DERIVATION OF KIRCHHOFF-TYPE PLATE THEORIES FOR ELASTIC MATERIALS WITH VOIDS

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Abstract. We rigorously derive a Blake-Zisserman-Kirchhoff theory for thin plates with material voids, starting from a three-dimensional model with elastic bulk and interfacial energy featuring a Willmore-type curvature penalization. The effective two-dimensional model comprises a classical elastic bending energy and surface terms which reflect the possibility that voids can persist in the limit, that the limiting plate can be broken apart into several pieces, or that the plate can be folded. Building upon and extending the techniques used in the authors' recent work [34] on the derivation of one-dimensional theories for thin brittle rods with voids, the present contribution generalizes the results of [57], by considering general geometries on the admissible set of voids and constructing recovery sequences for all admissible limiting configurations.

1. Introduction

The rigorous derivation of variational theories for lower dimensional elastic objects, e.g., membranes, plates, shells, beams, and rods, has been one of the fundamental and challenging questions in the mathematical development of continuum mechanics. A common aspect in the analysis is always an appropriate model of three-dimensional nonlinear elasticity, and the key feature for the resulting theory is the energy scaling with respect to the thickness parameter of the initial domain. Despite the longstanding interest in such questions [6, 7], early results were usually relying on some a priori *ansatzes*, often leading to contrasting theories. Over the last decades though, modern techniques from the Calculus of Variations and Applied Analysis have been implemented very successfully for the rigorous derivation of effective models for thin elastic objects. From a technical point of view, the fundamental ingredient to perform these rigorous justifications has been the celebrated geometric rigidity estimate in the seminal work by G. Friesecke, R.D. James, and S. Müller [36], which has had a striking number of applications in dimension-reduction problems in the context of pure (hyper)elasticity. The interested reader is referred for instance to [18, 25, 36, 37, 44, 47, 48, 52, 53, 58, 59], for a by far non-exhaustive list of references regarding dimension-reduction results related to plate or rod theories in the bending regime.

However, concerning the investigation of phenomena beyond the perfectly elastic regime, for instance the behavior of solids with defects and impurities such as *plastic slips, dislocations*, *cracks*, or *stress-induced voids*, the mathematical understanding is far less well settled. In the case of thin elastic materials with voids, the natural variational formulation involves energies driven by the competition between bulk elastic and interfacial energies of perimeter type. Such variational models describe *stress driven rearrangement instabilities* (SDRI) in elastic solids, and have recently been a focal point of considerable attention both from the mathematical and the physics community, see for example [11, 13, 20, 26, 32, 33, 34, 38, 40, 41, 45, 46, 56, 61].

As far as dimension-reduction results beyond the purely elastic setting are concerned, we now present a concise summary of some important recent developments. In the framework of plasticity, we refer the reader, e.g., to [15, 23, 24, 49, 51]. Regarding models for brittle fracture, despite a significant recent progress on brittle plates and shells in the linear setting [1, 3, 9, 39], the theory in the nonlinear framework is mainly restricted to static and evolutionary models in the membrane regime [2, 8, 14]. Smaller energy regimes are less well studied, the only rigorous available result for now appearing to be for a two-dimensional thin brittle beam [60]. The main result in that work is the derivation of an effective *Griffith-Euler-Bernoulli* energy defined on the midline of the possibly fractured beam, which takes into account possible jump discontinuities of the limiting deformation and its derivative. From a technical perspective, the key tool in [60] is an appropriate generalization of [36], namely a *quantitative piecewise geometric rigidity theorem for SBD functions* [35]. Up to now, such a general result is available only in two dimensions, which is the main obstruction for the generalization of dimension-reduction results to settings of three-dimensional fracture. Let us mention, however, that similar rigidity results in higher dimensions are available in models for nonsimple materials [31], where a singular perturbation term depending on the second gradient of the deformation is incorporated in the elastic energy.

In the setting of SDRI models for solids with material voids, a first analysis on plate theories with surface discontinuities in the bending (Kirchhoff) energy regime has been performed in [57]. However, the results therein are conditional in two aspects. Firstly, only voids with restrictive assumptions on their distribution and geometry (satisfying the so-called *minimal droplet assumption*) are considered, which allows to resort to the classical rigidity theorem of [36]. Secondly, recovery sequences are only constructed under a specific regularity property for the outer Minkowski-content of voids and discontinuities. In our recent work [34], a related result for thin rods without restriction on the void geometry was accomplished, generalizing the results of [52] from the purely elastic setting. The cornerstone of our approach was a novel piecewise rigidity result in the realm of SDRI-models [32], which is based on a curvature regularization of the surface term.

The goal of the present article is to extend the methods used in [34] in order to show that the *Blake-Zisserman-Kirchhoff* model of [57] is the Γ-limit of the three-dimensional model, without restricting the void geometries and without restricting to a special class of configurations for the construction of recovery sequences.

We now describe our setting in more detail. We consider a three-dimensional thin plate with reference configuration

$$\Omega_h := S \times (-\tfrac{h}{2}, \tfrac{h}{2}) \subset \mathbb{R}^3$$

of thickness $0 < h \ll 1$, where the midsurface is represented by a bounded Lipschitz domain $S \subset \mathbb{R}^2$. Variational models for thin plates describing the formation of material voids which are not a priori prescribed, fall into the framework of *free discontinuity problems* [4], leading to an energy of the form

$$\mathcal{F}^h_{\mathrm{el,per}}(v, E) := \int_{\Omega_h \setminus \overline{E}} W(\nabla v)\, \mathrm{d}x + \beta_h \int_{\partial E \cap \Omega_h} \varphi(\nu_E)\, \mathrm{d}\mathcal{H}^2 \,. \tag{1.1}$$

Here, $E \subset \Omega_h$ represents the (sufficiently regular) void set within an elastic plate with reference configuration $\Omega_h \subset \mathbb{R}^3$, and v is the corresponding elastic deformation. The first term in (1.1) represents the nonlinear elastic energy with density W (see Section 2 for details), whereas the second one depends on a parameter $\beta_h > 0$ and a possibly anisotropic norm φ evaluated at the outer unit normal ν_E to $\partial E \cap \Omega_h$. For purely expository reasons, we restrict our analysis to the isotropic case, i.e., $\varphi(\cdot) \equiv |\cdot|_2$, see Remark 2.4 for some comments.

At a heuristic level, it is well known that an elastic energy scaling of the order h^3 corresponds to the *bending Kirchhoff theory*, leaving the midsurface S unstretched. At the same time, the surface area of voids completely separating the plate is of order h. Now, depending on the choice of the parameter β_h, one can expect different limiting models, after rescaling (1.1) with $\max\{h^{-3}, (\beta_h h)^{-1}\}$: the case $\beta_h \gg h^2$ will result in a purely elastic plate model, while the choice $\beta_h \ll h^2$ will lead to a model of purely brittle fracture. The critical regime $\beta_h \sim h^2$ is the most interesting and mathematically most challenging one, since the elastic and surface contributions in this case compete at the same order.

Hence, from here on we set for simplicity $\beta_h := h^2$. After rescaling the total energy in (1.1) by h^{-3}, our aim is to rigorously derive effective two-dimensional theories by means of Γ-convergence [12, 21]. As in [34], the presence of a priori unprescribed voids in the model hinders the use of the classical rigidity result [36]. Indeed, the distribution of voids in the material might possibly exhibit highly complicated geometries, for instance densely packed thin spikes or microscopically small components with small surface measure on different length scales, see Figure 1.

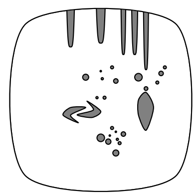

Figure 1. Densely packed thin spikes and microscopically small components leading to loss of rigidity. For simplicity, the figure illustrates a two-dimensional example.

To remedy these phenomena, motivated by our recent works [32, 34], we introduce a *curvature regularization* of the form

$$\mathcal{F}^h_{\rm curv}(E) := h^2\kappa_h \int_{\partial E\cap\Omega_h} |\boldsymbol{A}|^2 \,\mathrm{d}\mathcal{H}^2\,, \tag{1.2}$$

where $\boldsymbol{A}$ denotes the second fundamental form of $\partial E\cap\Omega_h$ and κ_h satisfies $\kappa_h \to 0^+$ as $h\to 0^+$ at a sufficient rate (see (2.5) for details). The presence of this additional Willmore-type energy penalization allows us to use the *piecewise rigidity estimate* [32, Theorem 2.1] in the analysis. It is a *singular perturbation* for the void set E and not for the deformation v, i.e., no higher-order derivatives of v are involved in the model. We also refer the interested reader to [33], where a related atomistic model is studied and additional explanations for the presence of a microscopic analog of the term in (1.2) are given, see [33, Subsection 2.5]. As mentioned therein, curvature regularizations of similar type are commonly used in the mathematical and physical literature of SDRI models, for instance in the description of heteroepitaxial growth of elastically stressed thin films or material voids, see [5, 27, 28, 42, 43, 55, 61]. Despite the possible modeling relevance, we mention that the presence of the curvature contribution in our model is only for mathematical reasons as a regularization term. In particular, it does not affect the structure of the effective limiting model.

The total energy of a pair (v,E) is then given by the sum of the two terms in (1.1) and (1.2), i.e.,

$$\mathcal{F}^h(v,E) := \mathcal{F}^h_{\rm el,per}(v,E) + \mathcal{F}^h_{\rm curv}(E)\,,$$

having set $\beta_h := h^2$ and $\varphi(\nu)\equiv 1$ for all $\nu\in\mathbb{S}^2$. The main outcome of this work is then Theorem 2.3, where we show that the rescaled energies $\big(h^{-3}\mathcal{F}^h(\cdot,\cdot)\big)_{h>0}$ Γ-converge in an appropriate topology to an effective two-dimensional model that is of the form

$$\frac{1}{24}\int_{S\setminus V} \mathcal{Q}_2\big(\mathrm{II}_y(x')\big)\,\mathrm{d}x' + \mathcal{H}^1\big(\partial^* V\cap S\big) + 2\mathcal{H}^1\big(J_{(y,\nabla' y)}\setminus\partial^* V\big)\,. \tag{1.3}$$

Here, $V\subset S$ denotes a set of finite perimeter in $S\subset\mathbb{R}^2$ and represents the void part in the limiting two-dimensional plate. As in [57], the limiting admissible deformations turn out to be *possibly fractured and creased flat isometric immersions*, i.e., $y\in SBV^{2,2}_{\rm isom}(S;\mathbb{R}^3)$, see Section 2.2 for precise definitions. The approximate gradient of y is denoted here by $\nabla' y$ and II_y denotes the induced second fundamental form on $y(S)$. The density of the limiting elastic energy, which should be conceived as a curvature energy on isometric immersions of the midsurface S, depends on a quadratic form $\mathcal{Q}_2$ which is defined through the quadratic form $D^2W(\mathrm{Id})$ of linearized elasticity via a suitable minimization problem, see (2.12) for its precise definition. The second term in (1.3) accounts for the presence of limiting voids by measuring the total length of their boundaries. The last term in (1.3) takes the fact into account that, in the limit, voids might collapse into discontinuities of the limiting deformation y or its derivative $\nabla' y$, corresponding to cracks or folds of the limiting plate, respectively. Exactly due to their origin, the length of those should be counted twice in the energy.

