ON A WEIGHTED VERSION OF THE BBM FORMULA

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Dedicated to Silvia, on the occasion of our marriage.

Abstract. We prove a weighted version of the Bourgain–Brezis–Mironescu (BBM) formula, both in the pointwise and Γ-convergence sense, together with a compactness criterion for energy-bounded sequences. The non-negative weights need only be L^∞ convergent to a bounded and uniformly continuous limit. We apply the BBM formula to show a Poincaré-type inequality and the stability of the first eigenvalues relative to the energies. Finally, we discuss a non-local analogue of the weighted BBM formula.

1. Introduction

This is a spin-off of the recent work [19] by L. Gennaioli and the author concerning *sharp* conditions for the validity of the Bourgain–Brezis–Mironescu (BBM, for short) formula [4, 12, 22]. Our main aim is to show that the *sufficiency* part of the BBM formula proved in [19] remains true even if the energies under consideration are modified in order to include some suitable *weights*. As an application of our results, we prove a Poincaré-type inequality and the stability of the first eigenvalues relative to the energies. Finally, we also discuss the validity of a non-local analogue of the weighted BBM formula.

1.1. **Weighted functionals.** Throughout the paper, given $p \in [1, \infty)$, we consider a family of kernels $(\rho_k)_{k\in\mathbb{N}} \subset L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $\rho_k \geq 0$ for every $k \in \mathbb{N}$ and, unless otherwise stated, we assume that

$$\sup_{R>0} \limsup_{k\to\infty} R^p \int_{\mathbb{R}^N} \frac{\rho_k(z)}{R^p + |z|^p} \,\mathrm{d}z < \infty \tag{1.1}$$

and, for some $\alpha \geq 0$,

$$\nu_k = \rho_k \mathscr{L}^N \overset{\star}{\rightharpoonup} \alpha\delta_0 \quad \text{in } \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^N) \text{ as } k \to \infty. \tag{1.2}$$

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Moreover, we fix a family of non-negative *weights* $(w_k)_{k\in\mathbb{N}} \subset L^\infty(\mathbb{R}^{2N})$ such that $w_k \to w$ in $L^\infty(\mathbb{R}^{2N})$ as $k\to\infty$ for some non-negative *limit weight* $w \in C_b(\mathbb{R}^{2N})$. We assume that

$$|w(x,y) - w(x',y')| \le \omega(|x-x'| + |y-y'|), \quad \text{for all } x,x',y,y' \in \mathbb{R}^N, \tag{1.3}$$

for some non-decreasing modulus of continuity $\omega\colon [0,\infty)\to[0,\infty)$ such that

$$\lim_{t\to 0^+} \omega(t) = \omega(0) = 0. \tag{1.4}$$

Observe that we do *not* impose any symmetry assumption on the weights $(w_k)_{k\in\mathbb{N}}$ and w, as one may be interested in possibly *non-symmetric* interactions [13].

With the above notation in force, we define the *weighted* functionals $\mathscr{F}^{w_k}_{k,p}\colon L^p(\mathbb{R}^N)\to[0,\infty]$ by letting

$$\begin{aligned} \mathscr{F}^{w_k}_{k,p}(u) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^p}\,\rho_k(x-y)\,w_k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^N} \frac{\|u(\cdot+z)-u\|^p_{L^p(w_k^z)}}{|z|^p}\,\rho_k(z)\,\mathrm{d}z, \end{aligned} \tag{1.5}$$

for all $u\in L^p(\mathbb{R}^N)$ and $k\in\mathbb{N}$, where, for each $z\in\mathbb{R}^N$,

$$\|u\|^p_{L^p(w_k^z)} = \int_{\mathbb{R}^d} |u(x)|^p\, w_k^z(x,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} |u(x)|^p\, w_k(x,x+z)\,\mathrm{d}x. \tag{1.6}$$

The functionals $(\mathscr{F}^{w}_{k,p})_{k\in\mathbb{N}}$, as well as the norm $\|\cdot\|_{L^p(w^z)}$, are analogously defined. The *unweighted* functionals in [19], where $w_k = w = 1$ for all $k\in\mathbb{N}$, are given by $(\mathscr{F}^{1}_{k,p})_{k\in\mathbb{N}}$.

1.2. **Local framework.** As proved in [19] (also refer to [14] for the case $p=2$ and to [18] for *radially symmetric* kernels), the unweighted functionals $(\mathscr{F}^{1}_{k,p})_{k\in\mathbb{N}}$ converge to

$$\mathscr{D}^{\mu}_{p,w}(u) = \int_{\mathbb{S}^{N-1}} \|\sigma\cdot Du\|^p_{L^p}\,\mathrm{d}\mu(\sigma) \tag{1.7}$$

in the pointwise and Γ-sense on $\mathcal{S}^p(\mathbb{R}^N)$ with respect to the L^p topology for some Radon measure $\mu\in\mathscr{M}(\mathbb{S}^{N-1})$ defined on the $(N-1)$-dimensional unit sphere $\mathbb{S}^{N-1}$ if and only if the non-negative kernels $(\rho_k)_{k\in\mathbb{N}}$ statisfy (1.1) and (1.2) (possibly, up to a subsequence). For the notion of Γ-convergence, refer to [19, Sec. 2.5] for an account and to [5, 11] for a complete treatment. Above and in the rest of the work, as in [19], we set

$$\mathcal{S}^p(\mathbb{R}^N) = \begin{cases} W^{1,p}(\mathbb{R}^N) & \text{for } p>1,\\ BV(\mathbb{R}^N) & \text{for } p=1,\end{cases}$$

and, for each $u\in\mathcal{S}^p(\mathbb{R}^N)$ and $\sigma\in\mathbb{S}^{N-1}$,

$$\|\sigma\cdot Du\|^p_{L^p} = \begin{cases} \displaystyle\int_{\mathbb{R}^N} |\sigma\cdot Du|^p\,\mathrm{d}x & \text{for } p>1,\\ |\sigma\cdot Du|(\mathbb{R}^N) & \text{for } p=1.\end{cases} \tag{1.8}$$

As customary, in (1.8) above Du denotes the distributional gradient of $u\in\mathcal{S}^p(\mathbb{R}^N)$. In particular, if $p=1$, then Du may be a finite (vectorial) Radon measure on $\mathbb{R}^N$.

We underline that the measure $\mu \in \mathscr{M}(\mathbb{S}^{N-1})$ appearing in (1.7) is uniquely determined by the family $(\rho_k)_{k\in\mathbb{N}}$ (possibly, up to a subsequence) and can be (formally) defined as

$$\mu(E) = \lim_{\delta\to0^+} \lim_{k\to\infty} \int_E \left(\int_0^\delta \rho_k(\sigma r)\, r^{N-1}\,\mathrm{d}r \right) \mathrm{d}\mathscr{H}^{N-1}(\sigma) \tag{1.9}$$

for every set $E \subset \mathbb{S}^{N-1}$ measurable with respect to the $(N-1)$-dimensional Hausdorff measure $\mathscr{H}^{N-1}$. We refer to [19, Lem. 2.9] for the precise construction of μ. Because of (1.9), the nature of the limit measure μ in (1.7) does *not* depend on the chosen weights $(w_k)_{k\in\mathbb{N}}$ and w. For this reason, from now on we tacitly assume that the measure $\mu \in \mathscr{M}(\mathbb{S}^{N-1})$ is uniquely identified by (1.1) and (1.2).

Our first main result, Theorem 1.1 below, proves that (1.1) and (1.2) are still *sufficient* for the pointwise and Γ-convergence of the functionals $(\mathscr{F}^{w_k}_{k,p})_{k\in\mathbb{N}}$ in (1.5) to a weighted version of the energy in (1.7); namely,

$$\mathscr{D}^\mu_{p,w}(u) = \int_{\mathbb{S}^{N-1}} \|\sigma\cdot Du\|^p_{L^p(w^0)}\,\mathrm{d}\mu(\sigma), \tag{1.10}$$

where $\mu \in \mathscr{M}(\mathbb{S}^{N-1})$ is as in (1.9) and, for every $\sigma \in \mathbb{S}^{N-1}$, as in (1.8),

$$\|\sigma\cdot Du\|^p_{L^p(w^0)} = \begin{cases} \displaystyle\int_{\mathbb{R}^N} w^0\,|\sigma\cdot Du|^p\,\mathrm{d}x & \text{for } p>1, \\ \displaystyle\int_{\mathbb{R}^N} w^0\,\mathrm{d}|\sigma\cdot Du| & \text{for } p=1. \end{cases} \tag{1.11}$$

