

On manifolds with almost non-negative Ricci curvature and integrally-positive k^{th} -scalar curvature

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Abstract

We consider manifolds with almost non-negative Ricci curvature and strictly positive integral lower bounds on the sum of the lowest k eigenvalues of the Ricci tensor.

If (M^n, g) is a Riemannian manifold satisfying such curvature bounds for $k = 2$, then we show that M is contained in a neighbourhood of controlled width of an isometrically embedded 1-dimensional sub-manifold. From this, we deduce several metric and topological consequences: M has at most linear volume growth and at most two ends, the first Betti number of M is bounded above by 1, and precise information on elements of infinite order in $\pi_1(M)$.

If (M^n, g) is a Riemannian manifold satisfying such bounds for $k \geq 2$ and additionally the Ricci curvature is asymptotically non-negative, then we show that M has at most $(k - 1)$ -dimensional behavior at large scales.

If $k = n = \dim(M)$, so that the integral lower bound is on the scalar curvature, assuming in addition that the $n - 2$ -Ricci curvature is asymptotically non-negative, then we prove that the dimension drop at large scales improves to $n - 2$.

From the above results we deduce topological restrictions, such as upper bounds on the first Betti number.

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1 Introduction

A renowned conjecture by Yau asserts that if (M^n, g, p) is a pointed Riemannian manifold of dimension n with $\text{Ric}_M \geq -\lambda$, then

$$\int_{B_1(p)} \text{R dVol} \leq C(\lambda, n),$$

where R denotes the scalar curvature. Under the assumption of non-negative sectional curvature, the aforementioned bound on the scalar curvature was obtained by Petrunin in [45]. A non-collapsed version of Yau's problem was then proposed by Naber in [44], where he conjectured that if (M^n, g, p) is a pointed Riemannian manifold with $\text{Ric}_M \geq -\lambda$ and $\text{Vol}(B_1(p)) \geq v$, then $\int_{B_1(p)} \text{R dVol} \leq C(\lambda, n, v)$. If true, Naber's version of Yau conjecture would allow to define a scalar curvature measure on non-collapsed Ricci limit spaces (see again [44]).

These conjectures are still open and, more generally, the geometric interplay between Ricci and scalar curvature bounds is still not fully understood. A step forward in this direction was carried out in a series of recent papers showing that 3-manifolds with non-negative Ricci curvature and scalar curvature bounded from below by a strictly positive constant have at most linear volume growth at infinity (see [43, 19]). This is a particular case of a more general conjecture by Gromov [29] stating that a non-compact pointed Riemannian manifold (M^n, g, p) of dimension n with non-negative Ricci curvature and scalar curvature bounded below by a strictly positive constant should satisfy

$$\limsup_{t \rightarrow +\infty} \frac{\text{Vol}(B_t(p))}{t^{n-2}} < +\infty. \quad (1)$$

Without the condition on the Ricci curvature, manifolds with uniformly positive scalar curvature can have arbitrarily large volume growth (by the surgery results in [31, 30]). Nevertheless such 3-manifolds have been shown to have bounded Urysohn 1-width (see [35, 32, 39, 40] and, assuming also non-negative Ricci curvature, [60]). This class of results enters in the framework of another more general conjecture by Gromov [32], stating that a manifold of dimension n with scalar curvature bounded below by a strictly positive constant should have bounded Urysohn $(n-2)$ -width.

Motivated and inspired by the above conjectures and results, we next discuss the framework and the main findings of the present work.

Given a manifold (M^n, g) and $k \in \mathbb{N}$, $k \leq n$, we denote by R_k the sum of the lowest k eigenvalues, counted with their multiplicity, of the Ricci tensor Ric_M . In analogy with the terminology used in [48], if $R_k \geq c$, we say that the k^{th} -scalar curvature of M is bounded below by c (note that this is different from the convention used in [58], where the Ricci curvature is said to be k -positive if $R_k \geq 0$). In this paper, we study the geometry of manifolds with almost non-negative Ricci curvature and k^{th} -scalar curvature bounded from below in integral sense by a positive constant (see [22, 37, 54, 24, 28] for previous literature on manifolds with $R_k \geq c$). As a byproduct, we also obtain new results on manifolds with almost non-negative Ricci curvature and a positive lower integral bound on the scalar curvature.

Broadly speaking, we show that a lower Ricci curvature bound combined with a positive lower bound on the k^{th} -scalar curvature forces the manifold to have $(k-1)$ -dimensional behavior at large scales. Moreover, if k is the dimension of the manifold (so that the lower bound is on the scalar curvature), the dimensional bound on the geometry at large scales can be improved to $(k-2)$, matching in this way the dimension-drop appearing in the aforementioned conjectures by Gromov. Our approach relies on techniques from metric geometry in which a limit space arises from a sequence of manifolds (see the foundational papers [9, 10, 11, 12]).

The first main result of the paper is Theorem 1.1, which shows the continuity of certain integral functionals of R_k under Gromov-Hausdorff convergence. We recall that, if Yau conjecture holds, given a sequence $(M_i, g_i, p_i) \rightarrow (X, d, x)$, one can define a scalar curvature measure μ on X as a weak limit $\mathbf{R}_{M_i} \text{Vol}_i \rightarrow \mu$ (see [44, Conjecture 2.18]). Hence, continuity results for $\int R$ under GH convergence would ensure that, if X is itself a smooth manifold, then $\mu = \mathbf{R}_X \text{Vol}$. Theorem 1.1 proves a weaker result along these lines when $X = \mathbb{R}^n$ (see Remark 3.5). Other continuity results for the integral of the scalar curvature were previously obtained in the presence of a lower bound on the sectional curvature in [46, Theorem B3] (after Perelman) and [38]. We refer to Definition 2.5 for the definition of (k, δ) -symmetric and δ -regular balls in a manifold. Given $a, b \in \mathbb{R}$, we denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

Theorem 1.1. *Let $L, k, n \in \mathbb{N}$ with $k \leq n$ and $s \in (0, 1)$ be fixed. Then*

$$\sup \left\{ \int_{B_1(x)} \mathbf{R}_k \wedge L \, d\text{Vol} : (M, g, x) \text{ has } \text{Ric}_M \geq -\delta, \text{ and } B_{10}(x) \text{ is } (k, \delta)\text{-symmetric} \right\},$$

$$\sup \left\{ \int_{B_1(x)} |\mathbf{R}|^s \, d\text{Vol} : (M, g, x) \text{ has } \text{Ric}_M \geq -\delta, \text{ and } B_{10}(x) \text{ is } \delta\text{-regular} \right\}$$

converge to 0 as $\delta \searrow 0$.

The previous result is the technical tool that will be used to study manifolds with a positive uniform lower bound on \mathbf{R}_k .

As a first application of Theorem 1.1, in Theorem 1.2 we show that manifolds of any dimension with integrally positive \mathbf{R}_2 (and almost non-negative Ricci curvature) are contained in a neighbourhood of controlled width of an isometrically embedded 1-dimensional manifold. The result applies also to 3-manifolds with integrally positive scalar curvature and, assuming either an integral sectional curvature lower bound or a strong non-collapsing assumption, gives a new precise description of the macroscopic behavior of these spaces. Let $v, \epsilon, \delta, L \in (0, +\infty)$, $n \in \mathbb{N}$, $s \in (0, 1)$, and consider the following sets of conditions on a manifold (M^n, g) .

(i) $n \geq 2$, $\text{Ric}_M \geq -\delta$, and

$$\int_{B_1(x)} \mathbf{R}_2 \wedge L \, d\text{Vol} \geq \epsilon, \quad \forall x \in M.$$

(ii) $n = 3$, $\text{Ric}_M \geq -\delta$, and

$$\int_{B_1(x)} \text{Sec}_M \wedge 0 \, d\text{Vol} \geq -\delta, \quad \int_{B_1(x)} \mathbf{R}_M \wedge L \, d\text{Vol} \geq \epsilon, \quad \forall x \in M.$$

(iii) $n = 3$, $\text{Ric}_M \geq -\delta$, and

$$\int_{B_1(x)} |\mathbf{R}_M|^s \, d\text{Vol} \geq \epsilon, \quad \text{Vol}(B_1(x)) \geq v, \quad \forall x \in M.$$

Given $r > 0$ and a subset $A \subset M$, the r -neighbourhood of A in M is the set of points whose distance from A is less than r .

Theorem 1.2. *Let $v, \epsilon, L \in (0, +\infty)$, $n \in \mathbb{N}$, $s \in (0, 1)$ be fixed. There exist $C, \delta > 0$ with the following property. If a manifold (M^n, g) satisfies one of the conditions (i), (ii), (iii), then there exists a 1-dimensional connected manifold I (possibly with boundary) and a distance preserving map $\phi : I \rightarrow M$, such that M is contained in the C -neighbourhood of $\phi(I)$.*

Theorem 1.2 has far reaching implications both for the metric and the topological properties of manifolds satisfying (i), (ii), or (iii). These are collected in Theorem 1.3 below. We denote by $b_1(M)$ the first Betti number of M . Given a loop γ in M , we denote by $[\gamma]$ its equivalence class in the fundamental group $\pi_1(M)$.

Theorem 1.3. *Let $v, \epsilon, L \in (0, +\infty)$, $n \in \mathbb{N}$, $s \in (0, 1)$ be fixed. There exist $C, \delta > 0$ with the following property. Let (M^n, g, p) be a manifold satisfying one of the conditions (i), (ii), (iii). Then,*

1. $\sup_{x \in M} \text{Vol}(B_t(x)) \leq Ct$, for all $t > 0$.
2. If $\text{Ric}_M \geq 0$, then $\inf_{x \in M} \text{Vol}(B_1(x)) > 0$.
3. M has at most two ends.
4. $b_1(M) \leq 1$.

5. $\pi_1(M)$ is infinite if and only if M is compact and its universal cover is non-compact.
6. If there exists a loop $\gamma \subset M$ such that $[\gamma] \in \pi_1(M)$ has infinite order, then M is contained in a C -neighbourhood of γ .

The previous two theorems are part of a series of recent results where positive lower bounds on the (scalar) curvature imply 1-dimensional behavior at large scales both from the metric and the topological viewpoint. For instance, a linear volume growth bound for 3-manifolds with $\text{Ric} \geq 0$ and $R \geq 1$ was obtained in [43], and was then generalized in [19, 4, 59]. Similarly, 3-manifolds with $R \geq 1$ have been recently shown to have bounded Urysohn 1-width (see [35, 32, 39, 40]).

We highlight that an analog of Theorem 1.2 is not to be expected in the higher dimensional case (see Remark 5.9) and that the result fails without the lower bounds on the Ricci curvature (see Remark 5.7). The proof of Theorem 1.2 relies on the study of a new geometric condition on length minimizing geodesics in a metric space (see Section 4).

We now turn our attention to manifolds with an integral lower bound on R_k , possibly for $k \geq 3$. Theorem 1.4 shows that asymptotically non-negative Ricci curvature (see Remark 3.9), coupled with a sufficiently slow asymptotic integral decay of R_k , implies a dimension-drop for the tangent cone at infinity. An analogous result (under different assumptions) was recently proved in [62, Theorem 1.6]. We recall that if a pointed non-compact manifold (M^n, g, p) has non-negative Ricci curvature, then Gromov's pre-compactness theorem ensures that, for every sequence $(M^n, g/r_j, p)$ as $r_j \rightarrow +\infty$, there exists a subsequence converging in pointed Gromov-Hausdorff sense to a metric space (X, d, p_∞) . Such space obtained via blow-down is called a *tangent cone at infinity* of M . A similar procedure is possible when the manifold has only asymptotically non-negative Ricci curvature (see again Remark 3.9). Finally, we refer to Section 3.2 for the definition of the $(n-2)^{\text{th}}$ -Ricci curvature Ric_{n-2} (after [48]).

Theorem 1.4. *Let (M^n, g, p) be a Riemannian manifold and let $L > 0$, $\alpha \in (0, 2)$.*

1. *Assume that*

$$\liminf_{x \rightarrow \infty} d(p, x)^2 \text{Ric}(x) \wedge 0 = 0 \quad \text{and} \quad \lim_{r \rightarrow +\infty} \int_{B_r(p)} 0 \vee (r^{2-\alpha} R_k) \wedge L \, d\text{Vol} = L.$$

Let (X, d, x_∞) be any tangent cone at infinity of M . If (\mathbb{R}^d, d_{eu}) is a tangent space to X at a point $x \neq x_\infty$, then $d \leq k - 1$.

2. *Assume that*

$$\liminf_{x \rightarrow \infty} d(p, x)^2 \text{Ric}_{n-2}(x) \wedge 0 = 0 \quad \text{and} \quad \lim_{r \rightarrow +\infty} \int_{B_r(p)} 0 \vee (r^{2-\alpha} R) \wedge L \, d\text{Vol} = L.$$

Let (X, d, x_∞) be any tangent cone at infinity of M . If (\mathbb{R}^d, d_{eu}) is a tangent space to X at a point $x \neq x_\infty$, then $d \leq n - 2$.

The previous theorem is sharp in the sense that it fails for $\alpha = 0$ (see Remark 3.16). A biproduct of Theorem 1.4 is that, if a manifold has asymptotically non-negative Ricci curvature and sufficiently slow integral asymptotic decay of the Ricci tensor, then it is compact (see Remark 3.17).

We now consider the topological implications of almost non-negative Ricci curvature and positive k^{th} -scalar curvature. This is the content of Theorems 1.5 and 1.6 (compare with [62, Theorems 1.3 and 1.7]).

Theorem 1.5. *Let (M^n, g, p) be a non-compact Riemannian manifold with $n \geq 3$ and let $\alpha \in (0, 2)$. Let $2 \leq k \leq n$ be a natural number.*

1. *If $\text{Ric}_M \geq 0$ and*

$$\limsup_{r \rightarrow +\infty} \left(r^{2-\alpha} \min_{B_r(p)} R_k \right) > 0,$$

then $b_1(M) \leq k - 2$.

2. If $\text{Ric}_{n-2} \geq 0$ and

$$\limsup_{r \rightarrow +\infty} \left(r^{2-\alpha} \min_{B_r(p)} \mathbf{R} \right) > 0,$$

then $b_1(M) \leq n - 3$.

The proof of Theorem 1.5 follows the lines of [62, Theorem 1.3] by replacing the use of [62, Theorem 1.6] with Theorem 1.4. We remark that Theorem 1.5, unlike [62, Theorem 1.3], allows for the scalar curvature to converge to zero at infinity.

Theorem 1.6. *Let $\epsilon, s, v \in (0, 1)$, let $L, D \in (0, +\infty)$, and let $k, n \in \mathbb{N}$ with $k \leq n$. There exists $\delta > 0$ such that for every manifold (M^n, g, p) with $\text{Ric}_M \geq -\delta$, $\text{diam}(M) \leq D$, $\text{Vol}(B_1(p)) \geq v$, the following holds.*

1. If $\int_{B_1(p)} \mathbf{R}_k \wedge L \, d\text{Vol} \geq \epsilon$, then $b_1(M) \leq k - 1$.

2. If $\int_{B_1(p)} |\mathbf{R}|^s \, d\text{Vol} \geq \epsilon$, then $b_1(M) \leq n - 2$.

If the manifold in the previous theorem has non-negative Ricci curvature, the previous result follows immediately from the splitting theorem [13]. We remark that Theorem 1.6, unlike [62, Theorem 1.7], only requires a local bound on the scalar curvature.

Finally, we mention that Theorem 1.1 can be combined with the results of [16] to prove a result from Jiang and Naber (see Theorem 1.7 below). This result was announced in [44] without proof; we sketch one in the appendix.

Theorem 1.7 (Jiang-Naber). *Let $K \in \mathbb{R}$, $n \in \mathbb{N}$, $v > 0$, $s \in (0, 1)$ be fixed and let (M^n, g, p) be a manifold with $\text{Ric}_M \geq K$ and $\text{Vol}(B_1(p)) \geq v$. There exists a constant $C(K, n, v, s) > 0$ such that*

$$\int_{B_1(x)} |\text{Ric}|^s \, d\text{Vol} \leq C(K, n, v, s).$$

The previous theorem is the best available result concerning the non-collapsed Yau conjecture mentioned at the beginning of the introduction.

The paper is organized as follows. Section 2 contains preliminaries. In Section 3, we prove Theorems 1.1, 1.4, 1.5, and 1.6. In Section 4, we study a property of length-minimizing geodesics in metric spaces which is then used to prove Theorem 1.2. Finally, in Section 5, we prove Theorems 1.2 and 1.3.

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2 Preliminaries

We start by recalling some basic notation and terminology about metric spaces. Throughout the section, (X, d) denotes a proper, complete, separable, length metric space. A metric measure spaces (X, d, \mathbf{m}) is a triple, where (X, d) is a metric space and \mathbf{m} is a non-negative Borel measure on X which is finite on bounded sets and whose support is the whole X . We denote by \mathbf{H}^k the k -dimensional Hausdorff measure induced by d on X . Given a smooth manifold (M, g) we denote by d and Vol respectively the Riemannian distance and volume measure induced by g . Given an open set $\Omega \subset X$, we denote by $\text{Lip}(\Omega)$, $\text{Lip}_{\text{loc}}(\Omega)$ and $\text{Lip}_c(\Omega)$ respectively the set of Lipschitz, locally Lipschitz and compactly supported Lipschitz functions in Ω . We denote the *slope* of $f \in \text{Lip}_{\text{loc}}(\Omega)$ at $x \in \Omega$ by

$$\text{lip}(f)(x) := \limsup_{y \rightarrow x} \frac{|f(x) - f(y)|}{d(x, y)}.$$

We say that two pointed metric measure spaces $(X, d_X, \mathbf{m}_X, \bar{x})$ and $(Y, d_Y, \mathbf{m}_Y, \bar{y})$ are *isomorphic* if there exists a bijective distance preserving map $i : X \rightarrow Y$ such that $i_{\#} \mathbf{m}_X = \mathbf{m}_Y$ and $i(\bar{x}) = \bar{y}$.

We next recall the main definitions concerning the pointed Gromov-Hausdorff convergence, referring to [7], [53], and [27] for an overview on the subject.

Definition 2.1 (δ -GH maps). Let (X, d_X, \bar{x}) and (Y, d_Y, \bar{y}) be pointed metric spaces and let $\delta > 0$. A map $f : X \rightarrow Y$ is a δ -GH map if:

- $f(\bar{x}) = \bar{y}$;
- the image of f is a δ -net in Y , i.e. for every $y \in Y$ there exists $x \in X$ such that $d_Y(y, f(x)) \leq \delta$;
- the distortion of f is less than δ , i.e.

$$\sup_{x_0, x_1 \in X} \left| d_X(x_0, x_1) - d_Y(f(x_0), f(x_1)) \right| \leq \delta.$$

Definition 2.2 (pGH-convergence). A sequence of pointed metric spaces (X_j, d_j, \bar{x}_j) converges to (X, d, \bar{x}) in *pointed Gromov-Hausdorff-sense* (pGH, for short) if for every $\delta, R > 0$ there exists $N > 0$ such that, for every $j \geq N$, there exists a δ -GH map $f_j^R : (\bar{B}_R(\bar{x}_j), d_j, \bar{x}_j) \rightarrow (\bar{B}_R(\bar{x}), d, \bar{x})$.

The previous definition of pointed Gromov Hausdorff convergence is slightly different from the one given in [7, p. 272]. Nevertheless, they coincide when considering the convergence of length spaces (as it will be, throughout the paper).

Definition 2.3 (pmGH-convergence). We say that a sequence of pointed metric measure spaces $(X_j, d_j, \mathbf{m}_j, \bar{x}_j)$ converges to $(X, d, \mathbf{m}, \bar{x})$ in *pointed measured Gromov-Hausdorff-sense* (pmGH, for short) if it converges in pointed Gromov-Hausdorff-sense and, for every $R > 0$, the maps $f_j^R : \bar{B}_R(\bar{x}_j) \rightarrow \bar{B}_R(\bar{x})$ given by Definition 2.2 satisfy $(f_j^R)_{\#}(\mathbf{m}_j \llcorner \bar{B}_R(\bar{x}_j)) \rightarrow \mathbf{m} \llcorner \bar{B}_R(\bar{x})$, as $j \rightarrow \infty$, weakly in duality with continuous boundedly supported functions on X .

Both pointed Gromov-Hausdorff and pointed measured Gromov-Hausdorff convergence are metrizable. The corresponding distances are denoted by d_{pGH} and d_{pmGH} . We recall that, in the case of a sequence of uniformly locally doubling metric measure spaces $(X_j, d_j, \mathbf{m}_j, \bar{x}_j)$ (as in the case of $\text{RCD}(K, N)$ spaces), pointed measured Gromov-Hausdorff convergence to (X, d, \mathbf{m}, x) can be equivalently characterized by asking for the existence of a proper metric space (Z, d_z) such that all the metric spaces (X_j, d_j) are isometrically embedded into (Z, d_z) , $\bar{x}_j \rightarrow x$ and $\mathbf{m}_j \rightarrow \mathbf{m}$ weakly in duality with continuous boundedly supported functions in Z (see [27]).

Next, we recall the fundamental notion of (k, δ) -splitting map (see [20, 9, 10, 17]) on a Riemannian manifold with Ricci curvature bounded below.

Definition 2.4 ((k, δ) -splitting maps). Let (M^n, g) be a Riemannian manifold and let $1 \leq k \leq n$. A harmonic map $u : B_r(x) \rightarrow \mathbb{R}^k$ is a (k, δ) -splitting map if:

1. For every $i, j \in \{1, \dots, k\}$, it holds $\int_{B_r(x)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}| \, d\text{Vol} \leq \delta$.
2. $\sup_{B_r(x)} |\nabla u| \leq 1 + \delta$.
3. $r^2 \int_{B_r(x)} |\text{Hess}(u)|^2 \, d\text{Vol} \leq \delta^2$.

