps $\kappa_1, \ldots, \kappa_n$ can be chosen to be $(\mathcal{H}^n \llcorner \mathrm{nor}(C))$-measurable and they are $(\mathcal{H}^n \llcorner \mathrm{nor}(C))$-almost uniquely determined.

Proof. The existence part of the statement and the measurability property are discussed in [51, Section 4] and [24, Section 3] (see in particular [24, Remark 3.7]). While uniqueness can be proved as in [42, Lemma 4.24]. □

Definition 1.4.21. *Suppose $C \subseteq \mathbb{R}^{n+1}$, we denote by $\kappa_{C,1}, \ldots, \kappa_{C,n}$ the $(\mathcal{H}^n \llcorner \mathrm{nor}(C))$-measurable maps given by Lemma 1.4.20.*

Definition 1.4.22 (Reach function of C)**.** *Suppose $C \subseteq \mathbb{R}^{n+1}$, we define the reach function of C, at $(x,u) \in \mathrm{nor}(C)$, as*

$$\boldsymbol{r}_C(x,u) := \sup\{r > 0 : \boldsymbol{\delta}_C(x+ru) = r\}.$$

Remark 1.4.23. Let $C \subseteq \mathbb{R}^{n+1}$. Given $(x,u) \in \widetilde{\mathrm{nor}}(C)$ and $r > 0$ such that $\boldsymbol{\delta}_C(x+ru) = r$, we have that (cf. [51, Lemma 4.8])

$$Q_C(x,u)(\tau,\tau) \geq -\frac{|\tau|^2}{r} \quad \text{whenever } \tau \in T_C(x,u).$$

[1] The proximal unit normal bundle of a closed set C, as defined in [51], is denoted with $N(C)$.

Therefore, for every $(x,u) \in \widetilde{\mathrm{nor}}(C)$, we deduce that

$$-\frac{1}{\boldsymbol{r}_C(x,u)} \le \kappa_{C,i}(x,u) \le +\infty \quad \text{for every } i \in \{1,\dots,n\}\,.$$

Definition 1.4.24. *For $\mathcal{H}^n$-a.e. $(x,u) \in \mathrm{nor}(C)$ we define*

$$\vec{\xi}_C(x,u) := \frac{\xi_{C,1}(x,u) \wedge \ldots \wedge \xi_{C,n}(x,u)}{|\xi_{C,1}(x,u) \wedge \ldots \wedge \xi_{C,n}(x,u)|} \in \bigwedge\nolimits_n(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})\,,$$

$$\zeta_C(x,u) := |\xi_{C,1}(x,u) \wedge \ldots \wedge \xi_{C,n}(x,u)|^{-1} \in (0,+\infty)\,,$$

where, for any $i \in \{1,\dots,n\}$ and with the notations of Lemma 1.4.20 and Definition 1.4.21, we set

$$\xi_{C,i}(x,u) := \begin{cases} (\tau_i(x,u), \kappa_{C,i}(x,u)\tau_i(x,u)) & \text{if } \kappa_{C,i}(x,u) < +\infty \\ (0, \tau_i(x,u)) & \text{if } \kappa_{C,i}(x,u) = +\infty \end{cases}$$

for $\mathcal{H}^n$-a.e. $(x,u) \in \mathrm{nor}(C)$.

Remark 1.4.25. The definitions of $\vec{\xi}_C$ and ζ_C does not depend on the choice of $\{\tau_1,\dots,\tau_n\}$, if the latter is consistent with Lemma 1.4.20. To prove this assertion let us assume that there exist two choices, $\{\tau_1,\dots,\tau_n\}$ and $\{\tau'_1,\dots,\tau'_n\}$, which satisfy Lemma 1.4.20 at $(x,u) \in \mathrm{nor}(C)$ and we show that

$$\bigwedge_{i=1}^{n} \xi_{C,i}(x,u) = \bigwedge_{i=1}^{n} \xi'_{C,i}(x,u)\,, \tag{1.4.18}$$

where the vectors $\{\xi_{C,1}(x,u),\dots,\xi_{C,n}(x,u)\}$ and $\{\xi'_{C,1}(x,u),\dots,\xi'_{C,n}(x,u)\}$, given by

$$\xi_{C,i} := \begin{cases} (\tau_i, \kappa_{C,i}\tau_i) & \text{if } \kappa_{C,i} < +\infty \\ (0,\tau_i) & \text{if } \kappa_{C,i} = +\infty \end{cases} \qquad \xi'_{C,i} := \begin{cases} (\tau'_i, \kappa_{C,i}\tau'_i) & \text{if } \kappa_{C,i} < +\infty \\ (0,\tau'_i) & \text{if } \kappa_{C,i} = +\infty \end{cases},$$

form two orthogonal bases of $\mathrm{Tan}^n(\mathcal{H}^n \llcorner Q, (x,u))$ (see statement of Lemma 1.4.20).

First, we assume that $\kappa_{C,1}(x,u) = +\infty$. Since $\bigwedge_n(\pi_1|\{0\}^{n+1} \times u^\perp)$ is injective and[2]

$$\begin{aligned}
\Big[\bigwedge\nolimits_n(\pi_1|\{0\}^{n+1} \times u^\perp)\Big]\Big(\bigwedge_{i=1}^{n} \xi_{C,i}(x,u)\Big) &= \bigwedge_{i=1}^{n} \pi_1(\xi_{C,i}(x,u)) \\
&= \bigwedge_{i=1}^{n} \tau_i(x,u) = *u = \bigwedge_{i=1}^{n} \tau'_i(x,u) \\
&= \bigwedge_{i=1}^{n} \pi_1(\xi'_{C,i}(x,u)) = \Big[\bigwedge\nolimits_n(\pi_1|\{0\}^{n+1} \times u^\perp)\Big]\Big(\bigwedge_{i=1}^{n} \xi'_{C,i}(x,u)\Big)\,,
\end{aligned}$$

we infer that $\bigwedge_{i=1}^n \xi_{C,i}(x,u) = \bigwedge_{i=1}^n \xi'_{C,i}(x,u)$. Now, assume that there exist $m \in \{1,\dots,n\}$ such that

$$\kappa_{C,m}(x,u) < +\infty \quad \text{and} \quad \kappa_{C,m+1}(x,u) = +\infty \;(\text{if } m \in \{1,\dots,n-1\})\,.$$

Therefore, for any $0 < t < \boldsymbol{r}_C(x,u)$ and $i \in \{1,\dots,m\}$, we have

$$-\frac{1}{t} < -\frac{1}{\boldsymbol{r}_C(x,u)} \le \kappa_{C,i}(x,u) < +\infty \quad \Rightarrow \quad 1 + t\,\kappa_{C,i}(x,u) \in (0,+\infty)\,.$$

Since $\bigwedge_n\big((\pi_0 + t\pi_1)|T\big)$ is injective, where $T := \mathrm{Tan}^n\big(\mathcal{H}^n \llcorner Q, (x,u)\big)$, and

$$\Big[\bigwedge\nolimits_n\big((\pi_0 + t\pi_1)|T\big)\Big]\Big(\bigwedge_{i=1}^{n} \xi_{C,i}(x,u)\Big)$$

---

[2] Let us consider $* : \mathbb{R}^{n+1} \to \bigwedge_n \mathbb{R}^{n+1}$ the Hodge-star operator, with respect to $\boldsymbol{e}_1 \wedge \ldots \wedge \boldsymbol{e}_{n+1}$ (cf. [14, 1.7.8] or [19, 4.1.3]). If $u \in \mathbb{S}^n$ and $\{\tau_1,\dots,\tau_n\}$ is an orthonomal basis of $u^\perp$, such that $u \wedge \tau_1 \wedge \ldots \wedge \tau_n = \boldsymbol{e}_1 \wedge \ldots \wedge \boldsymbol{e}_{n+1}$, then it follows from the shuffle formula [14, p. 18] that $*u = \tau_1 \wedge \ldots \wedge \tau_n$.

$$
\begin{aligned}
&= \bigwedge_{i=1}^{n} (\pi_0 + t\,\pi_1)\big(\xi_{C,i}(x,u)\big) = t^{n-m} \prod_{i=1}^{m} \big(1 + t\,\kappa_{C,i}(x,u)\big) \bigwedge_{i=1}^{n} \tau_i(x,u) \\
&= t^{n-m} \prod_{i=1}^{m} \big(1 + t\,\kappa_{C,i}(x,u)\big) *u = t^{n-m} \prod_{i=1}^{m} \big(1 + t\,\kappa_{C,i}(x,u)\big) \bigwedge_{i=1}^{n} \tau_i'(x,u) \\
&= \Big[\textstyle\bigwedge_n \big((\pi_0 + t\pi_1)|T\big) \Big] \Big(\displaystyle\bigwedge_{i=1}^{n} \xi'_{C,i}(x,u) \Big) ,
\end{aligned}
$$

we infer that $\bigwedge_{i=1}^{n} \xi_{C,i}(x,u) = \bigwedge_{i=1}^{n} \xi'_{C,i}(x,u)$.

The following Heintze-Karcher type inequality for arbitrary closed sets is proved in [24].

Theorem 1.4.26 (cf. [24, Theorem 3.20])**.** *Let $C \subset \mathbb{R}^{n+1}$ be a bounded closed set with non empty interior. Let $K = \mathbb{R}^{n+1} \setminus \mathrm{Int}(C)$ and assume that*

$$\sum_{i=1}^{n} \kappa_{K,i}(x,u) \le 0\,, \quad \text{for } \mathcal{H}^n\text{-a.e. } (x,u) \in \mathrm{nor}(K)\ .$$

Then

$$(n+1)\mathcal{L}^{n+1}\big(\mathrm{Int}(C)\big) \le \int_{\mathrm{nor}(K)} J_n^{\mathrm{nor}(K)} \pi_0(x,u)\, \frac{n}{|\sum_{i=1}^{n} \kappa_{K,i}(x,u)|}\, d\mathcal{H}^n(x,u)\,.$$

Moreover, if the equality holds and there exists $q < +\infty$ such that $|\sum_{i=1}^{n} \kappa_{K,i}(x,u)| \le q$ for $\mathcal{H}^n$-a.e. $(x,u) \in \mathrm{nor}(K)$, then $\mathrm{Int}(C)$ is a finite union of disjointed (possibly mutually tangent) open balls.

Lemma 1.4.27. *If $C \subseteq \mathbb{R}^{n+1}$ and α is the contact 1 form of $\mathbb{R}^{n+1}$ (cf. Definition 1.3.4), then*

$$\langle \xi, \alpha(x,u) \rangle = 0$$

for every $\xi \in \mathrm{Tan}^n\big(\mathcal{H}^n \llcorner Q, (x,u)\big)$ and for $\mathcal{H}^n$-a.e. $(x,u) \in Q$, if $Q \subseteq \mathrm{nor}(C)$ is a $\mathcal{H}^n$-measurable set with $\mathcal{H}^n(Q) < \infty$.

Proof. This follows from the definition of α and Lemma 1.4.20. □

Definition 1.4.28. *Given $C \subseteq \mathbb{R}^{n+1}$, we define the following subsets of C*

$$
\begin{aligned}
N_1(C) &:= \{x \in C : \mathcal{H}^0\big(\mathrm{nor}(C,x)\big) = 1\}\,, \\
N_2(C) &:= \{x \in C : \mathcal{H}^0\big(\mathrm{nor}(C,x)\big) = 2\}\,, \\
N_\infty(C) &:= \{x \in C : \mathcal{H}^0\big(\mathrm{nor}(C,x)\big) = \infty\}\,.
\end{aligned}
$$

Definition 1.4.29 (Distance cone of C)**.** *We define the distance cone of $C \subseteq \mathbb{R}^{n+1}$, at $x \in \pi_0\big(\mathrm{nor}(C)\big)$, as*

$$\mathrm{Dis}(C,x) := \{v \in \mathbb{R}^{n+1} : \boldsymbol{\delta}_C(x+v) = |v|\}\,.$$

Moreover, for $x \in \pi_0\big(\mathrm{nor}(C)\big)$, we denote by $\mathrm{aff}\,\mathrm{Dis}(C,x)$ the affine hull of $\mathrm{Dis}(C,x)$.

Remark 1.4.30. We notice that $\mathrm{Dis}(C,x)$ is a closed convex subset of $\mathrm{Nor}(C,x)$, for any choice of $x \in \pi_0\big(\mathrm{nor}(C)\big)$ (cf. [13, Theorem 4.8 (2)]). Furthermore, the following property holds

$$v \in \mathrm{Dis}(C,x) \quad \Rightarrow \quad tv \in \mathrm{Dis}(C,x)\,, \text{ for every } t \in [0,1] \tag{1.4.19}$$

indeed $B_{|tv|}(x+tv) \cap C \subseteq B_{|v|}(x+v) \cap C = \varnothing$ for any $v \in \mathrm{Dis}(C,x)$ and $t \in [0,1]$. Moreover

$$\mathrm{nor}(C,x) = \Big\{ \tfrac{u}{|u|} : u \in \mathrm{Dis}(C,x) \setminus \{0\} \Big\}\,. \tag{1.4.20}$$

Definition 1.4.31 (m-th stratum of C)**.** *For any $m \in \{0,\dots,n\}$ we define the m-th stratum of C by*

$$C^{(m)} := \{x \in C : 0 < \mathcal{H}^{n-m}\big(\mathrm{nor}(C,x)\big) < \infty\}\,.$$

Lemma 1.4.32. *Let $C \subseteq \mathbb{R}^n$ be a closed set. If there exists $m \in \{0, \dots, n-1\}$ such that*

$$\dim\left(\operatorname{aff}\operatorname{Dis}(C,x)\right) = n - m + 1 \quad \text{for } x \in \pi_0\left(\operatorname{nor}(C)\right), \tag{1.4.21}$$

then

$$\mathcal{H}^{n-m}\left(\operatorname{nor}(C,x)\right) \in (0,+\infty).$$

Similarly, if

$$\dim\left(\operatorname{aff}\operatorname{Dis}(C,x)\right) = 1 \quad \text{for } x \in \pi_0\left(\operatorname{nor}(C)\right), \tag{1.4.22}$$

then

$$\mathcal{H}^0\left(\operatorname{nor}(C,x)\right) \in \{1,2\}.$$

Proof. Let $x \in \pi_0\left(\operatorname{nor}(C)\right)$ such that (1.4.21) holds, first we prove that $\mathcal{H}^{n-m}\left(\operatorname{nor}(C,x)\right) < \infty$. We notice that

$$\operatorname{nor}(C,x) \subseteq \operatorname{aff}\operatorname{Dis}(C,x) \cap \mathbb{S}^n \cong \mathbb{S}^{n-m}$$

in fact, if $\nu \in \operatorname{nor}(C,x)$ then there exists $s > 0$ such that $\boldsymbol{\delta}_C(x+t\nu) = t$ for any $t \in [0,s]$, namely $\{t\nu : t \in [0,s]\} \subseteq \operatorname{Dis}(C,x)$. Therefore $\operatorname{span}\{\nu\} \subseteq \operatorname{aff}\operatorname{Dis}(C,x)$, from which the desidered result follows. Now we prove that $\mathcal{H}^{n-m}\left(\operatorname{nor}(C,x)\right) > 0$, let $\epsilon \in (0,1)$ and we introduce the Lipschitz mappings ρ_ϵ defined as

$$\rho_\epsilon : v \in \operatorname{Dis}_\epsilon(C,x) \mapsto \frac{v}{|v|} \in \operatorname{nor}(C,x), \quad \operatorname{Dis}_\epsilon(C,x) := \operatorname{Dis}(C,x) \cap \left(B_{\frac{1}{\epsilon}}(0) \setminus B_\epsilon(0)\right)$$

where $\operatorname{Dis}_\epsilon(C,x)$ is $\mathcal{H}^{n-m+1}$-rectifiable (cf. (1.4.21)) and with strictly positive $\mathcal{H}^{n-m+1}$-measure. The last property follows from the fact that $\operatorname{Dis}(C,x)$ is convex and $\operatorname{aff}\operatorname{Dis}(C,x)$ has dimension $n-m+1$, hence $\operatorname{Dis}(C,x)$ contains an $(n-m+1)$-dimensional simplex (cf. [30, Problems 3.5.4 (14)]). We notice that

$$\mathcal{H}^1\left(\rho_\epsilon^{-1}(u)\right) > 0 \quad \text{for every } u \in \operatorname{Im}(\rho_\epsilon)$$

indeed for each $u \in \operatorname{Im}(\rho_\epsilon)$ there exists $s \in (\epsilon, \epsilon^{-1})$ such that $su \in \operatorname{Dis}(C,x)$ and applying (1.4.19) we deduce that $\{tu : t \in [\epsilon, s]\} \subseteq \rho_\epsilon^{-1}(u)$, furthermore by [14, Theorem 3.2.31] we infer that $\operatorname{Im}(\rho_\epsilon)$ is also $\mathcal{H}^{n-m}$-rectifiable. Overall, by coarea formula for rectifiable sets (cf. [14, Theorem 3.2.22]), we obtain

$$\int_{\operatorname{Dis}_\epsilon(C,x)} J_{n-m}^{\operatorname{Dis}(C,x)}\rho_\epsilon \, d\mathcal{H}^{n-m+1} = \int_{\operatorname{Im}(\rho_\epsilon)} \mathcal{H}^1\left(\rho_\epsilon^{-1}(u)\right) d\mathcal{H}^{n-m}(u). \tag{1.4.23}$$

We remember that our goal is to prove that $\mathcal{H}^{n-m}\left(\operatorname{nor}(C,x)\right) > 0$ and to this purpose, since $\operatorname{Im}(\rho_\epsilon) \subseteq \operatorname{nor}(C,x)$, we use (1.4.23) to show that $\mathcal{H}^{n-m}\left(\operatorname{Im}(\rho_\epsilon)\right) > 0$. By virtue of the previous discussion, we have only to prove that

$$J_{n-m}^{\operatorname{Dis}(C,x)}\rho_\epsilon(z) > 0 \quad \text{for } \mathcal{H}^{n-m+1}\text{-a.e. } z \in \operatorname{Dis}_\epsilon(C,x).$$

By the locality property of the approximate tangent spaces (cf. (1.1.4)) and (1.4.21), applying [14, Lemma 3.2.17] we deduce that

$$\begin{aligned}\operatorname{Tan}^{n-m+1}\left(\mathcal{H}^{n-m+1} \llcorner \operatorname{Dis}_\epsilon(C,x), z\right) &= \operatorname{Tan}^{n-m+1}\left(\mathcal{H}^{n-m+1} \llcorner \operatorname{aff}\operatorname{Dis}(C,x), z\right) \\ &= \operatorname{aff}\operatorname{Dis}(C,x) \quad \text{for } \mathcal{H}^{n-m+1}\text{-a.e. } z \in \operatorname{Dis}_\epsilon(C,x)\end{aligned}$$

therefore

$$J_{n-m}^{\operatorname{Dis}(C,x)}\rho_\epsilon(z) = \sup\left\{\left|\left[\textstyle\bigwedge_{n-m} D\rho_\epsilon(z)\right](\xi)\right| : \xi \in \textstyle\bigwedge_{n-m} \operatorname{aff}\operatorname{Dis}(C,x), \, |\xi| = 1\right\}$$

for $\mathcal{H}^{n-m+1}$-a.e. $z \in \operatorname{Dis}_\epsilon(C,x)$. For every fixed $z \in \operatorname{Dis}_\epsilon(C,x)$, since $\ker D\rho_\epsilon(z) = \operatorname{span}\{z\}$, we deduce from (1.4.21) that there exist $v_1, \dots, v_{n-m} \in \operatorname{aff}\operatorname{Dis}(C,x)$ linearly indipendent and such that $v_i \perp z$ for every $i \in \{1, \dots, n-m\}$, hence

$$J_{n-m}^{\operatorname{Dis}(C,x)}\rho_\epsilon(z) \geq \left|\left[\textstyle\bigwedge_{n-m} D\rho_\epsilon(z)\right]\left(\frac{v_1 \wedge \ldots \wedge v_{n-m}}{|v_1 \wedge \ldots \wedge v_{n-m}|}\right)\right| > 0$$

for $\mathcal{H}^{n-m+1}$-a.e. $z \in \mathrm{Dis}_\epsilon(C,x)$. Overall we conclude that $\mathcal{H}^{n-m}(\mathrm{nor}(C,x)) \in (0,+\infty)$.

Let now $x \in \pi_0(\mathrm{nor}(C))$ such that (1.4.22) holds, we prove that $\mathcal{H}^0(\mathrm{nor}(C,x)) \in \{1,2\}$. Again by [30, Problems 3.5.4 (14)], we infer that $\mathrm{Dis}(C,x)$ contains a segment containing the origin. Namely, there exists $u \in \mathbb{S}^n$ such that either $\{tu : t \in [0,s]\} \subseteq \mathrm{Dis}(C,x)$ for some $s > 0$, or

$$\{tu : t \in [-s_1,s_2]\} \subseteq \mathrm{Dis}(C,x) \quad \text{for some } s_1,s_2 > 0\,.$$

In the first case we infer that $\mathcal{H}^0(\mathrm{nor}(C,x)) = 1$, while in the second $\mathcal{H}^0(\mathrm{nor}(C,x)) = 2$. □

Remark 1.4.33. We notice that the family $\{C^{(0)},\dots,C^{(n)}\}$ is disjoint. Moreover, it has been shown (cf. [51, Remark 5.2] and [38, Theorem 4.12]) that every $C^{(m)}$ is a Borel set and is also countably $\mathcal{H}^n$-rectifiable of class 2.

Corollary 1.4.34. *Let $C \subseteq \mathbb{R}^n$ be a closed set, then:*

(i) $\pi_0(\mathrm{nor}(C)) = \bigcup_{m=0}^n C^{(m)}$;

(ii) $N_1(C) \cup N_2(C) = C^{(n)}$;

(iii) $N_\infty(C) = \bigcup_{m=0}^{n-1} C^{(m)}$.

Proof. To prove *(i)*, since every $C^{(m)} \subseteq \pi_0(\mathrm{nor}(C))$, we show that $\pi_0(\mathrm{nor}(C)) \subseteq \bigcup_{m=0}^n C^{(m)}$. Let $x \in \pi_0(\mathrm{nor}(C))$, namely there exist $\nu \in \mathrm{nor}(C,x)$ and $s > 0$ so that $\boldsymbol{\delta}_C(x+t\nu) = t$ for any $t \in [0,s]$, therefore $\{t\nu : t \in [0,s]\} \subseteq \mathrm{Dis}(C,x)$. Hence $\dim(\mathrm{aff}\,\mathrm{Dis}(C,x)) \in \{1,\dots,n+1\}$, then by Lemma 1.4.32 the desidered result follows.

To prove *(ii)*, we notice that $N_1(C) \cup N_2(C) \subseteq C^{(n)}$. To show that $C^{(n)} \subseteq N_1(C) \cup N_2(C)$ assume that there exists $x \in C^{(n)} \setminus (N_1(C) \cup N_2(C))$, namely $\mathcal{H}^0(\mathrm{nor}(C,x)) \in \mathbb{N} \setminus \{0,1,2\}$. This means that $\mathrm{nor}(C,x)$ contains at least two linearly independent unit vectors $u_1,u_2 \in \mathbb{S}^n$, then from (1.4.19) we infer that $\{tu_i : t \in [0,s_i]\} \subseteq \mathrm{Dis}(C,x)$ for any $i \in \{1,2\}$ and for some $s_1,s_2 \in \mathbb{R}^+$. We deduce that $\dim(\mathrm{aff}\,\mathrm{Dis}(C,x)) \in \{2,\dots,n+1\}$ and by Lemma 1.4.32 we infer $\mathcal{H}^0(\mathrm{nor}(C,x)) = \infty$, which is a contradiction.

To prove *(iii)*, we notice that $\bigcup_{m=0}^{n-1} C^{(m)} \subseteq N_\infty(C)$. Now assume that $x \in N_\infty(C)$, again $\mathrm{nor}(C,x)$ contains at least two linearly indipendent unit vectors, so $\dim(\mathrm{aff}\,\mathrm{Dis}(C,x))$ is at least 2 and applying Lemma 1.4.32 we infer that $x \in \bigcup_{m=0}^{n-1} C^{(m)}$. □

Remark 1.4.35. A conseguence, of Lemma 1.4.32 and Corollary 1.4.34, is given by the statement:

if $x \in N_2(C)$, then there exists $u \in \mathbb{S}^n$ such that $\mathrm{nor}(C,x) = \{u,-u\}$.

Again, if there exist $u_1,u_2 \in \mathbb{S}^n$ linearly independent such that $\mathrm{nor}(C,x) = \{u_1,u_2\}$, then from (1.4.19) and by Lemma 1.4.32 we infer that $x \in C^{(m)}$ for some $m \in \{0,\dots,n-1\}$, which is a contradiction since $x \in C^{(n)}$ (cf. Corollary 1.4.34 *(ii)*).

Lemma 1.4.36. *Assume that $C \subseteq \mathbb{R}^{n+1}$ is closed, then the following statements hold.*

(i) $N_1(C)$, $N_2(C)$ and $N_\infty(C)$ are Borel subset of C.

(ii) Given F a C^2-diffeomorphism of $\mathbb{R}^{n+1}$, then

$$F(N_i(C)) = N_i(F(C)) \quad \text{for } i = 1,2,\infty\,.$$

(iii) The multivalued function

$$\mathrm{nor}(C,\cdot)|N_2(C) : N_2(C) \to \mathcal{P}(\mathbb{S}^n)$$

admits an $\mathcal{H}^n$-measurable selection $\nu_C : N_2(C) \to \mathbb{S}^n$ (namely $\nu_C(x) \in \mathrm{nor}(C,x)$ for every $x \in N_2(C)$). Moreover,

$$\nu_C(p) \in \mathrm{Nor}^n(\mathcal{H}^n \llcorner C, p) \quad \text{for every } p \in N_2(C)\,. \tag{1.4.24}$$

Proof. To prove *(i)*, we notice immediately that $N_\infty(C)$, $N_1(C)\cup N_2(C)$ and $\pi_0(\mathrm{nor}(C))$ are Borel subsets of C (cf. Remark 1.4.33 and Corollary 1.4.34). Then, if we consider a countable dense subset $\mathscr{D}$ of $\mathbb{S}^n$, our intention is to obtain the desidered result by proving that

$$X\setminus N_\infty(C)=N_2(C)\,, \tag{1.4.25}$$

where X is the Borel subset of C defined as

$$X:=\bigcup_{h=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{j=k}^{\infty}\bigcup_{(s,u,t,v)\in\mathcal{F}_{h,j}}C_{k,(s,u,t,v)}\,,$$

$$C_{k,(s,u,t,v)}:=\Big\{x\in C:\Big|\frac{\boldsymbol{\delta}_C(x+su)}{s}-1\Big|\leq k^{-1}\,,\quad \Big|\frac{\boldsymbol{\delta}_C(x-tv)}{t}-1\Big|\leq k^{-1}\Big\}\quad\text{for } k\in\mathbb{N}^+,\,(s,u,t,v)\in\mathcal{F}_{h,j}$$

and

$$\mathcal{F}_{h,j}:=\{(s,u,t,v)\in\mathbb{Q}^+\times\mathscr{D}\times\mathbb{Q}^+\times\mathscr{D}:h^{-1}\leq s,t\leq h\,,\quad |(s,u)-(t,v)|\leq j^{-1}\}\quad\text{for } h,j\in\mathbb{N}^+\,.$$

To prove (1.4.25), let $x\in N_2(C)$. Namely, we assume that there exists $u\in\mathbb{S}^n$ such that $\boldsymbol{\delta}_C(x\pm su)=s$ for some $s>0$. If we consider $\{s_k\}_{k\in\mathbb{N}}\subset\mathbb{Q}^+$ and $\{u_k\}_{k\in\mathbb{N}}\subset\mathscr{D}$ such that

$$s_k\xrightarrow[k\to\infty]{}s\quad,\quad u_k\xrightarrow[k\to\infty]{}u\quad\text{and}\quad\frac{\boldsymbol{\delta}_C(x\pm s_ku_k)}{s_k}\xrightarrow[k\to\infty]{}1\,,$$

we readily infer that $x\in X$. Since $x\notin N_\infty(C)$ we deduce that $N_2(C)\subseteq X\setminus N_\infty(C)$. Now, let $x\in X\setminus N_\infty(C)$. Namely there exists $h\in\mathbb{N}^+$ so that, for every $k\in\mathbb{N}^+$, there exist an integer $m_k\geq k$ and $(s_k,u_k,t_k,v_k)\in\mathbb{Q}^+\times\mathscr{D}\times\mathbb{Q}^+\times\mathscr{D}$ such that

$$\Big|\frac{\boldsymbol{\delta}_C(x+s_ku_k)}{s_k}-1\Big|\leq k^{-1}\quad,\quad\Big|\frac{\boldsymbol{\delta}_C(x-t_kv_k)}{t_k}-1\Big|\leq k^{-1}$$

$$h^{-1}\leq s_k,t_k\leq h\quad,\quad|(s_k,u_k)-(t_k,v_k)|\leq\frac{1}{m_k}\,.$$

Up to a subsequence we can assume that

$$u_k\xrightarrow[k\to\infty]{}u\in\mathbb{S}^n\quad,\quad v_k\xrightarrow[k\to\infty]{}v\in\mathbb{S}^n$$

$$s_k\xrightarrow[k\to\infty]{}s\in\mathbb{R}^+\quad,\quad t_k\xrightarrow[k\to\infty]{}t\in\mathbb{R}^+$$

then we infer that $\boldsymbol{\delta}_C(x\pm su)=s$. Since $x\notin N_1(C)\cap N_\infty(C)$ and

$$\pi_0(\mathrm{nor}(C))=N_1(C)\cup N_2(C)\cup N_\infty(C)\,, \tag{1.4.26}$$

we deduce that $x\in N_2(C)$, namely $X\setminus N_\infty(C)\subseteq N_2(C)$. From (1.4.25), since $N_\infty(C)$ and X are Borel subset of C, we conclude that $N_2(C)$ is a Borel subset of C. Then, from (1.4.26), also $N_1(C)$ is a Borel subset of C.

To prove *(ii)*, we notice that $F(C^{(m)})=F(C)^{(m)}$ for every $m\in\{0,\dots,n\}$ (cf. [54, Lemma 2.1]). Hence, from Corollary 1.4.34, we infer that

$$F(N_\infty(C))=N_\infty(F(C))\quad\text{and}\quad F(N_1(C))\cup F(N_2(C))=N_1(F(C))\cup N_2(F(C))$$

and if we prove that

$$F(N_1(C))\cap N_2(F(C))=\varnothing\,, \tag{1.4.27}$$

$$F(N_2(C))\cap N_1(F(C))=\varnothing\,, \tag{1.4.28}$$

we conclude also that $F\big(N_1(C)\big) = N_1\big(F(C)\big)$ and $F\big(N_2(C)\big) = N_2\big(F(C)\big)$. To prove (1.4.27) assume that there exist $x \in N_1(C)$, $u \in \mathbb{S}^n$ and $s > 0$ such that $B_s\big(F(x) \pm su\big) \cap F(C) = \emptyset$, namely $\Omega_\pm \cap C = \emptyset$ where $\Omega_\pm := F^{-1}\big[B_s\big(F(x) \pm su\big)\big]$ are two disjoint C^2-regular domains with $x \in \partial\Omega_+ \cap \partial\Omega_-$ and $\mathrm{Tan}(\partial\Omega_+, x) = DF^{-1}(x)(u^\perp) = \mathrm{Tan}(\partial\Omega_-, x) \in \mathbf{G}(n+1, n)$ by [14, 3.1.21]. Since $\Omega_\pm$ are C^2-regular domains, so they satisfy the interior sphere condition[3], there exists $r > 0$ such that

$$B_r(x + r\nu) \subseteq \Omega_+ \quad \text{and} \quad B_r(x - r\nu) \subseteq \Omega_-\,,$$

where $\nu \in \mathbb{S}^n$ is chosen so that $\nu^\perp = \mathrm{Tan}(\partial\Omega_+, x) = \mathrm{Tan}(\partial\Omega_-, x)$. Hence $B_r(x \pm r\nu) \cap C = \emptyset$, namely a contradiction since $x \in N_1(C)$. Similarly, to prove (1.4.28), assume that there exist $x \in C$, $v \in \mathbb{S}^n$ and $s > 0$ such that $B_s(x \pm sv) \cap C = \emptyset$ and $F(x) \in N_1\big(F(C)\big)$. Namely, $\Omega_\pm \cap F(C) = \emptyset$ where $\Omega_\pm := F\big[B_s(x \pm sv)\big]$ are two disjoint C^2-regular domains so that $F(x) \in \partial\Omega_+ \cap \partial\Omega_-$ and $\mathrm{Tan}\big(\partial\Omega_+, F(x)\big) = \mathrm{Tan}\big(\partial\Omega_-, F(x)\big) \in \mathbf{G}(n+1, n)$, again by the C^2-regularity of $\Omega_\pm$ we obtain a contradiction since $F(x) \in N_1\big(F(C)\big)$.

To prove *(iii)*, we intend to apply [10, Theorem *III*.6]. That is, if

$$N_2(C; U) := \big\{x \in N_2(C) : \mathrm{nor}(C, x) \cap U \neq \emptyset\big\}$$

is an $\mathcal{H}^n$-measurable subset of C for any open set U in $\mathbb{S}^n$, then the multivalued function $\mathrm{nor}(C, \cdot) : N_2(C) \to \mathcal{P}(\mathbb{S}^n)$ admits an $\mathcal{H}^n$-measurable selection $\nu_C : N_2(C) \to \mathbb{S}^n$ (in order to apply [10, Theorem *III*.6], we notice that for any $x \in N_2(C)$ the set $\mathrm{nor}(C, x)$ is compact and hence complete). Let U be an open set in $\mathbb{S}^n$, we notice that

$$N_2(C; U) = \pi_0\big(\mathrm{nor}(C) \cap (N_2(C) \times U)\big) \tag{1.4.29}$$

where $\mathrm{nor}(C) \cap (N_2(C) \times U)$ is a countably $\mathcal{H}^n$-rectifiable Borel subset of $\mathbb{R}^{n+1}$, hence with σ-finite $\mathcal{H}^n$-measure (cf. [36, Lemma 15.5 (1)]). Namely there exists a family $\{X_i\}_{i\in\mathbb{N}}$ of $\mathcal{H}^n$-measurable sets, with finite $\mathcal{H}^n$-measure, such that $\mathrm{nor}(C) \cap (N_2(C) \times U) = \bigcup_{i=1}^\infty X_i$. Since $\mathcal{H}^n$ is a Borel-regular outer measure, for any $i \in \mathbb{N}$ we have that (cf. [19, Proposition 6.2 *(ii)*])

$$\mathcal{H}^n(X_i) = \sup\big\{\mathcal{H}^n(F) : F \subseteq X_i\,,\, F \text{ closed}\big\}\,,$$

so there exists a sequence $\{F_{i,j}\}_{j\in\mathbb{N}}$ of closed subsets of X_i such that $\mathcal{H}^n(X_i \setminus \bigcup_{j=1}^\infty F_{i,j}) = 0$. Since any closed set in the Euclidean space is countable union of compact sets, there exists a sequence $\{K_{i,j}\}_{j\in\mathbb{N}}$ of compact subsets of X_i such that $\mathcal{H}^n(X_i \setminus \bigcup_{j=1}^\infty K_{i,j}) = 0$, overall

$$\mathcal{H}^n\Big(\big[\mathrm{nor}(C) \cap \big(N_2(C) \times U\big)\big] \setminus \bigcup_{i,j=1}^\infty K_{i,j}\Big) = 0\,.$$

We infer that $N_2(C; U)$ is an $\mathcal{H}^n$-measurable subset of C, indeed from (1.4.29) we obtain

$$N_2(C; U) = \bigcup_{i,j=1}^\infty \pi_0(K_{i,j}) \cup \pi_0\Big(\big[\mathrm{nor}(C) \cap \big(N_2(C) \times U\big)\big] \setminus \bigcup_{i,j=1}^\infty K_{i,j}\Big)\,,$$

where any $\pi_0(K_{i,j})$ is compact and $\pi_0\big(\big[\mathrm{nor}(C) \cap \big(N_2(C) \times U\big)\big] \setminus \bigcup_{i,j=1}^\infty K_{i,j}\big)$ is $\mathcal{H}^n$-negligible.