In (1.11) above, the function $w^0 \in C_b(\mathbb{R}^N)$ is defined as $w^0(x) = w(x, x+0) = w(x,x)$ for every $x \in \mathbb{R}^N$ according to the notation introduced in (1.6). Moreover, here and in the following, given any $R>0$, we define the (L^p closed) subspaces

$$L^p_R(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \operatorname{supp} u \subset \overline{B}_R \right\}, \quad \mathcal{S}^p_R(\mathbb{R}^N) = \mathcal{S}^p(\mathbb{R}^N) \cap L^p_R(\mathbb{R}^N).$$

Theorem 1.1 (Weighted BBM formula). *Let $p \in [1,\infty)$ and $(\rho_k)_{k\in\mathbb{N}}$ be as above. The following hold:*

(i) if $u \in \mathcal{S}^p(\mathbb{R}^N)$, then $\limsup_{k\to\infty} \mathscr{F}^{w_k}_{k,p}(u) \le \mathscr{D}^\mu_{p,w}(u)$;

(ii) if $(u_k)_{k\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ is such that $u_k \to u$ in $L^p(\mathbb{R}^N)$ as $k\to\infty$ for some $u \in \mathcal{S}^p(\mathbb{R}^N)$ and $\limsup_{k\to\infty} \mathscr{F}^1_{k,p}(u_k) < \infty$ then $\liminf_{k\to\infty} \mathscr{F}^{w_k}_{k,p}(u_k) \ge \mathscr{D}^\mu_{p,w}(u)$;

(iii) if $w>0$ on $\mathbb{R}^{2N}$, $R>0$ and $(u_k)_{k\in\mathbb{N}} \subset L^p_R(\mathbb{R}^N)$ is such that $u_k \to u$ in $L^p(\mathbb{R}^N)$ as $k\to\infty$ for some $u \in \mathcal{S}^p(\mathbb{R}^N)$, then $\limsup_{k\to\infty} \mathscr{F}^1_{k,p}(u_k) < \infty$ and therefore $\liminf_{k\to\infty} \mathscr{F}^{w_k}_{k,p}(u_k) \ge \mathscr{D}^\mu_{p,w}(u)$.

As a consequence, as $k\to\infty$, the functionals $(\mathscr{F}^{w_k}_{k,p})_{k\in\mathbb{N}}$ converge to $\mathscr{D}^\mu_{p,w}$ pointwise on $\mathcal{S}^p(\mathbb{R}^N)$ and in the Γ-sense on $\mathcal{S}^p_R(\mathbb{R}^N)$ for every $R>0$.

We underline that the functionals (1.5) are not invariant under translations, due to the presence of the weights $(w_k)_{k\in\mathbb{N}}$ and w. However, despite this lack of regularity, assumptions (1.1) and (1.2) are still *sufficient* for the validity of a BBM formula in this case. Besides, we observe that the assumptions (1.1) and (1.2) cannot be weakened as, in fact, by the main result of [19], both ones are also *necessary* for the validity of Theorem 1.1 in the unweighted case $w_k = w = 1$ for all $k \in \mathbb{N}$.

Finally, we remark that the validity of the statements (i), (ii) and (iii) in Theorem 1.1 for *some* measure $\mu \in \mathscr{M}(\mathbb{S}^{N-1})$ cannot imply the conditions (1.1) and (1.2), unless one imposes some additional (strong) assumptions on the weights, at least by requiring that $\inf_{\mathbb{R}^{2N}} w > 0$. Indeed, at the level of generality we are presently working, one may choose $w_k = w$ for all $k \in \mathbb{N}$ and $w(x,y) = \omega(|x-y|)$ for all $x, y \in \mathbb{R}^N$, where $\omega\colon [0,\infty) \to [0,\infty)$ is a bounded non-decreasing function satisfying (1.4). In this case, we may rewrite

$$\mathscr{F}^{w_k}_{k,p}(u) = \int_{\mathbb{R}^N} \frac{\|u(\cdot+z)-u\|^p_{L^p}}{|z|^p}\, \widetilde{\rho_k}(z)\,\mathrm{d}z = \widetilde{\mathscr{F}_{k,p}}(u)$$

where $\widetilde{\rho_k} = \rho_k\, \omega(|\cdot|)$ for all $k \in \mathbb{N}$. Note that the functionals $(\widetilde{\mathscr{F}_{k,p}})_{k\in\mathbb{N}}$ above are precisely the ones considered in [19], so the validity of the BBM formula (in our case, the validity of Theorem 1.1) is equivalent to the conditions (1.1) and (1.2) for the family $(\widetilde{\rho_k})_{k\in\mathbb{N}}$. In other words, by assuming (1.1) and (1.2), we are implicitly deciding the roles of the players in the game: $(\rho_k)_{k\in\mathbb{N}}$ play as the kernels and $(w_k)_{k\in\mathbb{N}}$ play as the weights.

1.3. **Compactness.** Our second main result, Theorem 1.3 below, is a simple compactness criterion for sequences of functions with bounded energy.

Definition 1.2. Let $p \in [1,\infty)$. A sequence $(u_k)_{k\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ is *p-energy bounded* if

$$\sup_{k\in\mathbb{N}} \Big(\|u_k\|_{L^p} + \mathscr{F}^1_{k,p}(u_k)\Big) < \infty.$$

Moreover, the family $(\rho_k)_{k\in\mathbb{N}}$ is *p-compact* if any p-energy bounded sequence in $L^p(\mathbb{R}^N)$ is compact in $L^p(E)$ for every compact set $E \subset \mathbb{R}^N$.

Theorem 1.3 (Compactness)**.** *Let $(\rho_k)_{k\in\mathbb{N}}$ be p-compact, $w > 0$ on $\mathbb{R}^{2N}$ and $R > 0$. If $(u_k)_{k\in\mathbb{N}} \subset L^p_R(\mathbb{R}^N)$ is such that*

$$\sup_{k\in\mathbb{N}} \Big(\|u_k\|_{L^p} + \mathscr{F}^{w_k}_{k,p}(u_k)\Big) < \infty,$$

then $(u_k)_{k\in\mathbb{N}}$ is compact in $L^p(\mathbb{R}^N)$.

Examples of p-compact families of kernels $(\rho_k)_{k\in\mathbb{N}}$ can be found for instance in [2,4,19,23]. Here we only refer to [4, Th. 5] and to [23, Ths. 1.2 and 1.3] for the case of *radially symmetric* kernels, and to [19, Th. 1.3] for possibly non-radially symmetric ones. We also refer to [3,18] for the purely non-local framework.

In the references above, the family $(\rho_k)_{k\in\mathbb{N}}$ is not only p-compact (for some or for all p), but also yields that any $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ limit of a p-energy bounded sequence is a more regular function; namely, it may belongs to $\mathcal{S}^p(\mathbb{R}^N)$. This motivates the following terminology.

Definition 1.4. Let $p \in [1,\infty)$. The family $(\rho_k)_{k\in\mathbb{N}}$ is *strongly p-compact* if it is p-compact and such that any $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ limit of a p-energy bounded sequence is in $\mathcal{S}^p(\mathbb{R}^N)$.

As well-known (e.g., see [22,23]), the family of standard *fractional* kernels, defined as

$$\rho_s(z) = \frac{1-s}{|z|^{N-(1-s)p}}, \quad \text{for } z \in \mathbb{R}^N \text{ and } s \in (0,1), \tag{1.12}$$

is an example of strongly p-compact family for every $p \in [1,\infty)$.

By combining Theorems 1.1 and 1.3 with Definition 1.4, we obtain the following result.

Corollary 1.5. *Let $p \in [1,\infty)$, $w > 0$ on $\mathbb{R}^{2N}$ and $R > 0$. Assume that $(\rho_k)_{k\in\mathbb{N}}$ is a strongly p-compact family. Then, the following hold:*

(i) if $(u_k)_{k\in\mathbb{N}} \subset L^p_R(\mathbb{R}^N)$ is such that

$$\sup_{k\in\mathbb{N}} \left(\|u_k\|_{L^p} + \mathscr{F}^{w_k}_{k,p}(u_k) \right) < \infty, \tag{1.13}$$

then $(u_k)_{k\in\mathbb{N}}$ is compact in $L^p(\mathbb{R}^N)$ and any of its $L^p(\mathbb{R}^N)$ limits is in $\mathcal{S}^p(\mathbb{R}^N)$;

(ii) if $(u_k)_{k\in\mathbb{N}} \subset L^p_R(\mathbb{R}^N)$ is such that $u_k \to u$ in $L^p(\mathbb{R}^N)$ as $k \to \infty$ for some $u \in L^p(\mathbb{R}^N)$, then $\liminf_{k\to\infty} \mathscr{F}^{w_k}_{k,p}(u_k) \ge \mathscr{D}^\mu_{p,w}(u)$;

(iii) if $u \in \mathcal{S}^p(\mathbb{R}^N)$, then $\limsup_{k\to\infty} \mathscr{F}^{w_k}_{k,p}(u) \le \mathscr{D}^\mu_{p,w}(u)$.