Definition 2.5 ((k, δ) -symmetric balls). Let (M^n, g) be a Riemannian manifold. A ball $B_r(p) \subset M$ is said to be (k, δ) -symmetric if there exists a pointed metric space (Y, d_Y, \bar{y}) such that $d_{\text{GH}}(B_r(p), B_r^{\mathbb{R}^k \times Y}(0, \bar{y})) \leq \delta r$. If a ball is (n, δ) -symmetric, we will say that it is δ -regular.

The next theorem corresponds to [15, Theorem 4.11] (after [9] and [10]).

Theorem 2.6. *Let $n \in \mathbb{N}$. For every $\epsilon > 0$, there exists $\delta_0(\epsilon, n) > 0$ with the following property. Let $\delta \in (0, \delta_0)$ and let (M^n, g, p) be a pointed Riemannian manifold with $\text{Ric}_M \geq -\delta$.*

1. *If there exists a (k, δ) -splitting map $u : B_2(p) \rightarrow \mathbb{R}^k$, then $B_1(p)$ is (k, ϵ) -symmetric.*
2. *If $B_2(p)$ is (k, δ) -symmetric, then there exists a (k, ϵ) -splitting map $u : B_1(p) \rightarrow \mathbb{R}^k$.*

The next result was proved in [9] (see also [2]).

Theorem 2.7. *Let $K \in \mathbb{R}$ and $n \in \mathbb{N}$. Let (M^n, g) be a Riemannian manifold with $\text{Ric}_M \geq K$. For every $p \in M$ and $R > 2r > 0$, there exists a function $\phi \in C_c^\infty(B_R(p))$ which is identically equal to 1 on $B_r(p)$, with support contained in $B_{2r}(p)$, and such that*

$$r^2 |\Delta \phi| + r |\nabla \phi| \leq C(K, n, R).$$

The next theorem follows from [10, 20, 9].

Theorem 2.8. *For every $\epsilon > 0$ there exists $\delta > 0$ such that if (M^n, g) has $\text{Ric}_M \geq -\delta$ and $B_{10}(x) \subset M$ is δ -regular, then $B_r(y)$ is ϵ -regular for every $r \in (0, 5)$ and every $y \in B_5(x)$.*

Definition 2.9 (Ricci limit space). A pointed metric measure space $(X, d, \mathbf{m}, \bar{x})$ is a *Ricci limit space* if there exists a sequence of pointed Riemannian manifolds (M_j^n, g_j, p_j) with $\text{Ric}_{M_j} \geq K$ for some fixed $K \in \mathbb{R}$ and $n \in \mathbb{N}$, such that

$$(M_j^n, d_{g_j}, \text{Vol}_j(B_1(p_j))^{-1} \text{Vol}_j, p_j) \longrightarrow (X, d, \mathbf{m}, \bar{x}) \quad \text{in pmGH-sense.}$$

The limit space is said to be *non-collapsed* if $\text{Vol}_j(B_1(p_j)) \geq v > 0$ for some fixed $v > 0$.

The next theorem was proved in [20, 10].

Theorem 2.10. *Let (M_j^n, g_j, p_j) be a sequence of pointed Riemannian manifolds with a uniform lower bound on the Ricci curvature converging in pGH-sense to a metric space (X, d, \bar{x}) . Then precisely one of the following holds.*

1. *Either: $\limsup_{j \rightarrow +\infty} \text{Vol}_j(B_1(p_j)) > 0$. In this case the limsup is a limit and the pGH convergence can be improved to pmGH convergence to $(X, d, \mathbf{H}^n, \bar{x})$.*
2. *Or: $\lim_{j \rightarrow +\infty} \text{Vol}_j(B_1(p_j)) = 0$. In this case the Hausdorff dimension of X is bounded above by $n - 1$.*

Definition 2.11 (Tangent space). Let $(X, d, \mathbf{m}, \bar{x})$ be a Ricci limit space. A metric space (Y, d_Y, y) is a *tangent space* at \bar{x} if there exists a sequence $1 > r_j > 0$, $r_j \searrow 0$ that satisfies

$$(X, d/r_j, \bar{x}) \xrightarrow{\text{pGH}} (Y, d_Y, y).$$

Gromov pre-compactness theorem implies that the set of tangent spaces at a point is always non-empty. Moreover, in [10] it was shown that for non-collapsed Ricci limit spaces all tangent spaces are metric cones.

Definition 2.12 (*k*-singular stratum). Let $(X, d, \mathbf{m}, \bar{x})$ be a Ricci limit space. For every $k \in \mathbb{N}$, the *k*-singular stratum is defined to be

$$\mathcal{S}_k = \{x \in X : \text{no tangent space is isometric to } \mathbb{R}^{k+1} \times Y, \text{ for any } (Y, d_Y)\}.$$

The next theorem was proved in [10].

Theorem 2.13. *Let (X, d, \bar{x}) be a non-collapsed Ricci limit space of dimension $n \geq 2$. Then $\mathcal{S}_{n-1} \setminus \mathcal{S}_{n-2} = \emptyset$.*

We next recall some properties of RCD spaces and of finite perimeter sets thereof, which will be useful later in the paper. We assume the reader to be familiar with the subject and we refer to the surveys [1, 51, 25] for an account of the theory. Let us just recall that an $\text{RCD}(K, N)$ space is a metric measure space (X, d, \mathbf{m}) with Ricci curvature bounded below by $K \in \mathbb{R}$ and dimension bounded above by $N \in [1, \infty]$ in a synthetic-sense, and whose Sobolev space $W^{1,2}$ is Hilbert. For the present work, it will suffice to consider the finite dimensional case; thus, if not otherwise specified, $N \geq 1$ will be a real number. Recall that such a class is stable under pmGH convergence, it contains Ricci limit spaces and finite dimensional Alexandrov spaces.

The next result follows from [41, 6] (after the seminal works for Ricci limits [10, 12, 21]).

Theorem 2.14. *Let (X, d, \mathbf{m}) be an $\text{RCD}(K, N)$ space. Then there exists $k \in \mathbb{N} \cap [1, N]$, called essential dimension of X , such that for \mathbf{m} -a.e. $x \in X$ all tangent spaces in x are isometric to $(\mathbb{R}^k, d_e, 0)$.*

The next proposition, from [36, Theorem 1.1], characterizes the RCD spaces having essential dimension 1.

Proposition 2.15. *Let (X, d, \mathbf{m}) be an $\text{RCD}(K, N)$ space of essential dimension equal to 1. Then (X, d) is isometric to a closed connected subset of \mathbb{R} or to $\mathbb{S}^1(r) := \{x \in \mathbb{R}^2 : |x| = r\}$.*

We now recall some topological properties of RCD spaces. The next proposition is a special case of [47, Theorem 5], in the non-collapsing scenario. We denote by $b_1(X)$ the first Betti number of X .

Proposition 2.16. *Let $((X_j, d_j, H_j^N, \bar{x}_j))_{j \in \mathbb{N}}$ be a sequence of pointed $\text{RCD}(0, N)$ spaces with $H_j^N(B_1(\bar{x}_j)) \geq v > 0$ converging in pGH-sense to a compact metric space (X, d) . If $b_1(X_j) \geq r$ for every j , then $b_1(X) \geq r$.*

The next proposition, proved in [55, Corollary 1.2] (see also [56, Theorem 1.2]), establishes the converse inequality in higher generality.

Proposition 2.17. *Let $((X_j, d_j, \mathbf{m}_j, \bar{x}_j))_{j \in \mathbb{N}}$ be a sequence of pointed $\text{RCD}(K, N)$ spaces converging in pmGH-sense to a compact metric measure space (X, d, \mathbf{m}) . If $b_1(X_j) \leq r$ for every j , then $b_1(X) \leq r$.*

Definition 2.18 (Covering space and universal cover). Let (X, d_X) be a connected metric space. We say that (Y, d_Y) is a *covering space* for X if there exists a continuous map $\pi : Y \rightarrow X$ such that for every point $x \in X$ there exists a neighborhood $U_x \subset X$ with the following property: $\pi^{-1}(U_x)$ is the disjoint union of open subsets of Y each of which is mapped homeomorphically onto U_x via π .

A connected metric space $(\tilde{X}, d_{\tilde{X}})$ is said to be a *universal cover* for (X, d_X) if it is a covering space with the following property: for any other connected covering space (Y, d_Y) of (X, d_X) , there exists a continuous map $f : \tilde{X} \rightarrow Y$ such that the triangle made with the projection maps onto X commutes.

The next result proved in [42] establishes the existence of the universal cover for an $\text{RCD}(K, N)$ space.

Proposition 2.19. *Let (X, d, m) be an $\text{RCD}(K, N)$ space. Then, it admits a universal cover $(\tilde{X}, d_{\tilde{X}})$. Moreover there exists a measure \tilde{m} on \tilde{X} such that $(\tilde{X}, d_{\tilde{X}}, \tilde{m})$ is an $\text{RCD}(K, N)$ space.*

Since $\text{RCD}(K, N)$ spaces are semi-locally simply connected thanks to [55] the universal is simply connected. The next proposition follows from [42, Theorem 1.3], after taking into account that the revised fundamental group mentioned in the reference coincides with the usual fundamental group thanks to [55].

Proposition 2.20. *Let (X, d, m) be a compact $\text{RCD}(0, N)$ space with $b_1(X) \geq k$. Then, its universal cover splits the Euclidean \mathbb{R}^k isomorphically as metric measure space.*

The next proposition from [50, Corollary 4.11] shows that the universal cover of a Ricci limit space is a Ricci limit space, as well.

Proposition 2.21. *Let $K \in \mathbb{R}$ and $n \in \mathbb{N}$. Let (M_j^n, g_j, \bar{x}_j) be a sequence of Riemannian manifolds with $\text{Ric}_{M_j} \geq K$ converging in pGH -sense to a compact metric space (X, d) . Let (\tilde{X}, \tilde{d}) be the universal cover of X and fix $\tilde{x} \in \tilde{X}$. Then, there is a sequence of coverings $(\tilde{M}_j, \tilde{g}_j, \tilde{x}_j)$ of M_j converging in pGH -sense to $(\tilde{X}, \tilde{d}, \tilde{x})$.*

The next proposition follows combining [3, Theorems 1.1 and 1.3] with the fact that each covering of a fixed manifold has volume growth controlled from above by the one of the universal cover.

Proposition 2.22. *Let $k, h \in \mathbb{N} \cup \{0\}$ with $k \leq h$. Let (M, g, \bar{x}) be a pointed Riemannian manifold with $\text{Vol}(B_t(\bar{x})) \geq c_1 t^k$ for some $c_1 > 0$ and whose universal cover $(\tilde{M}, \tilde{g}, \tilde{x})$ satisfies $\text{Vol}(B_t(\tilde{x})) \leq c_2 t^h$ for some $c_2 > 0$. Then $b_1(M) \leq h - k$.*

3 Integral estimates on scalar curvature and first geometric applications

3.1 Proof of Theorem 1.1

Throughout the section, (M^n, g) will be a smooth, complete (in some cases compact), n -dimensional Riemannian manifold, with $n \geq 2$. We denote by $\mathbf{R}_k : M \rightarrow \mathbb{R}$ the sum of the lowest k eigenvalues, counted with their multiplicity, of the Ricci tensor. We first prove a few lemmas that will be used to establish Theorem 1.1 from the Introduction. Given a matrix $A \in \mathbb{R}^{n \times k}$, we denote by $A^T \in \mathbb{R}^{k \times n}$ its transpose.

Lemma 3.1. *Let $k, n \in \mathbb{N}$ with $k \leq n$, and let $c, \epsilon > 0$ be fixed. There exist $c_1(n, \epsilon), \epsilon'(n, \epsilon) > 0$ arbitrarily small satisfying the following. Let $A = (a_{j,i}) \in \mathbb{R}^{n \times k}$ be a matrix with $|A^T A - I_k| \leq c_1$ and $|A|^2 \leq c$. Consider real numbers $-\epsilon' \leq \lambda_1 \leq \dots \leq \lambda_n$ with $\lambda_k \geq \epsilon/2k$. Then*

$$0 \vee \sum_{i=1}^k \sum_{j=1}^n \lambda_j a_{j,i}^2 \geq \sqrt{\epsilon'} (\lambda_1 + \dots + \lambda_k).$$

Proof. Assume by contradiction that the statement fails. Hence, we find sequences $c_h, \epsilon'_h \downarrow 0$ and $A_h = (a_{j,i}^h) \in \mathbb{R}^{n \times k}$ with $|A_h^T A_h - I_k| \leq c_h, |A_h|^2 \leq c$ and real numbers $-\epsilon'_h \leq \lambda_1^h \leq \dots \leq \lambda_n^h$ with $\lambda_k^h \geq \epsilon/2k$, with

$$0 \vee \sum_{i=1}^k \sum_{j=1}^n \lambda_j^h (a_{j,i}^h)^2 < \sqrt{\epsilon'_h} (\lambda_1^h + \dots + \lambda_k^h), \quad \text{for all } h \in \mathbb{N}.$$

In particular, dividing both sides by λ_k^h , it holds

$$\sum_{i=1}^k \sum_{j=1}^{k-1} \frac{\lambda_j^h}{\lambda_k^h} (a_{j,i}^h)^2 + \sum_{i=1}^k \sum_{j=k}^n (a_{j,i}^h)^2 \leq k \sqrt{\epsilon'_h}, \quad (2)$$

where the first summand in the l.h.s. is to be considered zero if $k = 1$.

Moreover, since $|A|^2 \leq c$, $-\epsilon'_h \leq \lambda_1^h \leq \dots \leq \lambda_n^h$ and $\lambda_k^h \geq \epsilon/2k$, it holds

$$\sum_{i=1}^k \sum_{j=1}^{k-1} \frac{\lambda_j^h}{\lambda_k^h} (a_{j,i}^h)^2 \geq -2k^2 c \frac{\epsilon'_h}{\epsilon}.$$

Combining with (2), we obtain

$$\sum_{i=1}^k \sum_{j=k}^n (a_{j,i}^h)^2 \leq k \sqrt{\epsilon'_h} + 2k^2 c \frac{\epsilon'_h}{\epsilon}.$$

Taking the limit as $h \rightarrow \infty$, we obtain a matrix $A = (a_{j,i}) \in \mathbb{R}^{n \times k}$ such that

$$A^T A = I_k \quad \text{and} \quad \sum_{i=1}^k \sum_{j=k}^n (a_{j,i})^2 = 0.$$

The last two identities are clearly in contradiction: indeed, the former states that the columns of A are k orthonormal vectors in \mathbb{R}^n , while the latter implies that their span is at most $k - 1$ dimensional. \square

The next lemma will be key to prove Theorem 1.1.

Lemma 3.2. *Let $k, n \in \mathbb{N}$, with $k \leq n$, and let $s \in (0, 1)$ be fixed. For every $\epsilon > 0$ there exists $\delta(n, \epsilon, s) > 0$ such that if (M^n, g) has $\text{Ric}_M \geq -\delta$ on $B_{20}(x)$, and $B_{10}(x) \subset M$ is (k, δ) -symmetric, then:*

- There exists $E \subset B_1(x)$ such that

$$\int_E \text{R}_k \, d\text{Vol} \leq \epsilon \text{Vol}(B_1(x)); \quad (3)$$

- There exist $x_j \in B_1(x)$ and $r_j \in [0, 1/2]$, with $j \in \mathbb{N}$, such that

$$B_1(x) \setminus E \subset \bigcup_{j \in \mathbb{N}} B_{r_j}(x_j) \quad \text{with} \quad \sum_{j \in \mathbb{N}} \text{Vol}(B_{r_j}(x_j)) r_j^{-2s} \leq \epsilon \text{Vol}(B_1(x)). \quad (4)$$

In particular, $\text{Vol}(B_1(x) \setminus E) \leq \epsilon \text{Vol}(B_1(x))$ and $\int_E \text{R}_k \, d\text{Vol} \leq \frac{\epsilon}{1-\epsilon}$.

Proof. Let $\epsilon > 0$ be fixed. Let $0 < \epsilon'(n, \epsilon) < \epsilon^2$ be given by Lemma 3.1 and let $0 < \delta < \epsilon'$ be small enough so that if $B_{10}(x) \subset M$ is (k, δ) -symmetric, there exists a harmonic (k, ϵ') -splitting map $u : B_5(x) \rightarrow \mathbb{R}^k$. Plugging the j -th entry u_j of the map u into the Bochner Formula yields

$$\Delta \frac{|\nabla u_j|^2 - 1}{2} = |\text{Hess}(u_j)|^2 + \text{Ric}(\nabla u_j, \nabla u_j), \quad \text{on } B_5(x). \quad (5)$$

Theorem 2.7 ensures the existence of a non-negative function $\phi \in C_c^\infty(B_2(x))$, with $\phi \equiv 1$ on $B_1(x)$ and such that

$$|\Delta \phi| + |\nabla \phi| \leq c(n).$$

Multiplying (5) by ϕ , integrating, and recalling Definition 2.4, we obtain

$$\int_{B_2(x)} \phi \text{Ric}(\nabla u_j, \nabla u_j) \, d\text{Vol} \leq c(n) \epsilon'.$$

Since $\text{Ric}_M \geq -\delta \geq -\epsilon'$ and $\sup_{B_5(x)} |\nabla u| \leq 2$, it holds

$$\int_{B_1(x)} \text{Ric}(\nabla u_j, \nabla u_j) \, d\text{Vol} \leq c(n) \int_{B_2(x)} \phi \text{Ric}(\nabla u_j, \nabla u_j) \, d\text{Vol} + c(n) \epsilon' \leq c(n) \epsilon'.$$

Summing over $j \in \{1, \dots, k\}$, we get

$$\int_{B_1(x)} \sum_{j=1}^k \text{Ric}(\nabla u_j, \nabla u_j) \, d\text{Vol} \leq c(n)\epsilon'. \quad (6)$$

We now define $E \subset B_1(x)$ to be the set of points where $\{\nabla u_j\}_{j=1}^k$ are almost orthonormal. For each point $p \in M$, we fix an orthonormal basis so that $T_p M \equiv \mathbb{R}^n$ and we use the convention that $\nabla u \in \mathbb{R}^{n \times k}$ (i.e., each ∇u_j is a column of ∇u). Let $c_1(n, \epsilon) > 0$ be the constant given by Lemma 3.1, and set

$$E := \{y \in B_1(x) : |\nabla u^T \nabla u - I_k| \leq c_1\}.$$

Observe that the condition in the previous definition is independent of the chosen orthonormal basis for $T_p M$. Fix now an orthonormal basis in $T_p M$ such that Ric_M in $T_p M$ is represented by the diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$, whose eigenvalues are $-\epsilon' \leq \lambda_1 \leq \dots \leq \lambda_n$.

Consider now the subsets of E defined by

$$E_+ := E \cap \{\text{R}_k \geq \epsilon/2\} \quad \text{and} \quad E_- := E \cap \{\text{R}_k < \epsilon/2\}.$$

At every point of E_+ , we have $\lambda_k \geq \epsilon/2k$. Hence, by Lemma 3.1, it holds

$$\begin{aligned} 0 \vee \sum_{j=1}^k \text{Ric}(\nabla u_j, \nabla u_j) &= 0 \vee \sum_{j=1}^k \nabla u_j^T \Lambda \nabla u_j \\ &= 0 \vee \sum_{j=1}^k \sum_{i=1}^n \lambda_i (\nabla u_{j,i})^2 \geq \sqrt{\epsilon'} (\lambda_1 + \dots + \lambda_k) = \sqrt{\epsilon'} \text{R}_k. \end{aligned}$$

Combining with (6), we obtain

$$\begin{aligned} \int_E \text{R}_k \, d\text{Vol} &= \int_{E_-} \text{R}_k \, d\text{Vol} + \int_{E_+} \text{R}_k \, d\text{Vol} \\ &\leq (\epsilon/2) \text{Vol}(B_1(x)) + \int_{E_+} \text{R}_k \, d\text{Vol} \leq (c(n)\sqrt{\epsilon'} + \epsilon/2) \text{Vol}(B_1(x)). \end{aligned} \quad (7)$$

Choosing $\epsilon' > 0$ small enough, we obtain (3).

To conclude the proof, we need to show that $B_1(x) \setminus E$ is contained in a union of sufficiently small balls. Let $\eta(\epsilon, n, s) > 0$ to be chosen later and consider the set $E' \subset B_1(x)$ given by

$$E' := \left\{ y \in B_1(x) : r^{2s} \int_{B_r(y)} |\text{Hess}(u)|^2 \, d\text{Vol} \leq \eta, \text{ for every } r \in (0, 1/10) \right\}.$$

We claim that

$$\begin{aligned} E' &\subset E \quad \text{and} \\ B_1(x) \setminus E' &\subset \bigcup_{i \in \mathbb{N}} B_{r_j}(x_j) \text{ with } \sum_{i \in \mathbb{N}} \text{Vol}(B_{r_j}(x_j)) r_j^{-2s} \leq \epsilon \text{Vol}(B_1(x)). \end{aligned} \quad (8)$$

The thesis (4) will follow by combining the claim (8) with (7).