Let us once again mention that one fundamental difference between our work and that of [57] lies in the assumptions on the admissible void sets. While we allow for voids with general geometry employing a mild curvature regularization, [57] is based on specific restrictive assumptions on the void geometry, namely the so-called *ψ-minimal droplet assumption*, cf. [57, Equation (6)]. This can be interpreted as an L^∞-diverging bound on the curvature of the boundary of the voids in the initial thin plate. In our setting, the curvature penalization term (1.2) can be thought of as imposing an L^2-diverging bound on the curvature: its nature allows for the voids to concentrate at arbitrarily small scales (independently of h), while also allowing for a diverging (with h) number of connected components of voids, see Remark 2.4(iv). Of course, our more general model comes at the expense of using more sophisticated geometric rigidity estimates [32] compared to the classical one of [36]. As in [34], our strategy relies on modifying the deformations and their gradients on a small part of the domain such that the new deformations are actually Sobolev except for the boundaries of a controllable number of cubes, with a good control on the elastic energy. Concerning the derivation of compactness and Γ-liminf, the estimate for the surface parts in (1.3) is the most delicate step and requires a fine control on the jump height of these modifications. By means of a contradiction argument based on a blow up method, we are able to reduce the problem to the setting of thin rods, which allows us to directly use the compactness result from [34, Theorem 2.1].

Besides the geometry of voids, our work differs from [57] by the fact that therein recovery sequences are only provided under specific regularity properties for the outer Minkowski-content of $\partial^* V$ and $J_{(y,\nabla' y)}$. By means of the coarea formula, the latter assumption allows to construct a sequence of three-dimensional voids with the required regularity, in particular satisfying the minimal droplet assumption. In the general case, however, a density result for boundaries of void sets and jump sets of SBV-functions appears to be required. Although many results are available in this direction, see, e.g., [19], to the best of our knowledge they are all incompatible with the isometry constraint, i.e., with $y \in SBV^{2,2}_{\rm isom}(S;\mathbb{R}^3)$. As approximation results are usually built on convolution techniques, it indeed cannot be expected to obtain a density result satisfying exactly an isometry constraint. Yet, we are able to obtain a density result which can control the deviation from an isometry in a controlled way, quantified in terms of the thickness h. This is then enough to construct recovery sequences by adapting the *ansatz* from the purely elastic case [36]. We regard this part as the most original technical novelty of the current paper, and we believe that the technique may be applicable also in other related settings.

1.1. **Organization of the paper.** Our paper is organized as follows. In Section 2 we introduce the model and state the main compactness and Γ-convergence results, i.e., Theorems 2.1 and 2.3, respectively, together with some additional modeling remarks. In Section 3, we collect the necessary technical ingredients for the proofs, namely the nonlinear piecewise rigidity estimates from [32] and the Korn-Poincaré inequality for SBV^2-functions with small jump set from [16] adapted to our setting, see Subsection 3.1, as well as the construction of almost Sobolev replacements for sequences of deformations with equibounded energy, see Subsections 3.2–3.3. Section 4 contains the proof of Theorem 2.1, while Sections 5 and 6 contain the proof of Theorem 2.3(i),(ii) respectively. Finally, in Appendices A and B we give the proofs of some auxiliary facts that are themselves not new but that are presented here for completeness only.

1.2. **Notation.** We close the introduction with some basic notation. Given $d \in \mathbb{N}$, $U \subset \mathbb{R}^d$ open, we denote by $\mathfrak{M}(U)$ the collection of all measurable subsets of U, and by $\mathcal{P}(U)$ the one of subsets of finite perimeter in U. Given $A, B \in \mathfrak{M}(U)$, we write χ_A for the characteristic function of A, $A \triangle B$ for their symmetric difference, $A \subset\subset B$ iff $\overline{A} \subset B$, and $\mathrm{dist}_{\mathcal{H}}(A, B)$ for the Hausdorff distance between A and B. For $v \in \mathbb{R}^d$ we denote by $|v|_\infty := \max\{|v_k| \colon k = 1, \dots, d\}$ its ℓ_∞-norm, while for the $|\cdot|_\infty$-distance of a point x (respectively a set B) to a set A, we write $\mathrm{dist}_\infty(x, A)$ (respectively $\mathrm{dist}_\infty(A, B)$). For every $A \subset \mathbb{R}^d$ and $\delta > 0$, we define

$$(A)_\delta := \{x \in \mathbb{R}^d \colon \mathrm{dist}(x, A) < \delta\} \,. \tag{1.4}$$

For $E \in \mathcal{P}(U)$ we denote by $\partial^* E$ the essential boundary of E, see [4, Definition 3.60]. For $d = 3$ we also denote by $\mathcal{A}_{\mathrm{reg}}(U)$ the collection of all open sets $E \subset U$ such that $\partial E \cap U$ is a two-dimensional C^2-surface in $\mathbb{R}^3$. Surfaces and functions of C^2-regularity will be called C^2-regular or just regular in the following. For $E \in \mathcal{A}_{\mathrm{reg}}(U)$ we denote by $\boldsymbol{A}$ the second fundamental form of $\partial E \cap U$, i.e., $|\boldsymbol{A}| = \sqrt{\kappa_1^2 + \kappa_2^2}$, where κ_1 and κ_2 are the corresponding principal curvatures. By ν_E we indicate the outer unit normal to $\partial E \cap U$.

The inner product of two vectors $a, b \in \mathbb{R}^3$ will be denoted by $a \cdot b$, and their exterior product by $a \wedge b$. We further write $\mathbb{S}^2 := \{\nu \in \mathbb{R}^3 \colon |\nu| = 1\}$. By id we denote the identity mapping on $\mathbb{R}^3$ and by $\mathrm{Id} \in \mathbb{R}^{3\times 3}$ the identity matrix. For each $F \in \mathbb{R}^{3\times 3}$ we let $\mathrm{sym}(F) := \frac{1}{2}\left(F + F^T\right)$ and we also introduce $SO(3) := \{F \in \mathbb{R}^{3\times 3} \colon F^T F = \mathrm{Id}, \det F = 1\}$. Moreover, we denote by $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$ and $\mathbb{R}^{3\times 3}_{\mathrm{skew}}$ the space of symmetric and skew-symmetric matrices, respectively. For $\sigma > 0$, we denote by T_σ the linear transformation in $\mathbb{R}^3$ with matrix representation given by

$$T_\sigma := \mathrm{diag}(1, 1, \sigma) \tag{1.5}$$

with respect to the canonical basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$. For $d, k \in \mathbb{N}$, we indicate by $\mathcal{L}^d$ and $\mathcal{H}^k$ the d-dimensional Lebesgue measure and the k-dimensional Hausdorff measure, respectively.

For $U \subset \mathbb{R}^n$ open, for $p \in [1, \infty]$ and $d, k \in \mathbb{N}$ we denote by $L^p(U; \mathbb{R}^d)$ and $W^{k,p}(U; \mathbb{R}^d)$ the standard Lebesgue and Sobolev spaces, respectively. Partial derivatives of a function $f \colon U \to \mathbb{R}^d$ will be denoted by $(\partial_i f)_{i=1,2,3}$. We use standard notation for SBV-functions, cf. [4, Chapter 4] for the definition and a detailed presentation of the properties of this space. In particular, for a function $u \in SBV(U; \mathbb{R}^d)$, we write ∇u for the approximate gradient, J_u for its jump set, and $u^\pm$ for the one-sided traces on J_u. Finally,

$$SBV^2(U; \mathbb{R}^d) := \Big\{u \in SBV(U; \mathbb{R}^d) \colon \int_U |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^{d-1}(J_u \cap U) < +\infty\Big\} \,.$$

2. The models and the main results

In this section we introduce the model and present our main results.

2.1. **The three-dimensional model.** We denote the reference configuration of the thin plate by

$$\Omega_h := S \times (-\tfrac{h}{2}, \tfrac{h}{2}) \subset \mathbb{R}^3 \,, \tag{2.1}$$

where $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain describing the midsurface of the thin plate, and $0 < h \ll 1$ denotes its infinitesimal thickness. For a large but fixed constant $M \gg 1$, the set of *admissible pairs* of function and set is given by

$$\mathcal{A}_h := \big\{(v, E) \colon \; E \in \mathcal{A}_{\mathrm{reg}}(\Omega_h), \; v \in W^{1,2}(\Omega_h \setminus \overline{E}; \mathbb{R}^3) \,, \; v|_E \equiv \mathrm{id} \,, \; \|v\|_{L^\infty(\Omega_h)} \le M\big\} \,. \tag{2.2}$$

The third condition in (2.2) is for definiteness only. While the last one therein is merely of technical nature to ensure compactness, it is also justified from a physical viewpoint since it corresponds to the assumption that the solid under consideration is confined in a bounded region. For each pair $(v, E) \in \mathcal{A}_h$, we consider the energy

$$\mathcal{F}^h(v, E) := \int_{\Omega_h \setminus \overline{E}} W(\nabla v)\, \mathrm{d}x + h^2 \mathcal{H}^2(\partial E \cap \Omega_h) + h^2 \kappa_h \int_{\partial E \cap \Omega_h} |\boldsymbol{A}|^2 \, \mathrm{d}\mathcal{H}^2 \,. \tag{2.3}$$

The first two terms correspond to the *elastic* and the *surface energy* of perimeter-type, respectively, while the third one is a *curvature regularization* of Willmore-type, where $\boldsymbol{A}$ denotes the second fundamental form of $\partial E \cap \Omega_h$ and κ_h is a suitable infinitesimal parameter specified in (2.5) below. The factor h^2 in front of the surface terms ensures that the elastic and the surface energy are of same order, since the elastic energy per unit volume is of order h^2, see the introduction for some heuristic explanations of the model.

The function $W \colon \mathbb{R}^{3\times 3} \to \mathbb{R}_+$ in (2.3) represents the *stored elastic energy density*, satisfying standard assumptions of nonlinear elasticity. In particular, we suppose that $W \in C^0(\mathbb{R}^{3\times 3}; \mathbb{R}_+)$ satisfies

(i) Frame indifference: $W(RF) = W(F)$ for all $R \in SO(3)$ and $F \in \mathbb{R}^{3\times 3}$,

(ii) Single energy-well structure: $\{W = 0\} \equiv SO(3)$,

(iii) Regularity: W is C^2-regular in a neighborhood of $SO(3)$,

(iv) Coercivity: There exists $c > 0$ such that for all $F \in \mathbb{R}^{3\times 3}$ it holds that (2.4)

$$W(F) \geq c \operatorname{dist}^2(F, SO(3)),$$

(v) Growth condition: There exists $C > 0$ such that for all $F \in \mathbb{R}^{3\times 3}$ it holds that

$$W(F) \leq C \operatorname{dist}^2(F, SO(3)).$$

We note that condition (v) excludes the natural assumption $W(F) \to +\infty$ as $\det F \to 0^+$, but it is needed in our analysis for the construction of recovery sequences. The choice of an isotropic perimeter energy is purely for simplicity of the exposition, and more general anisotropic perimeters can be chosen in the model without substantial changes in the proofs, see also Remark 2.4 below. As for the parameter $\kappa_h > 0$ in the curvature regularization, we require

$$\kappa_h h^{-2} \to 0 \,, \quad \kappa_h h^{-52/25} \to +\infty \quad \text{as } h \to 0 \,. \tag{2.5}$$

Similarly to its role in [34, Equation (2.5)], it is a technical assumption that has been chosen for simplicity rather than optimality. Its choice is related to the application of suitable piecewise rigidity results [32] and Korn inequalities [16], and will become apparent along the proof, see in particular (3.17).

As is by now customary in dimension-reduction problems, we perform an anisotropic change of variables to reformulate the problem in a fixed reference domain: recalling (1.5), we rescale our variables and set

$$\Omega := \Omega_1 \,, \quad V := T_{1/h}(E) = \{x \in \Omega : T_h x \in E\} \,. \tag{2.6}$$

Accordingly, the rescaled deformations are defined by $y \colon \Omega \to \mathbb{R}^3$ via

$$y(x) := v(T_h x) \,. \tag{2.7}$$

The total energy is rescaled by a factor h^3, hence we set

$$\mathcal{E}^h(y, V) := h^{-3} \mathcal{F}^h(v, E) \,, \tag{2.8}$$

where the pair (y, V) is related to (v, E) via (2.6) and (2.7). In this rescaling, one factor h corresponds to the change of volume and the other factor h^2 corresponds to the average elastic energy per unit volume in the bending regime for plates.