The proof of (1.4.24) follows immediately as a consequence of $\mathrm{nor}(C, x) \subseteq \mathrm{Nor}(C, x)$ (cf. [13, Theorem 4.8 (2)]) and $\mathrm{Tan}^n(\mathcal{H}^n \llcorner C, x) \subseteq \mathrm{Tan}(C, x)$ (cf. Remark 1.1.1 *(i)*), for any $x \in C$. The proof is complete. □

[3]**Interior sphere condition.** Given $\Omega \subseteq \mathbb{R}^n$ an open set, we say that Ω satisfies the *interior sphere condition* if for every $x \in \partial\Omega$ there exist $\nu \in \mathbb{S}^{n-1}$ and $r > 0$ such that $B_r(x + r\nu) \subseteq \Omega$, notice that $x \in \partial B_r(x + r\nu)$. The interior sphere condition always holds if Ω is a C^2-regular domain (cf. [21, Remark 4.3.8]), in this situation we have $\mathrm{Tan}(\partial\Omega, x) = \mathrm{Tan}(\partial B_r(x + r\nu), x) = \nu^\perp$ (the proof is the same as that performed for (4.1.7)).

Chapter 2

Fine properties of $W^{2,n}$-graphs

2.1 Legendrian cycles over $W^{2,n}$-graphs

Suppose $U \subseteq \mathbb{R}^n$ is an open set. We denote by $W^{k,p}(U)$ (resp. $W^{k,p}_{loc}(U)$) the usual Sobolev space of k-times weakly differentiable functions, whose distributional derivatives up to order k belong to the Lebesgue space $L^p(U)$ (resp. $L^p_{loc}(U)$); cf. [20, Chapter 7]. We denote by ∇f and $\mathbf{D}^i f$ the distributional gradient and the distributional i-differential of a Sobolev function f.

Definition 2.1.1. *Given $U \subseteq \mathbb{R}^n$ open set and $f \in C^0(U)$, we define $\Gamma^+(f,U)$ as the set of $x \in U$ for which there exists $p \in \mathbb{R}^n$, such that*

$$f(y) \leq f(x) + p \bullet (y - x) \quad \textit{for every } y \in U\,.$$

Definition 2.1.2. *Given $U \subseteq \mathbb{R}^n$ open set and $f \in C^0(U)$, we define $\mathcal{S}^*(f)$ ($\mathcal{S}_*(f)$, resp.) as the set of $x \in U$ where there exists a polynomial function P, of degree at most 2, such that $P(x) = f(x)$ and*

$$\limsup_{y\to x} \frac{f(y) - P(y)}{|y-x|^2} < \infty \qquad \left(\liminf_{y\to x} \frac{f(y) - P(y)}{|y-x|^2} > -\infty\,, \textit{ resp.} \right).$$

We set $\mathcal{S}(f)$ to be the set of $x \in U$ where there exists a polynomial function P of degree at most 2 such that $P(x) = f(x)$ and

$$\lim_{y\to x} \frac{f(y) - P(y)}{|y-x|^2} = 0\,.$$

Remark 2.1.3. (On the Borelianity of $\Gamma^+(f,U)$, $\mathcal{S}^*(f)$ and $\mathcal{S}_*(f)$).

To prove that $\Gamma^+(f,U) \in \mathcal{B}(U)$, we define the sets

$$U^+(f,\mu) := \big\{ a \in U : \exists\, L \in \hom(\mathbb{R}^n;\mathbb{R}),\ f \leq L \text{ on } U,\\ L(a) = f(a),\ \|\nabla L\|_\infty \leq \mu \big\} \quad \text{for } \mu \geq 0\,.$$

Since $\Gamma^+(f,U) = \bigcup_{i=1}^\infty U^+(f,i)$, we conclude if we show that $U^+(f,\mu) \in \mathcal{B}(U)$. To this end, we first introduce the class, which we may assume to be non-empty without loss of generality,

$$\mathcal{C}^+(f,\mu) := \big\{ L \in \hom(\mathbb{R}^n;\mathbb{R}) : f \leq L \text{ on } U,\ \|\nabla L\|_\infty \leq \mu \big\}\,,$$

then we define the map

$$\mathcal{L}^+[f,\mu] : x \in U \mapsto \inf \big\{ L(x) : L \in \mathcal{C}^+(f,\mu) \big\} \in \mathbb{R} \cup \{-\infty\}$$

and we notice that $\mathcal{L}^+[f,\mu]$ is upper semicontinuous. Indeed, the infimum of every collection of upper semicontinuous function, on a topological space, is upper semicontinuous (cf. [49, (c) p. 38]). Hence, we obtain the desidered result since one can readily prove that

$$U^+(f,\mu) = \big\{ x \in U : f(x) = \mathcal{L}^+[f,\mu](x) \big\}\,.$$

To show that $\mathcal{S}^*(f) \in \mathcal{B}(U)$, we assume $f \in C^0(U) \cap W^{2,n}_{loc}(U)$, since $(C^0 \cap W^{2,n}_{loc})$-functions are the object of our discussion. Then, by applying Lemma 2.1.9 and Lemma 2.1.17 (both of which are not related to the Borelianity of $\mathcal{S}^*(f)$), one can prove that (cf. Lemma 4.1.7 *(ii)*)

$$\overline{f}(\mathcal{S}^*(f)) = N_1(C_f)\,, \tag{2.1.1}$$

where $C_f := \{(x,u) \in U \times \mathbb{R} : u \leq f(x)\}$ is the *cato-graph of* f and

$$N_1(C_f) := \{x \in C_f : \mathcal{H}^0(\mathrm{nor}(C_f, x)) = 1\}\,.$$

From (2.1.1), since f is continuous and $N_1(C_f) \in \mathcal{B}(U \times \mathbb{R})$ (cf. Lemma 1.4.36 *(i)*), we obtain the desidered result. Similarly one can prove that $\mathcal{S}_*(f) \in \mathcal{B}(U)$.

Remark 2.1.4. (Generalized Stepanoff theorem)**.** The following generalization of Stepanoff's theorem (cf. [14, 3.1.9]) holds true. Recall some notation from [9]. Given $U \subset \mathbb{R}^n$ a bounded open set, $f : U \to \mathbb{R}$ bounded and $a \in U$, we say that $f \in T^2_\infty(a)$ if there exists an affine function $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$L(a) = f(a) \quad \text{and} \quad \limsup_{x \to a} \frac{|f(x) - L(x)|}{|x-a|^2} < \infty\,,$$

while we say that $f \in t^2_\infty(a)$ if $a \in \mathcal{S}(f)$. By [27, Theorem 1] the following assertion holds:

assuming that $f : \overline{U} \to \mathbb{R}$ *is bounded,*
if $f \in T^2_\infty(y)$ *for* $\mathcal{L}^n$*-a.e.* $y \in U$ *then* $f \in t^2_\infty(y)$ *for* $\mathcal{L}^n$*-a.e.* $y \in U$.

As an application of the generalized Stepanoff's theorem we give the following.

Lemma 2.1.5. *Given* $U \subseteq \mathbb{R}^n$ *an open set and* $f \in C^0(U)$ *such that*

$$\mathcal{L}^n\Big(U \setminus (\mathcal{S}^*(f) \cap \mathcal{S}_*(f))\Big) = 0\,,$$

then $\mathcal{L}^n(U \setminus \mathcal{S}(f)) = 0$.

Proof. Let us consider $x \in \mathcal{S}^*(f) \cap \mathcal{S}_*(f)$, then there exist $m_1, m_2 \in \mathbb{R}$ such that

$$m_1 \leq \frac{f(y) - f(x) - \nabla f(x) \bullet (y-x)}{|y-x|^2} \leq m_2 \quad \text{for every } y \in B_r(x) \setminus \{x\}\,.$$

Namely, the following implication holds

$$x \in \mathcal{S}^*(f) \cap \mathcal{S}_*(f) \quad \Rightarrow \quad f \in T^2_\infty(x)\,. \tag{2.1.2}$$

Since we are assuming that $\mathcal{S}^*(f) \cap \mathcal{S}_*(f)$ has full $\mathcal{L}^n$-measure in U, from (2.1.2) and by the generalized Stepanoff's theorem, we conclude that $\mathcal{S}(f)$ has full $\mathcal{L}^n$-measure in U. □

In the following lemma, using a standard approximation procedure, we extend the classical Alexandrov-Bakelmann-Pucci (ABP) estimate in [20, Theorem 9.2], which is stated for C^2-functions, to $W^{2,n}$-functions.

Lemma 2.1.6 (ABP estimate)**.** *Given* $U \subseteq \mathbb{R}^n$ *an open set and* $f \in C^0(\overline{U}) \cap W^{2,n}(U)$*, then*

$$\sup_U f \leq \sup_{\partial U} f + \frac{\mathrm{diam}(U)}{\boldsymbol{\alpha}(n)^{1/n}} \left(\int_{\Gamma^+(f,U)} |\det \mathbf{D}^2 f| \, d\mathcal{L}^n \right)^{1/n}.$$

Proof. We choose a sequence $\{f_k\}_{k \in \mathbb{N}} \subset C^\infty(\mathbb{R}^n)$ such that $f_k \to f$ in $W^{2,n}(V)$ and $f_k \to f$ uniformly on V, for every open set V compactly contained in U. Therefore, if V is an open set compactly contained in U, we use [20, Lemma 9.2] to estimate

$$\begin{aligned}\sup_V f &\leq \sup_V |f - f_k| + \sup_V f_k \\ &\leq \sup_V |f - f_k| + \sup_{\partial V} f_k + \frac{\mathrm{diam}(V)}{\boldsymbol{\alpha}(n)} \left(\int_{\Gamma^+(f_k,V)} |\det D^2 f_k| \, d\mathcal{L}^n \right)^{1/n}\end{aligned}$$

$$\leq \sup_V |f - f_k| + \sup_{\partial V} |f - f_k| + \sup_{\partial V} f + \frac{\operatorname{diam}(V)}{\boldsymbol{\alpha}(n)^{1/n}} \left(\int_V |\det D^2 f_k - \det \mathbf{D}^2 f| \, d\mathcal{L}^n + \int_{\Gamma^+(f_k, V)} |\det \mathbf{D}^2 f| \, d\mathcal{L}^n \right)^{1/n}.$$

Let $\mu := \mathcal{L}^n \llcorner |\det \mathbf{D}^2 f|$. Employing [8, Lemma A.1] we have that

$$\bigcap_{h \geq 1} \bigcup_{k \geq h} \Gamma^+(f_k, V) \subseteq \Gamma^+(f, V),$$

and

$$\mu(\Gamma^+(f,V)) \geq \mu\Big(\bigcap_{h\geq 1} \bigcup_{k \geq n} \Gamma^+(f_k, V) \Big) = \lim_{h\to\infty} \mu \Big(\bigcup_{k\geq h} \Gamma^+(f_k, V) \Big) \geq \limsup_{h\to\infty} \mu(\Gamma^+(f_h, V)).$$

Now we can pass to limit in the estimate above to conclude

$$\sup_V f \leq \sup_{\partial V} f + \frac{\operatorname{diam}(V)}{\boldsymbol{\alpha}(n)^{1/n}} \left(\int_{\Gamma^+(f,V)} |\det \mathbf{D}^2 f| \, d\mathcal{L}^n \right)^{1/n}$$

for every V compactly contained in U, and the same inequality holds with V replaced by U. The proof is complete. □

The next result asserts that a $W^{2,n}$-function is pointwise twice differentiable almost everywhere. In particular, can be deduced from a more general result of Calderon-Zygmund, asserting that functions in $W^{2,p}(U)$ with $p > \frac{n}{2}$ admit second-order Taylor expansion $\mathcal{L}^n$-a.e. on U (cf. [9] or [8, Proposition 2.2]). Alternatively, can be deduced from [15, Theorem 1.1], where almost everywhere twice differentiability is proved for a larger class of functions, namely the *strongly approximable functions*. Here we provide a different proof of this result for $W^{2,n}$-functions employing the method used by Trudinger in [59, Theorem 1] to treat the second order differentiability of viscosity solutions of second order elliptic PDE's. It is worth noting that both methods are based on the classical Alexandrov-Bakelmann-Pucci estimate.

Theorem 2.1.7. *Let $U \subseteq \mathbb{R}^n$ be an open set and let $f \in C^0(U) \cap W^{2,n}_{loc}(U)$. Then $\mathcal{L}^n(U \setminus \mathcal{S}(f)) = 0$.*

Proof. Fix $x \in U$ and $r > 0$ such that $\overline{B_r(x)} \subset U$. For every positive integer k, we define

$$v_k(y) := f(y) - k(|y-x|^2 - r^2) \quad \text{for } y \in U.$$

Noting that $v_k(y) = f(y)$ for $y \in \partial B_r(x)$ and $v_k \geq f + kr^2$ on U, we apply Lemma 2.1.6 (with f and U replaced by v_k and $B_r(x)$), obtaining

$$kr^2 + \sup_{B_r(x)} f \leq \sup_{\partial B_r(x)} f + c(n)\, r \left(\int_{\Gamma^+(v_k, B_r(x))} |\det \mathbf{D}^2 v_k| \, d\mathcal{L}^n \right)^{1/n}.$$

Since $\mathbf{D}^2 v_k(y) \leq 0$ for $\mathcal{L}^n$-a.e. $y \in \Gamma^+(v_k, B_r(x))$, we have that

$$\begin{aligned} |\det \mathbf{D}^2 v_k(y)| &\leq \frac{1}{n^n} |\operatorname{trace} \mathbf{D}^2 v_k(y)|^n \\ &= \frac{1}{n^n} |\operatorname{trace} \mathbf{D}^2 f(y) + 2kn|^n \\ &\leq c(n)\big(|\operatorname{trace} \mathbf{D}^2 f(y)|^n + k^n\big) \quad \text{for } \mathcal{L}^n\text{-a.e. } y \in \Gamma^+(v_k, B_r(x)). \end{aligned}$$

Therefore,

$$kr^2 + \sup_{B_r(x)} f \leq \sup_{\partial B_r(x)} f + c(n)\, r \left(\|\operatorname{trace} \mathbf{D}^2 f\|_{L^n(B_r(x))} + k\, \mathcal{L}^n\Big(\Gamma^+\big(v_k, B_r(x)\big)\Big)^{1/n} \right).$$

Noting that for $y \in \Gamma^+(v_k, B_r(x))$ there exists $p \in \mathbb{R}^n$ such that

$$f(z) \le f(y) + k|z-x|^2 - k|y-x|^2 + p \bullet (z-y) \quad \text{for } z \in B_r(x)$$

we easily infer that

$$\Gamma^+(v_k, B_r(x)) \subseteq \mathcal{S}^*(f) \cap B_r(x)$$

and

$$kr^2 + \sup_{B_r(x)} f \le \sup_{\partial B_r(x)} f + c(n)\, r \left(\|\operatorname{trace} \mathbf{D}^2 f\|_{L^n(B_r(x))} + k\, \mathcal{L}^n(\mathcal{S}^*(f) \cap B_r(x))^{1/n} \right).$$

Dividing both sides by kr^2 and letting $k \to \infty$, we conclude that

$$1 \le c(n) \frac{\mathcal{L}^n(\mathcal{S}^*(f) \cap B_r(x))}{r^n}. \tag{2.1.3}$$

Since (2.1.3) holds for every $x \in U$ and for every $r > 0$ small enough, it follows from [14, 2.10.19 (4)] that

$$\mathcal{L}^n(U \setminus \mathcal{S}^*(f)) = 0.$$

Since $\mathcal{S}_*(f) = \mathcal{S}^*(-f)$, we also infer that $\mathcal{L}^n(U \setminus \mathcal{S}_*(f)) = 0$. Therefore

$$\mathcal{L}^n\Big(U \setminus (\mathcal{S}^*(f) \cap \mathcal{S}_*(f))\Big) = 0,$$

whence the conclusion follows from Lemma 2.1.5. The proof is complete. □

Lemma 2.1.8. *Given $U \subseteq \mathbb{R}^n$ an open set and $f \in W^{1,1}_{loc}(U;\mathbb{R}^k)$, then f is $(\mathcal{L}^n \llcorner U)$-approximately differentiable at $\mathcal{L}^n$-a.e. $x \in U$ with $\operatorname{ap} Df(x) = \mathbf{D}f(x)$. In particular, there exists countably many $\mathcal{L}^n$-measurable sets $A_i \subseteq U$ such that $\mathcal{L}^n(U \setminus \bigcup_{i=1}^\infty A_i) = 0$ and $\operatorname{Lip}(f|A_i) < \infty$ for any $i \in \mathbb{N}_{\ge 1}$.*

Proof. This statement follows from classical pointwise differentiability properties of Sobolev functions. Indeed, by [61, Theorem 3.4.2] (or [12, Theorem 6.1]),

$$\lim_{r \to 0} r^{-n-1} \int_{B_r(x)} |f(y) - f(x) - \mathbf{D}f(x)(y-x)|\, d\mathcal{L}^n(y) = 0 \tag{2.1.4}$$

for $\mathcal{L}^n$-a.e. $x \in U$. Fix now $x \in U$ such that (2.1.4) holds, define also the affine function $L_x(y) := f(x) + \mathbf{D}f(x)(y-x)$ for $y \in \mathbb{R}^n$ and notice that

$$\frac{\epsilon\, \mathcal{L}^n(B_r(x) \cap \{y : |f(y) - L_x(y)| \ge \epsilon\, r\})}{r^n} \le r^{-n-1} \int_{B_r(x)} |f(y) - L_x(y)|\, d\mathcal{L}^n(y)$$

for every $\epsilon > 0$. Therefore, by (2.1.4) and [53, Lemma 2.7],

$$\Theta^n(\mathcal{L}^n \llcorner \{y : |f(y) - L_x(y)| \ge 2\epsilon\, |y-x|\}, x) = 0 \quad \text{for every } \epsilon > 0,$$

we conclude that f is $(\mathcal{L}^n \llcorner U)$-approximately differentiable at x with $\operatorname{ap} Df(x) = \mathbf{D}f(x)$ applying [14, p. 253] with ϕ and m replaced by $\mathcal{L}^n \llcorner U$ and n. Now we can use [14, 3.1.8] to infer the existence of the countable cover A_i. The proof is complete. □

We now recall a result due to Ulrich Menne (cf. [37, Appendix B]), which provides the basic oscillation lemma to establish the Lusin (N)-condition in Lemma 2.1.15.

Lemma 2.1.9 (Menne)**.** *Given $a \in \mathbb{R}^n$, $r > 0$, $f \in C^0(B_r(a)) \cap W^{2,n}(B_r(a))$, and $g \in C^2(B_r(a))$ such that*

$$g(a) = f(a) \quad \text{and} \quad f(x) \ge g(x), \quad \text{for every } x \in B_r(a).$$

Then f is pointwise differentiable at a with $Df(a) = Dg(a)$, and there exists a constant $c(n)$, depending only on n, such that

$$\|\mathbf{D}f - Df(a)\|_{L^n(B_r(a))} \le c(n)\, r \left(\|\mathbf{D}^2 f\|_{L^n(B_r(a))} + r\, \|D^2 g\|_{L^\infty(B_r(a))} \right), \tag{2.1.5}$$

$$\|f - L_a\|_{L^\infty(B_r(a))} \leq c(n)\, r \left(\|\mathbf{D}^2 f\|_{L^n(B_r(a))} + r\, \|D^2 g\|_{L^\infty(B_r(a))} \right), \tag{2.1.6}$$

where $L_a(x) := f(a) + Df(a)(x-a)$ *for* $x \in \mathbb{R}^n$.

Proof. In this proof we denote the average of a map h defined on (a subset of) $\mathbb{R}^n$, with values in some normed space Y, as follows

$$(h)_S := \frac{1}{\mathcal{L}^n(S)} \int_S h\, d\mathcal{L}^n \in Y.$$

We assume $g = 0$, noting that the general case easily follows from it. Therefore, $f(x) \geq 0$ for every $x \in B_r(a)$ and $f(a) = 0$. For each $0 < s \leq r$, noting that $(\mathbf{D}f)_{B_s(a)}$ is an homomorphism from $\mathbb{R}^n$ to $\mathbb{R}$, we define the affine functions $A_s : \mathbb{R}^n \to \mathbb{R}$ by

$$A_s(x) := (f)_{B_s(a)} + (\mathbf{D}f)_{B_s(a)}(x-a), \quad \text{for } x \in \mathbb{R}^n$$

and we notice that $(f - A_s)_{B_s(a)} = 0$ for each $0 < s \leq r$. Since $\mathbf{D}f - (\mathbf{D}f)_{B_s(a)} = \mathbf{D}(f - A_s)$, we apply Poincare inequality (cf. [20, (7.45) p. 164])

$$\|\mathbf{D}(f - A_s)\|_{L^n(B_s(a))} \leq c(n)\, s\, \|\mathbf{D}^2 f\|_{L^n(B_s(a))}$$

and

$$\|f - A_s\|_{L^n(B_s(a))} \leq c(n)\, s^2\, \|\mathbf{D}^2 f\|_{L^n(B_s(a))}$$

for $0 < s \leq r$. Let $T : W^{2,n}\big(B_s(a)\big) \to W^{2,n}\big(B_1(0)\big)$ be the linear map defined as

$$T(u)(x) := u(a + sx) \quad \text{for } x \in B_1(0)$$

and notice that

$$\|T(u)\|_{W^{2,n}(B_1(0))} = s^{-1}\|u\|_{L^n(B_s(a))} + \|\mathbf{D}u\|_{L^n(B_s(a))} + s\|\mathbf{D}^2 u\|_{L^n(B_s(a))}$$

for $u \in W^{2,n}(B_s(a))$. By Sobolev embedding theorem [20, Theorem 7.26] there exists a positive constant $c(n)$, depending only on n, such that

$$\|u\|_{L^\infty(B_1(0))} \leq c(n)\|u\|_{W^{2,n}(B_1(0))} \quad \text{for } u \in W^{2,n}(B_1(0))\,.$$

Therefore,

$$\begin{aligned}
&\|f - A_s\|_{L^\infty(B_s(a))} \\
&\quad = \|T(f - A_s)\|_{L^\infty(B_1(0))} \\
&\quad \leq c(n)\|T(f - A_s)\|_{W^{2,n}(B_1(0))} \\
&\quad \leq c(n)\Big(s^{-1}\|f - A_s\|_{L^n(B_s(a))} + \|\mathbf{D}(f - A_s)\|_{L^n(B_s(a))} + s\,\|\mathbf{D}^2(f - A_s)\|_{L^n(B_s(a))}\Big) \\
&\quad \leq c(n)\, s\, \|\mathbf{D}^2 f\|_{L^n(B_s(a))}\,.
\end{aligned}$$

For $0 < t \leq s$ we notice that

$$|A_s(a) - A_t(a)| \leq \|A_s - A_t\|_{L^\infty(B_t(a))} \leq c(n)\, s\, \|\mathbf{D}^2 f\|_{L^n(B_s(a))}\,.$$

Since $\lim_{t\to 0} A_t(a) = f(a) = 0$, we conclude

$$|A_s(a)| = \lim_{t\to 0} |A_s(a) - A_t(a)| \leq c(n)\, s\, \|\mathbf{D}^2 f\|_{L^n(B_s(a))}\,.$$

Moreover, noting that $DA_s(a)(a-x) = A_s(a) - A_s(x) \leq A_s(a) - A_s(x) + f(x)$ for $x \in B_r(a)$, we conclude

$$\|DA_s(a)\| = s^{-1} \sup_{x \in B_s(a)} DA_s(a)(a-x)$$

$$\leq s^{-1} \sup_{x\in B_s(a)} \left[A_s(a) + (f - A_s)(x)\right]$$
$$\leq s^{-1}|A_s(a)| + s^{-1}\|f - A_s\|_{L^\infty(B_s(a))}$$
$$\leq c(n)\|\mathbf{D}^2 f\|_{L^n(B_s(a))} .$$

In conclusion, for $0 < s \leq r$,

$$\begin{aligned}\|\mathbf{D}f\|_{L^n(B_s(a))} &\leq \|\mathbf{D}(f - A_s)\|_{L^n(B_s(a))} + \|\mathbf{D}A_s\|_{L^n(B_s(a))}\\ &\leq c(n)\, s\, \|\mathbf{D}^2 f\|_{L^n(B_s(a))} + c(n)\, s\, \|\mathbf{D}A_s(a)\|\\ &\leq c(n)\, s\, \|\mathbf{D}^2 f\|_{L^n(B_s(a))} ,\end{aligned}$$

$$\begin{aligned}\|f\|_{L^\infty(B_s(a))} &\leq \|f - A_s\|_{L^\infty(B_s(a))} + |A_s(a)| + \|DA_s(a)\|\\ &\leq c(n)\, s\, \|\mathbf{D}^2 f\|_{L^n(B_s(a))} .\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{s\to 0^+} \sup_{\tau\in \mathbb{S}^{n-1}} \frac{|f(a+\tau s) - f(a)|}{s} &\leq \lim_{s\to 0^+} \frac{\|f\|_{L^\infty(B_s(a))}}{s}\\ &\leq c(n) \lim_{s\to 0^+} \|\mathbf{D}^2 f\|_{L^n(B_s(a))} = 0 ,\end{aligned}$$

which implies that $Df(a) = 0$. The proof is complete. □

Remark 2.1.10. It follows from Lemma 2.1.9 that if $a \in \mathcal{S}^*(f) \cup \mathcal{S}_*(f)$, then a is a Lebesgue point of $\mathbf{D}f$, the map f is pointwise differentiable at a, and

$$Df(a) = \mathbf{D}f(a) .$$

Definition 2.1.11. *Let $\psi : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the function defined by*

$$\psi(y) := \frac{(-y, 1)}{\sqrt{1 + |y|^2}} .$$

Given an open set $U \subseteq \mathbb{R}^n$ and a function $f \in C^0(U) \cap W^{2,n}(U)$, we define

$$\Phi_f(x) := \Big(x, f(x), \psi(\nabla f(x))\Big) \quad \textit{for every } x \in \mathrm{Diff}(f) .$$

Remark 2.1.12. Let $\mathbb{S}^n_+ := \{(z, t) \in \mathbb{R}^n \times \mathbb{R} : |z|^2 + t^2 = 1, t > 0\}$. Notice that ψ is a diffeomorphism onto $\mathbb{S}^n_+$ with inverse given by

$$\varphi : (z, t) \in \mathbb{S}^n_+ \mapsto -\frac{z}{t} \in \mathbb{R}^n$$

and $\|D\psi(y)\| \leq 2$ for $y \in \mathbb{R}^n$.

Remark 2.1.13. Given $U \subseteq \mathbb{R}^n$ an open set and $f \in C^0(U) \cap W^{2,n}(U)$, we recall that $\boldsymbol{\nabla} f$ is $(\mathcal{L}^n \llcorner U)$-approximately differentiable at $\mathcal{L}^n$-a.e. $a \in U$, by Lemma 2.1.8. Moreover, $\nabla f = \boldsymbol{\nabla} f$ $\mathcal{L}^n$-a.e. in U, by Remark 2.1.10. If $a \in \mathrm{Diff}(f)$ and ∇f is $(\mathcal{L}^n \llcorner U)$-approximately differentiable at a, then Φ_f is $(\mathcal{L}^n \llcorner U)$-approximately differentiable at a and

$$\mathrm{ap}\, D\Phi_f(a)(\tau) = \Big(\tau, Df(a)(\tau), D\psi(\nabla f(a))\left[\mathrm{ap}\, D(\nabla f)(a)(\tau)\right]\Big)$$

for every $\tau \in \mathbb{R}^n$. In particular ap $D\Phi_f(a)$ is injective for $\mathcal{L}^n$-a.e. $a \in U$ and, recalling Remark 2.1.12 and noting that $\mathbf{D}(\nabla f) = \operatorname{ap} D(\nabla f)$ by Lemma 2.1.8, we infer that

$$\int_U \|\operatorname{ap} D\Phi_f\|^n \, d\mathcal{L}^n \leq c(n)\Big(\mathcal{L}^n(U) + \int_U \|Df\|^n \, d\mathcal{L}^n + \int_U \|\mathbf{D}^2 f\|^n \, d\mathcal{L}^n\Big).$$

It follows that Φ_f is a $W^{1,n}$-map over U.

Remark 2.1.14. The following basic fact from measure theory is used in the proof of Lemma 2.1.15. *Let $(X, \mathcal{M}, \mu)$ be a measure space, let C be a positive constant, and let $\{E_j\}_{j\in\mathbb{N}}$ be a countable family of sets in $\mathcal{M}$ such that $\mathcal{H}^0(\{j \in \mathbb{N} : E_j \cap E_i \neq \varnothing\}) \leq C$ for every $i \in \mathbb{N}$. Then, we have*

$$\sum_{i=1}^{\infty} \mu(E_i) \leq C\,\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big).$$

Indeed, let $J_i := \{j \in \mathbb{N} : E_i \cap E_j \neq \varnothing\}$ for $i \in \mathbb{N}$, and consider the disjoint family $\{A_i\}_{i\in\mathbb{N}}$ in $\mathcal{M}$ defined by $A_1 := E_1$, $A_i := E_i \setminus \bigcup_{j=1}^{i-1} A_j$ for $i \in \mathbb{N}_{\geq 2}$. Then $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} E_i$ and

$$\begin{aligned}
\sum_{j=1}^{\infty} \mu(E_j) &= \sum_{j=1}^{\infty} \mu\Big(E_j \cap \bigcup_{i=1}^{\infty} E_i\Big) = \sum_{j=1}^{\infty} \mu\Big(E_j \cap \bigcup_{i=1}^{\infty} A_i\Big) \\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \mu(E_j \cap A_i) = \sum_{i=1}^{\infty}\sum_{j\in J_i} \mu(E_j \cap A_i) \\
&\leq \sum_{i=1}^{\infty}\sum_{j\in J_i} \mu(A_i) \leq C\sum_{i=1}^{\infty} \mu(A_i) = C\,\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = C\,\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big).
\end{aligned}$$

The following result is proved using a Rado-Reichelderfer type argument; cf. [32] and references therein.

Lemma 2.1.15. *Given $U \subseteq \mathbb{R}^n$ an open set and $f \in C^0(U) \cap W^{2,n}_{loc}(U)$, then $\mathcal{H}^n(\Phi_f(Z)) = 0$ for every $Z \subset \mathcal{S}^*(f)$ such that $\mathcal{L}^n(Z) = 0$.*

Proof. Given $\mu > 0$ and $V \subseteq U$, we define $X(V,\mu)$ as the set of $x \in V$ for which there exists a polynomial function Q of degree at most 2 such that $f(y) \leq Q(y)$ for every $y \in V$, $f(x) = Q(x)$, $\|DQ(x)\| \leq \mu$ and $\|D^2Q\| \leq \mu$. If $D \subset U$ is a countable dense subset of U and $R(c) := \{s \in \mathbb{Q} : B_s(c) \subseteq U\}$ for every $c \in D$, then we notice that

$$\mathcal{S}^*(f) \subseteq \bigcup_{c\in D}\ \bigcup_{s\in R(c)}\ \bigcup_{i=1}^{\infty} X(B_s(c), i).$$

Hence, it is sufficient to show that $\mathcal{H}^n(\Phi_f(Z)) = 0$ whenever $Z \subseteq X(U,\mu)$ with $\mathcal{L}^n(Z) = 0$, for some $\mu > 0$. Notice that f is pointwise differentiable at each point of the set $X(U,\mu)$, by Lemma 2.1.9.

Now we prove the following claim:

there exists $c(n) \geq 0$ such that,

for any $\mu > 0$, $c \in U$ and $0 < r < 1$ such that $\overline{B_{3r}(c)} \subset U$, we have

$$\|Df(a) - Df(b)\| \leq c(n)\big(\|\mathbf{D}^2 f\|_{L^n(B_{3r}(c))} + \mu r\big), \tag{2.1.7}$$

$$|f(a) - f(b)| \leq c(n)\, r\,\big(\|\mathbf{D}^2 f\|_{L^n(B_{3r}(c))} + \mu\big), \tag{2.1.8}$$

for every $a, b \in X(U,\mu) \cap B_r(c)$.

We fix $a, b \in X(U,\mu) \cap B_r(c)$, $a \neq b$, and we define

$$s := \frac{|a-b|}{2} \quad \text{and} \quad d := \frac{a+b}{2}.$$

We notice that $s \le r$,

$$B_s(d) \subseteq B_{2s}(a) \cap B_{2s}(b) \quad \text{and} \quad B_{2s}(a) \cup B_{2s}(b) \subseteq B_{3r}(c)\,,$$

consequently it follows from (2.1.5) of Lemma 2.1.9 that

$$\begin{aligned}\|\mathbf{D}f - Df(e)\|_{L^n(B_s(d))} &\le \|\mathbf{D}f - Df(e)\|_{L^n(B_{2s}(e))} \\ &\le c(n)\, s\, (\|\mathbf{D}^2 f\|_{L^n(B_{2s}(e))} + \mu s) \\ &\le c(n)\, s\, (\|\mathbf{D}^2 f\|_{L^n(B_{3r}(c))} + \mu r)\end{aligned}$$

for $e \in \{a, b\}$, whence we infer

$$\begin{aligned}\boldsymbol{\alpha}(n)^{1/n}\, s\, \|Df(a) - Df(b)\| &\le \|\mathbf{D}f - Df(a)\|_{L^n(B_s(d))} + \|\mathbf{D}f - Df(b)\|_{L^n(B_s(d))} \\ &\le c(n)\, s\, (\|\mathbf{D}^2 f\|_{L^n(B_{3r}(c))} + \mu r)\end{aligned}$$

and (2.1.7) is proved. Moreover, combining (2.1.6) of Lemma 2.1.9 with (2.1.7)

$$\begin{aligned}&|f(a) - f(b)| \\ &\quad \le \|f - L_a\|_{L^\infty(B_s(d))} + \|f - L_b\|_{L^\infty(B_s(d))} + s\,\|Df(a)\| + s\,\|Df(b)\| \\ &\quad \le \|f - L_a\|_{L^\infty(B_{2s}(a))} + \|f - L_b\|_{L^\infty(B_{2s}(b))} + 2\mu r \\ &\quad \le c(n)\, r\, (\|\mathbf{D}^2 f\|_{L^n(B_{3r}(c))} + \mu)\end{aligned}$$

we have obtained (2.1.8).

We consider the function $\overline{f} \times \nabla f$ mapping $x \in \mathrm{Diff}(f)$ into $\big(\overline{f}(x), \nabla f(x)\big) \in \mathbb{R}^{2n+1}$. Then it follows from (2.1.7) and (2.1.8) that

$$\operatorname{diam}\big[(\overline{f} \times \nabla f)\big(B_r(c) \cap X(U,\mu)\big)\big] \le c(n,\mu)\big(\|\mathbf{D}^2 f\|_{L^n(B_{3r}(c))} + r\big)\,. \tag{2.1.9}$$

Let $Z \subseteq X(U,\mu)$ bounded and $\mathcal{L}^n(Z) = 0$. Given $\epsilon > 0$, we choose an open set $V \subseteq U$ such that $Z \subseteq V$, and

$$\mathcal{L}^n(V) \le \epsilon\,, \quad \|\mathbf{D}^2 f\|^n_{L^n(V)} \le \epsilon\,. \tag{2.1.10}$$

We define $R : Z \to \mathbb{R}$ and $\rho : Z \to \mathbb{R}$ as

$$R(x) := \inf\left\{1, \frac{\operatorname{dist}(x, \mathbb{R}^n \setminus V)}{4}\right\} \quad \text{for } x \in Z$$

$$\rho(x) := \operatorname{diam}\Big((\overline{f} \times \nabla f)\big(B_{R(x)}(x) \cap X(U,\mu)\big)\Big) \quad \text{for } x \in Z\,.$$

We notice that R is a Lipschitz function with $Lip(R) \le \frac{1}{4}$ and, noting that $B_{3R(x)}(x) \subseteq V$ for every $x \in Z$ and combining (2.1.9) and (2.1.10), we obtain

$$\rho(x) \le c(n,\mu)\big(\|\mathbf{D}^2 f\|_{L^n(B_{3R(x)}(x))} + R(x)\big) \le c(n,\mu)\,\epsilon^{1/n} \tag{2.1.11}$$

for $x \in Z$. We prove now the following claim:

there exists $C \subseteq Z$ countable such that
$\{B_{R(y)/5}(y) : y \in C\}$ is disjointed,
$Z \subseteq \bigcup_{y \in C} B_{R(y)}(y)$ and
$\mathcal{H}^0\big(\{y \in C : B_{3R(y)}(y) \cap B_{3R(x)}(x) \neq \varnothing\}\big) \le c(n)$, for every $x \in Z$.