We first prove that $E' \subset E$. It is sufficient to show that at every point of E' the quantity $|\nabla u^T \nabla u - I_k|$ is smaller than the constant $c_1(\epsilon, n)$ appearing in the definition of E . We prove it by a telescopic argument (cf. [5]). Set $\mathcal{E} := \sum_{i,j} |\nabla u_j \cdot \nabla u_j - \delta_{ij}|$. For every $j \in \mathbb{N}$ and $y \in B_1(x)$, we have

$$\begin{aligned} \int_{B_{2^{-j}}(y)} \mathcal{E} \, d\text{Vol} &= \left| \int_{B_1(y)} \mathcal{E} \, d\text{Vol} - \int_{B_1(y)} \mathcal{E} \, d\text{Vol} + \int_{B_{1/2}(y)} \mathcal{E} \, d\text{Vol} - \dots + \int_{B_{2^{-j}}(y)} \mathcal{E} \, d\text{Vol} \right| \\ &\leq c(n)\epsilon' + \sum_{j \in \mathbb{N}} \left| \int_{B_{2^{-j+1}}(y)} \mathcal{E} \, d\text{Vol} - \int_{B_{2^{-j}}(y)} \mathcal{E} \, d\text{Vol} \right| \\ &\leq c(n)\epsilon' + c(n) \sum_{j \in \mathbb{N}} 2^{-j} \int_{B_{2^{-j}}(y)} |\text{Hess}(u)| \, d\text{Vol}, \end{aligned}$$

where in the last inequality we used the Poincaré inequality. Using Hölder inequality and the definition of E' , we infer that

$$\begin{aligned} \int_{B_{2^{-j}}(y)} \mathcal{E} \, d\text{Vol} &\leq c(n)\epsilon' + c(n) \sum_{j \in \mathbb{N}} 2^{-j(1-s)} \left(2^{-2sj} \int_{B_{2^{-j}}(y)} |\text{Hess}(u)|^2 \, d\text{Vol} \right)^{1/2} \\ &\leq c(n)\epsilon' + c(n, s)\sqrt{\eta}. \end{aligned}$$

Choosing η and ϵ' to be sufficiently small, and letting j go to $+\infty$, we conclude that $E' \subset E$.

From now until the end of the proof, $\eta > 0$ is fixed by the previous step. To complete the proof, we need to show that

$$B_1(x) \setminus E' \subset \bigcup_{i \in \mathbb{N}} B_{r_j}(x_j)$$

with

$$\sum_{i \in \mathbb{N}} \text{Vol}(B_{r_j}(x_j)) r_j^{-2s} \leq \epsilon \text{Vol}(B_1(x)), \quad \text{and } r_j \in [0, 1/2].$$

To prove this, we use a Vitali covering argument coupled with a scale-picking (cf. [5]). For every $y \in B_1(x)$ such that

$$\sup_{0 < r < 1/10} r^{2s} \int_{B_r(y)} |\text{Hess}(u)|^2 \, d\text{Vol} > \eta,$$

we set $r_y \in [0, 1/10]$ to be the maximal radius such that

$$r_y^{2s} \int_{B_{r_y}(y)} |\text{Hess}(u)|^2 \, d\text{Vol} \geq \eta.$$

Clearly,

$$\int_{B_{r_y}(y)} |\text{Hess}(u)|^2 \, d\text{Vol} \geq \text{Vol}(B_{r_y}(y)) \eta r_y^{-2s}.$$

By Vitali covering theorem, we can cover E' with a countable collection of balls $\{B_{5r_{y_j}}(y_j)\}_{j \in \mathbb{N}}$ with $r_{y_j} \in (0, 1/10]$, such that $\{B_{r_{y_j}}(y_j)\}_{j \in \mathbb{N}}$ are disjoint. Hence

$$\begin{aligned} \sum_{j \in \mathbb{N}} \frac{\text{Vol}(B_{5r_{y_j}}(y_j))}{(5r_{y_j})^{2s}} &\leq \frac{c(n)}{\eta} \sum_{j \in \mathbb{N}} \int_{B_{r_{y_j}}(y_j)} |\text{Hess}(u)|^2 \, d\text{Vol} \\ &\leq \frac{c(n)}{\eta} \int_{B_2(x)} |\text{Hess}(u)|^2 \, d\text{Vol} \leq \frac{c(n)}{\eta} \text{Vol}(B_1(x)) \epsilon'. \end{aligned}$$

Observe that, by construction, $5r_j \in [0, 1/2]$ for every $j \in \mathbb{N}$. Since $\eta > 0$ is fixed, we can choose $\epsilon' = \epsilon \eta c(n)^{-1} > 0$ to conclude the proof. \square

Corollary 3.3. *Let $k, n \in \mathbb{N}$ with $k \leq n$ and $s \in (0, 1)$ be fixed. For every $\epsilon > 0$ there exists $\delta > 0$ such that if (M^n, g) has $\text{Ric}_M \geq -\delta$ and $B_{10}(x) \subset M$ is (k, δ) -symmetric, then there exists $E \subset B_1(x)$ such that*

$$\int_E |\text{R}_k|^s \, d\text{Vol} \leq \epsilon \text{Vol}(B_1(x))$$

with

$$B_1(x) \setminus E \subset \bigcup_{i \in \mathbb{N}} B_{r_j}(x_j) \quad \text{and}$$

$$\sum_{j \in \mathbb{N}} \text{Vol}(B_{r_j}(x_j)) r_j^{-2s} \leq \epsilon \text{Vol}(B_1(x)), \quad r_j \in [0, 1/2] \quad \text{for all } j \in \mathbb{N}.$$

Proof. Let $0 < \delta < \epsilon_1 < \epsilon$ be sufficiently small and let E be the set given by Lemma 3.2 relative to ϵ_1 . Since by assumption $\text{Ric}_M \geq -\delta$, then clearly

$$\mathbf{R}_k \geq -k\delta. \quad (9)$$

By Hölder inequality applied to the conjugate exponents $p = s^{-1} > 1$ and $q = (1 - s)^{-1} > 1$, using Lemma 3.2 and (9), we obtain

$$\int_E (\mathbf{R}_k \vee 0)^s \, d\text{Vol} \leq \left(\int_E \mathbf{R}_k \vee 0 \, d\text{Vol} \right)^s \text{Vol}(B_1(x))^{1-s} \leq \epsilon_1 \text{Vol}(B_1(x)).$$

Hence,

$$\int_E |\mathbf{R}_k|^s \, d\text{Vol} \leq \int_E (\mathbf{R}_k \vee 0)^s \, d\text{Vol} + (k\delta)^s \text{Vol}(B_1(x)) \leq (\epsilon_1 + (k\delta)^s) \text{Vol}(B_1(x)).$$

We conclude by choosing $\epsilon_1, \delta > 0$ such that $\epsilon_1 + (k\delta)^s \leq \epsilon$. \square

We now prove Theorem 1.1 from the Introduction.

Theorem 3.4. *Let $L, k, n \in \mathbb{N}$ with $k \leq n$ and $s \in (0, 1)$ be fixed. Then*

$$\sup \left\{ \int_{B_1(x)} \mathbf{R}_k \wedge L \, d\text{Vol} : (M, g, x) \text{ has } \text{Ric}_M \geq -\delta, \text{ and } B_{10}(x) \text{ is } (k, \delta)\text{-symmetric} \right\}, \quad (10)$$

$$\sup \left\{ \int_{B_1(x)} |\mathbf{R}|^s \, d\text{Vol} : (M, g, x) \text{ has } \text{Ric}_M \geq -\delta, \text{ and } B_{10}(x) \text{ is } \delta\text{-regular} \right\} \quad (11)$$

converge to 0 as $\delta \searrow 0$.

Proof. Let $k, n \in \mathbb{N}$ with $k < n$. We show that for every $0 < \epsilon < 1/2$ there exists $\delta > 0$ such that if (M^n, g) has $\text{Ric}_M \geq -\delta$ and $B_{10}(x) \subset M$ is (k, δ) -symmetric, then

$$\int_{B_1(x)} \mathbf{R}_k \wedge L \, d\text{Vol} \leq \epsilon. \quad (12)$$

This would imply the first part of the statement. Fix $\epsilon_1(\epsilon, n) > 0$ – to be chosen later – and let $0 < \delta(\epsilon_1) < \epsilon_1$ be given by Lemma 3.2 so that there exists $E \subset B_1(x)$ such that

$$\int_E \mathbf{R}_k \, d\text{Vol} \leq \epsilon_1 \text{Vol}(B_1(x)),$$

and $\text{Vol}(B_1(x) \setminus E) \leq \epsilon_1 \text{Vol}(B_1(x))$. Hence,

$$\begin{aligned} \int_{B_1(x)} \mathbf{R}_k \wedge L \, d\text{Vol} &\leq \int_E \mathbf{R}_k \, d\text{Vol} + \int_{B_1(x) \setminus E} 0 \vee \mathbf{R}_k \wedge L \, d\text{Vol} \\ &\leq (\epsilon_1 + L\epsilon_1) \text{Vol}(B_1(x)), \end{aligned}$$

which implies (12) if ϵ_1 is chosen small enough.

We next show (11). To this aim, fix $s \in (0, 1)$ and consider the case $k = n$. We want show that for every $0 < \epsilon < 1/2$ there exists $\delta > 0$ such that if (M^n, g, x) has $\text{Ric}_M \geq -\delta$ and $B_{10}(x) \subset M$ is δ -regular, then

$$\int_{B_1(x)} |\mathbf{R}|^s \, d\text{Vol} \leq \epsilon.$$

Let $\delta' > 0$ be as in Lemma 3.2 relative to ϵ . Let $0 < \delta < \delta'$ be small enough such that if (M^n, g, x) has $\text{Ric}_M \geq -\delta$ and $B_{10}(x) \subset M$ is δ -regular, then $B_r(y)$ is δ' -regular for every $r \in (0, 5)$ and every $y \in B_5(x)$. This is possible thanks to Theorem 2.8. Define the measure

$$\mu := |\mathbf{R}|^s \, d\text{Vol} \llcorner B_1(x).$$

Consider the set $E_1 \subset B_1(x)$ given by Corollary 3.3 such that

$$\mu(E_1) \leq \epsilon \text{Vol}(B_1(x)), \quad B_1(x) \setminus E_1 \subset \bigcup_{i \in \mathbb{N}} B_{r_i}(x_i)$$

with

$$\sum_{i \in \mathbb{N}} \text{Vol}(B_{r_i}(x_i)) r_i^{-2s} \leq \epsilon \text{Vol}(B_1(x)) \text{ and } r_i \in [0, 1/2].$$

In particular, it holds

$$\text{Vol}(B_1(x) \setminus E_1) \leq \epsilon \text{Vol}(B_1(x)).$$

In each ball $B_{r_i}(x_i)$, we can apply the scale invariant version of Corollary 3.3 to obtain a set $E_{2,i} \subset B_{r_i}(x_i)$ such that

$$\mu(E_{2,i}) \leq \epsilon r_i^{-2s} \text{Vol}(B_{r_i}(x_i)), \quad B_{r_i}(x_i) \setminus E_{2,i} \subset \bigcup_{j \in \mathbb{N}} B_{r_{i,j}}(x_{i,j})$$

with

$$\sum_{j \in \mathbb{N}} \text{Vol}(B_{r_{i,j}}(x_{i,j})) r_{i,j}^{-2s} \leq \epsilon r_i^{-2s} \text{Vol}(B_{r_i}(x_i)) \text{ and } r_{i,j} \in [0, r_i/2].$$

Defining

$$E_2 := E_1 \cup \bigcup_{i \in \mathbb{N}} E_{2,i},$$

we obtain

$$\begin{aligned} \mu(E_2) &\leq \epsilon \text{Vol}(B_1(x)) + \sum_{i \in \mathbb{N}} \mu(E_{2,i}) \\ &\leq \epsilon \text{Vol}(B_1(x)) + \epsilon \sum_{i \in \mathbb{N}} r_i^{-2s} \text{Vol}(B_{r_i}(x_i)) \leq (\epsilon + \epsilon^2) \text{Vol}(B_1(x)). \end{aligned}$$

Moreover, it holds

$$\text{Vol}(B_1(x) \setminus E_2) \leq \sum_{i,j \in \mathbb{N}} \text{Vol}(B_{r_{i,j}}(x_{i,j})) \leq \epsilon^2 \text{Vol}(B_1(x)).$$

If we keep iterating this argument, we obtain increasing sets E_k such that

$$\mu(E_k) \leq (\epsilon + \dots + \epsilon^k) \text{Vol}(B_1(x)).$$

Denoting by

$$E = \bigcup_{k \in \mathbb{N}} E_k,$$

we obtain that

$$\mu(E) \leq c \epsilon \text{Vol}(B_1(x)),$$

where $c > 0$ is a universal constant. To conclude the proof, it suffices to show that

$$\mu(B_1(x) \setminus E) = 0.$$

To this aim, observe that $\text{Vol}(B_1(x) \setminus E) = 0$, since $\text{Vol}(B_1(x) \setminus E_k) \leq \epsilon^k \text{Vol}(B_1(x))$. \square

Remark 3.5. Inspecting the proof of Lemma 3.2, one realizes that a priori L^p estimates on harmonic splitting maps for $p > 2$ would imply that

$$\sup \left\{ \int_{B_1(x)} |\mathbf{R}| \, d\text{Vol} : (M, g, x) \text{ has } \text{Ric}_M \geq -\delta \text{ and } B_{10}(x) \text{ is } \delta\text{-regular} \right\} \rightarrow 0$$

as $\delta \rightarrow 0$. This would allow to obtain a priori integral bounds on the scalar curvature outside of the quantitative singular set of a manifold with Ricci curvature bounded from below. Unfortunately, it is shown in [23] that such estimates cannot exist in such generality.

3.2 First geometric applications in almost non-negative Ricci and integrally-positive scalar curvature

We now obtain the first geometric applications of the results of the previous section. We introduce some notation first. After [48], we consider bounds on the k^{th} Ricci curvature.

Definition 3.6 (k^{th} Ricci curvature bounds). Let (M^n, g) be a manifold, let $1 \leq k \leq n-1$, let $x \in M$, and let $c \in \mathbb{R}$. We denote by $R_M(\cdot, \cdot, \cdot, \cdot)$ the Riemann curvature tensor of M . We say that $\text{Ric}_{M,k} \geq c$ (resp. $\leq c$) in $T_x M$ if, for all $(k+1)$ -dimensional subspaces $V \subset T_x M$, it holds

$$\sum_{i=1}^{k+1} R_M(e_i, v, v, e_i) \geq c \text{ (resp. } \leq c)$$

for every $v \in V$ and every orthonormal basis $\{e_1, \dots, e_{k+1}\}$ of V .

One can check that:

1. $\text{Ric}_{M,k} \geq c$ implies $\mathbf{R}_k \geq c$.
2. $\text{Ric}_{M,k} \geq c$ implies $\text{Ric}_{M,k+1} \geq \frac{k}{k-1}c$.
3. $\text{Ric}_{M,n-1} \geq c$ if and only if $\text{Ric}_M \geq c$.
4. $\text{Ric}_{1,M} \geq c$ if and only if $\text{Sec}_M \geq c$.

Definition 3.7 (Integral k^{th} Ricci curvature bounds). Let (M^n, g) be a manifold, let $1 \leq k \leq n-1$, let $A \subset M$ be measurable, let $c \in \mathbb{R}$, and let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function. We say that

$$\int_A F(\text{Ric}_{M,k}) d\text{Vol} \geq c$$

if there exists a measurable function $f : A \rightarrow \mathbb{R}$ such that:

1. $\int_A f d\text{Vol} \geq c$.
2. For all $x \in A$ and $(k+1)$ -dimensional subspaces $V \subset T_x M$, it holds

$$F\left(\sum_{i=1}^{k+1} R_M(e_i, v, v, e_i)\right) \geq f(x)$$

for every $v \in V$ and every orthonormal basis $\{e_1, \dots, e_{k+1}\}$ of V .

We say that $\int_A F(\text{Sec}_M) d\text{Vol} \geq c$ if $\int_A F(\text{Ric}_{1,M}) d\text{Vol} \geq c$.

The next theorem should be compared with [57, Theorem 1.1], where similar conclusions were obtained under stronger assumptions: non-negative Ricci curvature (below, almost non-negative Ricci curvature) and scalar curvature greater than 2 (below, a local integral lower bound on the scalar curvature). Whenever we consider a sequence of manifolds (M_j, g_j) , we write $\mathbf{R}_{M_j,k}$ instead of \mathbf{R}_k for the sake of clarity.

Theorem 3.8. *Let $s \in (0, 1)$, let $\epsilon, v, L \in (0, +\infty)$ and let $k, n \in \mathbb{N}$ with $k \leq n$.*

1. *There exists $\delta > 0$ such that the following holds. Let (M^n, g, p) be a manifold with $\text{Ric}_M \geq -\delta$ on M and*

$$\int_{B_1(p)} |\mathbf{R}_M|^s d\text{Vol} \geq \epsilon, \quad \text{Vol}(B_1(p)) \geq v.$$

Then $d_{\text{pGH}}(M, Y) \geq \delta$, for any metric space (Y, d_y, y) splitting \mathbb{R}^{n-1} isometrically.

2. There exists $\delta > 0$ such that the following holds. Let (M^n, g, p) be a manifold with $\text{Ric}_M \geq -\delta$ on M , and

$$\int_{B_1(p)} \text{Ric}_{M, n-2} \wedge 0 \, d\text{Vol} \geq -\delta, \quad \int_{B_1(p)} \mathbf{R}_M \wedge L \, d\text{Vol} \geq \epsilon.$$

Then $d_{\text{pGH}}(M, Y) \geq \delta$, for any metric space (Y, d_y, y) splitting \mathbb{R}^{n-1} isometrically.

Proof. Proof of 1. Assume by contradiction that there exist $\epsilon_0 > 0$, $v_0 > 0$, $\delta_j \downarrow 0$ and pointed Riemannian manifolds (M_j, g_j, p_j) with $\text{Ric}_{M_j} \geq -\delta_j$, $\text{Vol}_j(B_1(p_j)) \geq v_0$, and

$$\int_{B_1(p_j)} |\mathbf{R}_{M_j}|^s \, d\text{Vol}_j \geq \epsilon_0,$$

such that $M_j \rightarrow X$ in pGH-sense, where (X, d, p) splits \mathbb{R}^{n-1} isometrically. Thanks to Proposition 2.15 and Theorem 2.13, then either $X = \mathbb{R}^n$ or $X = \mathbb{R}^{n-1} \times S_r^1$, for some $r > 0$.

Case $X = \mathbb{R}^n$. Applying the second assertion of Theorem 1.1 to (M_j, g_j, p_j) , for j large enough, we reach a contradiction.

Case $X = \mathbb{R}^{n-1} \times S_r^1$. Let $s \in (0, 1)$. We claim that for every j there exists $p_j^s \in B_1(p_j)$ such that

$$\int_{B_s(p_j^s)} |\mathbf{R}_{M_j}|^s \, d\text{Vol}_j \geq c(n, s)\epsilon_0. \quad (13)$$

If (13) holds, we reach a contradiction by applying the second assertion of Theorem 1.1 to the rescaled sequence $(M_j, s^{-2}g_j, p_j)$, for $s \in (0, 1)$ small enough independent of j .

We prove (13). Let j be fixed. Let $\{q_i\}_{i=1}^l \subset B_1(p_j)$ be such that $\{B_s(q_i)\}_{i=1}^l$ covers $B_1(p_j)$, while the balls $\{B_{s/5}(q_i)\}_{i=1}^l$ are disjoint. Assume without loss of generality that

$$\int_{B_s(q_1)} |\mathbf{R}_{M_j}|^s \, d\text{Vol}_j = \max_i \int_{B_s(q_i)} |\mathbf{R}_{M_j}|^s \, d\text{Vol}_j.$$

Then,

$$\int_{B_1(p_j)} |\mathbf{R}_{M_j}|^s \, d\text{Vol}_j \leq l \int_{B_s(q_1)} |\mathbf{R}_{M_j}|^s \, d\text{Vol}_j.$$

Hence, to conclude, it suffices to show that $l \leq c(n, s) \frac{\text{Vol}_j(B_1(p_j))}{\text{Vol}_j(B_s(q_1))}$. To this aim, note that

$$\begin{aligned} \text{Vol}_j(B_1(p_j)) &\geq c(n) \text{Vol}_j(B_2(p_j)) \geq c(n) \sum_{i=1}^l \text{Vol}_j(B_{s/5}(q_i)) \\ &\geq c(n, s) l \text{Vol}_j(B_s(q_1)). \end{aligned}$$

This concludes the proof of item 1.

Proof of 2. By contradiction, assume there exist numbers $\delta_j \downarrow 0$ and pointed Riemannian manifolds (M_j, g_j, p_j) with $\text{Ric}_{M_j} \geq -\delta_j$ on M_j , and

$$\int_{B_1(p_j)} \text{Ric}_{M_j, n-2} \wedge 0 \, d\text{Vol}_j \geq -\delta_j, \quad \int_{B_1(p_j)} \mathbf{R}_{M_j} \wedge L \, d\text{Vol}_j \geq \epsilon, \quad (14)$$

such that $M_j \rightarrow X$ in pGH-sense, where (X, d, p) splits \mathbb{R}^{n-1} isometrically.