The corresponding rescaled gradient will be denoted as usual by

$$\nabla_h y(x) := \Big(\partial_1 y, \partial_2 y, \frac{1}{h}\partial_3 y\Big)(x) = \nabla v(T_h x)\,.$$

Therefore, by a change of variables we find

$$\mathcal{E}^h(y,V) = h^{-2}\int_{\Omega\setminus\overline{V}} W(\nabla_h y(x))\,\mathrm{d}x + \int_{\partial V\cap\Omega} \big|\big(\nu_V^1(z), \nu_V^2(z), h^{-1}\nu_V^3(z)\big)\big|\,\mathrm{d}\mathcal{H}^2(z) + \mathcal{E}^h_{\mathrm{curv}}(V)\,, \tag{2.9}$$

where $\nu_V(z) := \big(\nu_V^1(z), \nu_V^2(z), \nu_V^3(z)\big)$ denotes the outer unit normal to $\partial V \cap \Omega$ at the point z. Note that for the rescaling of the perimeter part of the energy, one can test with smooth functions and use the divergence theorem, as we also commented in [34, Equation (2.10)].

Regarding the term $\mathcal{E}^h_{\mathrm{curv}}(V)$ which denotes the curvature-related energetic contribution for the rescaled set V, its precise expression after the change of variables will not be of specific use in the subsequent analysis. Hence, we refrain from giving its explicit form.

In view of (1.5) and (2.2), the space of rescaled admissible pairs of deformations and voids is given by

$$\hat{\mathcal{A}}_h := \big\{(y,V)\colon V\in\mathcal{A}_{\mathrm{reg}}(\Omega)\,,\ y\in W^{1,2}(\Omega\setminus\overline{V};\mathbb{R}^3)\,,\ y|_V \equiv T_h(\mathrm{id})\,,\ \|y\|_{L^\infty(\Omega)}\le M\big\}\,. \tag{2.10}$$

2.2. **Limiting model and main result.** As in the purely elastic case considered in [36], the density of the limiting elastic energy will depend on the quadratic form $\mathcal{Q}_3\colon \mathbb{R}^{3\times 3}\to\mathbb{R}$, which is defined as

$$\mathcal{Q}_3(G) := D^2W(\mathrm{Id})[G,G]\,. \tag{2.11}$$

Due to (2.4), $\mathcal{Q}_3$ vanishes on $\mathbb{R}^{3\times3}_{\mathrm{skew}}$ and is strictly positive-definite on $\mathbb{R}^{3\times3}_{\mathrm{sym}}$. We also define $\mathcal{Q}_2\colon \mathbb{R}^{2\times2}\to\mathbb{R}$ as

$$\mathcal{Q}_2(A) := \min_{c\in\mathbb{R}^3} \mathcal{Q}_3(\hat{A} + c\otimes e_3)\,, \tag{2.12}$$

by minimizing over stretches in the x_3-direction. Here, for a matrix $A\in\mathbb{R}^{2\times2}$ we denote by $\hat{A}$ its extension to a (3×3)-matrix by adding zeros in the third row and column.

As in [57], where a similar model under more restrictive geometric assumptions on the voids was studied, the limiting energy will be defined on the space

$$\mathcal{A} := \Big\{(y,V)\in SBV^{2,2}_{\mathrm{isom}}(S;\mathbb{R}^3)\times\mathcal{P}(S)\colon y|_V = \mathrm{id}|_V\,, \|y\|_{L^\infty(\Omega)}\le M\Big\}\,, \tag{2.13}$$

where

$$SBV^{2,2}_{\mathrm{isom}}(S;\mathbb{R}^3) := \{y\in SBV^2(S;\mathbb{R}^3)\colon \nabla' y\in SBV^2(S;\mathbb{R}^{3\times2}), (\nabla' y, \partial_1 y\wedge\partial_2 y)\in SO(3)\text{ a.e.}\}\,,$$

and $\nabla' y := (\partial_1 y, \partial_2 y)$, i.e., the limiting admissible deformations are isometric away from the jump set

$$J_{(y,\nabla' y)} := J_y \cup J_{\nabla' y}\,. \tag{2.14}$$

We denote by $\widetilde{y}\colon \Omega\to\mathbb{R}^3$ maps of the form

$$\widetilde{y}(x) = y(x_1,x_2)\ \forall\, x\in\Omega\,,\ \text{for some } y\in SBV^{2,2}_{\mathrm{isom}}(S;\mathbb{R}^3)\,, \tag{2.15}$$

and, similarly, by $\widetilde{V}\subset\Omega$ sets of the form

$$\widetilde{V} = V\times\left(-\frac{1}{2},\frac{1}{2}\right)\quad\text{for some } V\in\mathcal{P}(S)\,. \tag{2.16}$$

In what follows, the pair $(\widetilde{y},\widetilde{V})$ will always be associated to (y,V) via (2.15) and (2.16) whenever it appears. For a mapping $y\in SBV^{2,2}_{\mathrm{isom}}(S;\mathbb{R}^3)$, we also introduce its second fundamental form via

$$\mathrm{II}_y := \big(\partial_i y\cdot\partial_j(\partial_1 y\wedge\partial_2 y)\big)_{1\le i,j\le 2}\,. \tag{2.17}$$

With the definitions (2.12) and (2.14) in mind, for each $(y,V)\in\mathcal{A}$, the limiting two-dimensional energy of Blake-Zisserman-Kirchhoff-type (cf. also [10, 17, 57]) is defined as

$$\mathcal{E}^0\big(y,V\big) := \frac{1}{24}\int_{S\setminus V}\mathcal{Q}_2(\mathrm{II}_y(x'))\,\mathrm{d}x' + \mathcal{H}^1\big(\partial^*V\cap S\big) + 2\mathcal{H}^1\big(J_{(y,\nabla' y)}\setminus\partial^*V\big)\,. \tag{2.18}$$

As mentioned also in the introduction, the limiting two-dimensional model features the classical bending-curvature energy derived in [36] and two surface terms related to the presence of voids. The middle term on the right-hand side of (2.18) corresponds to the energy contribution of the limiting void V, whereas the last one therein is associated to discontinuities or folds of the deformation, represented by J_y and $J_{\nabla' y}$, respectively. The origin of this term is due to the fact that voids may collapse into discontinuity curves in the limit, and thus appears with a factor 2, see Figure 2.

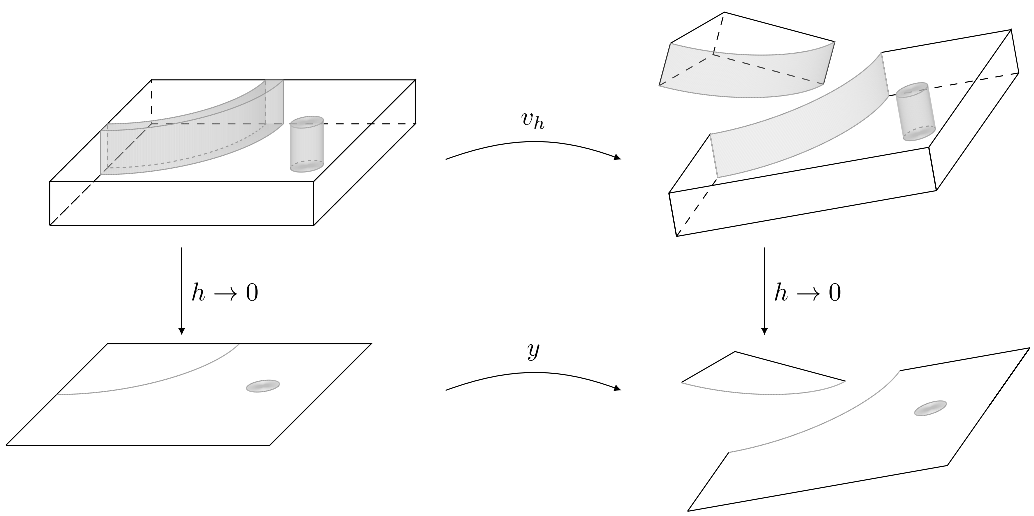

Figure 2. Bending a plate with voids and cracks: Collapsing voids lead to discontinuity curves for J_y. Folds corresponding to the presence of $J_{\nabla' y}$ are not depicted for simplicity.

With these definitions and notations, our main results can be summarized as follows.

Theorem 2.1. *(Compactness) Let $(h_j)_{j\in\mathbb{N}}\subset(0,\infty)$ with $h_j\searrow 0$ and $(y_{h_j},V_{h_j})\in\hat{\mathcal{A}}_{h_j}$ (cf. (2.10)) be such that*

$$\sup_{j\in\mathbb{N}}\mathcal{E}^{h_j}(y_{h_j},V_{h_j})<+\infty\,. \tag{2.19}$$

Then, there exists $(y,V)\in\mathcal{A}$ (cf. (2.13)) such that, up to a non-relabeled subsequence,

$$\begin{aligned} &\text{(i)} \quad \chi_{V_{h_j}}\longrightarrow\chi_{\widetilde{V}} \ \text{ in } L^1(\Omega)\,,\\ &\text{(ii)} \quad y_{h_j}\longrightarrow\widetilde{y} \text{ in } L^1(\Omega;\mathbb{R}^3)\,,\\ &\text{(iii)} \quad \nabla_{h_j}y_{h_j}\longrightarrow\big(\nabla'\widetilde{y},\partial_1\widetilde{y}\wedge\partial_2\widetilde{y}\big) \ \text{ strongly in } L^2(\Omega;\mathbb{R}^{3\times3})\,, \end{aligned} \tag{2.20}$$

where $(\widetilde{y},\widetilde{V})$ is associated to (y,V) via (2.15) and (2.16).

Definition 2.2. We say that $(y_{h_j},V_{h_j})\xrightarrow{\tau}\big(y,V\big)$ as $j\to\infty$ if and only if (2.20) holds.

Note that (2.10) implies that $\sup_{j\in\mathbb{N}}\|y_{h_j}\|_{L^\infty(\Omega)}\le M$, and therefore the convergence in (2.20)(ii) actually holds in $L^p(\Omega;\mathbb{R}^3)$ for every $p\in[1,+\infty)$. We are now ready to state the main Γ-convergence result.

Theorem 2.3. (*Γ-convergence*) *Let $(h_j)_{j\in\mathbb{N}} \subset (0,\infty)$ with $h_j \searrow 0$. The sequence of functionals $(\mathcal{E}^{h_j})_{j\in\mathbb{N}}$ $\Gamma(\tau)$-converges to the functional $\mathcal{E}^0$ of (2.18), i.e., the following two inequalities hold true.*
(i) (*Γ-liminf inequality*) *Whenever $(y_{h_j}, V_{h_j}) \xrightarrow{\tau} \big(y, V\big)$, then*

$$\liminf_{j\to+\infty} \mathcal{E}^{h_j}(y_{h_j}, V_{h_j}) \geq \mathcal{E}^0\big(y, V\big)\,.$$

(ii) (*Γ-limsup inequality*) *For every $\big(y, V\big) \in \mathcal{A}$ there exists $(y_{h_j}, V_{h_j}) \in \hat{\mathcal{A}}_{h_j}$ for each $j \in \mathbb{N}$ such that $(y_{h_j}, V_{h_j}) \xrightarrow{\tau} \big(y, V\big)$ and*

$$\limsup_{j\to+\infty} \mathcal{E}^{h_j}(y_{h_j}, V_{h_j}) \leq \mathcal{E}^0\big(y, V\big)\,.$$

Remark 2.4 (Possible extensions and variants)**.** (i) One could consider more general perimeter energies of the form

$$\beta_h \int_{\partial E\cap\Omega_h} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^2\,,$$

where $\lim_{h\to 0}(h^{-2}\beta_h) = \beta > 0$ and φ is a norm in $\mathbb{R}^3$. Similarly to [34], for simplicity of the exposition we have chosen $\beta_h := h^2$ and φ to be the standard isotropic Euclidean norm in $\mathbb{R}^3$. The general case is analogous in its treatment, where the limiting surface energy in (2.18) is given by

$$\int_{\partial^* V\cap S} \varphi_0(\nu_V)\,\mathrm{d}\mathcal{H}^2 + 2\int_{J_{(y,\nabla' y)}\setminus\partial^* V} \varphi_0(\nu_{J_{(y,\nabla' y)}})\,\mathrm{d}\mathcal{H}^2\,,$$

with

$$\varphi_0(x_1, x_2) := \min_{c\in\mathbb{R}} \varphi(x_1, x_2, c)\,,$$

see [57, Sections 2 and 6] for details, especially related to some technical adjustments needed in the construction of recovery sequences for the void sets in the presence of anisotropic surface energy densities.