A particular case of Corollary 1.5 has been recently obtained by A. Kubin, G. Saracco and the author in [21], with a completely different proof, for the fractional kernels (1.12) and weights defined as $w_k(x,y) = f_k(x)\, f_k(y)$ for all $k \in \mathbb{N}$ and $w(x,y) = f(x)\, f(y)$, for every $x,y \in \mathbb{R}^N$, where $(f_k)_{k\in\mathbb{N}} \subset L^\infty(\mathbb{R}^N;[0,\infty))$, $f \in \mathrm{Lip}_b(\mathbb{R}^N;(0,\infty))$ and $f_k \to f$ in $L^\infty(\mathbb{R}^N)$ as $k \to \infty$ (for the unweighted case, refer to [10, Th. 2.1] for $p = 2$ and to [6, Th. 3.1] for $p \in (1,\infty)$). For strictly related results, see [8, 15] for the fractional *Gaussian* case, and [20] for weights depending on negative powers of the distance from the boundary of the domain of integration.

Remark 1.6 (A generalization of [21, Th. 1.2])**.** Corollary 1.5 can be exploited to prove the stability of the gradient flows relative to the Hilbertian energies $(\mathscr{F}^{w_k}_{k,2})_{k\in\mathbb{N}}$ and $\mathscr{D}^\mu_{2,w}$, thus generalizing [21, Th. 1.2] to the present setting. Indeed, it is enough to follow the strategy outlined in [21, Sec. 4] line by line up to the natural (minor) adaptations.

1.4. **Poincaré inequality.** Motivated by [23], we exploit Corollary 1.5 to estalish a Poincaré-type inequality for the energies $(\mathscr{F}^{w_k}_{k,p})_{k\in\mathbb{N}}$, see Theorem 1.8 below.

In order to state our result, we need to introduce some notation. From now on, we fix a (non-empty) bounded open set $\Omega \subset \mathbb{R}^N$ and, for each $p \in [1,\infty)$, we define the subspaces

$$L^p_0(\Omega) = \left\{ u \in L^p(\mathbb{R}^N) : u = 0 \text{ a.e. on } \mathbb{R}^N \setminus \Omega \right\}, \quad \mathcal{S}^p_0(\Omega) = \left\{ u \in \mathcal{S}^p(\mathbb{R}^N) : u \in L^p_0(\Omega) \right\}. \tag{1.14}$$

As well known, if Ω has Lipschitz regular boundary, then $\mathcal{S}^p_0(\Omega)$ coincides with the closure of $C^\infty_c(\Omega)$ functions with respect to the energy $u \mapsto \|Du\|_{L^p}$, but we do not need such equivalence in the following.

Definition 1.7. Let $p \in [1,\infty)$. The measure $\mu \in \mathscr{M}(\mathbb{S}^{N-1})$ is (p,Ω)-*Poincaré* if there exists $C > 0$ such that, for every $u \in \mathcal{S}^p_0(\Omega)$, it holds

$$\|u\|^p_{L^p} \le C\, \mathscr{D}^\mu_p(u). \tag{1.15}$$

In this case, we let $A_{p,\mu,\Omega} > 0$ be the optimal constant for which (1.15) holds.

If $\mu \in \mathscr{M}(\mathbb{S}^{N-1})$ is (p,Ω)-Poincaré for some $p \in [1,\infty)$, then there exists $C > 0$, depending on $A_{p,\mu,\Omega}$ and $\inf_\Omega w^0 > 0$, such that

$$\|u\|^p_{L^p} \le C\, \mathscr{D}^\mu_{p,w}(u) \tag{1.16}$$

for all $u \in \mathcal{S}^p_0(\Omega)$. We thus let $A_{p,\mu,\Omega,w} > 0$ be the optimal constant for which (1.16) holds.

Theorem 1.8 (Poincaré inequality). *Let $p \in [1,\infty)$. Assume that $(\rho_k)_{k\in\mathbb{N}}$ is a strongly p-compact family and $\mu \in \mathscr{M}(\mathbb{S}^{N-1})$ is (p,Ω)-Poincaré. Then, for every $\varepsilon > 0$, then exists $k_\varepsilon \in \mathbb{N}$ such that*

$$\|u\|_{L^p}^p \le (A_{p,\mu,\Omega,w} + \varepsilon)\, \mathscr{F}_{k,p}^{w_k}(u)$$

for all $u \in L_0^p(\Omega)$ and $k \ge k_\varepsilon$.

It is worth observing that $\mu \in \mathscr{M}(\mathbb{S}^{N-1})$ is (p,Ω)-Poincaré as in Definition 1.7 for every $p \in [1,\infty)$ if $\operatorname{span}(\operatorname{supp}\mu) = \mathbb{R}^N$, due to [19, Lem. 2.1]. This latter property, in turn, is ensured for example by assuming that the family $(\rho_k)_{k\in\mathbb{N}}$ has *maximal rank* as in [19, Def. 2.8], see [19, Lem. 2.9] for the proof. As remarked in [19, Sec. 1.2], any radially symmetric family has maximal rank, but non-radially symmetric families with maximal rank are known (examples can be found in [22]). In particular, as a consequence, Theorem 1.8 applies to the radially symmetric family of fractional kernels (1.12).

1.5. **Spectral stability.** Inspired by [6], we make use of Corollary 1.5 to also show a stability result for the first eigenvalues relative to the energies $(\mathscr{F}_{k,p}^{w_k})_{k\in\mathbb{N}}$ and $\mathscr{D}_{p,w}^{\mu}$ and of the corresponding eigenfunctions, see Theorem 1.9 below.

Letting $p \in [1,\infty)$ and $\Omega \subset \mathbb{R}^N$ be as above, and keeping in mind the subspaces defined in (1.14), for each $k \in \mathbb{N}$ we let

$$\lambda_{k,p}^{w_k}(\Omega) = \inf\Big\{\mathscr{F}_{k,p}^{w_k}(u) : u \in L_0^p(\Omega) \text{ such that } \|u\|_{L^p} = 1\Big\} \in [0,\infty) \tag{1.17}$$

be the *first eigenvalue* relative to the energy $\mathscr{F}_{k,p}^{w_k}$ on Ω, and we call any $u \in L_0^p(\Omega)$ attaining the infimum in (1.17) an *eigenfunction* relative to $\lambda_{k,p}^{w_k}(\Omega)$. Analogously, we let

$$\lambda_{p,w}^{\mu}(\Omega) = \inf\Big\{\mathscr{D}_{p,w}^{\mu}(u) : u \in \mathcal{S}_0^p(\Omega) \text{ such that } \|u\|_{L^p} = 1\Big\} \in [0,\infty) \tag{1.18}$$

be the *first eigenvalue* relative to $\mathscr{D}_{p,w}^{\mu}$ on Ω, and we call any $u \in \mathcal{S}_0^p(\Omega)$ attaining the infimum in (1.18) an *eigenfunction* relative to $\lambda_{p,w}^{\mu}(\Omega)$.

It is worth observing that $\lambda_{k,p}^{w_k}(\Omega) < \infty$ for all $k \in \mathbb{N}$ because, owing to (1.1), we have that $\mathscr{F}_{k,p}^{w_k}(u) < \infty$ for any $u \in C_c^\infty(\Omega)$. Analogously, we also have that $\lambda_{p,w}^{\mu}(\Omega) < \infty$. In addition, if $\mu \in \mathscr{M}(\mathbb{S}^{N-1})$ is (p,Ω)-Poincaré as in Definition 1.7, then $\lambda_{p,w}^{\mu}(\Omega) > 0$ by (1.16) and also $\lambda_{k,p}^{w_k}(\Omega) > 0$ for all $k \in \mathbb{N}$ sufficiently large by Theorem 1.8.

Theorem 1.9 (Spectral stability). *Let $p \in [1,\infty)$ and assume that $(\rho_k)_{k\in\mathbb{N}}$ is a strongly p-compact family. Then, it holds*

$$\lim_{k\to\infty} \lambda_{k,p}^{w_k}(\Omega) = \lambda_{p,w}^{\mu}(\Omega). \tag{1.19}$$

Moreover, if $u_k \in L_0^p(\Omega)$ is an eigenfunction relative to $\lambda_{k,p}^{w_k}(\Omega)$ for each $k \in \mathbb{N}$, then there exist $(u_{k_j})_{j\in\mathbb{N}}$ and an eigenfunction $u \in \mathcal{S}_0^p(\Omega)$ relative to $\lambda_{p,w}^{\mu}(\Omega)$ such that

$$u_{k_j} \to u \text{ in } L^p(\mathbb{R}^N) \text{ as } j \to \infty. \tag{1.20}$$

In the unweighted setting $w_k = w = 1$ for all $k \in \mathbb{N}$, and assuming that $(\rho_k)_{k\in\mathbb{N}}$ is the family of fractional kernels (1.12), Theorem 1.9 corresponds to [6, Th. 1.2] for $m = 1$, the substantial (and natural) difference between the two results being that the convergence in (1.20) holds in a stronger sense in [6] (besides, observe that $p > 1$ in [6]).