By the first assertion of Theorem 1.1, it holds

$$\int_{B_1(p_j)} \mathbf{R}_{M_j, n-1} \wedge L \, d\text{Vol}_j \rightarrow 0, \quad \text{as } j \rightarrow \infty. \quad (15)$$

Consider an orthonormal base $\{e_1, \dots, e_n\}$ for the tangent space of the manifold M_j in a point $m \in M_j$, and assume that Ric_{M_j} in this basis is represented by a diagonal matrix $A = (a_{h,l}) \in$

$\mathbb{R}^{n \times n}$ such that $a_{h,h} \leq a_{h+1,h+1}$ for every $h \in \{1, \dots, n-1\}$. It holds

$$\begin{aligned} R_{M_j} &= \sum_{h \neq l} \text{Sec}_{M_j}(e_h, e_l) = R_{M_j, n-1} + \sum_{h=1}^{n-1} \text{Sec}_{M_j}(e_n, e_h) \\ &= 2R_{M_j, n-1} - \sum_{\substack{h \neq n \\ l \neq n}} \text{Sec}_{M_j}(e_h, e_l). \end{aligned}$$

Hence,

$$R_{M_j} \wedge L \leq 2(0 \vee R_{M_j, n-1} \wedge L) - \sum_{h=1}^{n-1} \left(\sum_{\substack{l \neq h \\ l \neq n}} \text{Sec}_{M_j}(e_h, e_l) \right) \wedge 0. \quad (16)$$

By the first inequality of (14), for every $h \in \{1, \dots, n-1\}$, it follows

$$\int_{B_1(p_j)} \left(\sum_{\substack{l \neq h \\ l \neq n}} \text{Sec}_{M_j}(e_h, e_l) \right) \wedge 0 \, d\text{Vol} \geq -\delta_j.$$

Combining with (16), it holds

$$\int_{B_1(p_j)} R_{M_j} \wedge L \, d\text{Vol}_j \leq 2 \int_{B_1(p_j)} R_{M_j, n-1} \wedge L \, d\text{Vol}_j + c(n)\delta_j.$$

From (15), we infer that

$$\limsup_{j \rightarrow +\infty} \int_{B_1(p_j)} R_{M_j} \wedge L \, d\text{Vol}_j = 0,$$

contradicting the second inequality of (14). \square

We now establish some first results of ‘‘dimension drop’’ in general dimension. Building on top of Section 4, in Section 5 we will obtain stronger conclusions in dimension 3, or in general dimensions under positivity conditions (in an integral sense) on R_k , $k \leq 2$.

We recall that if a pointed non-compact manifold (M, g, p) has non-negative Ricci curvature, then Gromov pre-compactness theorem ensures that, for every sequence $(M, g/r_j, p)$ as $r_j \rightarrow +\infty$, there exists a subsequence converging in pGH sense to a metric space (X, d, p_∞) . Such spaces obtained via blow-downs are called *tangent cones at infinity* of M .

Remark 3.9 (Manifolds with asymptotically non-negative Ricci curvature). Let (M, g, p) be a manifold with asymptotically non-negative Ricci curvature, i.e.

$$\liminf_{x \rightarrow \infty} d(p, x)^2 \text{Ric}(x) \wedge 0 = 0. \quad (17)$$

Let $r_i \uparrow +\infty$ be a sequence of scales. Let $c > 0$. In the rescaled spaces $(M_i, g/r_i^2, p)$ (where $M = M_i$ as sets), consider $x \notin B_c^i(p) \subset M_i$. Then, using $d(p, x) \geq cr_i$, it holds

$$\text{Ric}_{M_i}(x) \wedge 0 = r_i^2 \text{Ric}_M(x) \wedge 0 \geq c^{-2} d(p, x)^2 \text{Ric}_M(x) \wedge 0.$$

It follows from (17) that, for every $\delta > 0$, the rescaled spaces $(M_i, g/r_i^2, p)$ have $\text{Ric}_{M_i} \geq -\delta$ outside of $B_c^i(p)$ for i large enough.

In particular, for every $c > 0$, the spaces M_i are all uniformly locally doubling outside of $B_c^i(p)$. Hence, by Gromov’s precompactness Theorem, up to passing to a subsequence, there exists a pGH limit $(X_\infty, d_\infty, x_\infty)$. Hence, assumption (17) is enough to guarantee the existence of tangent cones at infinity.

Lemma 3.10. *Let (M^n, g, p) be a Riemannian manifold with asymptotically non-negative Ricci curvature, i.e.*

$$\liminf_{x \rightarrow \infty} d(p, x)^2 \text{Ric}(x) \wedge 0 = 0.$$

Let $s \in (0, 1)$. There exists $c(s, n) > 0$ such that

$$\limsup_{r \rightarrow +\infty} \frac{\text{Vol}(B_r(p))}{\text{Vol}(B_{rs}(p))} < c(s, n).$$

Proof. Let $r_i \uparrow +\infty$ and consider the rescaled spaces $(M_i, g/r_i^2, p)$, where $M_i = M$ as sets. The statement of the lemma is equivalent to proving

$$\limsup_{i \rightarrow +\infty} \frac{\text{Vol}_i(B_1^i(p))}{\text{Vol}_i(B_s^i(p))} < c(s, n).$$

By Remark 3.9, for every $\delta > 0$, for i large enough, it holds

$$\text{Ric}_{M_i} \geq -\delta \quad \text{on } M_i \setminus B_{s/2}^i(p). \quad (18)$$

Consider now a collection of balls $\{B_{(9/100)s}^i(a_{i,j})\}_j$ in M_i such that

1. $a_{i,j} \in B_{(101/100)s}^i(p) \setminus B_s^i(p)$.
2. $\{B_{(9/100)s}^i(a_{i,j})\}_j$ cover $B_{(101/100)s}^i(p) \setminus B_s^i(p)$.
3. $\{B_{(3/100)s}^i(a_{i,j})\}_j$ are disjoint.

By item 1, there exist points

$$b_{i,j} \in B_{(2/100)s}^i(a_{i,j}) \cap [B_{(99/100)s}^i(p) \setminus B_{s/2}^i(p)].$$

Since $B_{(9/100)s}^i(a_{i,j}) \subset B_{(11/100)s}^i(b_{i,j})$, by item 2, the collection $\{B_{(11/100)s}^i(b_{i,j})\}_j$ covers $B_{(101/100)s}^i(p) \setminus B_s^i(p)$. Hence,

$$\text{Vol}_i(B_{(101/100)s}^i(p) \setminus B_s^i(p)) \leq \sum_j \text{Vol}_i(B_{(11/100)s}^i(b_{i,j})). \quad (19)$$

Since $B_{(11/100)s}^i(b_{i,j}) \subset M_i \setminus B_{s/2}^i(p)$, by (18) and Bishop-Gromov's inequality, there exists a constant $c(n)$ such that, for i large enough,

$$\sum_j \text{Vol}_i(B_{(11/100)s}^i(b_{i,j})) \leq c(n) \sum_j \text{Vol}_i(B_{(1/100)s}^i(b_{i,j})). \quad (20)$$

Since $B_{(1/100)s}^i(b_{i,j}) \subset B_{(3/100)s}^i(a_{i,j})$, by item 3, the collection $\{B_{(1/100)s}^i(b_{i,j})\}_j$ consists of disjoint balls. Moreover, by our choice of $b_{i,j}$, it holds $B_{(1/100)s}^i(b_{i,j}) \subset B_s^i(p)$. Hence,

$$\sum_j \text{Vol}_i(B_{(1/100)s}^i(b_{i,j})) \leq \text{Vol}_i(B_s^i(p)). \quad (21)$$

Combining (19), (20), and (21), for i large enough, it holds

$$\text{Vol}_i(B_{(101/100)s}^i(p)) \leq c(n) \text{Vol}_i(B_s^i(p)).$$

Iterating this procedure a sufficient number of times, we find

$$\frac{\text{Vol}_i(B_1^i(p))}{\text{Vol}_i(B_s^i(p))} \leq c(n, s),$$

concluding the proof. \square

In order to work on manifolds with asymptotically non-negative Ricci curvature, we need slightly modified versions of Theorems 1.1 and 3.8 (i.e., Theorems 3.12 and 3.13). These will follow repeating the previous proofs, once Theorem 3.11 below is established.

Theorem 3.11 is a modified version of item 2 in Theorem 2.6, where the lower bound on the Ricci curvature is now only local, but the closedness to Euclidean space is required at large scales. Another difference is that the maps provided by Theorem 3.11 are not strictly speaking (k, δ) -splitting, since their gradient is bounded just by a dimensional constant which, a priori, may not be close to 1. Nevertheless, these maps are regular enough to prove Theorem 3.12.

Let us remark that Theorem 3.11 does not follow by repeating the proof of Theorem 2.6 (see [15, Section 4.9]), as the latter uses the lower semicontinuity of Cheeger energies under pmGH convergence of spaces, which relies on global Ricci curvature lower bounds. Instead, Theorem 3.11 can be proved along the lines of the Almost-Splitting Theorem from [9]. We only sketch the proof.

Theorem 3.11. *Let $k \leq n$ be natural numbers. For every $\epsilon > 0$, there exists $\delta > 0$ such that the following hold. If (M^n, g, p) is a manifold with $\text{Ric}_M \geq -\delta$ on $B_{\delta^{-1}}(p)$ and $d_{GH}(B_{\delta^{-1}}^M(p), B_{\delta^{-1}}^{\mathbb{R}^k}(0)) < \delta$, then there exists a harmonic map $\phi : B_1(p) \rightarrow \mathbb{R}^k$ with*

1. $\int_{B_1(p)} |\nabla \phi_i \cdot \nabla \phi_j - \delta_{ij}| d\text{Vol} \leq \epsilon$ for every i, j .
2. $|\nabla \phi| \leq c(n)$.
3. $\int_{B_1(p)} |\text{Hess}(\phi)|^2 d\text{Vol} \leq \epsilon$.

Proof. Fix $\epsilon > 0$ and let $0 < \delta \ll 1$ to be chosen later. Let $\{e_i\}_{i=1}^k$ be an orthonormal basis of \mathbb{R}^k . For every $i \in \{1, \dots, k\}$, let $p_i \in B_{\delta^{-1}}^M(p)$ be a point whose image via the δ -GH map from $B_{\delta^{-1}}^M(p)$ to $B_{\delta^{-1}}^{\mathbb{R}^k}(0)$ is δ -close to $\frac{\delta^{-1}}{2}e_i$. Consider the map $b_i : M \rightarrow \mathbb{R}$ defined by

$$b_i(x) := d(x, p_i) - d(p, p_i).$$

Observe that for every $R, \epsilon' > 0$, it holds $\Delta b_i \leq \epsilon'$ in $B_R^M(p)$ if δ is small enough. Consider then the harmonic replacement $h_i \in C^\infty(B_{100}^M(p))$ (assuming $\partial B_{100}^M(p)$ to be smooth, or approximating it with smooth sets otherwise) given by

$$\Delta h_i = 0 \quad \text{on } B_{100}^M(p), \quad h_i = b_i \quad \text{on } \partial B_{100}^M(p).$$

We claim that the restriction of the map $\phi := (h_1, \dots, h_k) : B_{100}^M(p) \rightarrow \mathbb{R}^k$ to $B_1^M(p)$ has the desired properties. By [8, Lemma 9.13] (after [9]), if $\delta > 0$ is small enough, we have

$$\int_{B_1(p)} |\text{Hess}(\phi)|^2 d\text{Vol} \leq \epsilon.$$

By [8, Lemma 9.10] (again after [9]) and [20, Lemma 2.9] (which holds even if $k < n$), if $\delta > 0$ is small enough, we have

$$\int_{B_1(p)} |\nabla h_i \cdot \nabla h_j - \delta_{ij}| d\text{Vol} \leq \epsilon.$$

The bound on the gradient of ϕ follows by Cheng-Yau's gradient estimates [18]. \square

The proof of the next result is the same as the one of Theorem 1.1, replacing the use of Theorem 2.6 with Theorem 3.11.

Theorem 3.12. *Let $L, k, n \in \mathbb{N}$ with $k \leq n$ be fixed. Then,*

$$\sup \left\{ \int_{B_1(x)} R_k \wedge L d\text{Vol} : (M, g, x) \text{ has } \text{Ric}_M \geq -\delta \text{ on } B_{\delta^{-1}}(x), \text{ and } d_{pGH}(M, \mathbb{R}^k) < \delta \right\}$$

converges to 0 as $\delta \searrow 0$.

By repeating the proof of item 2 in Theorem 3.8 (and using Theorem 3.12 instead of Theorem 1.1), one obtains the following result.

Theorem 3.13. *Let $\epsilon, L \in (0, +\infty)$ and let $n \in \mathbb{N}$. There exists $\delta > 0$, such that the following holds. Let (M^n, g, p) be a manifold with $\text{Ric}_M \geq -\delta$ on $B_{\delta^{-1}}(p)$, and*

$$\int_{B_1(p)} \text{Ric}_{M, n-2} \wedge 0 d\text{Vol} \geq -\delta, \quad \int_{B_1(p)} R_M \wedge L d\text{Vol} \geq \epsilon.$$

Then, $d_{pGH}(M, \mathbb{R}^{n-1}) \geq \delta$ and $d_{pGH}(M, \mathbb{R}^n) \geq \delta$.

We now prove Theorem 1.4 from the Introduction.

Theorem 3.14. *Let (M^n, g, p) be a Riemannian manifold and let $L > 0$ and $\alpha \in (0, 2)$.*

1. Assume that

$$\liminf_{x \rightarrow \infty} d(p, x)^2 \text{Ric}(x) \wedge 0 = 0 \quad \text{and} \quad \lim_{r \rightarrow +\infty} \int_{B_r(p)} 0 \vee (r^{2-\alpha} \mathbf{R}_k) \wedge L \, d\text{Vol} = L.$$

Let (X, d, x_∞) be any tangent cone at infinity of M . If (\mathbb{R}^d, d_{eu}) is a tangent space to X at a point $x \neq x_\infty$, then $d \leq k - 1$.

2. Assume that

$$\liminf_{x \rightarrow \infty} d(p, x)^2 \text{Ric}_{n-2}(x) \wedge 0 = 0 \quad \text{and} \quad \lim_{r \rightarrow +\infty} \int_{B_r(p)} 0 \vee (r^{2-\alpha} \mathbf{R}) \wedge L \, d\text{Vol} = L.$$

Let (X, d, x_∞) be any tangent cone at infinity of M . If (\mathbb{R}^d, d_{eu}) is a tangent space to X at a point $x \neq x_\infty$, then $d \leq n - 2$.

Proof. Proof of 1. Let $r_i \uparrow +\infty$ and consider the rescaled spaces $(M, g/r_i^2, p)$. We consider the manifolds (M_i, g_i, p_i) , where $M_i = M$, $g_i = g/r_i^2$, and $p_i = p$. We denote by $B_s^i(p_i)$ and Vol_i respectively balls and volumes in M_i . By the scaling properties of curvature, these manifolds satisfy

$$\int_{B_1^i(p_i)} 0 \vee (r_i^{-\alpha} \mathbf{R}_{M_i, k}) \wedge L \, d\text{Vol}_i = \int_{B_{r_i}(p)} 0 \vee (r_i^{2-\alpha} \mathbf{R}_{M, k}) \wedge L \, d\text{Vol},$$

so that by hypothesis

$$\lim_{i \rightarrow +\infty} \int_{B_1^i(p_i)} 0 \vee (r_i^{-\alpha} \mathbf{R}_{M_i, k}) \wedge L \, d\text{Vol}_i = L. \quad (22)$$

Let (X, d, x_∞) be a pGH limit of a (non-relabelled) subsequence of (M_i, g_i, p_i) . This limit exists thanks to Remark 3.9. Let $x \in B_1(x_\infty) \setminus x_\infty$ be a point where \mathbb{R}^d is a tangent space of X . We claim that $d \leq k - 1$. If this holds, then a rescaling argument implies the same dimensional bound on tangent spaces for every $x \neq x_\infty$.

Let $l := d(x_\infty, x)$, and let $R \gg 1$. Fix $\epsilon > 0$ and let $0 < s \ll (l/2R)$ be small enough so that

- $B_s^i(p'_i) \subset B_1^i(p_i)$ for i large enough.
- $(X, d/s, x)$ is ϵ -close in pGH sense to \mathbb{R}^d .

Then, there exists points $p'_i \in M_i \setminus B_{l/2}^i(p_i)$ converging to x such that, for i large enough, $(M_i, g_i/s^2, p'_i)$ is 2ϵ -close in pGH sense to \mathbb{R}^d .

Moreover, if $y \in B_{Rs}^i(p'_i)$, it holds $d_i(y, p_i) \geq l/2 - Rs > 0$, which is equivalent to

$$d(y, p) \geq r_i^2(l/2 - Rs) > 0. \quad (23)$$

Hence,

$$\text{Ric}_{M_i}(y) \wedge 0 \geq r_i^2 \text{Ric}_M(y) \wedge 0 \geq (l/2 - Rs)^{-2} d(p, y)^2 \text{Ric}_M(y) \wedge 0.$$

Since M has asymptotically non-negative Ricci curvature, combining (23) with the fact that $r_i \uparrow +\infty$, we deduce that, for every $\delta > 0$, it holds

$$\text{Ric}_{M_i} \geq -\delta \quad \text{on } B_{Rs}^i(p'_i) \text{ for } i \text{ large enough.} \quad (24)$$

We claim that, for i large enough, we also have

$$\int_{B_s^i(p'_i)} (s^2 \mathbf{R}_{M_i, k}) \wedge L \, d\text{Vol}_i \geq L/3. \quad (25)$$

If this is true, by choosing $\epsilon > 0$ small enough and combining with (24), we deduce that $d \leq k - 1$ by Theorem 3.12.

To prove (25), it is enough to show that

$$\frac{\text{Vol}_i(B_s^i(p'_i) \cap \{R_{M_i,k} \geq s^{-2}L/2\})}{\text{Vol}_i(B_s^i(p'_i))} \rightarrow 1, \quad \text{as } i \rightarrow +\infty. \quad (26)$$

Assume by contradiction that (26) fails. Then there exists $\delta > 0$ such that, up to a non-repeated subsequence:

$$\text{Vol}_i(B_s^i(p'_i) \cap \{R_{M_i,k} \geq s^{-2}L/2\}) < (1 - \delta)\text{Vol}_i(B_s^i(p'_i)).$$

Hence, for i large enough:

$$\begin{aligned} & \int_{B_1^i(p_i)} 0 \vee (r_i^{-\alpha} R_{M_i,k}) \wedge L \, d\text{Vol}_i \\ & \leq L \text{Vol}_i(B_1^i(p_i) \setminus B_s^i(p'_i)) + L(1 - \delta)\text{Vol}_i(B_s^i(p'_i)) + r_i^{-\alpha} s^{-2}L/2 \text{Vol}_i(B_s^i(p'_i)). \end{aligned}$$

Using that $B_s^i(p'_i) \subset B_1^i(p_i)$ for i large enough, we infer that

$$\begin{aligned} & \int_{B_1^i(p_i)} 0 \vee (r_i^{-\alpha} R_{M_i,k}) \wedge L \, d\text{Vol}_i \\ & \leq L \text{Vol}_i(B_1^i(p_i)) + (r_i^{-\alpha} s^{-2}L/2 - \delta L)\text{Vol}_i(B_s^i(p'_i)). \end{aligned}$$

Using Lemma 3.10 in the last inequality, we find a constant $c(s, n) > 0$ such that

$$\int_{B_1^i(p_i)} 0 \vee (r_i^{-\alpha} R_{M_i,k}) \wedge L \, d\text{Vol}_i \leq L + (r_i^{-\alpha} s^{-2}L/2 - \delta L)c(s, n).$$

Since $r_i \uparrow \infty$, this contradicts (22), proving item 1.

Proof of 2. Bounds as in (25) follow repeating the previous argument for the integral appearing in item 2. To conclude, one then uses Theorem 3.13 instead of Theorem 3.12. \square

Remark 3.15. In Theorem 3.14, if we replace the limit symbols with lim-sup symbols, we obtain that *there exists* a tangent cone satisfying the corresponding dimensional condition.

Moreover, one can replace $r^{2-\alpha}$ in the statement with an arbitrary positive measurable function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $f(r)/r^2 \rightarrow 0$ as $r \uparrow +\infty$.

The proofs of these facts are the same as the one of Theorem 3.14.

Remark 3.16. Theorem 3.14 fails for $\alpha = 0$. We give a counterexample for $k = 1$, and all the other cases follow by taking products with real factors. Indeed, consider the paraboloid $\Sigma := \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}$. The sectional curvature of Σ is $K(z) \sim 1/4z^2$ as $z \rightarrow +\infty$. For every $r > 0$, the subset $\{z \leq r\} \subset \Sigma$ is a ball of radius $f(r) \geq r$. Hence, on the paraboloid, for a positive constant $c > 0$, it holds $f(r)^2 K(z) \geq c > 0$ for all points in $\{z \leq r\}$. Hence,

$$\lim_{r \rightarrow +\infty} \int_{B_{f(r)}^\Sigma(0)} (f(r)^2 K) \wedge c \, \text{Vol} = c.$$

At the same time, the tangent cone at infinity of the paraboloid is a half line.

Remark 3.17. Let $L > 0$ and $\alpha \in (0, 2)$ be fixed. Another immediate consequence of Theorem 3.14 is that if a manifold (M, g) with asymptotically non-negative Ricci curvature has $\text{Ric}_M \geq f$ for a function f such that

$$\lim_{r \rightarrow +\infty} \int_{B_r(p)} 0 \vee (r^{2-\alpha} f) \wedge L \, d\text{Vol} = L,$$

then M is compact. A similar result was obtained in [14, Theorem 4.8], where also the case $\alpha = 2$ was addressed, under a pointwise (rather than integral) bound on Ric_M .

We now turn our attention to the topological implications of Theorem 3.14. The next result corresponds to Theorem 1.5 from the introduction.

Theorem 3.18. *Let (M^n, g, p) be a non-compact Riemannian manifold with $n \geq 3$ and let $\alpha \in (0, 2)$. Let $2 \leq k \leq n$ be a natural number.*

1. *If $\text{Ric}_M \geq 0$ and*

$$\limsup_{r \rightarrow +\infty} \left(r^{2-\alpha} \min_{B_r(p)} R_k \right) > 0, \quad (27)$$

then $b_1(M) \leq k - 2$.