(ii) Completely analogously to [34, Remark 2.1(ii)], any singular perturbation of the form

$$h^2\kappa_h \int_{\partial E\cap\Omega_h} |\boldsymbol{A}|^q\,\mathrm{d}\mathcal{H}^2$$

with $q \geq 2$ would be a legitimate choice for a curvature regularization, up to adjusting the condition for κ_h in (2.5) (which would then depend also on q). Let us nevertheless also mention here that the choice $q \geq 2$ is essential, see [32, Lemma 2.12 and Example 2.13]. For simplicity, we have chosen $q = 2$, which corresponds to a curvature regularization of classical Willmore-type.

(iii) We also remark that compressive boundary conditions and body forces can be included into the Γ-convergence statement. Although we omit the details here, we refer the interested reader to [36, Section 6] for a discussion in this direction, in the purely elastic case.

(iv) Finally, [34, Example 2.1] can easily be adjusted to our setting, the only difference coming from the energy rescaling by a factor h^{-3} instead of h^{-4} in this case. This allows to exhibit configurations $(v_h, E_h) \in \mathcal{A}_h$ with

$$\sup_{h>0} h^{-3}\mathcal{F}^h(v_h, E_h) < +\infty\,,$$

where E_h consists of balls which concentrate on arbitrarily small scales (independently of h), and whose number is diverging at a rate faster than h^{-1}. In contrast, in the setting of [57] and for void sets consisting of a disjoint union of balls, the minimal droplet assumption implies a lower bound of order h on the radius of each ball and an upper bound of order h^{-1} for the total number of balls, cf. [57, Remark 3.1].

3. Piecewise rigidity and Sobolev modification of deformations

In this section we collect some preparatory ingredients which are of utter importance in the proofs of the compactness result of Theorem 2.1 in Section 4 and the Γ-liminf inequality of Theorem 2.3 in Section 5. Our reasoning follows in spirit the analogous one in [34, Section 3] and relies on the approximation of a sequence of deformations with equibounded energy by mappings which enjoy good Sobolev bounds on a large portion (and asymptotically all) of the bulk domain of the thin plate, while still retaining a good control on their total jump set in the full domain. This will allow us to use tools from the theory of SBV functions, in particular *Ambrosio's compactness and lower semicontinuity theorems* (cf. [4, Theorems 4.7 and 4.8]), together with the proof strategy from [36, 57].

In this and the following sections, we will use the continuum subscript $h > 0$ instead of the sequential subscript notation $(h_j)_{j\in\mathbb{N}}$ purely for notational convenience. To state the main results of this section, we need to introduce some further notation. Recall the definition of Ω_h in (2.1). As the estimates provided by the rigidity result stated in Theorem 3.3 below are only local, we need to introduce a slightly smaller reference domain. To this end, for every $\rho \in (0,1)$, we define

$$\Omega_{h,\rho} := \big\{x' \in S\colon \mathrm{dist}(x',\partial S) > \rho\big\} \times \big(-\tfrac{(1-\rho)}{2}h, \tfrac{(1-\rho)}{2}h\big)\,. \tag{3.1}$$

Eventually, in Sections 4–5 we will send $\rho \to 0^+$, after the convergence $h \to 0^+$. In what follows, for every $i \in \mathbb{Z}^2$, we introduce cubes and cuboids of sidelengths proportional to $h > 0$, namely

$$Q_h(i) := (hi,0) + (-\tfrac{h}{2},\tfrac{h}{2})^3 \quad \text{ and } \quad \hat{Q}_h(i) := (hi,0) + (-\tfrac{3h}{2},\tfrac{3h}{2})^2 \times (-\tfrac{h}{2},\tfrac{h}{2})\,, \tag{3.2}$$

which will serve to cover Ω_h, up to the negligible skeleton of the covering. In particular, we set

$$\mathcal{Q}_h := \{Q_h(i)\colon \hat{Q}_h(i) \subset \Omega_h\}\,.$$

Similarly, for every $\rho \in (0,1)$ we set

$$Q_{h,\rho}(i) := (hi,0) + (1-\rho)(-\tfrac{h}{2},\tfrac{h}{2})^3\,, \quad \hat{Q}_{h,\rho}(i) := (hi,0) + (1-\rho)\big((-\tfrac{3h}{2},\tfrac{3h}{2})^2 \times (-\tfrac{h}{2},\tfrac{h}{2})\big)\,. \tag{3.3}$$

Given a measurable set $K \subset \mathbb{R}^3$ and $\gamma > 0$, we introduce the localized version of the total surface energy as

$$\mathcal{G}^{\gamma}_{\mathrm{surf}}(E;K) := \mathcal{H}^2(\partial E \cap K) + \gamma \int_{\partial E\cap K} |\boldsymbol{A}|^2 \,\mathrm{d}\mathcal{H}^2\,, \tag{3.4}$$

using a general parameter γ in place of κ_h for later purposes. The total rescaled energy is then given by

$$\mathcal{G}^h(v,E) := \frac{1}{h^3}\int_{\Omega_h\setminus\overline{E}} W(\nabla v)\,\mathrm{d}x + \frac{1}{h}\mathcal{G}^{\kappa_h}_{\mathrm{surf}}(E;\Omega_h) = h^{-3}\mathcal{F}^h(v,E)\,, \tag{3.5}$$

for $(v,E) \in \mathcal{A}_h$, cf. (2.2), where $\mathcal{F}^h$ is as in (2.3). Note again that one factor h in the rescaling of the elastic energy corresponds to the volume of Ω_h, while the extra factor h^2 corresponds to the average elastic energy per unit volume.

Proposition 3.1 (Sobolev modification of deformations and their gradients)**.** *Let $0 < \rho \le \rho_0$ for some universal $\rho_0 > 0$. Then, there exist constants $C := C(S,M) > 0$ and $h_0 = h_0(\rho) > 0$ such that for every sequence $(v_h,E_h)_{h>0}$ with $(v_h,E_h) \in \mathcal{A}_h$ and*

$$\sup_{h>0} \mathcal{G}^h(v_h,E_h) < +\infty\,, \tag{3.6}$$

and for every $0 < h \le h_0$ there exist fields $r_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^3)$ and $R_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^{3\times3})$ satisfying the following properties:

$$
\begin{aligned}
&\text{(i)} \quad \|r_h\|_{L^\infty(\Omega_{h,\rho})} + \|R_h\|_{L^\infty(\Omega_{h,\rho})} \le C\,,\\
&\text{(ii)} \quad \mathcal{H}^2(J_{r_h}) + \mathcal{H}^2(J_{R_h}) \le Ch\,,\\
&\text{(iii)} \quad \int_{\Omega_{h,\rho}} |\nabla r_h|^2 \,\mathrm{d}x + \int_{\Omega_{h,\rho}} |\nabla R_h|^2 \,\mathrm{d}x \le Ch\,,\\
&\text{(iv)} \quad h^{-1}\mathcal{L}^3(\Omega_{h,\rho}\cap\{|v_h(x) - r_h| > \theta_h\}) \to 0\,,\ h^{-1}\mathcal{L}^3(\Omega_{h,\rho}\cap\{|\nabla v_h - R_h| > \theta_h\}) \to 0\,,
\end{aligned}
\tag{3.7}
$$

where $(\theta_h)_{h>0} \subset (0,+\infty)$ is a sequence with

$$
\theta_h \to 0 \quad \text{and} \quad \theta_h h^{-1} \to \infty\,. \tag{3.8}
$$

Moreover, there exists $w_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^3)$ such that

$$
\begin{aligned}
&\text{(i)} \quad h^{-1}\mathcal{L}^3(\{w_h \neq v_h\}\cap\Omega_{h,\rho}) \to 0 \ \text{as } h \to 0\,,\\
&\text{(ii)} \quad \int_{\Omega_{h,\rho}} |\nabla w_h - R_h|^2 \,\mathrm{d}x \le Ch^3\,,\\
&\text{(iii)} \quad \|w_h\|_{L^\infty(\Omega_{h,\rho})} + h^{-1}\mathcal{H}^2(J_{w_h}) + h^{-1}\int_{\Omega_{h,\rho}} |\nabla w_h|^2 \,\mathrm{d}x \le C\,,\\
&\text{(iv)} \quad J_{w_h} \subset \Omega_{h,\rho} \cap \bigcup_{Q_h(i)\in\mathcal{Q}_{v_h}} \partial Q_h(i)\,,\ \text{for some } \mathcal{Q}_{v_h} \subset \mathcal{Q}_h \text{ with } \#\mathcal{Q}_{v_h} \le Ch^{-1}\,.
\end{aligned}
\tag{3.9}
$$

The maps r_h, R_h can be thought of as regularized piecewise affine/constant approximations of v_h, ∇v_h, respectively, see (3.7)(iv). These approximations enjoy good SBV^2-bounds provided by (3.7)(i)–(iii), which will be crucial later to employ compactness and lower semicontinuity results in SBV^2 on their appropriate $T_{1/h}$-rescalings. The role of the maps w_h in the second part of the proposition is analogous, with the extra advantage that w_h is obtained by changing the map v_h on an asymptotically vanishing portion of the volume, see (3.9)(i), while having a more precise information on the geometry of the jump set, the latter being in a sense *cubic*, see (3.9)(iv).

We note that r_h, R_h, and w_h depend on ρ which we do not include in the notation for simplicity. In the rest of this section, we focus on proving Proposition 3.1, so that starting from Section 4 we can give the proofs of our main compactness and Γ-convergence results. In the proofs, we will send the parameters h, ρ to zero in this order. Thus, for the sake of keeping the notation simple, generic constants which are independent of h, ρ will be denoted by C, and we will use a subscript notation in order to highlight the dependence of a particular constant on a specific parameter.

3.1. **Rigidity results.** This subsection is devoted to recalling some rigidity results which are the basis for our proofs, as was also the case for the derivation of effective theories for elastic rods with voids in our previous work [34].

Geometric rigidity in variable domains: We first recall the result [32, Theorem 2.1], see also [34, Section 3.1]. As mentioned in the introduction, the behavior of deformations v on connected components of $\Omega_h \setminus \overline{E}$ might fail to be rigid, cf. [32, Example 2.6]. The main result in [32] consists in showing that rigidity estimates can be obtained outside of a slightly thickened version of the voids. We omit the proofs of the next two results, since they are identical to the ones of [34, Proposition 3.3 and Theorem 3.1].