Applying Besicovitch's covering theorem [5, Theorem 2.17] (see also the remark at the beginning of page 52), there exists a positive constant $\xi(n)$ depending only on n, and there exist

$Z_1, \dots, Z_{\xi(n)} \subseteq Z$ such that

$$Z \subseteq \bigcup_{i=1}^{\xi(n)} \bigcup_{x \in Z_i} B_{R(x)/5}(x)\,,$$

and $\{B_{R(x)/5}(x) : x \in Z_i\}$ is disjointed for any $i \in \{1, \dots, \xi(n)\}$. We now apply [14, Lemma 3.1.12] with $S = Z_i$, $U = Z$, $h = \frac{R}{5}$, $\lambda = \frac{1}{20}$ and $\alpha = \beta = 15$, to infer that

$$\mathcal{H}^0\big(\{y \in Z_i : B_{3R(y)}(y) \cap B_{3R(x)}(x) \neq \varnothing\}\big) \leq c(n) \quad \text{for every } i \in \{1, \dots, \xi(n)\}\,.$$

We define $Z' := \bigcup_{i=1}^{\xi(n)} Z_i$, and we notice that $Z \subseteq \bigcup_{x \in Z'} B_{R(x)/5}(x)$ and

$$\mathcal{H}^0\big(\{y \in Z' : B_{3R(y)}(y) \cap B_{3R(x)}(x) \neq \varnothing\}\big) \leq \xi(n)c(n) \quad \text{for every } x \in Z\,.$$

Now, we apply Vitali's $5r$-covering theorem [6, Theorem 2.2.3] to find a countable set $C \subseteq Z'$ such that the family $\{B_{R(x)/5}(x) : x \in C\}$ is disjointed and

$$Z \subseteq \bigcup_{x \in C} B_{R(x)}(x)\,,$$

which proves the claim.

Denoting with ϕ_δ the size δ approximating measure of the n-dimensional Hausdorff measure $\mathcal{H}^n$ of $\mathbb{R}^{2n+1}$ (cf. [14, 2.10.1, 2.10.2]), and combining (2.1.11) with the claim above and with Remark 2.1.14, we have

$$\begin{aligned}
\phi_{2c(n,\mu)\epsilon^{1/n}}\big((\overline{f} \times \nabla f)(Z)\big) &\leq c(n) \sum_{y \in C} \rho(y)^n \\
&\leq c(n,\mu) \sum_{y \in C} \Big(\|\mathbf{D}^2 f\|_{L^n(B_{3R(y)}(y))} + R(y) \Big)^n \\
&\leq c(n,\mu) \sum_{y \in C} R(y)^n + c(n,\mu) \sum_{y \in C} \int_{B_{3R(y)}(y)} \|\mathbf{D}^2 f\|^n \, d\mathcal{L}^n \\
&\leq c(n,\mu)\, \mathcal{L}^n(V) + c(n,\mu) \int_V \|\mathbf{D}^2 f\|^n \, d\mathcal{L}^n \\
&\leq c(n,\mu)\, \epsilon\,.
\end{aligned}$$

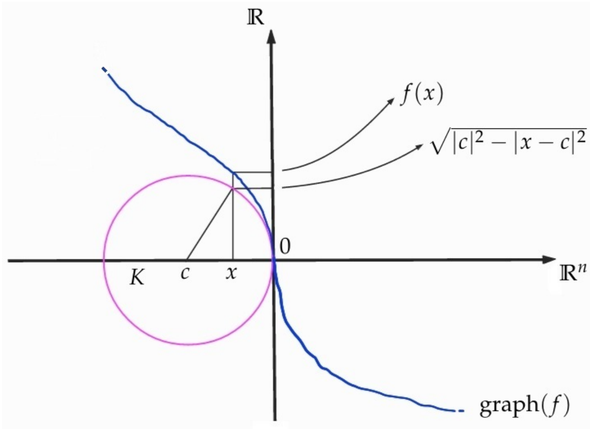
Letting $\epsilon \to 0$, we deduce that $\mathcal{H}^n\big((\overline{f} \times \nabla f)(Z)\big) = 0$. Since $\Phi_f = (\mathbf{1}_{\mathbb{R}^{n+1}} \times \psi) \circ (\overline{f} \times \nabla f)$ (cf. Remark 2.1.12), we conclude that $\mathcal{H}^n(\Phi_f(Z)) = 0$. The proof is complete. □

Remark 2.1.16. Given $U \subseteq \mathbb{R}^n$ an open set and a function $f \in C^0(U) \cap W^{2,n}_{loc}(U)$, from the proof of the above theorem we have that *$\overline{f} \times \nabla f$ satisfies the Lusin (N)-property on $\mathcal{S}^*(f)$.* It easly follows that the same property is satisfied by ∇f and $\overline{\nabla f}$. Furthermore, since $\mathcal{S}_*(f) = \mathcal{S}^*(-f)$, the previous results also hold on $\mathcal{S}_*(f)$.

Lemma 2.1.17. *Suppose $U \subseteq \mathbb{R}^n$ is open, $\gamma > \frac{1}{2}$, $f \in C^{0,\gamma}(U)$, $x \in U$, $v \in \mathbb{S}^n \subseteq \mathbb{R}^n \times \mathbb{R}$ such that $B^{n+1}_s(\overline{f}(x) + sv) \cap \overline{f}(U) = \varnothing$ for some $s > 0$.*
Then $v \notin \mathbb{R}^n \times \{0\}$. In particular, this is always true for $f \in C^0(U) \cap W^{2,n}_{loc}(U)$.

Proof. We prove the assertion by contradiction. Suppose $0 \in U$ and $f(0) = 0$, hence there exists $c \in U \setminus \{0\}$ such that

$$B^{n+1}_{|c|}(c) \cap G = \varnothing \quad \text{and} \quad K := \overline{B^{n+1}_{|c|}(c) \cap (\mathbb{R}^n \times \{0\})} \subseteq U\,.$$



Suppose $L > 0$ such that $|f(x) - f(y)| \leq L|x-y|^\gamma$ for every $x, y \in K$, and define

$$h_\pm(x) = \pm\sqrt{|c|^2 - |x-c|^2} = \pm\sqrt{2c \bullet x - |x|^2}$$

for $x \in K$. Therefore, either $h_+(x) \leq f(x)$ for every $x \in K$, or $f(x) \leq h_-(x)$ for every $x \in K$. In both cases, replacing x with tc and $0 < t < 1$, one obtains

$$t^{1-2\gamma}(2-t) \leq L^2|c|^{2\gamma-2}$$

for $0 < t < 1$. This is clearly impossible, since $1 - 2\gamma < 0$. The proof is complete. □

Definition 2.1.18. *Given $U \subseteq \mathbb{R}^n$ an open set and $f : U \to \mathbb{R}$, we define the cato-graph of f as*

$$C_f := \{(x,u) \in U \times \mathbb{R} : u \leq f(x)\}$$

and

$$N_f := \operatorname{nor}(C_f) \cap (U \times \mathbb{R} \times \mathbb{R}^{n+1}).$$

Lemma 2.1.19. *If $U \subset \mathbb{R}^n$ is a bounded open set and $f \in C^0(U) \cap W^{2,n}(U)$, then*

$$N_f \cap (A \times \mathbb{R} \times \mathbb{R}^{n+1}) = \Phi_f[A \cap \mathcal{S}^*(f)] \tag{2.1.12}$$

for every $A \subseteq U$, and

$$\int_{N_f} \beta \, d\mathcal{H}^n = \int_U \beta(\Phi_f(x)) \, J_n\Phi_f(x) \, d\mathcal{L}^n(x) \tag{2.1.13}$$

whenever $\beta : U \times \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is an $\mathcal{H}^n$-integrable function. In particular, $\mathcal{H}^n(N_f) < \infty$, and

$$\operatorname{ap} D\Phi_f(x)[\mathbb{R}^n] = \operatorname{Tan}^n(\mathcal{H}^n \llcorner N_f, \Phi_f(x)) \tag{2.1.14}$$

for $\mathcal{L}^n$-a.e. $x \in \mathcal{S}^*(f)$.

Proof. Suppose $(z,\nu) \in N_f$ such that $z = (x, f(x))$, with $x \in A$, and $B^{n+1}_s(z + s\nu) \cap C_f = \varnothing$, for some $s > 0$. Since $\nu \notin \mathbb{R}^n \times \{0\}$ by Lemma 2.1.17, we can easily see that there exists an open set $W \subseteq U \times \mathbb{R}$ with $z \in W$, an open set $V \subseteq U$ with $x \in V$, and a smooth function $g : V \to \mathbb{R}$ such that $f(x) = g(x)$ and

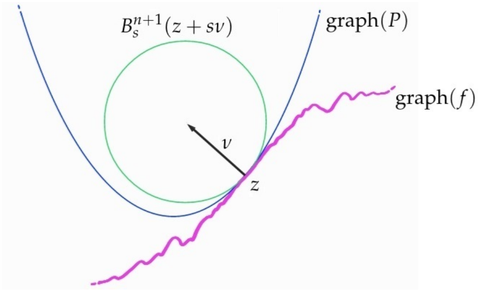
$$W \cap B^{n+1}_s(z + s\nu) = \{(y,u) \in U \times \mathbb{R} : x \in V, \ u > g(y)\},$$

in particular $f(y) \leq g(y)$ for every $y \in V$. It follows that $x \in \mathcal{S}^*(f)$, that $x \in \operatorname{Diff}(f)$ and $Df(x) = Dg(x)$ by Lemma 2.1.9. Noting that

$$\nu = \frac{(-\nabla g(x), 1)}{\sqrt{1 + |\nabla g(x)|^2}}$$

we conclude $(z,\nu) = \Phi_f(x)$ and $N_f \cap (A \times \mathbb{R} \times \mathbb{R}^{n+1}) \subseteq \Phi_f(\mathcal{S}^*(f) \cap A)$.

The opposite inclusion is clear, since for every $x \in \mathcal{S}^*(f)$ there exists a polynomial function P and an open neighbourhood V of x, such that $P(x) = f(x)$ and $P(y) \geq f(y)$ for every $y \in V$.



We prove now the area formula in (2.1.13). Since Φ_f is a $W^{1,n}$-map (cf. Remark 2.1.13), by the classical result in [9, Theorem 13], there exists a sequence

$$\{\Psi_i\}_{i\in\mathbb{N}} \subset C^1(\mathbb{R}^n; \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{n+1}) \cap Lip(\mathbb{R}^n; \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{n+1})$$

such that the following Lusin-type approximation holds

$$\mathcal{L}^n\big(U \setminus \textstyle\bigcup_{i=1}^\infty \{x \in U : \Psi_i(x) = \Phi_f(x)\}\big) = 0\,.$$

We define $\widetilde{A}_i$ as the set of $x \in \mathcal{S}^*(f)$ where $\Psi_i(x) = \Phi_f(x)$ and Φ_f is approximately differentiable at x, with $\mathrm{ap}\,D\Phi_f(x) = D\Psi_i(x)$. Then we set

$$A_1 := \widetilde{A}_1\,, \qquad A_i := \widetilde{A}_i \setminus \textstyle\bigcup_{j=1}^{i-1}\widetilde{A}_j \quad \text{for } i \geq 2\ .$$

By combining Theorem 2.1.7, Lemma 2.1.8, [12, Theorem 6.3], Lemma 2.1.15 and (2.1.12)

$$\mathcal{L}^n\big(U \setminus \textstyle\bigcup_{i=1}^\infty A_i\big) = 0 \quad \text{and} \quad \mathcal{H}^n\big(N_f \setminus \textstyle\bigcup_{i=1}^\infty \Phi_f(A_i)\big) = 0\,. \tag{2.1.15}$$

Let $\beta : U \times \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be an $\mathcal{H}^n$-measurable non-negative function. Employing [14, 2.4.8, 3.2.5], the injectivity of Φ_f and (2.1.15) we obtain

$$\begin{aligned}\int_U \beta(\Phi_f(x))\, J_n\Phi_f(x)\, d\mathcal{L}^n(x) &= \sum_{i=1}^\infty \int_{A_i} \beta(\Psi_i(x))\, J_n\Psi_i(x)\, d\mathcal{L}^n(x)\\ &= \sum_{i=1}^\infty \int_{\Phi_f(A_i)} \beta(y)\, d\mathcal{H}^n(y)\\ &= \int_{N_f} \beta(y)\, d\mathcal{H}^n(y)\,.\end{aligned}$$

While, if $\beta : U \times \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is $\mathcal{H}^n$-integrable, it is enough to apply the previous argument to the positive and negative part of β. Choosing $\beta = 1$ we conclude that $\mathcal{H}^n(N_f) < \infty$ and we deduce that $\mathrm{Tan}^n\big(\mathcal{H}^n \llcorner N_f, (z,\nu)\big)$ is a n-dimensional plane for $\mathcal{H}^n$-a.e. $(z,\nu) \in N_f$. Let D_i be the set of $x \in A_i$ such that $\Theta^n(\mathcal{L}^n \llcorner \mathbb{R}^n \setminus A_i, x) = 0$, $\mathrm{ap}\,D\Phi_f(x)$ is injective and $\mathrm{Tan}^n\big(\mathcal{H}^n \llcorner N_f, \Phi_f(x)\big)$ is a n-dimensional plane. Noting [14, 2.10.19] and Remark 2.1.13, we deduce that

$$\mathcal{H}^n(A_i \setminus D_i) = 0\,.$$

Since $\mathrm{Tan}^n(\mathcal{L}^n \llcorner A_i, x) = \mathbb{R}^n$ for $x \in D_i$ and noting that $\Psi|A_i$ is a bi-lipschitz homeomorphism onto $\Phi_f(A_i)$, we employ [51, Lemma B.2] to conclude

$$\begin{aligned}\mathrm{ap}\,D\Phi_f(x)[\mathbb{R}^n] &= D\Psi_i(x)\big[\mathrm{Tan}^n(\mathcal{L}^n \llcorner A_i, x)\big]\\ &\subseteq \mathrm{Tan}^n\big(\mathcal{H}^n \llcorner \Psi_i(A_i), \Phi_f(x)\big) \subseteq \mathrm{Tan}^n\big(\mathcal{H}^n \llcorner N_f, \Phi_f(x)\big)\end{aligned}$$

for every $x \in D_i$. Since $\operatorname{Tan}^n(\mathcal{H}^n \llcorner N_f, \Phi_f(x))$ is a n-dimensional plane and $\operatorname{ap} D\Phi_f(x)$ is injective for every $x \in D_i$, we conclude that

$$\operatorname{ap} D\Phi_f(x)[\mathbb{R}^n] = \operatorname{Tan}^n\big(\mathcal{H}^n \llcorner N_f, \Phi_f(x)\big) \quad \text{for every } x \in D_i\,.$$

The proof is complete. □

Theorem 2.1.20. *Given $U \subset \mathbb{R}^n$ a bounded open set and $f \in C^0(U) \cap W^{2,n}(U)$, then there exists a Borel n-vectorfield $\vec{\eta}$ on N_f such that*

$$(\mathcal{H}^n \llcorner N_f) \wedge \vec{\eta} \textit{ is a Legendrian cycle of } U \times \mathbb{R}$$

and, for $\mathcal{H}^n$-a.e. $(z,\nu) \in N_f$,

$$|\vec{\eta}(z,\nu)| = 1\,, \quad \vec{\eta}(z,\nu) \textit{ is simple}\,,$$

$$\operatorname{Tan}^n\big(\mathcal{H}^n \llcorner N_f, (z,\nu)\big) \textit{ is associated with } \vec{\eta}(z,\nu)$$

and (cf. (1.0.2))

$$\langle [\textstyle\bigwedge_n \pi_0]\,(\vec{\eta}(z,\nu)) \wedge \nu\,, dX_1 \wedge \ldots \wedge dX_{n+1}\rangle > 0\,.$$

Proof. Let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ be the standard basis of $\mathbb{R}^n$ and we use the symbols $D_1, \ldots D_n$ and $\operatorname{ap} D_1, \ldots, \operatorname{ap} D_n$ for the partial derivatives and the approximate partial derivatives with respect to $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. We notice by [14, 3.1.4] that $\operatorname{ap} D_i \Phi_f$ is a $(\mathcal{L}^n \llcorner U)$-measurable map for any $i \in \{1, \ldots, n\}$. Hence, by the classical Lusin theorem (cf. [14, 2.3.5, 2.3.6]) there exists a Borel map $\xi_i : U \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ such that ξ_i is $(\mathcal{L}^n \llcorner U)$-a.e. equal to $\operatorname{ap} D_i \Phi_f$ for $i \in \{1, \ldots, n\}$.

We define the n-current $T \in \mathcal{D}_n(U \times \mathbb{R} \times \mathbb{R}^{n+1})$ by

$$T(\phi) := \int_U \langle \xi_1(x) \wedge \ldots \wedge \xi_n(x), \phi(\Phi_f(x))\rangle \, d\mathcal{L}^n(x) \tag{2.1.16}$$

for every $\phi \in \mathcal{D}^n(U \times \mathbb{R} \times \mathbb{R}^{n+1})$, by [14, 1.7.6] and Remark 2.1.13 we notice that

$$|T(\phi)| \leq c(n)\big(\mathcal{L}^n(U) + \|f\|^n_{W^{2,n}(U)}\big)\, \|\phi\|_{L^\infty(U \times \mathbb{R} \times \mathbb{R}^{n+1})}\,.$$

We choose a sequence $\{f_k\}_{k \in \mathbb{N}} \subset C^\infty(U) \cap W^{2,n}(U)$ so that $f_k \to f$ in $W^{2,n}(U)$, $f_k(x) \to f(x)$ and $Df_k(x) \to \mathbf{D}f(x)$ for $\mathcal{L}^n$-a.e. $x \in U$; cf. [20, Theorem 7.9]. Since $\Phi_{f_k} : U \to U \times \mathbb{R} \times \mathbb{R}^{n+1}$ is a smooth proper map, if we consider the n-current $\mathbf{E}^n := \mathcal{L}^n \wedge (\boldsymbol{e}_1 \wedge \ldots \wedge \boldsymbol{e}_n)$ defined by Lebesgue integration (with the canonical orientation of $\mathbb{R}^n$), we define

$$T_k := (\Phi_{f_k})_\#(\mathbf{E}^n \llcorner U) \in \mathcal{D}_n(U \times \mathbb{R} \times \mathbb{R}^{n+1})\,,$$

and we prove that

$$T_k \to T \quad \text{in } \mathcal{D}_n(U \times \mathbb{R} \times \mathbb{R}^{n+1})\,. \tag{2.1.17}$$

Let $\phi \in \mathcal{D}^n(U \times \mathbb{R} \times \mathbb{R}^{n+1})$. Then,

$$T_k(\phi) = \int_U \langle D_1\Phi_{f_k}(x) \wedge \ldots \wedge D_n\Phi_{f_k}(x), \phi(\Phi_{f_k}(x))\rangle \, d\mathcal{L}^n(x)\,,$$

and we estimate

$$\begin{aligned} &|T_k(\phi) - T(\phi)| \\ &\leq \int_U \big|\langle D_1\Phi_{f_k}(x) \wedge \ldots \wedge D_n\Phi_{f_k}(x) - \xi_1(x) \wedge \ldots \wedge \xi_n(x), \phi(\Phi_{f_k}(x))\rangle\big| \, d\mathcal{L}^n(x) \\ &\quad + \int_U \big|\langle \xi_1(x) \wedge \ldots \wedge \xi_n(x), \phi(\Phi_{f_k}(x)) - \phi(\Phi_f(x))\rangle\big| \, d\mathcal{L}^n(x) \\ &\leq \|\phi\|_{L^\infty(U)} \int_U \big|D_1\Phi_{f_k}(x) \wedge \ldots \wedge D_n\Phi_{f_k}(x) - \xi_1(x) \wedge \ldots \wedge \xi_n(x)\big| \, d\mathcal{L}^n(x) \\ &\quad + \int_U \|\phi(\Phi_f(x)) - \phi(\Phi_{f_k}(x))\| \, J_n\Phi_f(x) \, d\mathcal{L}^n(x)\,. \end{aligned} \tag{2.1.18}$$

Noting that $J_n\Phi_f \in L^1(U)$ (cf. Remark 2.1.13) and

$$\|\phi(\Phi_f(x)) - \phi(\Phi_{f_k}(x))\|\, J_n\Phi_f(x) \le 2\,\|\phi\|_{L^\infty(U\times\mathbb{R}\times\mathbb{R}^n)}\, J_n\Phi_f(x)$$

for $\mathcal{L}^n$-a.e. $x \in U$ and for any $k \in \mathbb{N}$, applying the dominated convergence theorem, we infer

$$\lim_{k\to\infty}\int_U \|\phi(\Phi_f(x)) - \phi(\Phi_{f_k}(x))\|\, J_n\Phi_f(x)\, d\mathcal{L}^n(x) = 0\,. \tag{2.1.19}$$

We observe that

$$\begin{aligned}&D_1\Phi_{f_k}\wedge\ldots\wedge D_n\Phi_{f_k} - \xi_1\wedge\ldots\wedge\xi_n\\ &\quad= \sum_{i=1}^n \xi_1\wedge\ldots\wedge\xi_{i-1}\wedge\big(D_i\Phi_{f_k}-\xi_i\big)\wedge D_{i+1}\Phi_{f_k}\wedge\ldots\wedge D_n\Phi_{f_k}\end{aligned}$$

and we use the generalized Hölder's inequality to estimate

$$\begin{aligned}&\int_U \big|D_1\Phi_{f_k}\wedge\ldots\wedge D_n\Phi_{f_k} - \xi_1\wedge\ldots\wedge\xi_n\big|\, d\mathcal{L}^n \qquad (2.1.20)\\ &\quad\le \sum_{i=1}^n\int_U |\xi_1\wedge\ldots\wedge\xi_{i-1}|\cdot|\xi_i - D_i\Phi_{f_k}|\cdot|D_{i+1}\Phi_{f_k}\wedge\ldots\wedge D_n\Phi_{f_k}|\, d\mathcal{L}^n\\ &\quad\le \sum_{i=1}^n\int_U \|\mathrm{ap}\, D\Phi_f\|^{i-1}\cdot\|\mathrm{ap}\, D\Phi_f - D\Phi_{f_k}\|\cdot\|D\Phi_{f_k}\|^{n-i}\, d\mathcal{L}^n\\ &\quad\le \sum_{i=1}^n\left(\int_U\|\mathrm{ap}\, D\Phi_f\|^n\, d\mathcal{L}^n\right)^{\frac{i-1}{n}}\cdot\left(\int_U\|\mathrm{ap}\, D\Phi_f - D\Phi_{f_k}\|^n\, d\mathcal{L}^n\right)^{\frac{1}{n}}\cdot\left(\int_U\|D\Phi_{f_k}\|^n\, d\mathcal{L}^n\right)^{\frac{n-i}{n}}.\end{aligned}$$

Moreover, by (2.1.12),

$$\begin{aligned}&\int_U\|D\Phi_{f_k} - \mathrm{ap}\, D\Phi_f\|^n\, d\mathcal{L}^n\\ &\quad\le c(n)\left(\int_U\|D\overline{f}_k - \mathbf{D}\overline{f}\|^n\, d\mathcal{L}^n + \int_U\|D(\psi\circ\nabla f_k) - \mathbf{D}(\psi\circ\boldsymbol{\nabla} f)\|^n\, d\mathcal{L}^n\right)\\ &\quad\le c(n)\left(\int_U\|D\overline{f}_k - \mathbf{D}\overline{f}\|^n\, d\mathcal{L}^n + \int_U\|D\psi(\nabla f_k)\|^n\cdot\|D^2 f_k - \mathbf{D}^2 f_k\|^n\, d\mathcal{L}^n\right.\\ &\qquad\qquad\left.+\int_U\|D\psi(\nabla f_k) - D\psi(\boldsymbol{\nabla} f)\|^n\cdot\|\mathbf{D}^2 f\|^n\, d\mathcal{L}^n\right)\\ &\quad\le c(n)\left(\int_U\|D\overline{f}_k - \mathbf{D}\overline{f}\|^n\, d\mathcal{L}^n + \int_U\|D^2 f_k - \mathbf{D}^2 f_k\|^n\, d\mathcal{L}^n\right.\\ &\qquad\qquad\left.+\int_U\|D\psi(\nabla f_k) - D\psi(\boldsymbol{\nabla} f)\|^n\cdot\|\mathbf{D}^2 f\|^n\, d\mathcal{L}^n\right)\end{aligned}$$

and

$$\lim_{k\to\infty}\int_U\|D\psi(\nabla f_k) - D\psi(\boldsymbol{\nabla} f)\|^n\cdot\|\mathbf{D}^2 f\|^n\, d\mathcal{L}^n = 0$$

by dominated convergence theorem. Consequently $\|D\Phi_{f_k} - \mathbf{D}\Phi_f\|_{L^n(U)} \to 0$, and combining (2.1.18), (2.1.19) and (2.1.20) we obtain (2.1.17).

Since $\partial T_k = 0$ for every $k \in \mathbb{N}$, we readily infer from (2.1.17) that $\partial T = 0$.

Define $G := [\Phi_f|\mathcal{S}^*(f)]^{-1} : N_f \to U$ and notice that G is simply the restriction on N_f of the linear function that maps a point of $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ onto its first n coordinates, in particular G is a Borel map (recall that N_f is a Borel set). We employ Lemma 2.1.19 to see that

$$T(\phi) = \int_{N_f}\langle\vec{\xi}\,[G(z,\nu)], \phi(z,\nu)\rangle\, d\mathcal{H}^n(z,\nu) \quad \text{for any } \phi\in\mathcal{D}^n(U\times\mathbb{R}\times\mathbb{R}^{n+1}) \tag{2.1.21}$$

where $\vec{\xi} : U \to \bigwedge_n(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ is the Borel map defined as

$$\vec{\xi}(x) := \frac{\xi_1(x) \wedge \ldots \wedge \xi_n(x)}{|\xi_1(x) \wedge \ldots \wedge \xi_n(x)|} \quad \text{for } x \in U\,.$$

Then we define the Borel mapping

$$\vec{\eta} : y \in N_f \mapsto (\vec{\xi} \circ G)(y) \in \bigwedge_n(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})\,,$$

and we infer that $\mathrm{Tan}^n\big(\mathcal{H}^n \llcorner N_f, (z,\nu)\big)$ is associated with $\vec{\eta}(z,\nu)$ for $\mathcal{H}^n$-a.e. $(z,\nu) \in N_f$ by Lemma 2.1.19, and $(\mathcal{H}^n \llcorner N_f) \wedge \vec{\eta}$ is a Legendrian cycle by Lemma 1.4.27.
Finally, denoting by $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ the canonical basis of $\mathbb{R}^n$ and by $X_1, \ldots, X_{n+1}$ the coordinate functions of $\mathbb{R}^{n+1} \simeq \mathbb{R}^n \times \mathbb{R}$, we use Remark 2.1.13 and shuffle formula (cf. [14, p. 19]) to compute

$$\begin{aligned}
&\langle [\textstyle\bigwedge_n \pi_0]\,(\vec{\xi}(x)) \wedge \psi(\nabla f(x)), dX_1 \wedge \ldots \wedge dX_{n+1}\rangle \\
&\quad = \frac{1}{J_n\Phi_f(x)} \langle (\boldsymbol{e}_1, D_1 f(x)) \wedge \ldots \wedge (\boldsymbol{e}_n, D_n f(x)) \wedge \psi(\nabla f(x)), dX_1 \wedge \ldots \wedge dX_{n+1}\rangle \\
&\quad = \frac{dX_{n+1}\Big(\psi(\nabla f(x))\Big)}{J_n\Phi_f(x)} > 0 \quad \text{for } \mathcal{L}^n \text{ a.e. } x \in U\,.
\end{aligned}$$

The proof is complete. □

Remark 2.1.21. Let $U \subset \mathbb{R}^n$ be a bounded open set, $f \in C^0(U) \cap W^{2,n}(U)$ and $\Gamma := \mathrm{graph}(f)$. First, we denote by $\vec{\eta}_{N_f}$ the Borel n-vectorfield $\vec{\eta}$ introduced in Theorem 2.1.20, and by Φ_f^+ the map Φ_f given in Definition 2.1.11. Then, if we consider the *epi-graph* of f

$$E_f := \{(x,u) \in U \times \mathbb{R} : u \geq f(x)\}\,,$$

also the $W^{1,n}$-mapping

$$\Phi_f^-(x) := \Big(x, f(x), -\psi(\nabla f(x))\Big) \quad \text{for every } x \in \mathrm{Diff}(f)$$

and

$$M_f := \mathrm{nor}(E_f) \cap (U \times \mathbb{R} \times \mathbb{R}^{n+1})\,. \tag{2.1.22}$$

Similarly to Lemma 2.1.19 and Theorem 2.1.20, one can prove that

$$M_f \cap (A \times \mathbb{R} \times \mathbb{R}^{n+1}) = \Phi_f^-\big[A \cap \mathcal{S}_*(f)\big] \quad \text{for every } A \subseteq U \tag{2.1.23}$$

and there exists a Borel n-vectorfield $\vec{\eta}_{M_f}$ on M_f, such that

$$(\mathcal{H}^n \llcorner M_f) \wedge \vec{\eta}_{M_f} \textit{ is a Legendrian cycle of } U \times \mathbb{R}, \textit{ where } \mathcal{H}^n(M_f) < \infty\,. \tag{2.1.24}$$

Furthermore, for $\mathcal{H}^n$-a.e. $(z,\nu) \in M_f$, we have $|\vec{\eta}_{M_f}(z,\nu)| = 1$, $\vec{\eta}_{M_f}(z,\nu)$ is simple, also $\mathrm{Tan}^n\big(\mathcal{H}^n \llcorner M_f, (z,\nu)\big)$ is associated with $\vec{\eta}_{M_f}(z,\nu)$ and

$$\langle [\textstyle\bigwedge_n \pi_0]\,(\vec{\eta}_{M_f}(z,\nu)) \wedge \nu\,, dX_1 \wedge \ldots \wedge dX_{n+1}\rangle > 0\,.$$

Now, if we define

$$N(\Gamma) := \mathrm{nor}(\Gamma) \cap (U \times \mathbb{R} \times \mathbb{R}^{n+1})\,, \tag{2.1.25}$$

we readily infer that

$$N(\Gamma) = N_f \cup M_f \quad \text{and} \quad N_f \cap M_f = \emptyset \tag{2.1.26}$$

and, from (2.1.12) and (2.1.23), we deduce that

$$\mathrm{nor}(\Gamma) \cap (A \times \mathbb{R} \times \mathbb{R}^{n+1}) = \Phi_f^+\big[A \cap \mathcal{S}^*(f)\big] \cup \Phi_f^-\big[A \cap \mathcal{S}_*(f)\big] \tag{2.1.27}$$

for every $A \subseteq U$. In particular, given $Z \subset \Gamma$ such that $\mathcal{H}^n(Z) = 0$, then

$$\begin{aligned} N(\Gamma) \llcorner Z &= \mathrm{nor}(\Gamma) \llcorner (\pi(Z) \times \mathbb{R}) \\ &= \Phi_f^+ \big(\pi(Z) \cap \mathcal{S}^*(f)\big) \cup \Phi_f^- \big(\pi(Z) \cap \mathcal{S}_*(f)\big) \end{aligned}$$

where π is the canonical projection on the first n-components (hence $\mathcal{L}^n\big(\pi(Z)\big) = 0$). Then, since Φ_f^+ and Φ_f^- satisfies, respectively, the Lusin (N)-property on $\mathcal{S}^*(f)$ and $\mathcal{S}_*(f)$ (cf. Lemma 2.1.15 and Remark 2.1.16), we conclude that $\mathcal{H}^n(N(\Gamma) \llcorner Z) = 0$. Namely, $N(\Gamma)$ *satisfies the Lusin* (N)*-property on* Γ. Furthermore, the Borel n-vectorfield

$$\vec{\eta}_\Gamma := \vec{\eta}_{N_f} \mathbf{1}_{N_f} + \vec{\eta}_{M_f} \mathbf{1}_{M_f} \quad \text{on } N(\Gamma) \tag{2.1.28}$$

satisfies, for $\mathcal{H}^n$-a.e. $(x, \nu) \in N(\Gamma)$, the following properties:

$$\begin{gathered} |\vec{\eta}_\Gamma(z,\nu)| = 1\,, \quad \vec{\eta}_\Gamma(z,\nu) \text{ is simple}\,, \\ \mathrm{Tan}^n\big(\mathcal{H}^n \llcorner N(\Gamma), (z,\nu)\big) \text{ is associated with } \vec{\eta}_\Gamma(z,\nu)\,, \\ \langle [\textstyle\bigwedge_n \pi_0] \big(\vec{\eta}_\Gamma(z,\nu)\big) \wedge \nu \,, dX_1 \wedge \ldots \wedge dX_{n+1} \rangle > 0\,. \end{gathered} \tag{2.1.29}$$

Overall, from Theorem 2.1.20, (2.1.24) and (2.1.27), we conclude that:

$$\begin{gathered} \mathcal{N}_\Gamma := \big(\mathcal{H}^n \llcorner N(\Gamma)\big) \wedge \vec{\eta}_\Gamma \textit{ is a Legendrian cycle of } U \times \mathbb{R}\textit{, where} \\ N(\Gamma) \textit{ has finite } \mathcal{H}^n\textit{-measure and satisfies the Lusin } (N)\textit{-property on } \Gamma. \end{gathered} \tag{2.1.30}$$

We denote $\mathcal{N}_\Gamma$ as the *Legendrian cycle associated with* Γ.

2.2 The support of Legendrian cycles

Given $C \subseteq \mathbb{R}^{n+1}$, we define $\mathrm{Unp}(C)$ as the set of $x \in \mathbb{R}^{n+1} \setminus \overline{C}$ such that there exists an *unique* $y \in \overline{C}$ with $\boldsymbol{\delta}_C(x) = |y - x|$. We define also the *nearest point projection* $\boldsymbol{\xi}_C$ as the multivalued function mapping $x \in \mathbb{R}^{n+1}$ onto

$$\boldsymbol{\xi}_C(x) := \{a \in C : |a - x| = \boldsymbol{\delta}_C(x)\}\,.$$

Notice that $\boldsymbol{\xi}_C|\mathrm{Unp}(C)$ is single-valued and we define

$$\boldsymbol{\nu}_C(x) := \frac{x - \boldsymbol{\xi}_C(x)}{\boldsymbol{\delta}_C(x)} \quad \text{and} \quad \boldsymbol{\psi}_C(x) := \big(\boldsymbol{\xi}_C(x), \boldsymbol{\nu}_C(x)\big)\,,$$

for $x \in \mathrm{Unp}(C)$. It is well known that, if $x \in \mathbb{R}^{n+1} \setminus \overline{C}$ and $\boldsymbol{\delta}_C$ is pointwise differentiable at x, then $x \in \mathrm{Unp}(C)$ and $\nabla \boldsymbol{\delta}_C(x) = \boldsymbol{\nu}_C(x)$ (cf. [13, Theorem 4.8 (3)]). In particular, Rademacher theorem ensures that

$$\mathcal{L}^{n+1}\Big(\mathbb{R}^{n+1} \setminus \big(\overline{C} \cup \mathrm{Unp}(C)\big)\Big) = 0\,. \tag{2.2.31}$$

Moreover, the mappings $\boldsymbol{\xi}_C$, $\boldsymbol{\nu}_C$ and $\boldsymbol{\psi}_C$ are continuous functions over $\mathrm{Unp}(C)$ and it is easy to see that

$$\mathrm{nor}(C) = \boldsymbol{\psi}_C\big(\mathrm{Unp}(C)\big)\,. \tag{2.2.32}$$

We also define

$$\boldsymbol{\rho}_C(x) := \sup\big\{s > 0 : \boldsymbol{\delta}_C\big(a + s(x-a)\big) = s\,\boldsymbol{\delta}_C(x)\big\}\,,$$

for $x \in \mathbb{R}^{n+1} \setminus \overline{C}$ and $a \in \boldsymbol{\xi}_C(x)$. This definition does not depend on the choice of $a \in \boldsymbol{\xi}_C(x)$, the function $\boldsymbol{\rho}_C : \mathbb{R}^{n+1} \setminus \overline{C} \to [1, \infty]$ is upper-semicontinuous and we set

$$\mathrm{Cut}(C) := \big\{x \in \mathbb{R}^{n+1} \setminus \overline{C} : \boldsymbol{\rho}_C(x) = 1\big\}$$

(cf. [28, Remark 2.32 and Lemma 2.33]). Finally we define

$$S_t(C) := \big\{x \in \mathbb{R}^{n+1} : \boldsymbol{\delta}_C(x) = t\big\} \quad \text{for } t > 0$$

and we recall from [24, Lemma 4.2 (53)] that

$$\mathcal{H}^n\big(S_t(C)\cap \mathrm{Unp}(C)\cap \mathrm{Cut}(C)\big)=0 \quad \text{for every } t>0\,. \tag{2.2.33}$$

We are ready to prove the following lemma.