2. *If $\text{Ric}_{n-2} \geq 0$ and*

$$\limsup_{r \rightarrow +\infty} \left(r^{2-\alpha} \min_{B_r(p)} R \right) > 0, \quad (28)$$

then $b_1(M) \leq n - 3$.

Proof. Proof of 1: Let $(\tilde{M}, \tilde{g}, \tilde{p})$ be a covering of M , and let $\pi : \tilde{M} \rightarrow M$ be the covering map. We claim that \tilde{M} has a tangent cone at infinity of dimension $\leq k - 1$.

We first show that \tilde{M} satisfies condition (27). Let $r > 0$ and let $\tilde{x} \in \tilde{M}$ be a point with $\tilde{d}(\tilde{x}, \tilde{p}) < r$. Then, $\pi(\tilde{x}) \in B_r(p)$, so that

$$r^{2-\alpha} \min_{B_r(\tilde{p})} R_{\tilde{M}, k} \geq r^{2-\alpha} \min_{B_r(p)} R_{M, k}.$$

Hence, also \tilde{M} satisfies (27).

Let now $r_i \uparrow +\infty$ and let $\epsilon > 0$ be such that

$$\lim_{i \rightarrow +\infty} \left(r_i^{2-\alpha} \min_{B_{r_i}(\tilde{p})} R_{\tilde{M}, k} \right) = \epsilon.$$

It follows that

$$\lim_{i \rightarrow +\infty} \int_{B_{r_i}(\tilde{p})} (r_i^{2-\alpha} R_{\tilde{M}, k}) \wedge \epsilon \tilde{d}\text{Vol} = \epsilon.$$

As a consequence, by item 1 of Theorem 3.14 and Remark 3.15, \tilde{M} has a tangent cone at infinity of dimension $\leq k - 1$. The statement now follows by repeating the argument of [62, Theorem 1.3], where instead of using [62, Theorem 1.6] one uses that \tilde{M} has a tangent cone at infinity of dimension $\leq k - 1$.

Proof of 2: The argument is analogous to the proof of item 1, now using item 2 of Theorem 3.14. \square

Remark 3.19. In the proof of the previous result, one cannot weaken condition (27) to

$$\limsup_{x \rightarrow +\infty} d(p, x)^{2-\alpha} R_k(x) > 0. \quad (29)$$

Indeed, (29) does not pass to covering spaces. To see this, let $k = 3$ and consider a product $\mathbb{R} \times S^1 \times S^2$ with the standard metric. We can now perturb the metric in a compact set so that it has negative R_3 at a point. The universal cover of this perturbed space will then have a sequence of points with negative R_3 escaping to infinity. A similar argument holds for condition (28).

Next, we prove Theorem 1.6 from the Introduction.

Theorem 3.20. *Let $\epsilon, s, v \in (0, 1)$, let $L, D \in (0, +\infty)$, and let $k, n \in \mathbb{N}$ with $k \leq n$. There exists $\delta > 0$ such that for every manifold (M^n, g, p) with $\text{Ric}_M \geq -\delta$, $\text{diam}(M) \leq D$, $\text{Vol}(B_1(p)) \geq v$, the following holds.*

1. *If $\int_{B_1(p)} R_k \wedge L d\text{Vol} \geq \epsilon$, then $b_1(M) \leq k - 1$.*

2. *If $\int_{B_1(p)} |R|^s d\text{Vol} \geq \epsilon$, then $b_1(M) \leq n - 2$.*

Proof. We consider case 1 first. Assume by contradiction that the statement fails. Hence, there exist $v, D, \epsilon_0 > 0$, a sequence $\delta_j \downarrow 0$ and pointed Riemannian manifolds (M_j^n, g_j, p_j) with $\text{Ric}_{M_j} \geq -\delta_j$, $\text{diam}(M_j) \leq D$, $\text{Vol}_j(B_1(p_j)) \geq v$,

$$\int_{B_1(p_j)} \mathbf{R}_{M_j, k} \wedge L \, d\text{Vol}_j \geq \epsilon_0,$$

and $b_1(M_j) \geq k$. Up to a subsequence, M_j GH-converge to a metric space (X, d) . Thanks to the non-collapsing assumption, the metric measure space (X, d, \mathbf{H}^n) is an $\text{RCD}(0, n)$ space while, thanks to Proposition 2.16, it holds $b_1(X) \geq k$. By Proposition 2.20, the universal cover (\tilde{X}, \tilde{d}) of X splits \mathbb{R}^k isometrically.

Let $\tilde{p} \in \tilde{X}$. By Proposition 2.21, the pointed metric space $(\tilde{X}, \tilde{d}, \tilde{p})$ is the pGH-limit of a sequence $(\tilde{M}_j, \tilde{g}_j, \tilde{p}_j)$, where each \tilde{M}_j covers M_j . These covering spaces satisfy

$$\int_{B_1(\tilde{p}_j)} \mathbf{R}_{\tilde{M}_j, k} \wedge L \, d\tilde{\text{Vol}}_j \geq \int_{B_1(p_j)} \mathbf{R}_{M_j, k} \wedge L \, d\text{Vol}_j \geq v\epsilon_0 \geq c(n)v\epsilon_0 \tilde{\text{Vol}}_j(B_1(\tilde{p}_j)),$$

contradicting Theorem 1.1.

Assertion 2 follows in an analogous way, using Theorem 3.8 instead of Theorem 1.1. \square

4 The segment neighbourhood property (SNP) and some metric lemmas

The goal of this section is to show that metric spaces satisfying the segment neighbourhood property (see Definition 4.2), are contained in a neighbourhood of controlled width of an isometrically embedded 1-dimensional manifold (see Theorem 4.17). This will be a key step to prove Theorem 1.2. We mention that a similar (but unrelated) property for metric spaces was studied in [34].

Throughout the section, all metric spaces (X, d) will be assumed to be proper, complete, separable, and geodesic. If not otherwise specified, all curves are assumed to be of finite length and parametrized by arc-length. The length functional is denoted by $L(\cdot)$.

Definition 4.1 (Ray, segment, line). Let (X, d) be a metric space. A function $r : [0, +\infty) \rightarrow X$ is called a *ray* if $d(r(t), r(s)) = |t - s|$ for every $t, s \in [0, +\infty)$. Similarly $r : [a, b] \rightarrow X$ is called a *segment* if $d(r(t), r(s)) = |t - s|$ for every $t, s \in [a, b]$ and $r : \mathbb{R} \rightarrow X$ is called a *line* if $d(r(t), r(s)) = |t - s|$ for every $t, s \in \mathbb{R}$.

When considering a ray (resp. a segment or a line) r in X , by a slight abuse in order to keep notation short, we denote by r both the function $r : [0, +\infty) \rightarrow X$ and its image. Moreover, $[r(a), r(b)]$ denotes the set $r([a, b]) \subset X$, for every $0 < a < b$.

In this section we deal with metric spaces satisfying the following condition, called *segment neighbourhood property*.

To that aim, let us first fix some notation. Given a closed set $K \subset X$ and a point $x \in X$, we call *the projection* of x into K the subset $\pi_K(x) \subset K$ defined by

$$\pi_K(x) := \{y \in K : d(x, K) = d(x, y)\}.$$

Given a subset $A \subset X$, we call *the projection* of A into K the subset $\pi_K(A) \subset K$ defined by

$$\pi_K(A) := \bigcup_{x \in A} \pi_K(x).$$

Moreover, for every $y \in K$, we denote by $\pi_K^{-1}(y) \subset X$ the subset

$$\pi_K^{-1}(y) := \{x \in X : d(x, K) = d(x, y)\}. \quad (30)$$

Definition 4.2 (Segment neighbourhood property, (SNP)). We say that the metric space (X, d) satisfies the *segment neighbourhood property* (SNP, for short), if there exist $R, D > 0$, with $R \geq 20D$, such that the following holds. For every segment r of length greater than $2R$ in X and for every $t \in (R, L(r) - R)$ and $x \in \pi_r^{-1}(r(t))$, it holds $d(x, r) < D$.

Remark 4.3. Let (X, d) be a metric space satisfying the (SNP) (see Definition 4.2) and let r be a ray in X . If a point $x \in X$ has a projection on r which lies in $(r(R), r(\infty))$, then $d(x, r) \leq D$. This fact follows applying (SNP) to the segments $[r(0), r(T)]$ for $T > 2R$.

Remark 4.4. If (X, d) has (SNP) for $R, D > 0$, then it also satisfies (SNP) for $R', D' > 0$ for every $R' \geq R$ and $D' \leq D$.

Lemma 4.5. Let (X, d) be a metric space and let $C_1 \subset C_2 \subset X$ be closed sets. The set

$$K := \{x \in X : x \text{ has a projection on } C_2 \text{ lying in } C_1\}$$

is closed.

In particular, if $\alpha : [0, b] \rightarrow X$ is a curve, then the set

$$I := \{s \in [0, b] : \alpha(s) \text{ has a projection on } C_2 \text{ lying in } C_1\}$$

is closed as well.

Proof. Let $\{x_i\}_{i \in \mathbb{N}} \in K$ be a sequence converging to $x_\infty \in X$. We claim that $x_\infty \in K$. For every $i \in \mathbb{N}$, let γ_i be a segment from x_i to a point $\gamma_i(L(\gamma_i)) \in C_1$ such that $d(x_i, C_2) = L(\gamma_i)$. If $L(\gamma_i) \rightarrow 0$ as $i \rightarrow +\infty$, then $x_\infty \in C_1$, so that $x_\infty \in K$.

If instead $L(\gamma_i) \not\rightarrow 0$, modulo passing to a subsequence, the segments γ_i converge to a finite segment γ from x_∞ to a limit point of the sequence $\{\gamma_i(L(\gamma_i))\}_{i \in \mathbb{N}}$. Since C_1 is closed, all limit points of $\{\gamma_i(L(\gamma_i))\}_{i \in \mathbb{N}}$ lie in C_1 , so that also $\gamma(L(\gamma)) \in C_1$. By lower semicontinuity of the length:

$$d(x_\infty, C_2) \leq L(\gamma) \leq \liminf_{i \in \mathbb{N}} L(\gamma_i) = \liminf_{i \in \mathbb{N}} d(x_i, C_2) = d(x_\infty, C_2).$$

In particular, $d(x_\infty, C_2) = L(\gamma)$, so that $x_\infty \in K$. Hence, K is closed. Being the preimage of K via the continuous function α , also I is closed. \square

The next lemma contains the key technical properties of metric spaces satisfying the (SNP) (see Definition 4.2).

Lemma 4.6. Let (X, d) be a metric space satisfying the (SNP) (see Definition 4.2). Let $\alpha : [0, L(\alpha)] \rightarrow X$ be a unit speed curve.

1. Let l be a segment with $L(l) > 2R$. Let $t \in (R, L(l) - R)$, and let $u \in [0, R] \cup [L(l) - R, L(l)]$. If $l(t) \in \pi_l(\alpha(0))$ and $l(u) \in \pi_l(\alpha(L(\alpha)))$, then the set $\pi_l(\alpha)$ contains a $3D$ -net of either $[l(R), l(t)]$ or $[l(t), l(L(l) - R)]$.
2. Let l be a segment with $L(l) > 2R$. Let $t, u \in (R, L(l) - R)$ with $t < u$. If $l(t) \in \pi_l(\alpha(0))$ and $l(u) \in \pi_l(\alpha(L(\alpha)))$, then $\pi_l(\alpha)$ contains a $3D$ -net of either $[l(t), l(u)]$ or $[l(R), l(t)] \cup [l(u), l(L(l) - R)]$.
3. Let r be a ray, let $t > R$ and let $u \in [0, R]$. If $r(t) \in \pi_r(\alpha(0))$ and $r(u) \in \pi_r(\alpha(L(\alpha)))$, then $\pi_r(\alpha)$ contains a $3D$ -net of $[r(R), r(t)]$.

Proof. We consider item 1 first. Assume by contradiction that the claim fails. Hence, there are $x_1 \in [R, t]$ and $x_2 \in [t, L(l) - R]$ such that

$$\pi_l(\alpha) \cap B_{3D}(l(x_1)) = \pi_l(\alpha) \cap B_{3D}(l(x_2)) = \emptyset. \quad (31)$$

Consider

$$\tilde{s} := \max \left\{ s \in [0, L(\alpha)] : \alpha(s) \text{ has a projection on } l \right. \\ \left. \text{contained in } [l(x_1), l(x_2)] \right\}.$$

In order to show that \tilde{s} is well defined, it is sufficient to observe that the set appearing in its definition is:

- *closed*, thanks to Lemma 4.5;
- *non-empty*, since by assumption $l(t) \in \pi_l(\alpha)$.

We claim that $\tilde{s} < L(\alpha)$. If this is not the case, then, combining with (31), $\alpha(L(\alpha))$ admits a projection on l contained in $[l(x_1 + 3D), l(x_2 - 3D)]$. By the (SNP), $\alpha(L(\alpha))$ is D -close to $[l(x_1 + 3D), l(x_2 - 3D)]$, so that it cannot have a projection on l given by $l(u)$ for $u \in [0, R] \cup [L(l) - R, L(l)]$, a contradiction. This shows that $\tilde{s} < L(\alpha)$.

Hence, for every $\epsilon > 0$, there is a projection p_ϵ of $\alpha(\tilde{s} + \epsilon)$ on l . Let p be a projection of $\alpha(\tilde{s})$ on l lying in $[l(x_1), l(x_2)]$. By the (SNP), it holds $d(p, \alpha(\tilde{s})) < D$. It follows that $d(p_\epsilon, \alpha(\tilde{s} + \epsilon)) < D + \epsilon$. Combining these facts, it holds

$$d(p, p_\epsilon) \leq d(p, \alpha(\tilde{s})) + d(\alpha(\tilde{s}), \alpha(\tilde{s} + \epsilon)) + d(\alpha(\tilde{s}), p_\epsilon) \leq 2D + 2\epsilon.$$

This contradicts the fact that $p_\epsilon \in [l(0), l(x_1 - 3D)] \cup [l(x_2 + 3D), l(L(l))]$ by (31).

The proof of item 2 is very similar to the one of item 1, and for this reason it is only sketched. Assume by contradiction that the claim fails. Hence, there are $x_1 \in [t, u]$ and $x_2 \in [R, t] \cup [u, L(l) - R]$ such that, as in (31),

$$\pi_l(\alpha) \cap B_{3D}(l(x_1)) = \pi_l(\alpha) \cap B_{3D}(l(x_2)) = \emptyset.$$

One can now assume without loss of generality that $x_2 \geq x_1$ (the other case being analogous), and the proof can be carried out in the same way as for item 1.

We now consider item 3. Since the image of α is compact, $\pi_l(\alpha)$ is a bounded subset of l . Hence, $\pi_l(\alpha) \subset [r(0), r(T)]$ for some $T > 0$. We consider now the segment $l := [r(0), r(T + 2R)]$. It is easy to check that $\pi_l(\alpha) = \pi_r(\alpha)$. By item 1, $\pi_l(\alpha)$ contains a $3D$ -net of either $[l(R), l(t)]$ or $[l(t), l(L(l) - R)]$. This last case cannot happen as $[l(t), l(L(l) - R)] = [r(t), r(T + R)]$ and $\pi_l(\alpha) \subset [r(0), r(T)]$. Hence, $\pi_l(\alpha)$ (which again coincides with $\pi_r(\alpha)$) contains a $3D$ -net of $[l(R), l(t)] = [r(R), r(t)]$. \square

The next few lemmas are needed to prove Proposition 4.11, which gives the macroscopic description of non-compact spaces having the (SNP).

Lemma 4.7. *Let (X, d) be a metric space satisfying the (SNP) (see Definition 4.2). Let r be a ray in X and let $y \in X$ with $d(r, y) > D$. Let $t > 0$ and let γ be a curve connecting y and $r(t)$. Then, for every $s \in [0, t]$, it holds $d(r(s), \gamma) \leq 2R$.*

Proof. We parametrize γ by arc length in such a way that $\gamma(0) = r(t)$ and $\gamma(L(\gamma)) = y$. By the (SNP) and the fact that $d(y, r) > D$, it follows that $\pi_r(\gamma(L(\gamma))) \in [r(0), r(R)]$. By Lemma 4.6, $\pi_r(\gamma)$ contains a $3D$ -net of $[r(R), r(t)]$. Using again the (SNP), it follows that $[r(R), r(t)]$ is in a $7D$ -neighbourhood of γ . The statement then follows. \square

Definition 4.8 (Divergent rays). We say that two rays r_1, r_2 in a metric space (X, d) are *divergent* if

$$\limsup_{t \rightarrow +\infty} d(r_1(t), r_2(t)) = +\infty.$$

Lemma 4.9. *Let (X, d) be a metric space satisfying the (SNP) (see Definition 4.2). If there are two divergent rays in X , then X contains a line.*

Proof. Let r_1, r_2 be divergent rays. Consider a sequence $t_j \uparrow +\infty$ such that $d(r_1(t_j), r_2(t_j)) \rightarrow +\infty$ and let l_j be segments from $r_1(t_j)$ to $r_2(t_j)$. By Lemma 4.7, each one of these segments satisfies

$$d(l_j, r_1(0)) \leq c(R, D). \tag{32}$$

Since $d(r_1(0), r_1(t_j)) \rightarrow +\infty$ and $d(r_1(0), r_2(t_j)) \rightarrow +\infty$, condition (32) implies that l_j converges (modulo passing to a subsequence) to a line. This concludes the proof. \square

Lemma 4.10. *Let (X, d) be a metric space satisfying the (SNP) (see Definition 4.2). Let r be a ray in X . Let $y \in X$ be a point such that $d(y, r) > D$. Let r_y be a ray obtained as a limit of the segments connecting y and $r(t_j)$, for some sequence $t_j \rightarrow +\infty$. Then, r is contained in the $2R$ -neighbourhood of r_y .*

Proof. It follows immediately from Lemma 4.7. \square

The next proposition gives the macroscopic description of non-compact spaces satisfying the (SNP).

Proposition 4.11. *Let (X, d) be a metric space satisfying the (SNP) (see Definition 4.2). Assume that X is non-compact and it does not contain a line. There exists a ray r in X whose $10R$ -neighbourhood contains X .*

Proof. We first show that, given any ray r in X , there exists $D'(X, r) > 0$ such that X is contained in the D' -neighbourhood of r .

If the claim were false, we would find points p_j at arbitrarily large distance from r . For each point p_j , let r_j be a ray obtained as a limit of the segments from p_j to $r(t_i)$ for some sequence $t_i \rightarrow +\infty$. By Lemma 4.7, each ray r_j has a point q_j whose distance from $r(0)$ is less than $2R$. Since the distance between $p_j = r_j(0)$ and q_j goes to infinity as j increases, and the sequence (q_j) is contained in a compact subset of X , then, up to a subsequence, the rays r_j converge to a line l contained in X , a contradiction.

We now prove the statement of the lemma. If X is contained in the $10R$ -neighbourhood of X there is nothing to prove. Otherwise, we claim that there exists an element $y \in X$ at maximal distance from r . Let $y_i \in X$ be a sequence maximizing the distance from r . We can assume that $9R \leq d(y_i, r) \leq D'(X)$. By the (SNP), all projections of y_i on r are contained in $[r(0), r(R)]$. It follows that the sequence $\{y_i\}_{i \in \mathbb{N}}$ is precompact, so that there exists a limit point $y \in X$ at maximal distance from r .

Let r_y be the ray which is the limit of the segments joining y and $r(t_i)$ for a sequence $t_i \uparrow +\infty$. By Lemma 4.10, for every $t \geq 0$, it holds $d(r(t), r_y) \leq 2R$. Since $d(y, r) \geq 10R$, it follows that all projections of r on r_y are contained in $[r_y(5R), r_y(+\infty))$.

Assume now by contradiction that there exists $y' \in X$ such that $d(y', r_y) \geq 10R$. Let γ be a segment realizing the distance between y' and r . Let $r(t_\gamma)$ be the endpoint of γ . Let $r_y(t_\gamma^y)$ be a projection on r_y of $r(t_\gamma)$. By our previous remarks, $t_\gamma^y \geq 5R$. By Lemma 4.6, $\pi_{r_y}(\gamma)$ contains a $3D$ -neighbourhood of $[r_y(R), r_y(5R)]$. Hence, γ contains a point p_1 which is $7D$ -close to $r_y(R)$. Recall that we also have $d(y', r_y) \geq 10R$. Hence,

$$\begin{aligned} d(y', r) &= L(\gamma) = d(y', p_1) + d(p_1, r) \geq d(y', r_y) - 7D + d(y, r) - R - 7D \\ &\geq d(y, r) + 9R - 14D > d(y, r) \end{aligned}$$

Hence, y was not an element maximizing the distance from r , a contradiction. \square

We now turn our attention to the study of compact spaces having the (SNP). The key result in this sense is Proposition 4.16. We first need a few lemmas.

Lemma 4.12. *Let (X, d) be a metric space satisfying the (SNP). Let l be a segment in X and assume that $\text{length}(l) \geq 10R$. Let \bar{l} be a segment starting from $l(0)$ with $\pi_l(\bar{l}(L(\bar{l}))) \subset [l(0), l(R)] \cup [l(L(l) - R), l(L(l))]$. If \bar{l} has a projection on l lying in $[l(3R), l(L(l) - 3R)]$, then $\pi_l(\bar{l})$ contains a $3D$ -net of $[l(R), l(L(l) - R)]$.*

Proof. Let $t_0 \in [0, L(\bar{l})]$ be such that $\bar{l}(t_0)$ has a projection on l given by $l(t)$ for $t \in [3R, L(l) - 3R]$. Let $\bar{l}_1 := \bar{l}|_{[0, t_0]}$ and $\bar{l}_2 := \bar{l}|_{[t_0, L(\bar{l})]}$.