Remark 1.10 (A generalization of [6, Th. 1.2])**.** Corollary 1.5 allows to recover [6, Th. 1.2] also for $m \ge 2$ for the unweighted energies $(\mathscr{F}^1_{k,p})_{k\in\mathbb{N}}$ and $\mathscr{D}^\mu_p$ for $p \in (1,\infty)$. Indeed, up to assuming that $\rho_k/|\cdot|^p \notin L^1(\mathbb{R}^N)$ for all sufficiently large $k \in \mathbb{N}$, one can exploit the compactness criterion proved in [3, Th. 2.11] and follow the proof of [6, Th. 1.2] (based on the general results achieved in [16]) line by line up to the natural (minor) adaptations.

1.6. **Non-local framework.** We end up with a generalization of the non-local BBM formula obtained in [19, Ths. 1.4 and 1.5], see Theorem 1.11 below. Here and in the following, we let $A \subset \mathbb{R}^N$ be an open set and $J_k, J\colon A^2 \to [0,\infty]$, with $k \in \mathbb{N}$ and $A^2 = A \times A \subset \mathbb{R}^{2N}$, be measurable functions. We thus let

$$\mathscr{J}_{k,p}(u) = \int_A \int_A |u(x)-u(y)|^p \, J_k(x,y) \, \mathrm{d}x \, \mathrm{d}y$$

for all $k \in \mathbb{N}$ and, similarly,

$$\mathscr{J}_p(u) = \int_A \int_A |u(x)-u(y)|^p \, J(x,y) \, \mathrm{d}x \, \mathrm{d}y,$$

for every $u \in L^p(A)$, and we define

$$\mathcal{W}^{J,p}(A) = \{u \in L^p(A) : \mathscr{J}_p(u) < \infty\}.$$

Theorem 1.11 (Weighted non-local BBM formula)**.** *Let $p \in [1,\infty)$ and let $J_k, J\colon A^2 \to [0,\infty]$, $k \in \mathbb{N}$, be as above. The following hold.*

(i) Assume that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\inf_{k\in\mathbb{N}} J_k(x,y) \ge \frac{1}{\varepsilon\delta^N} \quad \text{for } \mathscr{L}^{2N}\text{-a.e. } (x,y) \in A^2 \text{ s.t. } |x-y| < \delta. \tag{1.21}$$

If $(u_k)_{k\in\mathbb{N}} \subset L^p(A)$ is such that

$$\sup_{k\in\mathbb{N}} \Big(\|u_k\|_{L^p(A)} + \mathscr{J}_{k,p}(u_k) \Big) < \infty, \tag{1.22}$$

then $(u_k)_{k\in\mathbb{N}}$ is compact in $L^p(E)$ for every compact set $E \subset A$.

(ii) Assume that

$$\liminf_{k\to\infty} J_k \ge J \quad \mathscr{L}^{2N}\text{-a.e. in } A^2. \tag{1.23}$$

If $(u_k)_{k\in\mathbb{N}} \subset L^p(A)$ is such that $u_k \to u$ in $L^p(A)$ as $k \to \infty$ for some $u \in L^p(A)$, then $\liminf_{k\to\infty} \mathscr{J}_{k,p}(u_k) \ge \mathscr{J}_p(u)$.

(iii) Assume that

$$J = \lim_{k\to\infty} J_k \quad \mathscr{L}^{2N}\text{-a.e. on } A^2 \tag{1.24}$$

and that either there exist $C > 0$ such that

$$\sup_{k\in\mathbb{N}} J_k \le CJ \quad \mathscr{L}^{2N}\text{-a.e. on } A^2, \tag{1.25}$$

or that there exist $H_+, H_- \subset A^2$, with $\mathscr{L}^{2N}(H_+ \cap H_-) = 0$ and $\mathscr{L}^{2N}(A^2 \setminus (H_+ \cup H_-)) = 0$, such that

$$J_k \le J_{k+1} \text{ a.e. on } H_+ \quad \text{and} \quad J_k \ge J_{k+1} \text{ a.e. on } H_- \tag{1.26}$$

for all $k \in \mathbb{N}$. If $u \in L^p(A)$ is such that $\mathscr{J}_p(u) < \infty$, then $\lim_{k\to\infty} \mathscr{J}_{k,p}(u) = \mathscr{J}_p(u)$.

Therefore, under (1.24) *and either* (1.25) *or* (1.26)*, as* $k \to \infty$*, the functionals* $(\mathscr{J}_{k,p})_{k\in\mathbb{N}}$ *converge pointwise and in the* Γ*-sense to* $\mathscr{J}$ *with respect to the* L^p *topology in* $\mathcal{W}^{J,p}(A)$*.*

Theorem 1.11 may be already known to experts, but we were not able to trace it in the literature. Some particular instances of Theorem 1.11 can be found in [1, Th. 1.1(iii)] and [9, Th. 5.9]. Moreover, note that the statements (ii) and (iii) of Theorem 1.11 can be *verbatim* stated and proved in any measure space $(A, \mathfrak{m})$, while the statement (i) requires some extra caution. In fact, we underline that (1.21) is only prototypical and can be relaxed or variated in many ways, although we do not insist on this technical point.

We exploit Theorem 1.11 to obtain a weighted counterpart of [19, Ths. 1.4 and 1.5], see Corollary 1.12 below (in which we do *not* assume (1.1) and (1.2)). Similarly as in [19], given a measurable function $\kappa\colon \mathbb{R}^N \to [0,\infty)$, we let

$$W^{\kappa,p}_w(\mathbb{R}^N) = \Big\{u \in L^p(\mathbb{R}^N) : [u]_{W^{\kappa,p}_w} < \infty\Big\},$$

where

$$[u]_{W^{\kappa,p}_w} = \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p\,\kappa(x-y)\,w(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}.$$

Corollary 1.12. *Let* $(\rho_k)_{k\in\mathbb{N}} \subset L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *be such that* $\rho_k \ge 0$ *for all* $k \in \mathbb{N}$*. Assume that, for every* $\varepsilon > 0$*, there exist* $\delta > 0$ *such that*

$$\inf_{k\in\mathbb{N}} \frac{\rho_k}{|\cdot|^p} \ge \frac{1}{\varepsilon\delta^N} \quad \mathscr{L}^N\text{-a.e. on } B_\delta, \tag{1.27}$$

and that, moreover, for some $C > 0$*, the family* $(\rho_k)_{k\in\mathbb{N}}$ *satisfies*

$$\sup_{k\in\mathbb{N}} \frac{\rho_k}{|\cdot|^p} \le C\kappa \quad \text{and} \quad \lim_{k\to\infty} \frac{\rho_k}{|\cdot|^p} = \kappa \quad \text{a.e. on } \mathbb{R}^N. \tag{1.28}$$

Then, the following hold:

(i) if $(u_k)_{k\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ *is such that*

$$\sup_{k\in\mathbb{N}} \Big(\|u_k\|_{L^p} + \mathscr{F}^{w_k}_{k,p}(u_k)\Big) < \infty,$$

then $(u_k)_{k\in\mathbb{N}}$ *is compact in* $L^p(B_R)$ *for every* $R > 0$ *and any of its* $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ *limit belongs to* $W^{\kappa,p}_w(\mathbb{R}^N)$*;*

(ii) if $(u_k)_{k\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ *is such that* $u_k \to u$ *in* $L^p(\mathbb{R}^N)$ *as* $k \to \infty$ *for some* $u \in L^p(\mathbb{R}^N)$*, then* $\liminf_{k\to\infty} \mathscr{F}^{w_k}_{k,p}(u_k) \ge [u]^p_{W^{\kappa,p}_w}$*;*

(iii) if $u \in W^{\kappa,p}_w(\mathbb{R}^N)$*, then* $\lim_{k\to\infty} \mathscr{F}^{w_k}_{k,p}(u) = [u]^p_{W^{\kappa,p}_w}$*.*

As a consequence, as $k \to \infty$*, the functionals* $(\mathscr{F}^{w_k}_{k,p})_{k\in\mathbb{N}}$ *converge pointwise and in the* Γ*-sense to* $[\,\cdot\,]^p_{W^{\kappa,p}_w}$ *with respect to the* L^p *topology in* $W^{\kappa,p}(\mathbb{R}^N)$*.*

2. Proofs of the results

The rest of the paper is dedicated to the proofs of the results. The main notation and the conventions adopted here are identical to the ones introduced in [19, Sec. 2].