Lemma 2.2.22. *If $C\subseteq \mathbb{R}^{n+1}$ and $W\subseteq \mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ is an open set such that $W\cap \mathrm{nor}(C)\neq\varnothing$, then $\mathcal{H}^n\big(W\cap \mathrm{nor}(C)\big)>0$.*

Proof. Since $\mathrm{nor}(C)\neq\varnothing$, from (2.2.32) we deduce that $\mathbb{R}^{n+1}\setminus\overline{C}\neq\varnothing$. Furthermore, it follows from the continuity of $\boldsymbol{\psi}_C$ that $\boldsymbol{\psi}_C^{-1}\big(W\cap\mathrm{nor}(C)\big)$ is relatively open in $\mathrm{Unp}(C)$. That is, there exists an open set $V\subseteq\mathbb{R}^{n+1}$ such that

$$\boldsymbol{\psi}_C^{-1}\big(W\cap\mathrm{nor}(C)\big)=V\cap\mathrm{Unp}(C)$$

from which it follows that the open set $V\cap(\mathbb{R}^{n+1}\setminus\overline{C})$ non-empty. Since we have that

$$\begin{aligned}V\cap(\mathbb{R}^{n+1}\setminus\overline{C})&=\big(V\cap\mathrm{Unp}(C)\big)\cup\Big(V\cap\big[\mathbb{R}^{n+1}\setminus\big(\overline{C}\cup\mathrm{Unp}(C)\big)\big]\Big)\\&=\boldsymbol{\psi}_C^{-1}\big(W\cap\mathrm{nor}(C)\big)\cup\Big(V\cap\big[\mathbb{R}^{n+1}\setminus\big(\overline{C}\cup\mathrm{Unp}(C)\big)\big]\Big)\end{aligned}$$

then, from (2.2.31), we infer

$$\mathcal{L}^{n+1}\Big(\boldsymbol{\psi}_C^{-1}\big(W\cap\mathrm{nor}(C)\big)\Big)>0\,.$$

Now we consider $\boldsymbol{\delta}_C\in Lip(\mathbb{R}^{n+1})$ and we notice that

$$J_1\boldsymbol{\delta}_C(x)=|\nabla\boldsymbol{\delta}_C(x)|=|\boldsymbol{\nu}_C(x)|=1 \quad \text{for } \mathcal{L}^{n+1}\text{-a.e. } x\in\mathbb{R}^{n+1}\setminus\overline{C}$$

then, if we define $T:=\boldsymbol{\psi}_C^{-1}\big(W\cap\mathrm{nor}(C)\big)$, by coarea formula (cf. [14, 3.2.11]) we obtain

$$\begin{aligned}0<\mathcal{L}^{n+1}(T)&=\int_T J_1\boldsymbol{\delta}_C(x)\,d\mathcal{L}^{n+1}(x)\\&=\int_0^{+\infty}\mathcal{H}^n\big(T\cap\{\boldsymbol{\delta}_C=t\}\big)\,d\mathcal{L}^1(t)=\int_0^{+\infty}\mathcal{H}^n\big(T\cap S_t(C)\big)\,d\mathcal{L}^1(t)\,.\end{aligned}$$

Then there exists $\tau>0$ such that $\mathcal{H}^n\big(T\cap S_\tau(C)\big)>0$ and we use (2.2.33) to conclude

$$\mathcal{H}^n\big((T\cap S_\tau(C))\setminus\mathrm{Cut}(C)\big)>0$$

namely

$$\mathcal{H}^n\big(T\cap S_\tau(C)\cap\{\boldsymbol{\rho}_C>1\}\big)=\lim_{k\to\infty}\mathcal{H}^n\Big(T\cap S_\tau(C)\cap\Big\{\boldsymbol{\rho}_C\geq 1+\tfrac{1}{k}\Big\}\Big)>0\,,$$

consequently there exists $s>\tau$ such that

$$\mathcal{H}^n\big(T\cap S_\tau(C)\cap\{\boldsymbol{\rho}_C\geq s/\tau\}\big)>0\,.$$

Since $\boldsymbol{\psi}_C|S_\tau(C)\cap\{\boldsymbol{\rho}_C\geq s/\tau\}$ is a bi-lipschitz homeomorphism by [24, Theorem 3.16], we conclude that

$$\mathcal{H}^n\Big(\boldsymbol{\psi}_C\big(T\cap S_\tau(C)\cap\{\boldsymbol{\rho}_C\geq s/\tau\}\big)\Big)>0$$

and if we notice that

$$\boldsymbol{\psi}_C\big(T\cap S_\tau(C)\cap\{\boldsymbol{\rho}_C\geq s/\tau\}\big)\subseteq\boldsymbol{\psi}_C(T)=W\cap\mathrm{nor}(C)$$

we obtain the desidered result. The proof is complete. □

Remark 2.2.23. Let us consider the positive measure $\mu := \mathcal{H}^n \llcorner \operatorname{nor}(C)$, where $C \subseteq \mathbb{R}^{n+1}$ is an arbitrary set. By virtue of Lemma 2.2.22, we deduce that

$$\mu\big(B_r^{2n+2}(z)\big) > 0 \quad \text{for every } z \in \operatorname{nor}(C) \text{ and } r > 0\,.$$

Overall, since

$$\operatorname{spt}(\mu) := (\mathbb{R}^{n+1} \times \mathbb{S}^n) \setminus \{z \in \mathbb{R}^{n+1} \times \mathbb{S}^n : \exists r > 0 \text{ such that } \mu\big(B_r^{2n+2}(z)\big) = 0\}\,,$$

we infer that $\operatorname{nor}(C) \subseteq \operatorname{spt}\mu$.

For the next proof we recall that, for a subset $C \subseteq \mathbb{R}^{n+1}$, the normal cone $\operatorname{Nor}(C,z)$ (cf. [14, p. 3.1.21]) coincides with the cone of *regular normals* of C at z introduced in [46, Definition 6.3] (and denoted there by $\widehat{N}_C(z)$), while $\operatorname{nor}(C,z)$ is the cone of *proximal normals* of unit length of C at z defined in [46, Example 6.16].

Lemma 2.2.24. *Suppose $U \subseteq \mathbb{R}^n$ is open and $f \in C^0(U)$ such that $\overline{\nabla f}\big(\operatorname{Diff}(f)\big)$ is a dense subset of $U \times \mathbb{R}^n$. Then $N(\Gamma)$ is dense in $\Gamma \times \mathbb{S}^n$, where $\Gamma := \operatorname{graph}(f)$ and $N(\Gamma)$ is given in (2.1.25).*

Proof. First, we observe that

$$\psi\big(\nabla f(x)\big) \in \operatorname{Nor}\big(\Gamma, \overline{f}(x)\big) \quad \text{for every } x \in \operatorname{Diff}(f)\,.$$

Since ψ is a diffeomorphism of $\mathbb{R}^n$ onto $\mathbb{S}^n_+$ and f is continuous, the set $\Phi_f(U)$ is dense in $\Gamma \times \mathbb{S}^n_+$ (ψ and Φ_f are given in Definition 2.1.11). Consequently we infer that N_f is dense in $\Gamma \times \mathbb{S}^n_+$ by standard approximation of regular normals (cf. [46, Exercise 6.18 *(a)*]), where N_f is given in Definition 2.1.18. With the same argument we deduce also that M_f is dense in $\Gamma \times \mathbb{S}^n_-$, where M_f is given in (2.1.22). Overall, from (2.1.26), we conclude that $N(\Gamma)$ is dense in $\Gamma \times \mathbb{S}^n$. □

Fu, in [15, p. 2260], observed that there exist a continuous function f, as in Lemma 2.2.24, that belong to $W^{2,n}(U)$. Consequently, combining Lemma 2.2.22, Lemma 2.2.24 and Remark 2.1.21, we conclude that:

there exists n-dimensional Legendrian cycles (of open subsets on $\mathbb{R}^{n+1}$),
whose support has positive $\mathcal{H}^{2n}$-measure.

To prove this assertion, first we consider $\Gamma = \operatorname{graph}(f)$ (as subset of $\mathbb{R}^{n+1}$) and we employ Remark 2.2.23 to deduce that

$$\operatorname{nor}(\Gamma) \subseteq \operatorname{spt}\big(\mathcal{H}^n \llcorner \operatorname{nor}(\Gamma)\big)\,.$$

Then, if we consider the Legendrian cycle of $U \times \mathbb{R}$, associated with $\Gamma = \operatorname{graph}(f)$ and given by (cf. (2.1.30) in Remark 2.1.21)

$$\mathcal{N}_\Gamma = \big(\mathcal{H}^n \llcorner N(\Gamma)\big) \wedge \vec{\eta}_\Gamma\,,$$

we readily infer that (cf. [56, p. 135])

$$\begin{aligned} N(\Gamma) &\subseteq \operatorname{spt}\big(\mathcal{H}^n \llcorner \operatorname{nor}(\Gamma)\big) \cap (U \times \mathbb{R} \times \mathbb{R}^{n+1}) \\ &= \operatorname{spt}\big(\mathcal{H}^n \llcorner N(\Gamma)\big) \\ &= \operatorname{spt}(\mathcal{N}_\Gamma) \subseteq \Gamma \times \mathbb{S}^n\,. \end{aligned}$$

In conclusion, since $N(\Gamma)$ is dense in $\Gamma \times \mathbb{S}^n$ (cf. Lemma 2.2.24), we infer that $\operatorname{spt}(\mathcal{N}_\Gamma) = \Gamma \times \mathbb{S}^n$ and applying Eilenberg's inequality (cf. [14, 2.10.25]) we obtain the desidered result, namely

$$\begin{aligned} \mathcal{H}^{2n}\big(\operatorname{spt}(\mathcal{N}_\Gamma)\big) &= \mathcal{H}^{2n}(\Gamma \times \mathbb{S}^n) \\ &\geq c(n) \int_{\mathbb{S}^n}^{*} \mathcal{H}^n\big(\overline{f}(U) \times \{u\}\big)\, d\mathcal{H}^n(u) \\ &\geq c(n)\, \mathcal{L}^n(U)\, \mathcal{H}^n(\mathbb{S}^n) > 0\,. \end{aligned}$$

This answers a question implicit in [43, Remark 2.3].

2.3 The Lagrangian cycle of $W^{2,n}$-functions

We introduce the notion of Monge-Ampère functions.

Definition 2.3.25 (Symplectic 2-form)**.** *The exterior derivative of the contact 1-form α of $\mathbb{R}^n$ is called the symplectic form $\omega := d\alpha$. It is a constant 2-form in $\mathbb{R}^n \times \mathbb{R}^n$, acting as*

$$\langle (y,v) \wedge (z,w), \omega \rangle := v \bullet z - w \bullet y \quad \text{for every } y,v,z,w \in \mathbb{R}^n .$$

Definition 2.3.26 (Lagrangian current)**.** *Let $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be an open set and let $k \geq 2$ be an integer. A current $S \in \mathcal{D}_k(\Omega)$ is called Lagrangian if $S \llcorner \omega = 0$.*

Definition 2.3.27 (Monge-Ampère functions)**.** *Given an open set $U \subseteq \mathbb{R}^n$, we say that a function $f \in W^{1,1}_{loc}(U)$ is Monge-Ampère if there exists an integer multiplicity locally rectifiable n-current S on $U \times \mathbb{R}^n$, such that:*

(i) $\partial S = 0$;

(ii) $S \llcorner \omega = 0$;

(iii) $\|S\|(K \times \mathbb{R}^n) < \infty$ *for every $K \subset U$ compact;*

(iv) $S(\phi\, dX_1 \wedge \ldots \wedge dX_n) = \int_U \phi(x, \boldsymbol{\nabla} f(x))\, d\mathcal{L}^n(x)$ *for every $\phi \in C^\infty_c(U \times \mathbb{R}^n)$.*

The following uniqueness theorem, proved by Fu in [16] for locally Lipschitzian functions and later generalized by Jerrard in [26], guarantees that the current S above is uniquely determined by the given conditions.

Theorem 2.3.28 ([16, Theorem 1.1], [26, Theorem 4.1])**.** *Suppose $U \subseteq \mathbb{R}^n$ is open and S is an integer-multiplicity n-dimensional rectifiable Lagrangian current in $U \times \mathbb{R}^n$, with no boundary in $U \times \mathbb{R}^n$, such that $\|S\|(K \times \mathbb{R}^n) < \infty$ for every $K \subseteq U$ and*

$$S(\phi\, dX_1 \wedge \ldots \wedge dX_n) = 0 \quad \text{for every } \phi \in C^\infty_c(U \times \mathbb{R}^n) .$$

Then $S = 0$.

Hence, if $f \in W^{1,1}_{loc}(U)$ is Monge-Ampère and S satisfies *(i)-(iv)* above, we write $[df] := S$. If $f \in C^2(U)$, then $[df] = \overline{\nabla f}_\#(\mathbf{E}^n \llcorner U)$ (where $\mathbf{E}^n := \mathcal{L}^n \wedge (\boldsymbol{e}_1 \wedge \ldots \wedge \boldsymbol{e}_n)$ denote the n- current in $\mathbb{R}^n$ defined by Lebesgue integration, with the canonical orientation of $\mathbb{R}^n$) and

$$\begin{aligned}
[df](\phi) &= \int_U \langle [\textstyle\bigwedge_n D(\overline{\nabla f})(x)](\boldsymbol{e}_1 \wedge \ldots \wedge \boldsymbol{e}_n), \phi(x, \nabla f(x)) \rangle \, d\mathcal{L}^n(x) \\
&= \int_U \langle (\boldsymbol{e}_1, D(\nabla f)(x)\boldsymbol{e}_1) \wedge \ldots \wedge (\boldsymbol{e}_n, D(\nabla f)(x)\boldsymbol{e}_n), \phi(x, \nabla f(x)) \rangle \, d\mathcal{L}^n(x)
\end{aligned}$$

for every $\phi \in \mathcal{D}^n(U \times \mathbb{R}^n)$.

Lemma 2.3.29. *Let $U \subseteq \mathbb{R}^n$ be an open set. Every $f \in W^{2,n}_{loc}(U)$ is a Monge-Ampère function and*

$$[df](\phi) := \int_U \langle (\boldsymbol{e}_1, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_1) \wedge \ldots \wedge (\boldsymbol{e}_n, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_n), \phi(x, \nabla f(x)) \rangle \, d\mathcal{L}^n(x)$$

for $\phi \in \mathcal{D}^n(U \times \mathbb{R}^n)$.

Proof. Given $f \in W^{2,n}_{loc}(U)$, first we notice that (cf. [14, 1.7.6])

$$\big| (\boldsymbol{e}_1, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_1) \wedge \ldots \wedge (\boldsymbol{e}_n, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_n) \big| \leq \big(1 + \|\mathbf{D}(\nabla f)(x)\|\big)^n \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in U .$$

Moreover the expression

$$S(\phi) := \int_U \langle (\boldsymbol{e}_1, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_1) \wedge \ldots \wedge (\boldsymbol{e}_n, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_n), \phi(x, \nabla f(x)) \rangle \, d\mathcal{L}^n(x) ,$$

for $\phi \in \mathcal{D}^n(U \times \mathbb{R}^n)$, defines a n-dimensional current $S \in \mathcal{D}_n(U \times \mathbb{R}^n)$. To prove it, we consider $\{\varphi_k\}_{k\in\mathbb{N}} \subset \mathcal{D}^n(U \times \mathbb{R}^n)$ and $\varphi \in \mathcal{D}^n(U \times \mathbb{R}^n)$ such that there exists a compact set $K \subset U \times \mathbb{R}^n$, where $\{\operatorname{spt}(\varphi_k)\}_{k\in\mathbb{N}} \subset \mathscr{P}(K)$, $\operatorname{spt}(\varphi) \subseteq K$ and

$$\|D^\alpha \varphi_k - D^\alpha \varphi\|_{L^\infty(K)} \xrightarrow[k\to\infty]{} 0 \quad \text{for any multi-index } \alpha \in \mathbb{N}^{\aleph_0}$$

(namely $\varphi_k \to \varphi$ with respect to the canonical LF-topology). Since

$$(\overline{\nabla f})^{-1}(K) \subseteq \pi_0(K) \subset\subset U\,,$$

if we define

$$\alpha[g,\phi](x) := \langle (\boldsymbol{e}_1, \mathbf{D}(\nabla g)(x)\boldsymbol{e}_1) \wedge \ldots \wedge (\boldsymbol{e}_n, \mathbf{D}(\nabla g)(x)\boldsymbol{e}_n), \phi(x, \nabla g(x))\rangle \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in U$$

whenever $g \in W^{2,n}_{loc}(U)$ and $\phi \in \mathcal{D}^n(U \times \mathbb{R}^n)$, we deduce that

$$\begin{aligned} |S(\varphi_k) - S(\varphi)| &\le \int_U \big|\alpha[f,\varphi_k](x) - \alpha[f,\varphi](x)\big|\, d\mathcal{L}^n(x) \\ &\le \|\varphi_k - \varphi\|_{L^\infty(K)} \int_{(\overline{\nabla f})^{-1}(K)} \big(1 + \|\mathbf{D}(\nabla f)(x)\|\big)^n\, d\mathcal{L}^n(x) \xrightarrow[k\to\infty]{} 0\,. \end{aligned}$$

We infer that S is a continuous linear operator with respect to the canonical LF-topology, namely $S \in \mathcal{D}_n(U \times \mathbb{R}^n)$.

Now we show that f is Monge-Ampère and $[df] = S$. We choose a sequence $\{f_k\}_{k\in\mathbb{N}}$ in $C^\infty(\mathbb{R}^n)$ such that $f_k \to f$ in $W^{2,n}_{loc}(U)$, $\nabla f_k(x) \to \nabla f(x)$ and $D(\nabla f_k)(x) \to \mathbf{D}(\nabla f)(x)$ for $\mathcal{L}^n$-a.e. $x \in U$, hence as in the proof of Theorem 2.1.20 (cf. estimate (2.1.18)) we infer that $[df_k](\phi) \to S(\phi)$ for every $\phi \in \mathcal{D}^n(U \times \mathbb{R}^n)$. Namely S is a Lagrangian cycle and satisfies *(iv)* above, by definition. To check the previous *(iii)*, we notice that given $\phi \in \mathcal{D}^n(U \times \mathbb{R}^n)$ and $V \subset\subset U$, an open set with $\operatorname{spt}(\phi) \subset V \times \mathbb{R}^n$, then

$$\begin{aligned} |S(\phi)| &\le \int_V \big|\alpha[f,\phi](x)\big|\, d\mathcal{L}^n(x) \\ &\le \|\phi\|_{L^\infty(U\times\mathbb{R}^n)} \int_V \big(1 + \|\mathbf{D}(\nabla f)(x)\|\big)^n\, d\mathcal{L}^n(x) < \infty\,. \end{aligned}$$

The proof is complete. □

Theorem 2.3.30. *Given $U \subseteq \mathbb{R}^n$ an open set and $f \in C^0(U) \cap W^{2,n}_{loc}(U)$, then the area formula holds for the mappings $\overline{\nabla f}$, namely*

$$\mathcal{H}^n\big(\overline{\nabla f}(E)\big) = \int_E J_n \overline{\nabla f}\, d\mathcal{L}^n \tag{2.3.34}$$

for every $\mathcal{L}^n$-measurable set $E \subseteq \mathcal{S}(f)$. Moreover $\overline{\nabla f}(\mathcal{S}(f) \cap K)$ is $\mathcal{H}^n$-rectifiable for every $K \subset U$ compact,

$$[df] = \big[\mathcal{H}^n \llcorner \overline{\nabla f}(\mathcal{S}(f))\big] \wedge (\vec{\eta}_f \circ \pi_0) \tag{2.3.35}$$

where the n-vectorfield $\vec{\eta}_f$ is defined as

$$\vec{\eta}_f(x) := \frac{(\boldsymbol{e}_1, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_1) \wedge \ldots \wedge (\boldsymbol{e}_n, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_n)}{|(\boldsymbol{e}_1, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_1) \wedge \ldots \wedge (\boldsymbol{e}_n, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_n)|} \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in U\,,$$

and

$$\mathcal{H}^n\Big(\overline{\nabla f}\Big(\big(\mathcal{S}^*(f) \cup \mathcal{S}_*(f)\big) \setminus \mathcal{S}(f)\Big)\Big) = 0\,. \tag{2.3.36}$$

Proof. First we prove (2.3.34). Since $\nabla f \in W^{1,n}_{loc}(U;\mathbb{R}^n)$, by the classical result in [9, Theorem 13], there exists a sequence $\{\Phi_i\}_{i\in\mathbb{N}} \subset C^1(\mathbb{R}^n;\mathbb{R}^n) \cap Lip(\mathbb{R}^n;\mathbb{R}^n)$ such that the following

Lusin-type approximation holds

$$\mathcal{L}^n\big(U \setminus \bigcup_{i=1}^{\infty}\{x \in U : \Phi_i(x) = \nabla f(x)\}\big) = 0\,.$$

We define $\widetilde{A}_i$ as the set of $x \in \mathcal{S}(f)$ where $\Phi_i(x) = \nabla f(x)$ and ∇f is approximately differentiable at x, with $\operatorname{ap} D\nabla f(x) = D\Phi_i(x)$. By combining Theorem 2.1.7, Lemma 2.1.8 and [12, Theorem 6.3], we deduce that $\widetilde{A}_i$ has full $\mathcal{L}^n$-measure in the set $\{x \in U : \Phi_i(x) = \nabla f(x)\}$. Now, if we define the following sequence of disjoint sets

$$A_1 := \widetilde{A}_1 \quad \text{and} \quad A_i := \widetilde{A}_i \setminus \bigcup_{j=1}^{i-1}\widetilde{A}_j \quad \text{for } i \geq 2\,,$$

it follows that $\mathcal{L}^n\big(U \setminus \bigcup_{i=1}^{\infty} A_i\big) = 0$. Moreover (taking into account of Lemma 2.1.8)

$$\begin{aligned} J_n\overline{\nabla f}(x) &= \big|(\boldsymbol{e}_1, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_1) \wedge \ldots \wedge (\boldsymbol{e}_n, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_n)\big| \\ &= \big|(\boldsymbol{e}_1, D\Phi_i(x)\boldsymbol{e}_1) \wedge \ldots \wedge (\boldsymbol{e}_n, D\Phi_i(x)\boldsymbol{e}_n)\big| = J_n\overline{\Phi}_i(x) \end{aligned} \tag{2.3.37}$$

for $\mathcal{L}^n$-a.e. $x \in A_i$ and for every $i \in \mathbb{N}$. Then, for every $\mathcal{L}^n$-measurable set $E \subseteq \mathcal{S}(f)$, since $\overline{\nabla f}$ satisfies the Lusin (N)-property on $\mathcal{S}(f)$ (cf. Remark 2.1.16), by applying [14, 3.2.3 (1) and 2.4.8] and (2.3.37), we obtain

$$\begin{aligned} \mathcal{H}^n\big(\overline{\nabla f}(E)\big) &= \mathcal{H}^n\big(\overline{\nabla f}(E \setminus \textstyle\bigcup_{i=1}^{\infty} A_i)\big) + \displaystyle\sum_{i=1}^{\infty} \mathcal{H}^n\big(\overline{\nabla f}(E \cap A_i)\big) \\ &= \sum_{i=1}^{\infty} \int_{E\cap A_i} J_n\overline{\Phi}_i \, d\mathcal{L}^n = \int_E J_n\overline{\nabla f}\, d\mathcal{L}^n\,. \end{aligned}$$

Again, since $\overline{\nabla f}$ satisfies the Lusin (N)-property on $\mathcal{S}^*(f) \cup \mathcal{S}_*(f)$ (cf. Remark 2.1.16), it follows that (2.3.36) is a consequence of Theorem 2.1.7.

Now, we prove that $\overline{\nabla f}\big(\mathcal{S}(f) \cap K\big)$ is $\mathcal{H}^n$-rectifiable for every compact set $K \subset U$. We consider the map $\Psi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{S}^n_+ \to \mathbb{R}^n \times \mathbb{R}^n$, defined as $\Psi(x,t,u) := \big(x, \varphi(u)\big)$, where φ is given in Remark 2.1.12. From (2.1.12), we obtain

$$\overline{\nabla f}\big(\mathcal{S}(f)\big) = (\Psi \circ \Phi_f)\big(\mathcal{S}(f)\big) \subseteq \Psi(N_f)$$

where N_f is countably $\mathcal{H}^n$-rectifiable. This implies that $\overline{\nabla f}\big(\mathcal{S}(f)\big)$ is countably $\mathcal{H}^n$-rectifiable. To show that $\mathcal{H}^n\big(\overline{\nabla f}(\mathcal{S}(f) \cap K)\big) < \infty$ for every compact set $K \subset U$, we apply the area formula (2.3.34).

To prove (2.3.35), by applying [14, 2.4.8 and 3.2.5] and (2.3.37), we obtain the following

$$\begin{aligned} [df](\phi) &= \int_{\mathcal{S}(f)} \langle (\boldsymbol{e}_1, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_1) \wedge \ldots \wedge (\boldsymbol{e}_n, \mathbf{D}(\nabla f)(x)\boldsymbol{e}_n), \phi(x, \nabla f(x))\rangle \, d\mathcal{L}^n(x) \\ &= \sum_{i=1}^{\infty} \int_{A_i} \langle \vec{\eta}_f\Big(\pi_0\big(\overline{\Phi}_i(x)\big)\Big), \phi\big(\overline{\Phi}_i(x)\big)\rangle \, J_n\overline{\Phi}_i(x)\, d\mathcal{L}^n(x) \\ &= \sum_{i=1}^{\infty} \int_{\overline{\Phi}_i(A_i)} \langle \vec{\eta}_f\big(\pi_0(y)\big), \phi(y)\rangle \, d\mathcal{H}^n(y) \\ &= \int_{\overline{\nabla f}(\mathcal{S}(f))} \langle \vec{\eta}_f\big(\pi_0(y)\big), \phi(y)\rangle \, d\mathcal{H}^n(y) \end{aligned}$$

for every $\phi \in \mathcal{D}^n(U \times \mathbb{R}^n)$. The proof is complete. □

Remark 2.3.31 (Roskovec example)**.** The Lusin (N)-property does not generally hold for Sobolev mappings in the critical $W^{1,n}$-case. In fact, Tomás Roskovec (cf. [48]), using a Cesari-type construction, provides an example of a function $f \in C^1([-1,1]^n)$ such that

$$\nabla f \in W^{1,n}\big((-1,1)^n; \mathbb{R}^n\big) \quad \text{and} \quad [-1,1]^n \subseteq \nabla f([-1,1] \times \{0\}^{n-1})\,.$$

In other words, ∇f is a $(C^0 \cap W^{1,n})$-regular vector field, but it does not satisfy the Lusin (N)-property since it maps a segment into an n-cube. Naturally, $\overline{\nabla f}([-1,1] \times \{0\}^{n-1})$ will also have positive $\mathcal{H}^n$-measure. Taking into account that $\overline{\nabla f}$ satisfies the Lusin (N)-property on $\mathcal{S}(f)$, we deduce that

$$\mathcal{H}^n\Big(\overline{\nabla f}((-1,1)^n) \setminus \overline{\nabla f}\big(\mathcal{S}(f)\big)\Big) \geq \mathcal{H}^n\Big(\overline{\nabla f}([-1,1] \times \{0\}^{n-1})\Big) > 0\,.$$

Overall, from representation (2.3.35), we conclude the following:

$[df]$ is carried by $\overline{\nabla f}\big(\mathcal{S}(f)\big)$,
and it is not possible to replace $\overline{\nabla f}\big(\mathcal{S}(f)\big)$ with $\overline{\nabla f}(U)$,
even if f were C^1-regular.

Chapter 3

Fine properties of $W^{2,n}$-domains

3.1 Reilly-type variational formulae for $W^{2,n}$-domains

In this section we study the structure of the unit normal bundle of a $W^{2,n}$-domain (see Theorem 3.1.7), and we prove the variational formulae for their mean curvature functions (see Theorem 3.1.15). The latter extends the well known variational formulae obtained by Reilly in [45] for smooth domains. As a corollary Minkowski-Hsiung formulae are also proved; cf. Theorem 3.1.17.

Definition 3.1.1 (Viscosity boundary). *Suppose $\Omega \subseteq \mathbb{R}^{n+1}$ be an open set. We define $\partial^v_+\Omega$ to be the set of all $p \in \partial\Omega$ such that there exists $\nu \in \mathbb{S}^n$ and $r > 0$ such that*

$$B^{n+1}_r(p + r\nu) \cap \Omega = \varnothing \quad \text{and} \quad B^{n+1}_r(p - r\nu) \subseteq \Omega\,.$$

[Notice $\{p\} = \partial B_r(p + r\nu) \cap \partial B_r(p - r\nu)$.] Clearly for each $p \in \partial^v_+\Omega$ the unit vector ν is unique. This defines an exterior unit-normal vector field on $\partial^v_+\Omega$,

$$\nu_\Omega : \partial^v_+\Omega \to \mathbb{S}^n.$$

We recall the notion of *second order rectifiability*. Suppose $X \subset \mathbb{R}^m$ and μ is a positive integer such that $\mathcal{H}^\mu(X) < \infty$. We say that X is *$\mathcal{H}^\mu$-rectifiable of class* 2 if and only if there exists countably many μ-dimensional submanifolds $\Sigma_i \subset \mathbb{R}^m$ of class 2 such that

$$\mathcal{H}^\mu\Big(X \setminus \bigcup_{i=1}^{\infty} \Sigma_i\Big) = 0\,.$$

Lemma 3.1.2. *Suppose $X \subset \mathbb{R}^{n+1}$ is $\mathcal{H}^n$-measurable and $\mathcal{H}^n$-rectifiable of class 2, and $\nu : X \to \mathbb{S}^n$ is a $(\mathcal{H}^n \llcorner X)$-measurable map such that*

$$\nu(a) \in \mathrm{Nor}^n(\mathcal{H}^n \llcorner X, a)\,, \quad \textit{for } \mathcal{H}^n\textit{-a.e. } a \in \Sigma\,. \tag{3.1.1}$$

Then there exist a countable family of measurable sets $X_i \subseteq X$ such that $\mathcal{H}^n\big(X \setminus \bigcup_{i=1}^{\infty} X_i\big) = 0$ and $\mathrm{Lip}(\nu|X_i) < \infty$; moreover, ν is $(\mathcal{H}^n \llcorner X)$-approximately differentiable at $\mathcal{H}^n$-a.e. $a \in X$ and $\mathrm{ap}\, D\nu(a)$ is a symmetric endomorphism of $\mathrm{Tan}^n(\mathcal{H}^n \llcorner X, a)$.

Proof. Let $\{\Sigma_i\}_{i\in\mathbb{N}}$ be a family of n-dimensional C^2-hypersurfaces such that

$$\mathcal{H}^n\big(X \setminus \textstyle\bigcup_{i=1}^{\infty}\Sigma_i\big) = 0\,. \tag{3.1.2}$$

For each $i \in \mathbb{N}$, let $\eta_i : \Sigma_i \to \mathbb{S}^n$ be a unit normal C^1-vectofield to Σ_i, which, whitout loss of generality, we assume to be Lipschitz continuous. We define

$$\Sigma_i^{\pm} := \big\{a \in \Sigma_i \cap X : \eta_i(a) = \pm\nu(a)\big\} \quad \text{for } i \in \mathbb{N}\,. \tag{3.1.3}$$

By the rectifiability and the locality property of approximate tangent spaces (cf. (1.1.4)) and Remark 1.1.1 *(i)*, for every $i \in \mathbb{N}$, we obtain

$$\mathrm{Tan}^n(\mathcal{H}^n \llcorner X, a) = \mathrm{Tan}^n(\mathcal{H}^n \llcorner \Sigma_i, a) = \mathrm{Tan}(\Sigma_i, a) \in \mathbf{G}(n+1, n) \tag{3.1.4}$$

for $\mathcal{H}^n$-a.e. $a \in \Sigma_i \cap X$. Finally, combining (3.1.1), (3.1.3) and (3.1.4), we deduce that

$$\mathcal{H}^n\big((\Sigma_i \cap X) \setminus (\Sigma_i^+ \cup \Sigma_i^-)\big) = 0 \quad \text{for every } i \in \mathbb{N}$$

and from (3.1.2), it follows that $\mathcal{H}^n\big(X \setminus \bigcup_{i=1}^{\infty}(\Sigma_i^+ \cup \Sigma_i^-)\big) = 0$.

From the rectifiability of the sets $\Sigma_i^{\pm}$ and since $Lip(\nu|\Sigma_i^{\pm}) < +\infty$, we apply [14, 3.2.19] to conclude that ν is $(\mathcal{H}^n \llcorner \Sigma_i^{\pm})$-approximately differentiable at $\mathcal{H}^n$-a.e. $a \in \Sigma_i^{\pm}$, where

$$\operatorname{ap} D\nu(a) : \operatorname{Tan}^n(\mathcal{H}^n \llcorner \Sigma_i^{\pm}, a) \to \operatorname{Tan}^n(\mathcal{H}^n \llcorner \Sigma_i^{\pm}, a)$$

and $\operatorname{Tan}^n(\mathcal{H}^n \llcorner \Sigma_i^{\pm}, a) \in \mathbf{G}(n+1, n)$. Furthermore, applying [14, 2.10.19 (4)], we infer that

$$\Theta^n(\mathcal{H}^n \llcorner X \setminus \Sigma_i^{\pm}, a) = 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } a \in \Sigma_i^{\pm}, \tag{3.1.5}$$

and we deduce that ν is $(\mathcal{H}^n \llcorner X)$-approximately differentiable at $\mathcal{H}^n$-a.e. $a \in \Sigma_i^{\pm}$, where

$$\operatorname{ap} D\nu(a) : \operatorname{Tan}^n(\mathcal{H}^n \llcorner X, a) \to \operatorname{Tan}^n(\mathcal{H}^n \llcorner X, a)$$

and $\operatorname{Tan}^n(\mathcal{H}^n \llcorner X, a) \in \mathbf{G}(n+1, n)$. Again by (3.1.5), we infer that

$$\Theta\big(\mathcal{H}^n \llcorner \{x \in \Sigma_i \cap X : \eta_i(x) \neq \pm\nu(x)\}, a\big) = 0$$

for $\mathcal{H}^n$-a.e. $a \in \Sigma_i^{\pm}$, where

$$\operatorname{ap} D\nu(a) = \pm D\eta_i(a)|\operatorname{Tan}^n(\mathcal{H}^n \llcorner X, a)\,.$$

Hence, the simmetry of $\operatorname{ap} D\nu(a)$ follows directly from the symmetry of the Weingarten map $D\eta_i(a)|\operatorname{Tan}(\Sigma_i, a)$. The proof is complete. □

We introduce now the class of $W^{2,n}$-domains.