Hence, both \bar{l}_1 and \bar{l}_2 have an endpoint whose projection on l lies in $[l(0), l(R)] \cup [l(L(l) - R), l(L(l))]$.

By item 1 of Lemma 4.6, $\pi_l(\bar{l}_1)$ contains a $3D$ -net of either $[l(R), l(t)]$ or $[l(t), l(L(l) - R)]$, and the same holds for $\pi_l(\bar{l}_2)$.

Assume by contradiction that both $\pi_l(\bar{l}_1)$ and $\pi_l(\bar{l}_2)$ contain a $3D$ -net of $[l(R), l(t)]$. In this case, there exist $t_1 \leq t_0 \leq t_2$ such that $\bar{l}(t_1)$ and $\bar{l}(t_2)$ are $7D$ -close to $l(R)$. Hence,

$$d(\bar{l}(t_1), \bar{l}(t_2)) \leq 14D. \quad (33)$$

Since $\bar{l}(t_0)$ is D -close to a point $l(t)$ in $[l(3R), l(L(l) - 3R)]$, it follows

$$t_0 - t_1 = d(\bar{l}(t_0), \bar{l}(t_1)) \geq d(l(R), l(t)) - 8D \geq 2R - 8D,$$

so that $d(\bar{l}(t_1), \bar{l}(t_2)) \geq 2R - 5D$. This violates (33).

By the same argument, it is not possible that both $\pi_l(\bar{l}_1)$ and $\pi_l(\bar{l}_2)$ contain a $3D$ -net of $[l(t), l(L(l) - R)]$, concluding the proof. \square

Lemma 4.13. *Let (X, d) be a metric space satisfying the (SNP). Let l be a segment in X and assume that $\text{length}(l) \geq 10R$. Let $p \in X$ be a point with $d(p, l) \geq 10R$ and let \bar{l} be a segment connecting p to $l(0)$. If \bar{l} has a projection on l lying in $[l(3R), l(L(l) - 3R)]$, then $L(\bar{l}) > L(l)$.*

Proof. Since $d(p, l) \geq 10R$, the (SNP) implies that $\pi_l(p) \subset [l(0), l(R)] \cup [l(L(l) - R), l(L(l))]$. By Lemma 4.12, there is a point $\bar{l}(s_0)$ which is $7D$ -close to $l(L(l) - R)$. Using triangle inequality, this implies

$$L(\bar{l}) = d(p, \bar{l}(s_0)) + d(\bar{l}(s_0), l(0)) \geq d(p, l) - 7D + d(l(0), l(L(l) - R)) - 7D.$$

Using that $d(p, l) \geq 10R$ and that l is a segment, it follows $L(\bar{l}) \geq 9R - 14D + L(l) > L(l)$. \square

Lemma 4.14. *Let (X, d) be a compact metric space satisfying the (SNP). Let l be a segment of maximal length in X and assume that $\text{length}(l) \geq 10R$. Let $p \in X$ be a point with $d(p, l) \geq 10R$ and let γ_1, γ_2 be segments connecting p respectively to $l(L(l))$ and $l(0)$. Let $b_1, b_2 \in \{\gamma_1, \gamma_2, l\}$ with $b_1 \neq b_2$. Then $\pi_{b_1}(b_2) \subset [b_1(0), b_1(3R)] \cup [b_1(L(b_1) - 3R), b_1(L(b_1))]$.*

Proof. If $b_1 = l$, the statement follows by Lemma 4.13.

We now consider the case $b_1 \in \{\gamma_1, \gamma_2\}$ and $b_2 = l$. If the statement fails, l has a projection on b_1 lying in $(b_1(3R), b_1(L(b_1) - 3R))$. Consider the orientation of l such that $l(0) \in b_1$. Then, $\pi_{b_1}(l(L(l))) \subset [b_1(0), b_1(R)] \cup [b_1(L(b_1) - R), b_1(L(b_1))]$ since otherwise, using the (SNP), it follows that l is shorter than b_1 . Hence, by Lemma 4.12, there exists $s > 0$ such that $l(s)$ is $R + 7D$ -close to p . This contradicts that $d(p, l) \geq 10R$.

Finally, we consider the case $b_1 = \gamma_1$ and $b_2 = \gamma_2$ (the case $b_2 = \gamma_1$ and $b_1 = \gamma_2$ is identical).

We consider orientations such that $\gamma_1(L(\gamma_1)) = \gamma_2(0)$, $\gamma_2(L(\gamma_2)) = l(0)$ and $l(L(l)) = \gamma_1(0)$.

Assume by contradiction that $\gamma_1(t)$, with $t \in (3R, L(\gamma_1) - 3R)$ is a projection of γ_2 on γ_1 . By Lemma 4.12, there exists $s > 0$ such that $\gamma_2(s)$ is $R + 7D$ -close to $\gamma_1(0) = l(L(l))$. It follows that

$$L(\gamma_2) = d(p, \gamma_2(s)) + d(\gamma_2(s), l(0)) \geq d(p, l) - R - 7D + L(l) - R - 7D > L(l).$$

This is a contradiction. \square

Corollary 4.15. *Let (X, d) and γ_1, γ_2, l be as in the previous lemma. Let $\{b_1, b_2, b_3\} = \{\gamma_1, \gamma_2, l\}$. If $b_1(0) \notin b_2$, the follows happen.*

1. $d(b_1(0), b_2(L(b_2)/2)) \geq \min\{L(b_1), L(b_3)\} + L(b_2)/2 - 10R$.
2. $d(b_1(L(b_1)/2), b_2(L(b_2)/2)) \geq L(b_1)/2 + L(b_2)/2 - 10R$.

Proof. We only prove the first point, the proof of the second one being analogous. Let γ be a segment connecting $b_1(0)$ and $b_2(L(b_2)/2)$. If γ has a point which is $5R$ -close to either $b_2(3R)$ or $b_2(L(b_2) - 3R)$, the statement follows by triangle inequality.

By the previous lemma, a projection of $b_1(0)$ on b_2 lies in $[b_2(3R), b_2(L(b_2) - 3R)]$. If such projection lies in $[b_2(R), b_2(3R)]$ or $[b_2(L(b_2) - 3R), b_2(L(b_2) - R)]$, then γ has a point which is $5R$ -close to either $b_2(3R)$ or $b_2(L(b_2) - 3R)$ by the (SNP), otherwise the same conclusion follows by Lemma 4.6. \square

Proposition 4.16. *Let (X, d) be a compact metric space satisfying the (SNP) (see Definition 4.2). Let l be a segment of maximal length in X and assume that $L(l) \geq 200R$. If X is not contained in a $200R$ -neighbourhood of l , then there exists $L > 0$ and a distance preserving map $\phi : S_L^1 \rightarrow X$, such that X is contained in the D -neighbourhood of $\phi(S_L^1)$. In particular, $L \geq 50R$.*

Proof. Let $p \in X$ with $d(p, X) > 200R$. Let γ_1 and γ_2 be segments from p to $l(L(l))$ and $l(0)$ respectively. Note that both these segments have length greater than $200R$. On γ_1 , we consider the subsegment

$$I^{\gamma_1} := [\gamma_1(L(\gamma_1)/2 - 5R), \gamma_1(L(\gamma_1)/2 + 5R)].$$

We also consider subsegments I^{γ_2} and I^l defined in the same way. Observe that these subsegments are independent of the chosen parametrization by arc-length of γ_1 , γ_2 and l .

To construct the map ϕ of the statement, we will consider the shortest loop passing near each one of the previously defined segments. To this aim, let

$$\begin{aligned} A := \left\{ \phi : [0, L(\gamma_1 + \gamma_2 + l)] \rightarrow X \mid \phi \text{ is 1-Lipschitz, } \phi(0) = \phi(L(\gamma_1 + \gamma_2 + l)), \right. \\ \left. \exists t^{\gamma_1}, t^{\gamma_2}, t^l \in [0, L(\gamma_1 + \gamma_2 + l)] \text{ such that} \right. \\ \left. \phi(t^b) \text{ has a projection on } b \text{ lying in } I^b \text{ for every } b \in \{\gamma_1, \gamma_2, l\} \right\}. \end{aligned}$$

Notice that $A \neq \emptyset$ since $\gamma_1 + \gamma_2 + l \in A$. We claim that if $\{\phi_i\}_{i \in \mathbb{N}} \subset A$ and $\phi_i \rightarrow \phi_\infty$ uniformly, then $\phi_\infty \in A$. First of all ϕ_∞ is 1-Lipschitz and satisfies $\phi_\infty(0) = \phi_\infty(L(\gamma_1 + \gamma_2 + l))$ trivially. Let $b \in \{\gamma_1, \gamma_2, l\}$. Consider for every $i \in \mathbb{N}$ the corresponding t_i^b . Modulo passing to a subsequence, there exists $t_\infty^b \in [0, L(\gamma_1 + \gamma_2 + l)]$ such that $t_i^b \rightarrow t_\infty^b$. By Lemma 4.5, $\phi_\infty(t_\infty^b)$ has a projection on b lying in I^b . It follows that $\phi_\infty \in A$ as claimed.

Consider now a sequence $\phi_i \in A$ minimizing the length functional. Each image of ϕ_i has a projection on γ_1 lying in $I_1^{\gamma_1}$, so that by the (SNP), it has a point which is D -close to $I_1^{\gamma_1}$. Hence, all the images of the maps ϕ_i are contained in a bounded set of X . Since the ϕ_i are all 1-Lipschitz, by Ascoli-Arzelà theorem, there exists (modulo passing to a subsequence) ϕ_∞ such that $\phi_i \rightarrow \phi_\infty$ uniformly. By our previous point $\phi_\infty \in A$. Since the length functional is lower semicontinuous w.r.t. uniform convergence, ϕ_∞ is an element of minimal length in A .

Let $T := L(\phi_\infty)$ and consider the reparametrization $\phi_\infty : [0, T] \rightarrow X$ by arc-length, and the induced map (denoted again by ϕ_∞) $\phi_\infty : S_L^1 \rightarrow X$, where $L := T/2\pi$. By Corollary 4.15, it follows that $T \geq 60R$, and that the distance in S_L^1 between any two points of $\{t^{\gamma_1}, t^{\gamma_2}, t^l\}$ is at least $20R$ (in particular they are all distinct).

We claim that $\phi_\infty : S_L^1 \rightarrow X$ is distance preserving and that X is contained in the D -neighbourhood of $\phi_\infty(S_L^1)$. We divide the proof of the claim in steps.

Step 1. We claim that, for $b \in \{\gamma_1, \gamma_2, l\}$, the projection $\pi_b(\phi_\infty(S_L^1))$ contains a $3D$ -net of $[b(3R), b(L(b) - 3R)]$.

Let $b_1, b_2 \in \{\gamma_1, \gamma_2, l\}$ with $b_1 \neq b_2$. Let $t^{b_1}, t^{b_2} \in [0, T]$ be the numbers given in the definition of A relative to ϕ_∞ . Modulo reparametrizing ϕ_∞ and $\gamma_1 + \gamma_2 + l$, we can assume that $b_1(L(b_1)) = b_2(0)$ and $t^{b_1} \leq t^{b_2}$ with $t^{b_3} \notin [t^{b_1}, t^{b_2}]$, i.e. $[t^{b_1}, t^{b_2}]$ corresponds to the arc of S_L^1 that connects t^{b_1} and t^{b_2} without crossing t^{b_3} .

Let $b_1(\bar{t}^{b_1}) \in I^{b_1}$ be a projection on b_1 of $\phi_\infty(t^{b_1})$. Similarly, let $b_2(\bar{t}^{b_2}) \in I^{b_2}$ be a projection on b_2 of $\phi_\infty(t^{b_2})$.

We show that $\pi_{b_1}(\phi_\infty(S_L^1))$ contains a $3D$ -net of $[b_1(\bar{t}^{b_1}), b_1(L(b_1) - 3R)]$, while $\pi_{b_2}(\phi_\infty(S_L^1))$ contains a $3D$ -net of $[b_2(3R), b_2(\bar{t}^{b_2})]$. The claim then follows by repeating the same argument for all pairs of segments in $\{\gamma_1, \gamma_2, l\}$.

By definition of ϕ_∞ , the restriction $\phi_\infty|_{[t^{b_1}, t^{b_2}]}$ is a segment. By Lemmas 4.14 and 4.6 (see also Remark 4.4), $\pi_{b_1}(\phi_\infty([t^{b_1}, t^{b_2}]))$ contains a $3D$ -net of either $[b_1(3R), b_1(\bar{t}^{b_1})]$ or $[b_1(\bar{t}^{b_1}), b_1(L(b_1) - 3R)]$. If the former case happens, $\phi_\infty([t^{b_1}, t^{b_2}])$ has a point which is $7D$ -close to $b_1(3R)$, so that it also contains a point q which is $5R$ -close to $b_1(0)$. By Corollary 4.15, $b_1(0)$ is at least

$\min\{L(b_1), L(b_3)\} + L(b_2)/2 - 20R$ distant from $\phi_\infty(t^{b_2})$. Hence,

$$\begin{aligned} L(\phi_\infty|_{[t^{b_1}, t^{b_2}]}) &= d(\phi_\infty(t^{b_1}), q) + d(q, b_2(t^{b_2})) \\ &\geq L(b_1)/2 + \min\{L(b_1), L(b_3)\} + L(b_2)/2 - 50R \\ &\geq L(b_1)/2 + L(b_2)/2 + 100R. \end{aligned}$$

This is a contradiction, because we would obtain a shorter curve between $\phi_\infty(t^{b_1})$ and $\phi_\infty(t^{b_2})$ by first joining them to their projections $b_1(\bar{t}^{b_1})$ and $b_1(\bar{t}^{b_2})$, and then joining the projections along b_1 and b_2 themselves.

Hence, $\pi_{b_1}(\phi_\infty([t^{b_1}, t^{b_2}]))$ contains a 3D-net of $[b_1(\bar{t}^{b_1}), b_1(L(b_1) - 3R)]$. The same argument shows that $\pi_{b_2}(\phi_\infty([t^{b_1}, t^{b_2}]))$ contains a 3D-net of $[b_2(3R), b_2(\bar{t}^{b_2})]$. This concludes the proof of Step 1.

Step 2. For every $b_1, b_2 \in \{\gamma_1, \gamma_2, l\}$, with $b_1 \neq b_2$, assuming as before that $t^{b_1} \leq t^{b_2}$ in $[0, T]$ with $t^{b_3} \notin [t^{b_1}, t^{b_2}]$, it holds $\pi_{\phi_\infty([t^{b_1}, t^{b_2}])}(\phi_\infty(t^{b_2} + 3R)) = \{\phi_\infty(t^{b_2})\}$.

Modulo reparametrizing, we assume $b_1(L(b_1)) = b_2(0)$ and $t^{b_1} \leq t^{b_2} \leq t^{b_3}$ in $[0, T]$. By Step 1, there exist $t_{1,2} \in [t^{b_1}, t^{b_2}]$ and $t_{2,3} \in [t^{b_2}, t^{b_3}]$, such that $\phi_\infty(t_{1,2})$ is 7D-close to $b_2(L(b_2) - 20R)$ and $\phi_\infty(t_{2,3})$ is 7D-close to $b_2(L(b_2) + 20R)$. Let γ be a segment connecting $\phi_\infty(t_{1,2})$ and $\phi_\infty(t_{2,3})$. By item 2 of Lemma 4.6 and the (SNP), either $\pi_{b_2}(\gamma)$ intersects I^{b_2} , or γ contains points that are 7D close to $b_2(R)$ and $b_2(L(b_2) - R)$. In the latter case, since $L(b_2) \geq 200R$, γ would not be a segment. Hence, $\pi_{b_2}(\gamma)$ intersects I^{b_2} . It follows that $\gamma + \phi_\infty|_{S_L^1 \setminus [t_{1,2}, t_{2,3}]}$ belongs to A . In particular, $\phi_\infty|_{[t_{1,2}, t_{2,3}]}$ is shorter than γ , so that it is itself a segment.

At the same time, since $\phi_\infty|_{[t^{b_1}, t^{b_2}]}$ is a segment, it follows

$$\pi_{\phi_\infty([t^{b_1}, t^{b_2}])}(\phi_\infty(t^{b_2} + 3R)) = \pi_{\phi_\infty([t_{1,2}, t_{2,3}])}(\phi_\infty(t^{b_2} + 3R)).$$

Combining with the fact that $\phi_\infty|_{[t_{1,2}, t_{2,3}]}$ is a segment, we obtain the statement.

Step 3: Let $b_1, b_2 \in \{\gamma_1, \gamma_2, l\}$ with $b_1 \neq b_2$. We claim that the two arcs of S_L^1 defined by $S_L^1 \setminus [B_{2R}(t^{b_1}) \cup B_{2R}(t^{b_2})]$ are sent by ϕ_∞ in different path connected components of $X \setminus [B_{2R}(\phi_\infty(t^{b_1})) \cup B_{2R}(\phi_\infty(t^{b_2}))]$.

As before, w.l.o.g. we can assume that $t^{b_1} \leq t^{b_2}$. Consider the segment $\phi_\infty|_{[t^{b_1}, t^{b_2}]}$. We assume by contradiction that there exists a curve α connecting $\phi_\infty(t^{b_1} + 3R)$ and $\phi_\infty(t^{b_2} + 3R)$ in X which does not intersect $B_{2R}(\phi_\infty(t^{b_1})) \cup B_{2R}(\phi_\infty(t^{b_2}))$. By item 1 of Lemma 4.6 applied to the segment $\phi_\infty([t^{b_1}, t^{b_2}])$ (this can be applied thanks to step 2), $\pi_{\phi_\infty([t^{b_1}, t^{b_2}])}(\alpha)$ contains a 3D-net of either $[\phi_\infty(t^{b_1} + R), \phi_\infty(t^{b_1} + 3R)]$ or $[\phi_\infty(t^{b_1} + 3R), \phi_\infty(t^{b_2} - R)]$. In both cases α intersects $B_{2R}(\phi_\infty(t^{b_1})) \cup B_{2R}(\phi_\infty(t^{b_2}))$, a contradiction.

Step 4. We claim that $\phi_\infty : S_L^1 \rightarrow X$ is distance preserving and that X lies in a D -neighbourhood of $\phi_\infty(S_L^1)$.

We show first that ϕ_∞ preserves distances. Assume that this is not the case. Then, there exist $x, y \in S_L^1$, and a segment \bar{l} , connecting $\phi_\infty(x)$ and $\phi_\infty(y)$, where $L(\bar{l})$ is less than the length of the shortest arc of S_L^1 connecting x and y . Let $\eta_1, \eta_2 \subset S_L^1$ be the two arcs connecting x and y .

We would like to show that one between the loops $\bar{l} + \phi_\infty|_{\eta_1}$ and $\bar{l} + \phi_\infty|_{\eta_2}$ belongs to A . If this is the case, we obtain a contradiction since ϕ_∞ was length minimizing.

By step 1, for every $b \in \{\gamma_1, \gamma_2, l\}$, we can assume that $\phi_\infty(t^b)$ is 7D-close to $b(L(b)/2)$. This implies that, if a loop has length less than $L(\gamma_1 + \gamma_2 + l)$, and it intersects $B_{2R}(\phi_\infty(t^b))$ for every $b \in \{\gamma_1, \gamma_2, l\}$, then it admits a reparametrization that belongs to A .

Hence, it suffices to show that one between $\bar{l} + \phi_\infty|_{\eta_1}$ and $\bar{l} + \phi_\infty|_{\eta_2}$ intersects all the balls $B_{2R}(\phi_\infty(t^b))$ for every $b \in \{\gamma_1, \gamma_2, l\}$.

We consider all the possible cases. If $\phi_\infty|_{\eta_1}$ intersects all the balls $B_{2R}(\phi_\infty(t^b))$, then there is nothing left to prove. Let now $\{b_1, b_2, b_3\} = \{\gamma_1, \gamma_2, \gamma_3\}$. Assume that $\phi_\infty|_{\eta_1}$ intersects only $B_{2R}(\phi_\infty(t^{b_1}))$ and $B_{2R}(\phi_\infty(t^{b_2}))$. By Step 3, either \bar{l} intersects $B_{2R}(\phi_\infty(t^{b_3}))$, or it intersects both $B_{2R}(\phi_\infty(t^{b_1}))$ and $B_{2R}(\phi_\infty(t^{b_2}))$. In either case, one between $\bar{l} + \phi_\infty|_{\eta_1}$ and $\bar{l} + \phi_\infty|_{\eta_2}$ intersects all the balls. The case when $\phi_\infty|_{\eta_1}$ intersects exactly one of the aforementioned balls is analogous.

It follows that $\phi_\infty : S_L^1 \rightarrow X$ preserves distances. Since $2\pi L > 2R$, the (SNP) implies that X is in the D -neighbourhood of $\phi_\infty(S_L^1)$. Since X contains a segment of length greater than $200R$, it follows that $L \geq 50R$, concluding the proof. \square

The next theorem, summarizes the results of this section.