Proposition 3.2 (Global thickening of sets)**.** *Let $h, \rho > 0$ and $\gamma \in (0,1)$. There exists a universal constant $C_0 > 0$ and $\eta_0 = \eta_0(\rho) \in (0,1)$, such that for every $\eta \in (0,\eta_0]$ the following holds:*

Given $E \in \mathcal{A}_{\rm reg}(\Omega_h)$, we can find an open set $E_{h,\eta,\gamma}$ such that $E \subset E_{h,\eta,\gamma} \subset \Omega_h$, $\partial E_{h,\eta,\gamma} \cap \Omega_h$ is a union of finitely many C^2-regular submanifolds, and

$$\begin{aligned} &\text{(i)} \quad \mathcal{L}^3(E_{h,\eta,\gamma} \setminus E) \le h\eta\gamma^{1/2}\mathcal{G}^{\gamma h^2}_{\rm surf}(E;\Omega_h), \qquad \mathrm{dist}_{\mathcal{H}}(E, E_{h,\eta,\gamma}) \le h\eta\gamma^{1/2}\,, \\ &\text{(ii)} \quad \mathcal{H}^2(\partial E_{h,\eta,\gamma} \cap \Omega_h) \le (1 + C_0\eta)\,\mathcal{G}^{\gamma h^2}_{\rm surf}(E;\Omega_h)\,. \end{aligned} \tag{3.10}$$

On the complement of $\Omega_{h,\rho} \setminus \overline{E_{h,\eta,\gamma}}$ one can obtain good quantitative piecewise rigidity estimates, as provided by the following theorem. For its statement, we recall (3.2)–(3.3), and mention that this version is specific to our purposes compared to the more general one of [34, Theorem 3.1], from which it actually follows by a direct application.

Theorem 3.3 (Geometric rigidity in variable domains)**.** *Let $h, \rho > 0$ and $\gamma \in (0,1)$. There exist a universal constant $C_0 > 0$, $\eta_0 = \eta_0(\rho) > 0$, and for each $\eta \in (0, \eta_0]$ there exists $C_\eta > 0$ such that the following holds:*
For every $E \in \mathcal{A}_{\rm reg}(\Omega_h)$, denoting by $E_{h,\eta,\gamma}$ the set of Proposition 3.2, for every $Q_h(i) \in \mathcal{Q}_h$, for the connected components $(U_j)_j$ of $\hat{Q}_{h,\rho}(i) \setminus \overline{E_{h,\eta,\gamma}}$ and for every $y \in W^{1,2}(\Omega_h \setminus \overline{E};\mathbb{R}^3)$ there exist corresponding rotations $(R_j)_j \subset SO(3)$ and vectors $(b_j)_j \subset \mathbb{R}^3$ such that

$$\begin{aligned} &\text{(i)} \quad \sum\nolimits_j \int_{U_j} \big|\mathrm{sym}\big((R_j)^T\nabla y - \mathrm{Id}\big)\big|^2 \,\mathrm{d}x \le C_0\big(1 + C_\eta\gamma^{-15/2}h^{-3}\varepsilon\big) \int_{\hat{Q}_h(i)\setminus\overline{E}} \mathrm{dist}^2(\nabla y, SO(3))\,\mathrm{d}x\,, \\ &\text{(ii)} \quad \sum\nolimits_j \int_{U_j} \big|(R_j)^T\nabla y - \mathrm{Id}\big|^2 \,\mathrm{d}x \le C_\eta\gamma^{-3} \int_{\hat{Q}_h(i)\setminus\overline{E}} \mathrm{dist}^2(\nabla y, SO(3))\,\mathrm{d}x\,, \\ &\text{(iii)} \quad \sum\nolimits_j \int_{U_j} \frac{1}{h^2}\big|y - (R_jx + b_j)\big|^2 \,\mathrm{d}x \le C_\eta\gamma^{-5} \int_{\hat{Q}_h(i)\setminus\overline{E}} \mathrm{dist}^2(\nabla y, SO(3))\,\mathrm{d}x\,, \end{aligned} \tag{3.11}$$

where for brevity $\varepsilon := \int_{\hat{Q}_h(i)\setminus\overline{E}} \mathrm{dist}^2(\nabla y, SO(3))\,\mathrm{d}x$.

In the subsequent proofs we will choose the parameters η and γ depending on the regime of the elastic energy ε such that $C_\eta\gamma^{-15/2}h^{-3}\varepsilon \le 1$ and $C_\eta\gamma^{-5} \le \varepsilon^{-\theta}$ for some $\theta > 0$ small. With these choices, we obtain a sharp control on symmetrized gradients with repect to ε, see (3.11)(i), while the estimate in (3.11)(ii) and the Poincaré-type estimate (3.11)(iii) yield a suboptimal control in the exponent, of the order $\varepsilon^{1-\theta}$.

Korn and Poincaré inequalities in SBV^2: As in [34], the issue of the suboptimal exponent in the gradient estimate can be remedied in case the surface area of the void set is small. This relies on sophisticated Korn and Poincaré inequalities in the space $GSBD^2$ [22]. Since we will here need the results only for SBV^2-functions, we formulate the corresponding statements of [16, Theorem 1.1, Theorem 1.2] in a simplified setting. In the sequel, we call a mapping $a\colon \mathbb{R}^3 \to \mathbb{R}^3$ an *infinitesimal rigid motion* if a is affine with $\mathrm{sym}(\nabla a) = 0$.

Theorem 3.4 (Korn-Poincaré inequality for functions with small jump set)**.** *Let $U \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then, there exists a constant $c = c(U,d) > 0$ such that for all $u \in SBV^2(U;\mathbb{R}^d)$ there exists a set of finite perimeter $\omega \subset U$ with*

$$\mathcal{H}^{d-1}(\partial^*\omega) \le c\mathcal{H}^{d-1}(J_u)\,, \qquad \mathcal{L}^d(\omega) \le c(\mathcal{H}^{d-1}(J_u))^{d/d-1}\,, \tag{3.12}$$

and an infinitesimal rigid motion a such that

$$(\mathrm{diam}(U))^{-1}\|u - a\|_{L^2(U\setminus\omega)} + \|\nabla u - \nabla a\|_{L^2(U\setminus\omega)} \le c\|\mathrm{sym}(\nabla u)\|_{L^2(U)}\,.$$

Moreover, there exists $v \in W^{1,2}(U;\mathbb{R}^d)$ such that $v \equiv u$ on $U \setminus \omega$ and

$$\|\mathrm{sym}(\nabla v)\|_{L^2(U)} \le c\|\mathrm{sym}(\nabla u)\|_{L^2(U)}\,.$$

Furthermore, if $u \in L^\infty(U;\mathbb{R}^d)$, one has $\|v\|_{L^\infty(U)} \le \|u\|_{L^\infty(U)}$.

Remark 3.5. *We refer the reader to the discussion below* [34, Theorem 3.2] *on the derivation of the above theorem from the results in* [16]*. Note that the result is indeed only relevant if $\mathcal{H}^{d-1}(J_u)$ is small, since otherwise $\omega = U$ is possible and the statement is empty. Moreover, it is easily seen that the constant $c = c(U,d)$ of Theorem 3.4 is invariant under dilations of the domain U.*

An easy consequence of the above theorem is a version of the Poincaré inequality in SBV in arbitrary codimension, namely the following.

Corollary 3.6. *Let $U \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $m \in \mathbb{N}$. Then, there exists a constant $c = c(U,d,m) > 0$ such that for all $u \in SBV^2(U;\mathbb{R}^m)$ there exists a set of finite perimeter $\omega \subset U$ with*

$$\mathcal{H}^{d-1}(\partial^*\omega) \le c\mathcal{H}^{d-1}(J_u)\,, \qquad \mathcal{L}^d(\omega) \le c(\mathcal{H}^{d-1}(J_u))^{d/(d-1)}\,, \tag{3.13}$$

and a constant vector $b \in \mathbb{R}^m$ such that

$$\big(\mathrm{diam}(U)\big)^{-1}\|u-b\|_{L^2(U\setminus\omega)} \le c\|\nabla u\|_{L^2(U)}\,. \tag{3.14}$$

Proof. The statement is a simple consequence of Theorem 3.4. Without restriction we can assume that $m \ge d$, as otherwise we add $(d-m)$-components to u which are identically zero. We denote by Σ the collection of all strictly increasing multi-indices of length d from $\{1,\dots,m\}$, so that $\#\Sigma = \binom{m}{d}$. For every $\sigma \in \Sigma$ and $t \in \mathbb{R}^m$, we denote by $\pi_\sigma t \in \mathbb{R}^m$ the orthogonal projection of t onto the components indicated by σ. In a similar fashion, we define $\pi_\sigma u\colon U \to \mathbb{R}^m$. We apply Theorem 3.4 on $\pi_\sigma u$ (which is essentially $\mathbb{R}^d$-valued) to obtain $\omega_\sigma \subset U$ satisfying (3.13) (for ω_σ in place of ω) and $b_\sigma \in \mathbb{R}^m$ such that

$$\big(\mathrm{diam}(U)\big)^{-1}\|\pi_\sigma u - b_\sigma\|_{L^2(U\setminus\omega_\sigma)} \le c\|\nabla u\|_{L^2(U)}\,.$$

We define $\omega := \bigcup_{\sigma\in\Sigma}\omega_\sigma$ and observe that (3.13) holds. Defining

$$b := \frac{1}{\binom{m}{d-1}}\sum_{\sigma\in\Sigma} b_\sigma \in \mathbb{R}^m\,,$$

it can be easily verified that (3.14) holds, noticing that also $u = \frac{1}{\binom{m}{d-1}}\sum_{\sigma\in\Sigma}\pi_\sigma u$. ☐

As in Theorem 3.4, we emphasize that the application of the previous corollary is only meaningful if $\mathcal{H}^{d-1}(J_u)$ is smaller than a sufficiently small constant depending on U.

Difference of affine maps: We close this subsection with the statement of the following elementary lemma, cf. [34, Lemma 3.1], whose proof can be found therein. By $B_r(x) \subset \mathbb{R}^3$ we denote the open ball centered at $x \in \mathbb{R}^3$ with radius $r > 0$.

Lemma 3.7 (Estimate on affine maps)**.** *Let $\delta > 0$. Then there exists a constant $C > 0$ only depending on δ such for every $G \in \mathbb{R}^{3\times3}$, $b \in \mathbb{R}^3$, $x \in \mathbb{R}^3$, and $E \subset B_r(x)$ for some $r > 0$ with $\mathcal{L}^3(E) \ge \delta r^3$,we have*

$$\|G\cdot+b\|_{L^\infty(B_r(x))} \le Cr^{-3/2}\|G\cdot+b\|_{L^2(E)}\,, \quad |G| \le Cr^{-5/2}\|G\cdot+b\|_{L^2(E)}\,.$$

3.2. **Local rigidity estimates and Sobolev replacement on good cubes.** In this subsection we first introduce some extra necessary notation and definitions for the rest of the section. We start by introducing the thickened void sets and then partition our reference domain $\Omega_{h,\rho}$, see (3.1) into cubes, where the partitioning is with respect to the surface area of the boundary of the thickened void and the size of the local elastic energy in each cuboid of the partition.