Definition 3.1.3. ($W^{2,n}$-domains). *An open set $\Omega \subseteq \mathbb{R}^{n+1}$ is a $W^{2,n}$-domain if there exists a pair (Ω', F), that satisfies the following properties:*

(i) $\Omega' \subseteq \mathbb{R}^{n+1}$ is an open set such that for each $p \in \partial\Omega'$ there exist $\epsilon > 0$, $\nu \in \mathbb{S}^n$, a bounded open set $U \subset \nu^{\perp}$ with $0 \in U$ and $f \in C^0(U) \cap W^{2,n}(U)$ with $f(0) = 0$ such that

$$\big\{p + b + \tau\nu : b \in U,\ -\epsilon < \tau \le f(b)\big\} = \overline{\Omega'} \cap \big\{p + b + \tau\nu : b \in U,\ -\epsilon < \tau < \epsilon\big\}\,;$$

(ii) F is a C^2-diffeomorphism defined over an open set $V \subseteq \mathbb{R}^{n+1}$, where $\overline{\Omega'} \subseteq V$;

(iii) $F(\Omega') = \Omega$.

Remark 3.1.4. This class of domains is invariant under images of C^2-diffeomorphisms, which is clearly a necessary condition in order to provide a natural framework to generalize Reilly's variational formulae. We do not know if we really need to introduce the diffeomorphism F in the definition above; in other words, if Ω' belongs to the class $\mathcal{S}$ of domains satisfying only condition *(i)* of Definition 3.1.3, is it true that $F(\Omega')$ belongs to $\mathcal{S}$ too?

We collect some basic properties of $W^{2,n}$-domains.

Lemma 3.1.5. *If $\Omega \subseteq \mathbb{R}^{n+1}$ is a $W^{2,n}$-domain, then the following statements hold.*

(i) $\mathcal{H}^n(\partial\Omega \setminus \partial^v_+\Omega) = 0$ and $K \cap \partial\Omega$ is $\mathcal{H}^n$-rectifiable of class 2 for every compact set $K \subseteq \mathbb{R}^{n+1}$.

(ii) For $\mathcal{H}^n$ a.e. $p \in \partial\Omega$,

$$\operatorname{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, p) = \operatorname{Tan}(\partial\Omega, p) = \nu_\Omega(p)^{\perp}.$$

(iii) For every $p \in \partial^v_+\Omega$,

$$\operatorname{Tan}^{n+1}(\mathcal{L}^{n+1} \llcorner \Omega, p) = \operatorname{Tan}(\Omega, p) = \{v \in \mathbb{R}^{n+1} : v \bullet \nu_\Omega(p) \le 0\}\,.$$

Proof. Suppose $\Omega = F(\Omega')$, where Ω' and F are as in Definition 3.1.3. Clearly, $F(\partial\Omega') = \partial\Omega$ and $F(\partial^v_+\Omega') = \partial^v_+\Omega$. Therefore, assertion *(i)* follows from Theorem 2.1.7, Theorem 5.3.3 and Remark 5.3.4.

If $p \in \partial^v_+\Omega$, we have $\mathrm{Tan}(\partial\Omega, p) \subseteq \nu_\Omega(p)^\perp$; since $\mathrm{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, p) \subseteq \mathrm{Tan}(\partial\Omega, p)$ for every $p \in \partial\Omega$ and $\mathrm{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, p)$ is a n-dimensional plane for $\mathcal{H}^n$-a.e. $p \in \partial\Omega$, we obtain *(ii)*. Finally it follows from definitions that $\mathrm{Tan}(\Omega, p) = \{v \in \mathbb{R}^{n+1} : v \bullet \nu_\Omega(p) \leq 0\}$ and $\mathrm{Tan}^{n+1}(\mathcal{L}^{n+1} \llcorner \Omega, p) = \{v \in \mathbb{R}^{n+1} : v \bullet \nu_\Omega(p) \leq 0\}$ for every $p \in \partial^v_+\Omega$. □

By Lemma 3.1.2 the map ν_Ω is $(\mathcal{H}^n \llcorner \partial\Omega)$-approximately differentiable with a symmetric approximate differential $\mathrm{ap}\, D\nu_\Omega(x)$ at $\mathcal{H}^n$-a.e. $x \in \partial\Omega$. Consequently we introduce the following definition.

Definition 3.1.6 (Approximate principal curvatures)**.** *Suppose $\Omega \subseteq \mathbb{R}^{n+1}$ is a $W^{2,n}$-domain. The approximate principal curvatures of Ω are the $\mathbb{R}$-valued $(\mathcal{H}^n \llcorner \partial\Omega)$-measurable maps*

$$\chi_{\Omega,1}, \ldots, \chi_{\Omega,n}$$

defined so that $\chi_{\Omega,1}(p) \leq \ldots \leq \chi_{\Omega,n}(p)$ are the eigenvalues of $\mathrm{ap}\, D\nu_\Omega(p)$ for $\mathcal{H}^n$-a.e. $p \in \partial\Omega$.

We prove now the main structure theorem for the unit normal bundle $\mathrm{nor}(\Omega)$ of a $W^{2,n}$-domain.

Theorem 3.1.7. *Given $\Omega \subseteq \mathbb{R}^{n+1}$ a $W^{2,n}$-domain, then the following statements hold.*

(i) $\mathcal{H}^n(\overline{\nu}_\Omega(Z)) = 0$ *whenever* $Z \subseteq \partial^v_+\Omega$ *with* $\mathcal{H}^n(Z) = 0$.

(ii) $\mathcal{H}^n\big(\mathrm{nor}(\Omega) \setminus \overline{\nu}_\Omega(\partial^v_+\Omega)\big) = 0$.

(iii) $\kappa_{\Omega,i}(x,u) = \chi_{\Omega,i}(x)$ *for every* $i \in \{1, \ldots, n\}$ *and for* $\mathcal{H}^n$*-a.e.* $(x,u) \in \mathrm{nor}(\Omega)$*. In particular,* $\kappa_{\Omega,i}(x,u) < \infty$ *for* $\mathcal{H}^n$*-a.e.* $(x,u) \in \mathrm{nor}(\Omega)$.

(iv) If $\partial\Omega$ *is compact, then* $\mathcal{H}^n\big(\mathrm{nor}(\Omega)\big) < \infty$ *and there exists an unique Legendrian cycle* T *of* $\mathbb{R}^{n+1}$ *such that*

$$T = \big(\mathcal{H}^n \llcorner \mathrm{nor}(\Omega)\big) \wedge \vec{\eta}\,,$$

where $\vec{\eta}$ *is a* $\big(\mathcal{H}^n \llcorner \mathrm{nor}(\Omega)\big)$*-measurable n-vectorfield such that*

$$|\vec{\eta}(x,u)| = 1\,, \quad \vec{\eta}(x,u) \text{ is simple}\,,$$

$$\mathrm{Tan}^n\big(\mathcal{H}^n \llcorner \mathrm{nor}(\Omega), (x,u)\big) \text{ is associated with } \vec{\eta}(x,u)$$

and

$$\langle [\textstyle\bigwedge_n \pi_0](\vec{\eta}(x,u)) \wedge u, dX_1 \wedge \ldots \wedge dX_{n+1} \rangle > 0\,,$$

for $\mathcal{H}^n$*-a.e.* $(x,u) \in \mathrm{nor}(\Omega)$*. In this case* $\vec{\eta} = \zeta_1 \wedge \ldots \wedge \zeta_n$*, where*

$$\zeta_i := \left(\frac{1}{\sqrt{1 + \kappa^2_{\Omega,i}}}\, \tau_i\,, \frac{\kappa_{\Omega,i}}{\sqrt{1 + \kappa^2_{\Omega,i}}}\, \tau_i \right), \quad \text{for } i \in \{1, \ldots, n\}$$

and $\{\tau_1(x,u), \ldots, \tau_n(x,u)\}$ *is an orthonormal basis of* $u^\perp$ *such that*

$$\tau_1(x,u) \wedge \ldots \wedge \tau_n(x,u) \wedge u = \boldsymbol{e}_1 \wedge \ldots \wedge \boldsymbol{e}_{n+1}\,,$$

for $\mathcal{H}^n$*-a.e.* $(x,u) \in \mathrm{nor}(\Omega)$.

Proof. Suppose $\Omega = F(\Omega')$, where Ω' and F are as in Definition 3.1.3. We recall the definition of Ψ_F from (1.3.13) and notice that

$$\Psi_F\big(\mathrm{nor}(\Omega')\big) = \mathrm{nor}(\Omega)\,, \tag{3.1.6}$$

by [54, Lemma 2.1]. Since $F(\partial^v_+\Omega') = \partial^v_+\Omega$, we readily infer from (3.1.6) that

$$\Psi_F\big(x, \nu_{\Omega'}(x)\big) = \Big(F(x), \nu_\Omega\big(F(x)\big)\Big) \quad \text{for every } x \in \partial^v_+\Omega'$$

and

$$\Psi_F\Big(\overline{\nu}_{\Omega'}(F^{-1}(S))\Big) = \overline{\nu}_\Omega(S) \quad \text{for every } S \subseteq \partial^v_+\Omega\,. \tag{3.1.7}$$

To prove the assertions in *(i)* and *(ii)* we notice, firstly, that they are true for Ω' as a consequence of Lemma 2.1.15 and (2.1.12) of Lemma 2.1.19; then we apply (3.1.6) and (3.1.7).

To prove *(iii)* we first employ Lemma 3.1.2 to find a countable family of $\mathcal{H}^n$-measurable sets $X_i \subseteq \partial^v_+\Omega$ such that $\mathcal{H}^n(\partial\Omega \setminus \bigcup_{i=1}^\infty X_i) = 0$ and $Lip(\nu_\Omega | X_i) < \infty$ for every $i \in \mathbb{N}$; then we define Y_i to be the set of $x \in X_i$ such that ν_Ω is $(\mathcal{H}^n \llcorner \partial\Omega)$-approximately differentiable at x, $\mathrm{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, x)$ and $\mathrm{Tan}^n\big(\mathcal{H}^n \llcorner \mathrm{nor}(\Omega), \overline{\nu}_\Omega(x)\big)$ are n-dimensional planes, and $\Theta^n(\mathcal{H}^n \llcorner \partial\Omega \setminus X_i, x) = 0$.
We notice that $\overline{\nu}_\Omega | X_i$ is bi-lipschitz and, since $\mathrm{Tan}^n\big(\mathcal{H}^n \llcorner \mathrm{nor}(\Omega), (x,u)\big)$ is a n-dimensional plane for $\mathcal{H}^n$-a.e. $(x,u) \in \mathrm{nor}(\Omega)$, we conclude that

$$\mathcal{H}^n(X_i \setminus Y_i) = 0 \qquad \text{for every } i \in \mathbb{N}\,.$$

It follows from *(i)* and *(ii)* that

$$\mathcal{H}^n\Big(\mathrm{nor}(\Omega) \setminus \bigcup_{i=1}^\infty \overline{\nu}_\Omega(Y_i)\Big) = 0\,. \tag{3.1.8}$$

We fix now $x \in Y_i$. Then there exists a map $g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ pointwise differentiable at x such that $g(x) = \overline{\nu}_\Omega(x)$, $\Theta^n(\mathcal{H}^n \llcorner \partial\Omega \setminus \{g = \overline{\nu}_\Omega\}, x) = 0$ and

$$\mathrm{ap}\, D\overline{\nu}_\Omega(x) = Dg(x) | \mathrm{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, x)\,.$$

Noting that $\mathrm{ap}\, D\overline{\nu}_\Omega(x)$ is injective, $g | X_i \cap \{g = \overline{\nu}_\Omega\}$ is bi-lipschitz and

$$\mathrm{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, x) = \mathrm{Tan}^n\big(\mathcal{H}^n \llcorner X_i \cap \{g = \overline{\nu}_\Omega\}, x\big)\,,$$

we readily infer by [51, Lemma B.2] that

$$\mathrm{ap}\, D\overline{\nu}_\Omega(x)\big[\mathrm{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, x)\big] = \mathrm{Tan}^n\big(\mathcal{H}^n \llcorner \mathrm{nor}(\Omega), \overline{\nu}_\Omega(x)\big)\,.$$

Hence, if $\{\tau_1, \ldots, \tau_n\}$ is an orthonormal basis of $\mathrm{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, x)$ with

$$\mathrm{ap}\, D\nu_\Omega(x)(\tau_i) = \chi_{\Omega,i}(x)\tau_i \qquad \text{for } i \in \{1, \ldots, n\}\,,$$

we conclude that

$$\left\{\left(\frac{1}{\sqrt{1+\chi_{\Omega,i}(x)^2}}\tau_i\,, \frac{\chi_{\Omega,i}(x)}{\sqrt{1+\chi_{\Omega,i}(x)^2}}\tau_i\right) : i \in \{1, \ldots, n\}\right\}$$

is an orthonormal basis of $\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathrm{nor}(\Omega), \overline{\nu}_\Omega(x))$. Since x is arbitrarily chosen in Y_i, thanks to (3.1.8), we deduce from the uniqueness stated in Lemma 1.4.20 that

$$\kappa_{\Omega,i}(x,u) = \chi_{\Omega,i}(x) \qquad \text{for } \mathcal{H}^n \text{ a.e. } (x,u) \in \mathrm{nor}(\Omega)\,.$$

Finally we prove *(iv)*. By Lemma 1.4.20 we can choose maps $\{\tau_1, \ldots, \tau_n\}$ defined $\mathcal{H}^n$-a.e. on $\mathrm{nor}(\Omega')$ such that $\{\tau_1(x,u), \ldots, \tau_n(x,u), u\}$ is an orthonormal basis of $\mathbb{R}^{n+1}$,

$$\tau_1(x,u) \wedge \ldots \wedge \tau_n(x,u) \wedge u = \boldsymbol{e}_1 \wedge \ldots \wedge \boldsymbol{e}_{n+1} \quad \text{for } \mathcal{H}^n\text{-a.e. } (x,u) \in \mathrm{nor}(\Omega') \tag{3.1.9}$$

and the vectors

$$\zeta'_i(x,u) := \left(\frac{1}{\sqrt{1+\kappa_{\Omega',i}(x,u)^2}}\tau_i(x,u), \frac{\kappa_{\Omega',i}(x,u)}{\sqrt{1+\kappa_{\Omega',i}(x,u)^2}}\tau_i(x,u)\right) \quad \text{for } i \in \{1, \ldots, n\}$$

form an orthonormal basis of $\operatorname{Tan}^n\big(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega'), (x,u)\big)$ for $\mathcal{H}^n$-a.e. $(x,u) \in \operatorname{nor}(\Omega')$. Then we define

$$\vec{\eta}\,' := \zeta_1' \wedge \ldots \wedge \zeta_n'$$

and notice that

$$|\vec{\eta}\,'(x,u)| = 1\,, \quad \vec{\eta}\,'(x,u) \text{ is simple}\,,$$
$$\operatorname{Tan}^n\big(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega'), (x,u)\big) \text{ is associated with } \vec{\eta}\,'(x,u)$$

and (cf. (1.0.2))

$$\langle [\textstyle\bigwedge_n \pi_0]\,(\vec{\eta}\,'(x,u)) \wedge u, dX_1 \wedge \ldots \wedge dX_{n+1}\rangle > 0 \quad \text{(by (3.1.9))} \tag{3.1.10}$$

for $\mathcal{H}^n$-a.e. $(x,u) \in \operatorname{nor}(\Omega')$. If $p \in \partial\Omega'$, $\epsilon > 0$, $\nu \in \mathbb{S}^n$, $U \subset \nu^\perp$ is a bounded open set with $0 \in U$ and $f \in W^{2,n}(U)$ is a continuous function with $f(0) = 0$ such that

$$\big\{p + b + \tau\nu : b \in U,\ -\epsilon < \tau \le f(b)\big\} = \overline{\Omega'} \cap C_{U,\epsilon}\,,$$

where $C_{U,t} := \{p + b + \tau\nu : b \in U,\ -t < \tau < t\}$ for any $0 < t \le +\infty$, then we observe that

$$N_f = \operatorname{nor}(\Omega') \cap (C_{U,\epsilon} \times \mathbb{S}^n)$$

where $N_f := \operatorname{nor}(C_f) \cap (C_{U,\infty} \times \mathbb{S}^n)$ and $C_f := \big\{p + b + \tau\nu : b \in U,\ -\infty < \tau \le f(b)\big\}$. It follows from (3.1.10) and Theorem 2.1.20 that $\vec{\eta}\,'\big|\big[\operatorname{nor}(\Omega') \cap (C_{U,\epsilon} \times \mathbb{S}^n)\big]$ is $\mathcal{H}^n$-a.e. equal to a Borel n-vectorfield defined over $\operatorname{nor}(\Omega') \cap (C_{U,\epsilon} \times \mathbb{S}^n)$ and

$$(\mathcal{H}^n \llcorner \big[\operatorname{nor}(\Omega') \cap (C_{U,\epsilon} \times \mathbb{S}^n)\big]) \wedge \vec{\eta}\,'$$

is a n-dimensional Legendrian cycle of $C_{U,\epsilon}$. Therefore, we define the integer multiplicity locally rectifiable n-current

$$T' := \big(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega')\big) \wedge \vec{\eta}\,'$$

and we conclude by Lemma 1.3.5 that T' is a Legendrian cycle of $\mathbb{R}^{n+1}$.

We define now $\psi := \Psi_F|\operatorname{nor}(\Omega')$ and, recalling (3.1.6) and noting that

$$\operatorname{ap} D\psi\big(\psi^{-1}(y,v)\big) = D\Psi_F\big(\psi^{-1}(y,v)\big) \tag{3.1.11}$$

for $\mathcal{H}^n$-a.e. $(y,v) \in \operatorname{nor}(\Omega)$, we define

$$\vec{\eta}(y,v) := \frac{\big[\textstyle\bigwedge_n \operatorname{ap} D\psi\big(\psi^{-1}(y,v)\big)\big]\,\vec{\eta}\,'\big(\psi^{-1}(y,v)\big)}{J_n^{\operatorname{nor}(\Omega')}\psi\big(\psi^{-1}(y,v)\big)}$$

for $\mathcal{H}^n$-a.e. $(y,v) \in \operatorname{nor}(\Omega)$. Since Ψ_F is a diffeomorphism we have that $\vec{\eta}(y,v) \neq 0$ for $\mathcal{H}^n$-a.e. $(y,v) \in \operatorname{nor}(\Omega)$. We now apply [14, 4.1.30] with U, K, W, ξ, G and g replaced by $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, $\partial\Omega' \times \mathbb{S}^n$, $\operatorname{nor}(\Omega')$, Ψ_F and ψ respectively. We infer that

$$(\Psi_F)_\#\big[\big(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega')\big) \wedge \vec{\eta}\,'\big] = \big(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega)\big) \wedge \vec{\eta}$$

and that $|\vec{\eta}(y,v)| = 1$ and $\operatorname{Tan}^n\big(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega), (y,v)\big)$ is associated with $\vec{\eta}(y,v)$ for $\mathcal{H}^n$-a.e. $(y,v) \in \operatorname{nor}(\Omega)$. Clearly $\big(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega)\big) \wedge \vec{\eta}$ is a cycle, and $\big[\big(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega)\big) \wedge \vec{\eta}\big] \llcorner \alpha = 0$ by Lemma 1.4.27. Finally, since $\tau_1(x,u) \wedge \ldots \wedge \tau_n(x,u) = (-1)^n * u$ and

$$\begin{aligned}
&[\textstyle\bigwedge_n \pi_0]\Big(\vec{\eta}\big(\Psi_F(x,u)\big)\Big)\\
&= \frac{1}{J_n^{\operatorname{nor}(\Omega')}\psi(x,u)}\left(\prod_{i=1}^n \frac{1}{\sqrt{1+\kappa_{\Omega',i}(x,u)^2}}\right)\big[DF(x)\big(\tau_1(x,u)\big) \wedge \ldots \wedge DF(x)\big(\tau_n(x,u)\big)\big]\\
&= \frac{(-1)^n}{J_n^{\operatorname{nor}(\Omega')}\psi(x,u)}\left(\prod_{i=1}^n \frac{1}{\sqrt{1+\kappa_{\Omega',i}(x,u)^2}}\right)\big[\textstyle\bigwedge_n DF(x)\big](*u)
\end{aligned}$$

for $\mathcal{H}^n$-a.e. $(x, u) \in \mathrm{nor}(\Omega')$, it follows by Remark 3.1.8 below and (3.1.6) that either

$$\langle [\textstyle\bigwedge_n \pi_0] (\vec{\eta}(y, v)) \wedge v, dX_1 \wedge \ldots \wedge dX_{n+1} \rangle > 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } (y, v) \in \mathrm{nor}(\Omega)$$

or

$$\langle [\textstyle\bigwedge_n \pi_0] (\vec{\eta}(y, v)) \wedge v, dX_1 \wedge \ldots \wedge dX_{n+1} \rangle < 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } (y, v) \in \mathrm{nor}(\Omega) .$$

This settles the existence part in statement *(iv)*. Uniqueness easily follows from the defining conditions of T and the representation of $\vec{\eta}$ follows from Lemma 1.4.20.
The proof is complete. □

Remark 3.1.8. Let $* : \mathbb{R}^{n+1} \to \bigwedge_n \mathbb{R}^{n+1}$ be the Hodge-star operator, taken with respect to $\boldsymbol{e}_1 \wedge \ldots \wedge \boldsymbol{e}_{n+1}$ (cf. [14, 1.7.8]). We notice that if $u \in \mathbb{S}^n$ and $\{\tau_1, \ldots, \tau_n\}$ is an orthonomal basis of $u^\perp$ such that $u \wedge \tau_1 \wedge \ldots \wedge \tau_n = \boldsymbol{e}_1 \wedge \ldots \wedge \boldsymbol{e}_{n+1}$, then it follows from the shuffle formula [14, p. 18] that

$$*u = \tau_1 \wedge \ldots \wedge \tau_n .$$

Using this remark, we prove that:

if $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a diffeomorphism, then either
$\langle [\bigwedge_n DF(x)] (*u) \wedge (DF(x)^{-1})^*(u), dX_1 \wedge \ldots \wedge dX_{n+1} \rangle > 0$
for every $(x, u) \in \mathbb{R}^{n+1} \times \mathbb{S}^n$, or
$\langle [\bigwedge_n DF(x)] (*u) \wedge (DF(x)^{-1})^*(u), dX_1 \wedge \ldots \wedge dX_{n+1} \rangle < 0$
for every $(x, u) \in \mathbb{R}^{n+1} \times \mathbb{S}^n$.

By contradiction, assume that there exists $(x, u) \in \mathbb{R}^{n+1} \times \mathbb{S}^n$ such that

$$\langle [\textstyle\bigwedge_n DF(x)] (*u) \wedge (DF(x)^{-1})^*(u), dX_1 \wedge \ldots \wedge dX_{n+1} \rangle = 0$$

and choose an orthonormal basis $\{\tau_1, \ldots, \tau_n\}$ of $u^\perp$ so that $u \wedge \tau_1 \wedge \ldots \wedge \tau_n = \boldsymbol{e}_1 \wedge \ldots \wedge \boldsymbol{e}_{n+1}$. Therefore, $DF(x)(\tau_1) \wedge \ldots \wedge DF(x)(\tau_n) \wedge (DF(x)^{-1})^*(u) = 0$ and, since $\{DF(x)(\tau_i)\}_{i=1}^n$ are linearly independent, we conclude that there exists $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$(DF(x)^{-1})^*(u) = \sum_{i=1}^n c_i \, DF(x)(\tau_i) .$$

Applying $DF(x)^{-1}$ to both sides and taking the scalar product with u, we get

$$\left[DF(x)^{-1} \circ (DF(x)^{-1})^* \right](u) \bullet u = 0 ,$$

whence we infer that $(DF(x)^{-1})^*(u) = 0$, a contradiction.

Definition 3.1.9. *Given $\Omega \subseteq \mathbb{R}^{n+1}$ a $W^{2,n}$-domain, we denote by $\mathcal{N}_\Omega$ the Legendrian cycle given by Theorem 3.1.7 (iv).*

Remark 3.1.10. The proof of Theorem 3.1.7 *(iv)* proves that if $F : U \to V$ is a C^2-diffeomorphism between open subsets of $\mathbb{R}^{n+1}$ and Ω is a bounded $W^{2,n}$-domain such that $\overline{\Omega} \subseteq U$, then

$$(\Psi_F)_\#(\mathcal{N}_\Omega) = \mathcal{N}_{F(\Omega)} .$$

Definition 3.1.11 (r-th elementary symmetric function)**.** *Suppose $r \in \{1, \ldots, n\}$. The r-th symmetric function $\sigma_r : \mathbb{R}^n \to \mathbb{R}$ is defined as*

$$\sigma_r(t_1, \ldots, t_n) := \frac{1}{\binom{n}{r}} \sum_{\lambda \in \Lambda_{n,r}} t_{\lambda(1)} \cdots t_{\lambda(r)} ,$$

where $\Lambda_{n,r}$ is the set of all increasing functions from $\{1, \ldots, r\}$ to $\{1, \ldots, n\}$. We set

$$\sigma_0(t_1, \ldots, t_n) := 1 \quad \textit{for } (t_1, \ldots, t_n) \in \mathbb{R}^n .$$

Definition 3.1.12 (r-th mean curvature function). *Suppose $\Omega \subseteq \mathbb{R}^{n+1}$ is a $W^{2,n}$-domain and $r \in \{0, \dots, n\}$. Then we define the r-th mean curvature function of Ω as*

$$H_{\Omega,r}(z) := \sigma_r(\chi_{\Omega,1}(z), \dots, \chi_{\Omega,n}(z)),$$

for $\mathcal{H}^n$ a.e. $z \in \partial\Omega$.

Lemma 3.1.13. *If $\Omega \subset \mathbb{R}^{n+1}$ is a bounded $W^{2,n}$-domain, then*

$$(\mathcal{N}_\Omega \llcorner \varphi_{n-k})(\phi) = \binom{n}{k} \int_{\partial\Omega} H_{\Omega,k}(x)\, \phi(x, \nu_\Omega(x))\, d\mathcal{H}^n(x)$$

for every $\phi \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ and $k \in \{0, \dots, n\}$.

Proof. We know by Theorem 3.1.7 *(iv)* that

$$\mathcal{N}_\Omega = (\mathcal{H}^n \llcorner \operatorname{nor}(\Omega)) \wedge (\zeta_1 \wedge \dots \wedge \zeta_n).$$

Noting that

$$J_n^{\operatorname{nor}(\Omega)} \pi_0(x,u) = \prod_{i=1}^{n} \frac{1}{\sqrt{1 + \kappa_{\Omega,i}(x,u)^2}} \qquad \text{for } \mathcal{H}^n\text{-a.e. } (x,u) \in \operatorname{nor}(\Omega),$$

we employ Theorem 3.1.7 *(iii)* to compute

$$(\mathcal{N}_\Omega \llcorner \varphi_{n-k})(\phi) = \binom{n}{k} \int_{\operatorname{nor}(\Omega)} J_n^{\operatorname{nor}(\Omega)} \pi_0(x,u)\, \phi(x,u)\, H_{\Omega,k}(x)\, d\mathcal{H}^n(x,u),$$

whence we conclude using area formula for rectifiable sets in combination with Theorem 3.1.7 *(ii)* and Lemma 3.1.5 *(i)*. ☐

Definition 3.1.14 (r-th total curvature measure). *If $\Omega \subset \mathbb{R}^{n+1}$ is a bounded $W^{2,n}$-domain and $r \in \{0, \dots, n\}$, we define*

$$\mathcal{A}_r(\Omega) := \int_{\partial\Omega} H_{\Omega,r}\, d\mathcal{H}^n.$$

Now we can quickly derive the following extension of Reilly's variational formulae (cf. [45]) to $W^{2,n}$-domain.

Theorem 3.1.15. *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a bounded $W^{2,n}$-domain and $\{F_t\}_{t \in (-\epsilon,\epsilon)}$ is a local variation of $\mathbb{R}^{n+1}$ with initial velocity vector field V. Then*

$$\frac{d}{dt}\mathcal{A}_{k-1}(F_t(\Omega))\Big|_{t=0} = (n-k+1) \int_{\partial\Omega} \nu_\Omega(x) \bullet V(x)\, H_{\Omega,k}(x)\, d\mathcal{H}^n \qquad \text{for } k \in \{1, \dots, n\}$$

and

$$\frac{d}{dt}\mathcal{A}_n(F_t(\Omega))\Big|_{t=0} = 0. \tag{3.1.12}$$

Proof. Combining Remark 3.1.10 and Lemma 3.1.13 we obtain

$$[(\Psi_{F_t})_\# \mathcal{N}_\Omega](\varphi_{n-k+1}) = \mathcal{N}_{F_t(\Omega)}(\varphi_{n-k+1}) = \binom{n}{k-1} \mathcal{A}_{k-1}(F_t(\Omega))$$

for $k \in \{1, \dots, n+1\}$. Hence we use Lemma 1.3.11 and again Lemma 3.1.13 to compute

$$\frac{d}{dt}[(\Psi_{F_t})_\# \mathcal{N}_\Omega](\varphi_{n-k+1})\Big|_{t=0} = k\binom{n}{k} \int_{\partial\Omega} V(x) \bullet \nu_\Omega(x)\, H_{\Omega,k}(x)\, d\mathcal{H}^n(x)$$

for $k \in \{1, \dots, n\}$ and

$$\frac{d}{dt}[(\Psi_{F_t})_\# \mathcal{N}_\Omega](\varphi_0)\Big|_{t=0} = 0.$$

The proof is complete. ☐

Remark 3.1.16. If Ω is a C^2-domain, then (3.1.12) follows from the Gauss-Bonnet theorem. The validity of the Gauss-Bonnet theorem for bounded $W^{2,n}$-domains is an interesting open question, and (3.1.12) seems to point to a possible positive answer.

The following integral formulae can be easily deduced from Theorem 3.1.15 by a standard procedure.

Corollary 3.1.17. *If $\Omega \subset \mathbb{R}^{n+1}$ is a bounded $W^{2,n}$-domain and $r \in \{1, \dots, n\}$ then*

$$\int_{\partial\Omega} H_{\Omega,r-1}(x)\, d\mathcal{H}^n(x) = \int_{\partial\Omega} x \bullet \nu_\Omega(x)\, H_{\Omega,r}(x)\, d\mathcal{H}^n(x)\,.$$

Proof. We consider the local variation $F_t(x) = e^t x$ for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and we notice that

$$\operatorname{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, x) = \operatorname{Tan}^n\big(\mathcal{H}^n \llcorner F_t(\partial\Omega), F_t(x)\big)\,,$$

$$\nu_{F_t(\Omega)}\big(F_t(x)\big) = \nu_\Omega(x) \quad \text{and} \quad \chi_{F_t(\Omega),i}\big(F_t(x)\big) = e^{-t}\,\chi_{\Omega,i}(x)\,,$$

for $\mathcal{H}^n$-a.e. $x \in \partial\Omega$ and $i \in \{1, \dots, n\}$. Therefore, we compute by area formula

$$\begin{aligned}
\mathcal{A}_{r-1}(F_t(\Omega)) &= \int_{\partial F_t(\Omega)} H_{F_t(\Omega),r-1}\, d\mathcal{H}^n \\
&= e^{-(r-1)t} \int_{F_t(\partial\Omega)} H_{\Omega,r-1}\big(F_t^{-1}(y)\big)\, d\mathcal{H}^n(y) \\
&= e^{(n-r+1)t} \int_{\partial\Omega} H_{\Omega,r-1}(x)\, d\mathcal{H}^n(x)
\end{aligned}$$

and we apply Theorem 3.1.15. □

Corollary 3.1.18. *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a bounded $W^{2,n}$-domain, $k \in \{1, \dots, n\}$ and*

$$H_{\Omega,i}(z) \geq 0 \quad \textit{for } i \in \{0, \dots, k-1\} \textit{ and for } \mathcal{H}^n\textit{-a.e. } z \in \partial\Omega\,. \tag{3.1.13}$$

Then there exists $P \subseteq \partial\Omega$ such that $\mathcal{H}^n(P) > 0$ and $H_{\Omega,k}(z) \neq 0$ for $z \in P$.

Proof. Suppose $H_{\Omega,k}(z) = 0$ for $\mathcal{H}^n$-a.e. $z \in \partial\Omega$. Then we can employ Corollary 3.1.17 (with $r = k$) and use (3.1.13) (for $i = k-1$) to infer that $H_{\Omega,k-1}(z) = 0$ for $\mathcal{H}^n$-a.e. $z \in \partial\Omega$. Now we repeat this argument with $r = k-1$ and $i = k-2$ to infer that $H_{\Omega,k-2}(z) = 0$ for $\mathcal{H}^n$-a.e. $z \in \partial\Omega$, and we continue until we obtain that $H_{\Omega,0}(z) = 0$ for $\mathcal{H}^n$-a.e. $z \in \partial\Omega$, which means $\mathcal{H}^n(\partial\Omega) = 0$. Since the latter is clearly impossible, we have proved the assertion. □

3.2 Sphere theorems for $W^{2,n}$-domains

The results in the previous section in combination with the Heintze-Karcher inequality proved below can be used to generalize classical sphere theorems to $W^{2,n}$-domains.

Theorem 3.2.19 (Heintze-Karcher inequality)**.** *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a bounded and connected $W^{2,n}$-domain such that $H_{\Omega,1}(z) \geq 0$ for $\mathcal{H}^n$-a.e. $z \in \partial\Omega$, then*

$$(n+1)\, \mathcal{L}^{n+1}(\Omega) \leq \int_{\partial\Omega} \frac{1}{H_{\Omega,1}(x)}\, d\mathcal{L}^n(x)\,.$$

Moreover, if $H_{\Omega,1}(z) \geq \frac{\mathcal{H}^n(\partial\Omega)}{(n+1)\,\mathcal{L}^{n+1}(\Omega)}$ for $\mathcal{H}^n$-a.e. $z \in \partial\Omega$ then Ω is a round ball.

Proof. We define $\Omega' := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ and notice that Ω' is a $W^{2,n}$-domain. Since $\partial^v_+\Omega' = \partial^v_+\Omega$ and $\nu_{\Omega'} = -\nu_\Omega$, it follows from Theorem 3.1.7 that

$$\mathcal{H}^n\big(\operatorname{nor}(\Omega') \setminus \{(z, -\nu_\Omega(z)) : z \in \partial^v_+\Omega\}\big) = 0\,,$$

and

$$-\chi_{\Omega,i}(z) = \chi_{\Omega',i}(z) = \kappa_{\Omega',i}\big(z, -\nu_\Omega(z)\big) \quad \text{for } \mathcal{H}^n\text{-a.e. } z \in \partial^v_+\Omega\,.$$

Therefore

$$\sum_{i=1}^{n} \kappa_{\Omega',i}(z,u) = -n\, H_{\Omega,1}(z) \leq 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } (z,u) \in \operatorname{nor}(\Omega') \tag{3.2.14}$$

we infer from Theorem 1.4.26 and area formula for rectifiable sets [14, Theorem 3.2.22 (3)]

$$\begin{aligned}(n+1)\,\mathcal{L}^{n+1}(\Omega) &\leq \int_{\operatorname{nor}(\Omega')} J_n^{\operatorname{nor}(\Omega')} \pi_0(z,u)\, \frac{n}{|\sum_{i=1}^{n} \kappa_{\Omega',i}(z,u)|}\, d\mathcal{H}^n(z,u) \\ &= \int_{\partial\Omega} \frac{1}{H_{\Omega,1}(z)}\, d\mathcal{H}^n(z)\,. \end{aligned}\tag{3.2.15}$$

We assume now that $H_{\Omega,1}(z) \geq \frac{\mathcal{H}^n(\partial\Omega)}{(n+1)\,\mathcal{L}^{n+1}(\Omega)}$ for $\mathcal{H}^n$-a.e. $z \in \partial\Omega$. Then, we observe that

$$\mathcal{H}^n\left(\left\{z \in \partial\Omega : H_{\Omega,1}(z) \geq (1+\epsilon)\,\frac{\mathcal{H}^n(\partial\Omega)}{(n+1)\,\mathcal{L}^{n+1}(\Omega)}\right\}\right) = 0 \quad \text{for every } \epsilon > 0\,,$$

otherwise we would obtain a contradiction with the inequality (3.2.15) (cf. proof of [24, Corollary 5.16]). This implies that

$$H_{\Omega,1}(z) = \frac{\mathcal{H}^n(\partial\Omega)}{(n+1)\,\mathcal{L}^{n+1}(\Omega)} \quad \text{for } \mathcal{H}^n\text{-a.e. } z \in \partial\Omega\,,$$

whence we infer that (3.2.15) holds with equality. Recalling (3.2.14) we deduce from Theorem 1.4.26 that Ω must be a round ball. □

Theorem 3.2.20. *Suppose $k \in \{1,\ldots,n\}$, $\lambda \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^{n+1}$ is a bounded and connected $W^{2,n}$-domain such that*

$$H_{\Omega,i}(z) \geq 0 \quad \text{for } i \in \{0,\ldots,k-1\}\,, \tag{3.2.16}$$

$$H_{\Omega,k}(z) = \lambda \tag{3.2.17}$$

for $\mathcal{H}^n$-a.e. $z \in \partial\Omega$. Then Ω is a round ball.