Theorem 4.17. *Let (X, d) be a metric space satisfying the (SNP). There exists a 1-dimensional manifold I , possibly with boundary, and a distance preserving map $\phi : I \rightarrow X$, such that X is contained in the $200R$ -neighbourhood of $\phi(I)$.*

Proof. If X is non-compact, the statement follows from Proposition 4.11. Let X be compact. By Proposition 4.16, either X is contained in the $200R$ neighbourhood of a segment, or it is contained in the D -neighbourhood of a loop which is mapped isometrically into X . In both cases, the statement holds. \square

5 Large scale geometry of almost non-negative Ricci and integrally-positive scalar curvature

In this section we draw the main geometric results of the paper, by combining the integral curvature estimates of Section 3 with the technical metric results of Section 4.

5.1 Sufficient curvature conditions for (SNP)

In this section, we show that manifolds with almost non-negative Ricci curvature and a positive lower integral bound on R_2 (or R , if the manifolds are 3-dimensional) fit into the framework of Section 4.

The next theorem studies spaces of the form $\mathbb{R}^{n-3} \times X$ arising as limits of n -manifolds with almost non-negative Ricci curvature and positive scalar curvature in integral sense. One should compare also this result with [57, Theorem 1.1], where limit spaces of the form $\mathbb{R}^{n-2} \times X$ were studied.

We refer to Definitions 3.7 and (30) for the notation used in Theorem 5.1.

Theorem 5.1. *Let $v, \epsilon, L \in (0, +\infty)$, $n, k \in \mathbb{N}$ with $2 \leq k \leq n$, $s \in (0, 1)$ be fixed. There exist $R, D, \delta > 0$ with $R \geq 10D$ satisfying the following. Let (X, d, p) be a metric space such that $\mathbb{R}^d \times X$ is a pGH limit of manifolds (M_j^n, g_j, p_j) satisfying one of the following conditions.*

1. $d = k - 2$, and

$$\text{Ric}_{M_j} \geq -\delta, \quad \int_{B_1(x)} R_{M_j, k} \wedge L \, d\text{Vol}_j \geq \epsilon, \quad \forall x \in M_j. \quad (34)$$

2. $n \geq 3$, $d = n - 3$, $\text{Ric}_{M_j} \geq -\delta$, and

$$\int_{B_1(x)} \text{Ric}_{M_j, n-2} \wedge 0 \, d\text{Vol}_j \geq -\delta, \quad \int_{B_1(x)} R_{M_j} \wedge L \, d\text{Vol}_j \geq \epsilon, \quad \forall x \in M_j. \quad (35)$$

3. $n \geq 3$, $d = n - 3$, and

$$\text{Ric}_{M_j} \geq -\delta, \quad \int_{B_1(x)} |R_{M_j}|^s \, d\text{Vol}_j \geq \epsilon, \quad \text{Vol}_j(B_1(x)) \geq v, \quad \forall x \in M_j. \quad (36)$$

Let r be a segment of length greater than $2R$ in X . For every $t \in (R, \text{length}(r) - R)$ and $x \in \pi_r^{-1}(r(t))$, it holds $d(x, r) < D$, i.e., (X, d) satisfies the segment neighbourhood property (SNP) (see Definition 4.2).

In particular, there exists a 1-dimensional connected manifold I (possibly with boundary) and a distance preserving map $\phi : I \rightarrow X$ such that X is contained in the $C(D, R)$ -neighbourhood of $\phi(I)$.

Proof. First, consider the condition (34). Suppose by contradiction that the statement fails. Hence, there exist $\epsilon_0 > 0$, $R_j, D_j \uparrow +\infty$, $\delta_j \downarrow 0$, metric spaces (X_j, d_j, x_j) , such that $\mathbb{R}^{k-2} \times X_j$ arise as limits of manifolds $(M_{i,j})_{i \in \mathbb{N}}$ satisfying

$$\text{Ric}_{M_{i,j}} \geq -\delta_j, \quad \int_{B_1(x)} \mathbf{R}_{M_{i,j},k} \wedge L \, d\text{Vol}_{i,j} \geq \epsilon_0, \quad \forall x \in M_{i,j}, \quad (37)$$

and segments $r_j \subset X_j$ with $L(r_j) \geq 2R_j$, such that $d_j(q_j, r_j) \geq D_j$ for some $q_j \in \pi_{r_j}^{-1}(r_j(R_j))$. In particular, there exist points q_j , whose distance from r_j is greater than D_j , and whose point realizing the distance from r_j is $r_j(R_j)$. Let γ_j be the segment connecting q_j and $r_j(R_j)$.

By Gromov pre-compactness theorem, up to subsequences, the sequence $(X_j, d_j, r_j(R_j))$ converges in pGH-sense to a Ricci limit space (X, d, p) . Moreover, there exists a measure $\mathbf{m} \in \mathcal{M}(X)$ – arising as a limit of the renormalized volume measures of $M_{i,j}$ – such that (X, d, \mathbf{m}, p) is an RCD(0, n) space.

In addition, X contains a line which arises as a limit of the segments r_j centered in $r_j(R_j)$, since $R_j \uparrow \infty$. Hence, by the splitting theorem [26], X splits isomorphically as a metric measure space as $X = Y \times \mathbb{R}$ for an RCD(0, $n-1$) space (Y, d_y, \mathbf{m}_y) . Moreover, Y cannot be compact since the segments γ_j , which realize the distance from r_j , have arbitrarily large length. In particular, Y contains itself a ray. Hence, repeating the argument above, there exists a sequence $p'_j \in Y$, such that (Y, d_y, p'_j) converges in pGH-sense to a metric space $\mathbb{R} \times Y'$. In particular, there exists a sequence $p''_j \in X_j$ such that (X_j, d_j, p''_j) converges in pGH-sense to $\mathbb{R}^2 \times Y'$. Applying Theorem 1.1 to $M_{i,j}$ for i and j large enough, we arrive to a contradiction, since $\delta_j \downarrow 0$ and $\epsilon_0 > 0$ is fixed in (37).

If we assume conditions (35) or (36), the proof is analogous, by using Theorem 3.8 instead of Theorem 1.1.

Finally, by Theorem 4.17, there exists a 1-dimensional connected manifold I (possibly with boundary) and a distance preserving map $\phi : I \rightarrow X$ such that X is contained in the $C(D, R)$ -neighbourhood of $\phi(I)$. \square

As an immediate consequence of the previous theorem, we deduce the following key corollary.

Corollary 5.2. *Let $v, \epsilon, L \in (0, +\infty)$, $n \in \mathbb{N}$, $s \in (0, 1)$ be fixed. There exist $R, D, \delta > 0$ with $R \geq 10D$ satisfying the following. Let (X, d, p) be a pGH limit of manifolds (M_j^n, g_j, p_j) satisfying one of the following conditions.*

1. $n \geq 2$ and

$$\text{Ric}_{M_j} \geq -\delta, \quad \int_{B_1(x)} \mathbf{R}_{M_j,2} \wedge L \, d\text{Vol}_j \geq \epsilon, \quad \forall x \in M_j. \quad (38)$$

2. $n = 3$, $\text{Ric}_{M_j} \geq -\delta$ and

$$\int_{B_1(x)} \text{Sec}_{M_j} \wedge 0 \, d\text{Vol}_j \geq -\delta, \quad \int_{B_1(x)} \mathbf{R}_{M_j} \wedge L \, d\text{Vol}_j \geq \epsilon, \quad \forall x \in M_j. \quad (39)$$

3. $n = 3$ and

$$\text{Ric}_{M_j} \geq -\delta, \quad \int_{B_1(x)} |\mathbf{R}_{M_j}|^s \, d\text{Vol}_j \geq \epsilon, \quad \text{Vol}_j(B_1(x)) \geq v, \quad \forall x \in M_j. \quad (40)$$

Let r be a segment of length greater than $2R$ in X . For every $t \in (R, \text{length}(r) - R)$ and $x \in \pi_r^{-1}(r(t))$, it holds $d(x, r) < D$, i.e., (X, d) satisfies the segment neighbourhood property (SNP) (see Definition 4.2).

In particular, there exists a 1-dimensional connected manifold I (possibly with boundary) and a distance preserving map $\phi : I \rightarrow X$ such that X is contained in the $C(D, R)$ -neighbourhood of $\phi(I)$.

The remaining part of the section is devoted to showing that if a manifold satisfies one of the conditions (34), (35), (36), so do its coverings (modulo a rescaling by dimensional constants). This is the content of Corollaries 5.5.

Proposition 5.3. *Let $n \in \mathbb{N}$ and let $K \in \mathbb{R}$. Let (M^n, g, p) be a pointed Riemannian manifold with $\text{Ric}_M \geq K$. Let $(\tilde{M}^n, \tilde{g}, \tilde{p})$ be a Riemannian covering, with covering map $\pi : \tilde{M} \rightarrow M$ such that $\pi(\tilde{p}) = p$. Let $F : M \rightarrow \mathbb{R}_+$ be a measurable locally bounded function and let $\tilde{F} := F \circ \pi$.*

1. *It holds*

$$\int_{\tilde{B}_5(\tilde{p})} \tilde{F} \, d\tilde{\text{Vol}} \geq c(K, n) \int_{B_1(p)} F \, d\text{Vol}.$$

2. *There exists $p' \in M$ such that*

$$\int_{\tilde{B}_5(\tilde{p})} \tilde{F} \, d\tilde{\text{Vol}} \leq c(K, n) \int_{B_1(p')} F \, d\text{Vol}.$$

Proof. We first claim that

$$\max_{x \in B_1(p)} \#(\pi^{-1}(x) \cap \tilde{B}_1(\tilde{p})) \leq \min_{y \in B_1(p)} \#(\pi^{-1}(y) \cap \tilde{B}_3(\tilde{p})) < +\infty. \quad (41)$$

To prove the claim, let $x, y \in B_1(p)$. Since π is a covering map, $\#(\pi^{-1}(x) \cap \tilde{B}_1(\tilde{p}))$ is finite and positive. Let $l \in \mathbb{N}$ be such that

$$\pi^{-1}(x) \cap \tilde{B}_1(\tilde{p}) = \{\tilde{x}_1, \dots, \tilde{x}_l\}.$$

We recall that M is the quotient of \tilde{M} by a group of isometries acting in a proper and discontinuous way, and that $\pi : \tilde{M} \rightarrow M$ is the projection on such a quotient. Let $\gamma_1, \dots, \gamma_l$ be the isometries of \tilde{M} (as above) such that $\gamma_i(\tilde{x}_i) = \tilde{x}_1$. Let $\tilde{y} \in \pi^{-1}(y) \cap \tilde{B}_1(\tilde{p})$ (\tilde{y} exists, as before, because π is a covering map). Denoting by \tilde{d} the distance induced by \tilde{g} on \tilde{M} , then

$$\tilde{d}(\gamma_i(\tilde{y}), \tilde{p}) \leq \tilde{d}(\gamma_i(\tilde{y}), \gamma_i(\tilde{x}_i)) + \tilde{d}(\gamma_i(\tilde{x}_i), \tilde{p}) = \tilde{d}(\tilde{y}, \tilde{x}_i) + \tilde{d}(\tilde{x}_i, \tilde{p}) \leq 3.$$

Hence, $\{\gamma_1(\tilde{y}), \dots, \gamma_l(\tilde{y})\} \subset \pi^{-1}(y) \cap \tilde{B}_3(\tilde{p})$. Since $\gamma_i(\tilde{y}) \neq \gamma_j(\tilde{y})$ if $i \neq j$, the claim (41) follows.

Proof of 1. Let $\{A_1, \dots, A_l\}$ be a partition of $B_1(p)$ such that each A_i is contained in a covering chart of π . For every $i \in \{1, \dots, l\}$, we set

$$N_i := \max_{y \in A_i} \#(\pi^{-1}(y) \cap \tilde{B}_3(\tilde{p})).$$

Since π is a covering, each N_i is finite and well defined. We also set $N := \min_i N_i$. From (41) we know that

$$1 \leq \max_{x \in B_1(p)} \#(\pi^{-1}(x) \cap \tilde{B}_1(\tilde{p})) \leq N \leq N_i, \quad \text{for all } i = 1, \dots, l. \quad (42)$$

Each $\pi^{-1}(A_i) \cap \tilde{B}_5(\tilde{p})$ contains at least N_i disjoint isometric copies of A_i . Hence,

$$\int_{\tilde{B}_5(\tilde{p})} F(\mathbf{R}_{\tilde{M},k} \vee 0) \, d\tilde{\text{Vol}} \geq \sum_{i=1}^l N_i \int_{A_i} F(\mathbf{R}_{M,k} \vee 0) \, d\text{Vol} \geq N \int_{B_1(p)} F(\mathbf{R}_{M,k} \vee 0) \, d\text{Vol}.$$

To conclude, it suffices to show that

$$N \geq c(K, n) \frac{\text{Vol}(\tilde{B}_5(\tilde{p}))}{\text{Vol}(B_1(p))}. \quad (43)$$

For every $i \in \{1, \dots, l\}$, consider the isometric copies of A_i contained in $\pi^{-1}(A_i)$ that intersect $\tilde{B}_1(\tilde{p})$. By (42), these are less than N for every i . In particular, $\tilde{B}_1(\tilde{p})$ is covered by the union of at most N isometric copies of A_i as i varies. Hence

$$\text{Vol}(\tilde{B}_1(\tilde{p})) \leq \sum_{i=1}^l N \text{Vol}(A_i) = N \text{Vol}(B_1(p)).$$

Condition (43) then follows using Bishop-Gromov's inequality on \tilde{M} .

Proof of 2. Let $\bar{p} \in \tilde{B}_5(\bar{p})$ be such that

$$\int_{\tilde{B}_1(\bar{p})} \tilde{F} d\tilde{\text{Vol}} = \max_{x \in \tilde{B}_5(\bar{p})} \int_{\tilde{B}_1(x)} \tilde{F} d\tilde{\text{Vol}}.$$

We claim that it holds

$$\int_{\tilde{B}_5(\bar{p})} \tilde{F} d\tilde{\text{Vol}} \leq c(n, K) \int_{\tilde{B}_1(\bar{p})} \tilde{F} d\tilde{\text{Vol}}. \quad (44)$$

To prove the claim, let $\{x_i\}_{i=1}^l \subset \tilde{B}_5(\bar{p})$ be such that $\{\tilde{B}_1(x_i)\}_{i=1}^l$ is a covering of $\tilde{B}_5(\bar{p})$ while the balls $\{\tilde{B}_{1/5}(x_i)\}_{i=1}^l$ are all disjoint. Since the balls $\{\tilde{B}_{1/5}(x_i)\}_{i=1}^l$ are all disjoint, using the local doubling property under Ricci bounds, it holds

$$\tilde{\text{Vol}}(\tilde{B}_5(\bar{p})) \geq \sum_{i=1}^l \tilde{\text{Vol}}(\tilde{B}_{1/5}(x_i)) \geq lc(K, n) \tilde{\text{Vol}}(\tilde{B}_5(\bar{p})).$$

Hence, $l \leq c(K, n)$. It follows

$$\int_{\tilde{B}_5(\bar{p})} \tilde{F} d\tilde{\text{Vol}} \leq l \int_{\tilde{B}_1(\bar{p})} \tilde{F} d\tilde{\text{Vol}} \leq c(K, n) \int_{\tilde{B}_1(\bar{p})} \tilde{F} d\tilde{\text{Vol}},$$

which then implies (44) again by the local doubling property.

We now prove item 2. Set $p' := \pi(\bar{p})$. Let $\{A_1, \dots, A_l\}$ be a partition of $B_1(p')$ such that each A_i is contained in a covering chart of π . We set

$$N_i := \max_{y \in A_i} \#(\pi^{-1}(y) \cap \tilde{B}_1(\bar{p})).$$

and $N := \max_i N_i$. For every i , let $\{\tilde{A}_{i,j}\}_{j=1}^{N_i}$ be the isometric copies of A_i in $\pi^{-1}(A_i)$ which have non-empty intersection with $\tilde{B}_1(\bar{p})$. Since $\{\tilde{A}_{i,j}\}_{j=1}^{N_i}$ covers $\tilde{B}_1(\bar{p})$, it holds

$$\begin{aligned} \int_{\tilde{B}_1(\bar{p})} \tilde{F} d\tilde{\text{Vol}} &\leq \sum_{i,j} \int_{\tilde{A}_{i,j}} \tilde{F} d\tilde{\text{Vol}} \\ &\leq N \sum_i \int_{A_i} F d\text{Vol} \leq N \int_{B_1(p')} F d\text{Vol}. \end{aligned}$$

Taking into account (44), to conclude, it is sufficient to show that

$$N \leq c(K, n) \frac{\tilde{\text{Vol}}(\tilde{B}_1(\bar{p}))}{\text{Vol}(B_1(p'))}. \quad (45)$$

For every $i \in \{1, \dots, l\}$, consider the isometric copies of A_i contained in $\pi^{-1}(A_i)$ that intersect $\tilde{B}_3(\bar{p})$. By (42), these are more than N for every i . In particular, $\tilde{B}_3(\bar{p})$ is covered by the union of at least N isometric copies of A_i as i varies. Hence

$$\tilde{\text{Vol}}(\tilde{B}_3(\bar{p})) \geq \sum_{i=1}^l N \text{Vol}(A_i) = N \text{Vol}(B_1(p')).$$

Condition (45) then follows using Bishop-Gromov's inequality on \tilde{M} . \square

Proposition 5.3, together with the fact that covering maps reduce volumes of balls, imply the following proposition.

Proposition 5.4. *Let $v, \epsilon, \delta, L \in (0, +\infty)$, $n \in \mathbb{N}$, $s \in (0, 1)$ be fixed. Let (M^n, g, p) be a manifold satisfying one of the conditions (34), (35), (36). Let $(\tilde{M}^n, \tilde{g}, \bar{p})$ be a Riemannian covering, with covering map $\pi : \tilde{M} \rightarrow M$ such that $\pi(\bar{p}) = p$. Then $(\tilde{M}, \tilde{g}/5^2, \bar{p})$ satisfies the corresponding condition among (34), (35), (36), where the constants have been rescaled by factors depending only on the dimension n .*

Corollary 5.5. *Let $v, \epsilon, L \in (0, +\infty)$, $n \in \mathbb{N}$, $s \in (0, 1)$ be fixed. There exist $R, D, \delta > 0$ with $R \geq 10D$ satisfying the following. Let (M^n, g) be a manifold satisfying one of the conditions (38), (39), (40). Let $(\tilde{M}, \tilde{g}, \tilde{p})$ be a covering of M and let $\pi : \tilde{M} \rightarrow M$ be the covering map. Then, both (M, g) and (\tilde{M}, \tilde{g}) satisfy the segment neighbourhood property (SNP) (see Definition 4.2) w.r.t. R and D .*

Proof. The statement follows combining Proposition 5.4 and Corollary 5.2 (and taking into account Remark 4.4). \square

5.2 Applications of (SNP) to large scale geometry of manifolds

In this section, relying on Theorem 4.17, we prove metric and topological properties of manifolds satisfying one of the conditions (38), (39), (40). The next result is an immediate consequence of Theorem 4.17 and Corollary 5.2.

Theorem 5.6. *Let $v, \epsilon, L \in (0, +\infty)$, $n \in \mathbb{N}$, $s \in (0, 1)$ be fixed. There exist $C, \delta > 0$ with the following property. Let (X, d, p) be a limit of pointed Riemannian manifolds satisfying one of the conditions (38), (39), (40). Then, there exists a 1-dimensional connected manifold I (possibly with boundary) and a distance preserving map $\phi : I \rightarrow X$ such that X is contained in the C -neighbourhood of $\phi(I)$.*

Remark 5.7. The previous result fails if we remove the lower bounds on the Ricci curvature in conditions (38) and (40), thanks to the gluing results in [31, 30, 58]. Indeed, consider the gluing of four copies of $S^2 \times [0, +\infty)$ to S^3 minus four disjoint small balls. By [58], we can equip this space with a Riemannian metric g with the following properties:

1. g coincides with the metric on S^3 and with the metric on each of the branches $S^2 \times [0, +\infty)$ outside of a small neighbourhood of the surgery.
2. g has $R_2 \geq 0$.

Hence, there exists $\epsilon > 0$ such that the glued manifold M satisfies

$$\int_{B_1(x)} R_2 d\text{Vol} \geq \epsilon \quad \text{for every } x \in M.$$

At the same time, Theorem 5.6 fails on M . With a similar construction using [31, 30] one shows that also the Ricci bound in (40) cannot be removed.

The next result concerns volumes of manifolds satisfying one of the conditions (38), (39), or (40).

Theorem 5.8. *Let $v, \epsilon, L \in (0, +\infty)$, $n \in \mathbb{N}$, $s \in (0, 1)$ be fixed. There exist $C, \delta > 0$ satisfying the following. Let (M^n, g, p) be a manifold satisfying one of the conditions (38), (39), or (40). Then*

1. $\sup_{x \in M} \text{Vol}(B_t(x)) \leq Ct$, for all $t > 0$.
2. If $\text{Ric}_M \geq 0$, then $\inf_{x \in M} \text{Vol}(B_1(x)) > 0$.

Proof. By Theorem 5.6, there exists a 1-dimensional connected manifold I (possibly with boundary), a distance preserving map $\phi : I \rightarrow X$, and a constant $C' > 1$, such that X is contained in the C' -neighbourhood of $\phi(I)$.

Proof of 1. For $t \in (0, 1]$, the claim is a consequence of the classical volume comparison results for manifolds with Ricci curvature bounded from below by -1 . Hence, we only consider the case $t > 1$.