2.1. **Proof of Theorem 1.1.** We begin by observing that it is enough to prove the statements (i) and (ii) of Theorem 1.1 only in the case $w_k = w$ for all $k \in \mathbb{N}$. Indeed, this follows by combining [19, Th. 1.1] with the simple inequality

$$\left|\mathscr{F}^{w_k}_{k,p}(u) - \mathscr{F}^{w}_{k,p}(u)\right| \le \mathscr{F}^{1}_{k,p}(u)\, \|w_k - w\|_{L^\infty} \tag{2.1}$$

valid for all $k \in \mathbb{N}$ and $u \in L^p(\mathbb{R}^N)$. Moreover, since we can rescale all the functionals by a positive constant factor, it is not restrictive to further assume that $\|w\|_{L^\infty} = 1$.

We are going to take advantage of the following generalization of [19, Lem. 2.2], whose proof is briefly detailed below for the ease of the reader.

Lemma 2.1. *If $u \in \mathcal{S}^p(\mathbb{R}^N)$ and $z \in \mathbb{R}^N$, then the following hold:*

(i) $\|u(\cdot + z) - u\|^p_{L^p(w^z)} \le \|z \cdot Du\|^p_{L^p(w^0)} + \omega(|z|)\, |z|^p\, \|Du\|^p_{L^p}$;

(ii) $\left|\|u(\cdot + z) - u\|_{L^p(w^z)} - \|z \cdot Du\|_{L^p(w^0)}\right| \le \dfrac{|z|^2}{2}\, \|D^2u\|_{L^p} + |z|\, \omega(|z|)^{1/p}\, \|Du\|_{L^p}$.

In the proof of Lemma 2.1 we exploit the following result. Here and below, we let $(\eta_j)_{j\in\mathbb{N}} \subset C^\infty_c(\mathbb{R}^N)$, $\eta_j = j^N \eta(j\,\cdot\,)$ for all $j \in \mathbb{N}$, be a family of non-negative mollifiers, with $\eta \in C^\infty_c(\mathbb{R}^N)$ such that $\eta \ge 0$, $\operatorname{supp}\eta \subset B_1$ and $\int_{\mathbb{R}^N} \eta \,\mathrm{d}x = 1$.

Lemma 2.2. *If $z \in \mathbb{R}^N$, $u \in \mathcal{S}^p(\mathbb{R}^N)$ and $u^j = u * \eta_j$ for all $j \in \mathbb{N}$, then*

$$\lim_{j\to\infty} \|z \cdot Du^j\|^p_{L^p(w^0)} = \|z \cdot Du\|^p_{L^p(w^0)}. \tag{2.2}$$

Proof. Let us set $D_z u = z \cdot Du$ for brevity. For $p = 1$, the validity of (2.2) follows from the fact that $|\eta_j * D_z u|\mathscr{L}^N \overset{\star}{\rightharpoonup} |D_z u|$ in $\mathscr{M}(\mathbb{R}^N)$ as $j \to \infty$ and $w^0 \in C_b(\mathbb{R}^N)$. For $p > 1$, instead, on the one hand, owing to Jensen's inequality and the Dominated Convergence Theorem, we easily infer that

$$\limsup_{j\to\infty} \|D_z u^j\|^p_{L^p(w^0)} \le \lim_{j\to\infty} \int_{\mathbb{R}^N} (\eta^j * w^0)\, |D_z u|^p \,\mathrm{d}x = \|D_z u\|^p_{L^p(w^0)}.$$

On the other hand, since $(\eta_j * D_z u)(x) \to D_z u(x)$ as $j \to \infty$ for a.e. $x \in \mathbb{R}^N$, owing to Fatou's Lemma, we also get that

$$\liminf_{j\to\infty} \int_{\mathbb{R}^N} |\eta_j * D_z u|^p\, w^0 \,\mathrm{d}x \ge \int_{\mathbb{R}^N} |D_z u|^p\, w^0 \,\mathrm{d}x,$$

concluding the proof of (2.2) in the case $p > 1$. □

Proof of Lemma 2.1. By Lemma 2.2 and an approximation argument, we can assume $u \in C^\infty(\mathbb{R}^N) \cap \mathcal{S}^p(\mathbb{R}^N)$ without loss of generality. We prove the two statements separately.

Proof of (i). We can estimate

$$\begin{aligned}
\int_{\mathbb{R}^N} |u(x+z) - u(x)|^p\, w^z(x) \,\mathrm{d}x &\le \int_0^1 \int_{\mathbb{R}^N} |Du(x+tz) \cdot z|^p\, w(x, x+z) \,\mathrm{d}x \,\mathrm{d}t \\
&\le \int_0^1 \int_{\mathbb{R}^N} |Du(x+tz) \cdot z|^p\, w(x+tz, x+tz) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \int_0^1 \int_{\mathbb{R}^N} |Du(x+tz) \cdot z|^p\, |w(x, x+z) - w(x+tz, x+tz)| \,\mathrm{d}x \,\mathrm{d}t \\
&\le \|z \cdot Du\|^p_{L^p(w^0)} + \omega(|z|)\, |z|^p \|Du\|^p_{L^p}
\end{aligned}$$

by the Fundamental Theorem of Calculus, Jensen's inequality and (1.3).

Proof of (ii). Letting $\phi(t) = u(x+tz)$ for every $t \in [0,1]$, we can write

$$\phi(1) = \phi(0) + \phi'(0) + \int_0^1 \phi''(t)\,(1-t)\,\mathrm{d}t.$$

As a consequence, we infer that

$$\begin{aligned}\Big| \|u(\cdot+z) - u\|_{L^p(w^0)} - \|z\cdot Du\|_{L^p(w^0)} \Big|^p &\le \|u(\cdot+z) - u - z\cdot Du\|^p_{L^p(w^0)} \\ &= \frac{1}{2^p}\int_{\mathbb{R}^N} \left| \int_0^1 \Big(D^2u(x+tz)[z]\cdot z\Big)\, 2(1-t)\,\mathrm{d}t\right|^p w^0(x)\,\mathrm{d}x \\ &\le \frac{|z|^{2p}}{2^p}\int_0^1 \Big(\int_{\mathbb{R}^N} |D^2u(x+tz)|^p\,\mathrm{d}x\Big)\, 2(1-t)\,\mathrm{d}t = \frac{|z|^{2p}}{2^p}\,\|D^2u\|^p_{L^p}.\end{aligned}$$

by Jensen's inequality. The conclusion hence follows by combining the simple inequality

$$\Big| \|u(\cdot+z) - u\|_{L^p(w^z)} - \|u(\cdot+z) - u\|_{L^p(w^0)} \Big| \le \|u(\cdot+z) - u\|_{L^p} \|w^z - w^0\|^{1/p}_{L^\infty}$$

with (1.3) and [19, Lem. 2.2(i)] (or, equivalently, statement (i) for $w = 1$). □

We can now prove the pointwise lim sup inequality in Theorem 1.1(i) in the case $w_k = w$ for every $k \in \mathbb{N}$ and $\|w\|_{L^\infty} = 1$.

Proof of Theorem 1.1(i). It is enough to show how the proof of [19, Th. 3.2(ii)] adapts to the present setting. From now on, the notation is the same one used in [19], with the obvious modifications. In particular, we let

$$\mathscr{F}^{w_k}_{k,p}(u;A) = \int_A \frac{\|u(\cdot+z) - u\|^p_{L^p(w^z_k)}}{|z|^p}\,\rho_k(z)\,\mathrm{d}z \tag{2.3}$$

for every $k \in \mathbb{N}$ and every measurable set $A \subset \mathbb{R}^N$. By Lemma 2.1(i), we can estimate

$$\mathscr{F}^{w}_{k,p}(u;B_{\delta_l}) \le \int_{\mathbb{S}^{N-1}} \|\sigma\cdot Du\|^p_{L^p(w^0)}\,\mathrm{d}\mu^l_k(\sigma) + \omega(\delta_l)\,\|Du\|^p_{L^p}$$

for every $k, l \in \mathbb{N}$ (recall the notation in [19, Lem. 2.9]). Since $\sigma \mapsto \|\sigma\cdot Du\|^p_{L^p(w^0)} \in C(\mathbb{S}^{N-1})$ and $\delta_l \to 0^+$ as $l \to \infty$, we then get

$$\lim_{l\to\infty}\limsup_{k\to\infty} \mathscr{F}^{w}_{k,p}(u;B_{\delta_l}) \le \lim_{l\to\infty}\lim_{k\to\infty}\int_{\mathbb{S}^{N-1}} \|\sigma\cdot Du\|^p_{L^p(w^0)}\,\mathrm{d}\mu^l_k(\sigma) = \mathscr{D}^{\mu}_{p,w}(u).$$