Proof. Combining Theorem 3.1.17 and divergence Theorem for sets of finite perimeter (it is clear by Lemma 3.1.5 that Ω is a set of finite perimeter, whose reduced boundary is $\mathcal{H}^n$-a.e. equal to the topological boundary) we obtain

$$\int_{\partial\Omega} H_{\Omega,k-1}\, d\mathcal{H}^n = \lambda \int_{\partial\Omega} x \bullet \nu_\Omega(x)\, d\mathcal{H}^n(x) = \lambda(n+1)\,\mathcal{L}^{n+1}(\Omega) \tag{3.2.18}$$

and we infer that $\lambda \geq 0$. Hence we deduce from [24, Lemma 2.2] and Corollary 3.1.18 that

$$H_{\Omega,1}(z) \geq \ldots \geq H_{\Omega,k-1}(z)^{\frac{1}{k-1}} \geq H_{\Omega,k}(z)^{\frac{1}{k}} = \lambda^{\frac{1}{k}} > 0 \tag{3.2.19}$$

for $\mathcal{H}^n$-a.e. $z \in \partial\Omega$. By (3.2.19)

$$\int_{\partial\Omega} H_{\Omega,k-1}(z)\, d\mathcal{H}^n(z) \geq \lambda^{\frac{k-1}{k}}\, \mathcal{H}^n(\partial\Omega)$$

and combining with (3.2.18) we obtain

$$\lambda(n+1)\,\mathcal{L}^{n+1}(\Omega) \geq \lambda^{\frac{k-1}{k}}\, \mathcal{H}^n(\partial\Omega)\,.$$

Since $\lambda > 0$, we obtain from (3.2.19) that

$$H_{\Omega,1}(z) \geq \frac{\mathcal{H}^n(\partial\Omega)}{(n+1)\,\mathcal{L}^{n+1}(\Omega)} \quad \text{for } \mathcal{H}^n\text{-a.e. } z \in \partial\Omega \tag{3.2.20}$$

and we conclude applying Theorem 3.2.19. □

Remark 3.2.21. Hypothesis (3.2.16) in Theorem 3.2.20 can be replaced by the following assumption:

$$\partial_i \sigma_k\big(\chi_{\Omega,1}(z), \ldots, \chi_{\Omega,n}(z)\big) \geq 0 \quad \textit{for every } i \in \{1, \ldots, n\} \textit{ and for } \mathcal{H}^n\textit{-a.e. } z \in \partial\Omega\,. \tag{3.2.21}$$

Assume (3.2.21) in place of (3.2.16). First, using [50, eq. (1.15)], we obtain

$$H_{\Omega,k-1}(z) = \frac{1}{k}\sum_{i=1}^{n} \partial_i \sigma_k(\chi_{\Omega,1}(z), \ldots, \chi_{\Omega,n}(z)) \geq 0$$

for $\mathcal{H}^n$-a.e. $z \in \partial\Omega$. Then, as in (3.2.18), we deduce that $\lambda \geq 0$. Finally, applying [50, Proposition 1.3.2], we recover (3.2.16).

Chapter 4

Fine properties of $F_nW^{2,n}$-sets

4.1 Introduction to $F_nW^{2,n}$-sets

We introduce the class of $F_nW^{2,n}$-functions following Ambrosio, Gobbino and Pallara (cf. [4]), who developed an idea of De Giorgi.

Definition 4.1.1. ($F_nW^{2,n}$-functions). *Given $\Omega \subseteq \mathbb{R}^{n+1}$ an open set and a function $\boldsymbol{\iota} : \Omega \to \mathbb{N}$, we say that $\boldsymbol{\iota} \in F_nW^{2,n}(\Omega)$ if for any $z \in \{\boldsymbol{\iota} > 0\}$ there exist an open neighborhood $U \subseteq \Omega$ of z, also a positive integer $q(z)$ and a family $\{\Gamma_i\}_{i=1}^{q(z)}$ of subset of $\mathbb{R}^{n+1}$, such that*

$$\boldsymbol{\iota}(x) = \sum_{i=1}^{q(z)} \mathbf{1}_{\Gamma_i}(x) \quad \text{for any } x \in U \tag{4.1.1}$$

where every Γ_i satisfies the following property:

there exist $p_i \in \Gamma_i \cap U$, $\eta_i \in \mathbb{S}^n$, a set V_i open in $\eta_i^\perp$ so that $0 \in V_i$, $f_i \in C^0(V_i) \cap W^{2,n}(V_i)$ such that $f_i(0) = 0$ which also induces the map

$$\overline{f_i} : x \in V_i \mapsto x + f_i(x)\eta_i \in \mathbb{R}^{n+1},$$

in such a way that

$$\Gamma_i \cap U = \overline{f_i}(V_i) + p_i\,. \tag{4.1.2}$$

We say f_i a graph function of $\Gamma_i \cap U$, on V_i.

Remark 4.1.2. Given $U \subset \mathbb{R}^{n+1}$ a bounded open set and $f \in C^0(U) \cap W^{2,n}(U)$, we consider $\Gamma := \operatorname{graph}(f)$. Since $\overline{f}$ satisfies the Lusin (N)-property (cf. Remark 5.3.4), then applying Theorem 5.3.3 we infer that Γ is $\mathcal{H}^n$-rectifiable of class 2.

Definition 4.1.3. ($F_nW^{2,n}$-sets). *We say that a closed set $\mathcal{S} \subset \mathbb{R}^{n+1}$ is a $F_nW^{2,n}$-set if*

$$\mathcal{S} = \{\boldsymbol{\iota} > 0\}\,,$$

for some $\boldsymbol{\iota} \in F_nW^{2,n}(\mathbb{R}^{n+1})$. Moreover, we say $\boldsymbol{\iota}$ multiplicity function of $\mathcal{S}$.

Remark 4.1.4. Equivalently, a closed set $\mathcal{S} \subset \mathbb{R}^{n+1}$ is a $F_nW^{2,n}$-set if for every $z \in \mathcal{S}$ there exist a positive integer $q(z)$, an open neighborhood $U \subseteq \mathbb{R}^{n+1}$ of z (it is not restrictive to assume that U is bounded) and a family $\{\Gamma_i\}_{i=1}^{q(z)}$ of subset of $\mathbb{R}^{n+1}$ such that

$$\mathcal{S} \cap U = \bigcup_{i=1}^{q(z)} (\Gamma_i \cap U)\,, \tag{4.1.3}$$

where every $\Gamma_i \cap U$ coincides with the graph of a $(C^0 \cap W^{2,n})$-function.

Definition 4.1.5. ($\mathscr{W}^{2,n}$-sets). *We say that that a closed set $\mathcal{S} \subset \mathbb{R}^{n+1}$ is a $\mathscr{W}^{2,n}$-set if there exists a pair $(\mathcal{S}', F)$, that satisfies the following properties:*

(i) *$\mathcal{S}'$ is a $F_nW^{2,n}$-set;*

(ii) *$F(\mathcal{S}') = \mathcal{S}$, where F is a C^2-diffeomorphism of $\mathbb{R}^{n+1}$.*

Remark 4.1.6. Given $\mathcal{S}$ a $\mathscr{W}^{2,n}$-set with associated pair $(\mathcal{S}', F)$, let us assume that in an open neighborhood $U \subseteq \mathbb{R}^{n+1}$ of $z \in \mathcal{S}'$ holds (4.1.3). Then, we have

$$\mathrm{Tan}\big(\mathcal{S}, F(z)\big) = \bigcup_{i=1}^{q(z)} \mathrm{Tan}\big(F(\Gamma_i), F(z)\big) \tag{4.1.4}$$

where every $\Gamma_i \cap U$ coincides with the graph of a $(C^0 \cap W^{2,n})$-function. To prove (4.1.4), first we notice that $\bigcup_{i=1}^{q(z)} \mathrm{Tan}\big(F(\Gamma_i), F(z)\big) \subseteq \mathrm{Tan}\big(\mathcal{S}, F(z)\big)$. In fact, from (4.1.3) it immediately follows that $\mathrm{Tan}\big(F(\Gamma_i), F(z)\big) \subseteq \mathrm{Tan}\big(\mathcal{S}, F(z)\big)$ for every $i \in \{1, \dots, q(z)\}$. Now let us consider $u \in \mathrm{Tan}\big(\mathcal{S}, F(z)\big) \cap \mathbb{S}^n$, hence there exists $\{z_j\}_{j\in\mathbb{N}} \subset \mathcal{S} \setminus \{F(z)\}$ such that

$$z_j \xrightarrow[j\to\infty]{} F(z) \quad \text{and} \quad \frac{z_j - F(z)}{|z_j - F(z)|} \xrightarrow[j\to\infty]{} u\,.$$

Since $\mathcal{S} \cap F(U)$ is a finite union of $F(\Gamma_i) \cap F(U)$, it follows that there exist $h \in \{1, \dots, q(z)\}$ and $\{z_{j_k}\}_{k\in\mathbb{N}} \subseteq \{z_j\}_{j\in\mathbb{N}}$ such that $\{z_{j_k}\}_{k\in\mathbb{N}} \subset F(\Gamma_h) \setminus \{F(z)\}$ where

$$z_{j_k} \xrightarrow[k\to\infty]{} F(z) \quad \text{and} \quad \frac{z_{j_k} - F(z)}{|z_{j_k} - F(z)|} \xrightarrow[k\to\infty]{} u\,,$$

namely $u \in \mathrm{Tan}\big(F(\Gamma_h), F(z)\big)$. Hence $\mathrm{Tan}\big(\mathcal{S}, F(z)\big) \subseteq \bigcup_{i=1}^{q(z)} \mathrm{Tan}\big(F(\Gamma_i), F(z)\big)$.

Lemma 4.1.7. *Given $U \subseteq \mathbb{R}^n$ an open set, $f \in C^0(U) \cap W^{2,n}(U)$, $\Gamma := \mathrm{graph}(f)$ and $x \in U$, then the following properties hold.*

(i) Let $g \in C^2(U)$, assume that $f(x) = g(x)$ and Γ is contained either in the epi-graph or in the cato-graph of g. Then, if we introduce $\Sigma := \mathrm{graph}(g)$, we have that

$$\mathrm{Tan}(\Gamma, z) = \mathrm{Tan}(\Sigma, z) \quad \textit{where } z := \overline{f}(x) = \overline{g}(x)\,. \tag{4.1.5}$$

(ii) If we consider E_f and C_f, respectively the epi-graph and the cato-graph of f, we have that

$$\overline{f}\big(\mathcal{S}^*(f) \cap \mathcal{S}_*(f)\big) = N_2(\Gamma)\,, \tag{4.1.6}$$

$$\overline{f}\big(\mathcal{S}^*(f)\big) = N_1(C_f) \quad \textit{and} \quad \overline{f}\big(\mathcal{S}_*(f)\big) = N_1(E_f)\,.$$

Proof. About the proof of *(i)*, since Σ is a C^2-regular graph and therefore satisfies the two-sides sphere condition[1], we infer that there exists $\nu \in \mathbb{S}^n$ and $s > 0$ such that

$$B_s(z + s\nu) \cap \Gamma = \emptyset \quad \text{and} \quad B_s(z + s\nu) \cap \Sigma = \emptyset$$

more specifically, to prove (4.1.5), we show that

$$\mathrm{Tan}(\Gamma, z) = \mathrm{Tan}(\Sigma, z) = \nu^{\perp}. \tag{4.1.7}$$

First, applying Lemma 2.1.17, we deduce that $\nu \notin \mathbb{R}^n \times \{0\}$. Then, by the implicit function theorem, there exist $\delta > 0$, $h \in C^\infty\big(B_\delta(x)\big)$ and an open neighborhood $V \subset U \times \mathbb{R}$ of z, such that $V \cap \partial B_s(z + s\nu) = \mathrm{graph}(h)$, $\overline{h}(x) = z$ and (up to a sign)

$$\nu = \frac{(-\nabla h(x), 1)}{\sqrt{1 + |\nabla h(x)|^2}}\,.$$

[1]**Two-sides sphere condition.** We say that the graph Γ, of a continuos function f, satisfies the *two-sides sphere condition* if for every $x \in \Gamma$ there exist $\nu \in \mathbb{S}^{n-1}$ and $r > 0$ such that $B_r(x + r\nu) \subseteq E_f$ and $B_r(x - r\nu) \subseteq C_f$, where E_f and C_f are, respectively, the epi-graph and the cato-graph of f. The two-sides sphere condition always holds if Γ is a C^2-regular graph (cf. [21, Remark 4.3.8]), in this situation we have $\mathrm{Tan}(\Gamma, x) = \mathrm{Tan}(\partial B_r(x \pm r\nu), x) = \nu^{\perp}$ (the proof is the same as that performed for (4.1.7)).

Therefore, by Lemma 2.1.9 and [14, 3.1.21], we conclude that

$$\operatorname{Tan}(\Gamma,z) = D\overline{f}(x)[\mathbb{R}^n] = D\overline{g}(x)[\mathbb{R}^n] = \operatorname{Tan}(\Sigma,z)$$

and

$$D\overline{f}(x)[\mathbb{R}^n] = D\overline{h}(x)[\mathbb{R}^n] = \{(v,\nabla h(x)\bullet v) : v\in\mathbb{R}^n\} = \nu^\perp .$$

About the proof of *(ii)*, we only show (4.1.6) and similarly one can prove that

$$\overline{f}(\mathcal{S}^*(f)) = N_1(C_f) \quad \text{and} \quad \overline{f}(\mathcal{S}_*(f)) = N_1(E_f).$$

To prove $\overline{f}(\mathcal{S}^*(f)\cap\mathcal{S}_*(f)) \subseteq N_2(\Gamma)$, we consider an arbitrary $x\in\mathcal{S}^*(f)\cap\mathcal{S}_*(f)$ and we set $z := \overline{f}(x)$. Then, from the definitions of $\mathcal{S}^*(f)$ and $\mathcal{S}_*(f)$, we infer that $z\in N_2(\Gamma)\cup N_\infty(\Gamma)$ and, by the statement *(i)*, we deduce also that $\operatorname{Tan}(\Gamma,z)\in\mathbf{G}(n+1,n)$. Therefore, we obtain a contradiction if $z\in N_\infty(\Gamma)$. In this situation, in fact, since $\operatorname{nor}(\Gamma,z)\subseteq\operatorname{Nor}(\Gamma,z)\cap\mathbb{S}^n$ (cf. [13, Theorem 4.8 (2)]), we infer that

$$\infty = \mathcal{H}^0(\operatorname{nor}(\Gamma,z)) \le \mathcal{H}^0(\operatorname{Nor}(\Gamma,z)\cap\mathbb{S}^n) = 2.$$

Overall $z\in N_2(\Gamma)$, which is the desidered result.

Now we show that $N_2(\Gamma)\subseteq\overline{f}(\mathcal{S}^*(f)\cap\mathcal{S}_*(f))$. Assume that $x\in N_2(\Gamma)$, namely $x\in U$ and there exist $\nu\in\mathbb{S}^n$ and $s>0$ such that

$$B^{n+1}_s(z\pm s\nu)\cap\Gamma=\varnothing \quad \text{and} \quad \pi'(B^{n+1}_s(z\pm s\nu))\subseteq U \tag{4.1.8}$$

where $z := \overline{f}(x)$ and π' denotes the canonical projection on the first n components (in the continuation, we denote by π'' the canonical projection on the last component). Applying Lemma 2.1.17 we also deduce that $\nu\in\mathbb{S}^n\setminus(\mathbb{R}^n\times\{0\})$, namely $\pi''(\nu)\neq 0$. Our goal is to prove that

$$B^{n+1}_s(z-s\nu)\subset C_f \quad \text{and} \quad B^{n+1}_s(z+s\nu)\subset E_f \quad \text{(or vice versa)}, \tag{4.1.9}$$

where, respectively, C_f and E_f are the cato-graph and the epi-graph of f. Indeed from (4.1.9), by the implicit function theorem (remember that $\pi''(\nu)\neq 0$) and the Taylor expansion, we infer $N_2(\Gamma)\subseteq\overline{f}(\mathcal{S}^*(f)\cap\mathcal{S}_*(f))$.

If (4.1.9) does not hold, we deduce that $B^{n+1}_s(z-s\nu)$ and $B^{n+1}_s(z+s\nu)$ are both contained in C_f or in E_f, otherwise we contradict (4.1.8). So, it si not restrictive to assume that

$$B^{n+1}_s(z\pm s\nu)\subset E_f \quad \text{and} \quad \pi''(\nu)>0.$$

Then, if we focus our attention on $B^{n+1}_s(z-s\nu)$, we notice that:

1. $x=\pi'(z)\in\pi'(B^{n+1}_s(z-s\nu))$;
2. for every $\xi\in\{y\in\partial B^{n+1}_s(z-s\nu) : (y-z+s\nu)\bullet\boldsymbol{e}_{n+1}\le 0\}$ we have
$$f(x)=\pi''(z)=\pi''(z-s\nu)+s\,\pi''(\nu)>\pi''(z-s\nu)\ge\pi''(\xi);$$
3. since $B^{n+1}_s(z-s\nu)\subset E_f$, we have
$$f(y)\le\min\{t\in\mathbb{R} : y+t\,\boldsymbol{e}_{n+1}\in\overline{B}^{n+1}_s(z-s\nu)\} \quad \text{for every } y\in\pi'(\overline{B}^{n+1}_s(z-s\nu));$$

hence, if we consider $\widehat{z}\in\partial B^{n+1}_s(z-s\nu)$ such that

$$\{y\in\partial B^{n+1}_s(z-s\nu) : (y-z+s\nu)\bullet\boldsymbol{e}_{n+1}\le 0\}\cap(\pi')^{-1}(x)=\{\widehat{z}\},$$

we contradict the graphicability of f since $f(x)>\pi''(\widehat{z})\ge f(x)$. The proof is complete. □

Definition 4.1.8. *Let $\mathcal{S}$ be a $F_nW^{2,n}$-set and let us assume that (4.1.3) holds in an open neighborhood $U \subseteq \mathbb{R}^{n+1}$ of $z \in \mathcal{S}$. We define the map*

$$\boldsymbol{\mathcal{I}} : x \in \mathcal{S} \cap U \mapsto \big\{ i \in \{1, \dots, q(z)\} : x \in \Gamma_i \cap U \big\} \in \mathbb{N} . \tag{4.1.10}$$

Remark 4.1.9. We notice that $\mathcal{H}^0\big(\boldsymbol{\mathcal{I}}(x)\big) = \boldsymbol{\iota}(x)$ for every $x \in \mathcal{S} \cap U$.

Now we collect some fine properties of $\mathscr{W}^{2,n}$-sets.

Lemma 4.1.10. *Given $\mathcal{S}$ a $\mathscr{W}^{2,n}$-set with associated pair $(\mathcal{S}', F)$ and assume, in an open neighborhood $U \subseteq \mathbb{R}^{n+1}$ of $z \in \mathcal{S}'$, that there exists a family $\{\Gamma_i\}_{i=1}^{q(z)}$ of subset of $\mathbb{R}^{n+1}$ such that*

$$\mathcal{S}' \cap U = \bigcup_{i=1}^{q(z)} (\Gamma_i \cap U) , \tag{4.1.11}$$

where every $\Gamma_i \cap U$ coincides with the graph of a $(C^0 \cap W^{2,n})$-function. Then:

(i) $K \cap \mathcal{S}$ *is $\mathcal{H}^n$-rectifiable of class 2 for every compact set $K \subset \mathbb{R}^{n+1}$;*

(ii) $\mathrm{nor}(\mathcal{S})$ *satisfies the Lusin (N)-property, namely*

$$Z \subset \mathcal{S} \text{ s.t. } \mathcal{H}^n(Z) = 0 \ \Rightarrow\ \mathcal{H}^n\big(\mathrm{nor}(\mathcal{S}) \llcorner Z\big) = 0 ;$$

(iii) *If $(w, \nu) \in \mathrm{nor}(\mathcal{S})$, then $\mathrm{Tan}(\mathcal{S}, w) = \nu^{\perp}$;*

(iv) $N_\infty(\mathcal{S}) = \emptyset$;

(v) $\mathcal{H}^n\big(N_1(\mathcal{S})\big) = 0$;

(vi) $\mathcal{H}^n\Big(\big(\mathcal{S}' \cap U\big) \setminus \bigcup_{i=1}^{q(z)} \big(N_2(\Gamma_i) \cap U\big)\Big) = 0$;

(vii) *given $i \in \{1, \dots, q(z)\}$, we define $\nu_i : N_2(\Gamma_i) \cap U \to \mathbb{S}^n$ in such a way that the following is satisfied*

$$\mathrm{nor}(\Gamma_i, x) = \big\{\nu_i(x), -\nu_i(x)\big\} \quad \text{for } x \in N_2(\Gamma_i) \cap U .$$

Then, for $\mathcal{H}^n$-a.e. $x \in \mathcal{S}' \cap U$, we infer that

$$\nu_i(x) = \pm\nu_j(x) \quad \text{and} \quad B_s^{n+1}\big(x \pm \nu_i(x)\big) \cap \mathcal{S}' = \emptyset \quad \text{for some } s > 0$$

whenever $i, j \in \boldsymbol{\mathcal{I}}(x)$. Moreover, for $\mathcal{H}^n$-a.e. $(x, u) \in \mathrm{nor}(\mathcal{S}') \llcorner U$, we have

$$\sum_{i=1}^{q(z)} \mathbf{1}_{\mathrm{nor}(\Gamma_i)}(x, u) = \boldsymbol{\iota}(x)\, \mathbf{1}_{\mathrm{nor}(\mathcal{S}')}(x, u) \tag{4.1.12}$$

therefore

$$\mathcal{H}^n\Big(\Big[\bigcup_{i=1}^{q(z)} \mathrm{nor}(\Gamma_i) \llcorner U\Big] \setminus \big[\mathrm{nor}(\mathcal{S}') \llcorner U\big]\Big) = 0 ; \tag{4.1.13}$$

(viii) $\mathcal{H}^n\big(\mathcal{S} \setminus N_2(\mathcal{S})\big) = 0$.

Proof. Since $F(\mathcal{S}') = \mathcal{S}$, the assertion *(i)* follows from (4.1.11) and Remark 4.1.2.

To prove *(ii)*, we consider the C^1-diffeomorphism

$$\Psi_F : (x, y) \in \mathbb{R}^{n+1} \times \mathbb{S}^n \mapsto \left(F(x), \frac{(DF(x)^{-1})^*(y)}{|(DF(x)^{-1})^*(y)|}\right) \in \mathbb{R}^{n+1} \times \mathbb{S}^n$$

for which we have (cf. [54, Lemma 2.1])

$$\Psi_F\big(\mathrm{nor}(\mathcal{S}')\big) = \mathrm{nor}(\mathcal{S}) .$$

Hence, for every set $Z \subseteq \mathbb{R}^{n+1}$ we deduce that

$$\mathcal{H}^n\big(\mathrm{nor}(\mathcal{S}) \llcorner Z\big) = \mathcal{H}^n\Big(\Psi_F\big(\mathrm{nor}(\mathcal{S}') \llcorner F^{-1}(Z)\big)\Big)$$

then, to prove the Lusin (N)-property on $\mathrm{nor}(\mathcal{S})$, it is enough to prove the same property on $\mathrm{nor}(\mathcal{S}')$. To this aim, let us consider an arbitrary $z \in \mathcal{S}'$ and we choose an open neighborhood $U \subseteq \mathbb{R}^{n+1}$ of z such that (4.1.11) holds, then we readily infer that

$$\mathrm{nor}(\mathcal{S}') \llcorner U \subseteq \bigcup_{i=1}^{q(z)} \mathrm{nor}(\Gamma_i) \llcorner U\,. \tag{4.1.14}$$

Hence, from (4.1.14) and (2.1.30), we deduce that

$$\mathcal{H}^n\big(\mathrm{nor}(\mathcal{S}') \llcorner (U \cap Z)\big) \le \sum_{i=1}^{q(z)} \mathcal{H}^n\big(\mathrm{nor}(\Gamma_i) \llcorner (U \cap Z)\big) = 0$$

for every $\mathcal{H}^n$-negligible set $Z \subset \mathcal{S}$. Since U is arbitrarily chosen, the Lusin (N)-property on $\mathrm{nor}(\mathcal{S}')$ easily follows.

Now we prove *(iii)*, hence we choose $(w, \nu) \in \mathrm{nor}(\mathcal{S})$, where $w = F(z)$. There exist an open neighborhood U of z and $\{\Gamma_i\}_{i=1}^{q(z)}$ such that (4.1.11) holds and

$$\mathcal{S} \cap F(U) = \bigcup_{i=1}^{q(z)} \big(F(\Gamma_i) \cap F(U)\big)\,,$$

where every $\Gamma_i \cap U$ is the graph of a $(C^0 \cap W^{2,n})$-function. We claim that, if $i \in \{1, \ldots, q(z)\}$ such that $z \in \Gamma_i \cap U$, then $\mathrm{Tan}\big(F(\Gamma_i), w\big) = \nu^\perp$. This clearly proves *(iii)*, since (cf. (4.1.4))

$$\mathrm{Tan}(\mathcal{S}, w) = \bigcup_{i=1}^{q(z)} \mathrm{Tan}\big(F(\Gamma_i), w\big)$$

and $\mathrm{Tan}\big(F(\Gamma_i), w\big) = \varnothing$ if $w \notin F(\Gamma_i)$ (indeed $w \in F(U)$ and $F(\Gamma_i)$ is a closed set in $F(U)$). If $z \in \Gamma_i \cap U$, then there exists $r > 0$ such that

$$B_r^{n+1}(w + r\nu) \cap \big(F(\Gamma_i) \cap F(U)\big) = \varnothing\,.$$

The domain $\Omega := F^{-1}\big[B_r^{n+1}(w + r\nu)\big]$ is C^2-regular, $z \in \partial\Omega$ and $\mathrm{Tan}(\partial\Omega, z) = DF^{-1}(z)(\nu^\perp)$ by [14, 3.1.21]. Since $\Omega \cap \Gamma_i \cap U = \varnothing$, it follows from Lemma 4.1.7 *(i)* that

$$\mathrm{Tan}(\Gamma_i, z) = \mathrm{Tan}(\partial\Omega, z)\,,$$

namely $\mathrm{Tan}(\Gamma_i, z) = DF^{-1}(z)(\nu^\perp)$. Again by [14, 3.1.21], we infer that

$$\mathrm{Tan}\big(F(\Gamma_i), w\big) = DF(z)\big[\mathrm{Tan}(\Gamma_i, z)\big] = \nu^\perp\,.$$

Clearly *(iv)* follows from *(iii)*.

Now we prove *(v)* and *(vi)*. To show that $\mathcal{H}^n(N_1(\mathcal{S})) = 0$, since $N_1(\mathcal{S}) = F(N_1(\mathcal{S}'))$ (cf. Lemma 1.4.36 *(ii)*), it is enough to prove that $\mathcal{H}^n(N_1(\mathcal{S}')) = 0$. Let us consider an arbitrary $z \in \mathcal{S}'$ and we choose a bounded open neighborhood $U \subset \mathbb{R}^{n+1}$ of z such that

$$\mathcal{S}' \cap U = \bigcup_{i=1}^{q(z)} (\Gamma_i \cap U)\,,$$

where every

$$\Gamma_i \cap U = \overline{f_i}(V_i) + p_i$$

for some V_i open in $\eta_i^\perp$ with $0 \in V_i$, $p_i \in \Gamma_i \cap U$, $f_i \in C^0(V_i) \cap W^{2,n}(V_i)$ with $f_i(0) = 0$ and

$$\overline{f}_i : x \in V_i \mapsto x + f_i(x)\eta_i \in \mathbb{R}^{n+1} .$$

Therefore, for every $i \in \{1, \dots, q(z)\}$, by Lemma 4.1.7 *(ii)* we have that

$$N_1(\Gamma_i) \cap U \subseteq \overline{f}_i\Big(\big(\mathcal{S}^*(f_i) \cup \mathcal{S}_*(f_i)\big) \setminus \big(\mathcal{S}^*(f_i) \cap \mathcal{S}_*(f_i)\big) \Big) + p_i$$

and by Theorem 2.1.7 and Remark 5.3.4 we infer

$$\mathcal{H}^n\big(N_1(\Gamma_i) \cap U\big) \leq \mathcal{H}^n\bigg(\overline{f}_i\Big(\big(\mathcal{S}^*(f_i) \cup \mathcal{S}_*(f_i)\big) \setminus \big(\mathcal{S}^*(f_i) \cap \mathcal{S}_*(f_i)\big) \Big) \bigg) = 0 .$$

Noting that $N_1(\mathcal{S}') \cap U \subseteq \bigcup_{i=1}^{q(z)} N_1(\Gamma_i) \cap U$, we obtain that $\mathcal{H}^n\big(N_1(\mathcal{S}') \cap U\big) = 0$. Since U is arbitrarily chosen we conclude that $\mathcal{H}^n(N_1(\mathcal{S}')) = 0$. To prove *(vi)*, applying Lemma 4.1.7 *(ii)* we deduce that

$$\begin{aligned}(\mathcal{S}' \cap U) \setminus \bigcup_{i=1}^{q(z)} (N_2(\Gamma_i) \cap U) &\subseteq \bigcup_{i=1}^{q(z)} \Big[(\Gamma_i \setminus N_2(\Gamma_i)) \cap U \Big] \\ &= \bigcup_{i=1}^{q(z)} \Big[\overline{f}_i\Big(V_i \setminus \big(\mathcal{S}^*(f_i) \cap \mathcal{S}_*(f_i)\big)\Big) + p_i \Big]\end{aligned}$$

then, by Theorem 2.1.7 and Remark 5.3.4, we obtain the desidered result.

To prove *(vii)*, first we recall that the set $\bigcup_{i=1}^{q(z)} (N_2(\Gamma_i) \cap U)$ has full $\mathcal{H}^n$-measure in $\mathcal{S}' \cap U$ (cf. statement *(vi)*). Then, for $\mathcal{H}^n$-a.e. $x \in N_2(\Gamma_i) \cap N_2(\Gamma_j) \cap U$ where $i, j \in \{1, \dots, q(z)\}$, by Lemma 4.1.7 *(i)* (cf. (4.1.7)) and by the locality property of the approximate tangent spaces (cf. (1.1.4)) we have that

$$\begin{aligned}\nu_i(x)^\perp &= \mathrm{Tan}(\Gamma_i, x) = \mathrm{Tan}^n(\mathcal{H}^n \llcorner \Gamma_i, x) \\ &= \mathrm{Tan}^n(\mathcal{H}^n \llcorner \Gamma_j, x) = \mathrm{Tan}(\Gamma_j, x) = \nu_j(x)^\perp ,\end{aligned}$$

hence

$$\nu_i(x) = \pm \nu_j(x) \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in N_2(\Gamma_i) \cap N_2(\Gamma_j) \cap U .$$

So there exists a map ν, with values in $\mathbb{S}^n$ and defined $\mathcal{H}^n$-a.e. on $\mathcal{S}' \cap U$ in such a way that

$$\nu(x) \in \{\nu_i(x), -\nu_i(x)\} \quad \text{if } x \in N_2(\Gamma_i) \cap U$$

for $i \in \{1, \dots, q(z)\}$, such that for $\mathcal{H}^n$-a.e. $x \in \mathcal{S}' \cap U$ we have

$$B_s^{n+1}(x \pm s\nu(x)) \cap \mathcal{S}' = \varnothing \quad \text{for some } s > 0$$

namely (cf. statement *(iv)*)

$$\mathcal{H}^n\big((\mathcal{S}' \cap U) \setminus N_2(\mathcal{S}')\big) = 0 . \tag{4.1.15}$$

Hence, for every $i \in \{1, \dots, q(z)\}$, we have that

$$\mathrm{nor}(\mathcal{S}', x) = \mathrm{nor}(\Gamma_i, x) = \{\nu(x), -\nu(x)\} \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \Gamma_i \cap U$$

thus, by the Lusin (N)-property on $\mathrm{nor}(\mathcal{S}')$

$$\mathbf{1}_{\mathrm{nor}(\Gamma_i)}(x, u) = \mathbf{1}_{\mathrm{nor}(\mathcal{S}')}(x, u)\, \mathbf{1}_{\Gamma_i}(x) \quad \text{for } \mathcal{H}^n\text{-a.e. } (x, u) \in \mathrm{nor}(\mathcal{S}') \llcorner U .$$

Therefore, from the definition of $\boldsymbol{\iota}$ (cf. (4.1.1)), we conclude that

$$\sum_{i=1}^{q(z)} \mathbf{1}_{\mathrm{nor}(\Gamma_i)}(x, u) = \boldsymbol{\iota}(x)\, \mathbf{1}_{\mathrm{nor}(\mathcal{S}')}(x, u) \quad \text{for } \mathcal{H}^n\text{-a.e. } (x, u) \in \mathrm{nor}(\mathcal{S}') \llcorner U$$

hence

$$\mathcal{H}^n\Big(\Big[\bigcup_{i=1}^{q(z)} \operatorname{nor}(\Gamma_i) \llcorner U\Big] \setminus \big[\operatorname{nor}(\mathcal{S}') \llcorner U\big]\Big) = 0\,.$$

To prove *(viii)*, as usually it is sufficient to show the assertion on $\mathcal{S}'$ (cf. Lemma 1.4.36 *(ii)*) and so we conclude from (4.1.15).The proof is complete. □

4.2 Curvature notions on $\mathscr{W}^{2,n}$-sets

Let $\mathcal{S}$ be a $\mathscr{W}^{2,n}$-set , since $N_2(\mathcal{S})$ has full $\mathcal{H}^n$-measure in $\mathcal{S}$, from Lemma 1.4.36 *(iii)* we infer that the multivalued function

$$\operatorname{nor}(\mathcal{S},\cdot) : \mathcal{S} \to \mathcal{P}(\mathbb{S}^n)$$

admits an $\mathcal{H}^n$-measurable selection $\nu_{\mathcal{S}} : \mathcal{S} \to \mathbb{S}^n$, moreover (cf. (1.4.24))

$$\nu_{\mathcal{S}}(p) \in \operatorname{Nor}^n(\mathcal{H}^n \llcorner \mathcal{S}, p) \quad \text{for } \mathcal{H}^n\text{-a.e. } p \in \mathcal{S}\,.$$

Applying Lemma 3.1.2 we infer that $\nu_{\mathcal{S}}$ is $(\mathcal{H}^n \llcorner \mathcal{S})$-approximately differentiable at $\mathcal{H}^n$-a.e. $p \in \mathcal{S}$ and $\operatorname{ap} D\nu_{\mathcal{S}}(p)$ is a symmetric endomorphism of $\operatorname{Tan}^n(\mathcal{H}^n \llcorner \mathcal{S}, p)$, moreover we refer to $\nu_{\mathcal{S}}$ as *selected unit-normal vector field* on $\mathcal{S}$.