Let us start with a sequence $(v_h, E_h)_{h>0}$ of admissible deformations and void sets in the thin plate Ω_h. Recalling (3.5), we suppose that

$$\sup_{h>0} \mathcal{G}^h(v_h, E_h) < +\infty \,. \tag{3.15}$$

We fix

$$0 < \rho \leq \rho_0 := 1 - (127/128)^{1/3} \tag{3.16}$$

as in Proposition 3.1 and recall the choice of $(\kappa_h)_{h>0}$ in (2.5). Let $\eta_0 = \eta_0(\rho) \in (0,1)$ be the minimum of the constants in Proposition 3.2 and Theorem 3.3. In view of (2.5), we can choose a sequence $(\eta_h)_{h>0} \subset (0, \eta_0)$ converging slowly enough to zero, so that the constant C_{η_h} in (3.11), obtained by applying Theorem 3.3 for ρ, $\eta = \eta_h \in (0, \eta_0(\rho))$, and $\gamma = \kappa_h/h^2$, satisfies

$$\limsup_{h\to 0} \, C_{\eta_h} \Big(\frac{h^2}{\kappa_h}\Big)^5 h^{2/5} \leq 1 \,. \tag{3.17}$$

Applying Proposition 3.2 with the above choice of ρ, η and γ, for all $h > 0$ we find open sets E_h^* with $E_h \subset E_h^* \subset \Omega_h$ such that $\partial E_h^* \cap \Omega_h$ is a union of finitely many C^2-regular submanifolds and

$$\begin{aligned} &\text{(i)} \quad h^{-2}\mathcal{L}^3(E_h^* \setminus E_h) \to 0, \qquad h^{-1}\,\mathrm{dist}_{\mathcal{H}}(E_h^*, E_h) \to 0 \quad \text{as } h \to 0\,, \\ &\text{(ii)} \quad \liminf_{h\to 0} h^{-1}\mathcal{H}^2(\partial E_h^* \cap \Omega_h) \leq \liminf_{h\to 0} \, h^{-1}\mathcal{G}^{\kappa_h}_{\mathrm{surf}}(E_h; \Omega_h)\,. \end{aligned} \tag{3.18}$$

Here, we made use of (3.10), $\eta_h \to 0$, (2.5), and the fact that $h^{-1}\mathcal{G}^{\gamma h^2}_{\mathrm{surf}}(E_h; \Omega_h) = h^{-1}\mathcal{G}^{\kappa_h}_{\mathrm{surf}}(E_h; \Omega_h)$ is uniformly bounded by (3.5) and (3.15). In the estimates (3.11), the behavior of the deformation inside E_h^* cannot be controlled. Thus, in accordance to (2.2), for definiteness only we can without restriction assume that the deformation is the identity also inside E_h^*, i.e., we define $v_h^* \colon \Omega_h \to \mathbb{R}^3$ by

$$v_h^*(x) := \begin{cases} v_h(x) & \text{if } x \in \Omega_h \setminus E_h^*\,, \\ x & \text{if } x \in E_h^*\,. \end{cases} \tag{3.19}$$

In view of (2.2) and (3.18)(i), we get

$$h^{-2}\mathcal{L}^3(\{v_h^* \neq v_h\}) \leq h^{-2}\mathcal{L}^3(E_h^* \setminus E_h) \to 0 \quad \text{ as } h \to 0\,. \tag{3.20}$$

In view of Remark 3.5, set $c_{\mathrm{KP}} := c(\hat{Q}_1(0))$ and also introduce the (small) parameter

$$\alpha := \Big(\frac{128}{9}\max\{2c_{\mathrm{isop}}, c_{\mathrm{KP}}\}\Big)^{-2/3}\,, \tag{3.21}$$

where c_{isop} denotes the relative isoperimetric constant of the cuboid $\hat{Q}_1(0)$ in $\mathbb{R}^3$, the latter being also scaling invariant, cf. [4, Equation (3.43)] for a version stated on balls instead of cuboids. For every $Q_h(i) \in \mathcal{Q}_h$, we also introduce the localized elastic energy

$$\varepsilon_{i,h} := \int_{\hat{Q}_h(i)\setminus\overline{E_h}} \mathrm{dist}^2(\nabla v_h, SO(3))\,\mathrm{d}x\,. \tag{3.22}$$

We now divide the family of cubes $\mathcal{Q}_h$ into two subfamilies: first, we consider the family of indices associated to *good cubes*, defined by

$$I^h_{\mathrm{g}} := \Big\{ i \in \mathbb{Z}^2 : Q_h(i) \in \mathcal{Q}_h,\ \mathcal{H}^2(\partial E_h^* \cap \hat{Q}_{h,\rho}(i)) \leq \alpha h^2\,,\ \varepsilon_{i,h} \leq h^4 \Big\}\,. \tag{3.23}$$

For each cube in this subfamily, Theorem 3.4 is applicable without introducing a too large exceptional set, cf. (3.12). The complementary family of indices will correspond to the *bad cubes*, namely we set

$$I^h_{\mathrm{b}} := \{ i \in \mathbb{Z}^2 \setminus I^h_{\mathrm{g}} \colon Q_h(i) \in \mathcal{Q}_h\}\,.$$

We also remark that, for each $i \in \mathbb{Z}^2$,

$$\#\big\{i' \in \mathbb{Z}^2 \colon \hat{Q}_{h,\rho}(i) \cap \hat{Q}_{h,\rho}(i') \neq \emptyset\big\} \leq 25\,. \tag{3.24}$$

By (3.23)–(3.24), (3.18)(ii), (2.4)(iv), (3.5), and (3.15), for $h > 0$ small enough, we obtain

$$\begin{aligned}\#I^h_{\rm b} &\leq \alpha^{-1}h^{-2}\sum_{i\in I^h_{\rm b}}\mathcal{H}^2(\partial E^*_h \cap \hat{Q}_{h,\rho}(i)) + h^{-4}\sum_{i\in I^h_{\rm b}}\varepsilon_{i,h}\\ &\leq C\alpha^{-1}h^{-2}\mathcal{H}^2(\partial E^*_h \cap \Omega_h) + Ch^{-4}\int_{\Omega_h\setminus\overline{E}} W(\nabla v_h)\,\mathrm{d}x \leq Ch^{-1}\mathcal{G}^h(v_h,E_h) \leq Ch^{-1}\,,\end{aligned}$$

i.e., we deduce that

$$\#I^h_{\rm b} \leq Ch^{-1} \tag{3.25}$$

for an absolute constant $C = C(\alpha) > 0$, independent of h. Moreover, by using again (2.4)(iv), (3.24), (3.5), and (3.15), we also obtain the estimate

$$\sum_{i\in I^h_{\rm g}\cup I^h_{\rm b}}\varepsilon_{i,h} \leq C\int_{\Omega_h\setminus\overline{E_h}}\mathrm{dist}^2(\nabla v_h, SO(3))\,\mathrm{d}x \leq Ch^3\,. \tag{3.26}$$

Proposition 3.8 (Local rigidity and Sobolev approximation)**.** *Let $0 < \rho \leq \rho_0$. There exist an absolute constant $C > 0$ independent of h and $h_0 = h_0(\rho) > 0$ such that for all $0 < h \leq h_0$ and for every $i \in I^h_{\rm g}$ there exists a set of finite perimeter $D_{i,h} \subset \hat{Q}_{h,\rho}(i)$, satisfying*

$$\mathcal{L}^3\big(\hat{Q}_{h,\rho}(i)\setminus D_{i,h}\big) \leq Ch\mathcal{H}^2\big(\partial E^*_h \cap \hat{Q}_h(i)\big)\,,\ \mathcal{L}^3(\hat{Q}_h(i)\setminus D_{i,h}) \leq \frac{1}{32}\mathcal{L}^3(\hat{Q}_h(i))\,, \tag{3.27}$$

$$\mathcal{H}^2(\partial^* D_{i,h} \cap \hat{Q}_{h,\rho}(i)) \leq C\mathcal{H}^2\big(\partial E^*_h \cap \hat{Q}_{h,\rho}(i)\big)\,, \tag{3.28}$$

and a corresponding rigid motion $r_{i,h}(x) := R_{i,h}x + b_{i,h}$, where $R_{i,h} \in SO(3)$ and $b_{i,h} \in \mathbb{R}^3$ with $|b_{i,h}| \leq CM$ (recall (2.2)), such that

$$h^{-2}\int_{D_{i,h}}\big|v^*_h(x) - r_{i,h}(x)\big|^2\,\mathrm{d}x + \int_{D_{i,h}}\big|\nabla v^*_h(x) - R_{i,h}\big|^2\,\mathrm{d}x \leq C\varepsilon_{i,h}\,, \tag{3.29}$$

with v^*_h defined as in (3.19).

Furthermore, there exists a Sobolev map $z_{i,h} \in W^{1,2}(\hat{Q}_{h,\rho}(i);\mathbb{R}^3)$ such that

(i) $z_{i,h} \equiv v^*_h$ *on* $D_{i,h}\,,$

(ii) $$h^{-2}\int_{\hat{Q}_{h,\rho}(i)}\big|z_{i,h}(x) - r_{i,h}(x)\big|^2\,\mathrm{d}x + \int_{\hat{Q}_{h,\rho}(i)}\big|\nabla z_{i,h}(x) - R_{i,h}\big|^2\,\mathrm{d}x \leq C\varepsilon_{i,h}\,, \tag{3.30}$$

(iii) $\|z_{i,h}\|_{L^\infty(\hat{Q}_{h,\rho}(i))} \leq CM\,.$

Since $\mathcal{L}^3(\hat{Q}_h(i)\setminus D_{i,h})$ is small, we will be hereafter referring to $D_{i,h}$ as the *dominant component*, and in a similar fashion $r_{i,h}$ will be referred to as the *dominant rigid motion* which approximates v^*_h in $\hat{Q}_{h,\rho}(i)$. Of course, it is still possible that $D_{i,h} \subset E^*_h$, which should be interpreted as the void having large volume in $\hat{Q}_{h,\rho}(i)$.

The reader might have already noticed that the estimate (3.29) for the full gradient and also for the deformations v^*_h comes with the optimal exponent in the local elastic energy $\varepsilon_{i,h}$, which at first glimpse might be in contrast to (3.11), as $\gamma^{-1} = (\kappa_h/h^2)^{-1}$ is diverging with $h \to 0^+$, cf. (2.5). As shown in the proof, such an improvement is possible by applying the Korn-Poincaré inequality of Theorem 3.4, in the case of void sets with small surface area.

The proof of Proposition 3.8 is basically a repetition of the analogous one in [34, Proposition 3.5]. For the sake of completeness, we include it in Appendix A. The main difference is the additional requirement $\varepsilon_{i,h} \leq h^4$ in the definition of $I^h_{\rm g}$ in (3.23). We refer to Remark A.1 for a comment in this direction.

By a standard argument, which is again repeated in Appendix A, an immediate consequence of the previous proposition is an optimal estimate for the difference of two dominant rigid motions on neighboring good cubes.

Corollary 3.9 (Difference of rigid motions)**.** *Let $i, i' \in I^h_{\rm g}$ be such that $|i - i'|_\infty \leq 1$. The rigid motions $r_{i,h}$, $r_{i',h}$ of Proposition 3.8 satisfy*

$$h^{-2}\|r_{i,h} - r_{i',h}\|^2_{L^\infty(\hat{Q}_h(i)\cup\hat{Q}_h(i'))} + \big|R_{i,h} - R_{i',h}\big|^2 \leq Ch^{-3}(\varepsilon_{i,h} + \varepsilon_{i',h})\,. \tag{3.31}$$

3.3. **Construction of almost Sobolev replacements and proof of Proposition 3.1.** In this subsection we construct the fields $(r_h)_{h>0}$, $(R_h)_{h>0}$, and $(w_h)_{h>0}$ of Proposition 3.1, and afterwards give the proof of the proposition.

First of all, recalling (3.1) and (3.23), we introduce index sets related to *interior good cubes* by

$$I^h_{\rm int} := \big\{i \in I^h_{\rm g} \colon i' \in I^h_{\rm g} \ \text{ for every } i' \in \mathbb{Z}^2 \text{ with } |i' - i|_\infty \leq 1\big\}\,, \tag{3.32}$$

and *exterior good cubes* by

$$I^{h,\rho}_{\rm ext} := \big\{i \in I^h_{\rm g} \setminus I^h_{\rm int} \colon Q_h(i) \cap \Omega_{h,\rho} \neq \emptyset\big\}\,.$$

Note that, for $h > 0$ small enough, if $i \in I^{h,\rho}_{\rm ext}$, then there exists at least one $i' \in \mathbb{Z}^2$ with $|i' - i|_\infty \leq 1$ such that $i' \in I^h_{\rm b}$. By (3.25) this yields

$$\# I^{h,\rho}_{\rm ext} \leq Ch^{-1}\,, \tag{3.33}$$

for a universal constant $C > 0$.