Moreover, since

$$z \mapsto f_{u,w}(z) = \frac{\|u(\,\cdot+z) - u\|^p_{L^p(w^z)}}{|z|^p} \in C(\mathbb{R}^N \setminus \{0\}), \tag{2.4}$$

we get that (observe that now $\nu = \alpha\delta_0$ for some $\alpha \ge 0$ by (1.2))

$$\lim_{l\to\infty}\limsup_{k\to\infty} \mathscr{F}^{w}_{k,p}(u;A_l) \le \lim_{l\to\infty}\lim_{k\to\infty}\int_{A_l} f_{u,w}\,\mathrm{d}\nu_k = \alpha\int_{\mathbb{R}^N\setminus\{0\}} f_{u,w}\,\mathrm{d}\delta_0 = 0.$$

Finally, we can estimate

$$\lim_{l\to\infty}\limsup_{k\to\infty} \mathscr{F}^{w}_{k,p}(u;B^c_{1/\delta_l}) \le 2^p\,\|u\|^p_{L^p}\lim_{l\to\infty}\limsup_{k\to\infty}\int_{B^c_{1/\delta_l}} \frac{\rho_k(z)}{|z|^p}\,\mathrm{d}z = 0,$$

and the conclusion readily follows. □

We now pass to proof of the Γ-lim inf inequality in Theorem 1.1(ii) in the case $w_k = w$ for every $k \in \mathbb{N}$ and $\|w\|_{L^\infty} = 1$. We let

$$w[\eta_j](x,y) = \int_{\mathbb{R}^N} w(x+z, y+z)\,\eta_j(z)\,\mathrm{d}z$$

for all $j \in \mathbb{N}$ and $x, y \in \mathbb{R}^N$. It is not difficult to see that, by (1.3),

$$\|w[\eta_j] - w\|_{L^\infty} \le |B_1|\,\|\eta\|_{L^\infty}\,\omega\left(\tfrac{2}{j}\right) \quad \text{for all } j \in \mathbb{N}. \tag{2.5}$$

In the proof of Theorem 1.1(ii), we also need the following lower semicontinuity result, which immediately follows from (2.2) owing to Fatou's Lemma.

Lemma 2.3. *If $u \in \mathcal{S}^p(\mathbb{R}^N)$ and $u^j = u * \eta_j$ for all $j \in \mathbb{N}$, then*

$$\liminf_{j\to\infty} \mathscr{D}^{\mu}_{p,w}(u^j) \ge \mathscr{D}^{\mu}_{p,w}(u).$$

Proof of Theorem 1.1(ii). It is enough to show how the proof of [19, Th. 3.2(iii)] adapts to the present setting. From now on, the notation is the same one used in [19], with the obvious modifications (also recall the notation (2.3)). We set $u^j_k = u_k * \eta_j$ and $u^j = u * \eta_j$ for every $k, j \in \mathbb{N}$. We replace the use of Minkowski's inequality at the beginning of the proof of [19, Th. 3.2(iii)] with

$$\mathscr{F}^w_{k,p}(u_k) \ge \mathscr{F}^w_{k,p}(u^j_k) - |B_1|\,\|\eta\|_{L^\infty}\,\omega\left(\tfrac{2}{j}\right)\mathscr{F}^1_{k,p}(u_k)$$

for all $k, j \in \mathbb{N}$, which readily follows by combining (2.1) and (2.5). Hence

$$\liminf_{k\to\infty} \mathscr{F}^w_{k,p}(u_k) \ge \liminf_{k\to\infty} \mathscr{F}^w_{k,p}(u^j_k) - C_\eta\,\omega\left(\tfrac{2}{j}\right)\limsup_{k\to\infty} \mathscr{F}^1_{k,p}(u_k) \tag{2.6}$$

for all $j \in \mathbb{N}$. As in [19], we now just have to show that

$$\lim_{k\to\infty} \mathscr{F}^w_{k,p}(u^j_k) = \mathscr{D}^{\mu}_{p,w}(u^j) \tag{2.7}$$

for every $j \in \mathbb{N}$, from which the conclusion follows by letting $j \to \infty$ in (2.6), thanks to Lemma 2.3. The proof of (2.7) goes as in [19] so we only sketch it. We first notice that

$$\lim_{l\to\infty}\lim_{k\to\infty} \mathscr{F}^w_{k,p}(u^j_k; B^c_{1/\delta_l}) \le 2^p\,M\,\|u^j\|^p_{L^p}\lim_{l\to\infty}\delta^p_l = 0$$

for every $j \in \mathbb{N}$, where $M > 0$ depends on (1.1) only. Moreover, with the same notation in (2.4), since $f_{u^j_k,w} \to f_{u^j,w}$ locally uniformly on $\mathbb{R}^N \setminus \{0\}$ as $k \to \infty$, we also have that (recall that now $\nu = \alpha\delta_0$ for some $\alpha \ge 0$ by (1.2))

$$\lim_{l\to\infty}\lim_{k\to\infty} \mathscr{F}^w_{k,p}(u^j_k; A_l) = \lim_{l\to\infty}\lim_{k\to\infty}\int_{A_l} f_{u^j_k}\,\mathrm{d}\nu_k = \alpha\int_{\mathbb{R}^N\setminus\{0\}} f_{u^j}\,\mathrm{d}\delta_0 = 0$$

for every $j \in \mathbb{N}$. Besides, since $\sigma \mapsto \|\sigma \cdot Du^j_k\|^p_{L^p(w^0)}$ converges uniformly on $\mathbb{S}^{N-1}$ to $\sigma \mapsto \|\sigma \cdot Du^j\|^p_{L^p(w^0)}$ as $k \to \infty$ for every $j \in \mathbb{N}$, we can infer that

$$\begin{aligned}\lim_{l\to\infty}\lim_{k\to\infty}\int_{B_{\delta_l}} \left\|\tfrac{z}{|z|}\cdot Du^j_k\right\|^p_{L^p(w^0)}\mathrm{d}\nu_k(z) &= \lim_{l\to\infty}\lim_{k\to\infty}\int_{\mathbb{S}^{N-1}} \|\sigma\cdot Du^j_k\|^p_{L^p(w^0)}\,\mathrm{d}\mu^l_k(\sigma)\\ &= \lim_{l\to\infty}\int_{\mathbb{S}^{N-1}} \|\sigma\cdot Du^j\|^p_{L^p(w^0)}\,\mathrm{d}\mu^l(\sigma) = \int_{\mathbb{S}^{N-1}} \|\sigma\cdot Du^j\|^p_{L^p(w^0)}\,\mathrm{d}\mu(\sigma)\end{aligned}$$

for every $j \in \mathbb{N}$ (recall the notation in [19, Lem. 2.9]). Hence the claim in (2.7) reduces to

$$\lim_{l\to\infty}\lim_{k\to\infty} \mathscr{F}^w_{k,p}(u^j_k; B_{\delta_l}) = \int_{\mathbb{S}^{N-1}} \|\sigma\cdot Du^j\|^p_{L^p(w^0)}\,\mathrm{d}\mu(\sigma)$$

or, equivalently,

$$\lim_{l\to\infty}\lim_{k\to\infty}\left|\mathscr{F}^w_{k,p}(u^j_k; B_{\delta_l}) - \int_{B_{\delta_l}} \left\|\tfrac{z}{|z|}\cdot Du^j_k\right\|^p_{L^p(w^0)}\,\mathrm{d}\nu_k(z)\right| = 0 \tag{2.8}$$

for every $j \in \mathbb{N}$. To show (2.8), we observe that, by Lemma 2.1(ii), we can estimate

$$\left|\frac{\|u^j_k(\,\cdot+z) - u^j_k\|_{L^p(w^z)}}{|z|} - \left\|\tfrac{z}{|z|}\cdot Du^j_k\right\|_{L^p(w^0)}\right| \le \frac{|z|}{2}\,\|D^2u^j_k\|_{L^p} + \omega(|z|)^{1/p}\,\|Du^j_k\|_{L^p}$$

for every $z \in \mathbb{R}^N$. Hence, since $|a^p - b^p| \le p\max\{a,b\}^{p-1}|a-b|$, for all $a, b \ge 0$, and

$$\max\left\{\frac{\|u^j_k(\,\cdot+z) - u^j_k\|_{L^p(w^z)}}{|z|}, \left\|\tfrac{z}{|z|}\cdot Du^j_k\right\|_{L^p(w^0)}\right\} \le \|Du^j_k\|_{L^p}$$