Definition 4.2.11 (Approximate principal curvatures). *Given $\mathcal{S}$ a $\mathscr{W}^{2,n}$-set and $\nu_{\mathcal{S}}$ a selected unit-normal vector field on $\mathcal{S}$. The approximate principal curvatures of $\mathcal{S}$, with respect to $\nu_{\mathcal{S}}$, are the $\mathbb{R}$-valued $(\mathcal{H}^n \llcorner \mathcal{S})$-measurable maps*

$$\chi_{\mathcal{S},1}, \ldots, \chi_{\mathcal{S},n}$$

defined so that $\chi_{\mathcal{S},1}(p) \le \ldots \le \chi_{\mathcal{S},n}(p)$ are the eigenvalues of $\operatorname{ap} D\nu_{\mathcal{S}}(p)$*, for $\mathcal{H}^n$-a.e. $p \in \mathcal{S}$.*

Definition 4.2.12 (k-th mean curvature function). *Given $\mathcal{S}$ a $\mathscr{W}^{2,n}$-set, $\nu_{\mathcal{S}}$ a selected unit-normal vector field on $\mathcal{S}$ and $k \in \{0,\ldots,n\}$. The k-th mean curvature function of $\mathcal{S}$, respect to $\nu_{\mathcal{S}}$, is defined as follows*

$$\begin{aligned} H_{\mathcal{S},k}(p) &:= \sigma_k(\chi_{\mathcal{S},1}(p), \ldots, \chi_{\mathcal{S},n}(p)) \\ &= \frac{1}{\binom{n}{k}} \sum_{\lambda \in \Lambda(n,k)} \chi_{\mathcal{S},1}(p) \ldots \chi_{\mathcal{S},k}(p) \qquad \text{for } \mathcal{H}^n\text{-a.e. } p \in \mathcal{S}\,. \end{aligned}$$

Definition 4.2.13. *Given $\mathcal{S}$ a $\mathscr{W}^{2,n}$-set, with associated pair $(\mathcal{S}', F)$ where*

$$\mathcal{S}' = \{\boldsymbol{\iota} > 0\} \quad \text{for } \boldsymbol{\iota} \in F_n W^{2,n}(\mathbb{R}^{n+1})\,,$$

and $\nu_{\mathcal{S}}$ a selected unit-normal vector field on $\mathcal{S}$. We define

$$\mathcal{A}_k(\mathcal{S}) := \int_{\mathcal{S}} H_{\mathcal{S},k}(x)\, \boldsymbol{\iota}(F^{-1}(x))\, d\mathcal{H}^n(x) \quad \text{for } k \in \{0,\ldots,n\}\,.$$

Lemma 4.2.14. *Given $\mathcal{S}$ a compact $\mathscr{W}^{2,n}$-set, with associated pair $(\mathcal{S}', F)$, and $\nu_{\mathcal{S}}$ a selected unit-normal vector field on $\mathcal{S}$. Then, the following statements hold:*

(i) $\mathcal{H}^n(\overline{\nu}_{\mathcal{S}}(Z)) = 0$ *for any* $Z \subseteq N_2(\mathcal{S})$ *such that* $\mathcal{H}^n(Z) = 0$;

(ii) $\mathcal{H}^n\big(\operatorname{nor}(\mathcal{S}) \setminus [\overline{\nu}_{\mathcal{S}}(N_2(\mathcal{S})) \cup \overline{-\nu}_{\mathcal{S}}(N_2(\mathcal{S}))]\big) = 0$ *and* $\mathcal{H}^n\big(\operatorname{nor}(\mathcal{S})\big) < \infty$;

(iii) for any $i \in \{1,\ldots,n\}$ we have (cf. Definition 1.4.21)

$$\chi_{\mathcal{S},i}(x) = \kappa_{\mathcal{S},i}(x, \nu_{\mathcal{S}}(x)) = -\kappa_{\mathcal{S},i}(x, -\nu_{\mathcal{S}}(x)) \quad \text{for } \mathcal{H}^n \text{ a.e. } x \in \mathcal{S}\,, \tag{4.2.16}$$

in particular $\kappa_{\mathcal{S},i}(x,u) < +\infty$ for $\mathcal{H}^n$-a.e. $(x,u) \in \operatorname{nor}(\mathcal{S})$.

Proof. Assertion *(i)* is a readily consequence of the Lusin (N)-property on $\operatorname{nor}(\mathcal{S})$ (cf. Lemma 4.1.10 *(ii)*).

To prove *(ii)* we recall the definition of Ψ_F from (1.3.13) and notice that (cf. [54, Lemma 2.1])

$$\Psi_F\big(\mathrm{nor}(\mathcal{S}')\big) = \mathrm{nor}(\mathcal{S})\,,$$

hence we deduce that $\mathcal{H}^n\big(\mathrm{nor}(\mathcal{S})\big) < \infty$ from (4.1.3) and (2.1.30). Moreover, since $N_2(\mathcal{S})$ has full $\mathcal{H}^n$-measure in $\mathcal{S}$ (cf. Lemma 4.1.10 *(vii)*), by the Lusin (N)-property on $\mathrm{nor}(\mathcal{S})$ we obtain

$$\mathcal{H}^n\big(\mathrm{nor}(\mathcal{S}) \setminus \big[\overline{\nu}_{\mathcal{S}}\big(N_2(\mathcal{S})\big) \cup \overline{-\nu}_{\mathcal{S}}\big(N_2(\mathcal{S})\big)\big]\big) = \mathcal{H}^n\big(\mathrm{nor}(\mathcal{S}) \setminus \mathrm{nor}(\mathcal{S}) \llcorner N_2(\mathcal{S})\big) = 0\,.$$

To prove *(iii)* we first employ Lemma 3.1.2 to find a countable family of $\mathcal{H}^n$-measurable sets $X_i \subseteq N_2(\mathcal{S})$ such that $\mathcal{H}^n\big(\mathcal{S} \setminus \bigcup_{i=1}^{\infty} X_i\big) = 0$ and $Lip(\nu_{\mathcal{S}}|X_i) < \infty$ for every $i \in \mathbb{N}$; then we define Y_i to be the set of $x \in X_i$ such that $\nu_{\mathcal{S}}$ is $(\mathcal{H}^n \llcorner \mathcal{S})$-approximately differentiable at x, $\Theta^n\big(\mathcal{H}^n \llcorner (\mathcal{S} \setminus X_i), x\big) = 0$ (this property holds $\mathcal{H}^n$-a.e. on X_i, cf. [14, 2.10.19 (4)]), moreover $\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathcal{S}, x) \in \mathbf{G}(n+1, n)$ and also

$$\mathrm{Tan}^n\big(\mathcal{H}^n \llcorner \mathrm{nor}(\mathcal{S}), \overline{\nu}_{\mathcal{S}}(x)\big) = \mathrm{Tan}^n\big(\mathcal{H}^n \llcorner \overline{\nu}_{\mathcal{S}}\big(N_2(\mathcal{S})\big), \overline{\nu}_{\mathcal{S}}(x)\big) \in \mathbf{G}(2n+2, n)\,,$$

$$\mathrm{Tan}^n\big(\mathcal{H}^n \llcorner \mathrm{nor}(\mathcal{S}), \overline{-\nu}_{\mathcal{S}}(x)\big) = \mathrm{Tan}^n\big(\mathcal{H}^n \llcorner \overline{-\nu}_{\mathcal{S}}\big(N_2(\mathcal{S})\big), \overline{-\nu}_{\mathcal{S}}(x)\big) \in \mathbf{G}(2n+2, n)\,.$$

Since $\overline{\nu}_{\mathcal{S}}|X_i$ and $\overline{-\nu}_{\mathcal{S}}|X_i$ are bi-lipschitz and $\mathrm{Tan}^n\big(\mathcal{H}^n \llcorner \overline{\pm\nu}_{\mathcal{S}}\big(N_2(\mathcal{S})\big), (x,u)\big) \in \mathbf{G}(2n+2, n)$ for $\mathcal{H}^n$-a.e. $(x,u) \in \overline{\pm\nu}_{\mathcal{S}}\big(N_2(\mathcal{S})\big)$ (cf. [14, Theorem 3.2.19]), by the locality property of approximate tangent spaces (cf. (1.1.4)) we infer

$$\mathcal{H}^n(X_i \setminus Y_i) = 0 \quad \text{for every } i \in \mathbb{N}\,.$$

It follows from *(i)* and *(ii)* that

$$\begin{aligned}
&\mathcal{H}^n\Big(\mathrm{nor}(\mathcal{S}) \setminus \bigcup_{i=1}^{\infty} \big[\overline{\nu}_{\mathcal{S}}(Y_i) \cup \overline{-\nu}_{\mathcal{S}}(Y_i)\big]\Big) \\
&= \mathcal{H}^n\Big(\big[\overline{\nu}_{\mathcal{S}}\big(N_2(\mathcal{S})\big) \cup \overline{-\nu}_{\mathcal{S}}\big(N_2(\mathcal{S})\big)\big] \setminus \bigcup_{j=1}^{\infty} \big[\overline{\nu}_{\mathcal{S}}(Y_j) \cup \overline{-\nu}_{\mathcal{S}}(Y_j)\big]\Big) \\
&= \mathcal{H}^n\Big(\bigcup_{i=1}^{\infty} \big[\overline{\nu}_{\mathcal{S}}(X_i) \cup \overline{-\nu}_{\mathcal{S}}(X_i)\big] \setminus \bigcup_{j=1}^{\infty} \big[\overline{\nu}_{\mathcal{S}}(Y_j) \cup \overline{-\nu}_{\mathcal{S}}(Y_j)\big]\Big) \\
&\leq \sum_{i=1}^{\infty} \mathcal{H}^n\Big(\big[\overline{\nu}_{\mathcal{S}}(X_i) \cup \overline{-\nu}_{\mathcal{S}}(X_i)\big] \setminus \big[\overline{\nu}_{\mathcal{S}}(Y_i) \cup \overline{-\nu}_{\mathcal{S}}(Y_i)\big]\Big) \\
&\leq \sum_{i=1}^{\infty} \mathcal{H}^n\big(\overline{\nu}_{\mathcal{S}}(X_i) \setminus \overline{\nu}_{\mathcal{S}}(Y_i)\big) + \sum_{i=1}^{\infty} \mathcal{H}^n\big(\overline{-\nu}_{\mathcal{S}}(X_i) \setminus \overline{-\nu}_{\mathcal{S}}(Y_i)\big) = 0
\end{aligned} \tag{4.2.17}$$

furthermore

$$\mathcal{H}^n\Big(\mathcal{S} \setminus \bigcup_{i=1}^{\infty} Y_i\Big) \leq \sum_{j=1}^{\infty} \mathcal{H}^n(X_j \setminus Y_j) = 0\,. \tag{4.2.18}$$

We fix now $x \in Y_i$. Then there exists a map $g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ pointwise differentiable at x such that $g(x) = \nu_{\mathcal{S}}(x)$, $\Theta^n(\mathcal{H}^n \llcorner \mathcal{S} \setminus \{g = \nu_{\mathcal{S}}\}, x) = 0$ and

$$\mathrm{ap}\, D\overline{\nu}_{\mathcal{S}}(x) := D\overline{g}(x)|\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathcal{S}, x) \quad , \quad \mathrm{ap}\, D\overline{-\nu}_{\mathcal{S}}(x) := D\overline{-g}(x)|\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathcal{S}, x)\,.$$

Since $\overline{g}|X_i \cap \{\overline{g} = \overline{\nu}_{\mathcal{S}}\}$ and $\overline{-g}|X_i \cap \{\overline{-g} = \overline{-\nu}_{\mathcal{S}}\}$ are bi-lipschitz, $\mathrm{ap}\, D\overline{\nu}_{\mathcal{S}}(x)$ and $\mathrm{ap}\, D\overline{-\nu}_{\mathcal{S}}(x)$ are injective and

$$\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathcal{S}, x) = \mathrm{Tan}^n\big(\mathcal{H}^n \llcorner X_i \cap \{\overline{g} = \overline{\nu}_{\mathcal{S}}\}, x\big),$$

we infer from [51, Lemma B.2] that

$$\begin{aligned}
\mathrm{ap}\, D\overline{\nu}_{\mathcal{S}}(x)\big[\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathcal{S}, x)\big] &= D\overline{g}(x)\big[\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathcal{S}, x)\big] \\
&= D\overline{g}(x)\big[\mathrm{Tan}^n\big(\mathcal{H}^n \llcorner X_i \cap \{\overline{g} = \overline{\nu}_{\mathcal{S}}\}, x\big)\big]
\end{aligned}$$

$$\begin{aligned}&\subseteq \operatorname{Tan}^n\big(\mathcal{H}^n \llcorner \overline{g}(X_i \cap \{\overline{g} = \overline{\nu}_{\mathcal{S}}\}), \overline{g}(x)\big)\\&\subseteq \operatorname{Tan}^n\big(\mathcal{H}^n \llcorner \overline{\nu}_{\mathcal{S}}(N_2(\mathcal{S})), \overline{\nu}_{\mathcal{S}}(x)\big)\\&= \operatorname{Tan}^n\big(\mathcal{H}^n \llcorner \operatorname{nor}(\mathcal{S}), \overline{\nu}_{\mathcal{S}}(x)\big) \in \mathbf{G}(2n+2,n)\end{aligned}$$

and similarly

$$\operatorname{ap} D\overline{-\nu}_{\mathcal{S}}(x)\big[\operatorname{Tan}^n(\mathcal{H}^n \llcorner \mathcal{S}, x)\big] \subseteq \operatorname{Tan}^n\big(\mathcal{H}^n \llcorner \operatorname{nor}(\mathcal{S}), \overline{-\nu}_{\mathcal{S}}(x)\big) \in \mathbf{G}(2n+2,n)\,.$$

Overall, since $\operatorname{Tan}^n(\mathcal{H}^n \llcorner \mathcal{S}, x) \in \mathbf{G}(n+1,n)$ and $\operatorname{ap} D\overline{\pm\nu}_{\mathcal{S}}(x)$ are injective, we infer

$$\operatorname{ap} D\overline{\nu}_{\mathcal{S}}(x)\big[\operatorname{Tan}^n(\mathcal{H}^n \llcorner \mathcal{S}, x)\big] = \operatorname{Tan}^n\big(\mathcal{H}^n \llcorner \operatorname{nor}(\mathcal{S}), \overline{\nu}_{\mathcal{S}}(x)\big)\,,$$

$$\operatorname{ap} D\overline{-\nu}_{\mathcal{S}}(x)\big[\operatorname{Tan}^n(\mathcal{H}^n \llcorner \mathcal{S}, x)\big] = \operatorname{Tan}^n\big(\mathcal{H}^n \llcorner \operatorname{nor}(\mathcal{S}), \overline{-\nu}_{\mathcal{S}}(x)\big)\,.$$

Therefore, if $\{\tau_1, \dots, \tau_n\}$ is an orthonormal basis of $\operatorname{Tan}^n(\mathcal{H}^n \llcorner \mathcal{S}, x)$ with

$$\operatorname{ap} D\nu_{\mathcal{S}}(x)(\tau_i) = \chi_{\mathcal{S},i}(x)\tau_i \quad \text{for } i \in \{1, \dots, n\}\,,$$

we conclude that

$$\left\{\left(\frac{1}{\sqrt{1+\chi_{\mathcal{S},i}(x)^2}}\tau_i\,, \frac{\chi_{\mathcal{S},i}(x)}{\sqrt{1+\chi_{\mathcal{S},i}(x)^2}}\tau_i\right) : i \in \{1, \dots, n\}\right\},$$

$$\left\{\left(\frac{1}{\sqrt{1+\chi_{\mathcal{S},i}(x)^2}}\tau_i\,, \frac{-\chi_{\mathcal{S},i}(x)}{\sqrt{1+\chi_{\mathcal{S},i}(x)^2}}\tau_i\right) : i \in \{1, \dots, n\}\right\}$$

are orthonormal basis of $\operatorname{Tan}^n\big(\mathcal{H}^n \llcorner \operatorname{nor}(\mathcal{S}), \overline{\nu}_{\mathcal{S}}(x)\big)$ and $\operatorname{Tan}^n\big(\mathcal{H}^n \llcorner \operatorname{nor}(\mathcal{S}), \overline{-\nu}_{\mathcal{S}}(x)\big)$, respectively. Since x is arbitrarily chosen in Y_i, thanks to (4.2.18), we deduce from the uniqueness stated in Lemma 1.4.20 (cf. Definition 1.4.21) that

$$\chi_{\mathcal{S},i}(x) = \kappa_{\mathcal{S},i}(x, \nu_{\mathcal{S}}(x)) = -\kappa_{\mathcal{S},i}(x, -\nu_{\mathcal{S}}(x)) \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \mathcal{S}\,.$$

Moreover, from (4.2.17), we deduce that

$$\kappa_{\mathcal{S},i}(x,u) \in \big\{\chi_{\mathcal{S},i}(x), -\chi_{\mathcal{S},i}(x)\big\} \quad \text{for } \mathcal{H}^n\text{-a.e. } (x,u) \in \operatorname{nor}(\mathcal{S})\,.$$

The proof is complete. □

Definition 4.2.15. *Let $\mathcal{S}$ be a compact $\mathscr{W}^{2,n}$-set, we define*

$$\operatorname{nor}(\mathcal{S})^{(n)} := \big\{(x,u) \in \operatorname{nor}(\mathcal{S}) : \kappa_{\mathcal{S},n}(x,u) < +\infty\big\}\,.$$

Remark 4.2.16. From Lemma 4.2.14 *(iii)*, we infer that

$$\mathcal{H}^n\big(\operatorname{nor}(\mathcal{S}) \setminus \operatorname{nor}(\mathcal{S})^{(n)}\big) = 0\,. \tag{4.2.19}$$

Furthermore if $\pi_0 : \mathcal{S} \times \mathbb{S}^n \to \mathcal{S}$ is the canonical projection on the first factor, then

$$\begin{aligned}J_n^{\operatorname{nor}(\mathcal{S})}\pi_0(x,u) &= \left|\Big[\textstyle\bigwedge_n D^{\operatorname{nor}(\mathcal{S})}\pi_0(x,u)\Big]\left(\dfrac{\xi_{\mathcal{S},1}(x,u) \wedge \ldots \wedge \xi_{\mathcal{S},n}(x,u)}{|\xi_{\mathcal{S},1}(x,u) \wedge \ldots \wedge \xi_{\mathcal{S},n}(x,u)|}\right)\right|\\&= \zeta_{\mathcal{S}}(x,u)\,\big|\pi_0\big(\xi_{\mathcal{S},1}(y,u)\big) \wedge \ldots \wedge \pi_0\big(\xi_{\mathcal{S},n}(x,u)\big)\big|\\&= \zeta_{\mathcal{S}}(x,u)\\&= \textstyle\prod_{i=1}^n\big(1+\kappa_{\mathcal{S},i}(x,u)^2\big)^{-\frac{1}{2}} > 0\end{aligned} \tag{4.2.20}$$

for $\mathcal{H}^n$-a.e. $(x,u) \in \operatorname{nor}(\mathcal{S})$, where $\zeta_{\mathcal{S}}$ and $\{\xi_{\mathcal{S},1}, \dots, \xi_{\mathcal{S},n}\}$ are given in Definition 1.4.24.

Is well posed the following.

Definition 4.2.17 (k-th mean curvature function of $\mathrm{nor}(\mathcal{S})$). *Given $\mathcal{S}$ a compact $\mathscr{W}^{2,n}$-set and $k \in \{0,\dots,n\}$, the k-th mean curvature function of* $\mathrm{nor}(\mathcal{S})$ *is given by*

$$\begin{aligned} H_{\mathrm{nor}(\mathcal{S}),k}(x,u) &:= \sigma_k\big(\kappa_{\mathcal{S},1}(x,u),\dots,\kappa_{\mathcal{S},n}(x,u)\big) \\ &= \frac{1}{\binom{n}{k}} \sum_{\lambda\in\Lambda(n,k)} \kappa_{\mathcal{S},1}(x,u)\dots\kappa_{\mathcal{S},n}(x,u) \quad \text{for } \mathcal{H}^n\text{-a.e. } (x,u)\in \mathrm{nor}(\mathcal{S}), \end{aligned}$$

where $\kappa_{\mathcal{S},1},\dots,\kappa_{\mathcal{S},n}$ are given in Definition 1.4.21.

4.3 Reilly-type variational formulae

In this section let $\mathcal{S}$ be a compact $\mathscr{W}^{2,n}$-set with associated pair $(\mathcal{S}',F)$, namely $\mathcal{S}=F(\mathcal{S}')$ where $\mathcal{S}'=\{\boldsymbol{\iota}>0\}$ for some $\boldsymbol{\iota}\in F_nW^{2,n}(\mathbb{R}^{n+1})$. Then, there exists $\{U_i\}_{i=1}^N$ a family of bounded open neighborhoods of $\{z_i\}_{i=1}^N \subset \mathcal{S}$ such that

$$\mathcal{S}' = \bigcup_{i=1}^{N} (\mathcal{S}'\cap U_i) = \bigcup_{i=1}^{N}\bigcup_{j=1}^{q(z_i)} (\Gamma_j^{(i)}\cap U_i)\,, \tag{4.3.21}$$

where every $\Gamma_j^{(i)}\cap U_i$ coincides with the graph of a $(C^0\cap W^{2,n})$-function (cf. Definition 4.1.1). Namely, for a fixed $i\in\{1,\dots,N\}$ and for each $j\in\{1,\dots,q(z_i)\}$ there exist $p_j^{(i)}\in\Gamma_j^{(i)}\cap U_i$ and $\eta_j^{(i)}\in\mathbb{S}^n$ so that

$$\Gamma_j^{(i)}\cap U_i = \overline{f}_j^{(i)}(V_j^{(i)}) + p_j^{(i)},$$

for some sets $V_j^{(i)}$ open in $(\eta_j^{(i)})^\perp$ such that $0\in V_j^{(i)}$, and some graph functions $f_j^{(i)}$ on $V_j^{(i)}$.

We associate to $\mathrm{nor}(\mathcal{S}')$ and $\mathrm{nor}(\mathcal{S})$, respectively, the n-vectorfields $\vec{\xi}_{\mathcal{S}'}$ and $\vec{\xi}_{\mathcal{S}}$ defined in according to Definition 1.4.24 and Lemma 1.4.20. In particular, for $\mathcal{H}^n$-a.e. $(x,u)\in\mathrm{nor}(\mathcal{S}')$

$$\vec{\xi}_{\mathcal{S}'}(x,u) := \frac{\xi_{\mathcal{S}',1}(x,u)\wedge\dots\wedge\xi_{\mathcal{S}',n}(x,u)}{|\xi_{\mathcal{S}',1}(x,u)\wedge\dots\wedge\xi_{\mathcal{S}',n}(x,u)|} \in {\textstyle\bigwedge_n}(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})\,,$$

$$\zeta_{\mathcal{S}'}(x,u) := \frac{1}{|\xi_{\mathcal{S}',1}(x,u)\wedge\dots\wedge\xi_{\mathcal{S}',n}(x,u)|} \in (0,+\infty)$$

where each $\xi_{\mathcal{S}',i}$ is given by (notice that $\mathrm{nor}(\mathcal{S}')^{(n)}$ has full $\mathcal{H}^n$-measure in $\mathrm{nor}(\mathcal{S}')$; cf. (4.2.19))

$$\xi_{\mathcal{S}',i}(x,u) = \big(\tau_i(x,u), \kappa_{\mathcal{S}',i}(x,u)\tau_i(x,u)\big)$$

for $\mathcal{H}^n$-a.e. $(x,u)\in\mathrm{nor}(\mathcal{S}')$. The maps $\{\tau_1,\dots,\tau_n\}$ are defined $\mathcal{H}^n$-a.e. on $\mathrm{nor}(\mathcal{S}')$ in such a way that $\{\tau_1(x,u),\dots,\tau_n(x,u),u\}$ form a positively oriented orthonormal basis of $\mathbb{R}^{n+1}$ for $\mathcal{H}^n$-a.e. $(x,u)\in\mathrm{nor}(\mathcal{S}')$ (cf. Lemma 1.4.20), namely $\langle\bigwedge_{i=1}^n\tau_i(x,u)\wedge u, dX_1\wedge\dots\wedge dX_{n+1}\rangle = 1$.

Lemma 4.3.18. *$\vec{\xi}_{\mathcal{S}'}$ is an $\big(\mathcal{H}^n \llcorner \mathrm{nor}(\mathcal{S}')\big)$-measurable n-vectorfield.*

Proof. For every fixed $i\in\{1,\dots,N\}$ and for $j\in\{1,\dots,q(z_i)\}$, we prove that

$$\vec{\xi}_{\mathcal{S}'}(y,u) = \vec{\eta}_{\Gamma_j^{(i)}\cap U_i}(y,u) \quad \text{for } \mathcal{H}^n\text{-a.e. } (y,u)\in\mathrm{nor}(\Gamma_j^{(i)})\llcorner U_i \tag{4.3.22}$$

where the n-vectorfield, on the right hand-side of the previous one, is the Borel n-vectorfield given in (2.1.28) (cf. Remark 2.1.21). By the $\mathcal{H}^n$-rectifiability of the unit normal bundle, the locality property of the approximate tangent spaces (cf. (1.1.4)) and (4.1.13), we infer

$$\begin{aligned} &\mathrm{Tan}^n\big(\mathcal{H}^n\llcorner\mathrm{nor}(\mathcal{S}'),(y,u)\big) \\ &\qquad = \mathrm{Tan}^n\big(\mathcal{H}^n\llcorner\mathrm{nor}(\Gamma_j^{(i)}),(y,u)\big) \in \mathbf{G}(2n+2,n)\,, \end{aligned}$$

for $\mathcal{H}^n$-a.e. $(y,u) \in \mathrm{nor}(\Gamma_j^{(i)}) \llcorner U_i$. For such points (y,u), we have

$$\begin{aligned}&\textstyle\bigwedge_n \left[\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathrm{nor}(\mathcal{S}'),(y,u))\right]\\ &\qquad = \textstyle\bigwedge_n \left[\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathrm{nor}(\Gamma_j^{(i)}),(y,u))\right]\end{aligned}$$

and

$$\begin{aligned}&\dim\left(\textstyle\bigwedge_n \left[\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathrm{nor}(\mathcal{S}'),(y,u))\right]\right)\\ &\qquad = \dim\left(\textstyle\bigwedge_n \left[\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathrm{nor}(\Gamma_j^{(i)}),(y,u))\right]\right) = 1\,,\end{aligned}$$

therefore we readily deduce that

$$\vec{\xi}_{\mathcal{S}'}(y,u) = \pm\vec{\eta}_{\Gamma_j^{(i)}\cap U_i}(y,u) \quad \text{for } \mathcal{H}^n\text{-a.e. } (y,u) \in \mathrm{nor}(\Gamma_j^{(i)}) \llcorner U_i\,. \tag{4.3.23}$$

Now we introduce the sets

$$\begin{aligned}&Z := \left\{(y,u) \in \mathrm{nor}(\Gamma_j^{(i)}) \llcorner U_i : \vec{\xi}_{\mathcal{S}'}(y,u) = -\vec{\eta}_{\Gamma_j^{(i)}\cap U_i}(y,u)\right\},\\ &Z' := Z \cap \mathrm{nor}(\mathcal{S}')^{(n)},\end{aligned}$$

with the aim to prove that $\mathcal{H}^n(Z) = 0$. In particular, since $\mathcal{H}^n(Z \setminus Z') = 0$ (cf. (4.2.19) and (4.1.13)), we obtain the desidered result if $\mathcal{H}^n(Z') = 0$. First of all we notice that

$$\begin{aligned}\left[\textstyle\bigwedge_n \pi_0\right](\vec{\xi}_{\mathcal{S}'}(y,u)) &= \frac{\pi_0(\xi_{\mathcal{S}',1}(y,u)) \wedge \ldots \wedge \pi_0(\xi_{\mathcal{S}',n}(y,u))}{|\xi_{\mathcal{S}',1}(y,u) \wedge \ldots \wedge \xi_{\mathcal{S}',n}(y,u)|}\\ &= \zeta_{\mathcal{S}'}(y,u) \cdot \tau_1(y,u) \wedge \ldots \wedge \tau_n(y,u) \quad \text{for } \mathcal{H}^n\text{-a.e. } (y,u) \in Z'\end{aligned}$$

and, since $\{\tau_1(x,u),\ldots,\tau_n(x,u),u\}$ form a positively oriented orthonormal basis of $\mathbb{R}^{n+1}$, we obtain

$$\langle [\textstyle\bigwedge_n \pi_0](\vec{\xi}_{\mathcal{S}'}(y,u)) \wedge u, dX_1 \wedge \ldots \wedge dX_{n+1}\rangle = \zeta_{\mathcal{S}'}(y,u) > 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } (y,u) \in Z'\,.$$

This inequality gives a contradiction, from which the desidered result follows. Indeed

$$[\textstyle\bigwedge_n \pi_0](\vec{\xi}_{\mathcal{S}'}(y,u)) = -[\textstyle\bigwedge_n \pi_0](\vec{\eta}_{\Gamma_j^{(i)}\cap U_i}(y,u)) \quad \text{for any } (y,u) \in Z'$$

where (cf. (2.1.29))

$$\langle [\textstyle\bigwedge_n \pi_0](\vec{\eta}_{\Gamma_j^{(i)}\cap U_i}(y,u)) \wedge u, dX_1 \wedge \ldots \wedge dX_n\rangle > 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } (y,u) \in Z'\,.$$

The proof is complete. □

Definition 4.3.19. *We define $\mathcal{N}_{\mathcal{S}'} \in \mathcal{D}_n(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ as follows*

$$\mathcal{N}_{\mathcal{S}'} := (\boldsymbol{\iota} \circ \pi_0)(\mathcal{H}^n \llcorner \mathrm{nor}(\mathcal{S}')) \wedge \vec{\xi}_{\mathcal{S}'}\,, \tag{4.3.24}$$

where $\pi_0 : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the canonical projection on the first factor.

Theorem 4.3.20. *$\mathcal{N}_{\mathcal{S}'}$ is a Legendrian cycle of $\mathbb{R}^{n+1}$, we denote it as the Legendrian cycle associated with $\mathcal{S}'$. Moreover, for a selected unit-normal vector field $\nu_{\mathcal{S}'}$ on $\mathcal{S}'$, the following relations hold*

$$\begin{aligned}&(\mathcal{N}_{\mathcal{S}'} \llcorner \varphi_{n-k})(\phi)\\ &\qquad = \binom{n}{k} \int_{\mathcal{S}'} \left[\phi(x,\nu_{\mathcal{S}'}(x)) + (-1)^k \phi(x,-\nu_{\mathcal{S}'}(x))\right] H_{\mathcal{S}',k}(x)\, \boldsymbol{\iota}(x)\, d\mathcal{H}^n(x)\,,\end{aligned} \tag{4.3.25}$$

for any $\phi \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ and any $k \in \{0,\ldots,n\}$.

Proof. To prove that $\mathcal{N}_{\mathcal{S}'}$ is a Legendrian cycle of $\mathbb{R}^{n+1}$, we notice that $\mathcal{N}_{\mathcal{S}'} \llcorner (U_i \times \mathbb{R}^{n+1})$ is a Legendrian cycle of U_i for any $i \in \{1,\ldots,N\}$. Indeed, for every $\varphi \in \mathcal{D}^n(U_i \times \mathbb{R}^{n+1})$, from

(4.1.12) and (4.3.22) we obtain

$$\begin{aligned}\big[\mathcal{N}_{\mathcal{S}'} \llcorner (U_i \times \mathbb{R}^{n+1})\big](\varphi) &= \int_{\mathrm{nor}(\mathcal{S}') \llcorner U_i} \langle \vec{\xi}_{\mathcal{S}'}, \varphi \rangle \, \boldsymbol{\iota} \circ \pi_0 \, d\mathcal{H}^n \\ &= \sum_{j=1}^{q(z_i)} \int_{\mathrm{nor}(\Gamma_j^{(i)}) \llcorner U_i} \langle \vec{\xi}_{\mathcal{S}'}, \varphi \rangle \, d\mathcal{H}^n \\ &= \sum_{j=1}^{q(z_i)} \int_{\mathrm{nor}(\Gamma_j^{(i)}) \llcorner U_i} \langle \vec{\eta}_{\Gamma_j^{(i)} \cap U_i}, \varphi \rangle \, d\mathcal{H}^n \\ &= \sum_{j=1}^{q(z_i)} \mathcal{N}_{\Gamma_j^{(i)} \cap U_i}(\varphi) \,,\end{aligned}$$

where every

$$\mathcal{N}_{\Gamma_j^{(i)} \cap U_i} \in \mathcal{D}_n(U_i \times \mathbb{R}^{n+1})$$

is a Legendrian cycle of U_i (cf. (2.1.30) in Remark 2.1.21). Overall, applying Lemma 1.3.5, we infer that $\mathcal{N}_{\mathcal{S}}$ is a Legendrian cycle of $\mathbb{R}^{n+1}$.

To prove (4.3.25), we recall that (cf. (4.2.20))

$$J_n^{\mathrm{nor}(\mathcal{S}')} \pi_0(y,u) = \zeta_{\mathcal{S}'}(y,u) > 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } (y,u) \in \mathrm{nor}(\mathcal{S}')^{(n)} \,.$$

If we apply Lemma 1.3.17, Lemma 4.1.10 *(ii)* and *(vii)*, Lemma 4.2.14 *(iii)*, (4.2.19) and the area formula for rectifiable sets [14, Theorem 3.2.22 (3)], we infer (notice that $\mathrm{spt}(\mathcal{N}_{\mathcal{S}'})$ is compact)

$$\begin{aligned}(\mathcal{N}_{\mathcal{S}'} \llcorner \varphi_{n-k})(\phi) &= \binom{n}{k} \int_{\mathrm{nor}(\mathcal{S}') \llcorner N_2(\mathcal{S}')} \phi(y,u) \, \boldsymbol{\iota}(y) \, H_{\mathrm{nor}(\mathcal{S}'),k}(y,u) \, J_n^{\mathrm{nor}(\mathcal{S}')} \pi_0(y,u) \, d\mathcal{H}^n(y,u) \\ &= \binom{n}{k} \int_{N_2(\mathcal{S}')} \sum_{(y,u) \in (\pi_0 | \mathrm{nor}(\mathcal{S}') \llcorner N_2(\mathcal{S}'))^{-1}(x)} \big[\phi(y,u) \, H_{\mathrm{nor}(\mathcal{S}'),k}(y,u) \, \boldsymbol{\iota}(y)\big] \, d\mathcal{H}^n(x) \\ &= \binom{n}{k} \int_{\mathcal{S}'} \big[\phi(x, \nu_{\mathcal{S}'}(x)) + (-1)^k \phi(x, -\nu_{\mathcal{S}'}(x))\big] H_{\mathcal{S}',k}(x) \, \boldsymbol{\iota}(x) \, d\mathcal{H}^n(x) \,,\end{aligned}$$

for any $\phi \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ and $k \in \{0, \ldots, n\}$. The proof is complete. □

Now we consider again the C^1-diffeomorphism

$$\Psi_F : (x,y) \in \mathbb{R}^{n+1} \times \mathbb{S}^n \mapsto \left(F(x), \frac{(DF(x)^{-1})^*(y)}{|(DF(x)^{-1})^*(y)|} \right) \in \mathbb{R}^{n+1} \times \mathbb{S}^n \,,$$

for which we have (cf. [54, Lemma 2.1])

$$\Psi_F\big(\mathrm{nor}(\mathcal{S}')\big) = \mathrm{nor}(\mathcal{S}) \,. \tag{4.3.26}$$

Definition 4.3.21. *We define $\mathcal{N}_{\mathcal{S}} \in \mathcal{D}_n(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ as follows*

$$\mathcal{N}_{\mathcal{S}} := (\Psi_F)_\#(\mathcal{N}_{\mathcal{S}'}) \,.$$

Theorem 4.3.22. *$\mathcal{N}_{\mathcal{S}}$ is a Legendrian cycle of $\mathbb{R}^{n+1}$ and we denote it as the Legendrian cycle associated with $\mathcal{S}$, in particular*

$$\mathcal{N}_{\mathcal{S}} = \big(\boldsymbol{\iota} \circ \pi_0 \circ (\Psi_F | \mathrm{nor}(\mathcal{S}'))^{-1}\big) \, \big(\mathcal{H}^n \llcorner \mathrm{nor}(\mathcal{S})\big) \wedge \vec{\xi}_{\mathcal{S}} \,. \tag{4.3.27}$$

Moreover, for a selected unit-normal vector field $\nu_{\mathcal{S}}$ on $\mathcal{S}$, the following relations hold

$$(\mathcal{N}_{\mathcal{S}} \llcorner \varphi_{n-k})(\phi) \tag{4.3.28}$$

$$= \binom{n}{k} \int_{\mathcal{S}} \left[\phi(x, \nu_{\mathcal{S}}(x)) + (-1)^k \phi(x, -\nu_{\mathcal{S}}(x))\right] H_{\mathcal{S},k}(x)\, \boldsymbol{\iota}(F^{-1}(x))\, d\mathcal{H}^n(x)\,,$$

for any $\phi \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ *and* $k \in \{0, \dots, n\}$.