Let $x \in X$ and let $\phi(s_x)$ be a projection of x on $\phi(I)$. Let $\{s_i\}_{i=1}^{\lceil t \rceil}$ be a 1-net of $B_t^I(s_x)$. By triangular inequality, $B_t(x)$ is contained in the $10C'$ -neighbourhood of $\phi(B_t^I(s_x))$. Hence, again by triangle inequality and our choice of $\{s_i\}_{i=1}^{\lceil t \rceil}$, it follows

$$B_t(x) \subset \bigcup_{i=1}^{\lceil t \rceil} B_{20C'}(\phi(s_i)).$$

By the aforementioned volume bounds for manifolds with Ricci curvature bounded from below by -1 , there exists a constant $C > 0$ (depending only on the dimension), such that $\text{Vol}(B_t(x)) \leq C[t] \leq 2Ct$.

Proof of 2. If the manifold M is compact, there is nothing to prove. If M is non-compact and it contains a line, then $M = \mathbb{R} \times N$, where N is a compact manifold. Hence, also in this case the statement follows.

So we only need to consider the case when M is non-compact and it is contained in the C' neighbourhood of a ray. Fix $x \in M$ and let $t_x \geq 0$ be such that $x \in B_{C'}(r(t_x))$. By Brunn-Minkowski inequality [52],

$$\text{Vol}(A_{1/2}(2t_x)) \geq (1/2)^n \text{Vol}(B_1(r(0))), \quad (46)$$

where $A_{1/2}(2t_x)$ is the set consisting of the intermediate points of segments connecting points in $B_1(r(0))$ and in $B_1(r(2t_x))$.

We claim that $A_{1/2}(2t_x)$ is contained in $B_{10C'}(r(t_x))$. Assume for the moment that the claim holds. In this case, by the doubling property and (46), it holds $\text{Vol}(B_1(x)) \geq v' > 0$ for some v' depending on $\text{Vol}(B_1(r(0)))$, n and C' .

To conclude the proof, we are left to show that $A_{1/2}(2t_x)$ is contained in $B_{10C'}(r(t_x))$. To this aim, consider a segment l connecting points $a \in B_1(r(0))$ and $b \in B_1(r(2t_x))$. Let $t_l \geq 0$ be such that $r(t_l)$ is a projection of $l(L(l)/2)$ on r . Since X is in the C' -neighbourhood of r , by triangle inequality, it holds

$$d(r(0), r(t_l)) \leq 1 + C' + L(l)/2, \quad d(r(t_x), r(t_l)) \leq 1 + C' + L(l)/2.$$

Since $L(l) \leq t_x + 2$ again by triangle inequality, it follows that $t_l \in [t_x/2 - 4C', t_x/2 + 4C']$. Since $l(L(l)/2)$ is C' -close to $r(t_l)$, it follows $l(L(l)/2) \in B_{10C'}(r(t_x))$, as claimed. \square

The previous result, in particular, shows that the manifolds in question, when $\text{Ric}_M \geq 0$, fit into the framework of [61].

Remark 5.9. An analog of Theorem 5.8 is not to be expected in higher dimension. Indeed a manifold with $R_3 \geq 1$ and almost non-negative Ricci curvature might have exponential volume growth at infinity (consider rescalings of products of the hyperbolic plane and a sufficiently small 2-sphere). Similarly, there exists a manifold with $R_5 \geq 1$ and non-negative Ricci curvature with unit balls of arbitrarily small volume (consider a product of a 2-sphere and the manifold constructed in [49, Example 26]).

Let us recall some properties of covering spaces. Let $\pi : \tilde{X} \rightarrow X$ be a covering between metric spaces. Let $p \in X$ and $\tilde{p} \in \pi^{-1}(p)$. Given a loop γ in p , we can consider its lift $\tilde{\gamma}$ on \tilde{X} with starting point \tilde{p} . This lift is unique, and homotopic loops give rise to homotopic lifts (see [33, p. 60]).

In particular, one can consider the endpoint of $\tilde{\gamma}$, which we will denote $M(\tilde{p}, \gamma)$. This way of obtaining new points, given an element $[\gamma] \in \pi_1(X)$ and a point $\tilde{p} \in \tilde{X}$, is called the *monodromy action* of γ on \tilde{p} . Since homotopic loops give rise to the same monodromy action, if $[\gamma]$ is the identity in $\pi_1(X)$, then $M(\tilde{p}, \gamma) = \tilde{p}$.

Theorem 5.10. *Let $v, \epsilon, L \in (0, +\infty)$, $n \in \mathbb{N}$, $s \in (0, 1)$ be fixed. There exist $C, \delta > 0$ satisfying the following. Let (M^n, g, p) be a manifold satisfying one of the conditions (38), (39), or (40). The following hold.*

1. M has at most two ends.
2. $b_1(M) \leq 1$.
3. $\pi_1(M)$ is infinite if and only if M is compact and its universal cover is non-compact.
4. If there exists a loop $\gamma \subset M$ such that $[\gamma] \in \pi_1(M)$ has infinite order, then M is contained in a C -neighbourhood of γ .

Proof. Proof of 1. Let $C > 0$ be the constant given by Theorem 5.6. If X has more than one end, then it is contained in a C -neighbourhood of a line by Theorem 5.6 and cannot have a third end.

Proof of 3. Let \tilde{M} the universal cover of M . By Proposition 5.4 and Theorem 5.8, the volume of balls in \tilde{M} grows at most linearly, so that by Proposition 2.22, it holds $b_1(M) \leq 1$.

Proof of 3. It is clear that if M is compact and its universal cover \tilde{M} is non-compact, then $\pi_1(M)$ is infinite. We next show the converse. Since $\pi_1(M)$ has infinitely many elements, the universal cover \tilde{M} is not compact. Hence, it suffices to show that M is compact.

Assume by contradiction that M is not compact. We denote by \mathbf{d} and $\tilde{\mathbf{d}}$ respectively distances in M and \tilde{M} . Let $C > 0$ be the constant given by Theorem 5.6, and let $r_M \subset M$ and $r_{\tilde{M}} \subset \tilde{M}$ be the rays or lines whose C -neighbourhoods contain respectively M and \tilde{M} .

Let $p \in \tilde{M}$ be a preimage of $r_M(0)$ in \tilde{M} via the covering map. Since $\pi_1(M)$ has infinitely many elements, there exists a sequence $\{\gamma_i\}_{i \in \mathbb{N}} \subset \pi_1(M)$ such that $\tilde{\mathbf{d}}(M(p, \gamma_i), p) \rightarrow +\infty$. Combining this with the fact that \tilde{M} is contained in the C -neighbourhood of $r_{\tilde{M}}$, we get that there exist t_0, t_1, t_2 in the domain of $r_{\tilde{M}}$ (i.e., \mathbb{R} if $r_{\tilde{M}}$ is a line and \mathbb{R}_+ if $r_{\tilde{M}}$ is a ray) such that $t_2 \geq 100C + t_1 \geq t_0 + 200C$, and $\gamma_1, \gamma_2 \in \pi_1(M)$ with

$$\tilde{\mathbf{d}}(p, r_{\tilde{M}}(t_0)) \leq C, \quad \tilde{\mathbf{d}}(M(p, \gamma_1), r_{\tilde{M}}(t_1)) \leq C, \quad \tilde{\mathbf{d}}(M(p, \gamma_2), r_{\tilde{M}}(t_2)) \leq C. \quad (47)$$

Consider a lift \tilde{r}_M of r_M to \tilde{M} such that $\tilde{r}_M(0) = M(p, \gamma_1)$. Clearly, \tilde{r}_M is a ray in \tilde{M} . By (47) and Lemma 4.6 (\tilde{M} trivially has the (SNP)) and our choice of t_0, t_1, t_2 , there exists $t \geq 10C$ such that $\tilde{r}_M(t)$ is $2C$ -close to either $r_{\tilde{M}}(t_0)$ or $r_{\tilde{M}}(t_2)$. Assume without loss of generality that

$$\tilde{\mathbf{d}}(\tilde{r}_M(t), r_{\tilde{M}}(t_0)) \leq 2C. \quad (48)$$

Recall that each $\gamma \in \pi_1(M)$ induces an isometry ϕ of \tilde{M} defined by $\phi(\tilde{x}) = M(\tilde{x}, \gamma)$ for every $\tilde{x} \in \tilde{M}$, and that M is isometric to the quotient of \tilde{M} by such isometries. Hence:

$$10C \leq t = \mathbf{d}(r_M(0), r_M(t)) \leq \tilde{\mathbf{d}}(p, \tilde{r}_M(t)) + \tilde{\mathbf{d}}(M(p, \gamma_1), \tilde{r}_M(0)) = \tilde{\mathbf{d}}(p, \tilde{r}_M(t)).$$

Since, by (47) and (48),

$$\tilde{\mathbf{d}}(p, \tilde{r}_M(t)) \leq \tilde{\mathbf{d}}(p, r_{\tilde{M}}(t_0)) + \tilde{\mathbf{d}}(r_{\tilde{M}}(t_0), \tilde{r}_M(t)) \leq 3C$$

the combination of the last two inequalities yields a contradiction. Hence, M is compact.

Proof of 4. Let $\gamma \subset M$ be a loop such that $[\gamma] \in \pi_1(M)$ has infinite period. Let $\tilde{\gamma}$ be a lift of γ to \tilde{M} , and let $\tilde{\gamma}^{\mathbb{N}} : [0, +\infty) \rightarrow \tilde{M}$ be a lift of the infinite curve $\sum_{i \in \mathbb{N}} \gamma$. Since $[\gamma] \in \pi_1(M)$ has infinite period, it holds

$$\limsup_{t \rightarrow +\infty} \tilde{\mathbf{d}}(\tilde{\gamma}^{\mathbb{N}}(0), \tilde{\gamma}^{\mathbb{N}}(t)) = +\infty. \quad (49)$$

Let $r_{\tilde{M}}$ be the ray or the line in \tilde{M} whose C' -neighbourhood contains \tilde{M} (which exists combining Theorem 5.6 and Proposition 5.4). By (49) and Lemma 4.6, there exist t_1, t_2 in the domain of $r_{\tilde{M}}$ with $t_2 \geq 100C' + 100L(\gamma) + t_1$ such that $\pi_{r_{\tilde{M}}}(\tilde{\gamma}^{\mathbb{N}})$ contains a $3C'$ -net of $[r_{\tilde{M}}(t_1), r_{\tilde{M}}(t_2)]$. Assume now that there is a point $p \in M$ such that $\mathbf{d}(p, \gamma) \geq 10C'$. Let l be a segment realizing the distance from p to γ and let p_l be the footpoint of l on γ . The set of preimages of p_l in \tilde{M} via the covering map form a $L(\gamma)$ -net of $\tilde{\gamma}^{\mathbb{N}}$ in \tilde{M} . In particular, there is a preimage \tilde{p}_l which is $10C' + L(\gamma)$ close to $r_{\tilde{M}}((t_1 + t_2)/2)$. Let \tilde{l} be the lift of l to \tilde{M} starting from \tilde{p}_l (we are orienting l from p_l to p). Then,

$$\tilde{\mathbf{d}}(\tilde{p}_l, \tilde{\gamma}^{\mathbb{N}}) = \tilde{\mathbf{d}}(\tilde{l}(L(\tilde{l})), \tilde{\gamma}^{\mathbb{N}}) = \tilde{\mathbf{d}}(l(L(l), \gamma)) \geq 10C'. \quad (50)$$

At the same time, since \tilde{p}_l is $10C' + L(\gamma)$ close to $r_{\tilde{M}}((t_1 + t_2)/2)$, it has a projection $r_{\tilde{M}}(t)$ on $r_{\tilde{M}}$ for $t \in [t_1, t_2]$. Using that $r_{\tilde{M}}(t)$ is $7C'$ -close to a point in $\tilde{\gamma}^{\mathbb{N}}$ and (50), it holds

$$\tilde{\mathbf{d}}(\tilde{p}_l, r_{\tilde{M}}) = \tilde{\mathbf{d}}(\tilde{p}_l, r_{\tilde{M}}(t)) \geq \tilde{\mathbf{d}}(\tilde{p}_l, \tilde{\gamma}^{\mathbb{N}}) - 7C' \geq 3C'.$$

This is a contradiction. \square

Appendix

The appendix shows how to obtain a result by Jiang-Naber [44, Theorem 2.17]. Since the result was claimed in [44] without proof, for the reader's convenience, we provide an argument consisting in a combination of Theorem 1.1 with [16, Theorem 1.3]. The result is not used anywhere else in the paper, and it is included as a further application of the techniques developed in the manuscript.

Definition 5.11. Let (M^n, g, x) be a pointed Riemannian manifold. The quantitative singular strata are defined, for $r \in (0, 1)$, as

$$\mathcal{S}_{\delta, r}^k := \{y \in B_1(x) : B_s(y) \text{ is not } (k+1, \delta)\text{-symmetric, for any } s \in (r, 1)\},$$

$$\mathcal{S}_\delta^k := \{y \in B_1(x) : B_s(y) \text{ is not } (k+1, \delta)\text{-symmetric for any } s \in (0, 1)\}.$$

We use the abbreviated notation $\mathcal{S}_{\delta, r}$ for $\mathcal{S}_{\delta, r}^{n-1}$, and \mathcal{S}_δ for \mathcal{S}_δ^{n-1} . The set $B_1(x) \setminus \mathcal{S}_{\delta, r}$ is denoted by $\mathcal{R}_{\delta, r}$.

Proposition 5.12. Let $K \in \mathbb{R}$ and $n \in \mathbb{N}$ be fixed and let (M^n, g, x) be a manifold with $\text{Ric}_M \geq K$. For every $\delta > 0$ there exists $\eta(K, n, \delta) > 0$ such that $\mathcal{S}_{\delta, r} \subset \mathcal{S}_{\eta, r}^{n-2}$, for every $r \in (0, 1]$.

Proof. The statement follows arguing by contradiction and using Theorem 2.13. \square

The next theorem corresponds to [16, Theorem 1.3].

Theorem 5.13 (Cheeger-Naber). Let $K \in \mathbb{R}$, $n \in \mathbb{N}$, $v, \delta, \eta > 0$ be fixed and let (M^n, g, x) be a manifold with $\text{Ric}_M \geq K$ and $\text{Vol}(B_1(x)) \geq v$. Denoting by $T_r(\mathcal{S}_{\delta, r}^k)$ the r -neighbourhood of $\mathcal{S}_{\delta, r}^k$, there exists $C = C(n, v, \delta, \eta) > 0$ such that

$$\text{Vol}(T_r(\mathcal{S}_{\delta, r}^k)) \leq Cr^{n-k-\eta}.$$

Theorem 5.14. Let $n \in \mathbb{N}$, $v > 0$, $s \in (0, 1)$ be fixed. There exist $\delta(n, s), C(n, v, s) > 0$ such that, if (M^n, g, x) has $\text{Ric}_M \geq -\delta$ and $\text{Vol}(B_1(x)) \geq v$, then

$$\int_{B_1(x)} |R|^s \, d\text{Vol} \leq C(n, v, s).$$

Proof. By Theorem 1.1 and a standard rescaling, there exists $\delta(n, s) > 0$ such that if (M, g, x) has $\text{Ric}_M \geq -\delta$, then any δ -regular ball $B_{10r}(y) \subset M$ with $y \in B_1(x)$ and $r \in (0, 1]$ satisfies

$$\int_{B_r(y)} |R|^s \, d\text{Vol} \leq r^{-2s}. \quad (51)$$

Without loss of generality, for the rest of the proof we will assume that $\delta \leq (n-1)$. Let $\alpha \in (0, 1)$ be fixed, and consider the disjoint union

$$B_1(x) = \mathcal{R}_{\delta, \alpha} \cup \bigcup_{k \in \mathbb{N}} (\mathcal{S}_{\delta, \alpha^k} \setminus \mathcal{S}_{\delta, \alpha^{k+1}}).$$

Consider a covering $\{B_{r_j/10}(x_j)\}_{j \in \mathbb{N}}$ of $B_1(x)$ such that each $B_{r_j}(x_j)$ is δ -regular and satisfies the following.

1. If $x_j \in \mathcal{R}_{\delta, \alpha}$, then $r_j \in [\alpha, 1)$.
2. If $x_j \in \mathcal{S}_{\delta, \alpha^k} \setminus \mathcal{S}_{\delta, \alpha^{k+1}}$, then $r_j \in [\alpha^{k+1}, \alpha^k]$.

By refining the covering, we may assume that $\{B_{r_j/50}(x_j)\}_{j \in \mathbb{N}}$ consists of disjoint balls. We will bound

$$\sum_{x_j \in \mathcal{R}_{\delta, \alpha}} \int_{B_{r_j/10}(x_j)} |\mathbf{R}|^s \, d\text{Vol} \quad \text{and} \quad \sum_{k \in \mathbb{N}} \sum_{x_j \in \mathcal{S}_{\delta, \alpha^k} \setminus \mathcal{S}_{\delta, \alpha^{k+1}}} \int_{B_{r_j/10}(x_j)} |\mathbf{R}|^s \, d\text{Vol}$$

separately.

We treat the case $x_j \in \mathcal{R}_{\delta, \alpha}$ first. Combining the assumption that $0 < v \leq \text{Vol}(B_1(x)) \leq C(n)$ and condition **1** with Bishop-Gromov's volume monotonicity, we deduce that the number of balls in the aforementioned covering such that $x_j \in \mathcal{R}_{\delta, \alpha}$ is bounded above by $c(n, v, s)$. Hence, (51) yields

$$\sum_{x_j \in \mathcal{R}_{\delta, \alpha}} \int_{B_{r_j/10}(x_j)} |\mathbf{R}|^s \, d\text{Vol} \leq 10^{2s} \alpha^{-2s} \sum_{x_j \in \mathcal{R}_{\delta, \alpha}} \text{Vol}(B_{r_j/10}(x_j)) \leq c(n, v, s). \quad (52)$$

We now consider the case $x_j \in \mathcal{S}_{\delta, \alpha^k} \setminus \mathcal{S}_{\delta, \alpha^{k+1}}$. Again using (51) and Bishop-Gromov's volume monotonicity, we infer that

$$\begin{aligned} \sum_{x_j \in \mathcal{S}_{\delta, \alpha^k} \setminus \mathcal{S}_{\delta, \alpha^{k+1}}} \int_{B_{r_j/10}(x_j)} |\mathbf{R}|^s \, d\text{Vol} &\leq 10^{2s} \sum_{x_j \in \mathcal{S}_{\delta, \alpha^k} \setminus \mathcal{S}_{\delta, \alpha^{k+1}}} \text{Vol}(B_{r_j/10}(x_j)) r_j^{-2s} \\ &\leq \sum_{x_j \in \mathcal{S}_{\delta, \alpha^k} \setminus \mathcal{S}_{\delta, \alpha^{k+1}}} c(n) \text{Vol}(B_{r_j/50}(x_j)) \alpha^{-2s(k+1)}. \end{aligned} \quad (53)$$

We note that the balls $\{B_{r_j/50}(x_j)\}_{j \in \mathbb{N}}$ are disjoint and

$$\bigcup_{x_j \in \mathcal{S}_{\delta, \alpha^k} \setminus \mathcal{S}_{\delta, \alpha^{k+1}}} B_{r_j}(x_j) \subset T_{\alpha^k}(\mathcal{S}_{\delta, \alpha^k}).$$

Hence, choosing $\eta := 1 - s$ in Theorem 5.13, and combining with Proposition 5.12 and (53), we deduce

$$\sum_{x_j \in \mathcal{S}_{\delta, \alpha^k} \setminus \mathcal{S}_{\delta, \alpha^{k+1}}} \int_{B_{r_j/10}(x_j)} |\mathbf{R}|^s \, d\text{Vol} \leq c(n, v, s) \alpha^{(1-s)k}.$$

Hence,

$$\sum_{k \in \mathbb{N}} \sum_{x_j \in \mathcal{S}_{\delta, \alpha^k} \setminus \mathcal{S}_{\delta, \alpha^{k+1}}} \int_{B_{r_j/10}(x_j)} |\mathbf{R}|^s \, d\text{Vol} \leq c(n, v, s),$$

giving the statement. \square

One can now deduce Jiang-Naber's result [44, Theorem 2.17].

Theorem 5.15 (Jiang-Naber). *Let $K \in \mathbb{R}$, $n \in \mathbb{N}$, $v > 0$, $s \in (0, 1)$ be fixed and let (M^n, g, x) be a manifold with $\text{Ric}_M \geq K$ and $\text{Vol}(B_1(x)) \geq v$. There exists a constant $C(K, n, v, s) > 0$ such that*

$$\int_{B_1(x)} |\text{Ric}|^s \, d\text{Vol} \leq C(K, n, v, s). \quad (54)$$

Proof. By Theorem 5.14 and a standard rescaling, there exists $r_0(K, n, v, s) > 0$ such that

$$\int_{B_{r_0}(x)} |\mathbf{R}|^s \, d\text{Vol} \leq c(K, n, v, s), \quad \text{for all } y \in B_1(x). \quad (55)$$

Consider a covering $\{B_{r_0}(x_j)\}_{j \in \mathbb{N}}$ of $B_1(x)$ such that the balls $\{B_{r_0/3}(x_j)\}_{j \in \mathbb{N}}$ are disjoint. Then, the combination of (55) with Bishop-Gromov's volume monotonicity yields

$$\begin{aligned} \int_{B_1(x)} |\mathbf{R}|^s \, d\text{Vol} &\leq \sum_{j \in \mathbb{N}} \int_{B_{r_0}(x_j)} |\mathbf{R}|^s \, d\text{Vol} \leq c(n, K, v, s) \sum_{j \in \mathbb{N}} \text{Vol}(B_{r_0/3}(x_j)) \\ &\leq c(n, K, v, s) \text{Vol}(B_1(x)). \end{aligned} \quad (56)$$

The assumption that $\text{Ric} \geq K$ allows to promote the integral bound (56) on the scalar curvature into the claimed integral bound (54) on the Ricci curvature. \square

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