for every $k, j \in \mathbb{N}$ and $z \in \mathbb{R}^N$ by Lemma 2.1(i), we can estimate

$$\begin{aligned}
&\left|\frac{\|u^j_k(\,\cdot+z) - u^j_k\|^p_{L^p(w^z)}}{|z|^p} - \left\|\tfrac{z}{|z|}\cdot Du^j_k\right\|^p_{L^p(w^0)}\right| \\
&\qquad\le p\,\|Du^j_k\|^{p-1}_{L^p}\left|\frac{\|u^j_k(\,\cdot+z) - u^j_k\|_{L^p(w^z)}}{|z|} - \left\|\tfrac{z}{|z|}\cdot Du^j_k\right\|_{L^p(w^0)}\right| \\
&\qquad\le \frac{p|z|}{2}\,\|Du^j_k\|^{p-1}_{L^p}\,\|D^2u^j_k\|_{L^p} + p\,\omega(|z|)^{1/p}\,\|Du^j_k\|^p_{L^p},
\end{aligned}$$

for every $i, j \in \mathbb{N}$ and $z \in \mathbb{R}^N$. We thus obtain that

$$\begin{aligned}
\left|\mathscr{F}_{t_k,p}(u^j_k; B_{\delta_l}) - \int_{B_{\delta_l}} \left\|\tfrac{z}{|z|}\cdot Du^j_k\right\|^p_{L^p}\,\mathrm{d}\nu_k(z)\right| &\le \frac{p}{2}\,\|Du^j_k\|^{p-1}_{L^p}\,\|D^2u^j_k\|_{L^p}\,\delta_l\,\nu_k(B_{\delta_l}) \\
&\quad + p\,\omega(\delta_l)^{1/p}\|Du^j_k\|^p_{L^p}\,\nu_k(B_{\delta_l})
\end{aligned}$$

for every $k, j, l \in \mathbb{N}$, from which the claimed (2.8) readily follows, concluding the proof. □

We finally prove Theorem 1.1(iii) by reducing to Theorem 1.1(ii) by means of the following simple estimate.

Lemma 2.4. *Assume that $w > 0$ on $\mathbb{R}^{2n}$, let $R > 0$ and set*

$$\ell_R = \inf\{w(x, x+z) : x \in B_R,\ z \in B_{2R}\} > 0. \tag{2.9}$$

If $u \in L^p_R(\mathbb{R}^N)$ and $k \in \mathbb{N}$ is such that $\|w_k - w\|_{L^\infty} \le \frac{\ell_R}{2}$, then

$$\mathscr{F}^1_{k,p}(u) \le \frac{2}{\ell_R}\,\mathscr{F}^{w_k}_{k,p}(u) + 2^p\|u\|^p_{L^p}\int_{B^c_{2R}} \frac{\rho_k(z)}{|z|^p}\,\mathrm{d}z.$$

Proof. Note that, if $z \in B_{2R}$ and $x \in B^c_R$, then also $x + z \in B^c_R$. Therefore, we have

$$\begin{aligned}
\mathscr{F}^1_{k,p}(u) &\le \int_{B_{2R}} \frac{\|u(\cdot+z) - u\|^p_{L^p}}{|z|^p}\,\rho_k(z)\,\mathrm{d}z + 2^p\|u\|^p_{L^p}\int_{B^c_{2R}} \frac{\rho_k(z)}{|z|^p}\,\mathrm{d}z \\
&= \int_{B_{2R}} \frac{\|u(\cdot+z) - u\|^p_{L^p(B_R)}}{|z|^p}\,\rho_k(z)\,\mathrm{d}z + 2^p\|u\|^p_{L^p}\int_{B^c_{2R}} \frac{\rho_k(z)}{|z|^p}\,\mathrm{d}z.
\end{aligned}$$

The conclusion hence follows in virtue of the definition in (2.9), since the assumption that $\|w_k - w\|_{L^\infty} \le \frac{\ell_R}{2}$ implies that

$$w_k(x, x+z) \ge w(x, x+z) - |w_k(x, x+z) - w(x, x+z)| \ge \ell_R - \frac{\ell_R}{2} = \frac{\ell_R}{2}$$

for every $x \in B_R$ and $z \in B_{2R}$. $\square$

Proof of Theorem 1.1(iii). Without loss of generality and possibly passing to a non-relabelled subsequence, we can assume that

$$\liminf_{k\to\infty} \mathscr{F}^{w_k}_{k,p}(u_k) = \lim_{k\to\infty} \mathscr{F}^{w_k}_{k,p}(u_k) < \infty.$$

By applying Lemma 2.4 to each $u_k \in L^p_R(\mathbb{R}^N)$ for every $k \in \mathbb{N}$ sufficiently large depending on $R > 0$, we get that (where $\ell_R > 0$ is defined as in (2.9) in Lemma 2.4)

$$\limsup_{k\to\infty} \mathscr{F}^1_{k,p}(u_k) \le \frac{2}{\ell_R} \lim_{k\to\infty} \mathscr{F}^{w_k}_{k,p}(u_k) + \frac{M}{R^p} \|u\|^p_{L^p} < \infty,$$

where $M > 0$ depends on (1.1) only. The conclusion hence follows by Theorem 1.1(ii). $\square$

2.2. **Proof of Theorem 1.3.** The proof of Theorem 1.3 is a simple combination of Definition 1.2 and Lemma 2.4.

Proof of Theorem 1.3. Let $R > 0$ be fixed and let $\ell_R > 0$ be given by (2.9). Let $k_R \in \mathbb{N}$ be such that $\|w_k - w\|_{L^\infty} \le \frac{\ell_R}{2}$ for every $k \ge k_R$. By Lemma 2.4, we can thus estimate

$$\sup_{k\ge k_R} \mathscr{F}^1_{k,p}(u_k) \le \frac{2}{\ell_R} \sup_{k\ge k_R} \mathscr{F}^{w_k}_{k,p}(u_k) + M \sup_{k\in\mathbb{N}} \|u_k\|_{L^p} < \infty,$$

where $M > 0$ depends on (1.1) only. Therefore, we get that

$$\sup_{k\in\mathbb{N}} \left(\|u_k\|_{L^p} + \mathscr{F}^1_{k,p}(u_k) \right) < \infty$$

and thus, by Definition 1.2, we infer that $(u_k)_{k\in\mathbb{N}}$ is compact in $L^p(E)$ for every compact set $E \subset \mathbb{R}^N$ and any of its $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ limits is in $\mathcal{S}^p(\mathbb{R}^N)$. The conclusion hence immediately follows by recalling that $(u_k)_{k\in\mathbb{N}} \subset \mathcal{S}^p_R(\mathbb{R}^N)$. $\square$

2.3. **Proof of Theorem 1.8.** We establish Theorem 1.8 by exploiting Corollary 1.5.

Proof of Theorem 1.8. By contradiction, we can assume that there exists $\varepsilon > 0$ and $(u_k)_{k\in\mathbb{N}} \subset L^p_0(\Omega)$ such that $\|u_k\|^p_{L^p} = 1$ for all $k \in \mathbb{N}$ and

$$\mathscr{F}^{w_k}_{k,p}(u_k) < \frac{1}{A_{p,\mu,\Omega,w} + \varepsilon}$$

for all $k \in \mathbb{N}$. By (i) and (ii) in Corollary 1.5, up to passing to a subsequence, there exists $u \in \mathcal{S}^p_0(\Omega)$ such that $u_k \to u$ in $L^p(\mathbb{R}^N)$ as $k \to \infty$ and

$$\mathscr{D}^\mu_{p,w}(u) \le \liminf_{k\to\infty} \mathscr{F}^{w_k}_{k,p}(u_k) \le \frac{1}{A_{p,\mu,\Omega,w} + \varepsilon}.$$

This, in turn, in combination with (1.16) with the optimal constant $C = A_{p,\mu,\Omega,w}$, implies that $A_{p,\mu,\Omega,w} + \varepsilon < A_{p,\mu,\Omega,w}$, which is a contradiction, since $\varepsilon > 0$ by assumption. $\square$

2.4. **Proof of Theorem 1.9.** The proof of Theorem 1.9 is a plain application of Corollary 1.5 and exploits a simple argument which is at heart of many results of [17, 24].

Proof of Theorem 1.9. We split the proof into two parts, proving the convergence of the eigenvalues in (1.19) and of the eigenfunctions in (1.20) separately.