For the construction of the sequences $(r_h)_{h>0}$, $(R_h)_{h>0}$, and $(w_h)_{h>0}$, we consider a partition of unity subordinate to $I^h_{\rm int}$, cf. (3.32). For every $h > 0$, we introduce $(\psi^i_h)_{i\in I^h_{\rm int}} \subset C^\infty_c(\mathbb{R}^3)$ such that

$$\sum_{i\in I^h_{\rm int}} \psi^i_h = 1\,, \tag{3.34}$$

and, for every $i \in I^h_{\rm int}$,

$$0 \leq \psi^i_h \leq 1\,, \quad \psi^i_h \equiv 1 \text{ on } \widetilde{Q}_{9h/10}(i)\,, \quad \psi^i_h \equiv 0 \text{ on } \mathbb{R}^2 \setminus \widetilde{Q}_{11h/10}(i)\,, \quad \|\nabla\psi^i_h\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{h}\,, \tag{3.35}$$

where for $s > 0$ we have used the notation

$$\widetilde{Q}_{sh}(i) := (hi, 0) + \Big(-\frac{sh}{2}, \frac{sh}{2}\Big)^2 \times \Big(-\frac{h}{2}, \frac{h}{2}\Big)\,.$$

Then, we define $r_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^3)$, $R_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^{3\times3})$ and $w_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^3)$ as

$$r_h(x) := \begin{cases} x & \text{if } x \in Q_h(i) \cap \Omega_{h,\rho} \text{ for some } i \notin I^h_{\rm int}\,, \\ \sum_{i\in I^h_{\rm int}} \psi^i_h(x) r_{i,h}(x) & \text{otherwise}\,, \end{cases} \tag{3.36}$$

$$R_h(x) := \begin{cases} \mathrm{Id} & \text{if } x \in Q_h(i) \cap \Omega_{h,\rho} \ \text{ for some } i \notin I^h_{\rm int}\,, \\ \sum_{i\in I^h_{\rm int}} \psi^i_h(x) R_{i,h} & \text{otherwise}\,, \end{cases} \tag{3.37}$$

$$w_h(x) := \begin{cases} x & \text{if } x \in Q_h(i) \cap \Omega_{h,\rho} \text{ for some } i \notin I^h_{\rm int}\,, \\ \sum_{i\in I^h_{\rm int}} \psi^i_h(x) z_{i,h}(x) & \text{otherwise}\,, \end{cases} \tag{3.38}$$

where the fields $r_{i,h}$, $R_{i,h}$, and $z_{i,h}$ are given in Proposition 3.8. The construction and the definition of $\Omega_{h,\rho}$ in (3.1) implies that $r_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^3)$, $R_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^{3\times 3})$, $w_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^3)$, and the jump sets satisfy, for $h>0$ sufficiently small,

$$J_{r_h} \cup J_{R_h} \cup J_{w_h} \subset \Omega_{h,\rho} \cap \bigcup_{i\in I^h_{\rm b}\cup I^{h,\rho}_{\rm ext}} \partial Q_h(i)\,. \tag{3.39}$$

We are now ready to give the proof of Proposition 3.1, following the strategy of the proof of the corresponding result in the setting of rods, namely [34, Proposition 3.1].

Proof of Proposition 3.1. First of all, the bounds in (3.7)(i) follow from the definitions (3.36)–(3.37), the bound $|b_{i,h}| \le CM$ for $i \in I^h_{\rm g}$, and the fact that $SO(3) \subset \mathbb{R}^{3\times 3}$ is compact. Next, (3.7)(ii) follows directly from (3.39), (3.25), and (3.33).

We proceed to show (3.7)(iii), which we verify first for the fields $(R_h)_{h>0}$. For this, we first obtain a control on $\|\nabla R_h\|_{L^\infty(Q_h(i)\cap\Omega_{h,\rho})}$ for every $i \in I^h_{\rm int}$. Indeed, in view of (3.37), (3.34)–(3.35), and (3.31), for $i\in I^h_{\rm int}$ fixed, $x \in Q_h(i)\cap\Omega_{h,\rho}$, and $k\in\{1,2,3\}$, we can estimate

$$\begin{aligned}|\partial_k R_h(x)| &= \Big|\sum_{j\in I^h_{\rm int}} \partial_k\psi^j_h(x) R_{j,h}\Big| = \Big|\sum_{j\in I^h_{\rm int}} \partial_k\psi^j_h(x)(R_{j,h}-R_{i,h})\Big|\\ &\le \sum_{j\in\mathcal{N}(i)} \|\partial_k\psi^j_h\|_{L^\infty}|R_{j,h}-R_{i,h}| \le \frac{C}{h}h^{-3/2}\sum_{j\in\mathcal{N}(i)}\varepsilon^{1/2}_{j,h}\,,\end{aligned}$$

where

$$\mathcal{N}(i) := \{j\in\mathbb{Z}^2\colon |j-i|_\infty \le 1\}\,. \tag{3.40}$$

Therefore,

$$\|\nabla R_h\|^2_{L^\infty(Q_h(i)\cap\Omega_{h,\rho})} \le Ch^{-5}\sum_{j\in\mathcal{N}(i)}\varepsilon_{j,h}\,. \tag{3.41}$$

Using (3.37), (3.41), and (3.26), we can thus estimate

$$\int_{\Omega_{h,\rho}} |\nabla R_h|^2\,{\rm d}x \le \sum_{i\in I^h_{\rm int}}\int_{Q_h(i)\cap\Omega_{h,\rho}}|\nabla R_h|^2\,{\rm d}x \le Ch^{-5}h^3\sum_{i\in I^h_{\rm g}}\varepsilon_{i,h} \le Ch\,. \tag{3.42}$$

Analogously, for $i\in I^h_{\rm int}$, $x\in Q_h(i)\cap\Omega_{h,\rho}$, and $k\in\{1,2,3\}$, using (3.36), (3.34)–(3.35), and (3.31), we estimate

$$\begin{aligned}|\partial_k r_h(x)| &= \Big|\sum_{j\in I^h_{\rm int}}(\partial_k\psi^j_h(x)r_{j,h}(x) + \psi^j_h(x)R_{j,h}e_k)\Big| \le \Big|\sum_{j\in I^h_{\rm int}}\partial_k\psi^j_h(x)(r_{j,h}(x)-r_{i,h}(x))\Big| + 1\\ &\le \sum_{j\in\mathcal{N}(i)}\|\partial_k\psi^j_h\|_{L^\infty}\|r_{j,h}-r_{i,h}\|_{L^\infty(Q_h(i))} + 1 \le Ch^{-3/2}\sum_{j\in\mathcal{N}(i)}\varepsilon^{1/2}_{j,h} + 1\,.\end{aligned}$$

Thus,

$$\|\nabla r_h\|^2_{L^\infty(Q_h(i)\cap\Omega_{h,\rho})} \le Ch^{-3}\sum_{j\in\mathcal{N}(i)}\varepsilon_{j,h} + 1\,. \tag{3.43}$$

In a similar fashion as before, using (3.36), (3.25), (3.33), (3.43), and (3.26), we can estimate

$$\begin{aligned}\int_{\Omega_{h,\rho}}|\nabla r_h|^2\,{\rm d}x &= 3\sum_{i\in I^h_{\rm b}\cup I^{h,\rho}_{\rm ext}}\mathcal{L}^3(Q_h(i)\cap\Omega_{h,\rho}) + \sum_{i\in I^h_{\rm int}}\int_{Q_h(i)\cap\Omega_{h,\rho}}|\nabla r_h|^2\,{\rm d}x\\ &\le C\Big[\big(\#I^h_{\rm b} + \#I^{h,\rho}_{\rm ext}\big)h^3 + \sum_{i\in I^h_{\rm g}}\varepsilon_{i,h} + h\Big] \le C(h^2+h^3+h) \le Ch\,.\end{aligned} \tag{3.44}$$

Collecting (3.42) and (3.44), we conclude the proof of (3.7)(iii).

We now proceed to verify (3.7)(iv). Using (3.25), (3.33), (3.27), (3.24), (3.15), (3.18), (3.4), and (3.5) we first estimate for $h > 0$ sufficiently small,

$$\begin{aligned}\mathcal{L}^3\Big(\Omega_{h,\rho} \setminus \bigcup_{i \in I^h_{\rm int}} D_{i,h}\Big) &\le \sum_{i \in I^h_{\rm b} \cup I^{h,\rho}_{\rm ext}} \mathcal{L}^3(\hat{Q}_{h,\rho}(i)) + \sum_{i \in I^h_{\rm int}} \mathcal{L}^3(\hat{Q}_{h,\rho}(i) \setminus D_{i,h}) \\ &\le C\big(\#I^h_{\rm b} + \#I^{h,\rho}_{\rm ext}\big)h^3 + Ch \sum_{i \in I^h_{\rm int}} \mathcal{H}^2(\partial E^*_h \cap \hat{Q}_h(i)) \\ &\le Ch^2 + Ch\mathcal{H}^2(\partial E^*_h \cap \Omega_h) \le Ch^2 \,.\end{aligned} \tag{3.45}$$

Choose a sequence $(\theta_h)_{h>0} \subset (0, +\infty)$ as in (3.8). Then, using Chebyshev's inequality, (3.36), (3.34)–(3.35), (3.29), (3.31), and (3.26), we find

$$\begin{aligned}\mathcal{L}^3\Big(\{|v^*_h(x) - r_h| > \theta_h\} \cap \bigcup_{i \in I^h_{\rm int}} D_{i,h}\Big) &\le \theta_h^{-2} \sum_{i \in I^h_{\rm int}} \int_{D_{i,h}} \Big|\sum_{j \in \mathcal{N}(i)} \psi^j_h (v^*_h(x) - r_{j,h})\Big|^2 \,\mathrm{d}x \\ &\le C\theta_h^{-2} \sum_{i \in I^h_{\rm int}} \sum_{j \in \mathcal{N}(i)} \int_{D_{i,h}} \big|v^*_h(x) - r_{i,h}\big|^2 \,\mathrm{d}x \\ &\quad + C\theta_h^{-2} \sum_{i \in I^h_{\rm int}} \sum_{j \in \mathcal{N}(i)} \int_{D_{i,h}} \big|r_{i,h} - r_{j,h}\big|^2 \,\mathrm{d}x \\ &\le C\theta_h^{-2} h^2 \sum_{i \in I^h_{\rm g}} \varepsilon_{i,h} \le Ch^5 \theta_h^{-2} \,.\end{aligned} \tag{3.46}$$

Combining now (3.20), (3.45), and (3.46), we obtain

$$\begin{aligned}\mathcal{L}^3\big(\Omega_{h,\rho} \cap \{|v_h(x) - r_h| > \theta_h\}\big) &\le \mathcal{L}^3\big(\{v^*_h \neq v_h\}\big) + \mathcal{L}^3\big(\Omega_{h,\rho} \setminus \bigcup_{i \in I^h_{\rm int}} D_{i,h}\big) \\ &\quad + \mathcal{L}^3\big(\{|v^*_h(x) - r_h| > \theta_h\} \cap \bigcup_{i \in I^h_{\rm int}} D_{i,h}\big) \\ &\le Ch^2 + Ch^5 \theta_h^{-2} \,.\end{aligned} \tag{3.47}$$

Using the same estimates as for (3.46), we also get

$$\mathcal{L}^3\Big(\{|\nabla v^*_h(x) - R_h| > \theta_h\} \cap \bigcup_{i \in I^h_{\rm int}} D_{i,h}\Big) \le Ch^3 \theta_h^{-2} \,,$$

so that, in the same way as for (3.47), we obtain

$$\mathcal{L}^3\big(\Omega_{h,\rho} \cap \{|\nabla v_h(x) - R_h| > \theta_h\}\big) \le Ch^2 + Ch^3 \theta_h^{-2} \,. \tag{3.48}$$