Proof. We introduce $\psi := \Psi_F | \mathrm{nor}(\mathcal{S}')$, recalling (4.3.26) and since

$$\mathrm{ap}\, D\psi(\psi^{-1}(y,v)) = D\Psi_F(\psi^{-1}(y,v)) | \mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathrm{nor}(\mathcal{S}'), \psi^{-1}(y,v)) \tag{4.3.29}$$

for $\mathcal{H}^n$-a.e. $(y,v) \in \mathrm{nor}(\mathcal{S})$, we define the simple $(\mathcal{H}^n \llcorner \mathrm{nor}(\mathcal{S}))$-measurable n-vectorfield

$$\begin{aligned} \vec{\eta}(y,v) &:= \frac{\left[\bigwedge_n \mathrm{ap}\, D\psi(\psi^{-1}(y,v))\right] \vec{\xi}_{\mathcal{S}'}(\psi^{-1}(y,v))}{\left|\left[\bigwedge_n \mathrm{ap}\, D\psi(\psi^{-1}(y,v))\right] \vec{\xi}_{\mathcal{S}'}(\psi^{-1}(y,v))\right|} \\ &= \frac{\left[\bigwedge_n \mathrm{ap}\, D\psi(\psi^{-1}(y,v))\right] \vec{\xi}_{\mathcal{S}'}(\psi^{-1}(y,v))}{J_n^{\mathrm{nor}(\mathcal{S}')}\psi(\psi^{-1}(y,v))} \quad \text{for } \mathcal{H}^n\text{-a.e. } (y,v) \in \mathrm{nor}(\mathcal{S})\,. \end{aligned}$$

Then, by the area formula for rectifiable currents (cf. [14, 4.1.30] or [29, p. 197]) we obtain

$$(\Psi_F)_\#(\mathcal{N}_{\mathcal{S}'}) = (\boldsymbol{\iota} \circ \pi_0 \circ \psi^{-1})(\mathcal{H}^n \llcorner \mathrm{nor}(\mathcal{S})) \wedge \vec{\eta}\,,$$

moreover $|\vec{\eta}(y,v)| = 1$ and $\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathrm{nor}(\mathcal{S}), (y,v))$ is associated with $\vec{\eta}(y,v)$ for $\mathcal{H}^n$-a.e. $(y,v) \in \mathrm{nor}(\mathcal{S})$. Clearly $(\Psi_F)_\#(\mathcal{N}_{\mathcal{S}'})$ is a cycle, furthermore by the shuffle formula (cf. [14, 1.4.2]) and Lemma 1.4.27 we deduce that is also Legendrian.
Since $\tau_1(x,u) \wedge \dots \wedge \tau_n(x,u) = (-1)^n * u$ for $\mathcal{H}^n$-a.e. $(x,u) \in \mathrm{nor}(\mathcal{S}')$, we infer (cf. (4.2.19))

$$\begin{aligned} &\left[\textstyle\bigwedge_n \pi_0\right]\left(\vec{\eta}(\Psi_F(x,u))\right) \\ &= \frac{\prod_{i=1}^n (1 + \kappa_{\mathcal{S}',i}(x,u)^2)^{-\frac{1}{2}}}{J_n^{\mathrm{nor}(\mathcal{S}')}\psi(x,u)} \left[\textstyle\bigwedge_n DF(x)\right](\tau_1(x,u) \wedge \dots \wedge \tau_n(x,u)) \\ &= (-1)^n \frac{\prod_{i=1}^n (1 + \kappa_{\mathcal{S}',i}(x,u)^2)^{-\frac{1}{2}}}{J_n^{\mathrm{nor}(\mathcal{S}')}\psi(x,u)} \left[\textstyle\bigwedge_n DF(x)\right](*u) \quad \text{for } \mathcal{H}^n\text{-a.e. } (x,u) \in \mathrm{nor}(\mathcal{S}')\,. \end{aligned}$$

Therefore, from (4.3.26) and by Remark 3.1.8, it follows that either

$$\langle \left[\textstyle\bigwedge_n \pi_0\right](\vec{\eta}(y,v)) \wedge v, dX_1 \wedge \dots \wedge dX_{n+1}\rangle > 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } (y,v) \in \mathrm{nor}(\mathcal{S})$$

or

$$\langle \left[\textstyle\bigwedge_n \pi_0\right](\vec{\eta}(y,v)) \wedge v, dX_1 \wedge \dots \wedge dX_{n+1}\rangle < 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } (y,v) \in \mathrm{nor}(\mathcal{S})$$

furthermore, for $\mathcal{H}^n$-a.e. $(y,v) \in \mathrm{nor}(\mathcal{S})$, $\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathrm{nor}(\mathcal{S}), (y,v))$ is associated with both the n-vectorfields $\vec{\eta}(y,v)$ and $\vec{\xi}_{\mathcal{S}}(y,v)$, where (cf. Definition 1.4.24 and Lemma 1.4.20)

$$\langle \left[\textstyle\bigwedge_n \pi_0\right](\vec{\xi}_{\mathcal{S}}(y,v)) \wedge v, dX_1 \wedge \dots \wedge dX_{n+1}\rangle > 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } (y,v) \in \mathrm{nor}(\mathcal{S})\,.$$

Overall, up to a change of sign, we deduce that

$$\vec{\eta}(y,v) = \vec{\xi}_{\mathcal{S}}(y,v) \quad \text{for } \mathcal{H}^n\text{-a.e. } (y,v) \in \mathrm{nor}(\mathcal{S})$$

namely $(\Psi_F)_\#(\mathcal{N}_{\mathcal{S}'}) = \mathcal{N}_{\mathcal{S}}$.

To prove (4.3.28), as before we apply Lemma 1.3.17, Lemma 4.1.10 *(ii)* and *(vii)*, Lemma 4.2.14 *(iii)*, (4.2.19) and the area formula for rectifiable sets [14, Theorem 3.2.22 (3)] to infer

$$(\mathcal{N}_{\mathcal{S}} \llcorner \varphi_{n-k})(\phi) = \binom{n}{k} \int_{\mathrm{nor}(\mathcal{S}) \llcorner N_2(\mathcal{S})} \phi(y,u)\, \boldsymbol{\iota}(F^{-1}(y))\, H_{\mathrm{nor}(\mathcal{S}),k}(y,u)\, J_n^{\mathrm{nor}(\mathcal{S})}\pi_0(y,u)\, d\mathcal{H}^n(y,u)$$

$$= \binom{n}{k} \int_{N_2(\mathcal{S})} \sum_{(y,u)\in(\pi_0|\mathrm{nor}(\mathcal{S}) \llcorner N_2(\mathcal{S}))^{-1}(x)} [\phi(y,u)\, H_{\mathrm{nor}(\mathcal{S}),k}(y,u)\, \boldsymbol{\iota}(F^{-1}(y))]\, d\mathcal{H}^n(x)$$
$$= \binom{n}{k} \int_{\mathcal{S}} [\phi(x,\nu_{\mathcal{S}}(x)) + (-1)^k \phi(x,-\nu_{\mathcal{S}}(x))]\, H_{\mathcal{S},k}(x)\, \boldsymbol{\iota}(F^{-1}(x))\, d\mathcal{H}^n(x)\,,$$

for any $\phi \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ and $k \in \{0,\dots,n\}$. The proof is complete. □

Remark 4.3.23. Given G a C^2-diffeomorphism of $\mathbb{R}^{n+1}$, then

$$(\Psi_G)_\#(\mathcal{N}_{\mathcal{S}}) = (\Psi_{G\circ F})_\#(\mathcal{N}_{\mathcal{S}'}) = \mathcal{N}_{G(\mathcal{S})}\,. \tag{4.3.30}$$

Moreover, for a selected unit-normal vector field $\nu_{G(\mathcal{S})}$ on $G(\mathcal{S})$, from (4.3.28) we obtain

$$\mathcal{N}_{G(\mathcal{S})}(\varphi_{n-k}) = \binom{n}{k} \int_{G(\mathcal{S})} (1+(-1)^k)\, H_{G(\mathcal{S}),k}(x)\, \boldsymbol{\iota}((G\circ F)^{-1}(x))\, d\mathcal{H}^n(x)$$
$$= \begin{cases} 2\binom{n}{k} \mathcal{A}_k(G(\mathcal{S})) & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}, \quad \text{for every } k \in \{0,\dots,n\} \tag{4.3.31}$$

where $H_{G(\mathcal{S}),k}$ is the k-th mean curvature of $G(\mathcal{S})$, with respect to $\nu_{G(\mathcal{S})}$.

Now we derive the following extension of Reilly's variational formulae to $\mathscr{W}^{2,n}$-sets.

Theorem 4.3.24. *Let $\{F_t\}_{t\in(-\epsilon,\epsilon)}$ be a local variation of $\mathbb{R}^{n+1}$, with initial velocity vector field V. If $k \in \{1,\dots,n\}$ is odd, then*

$$\frac{d}{dt}\mathcal{A}_{k-1}(F_t(\mathcal{S}))\Big|_{t=0} = (n-k+1)\int_{\mathcal{S}} V(x)\bullet\nu_{\mathcal{S}}(x)\, H_{\mathcal{S},k}(x)\, \boldsymbol{\iota}(F^{-1}(x))\, d\mathcal{H}^n(x)\,.$$

Moreover, if n is even

$$\frac{d}{dt}\mathcal{A}_n(F_t(\mathcal{S}))\Big|_{t=0} = 0\,.$$

Proof. Combining (4.3.30) and (4.3.31), we obtain

$$[(\Psi_{F_t})_\#(\mathcal{N}_{\mathcal{S}})](\varphi_{n-k+1}) = \mathcal{N}_{F_t(\mathcal{S})}(\varphi_{n-k+1})$$
$$= \begin{cases} 2\binom{n}{k-1}\mathcal{A}_{k-1}(F_t(\mathcal{S})) & \text{if } k \in \{1,\dots,n+1\} \text{ is odd} \\ 0 & \text{if } k \in \{1,\dots,n+1\} \text{ is even}\,. \end{cases} \tag{4.3.32}$$

From (4.3.32), if $k \in \{1,\dots,n\}$ is odd and $\theta_V(x,y) := V(x)\bullet y$ for $x,y \in \mathbb{R}^{n+1}$, applying Lemma 1.3.11 and (4.3.28) we obtain

$$2\binom{n}{k-1}\frac{d}{dt}\mathcal{A}_{k-1}(F_t(\mathcal{S}))\Big|_{t=0}$$
$$= \frac{d}{dt}[(\Psi_{F_t})_\#(\mathcal{N}_{\mathcal{S}})](\varphi_{n-k+1})\Big|_{t=0} = k\,\mathcal{N}_{\mathcal{S}}(\theta_V\,\varphi_{n-k})$$
$$= (1+(-1)^{k+1})\,k\binom{n}{k}\int_{\mathcal{S}} V(x)\bullet\nu_{\mathcal{S}}(x)\, H_{\mathcal{S},k}(x)\,\boldsymbol{\iota}(F^{-1}(x))\, d\mathcal{H}^n(x)$$
$$= 2\,(n-k+1)\binom{n}{k-1}\int_{\mathcal{S}} V(x)\bullet\nu_{\mathcal{S}}(x)\, H_{\mathcal{S},k}(x)\,\boldsymbol{\iota}(F^{-1}(x))\, d\mathcal{H}^n(x)\,.$$

Moreover, if n is even (namely $k = n+1$ odd), from Lemma 1.3.11 we conclude

$$\frac{d}{dt}\mathcal{A}_n(F_t(\mathcal{S}))\Big|_{t=0} = \frac{d}{dt}[(\Psi_{F_t})_\#(\mathcal{N}_{\mathcal{S}})](\varphi_0)\Big|_{t=0} = 0\,.$$

The proof is complete. □

Chapter 5

Nabelpunksatz for Sobolev graphs

In this final chapter we extend the Nabelpunktsatz to graphs of twice weakly differentiable functions in terms of the approximate curvatures of their graphs. In particular, Theorem 5.3.3 provides a general version of the Nabelpunktsatz for $W^{2,1}$-graphs. In view of well known examples of convex functions, this result is sharp; cf. Remark 5.3.5. In this chapter we use the symbols $\mathbf{D}_i$ and $\mathbf{D}^2_{ij}$ (respectively D_i and D^2_{ij}) for the distributional partial derivatives of a Sobolev function (respectively the classical partial derivatives of a function) with respect to the standard basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ of $\mathbb{R}^n$.

5.1 Nabelpunksatz for C^2-graphs

Let $U \subseteq \mathbb{R}^n$ be a connected open set and let $f \in C^2(U)$. We define $\Gamma := \{(x, f(x)) : x \in U\}$, and $\nu : \Gamma \to \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ so that

$$\nu(\overline{f}(x)) := \frac{(-\nabla f(x), 1)}{\sqrt{1 + |\nabla f(x)|^2}} \tag{5.1.1}$$

for every $x \in U$. Differentiating (5.1.1) we get

$$D\nu(\overline{f}(x))\,(v, Df(x)(v)) = \frac{(-D(\nabla f)(x)(v), 0)}{\sqrt{1 + |\nabla f(x)|^2}} - \frac{\nabla f(x) \bullet D(\nabla f)(x)(v)}{1 + |\nabla f(x)|^2}\,\nu(\overline{f}(x))$$

for any $v \in \mathbb{R}^n$. We recall that Γ is umbilical if and only if there exists a function $\lambda : \Gamma \to \mathbb{R}$ such that

$$D\nu(z) = \lambda(z)\, \mathrm{id}|\mathrm{Tan}(\Gamma, z) \quad \text{for every } z \in \Gamma\,.$$

Therefore, since $\mathrm{Tan}(\Gamma, \overline{f}(x)) = \{(v, Df(x)(v)) : v \in \mathbb{R}^n\}$, we conclude that:

Γ is umbilical if and only if

$$\lambda(\overline{f}(x))\,[\boldsymbol{e}_i \bullet \boldsymbol{e}_j + D_i f(x) D_j f(x)] = -\frac{D^2_{ij} f(x)}{\sqrt{1 + |\nabla f(x)|^2}}\,, \tag{5.1.2}$$

for every $x \in U$ and for every $i, j \in \{1, \ldots, n\}$.

It follows from [57] (see also [34]) that:

if $U \subseteq \mathbb{R}^n$ is a connected open set,
$f \in C^2(U)$ and $\lambda : \Gamma \to \mathbb{R}$ is a function such that (5.1.2) holds for every $x \in U$,
then either $\overline{f}(U)$ is contained in an n-dimensional plane,
or $\overline{f}(U)$ is contained in an n-dimensional sphere.

The next theorem extends this result to functions $f \in W^{2,1}_{loc}(U)$.

5.2 Nabelpunksatz for $W^{2,1}_{loc}$-functions

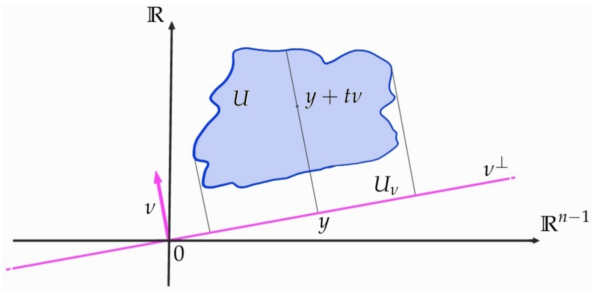
Suppose $U \subseteq \mathbb{R}^n$ is an open set, $\nu \in \mathbb{S}^{n-1}$ and π_ν is the orthogonal projection onto $\nu^\perp$. Then, we define

$$U_\nu := \pi_\nu(U)$$

and

$$U^\nu_y := \{t \in \mathbb{R} : y + t\nu \in U\} \subseteq \mathbb{R} \quad \text{for } y \in U_\nu\,.$$

Notice that U_ν is an open subset of $\nu^\perp$ and U^ν_y is an open subset of $\mathbb{R}$ for every $y \in U_\nu$.



Lemma 5.2.1. *Suppose $U \subseteq \mathbb{R}^n$ be an open set, $g \in W^{1,1}_{loc}(U)$ and $k \in \{1, \ldots, n\}$ such that*

$$\mathbf{D}_k g(x) = 0 \quad \text{for } \mathcal{L}^n \text{ a.e. } x \in U\,.$$

Then, for $\mathcal{L}^{n-1}$-a.e. $y \in U_{e_k}$, the function mapping $t \in U^{e_k}_y$ into $g(y + te_k)$ is $\mathcal{L}^1$-a.e. equal to a constant function.

Proof. It follows from [61, Theorem 2.1.4] that there exists a representative $\widetilde{g}$ of g such that the restriction of $\widetilde{g}$ on $U^{e_k}_y$ is absolutely continuous and

$$\mathbf{D}_k g(y + te_k) = \frac{d}{dt}\widetilde{g}(y + te_k) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in U^{e_k}_y\,,$$

for $\mathcal{L}^{n-1}$-a.e. $y \in U_{e_k}$. By the hypothesis, we have

$$\frac{d}{dt}\widetilde{g}(y + te_k) = 0$$

for $\mathcal{L}^1$-a.e. $t \in U^{e_k}_y$ and for $\mathcal{L}^{n-1}$-a.e. $y \in U_{e_k}$, and we readily obtain the conclusion from the absolute continuity hypothesis of $\widetilde{g}$. □

We now prove the first result of this chapter.

Theorem 5.2.2. *Suppose that $U \subseteq \mathbb{R}^n$ is a connected open set, $f \in W^{2,1}_{loc}(U)$ and $\mu : U \to \mathbb{R}$ is a function satisfying*

$$\mu(x)\left[\boldsymbol{e}_i \bullet \boldsymbol{e}_j + \mathbf{D}_i f(x)\mathbf{D}_j f(x)\right] = -\frac{\mathbf{D}^2_{ij} f(x)}{\sqrt{1 + |\boldsymbol{\nabla} f(x)|^2}} \tag{5.2.3}$$

for $\mathcal{L}^n$-a.e. $x \in U$ and for every $i, j \in \{1, \ldots, n\}$.

Then, either f is $\mathcal{L}^n$-a.e. equal to a linear function on U, or there exists a n-dimensional sphere S in $\mathbb{R}^{n+1}$ such that $\overline{f}(x) \in S$ for $\mathcal{L}^n$-a.e. $x \in U$.

Proof. Recall the diffeomorphism ψ from Remark 2.1.12 and define $\eta := \psi \circ \boldsymbol{\nabla} f$. By the classical chain-rule formula for Sobolev mappings (cf. [20]), $\eta \in W^{1,1}_{loc}(U, \mathbb{R}^{n+1})$ and

$$\begin{aligned}\mathbf{D}\eta(x)(v) &= \left[D\psi(\boldsymbol{\nabla} f(x)) \circ \mathbf{D}(\boldsymbol{\nabla} f)(x)\right](v)\\ &= \frac{(-\mathbf{D}(\boldsymbol{\nabla} f)(x)(v), 0)}{\sqrt{1 + |\boldsymbol{\nabla} f(x)|^2}} - \frac{\boldsymbol{\nabla} f(x) \bullet \mathbf{D}(\boldsymbol{\nabla} f)(x)(v)}{1 + |\boldsymbol{\nabla} f(x)|^2}\,\eta(x)\end{aligned}$$

for $\mathcal{L}^n$-a.e. $x \in U$. In particular, noting that $\eta(x) \bullet (\boldsymbol{e}_j, \mathbf{D}_j f(x)) = 0$ for every $j \in \{1, \dots, n\}$ and for $\mathcal{L}^n$-a.e. $x \in U$, we employ the umbilicality condition to obtain

$$\begin{aligned} \mathbf{D}_i \eta(x) \bullet (\boldsymbol{e}_j, \mathbf{D}_j f(x)) &= -\frac{\mathbf{D}^2_{ij} f(x)}{\sqrt{1 + |\boldsymbol{\nabla} f(x)|^2}} \\ &= \mu(x)\,(\boldsymbol{e}_i, \mathbf{D}_i f(x)) \bullet (\boldsymbol{e}_j, \mathbf{D}_j f(x)) \end{aligned}$$

for $\mathcal{L}^n$-a.e. $x \in U$ and for every $i, j \in \{1, \dots, n\}$. Consequently, for every $i \in \{1, \dots, n\}$ and for $\mathcal{L}^n$-a.e. $x \in U$, there exists $\lambda_i(x) \in \mathbb{R}$ such that

$$\mathbf{D}_i \eta(x) - \mu(x)\,(\boldsymbol{e}_i, \mathbf{D}_i f(x)) = \lambda_i(x)\,\eta(x)\,. \tag{5.2.4}$$

On the other hand, since η is a unit-length vector, it follows (again from the chain-rule formula for Sobolev mappings) that $\eta(x) \bullet \mathbf{D}_i \eta(x) = 0$ for $\mathcal{L}^n$-a.e. $x \in U$ and for $i \in \{1, \dots, n\}$. Thus, from (5.2.4), we deduce that $\lambda_i(x) = 0$ and

$$\mathbf{D}_i \eta(x) = \mu(x)\,(\boldsymbol{e}_i, \mathbf{D}_i f(x)) = \mu(x)\,\mathbf{D}_i \overline{f}(x) \tag{5.2.5}$$

for $\mathcal{L}^n$-a.e. $x \in U$. For $k \in \{1, \dots, n\}$, let $g_k \in W^{1,1}_{loc}(U)$ be defined as

$$g_k := -\frac{\mathbf{D}_k f}{\sqrt{1 + |\boldsymbol{\nabla} f|^2}}\,.$$

From (5.2.5), we observe that

$$\mathbf{D}_i g_j = 0 \qquad \text{whenever } i, j \in \{1, \dots, n\} \text{ and } i \neq j\,, \tag{5.2.6}$$

$$\mathbf{D}_i g_i = \mu \qquad \text{whenever } i \in \{1, \dots, n\}\,. \tag{5.2.7}$$

Now, we fix an open cube $Q \subset U$ with sides parallel to the coordinate axes, a function $\phi \in C^\infty_c(Q)$ and fix $k \in \{1, \dots, n\}$, we prove that

$$\int_Q \mu\, D_k \phi \, d\mathcal{L}^n = 0\,. \tag{5.2.8}$$

Let $j \in \{1, \dots, n\}$ be chosen with $k \neq j$. Since, by (5.2.6), we have $\mathbf{D}_k g_j = 0$, it follows from Lemma 5.2.1 that for $\mathcal{L}^{n-1}$-a.e. $y \in U_{\boldsymbol{e}_k}$ there exists $v_j(y) \in \mathbb{R}$ such that

$$g_j(y + t e_k) = v_j(y) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in U^{\boldsymbol{e}_k}_y.$$

Next, we use (5.2.7) to obtain

$$\begin{aligned} \int_Q \mu\, D_k \phi \, d\mathcal{L}^n &= \int_Q \mathbf{D}_j g_j \, D_k \phi \, d\mathcal{L}^n \\ &= -\int_Q g_j \, D_j(D_k \phi)\, d\mathcal{L}^n \\ &= -\int_{Q_{e_k}} v_j(y) \int_{Q^y_{\boldsymbol{e}_k}} D_k(D_j \phi)(y + t e_k)\, d\mathcal{L}^1(t)\, d\mathcal{L}^{n-1}(y) = 0\,. \end{aligned}$$

The last equality follows since the function mapping $t \in Q^{\boldsymbol{e}_k}_y$ into $D_j \phi(y + t e_k)$ has compact support in $Q^{\boldsymbol{e}_k}_y$.

Since (5.2.8) holds for every open cube Q with sides parallel to the coordinate axes and for every $\phi \in C^\infty_c(Q)$, and since U is connected, we infer from [5, Proposition 3.2 *(a)*] that

$$\mu \text{ is } \mathcal{L}^n\text{-a.e. equal to a constant function on } U. \tag{5.2.9}$$

Considering that U is connected, we combine (5.2.9) and (5.2.5) to infer that there exists $c \in \mathbb{R}$ and $w \in \mathbb{R}^{n+1}$ such that

$$\eta(x) - c\overline{f}(x) = w \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in U\,.$$

If $c \neq 0$ the last equation evidently implies that $\overline{f}(x) \in \partial B^{n+1}_{1/|c|}(-w/c)$ for $\mathcal{L}^n$-a.e. $x \in U$. If $c = 0$, we have that $w \bullet e_{n+1} = (1 + |\nabla f|^2)^{-1/2}$ and

$$\mathbf{D}_i f(x) = -\frac{w \bullet e_i}{w \bullet e_{n+1}} \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in U \text{ and } i \in \{1, \dots, n\}\,.$$

This implies that f is $\mathcal{L}^n$-a.e. equal to linear function on U, since U is connected. □

5.3 Nabelpunksatz for Sobolev graphs

Suppose $X \subset \mathbb{R}^{n+1}$ is $\mathcal{H}^n$-measurable and $\mathcal{H}^n$-rectifiable of class 2. We say that X is *approximate totally umbilical* if there exists an $(\mathcal{H}^n \llcorner X)$-measurable mapping ν such that $\nu(x) \in \mathrm{Nor}^n(\mathcal{H}^n \llcorner X, x) \cap \mathbb{S}^n$ and there exists a function $\mu : X \to \mathbb{R}$ such that

$$\mathrm{ap}\, D\nu(x)(\tau) = \mu(x)\tau \quad \text{for every } \tau \in \mathrm{Tan}^n(\mathcal{H}^n \llcorner X, x)\,, \tag{5.3.10}$$

for $\mathcal{H}^n$-a.e. $x \in X$ (keep in mind Lemma 3.1.2).

Moreover, if $U \subseteq \mathbb{R}^n$ is an open set and $g : U \to \mathbb{R}^k$ ($k \geq n$) then we say that g satisfies the *Lusin's* (N) *condition* if $\mathcal{H}^n\big(g(Z)\big) = 0$ for every $Z \subset U$ with $\mathcal{L}^n(Z) = 0$.

We are now ready to prove the second result of this chapter.

Theorem 5.3.3. *Suppose $U \subset \mathbb{R}^n$ is a bounded and connected open set, $f \in W^{2,1}_{loc}(U)$, and $\overline{f}$ satisfies the Lusin's (N)-condition, with $\Gamma := \overline{f}(U)$.*

Then Γ is $\mathcal{H}^n$-rectifiable of class 2. Moreover, if Γ is approximate totally umbilical, then, up to a $\mathcal{H}^n$-negligible set, either Γ is a subset of a n-dimensional plane or a subset of an n-dimensional sphere.

Proof. By virtue of [9, Theorem 13] and [14, 2.10.19 (4), 2.10.43] we can find a sequence $\{g_i\}_{i\in\mathbb{N}} \subset C^2(\mathbb{R}^n)$ such that $\mathcal{L}^n(U \setminus \bigcup_{i=1}^\infty \{g_i = f\}) = 0$ and $Lip(g_i) < \infty$ for every $i \in \mathbb{N}$. Hence, thanks to the Lusin's (N) condition, we readily infer that Γ is $\mathcal{H}^n$-rectifiable of class 2. We define V_i as the set of $x \in \{g_i = f\}$ such that

$$\Theta^n(\mathcal{L}^n \llcorner U \setminus \{g_i = f\}, x) = 0\,, \quad Dg_i(x) = \mathbf{D}f(x)$$

and $\mathrm{Tan}^n\big(\mathcal{H}^n \llcorner \Gamma, \overline{f}(x)\big)$ is a n-dimensional plane. Since $Dg_i(x) = \mathrm{ap}\, Df(x)$ for every $x \in V_i$, it follows from [14, 2.10.19 (4), 3.2.19] and Lemma 2.1.8 that

$$\mathcal{L}^n(\{g_i = f\} \setminus V_i) = 0 \quad \text{for every } i \in \mathbb{N}\,.$$

Since $\mathrm{Tan}^n(\mathcal{L}^n \llcorner \{g_i = f\}, x) = \mathbb{R}^n$ for every $x \in V_i$, and noting that $\overline{g}_i : \mathbb{R}^n \to \overline{g}_i(\mathbb{R}^n)$ is a bi-lipschitz homeomorphism, we use [51, Lemma B.2] to conclude

$$\mathbf{D}\overline{f}(x)[\mathbb{R}^n] = D\overline{g}_i(x)\big[\mathrm{Tan}^n(\mathcal{L}^n \llcorner \{g_i = f\}, x)\big] \subseteq \mathrm{Tan}^n\big(\mathcal{H}^n \llcorner \Gamma, \overline{f}(x)\big)\,,$$

for every $x \in V_i$. Since $\mathbf{D}\overline{f}(x)$ is injective whenever it exists, we conclude that

$$\mathbf{D}\overline{f}(x)[\mathbb{R}^n] = \mathrm{Tan}^n\big(\mathcal{H}^n \llcorner \Gamma, \overline{f}(x)\big)$$

and

$$\psi(\nabla f(x)) \in \mathrm{Nor}^n\big(\mathcal{H}^n \llcorner \Gamma, \overline{f}(x)\big)\,,$$

for every $x \in V_i$ and $i \in \mathbb{N}$. Let $V := \bigcup_{i=1}^\infty V_i$ and notice that $\mathcal{H}^n\big(\Gamma \setminus \overline{f}(V)\big) = 0$ (again by Lusin's (N) condition). Let ν be the $(\mathcal{H}^n \llcorner \Gamma)$-measurable map defined by

$$\nu := \psi \circ \nabla f \circ (\pi|\Gamma)\,,$$

where $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the orthogonal projection onto $\mathbb{R}$. We observe that if $z \in \overline{f}(V_i)$ and $\Theta^n\big(\mathcal{H}^n \llcorner \Gamma \setminus \overline{f}(V_i), z\big) = 0$, then ν is $(\mathcal{H}^n \llcorner \Gamma)$-approximately differentiable at z $\big($since $\nu|\overline{f}(V_i) = (\psi \circ \nabla g_i \circ \pi)|\overline{f}(V_i)\big)$ and

$$\mathrm{ap}\, D\nu(z) = D(\psi \circ \nabla g_i \circ \pi)(z)$$

$$
\begin{aligned}
&= D(\psi \circ \nabla g_i)(\pi(z)) \circ (\pi|\mathrm{Tan}^n(\mathcal{H}^n \llcorner \Gamma, z)) \\
&= \mathrm{ap}\, D(\psi \circ \boldsymbol{\nabla} f)(\pi(z)) \circ (\pi|\mathrm{Tan}^n(\mathcal{H}^n \llcorner \Gamma, z)) \\
&= \mathbf{D}(\psi \circ \boldsymbol{\nabla} f)(\pi(z)) \circ (\pi|\mathrm{Tan}^n(\mathcal{H}^n \llcorner \Gamma, z)),
\end{aligned}
$$

whence we infer

$$
\mathrm{ap}\, D\nu(z) \circ \mathbf{D}\overline{f}(\pi(z)) = \mathbf{D}(\psi \circ \boldsymbol{\nabla} f)(\pi(z)). \tag{5.3.11}
$$

By [14, 2.10.19 (4)] we conclude that (5.3.11) is true for $\mathcal{H}^n$-a.e. $z \in \Gamma$.

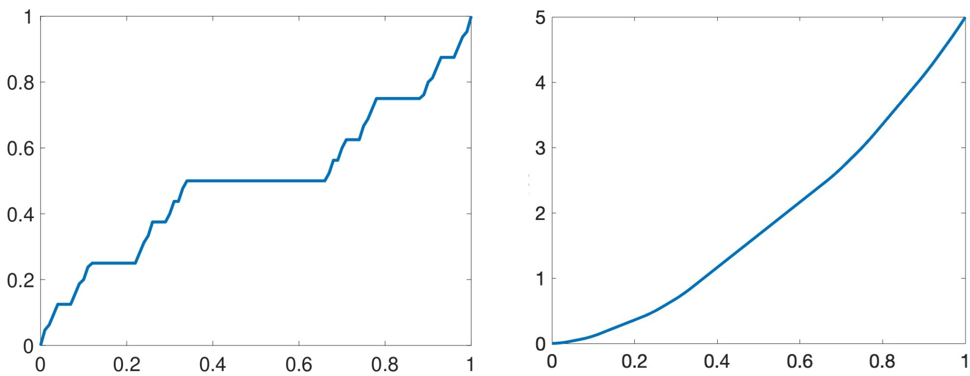
If Γ is approximate totally umbilical, it is easy to see that the unit normal vector field ν defined above fulfils the umbilicalilty condition in (5.3.10) with some function μ. Hence

$$
\begin{aligned}
\mu(\overline{f}(x))(\boldsymbol{e}_i \bullet \boldsymbol{e}_j + \mathbf{D}_i f(x)\mathbf{D}_j f(a)) &= (\mathrm{ap}\, D\nu(\overline{f}(x)) \circ \mathbf{D}\overline{f}(x))(\boldsymbol{e}_i) \bullet (\boldsymbol{e}_j, \mathbf{D}_j f(x)) \\
&= \mathbf{D}_i(\psi \circ \nabla f)(x) \bullet (\boldsymbol{e}_j, \mathbf{D}_j f(x)) \\
&= -\frac{\mathbf{D}^2_{ij} f(x)}{\sqrt{1 + |\boldsymbol{\nabla} f(a)|^2}},
\end{aligned}
$$

for every $i, j \in \{1, \dots, n\}$ and for $\mathcal{L}^n$-a.e. $x \in U$. By Theorem 5.2.2 and by the Lusin's (N)-property we deduce that, up to a $\mathcal{H}^n$-negligible set, Γ is either a subset of a n-dimensional plane, or a subset of a n-dimensional sphere of $\mathbb{R}^{n+1}$. The proof is complete. □

Remark 5.3.4. If $f \in W^{2,p}_{loc}(U)$ with $\frac{n}{2} < p < n$, the Sobolev embedding theorem [20, Theorem 7.26] ensures that $f \in W^{1,p^*}_{loc}(U)$ with $p^* > n$. Therefore, $\overline{f}$ satisfies the Lusin's (N)-condition by [35, Theorem 1.1].

Remark 5.3.5. It is easy to find convex functions $f \in C^{1,\alpha}(\mathbb{R}^n)$ such that the approximate principal curvatures of their graph are zero $\mathcal{H}^n$-almost everywhere, and the conclusion of Theorem 5.3.3 fails (notice that such a graph is $\mathcal{H}^n$-rectifiable of class 2 and $\overline{f}$ satisfies the Lusin's (N)-condition). Indeed, the gradient of these functions are continuous maps of bounded variation, whose distributional derivative is not a function. An example of such a functions is given by the primitive of the ternary Cantor function. Let $\mathcal{C} \subset [0,1]$ be the Cantor ternary set and $f : [0,1] \to [0,1]$ the Cantor-Vitali function. Recall that $f \in C^{0,\alpha}([0,1])$ with $\alpha = \log_3 2$ and $f(\mathcal{C}) = [0,1]$, in particular it is also increasing with $f(0) = 0$, $f(1) = 1$ and, finally, $f'(x) = 0$ for every x in the open set $[0,1] \setminus \mathcal{C}$. The function f provides an example of a BV-function that is not absolutely continuous, namely $f \in BV(0,1) \setminus W^{1,1}(0,1)$. In fact, since f is increasing, we deduce that the total variation of f is 1. Furthermore, since $\mathcal{L}^1(\mathcal{C}) = 0$ and $f(\mathcal{C}) = [0,1]$, we infer that f does not satisfy the Lusin (N)-property and hence is not absolutely continuous (cf. [31, Theorem 3.41]). If we now consider the primitive of f, denoted by F, we deduce that $F \in C^{1,\alpha}([0,1]) \setminus W^{2,1}(0,1)$ and since F is piecewise linear in $[0,1] \setminus \mathcal{C}$, we infer that the approximate principal curvature of graph(F) are zero $\mathcal{H}^1$-a.e., but the conclusion of Theorem 5.3.3 fails.



Graph of the ternary Cantor function (left) and its primitive (right).

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