Part 1: proof of (1.19). On the one hand, given $\varepsilon > 0$, we can find $u \in \mathcal{S}^p_0(\Omega)$ such that $\|u\|_{L^p} = 1$ and

$$\mathscr{D}^\mu_{p,w}(u) \le \lambda^\mu_{p,w}(\Omega) + \varepsilon.$$

Since $\lambda^{w_k}_{k,p}(\Omega) \le \mathscr{F}^{w_k}_{k,p}(u)$ for all $k \in \mathbb{N}$, by Corollary 1.5(iii) we thus infer that

$$\limsup_{k\to\infty} \lambda^{w_k}_{k,p}(\Omega) \le \limsup_{k\to\infty} \mathscr{F}^{w_k}_{k,p}(u) \le \mathscr{D}^\mu_{p,w}(u) \le \lambda^\mu_{p,w}(\Omega) + \varepsilon$$

from which we immediately get that

$$\limsup_{k\to\infty} \lambda^{w_k}_{k,p}(\Omega) \le \lambda^\mu_{p,w}(\Omega). \tag{2.10}$$

In particular, we deduce that

$$\sup_{k\in\mathbb{N}} \lambda^{w_k}_{k,p}(\Omega) < \infty. \tag{2.11}$$

On the other hand, for each $k \in \mathbb{N}$, we can find $u_k \in L^p_0(\Omega)$ with $\|u_k\|_{L^p} = 1$ such that

$$\mathscr{F}^{w_k}_{k,p}(u_k) \le \lambda^{w_k}_{k,p}(\Omega) + \frac{1}{k}. \tag{2.12}$$

Owing to (2.11), we thus get that $(u_k)_{k\in\mathbb{N}} \subset L^p_0(\Omega)$ satisfies (1.13). Therefore, by (i) and (ii) in Corollary 1.5, there exist $(u_{k_j})_{j\in\mathbb{N}}$ and $u \in \mathcal{S}^p(\Omega)$ such that $u_{k_j} \to u$ in $L^p(\mathbb{R}^N)$ as $k \to \infty$, and thus $u \in \mathcal{S}^p_0(\Omega)$ with $\|u\|_{L^p} = 1$, and, also owing to (2.12) and (2.10),

$$\mathscr{D}^\mu_{p,w}(u) \le \liminf_{j\to\infty} \mathscr{F}^{w_{k_j}}_{k_j,p}(u_{k_j}) \le \liminf_{k\to\infty} \mathscr{F}^{w_k}_{k,p}(u_k) \le \liminf_{k\to\infty} \lambda^{w_k}_{k,p}(\Omega) \le \lambda^\mu_{p,w}(\Omega). \tag{2.13}$$

Since $u \in \mathcal{S}^p_0(\Omega)$ is such that $\|u\|_{L^p} = 1$, we must have that $\lambda^\mu_{p,w}(\Omega) \le \mathscr{D}^\mu_{p,w}(u)$ and thus all inequalities in (2.13) are equalities, yielding

$$\lambda^\mu_{p,w}(\Omega) = \liminf_{k\to\infty} \lambda^{w_k}_{k,p}(\Omega) \tag{2.14}$$

which, combined with (2.10), yields (1.19).

Part 2: proof of (1.20). By assumption, we have $\|u_k\|_{L^p} = 1$ and $\mathscr{F}^{w_k}_{k,p}(u_k) = \lambda^{w_k}_{k,p}(\Omega)$ for all $k \in \mathbb{N}$. Owing to (1.19), we thus infer the validity of (1.13), and so, by Corollary 1.5(i), there exist $(u_{k_j})_{j\in\mathbb{N}}$ and $u \in \mathcal{S}^p_0(\Omega)$ satisfying (1.20). We are thus left to prove that $u \in \mathcal{S}^p_0(\Omega)$ is an eigenfunction relative to $\lambda^\mu_{p,w}(\Omega)$. Indeed, by (1.20), we know that $\|u\|_{L^p} = 1$. Moreover, by Corollary 1.5(ii), we also infer that

$$\mathscr{D}^\mu_{p,w}(u) \le \liminf_{j\to\infty} \mathscr{F}^{w_{k_j}}_{k_j,p}(u_{k_j}) \le \liminf_{k\to\infty} \mathscr{F}^{w_k}_{k,p}(u_k) = \liminf_{k\to\infty} \lambda^{w_k}_{k,p}(\Omega) = \lambda^\mu_{p,w}(\Omega), \tag{2.15}$$

from which we obtain $\mathscr{D}^\mu_{p,w}(u) = \lambda^\mu_{p,w}(\Omega)$, concluding the proof. □

2.5. **Proof of Theorem 1.11.** We now pass to the proof of Theorem 1.11. Since the argument follows the one of the proof of [19, Th. 5.1] with minor changes, we only sketch it for the ease of the reader.

Proof of Theorem 1.11. We prove the three statements separately.

Proof of (i). Let $E \subset A$ be a compact set and let $\varepsilon > 0$. Let $\delta > 0$ be given by (1.21) and also such that $E_\delta \subset A$, where

$$E_\delta = \bigcup_{x \in E} B_\delta(x).$$

Letting $\eta_\delta = \chi_{B_\delta}/|B_\delta|$ and arguing as in the proof of [19, Lem. 5.3], we can estimate

$$\begin{aligned} \|\eta_\delta * v - v\|^p_{L^p(E)} &\le \frac{1}{|B_\delta|} \int_E \int_{B_\delta} |v(x+z) - v(x)|^p \,\mathrm{d}z \,\mathrm{d}x \\ &= \frac{1}{|B_1|} \int_E \int_{B_\delta(x)} \frac{|v(y) - v(x)|^p}{\delta^N} \,\mathrm{d}y \,\mathrm{d}x \\ &\le \frac{\varepsilon}{|B_1|} \int_E \int_{B_\delta(x)} |v(y) - v(x)|^p \, J_k(x,y) \,\mathrm{d}y \,\mathrm{d}x \le \varepsilon \mathscr{J}_{k,p}(v) \end{aligned} \tag{2.16}$$

for every $v \in L^p(\mathbb{R}^N)$ and $k \in \mathbb{N}$, owing to (1.21). Now, if $(u_k)_{k\in\mathbb{N}} \subset L^p(A)$ satisfies (1.22), then the sequence $(v_k^E)_{k\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$, where, for all $k \in \mathbb{N}$,

$$v_k^E = \begin{cases} u_k & \text{on } E, \\ 0 & \text{otherwise,} \end{cases}$$

is bounded in $L^p(\mathbb{R}^N)$. Therefore, by [7, Cor. 4.28], for every $\delta > 0$ sufficiently small, $(\eta_\delta * v_k^E)_{k\in\mathbb{N}}$ is compact in $L^p(E)$, and thus totally bounded in $L^p(E)$. By (2.16), also $(u_k)_{k\in\mathbb{N}} \subset L^p(E)$ is thus totally bounded in $L^p(E)$, and therefore compact in $L^p(E)$.

Proof of (ii). If $(u_k)_{k\in\mathbb{N}} \subset L^p(A)$ is such that $u_k \to u$ in $L^p(A)$ as $k \to \infty$, then, up to a subsequence, by (1.23) we get that

$$\liminf_{k\to\infty} |u_k(x) - u_k(y)|^p \, J_k(x,y) \ge |u(x) - u(y)|^p \, J(x,y)$$

for $\mathscr{L}^{2n}$-a.e. $(x,y) \in A^2$, so that the conclusion follows by Fatou's Lemma.

Proof of (iii). If $u \in L^p(A)$ is such that $\mathscr{J}_p(u) < \infty$, then

$$(x,y) \mapsto |u(x) - u(y)|^p J_p(x,y) \in L^1(A^2).$$

Therefore, by (1.24), the conclusion follows either by the Dominated Convergence Theorem under (1.25), or by the Monotone Convergence Theorem under (1.26). ∎

2.6. **Proof of Corollary 1.12.** We conclude with a brief sketch the proof of Corollary 1.12 for the ease of the reader.

Proof of Corollary 1.12. To prove statements (ii) and (iii) of Corollary 1.12, we can directly apply the corresponding statements (ii) and (iii) of Theorem 1.11 with the choices

$$A = \mathbb{R}^N, \quad J_k(x,y) = \frac{\rho_k(x-y)}{|x-y|^p} \, w_k(x,y) \quad \text{and} \quad J(x,y) = \kappa(x-y) \, w(x,y)$$

for all $x, y \in \mathbb{R}^N$ and $k \in \mathbb{N}$, since clearly (1.28) implies the validity of (1.24) and (1.25). To prove statement (i) of Corollary 1.12, instead, it is enough to observe that

$$\inf\{w_k(x,y) : x \in B_R,\ y \in B_1(x)\} \geq \frac{1}{2} \inf\{w(x,y) : x \in B_R,\ y \in B_1(x)\} > 0$$

for all $k \in \mathbb{N}$ sufficiently large whenever $R > 0$ is fixed, and then argue as in the proof of statement (i) of Theorem 1.11. We omit the analogous computations. □

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