Now, (3.7)(iv) follows from (3.47)–(3.48) and (3.8). Similarly, we observe that

$$\{w_h \neq v_h\} \cap \Omega_{h,\rho} \subset \{v_h \neq v^*_h\} \cup \bigcup_{i \in I^h_{\rm b} \cup I^{h,\rho}_{\rm ext}} \hat{Q}_{h,\rho}(i) \cup \bigcup_{i \in I^h_{\rm int}} \big(\hat{Q}_{h,\rho}(i) \setminus D_{i,h}\big) \,,$$

so that (3.9)(i) follows from (3.20) and (3.45). We proceed with (3.9)(ii). By (3.37), (3.38), (3.34)–(3.35), and (3.40), we can estimate

$$\begin{aligned}\int_{\Omega_{h,\rho}} |\nabla w_h - R_h|^2 \,\mathrm{d}x &\le \sum_{i\in I^h_{\rm int}} \int_{\hat{Q}_{h,\rho}(i)} \Big| \sum_{j\in\mathcal{N}(i)} z_{j,h}\otimes\nabla\psi^j_h + \sum_{j\in\mathcal{N}(i)} \psi^j_h(\nabla z_{j,h} - R_{j,h})\Big|^2 \\ &= \sum_{i\in I^h_{\rm int}} \int_{\hat{Q}_{h,\rho}(i)} \Big| \sum_{j\in\mathcal{N}(i)} (z_{j,h}-r_{i,h})\otimes\nabla\psi^j_h + \sum_{j\in\mathcal{N}(i)} \psi^j_h(\nabla z_{j,h} - R_{j,h})\Big|^2 \\ &\le Ch^{-2} \sum_{i\in I^h_{\rm int}} \sum_{j\in\mathcal{N}(i)} \int_{\hat{Q}_{h,\rho}(i)} |z_{j,h}-r_{i,h}|^2\,\mathrm{d}x + C\sum_{i\in I^h_{\rm g}} \int_{\hat{Q}_{h,\rho}(i)} |\nabla z_{i,h} - R_{i,h}|^2\,\mathrm{d}x \\ &\le Ch^{-2} \sum_{i\in I^h_{\rm int}} \sum_{j\in\mathcal{N}(i)} h^3\|r_{i,h}-r_{j,h}\|^2_{L^\infty(\hat{Q}_h(i)\cup\hat{Q}_h(j))} \\ &\qquad + C\sum_{i\in I^h_{\rm g}} \int_{\hat{Q}_{h,\rho}(i)} \big(h^{-2}|z_{i,h}-r_{i,h}|^2 + |\nabla z_{i,h}-R_{i,h}|^2\big)\,\mathrm{d}x\,. \end{aligned} \tag{3.49}$$

Using (3.30)(ii), (3.24), (3.31), and (3.26), we therefore estimate

$$\int_{\Omega_{h,\rho}} |\nabla w_h - R_h|^2\,\mathrm{d}x \le C\sum_{i\in I^h_{\rm g}} \varepsilon_{i,h} \le Ch^3\,,$$

which proves (3.9)(ii). Then, the gradient bound in (3.9)(iii) follows from (3.9)(ii) and (3.7)(i). The L^∞-bound in (3.9)(iii) follows from the definition (3.38) and the uniform control in (3.30)(iii).The surface area bound therein is a consequence of (3.39), (3.25), and (3.33). Finally, (3.9)(iv) follows directly from (3.39) for $\mathcal{Q}_{v_h} := \{Q_h(i)\in\mathcal{Q}_h : i\in I^h_{\rm b}\cup I^{h,\rho}_{\rm ext}\}$, where we recall once again (3.25) and (3.33). The proof is now complete. □

4. Proof of Theorem 2.1

We again use the continuum subscript $h>0$ instead of the sequential subscript notation $(h_j)_{j\in\mathbb{N}}$ for convenience.

Proof of Theorem 2.1. We split the proof into two steps.

Step (1): Compactness for the voids. By the energy bound (2.19) and (2.9) we have that

$$h^{-2}\int_{\Omega\setminus\overline{V_h}} W(\nabla_h y_h(x))\,\mathrm{d}x + \int_{\partial V_h\cap\Omega} \big|\big(\nu^1_{V_h}(z),\nu^2_{V_h}(z),h^{-1}\nu^3_{V_h}(z)\big)\big|\,\mathrm{d}\mathcal{H}^2(z) \le C\,, \tag{4.1}$$

where $\nu_{V_h}(z) := (\nu^1_{V_h}(z),\nu^2_{V_h}(z),\nu^3_{V_h}(z))$ denotes the outward pointing unit normal to $\partial V_h\cap\Omega$ at the point z. Note that (4.1) implies

$$\sup_{h>0}\big(\mathcal{L}^3(V_h) + \mathcal{H}^2(\partial V_h\cap\Omega)\big) \le C\,.$$

Hence, by the standard compactness result for sets of finite perimeter, cf. [4, Theorem 3.39], there exists $\widetilde{V}\in\mathcal{P}(\Omega)$ such that, up to a non-relabeled subsequence, we have

$$\chi_{V_h}\to\chi_{\widetilde{V}} \ \text{ in } L^1(\Omega)\,. \tag{4.2}$$

Invoking also *Reshetnyak's lower semicontinuity theorem*, cf. [4, Theorem 2.38], applied to the lower semicontinuous, positively 1-homogeneous, convex function $\phi:\mathbb{R}^2\to[0,+\infty)$ with $\phi(\nu) := |\nu^3|$, and using again (4.1), we obtain

$$\int_{\partial^* \widetilde{V} \cap \Omega} |\nu^3_{\widetilde{V}}| \, \mathrm{d}\mathcal{H}^2 \le \liminf_{h\to 0} \int_{\partial V_h \cap \Omega} |\nu^3_{V_h}| \, \mathrm{d}\mathcal{H}^2 \le C \liminf_{h\to 0} h = 0 \,,$$

where $\nu_{\widetilde{V}}$ denotes the measure-theoretic outer unit normal to $\partial^* \widetilde{V}$. This implies that

$$\nu^3_{\widetilde{V}}(x) = 0 \ \text{ for } \mathcal{H}^2\text{-a.e. } x \in \partial^* \widetilde{V} \cap \Omega \,. \tag{4.3}$$

In order to prove that the set $\widetilde{V}$ is cylindrical over a two-dimensional set, we proceed as follows. Let

$$V := \big\{(x_1,x_2) \in S \colon \mathcal{H}^1\big(((x_1,x_2) \times \mathbb{R}) \cap \widetilde{V}\big) > 0\big\} \,. \tag{4.4}$$

Our aim is to prove that

$$\mathcal{L}^3\Big(\widetilde{V} \triangle \big(V \times (-\tfrac{1}{2}, \tfrac{1}{2})\big)\Big) = 0 \,. \tag{4.5}$$

In view of Fubini's theorem, for the verification of (4.5) it is enough to show that

$$\mathcal{H}^1\Big(\big[\widetilde{V} \triangle \big(V \times (-\tfrac{1}{2}, \tfrac{1}{2})\big)\big] \cap \big((x_1,x_2) \times \mathbb{R}\big)\Big) = 0 \text{ for } \mathcal{L}^2\text{-a.e. } (x_1,x_2) \in S \,. \tag{4.6}$$

Trivially, for every $(x_1,x_2) \notin V$ we have by (4.4)

$$\mathcal{H}^1\big(\big[\widetilde{V} \triangle \big(V \times (-\tfrac{1}{2}, \tfrac{1}{2})\big)\big] \cap \big((x_1,x_2) \times \mathbb{R}\big)\big) \le \mathcal{H}^1\big(\widetilde{V} \cap \big((x_1,x_2) \times \mathbb{R}\big)\big) = 0 \,. \tag{4.7}$$

On the other hand, if $(x_1,x_2) \in V$, we can use (4.3) and the coarea formula, cf. [50, Formula 4.36], to obtain

$$0 = \int_{\partial^* \widetilde{V} \cap \Omega} |\nu^3_{\widetilde{V}}| \, \mathrm{d}\mathcal{H}^2 = \int_S \mathcal{H}^0\big((\partial^* \widetilde{V} \cap \Omega) \cap \big((x_1,x_2) \times \mathbb{R}\big)\big) \, \mathrm{d}x_1 \mathrm{d}x_2 \,.$$

In particular, for $\mathcal{L}^2$-a.e. $(x_1,x_2) \in V$ we find that $\mathcal{H}^0((\partial^* \widetilde{V} \cap \Omega) \cap ((x_1,x_2) \times \mathbb{R})) = 0$, which further implies that

$$\mathcal{H}^1\big(\big[\widetilde{V} \triangle \big(V \times (-\tfrac{1}{2}, \tfrac{1}{2})\big)\big] \cap \big((x_1,x_2) \times \mathbb{R}\big)\big) = 0. \tag{4.8}$$

Now, (4.7)–(4.8) imply (4.6). As discussed above, this in turn gives (4.5), which in particular implies that $V \in \mathcal{P}(S)$. Then, (4.2) yields (2.20)(i).

Step (2): Compactness for the deformations. Let $(v_h, E_h)_{h>0}$ be the sequence related to the sequence $(y_h, V_h)_{h>0}$ via (2.6)-(2.7), and let us fix $\rho > 0$ sufficiently small. In order to show (2.20)(ii),(iii) for the sequence $(y_h)_{h>0}$, we first consider the fields $(\widetilde{r}_h)_{h>0} \subset SBV^2(\Omega_{1,\rho};\mathbb{R}^3)$ and $(\widetilde{R}_h)_{h>0} \subset SBV^2(\Omega_{1,\rho};\mathbb{R}^{3\times 3})$, defined via

$$\widetilde{r}_h(x) := r_h(T_h x) \ \text{ and } \ \widetilde{R}_h(x) := R_h(T_h x) \,, \tag{4.9}$$

where we recall (3.36), (3.37), and (1.5). Let us mention once again that $\widetilde{r}_h$, $\widetilde{R}_h$ may depend also on ρ, which we do not include in the notation for simplicity. By (3.7)(i),(iii) we can estimate

$$\begin{aligned} \|\widetilde{r}_h\|_{L^\infty(\Omega_{1,\rho})} + \|\nabla \widetilde{r}_h\|_{L^2(\Omega_{1,\rho})} &\le \|\widetilde{r}_h\|_{L^\infty(\Omega_{1,\rho})} + \|\nabla_h \widetilde{r}_h\|_{L^2(\Omega_{1,\rho})} \\ &\le \|r_h\|_{L^\infty(\Omega_{h,\rho})} + h^{-\frac{1}{2}} \|\nabla r_h\|_{L^2(\Omega_{h,\rho})} \le C \,. \end{aligned} \tag{4.10}$$

By (3.7)(ii) and a simple change of variables (analogously to (4.1)), we also get

$$\mathcal{H}^2(J_{\widetilde{r}_h}) \le \int_{J_{\widetilde{r}_h}} |(\nu^1_{J_{\widetilde{r}_h}}, \nu^2_{J_{\widetilde{r}_h}}, h^{-1}\nu^3_{J_{\widetilde{r}_h}})| \, \mathrm{d}\mathcal{H}^2 = h^{-1}\mathcal{H}^2(J_{r_h}) \le C \,. \tag{4.11}$$

Exactly in the same fashion, we also have

$$\|\widetilde{R}_h\|_{L^\infty(\Omega_{1,\rho})} + \|\nabla_h \widetilde{R}_h\|_{L^2(\Omega_{1,\rho})} + \mathcal{H}^2(J_{\widetilde{R}_h}) \le C \,. \tag{4.12}$$

Thus, we can apply *Ambrosio's SBV compactness theorem*, cf. [4, Theorems 4.8], to obtain fields $r_\rho \in SBV^2(\Omega_{1,\rho};\mathbb{R}^3)$ and $R_\rho \in SBV^2(\Omega_{1,\rho};\mathbb{R}^{3\times3})$ such that, up to subsequences (not relabeled),

$$\begin{aligned} &\text{(i)} \quad \widetilde{r}_h \to r_\rho \text{ strongly in } L^2(\Omega_{1,\rho};\mathbb{R}^3)\,, \qquad \nabla\widetilde{r}_h \rightharpoonup \nabla r_\rho \ \text{ weakly in } L^2(\Omega_{1,\rho};\mathbb{R}^{3\times3}) \\ &\text{(ii)} \quad \widetilde{R}_h \to R_\rho \text{ strongly in } L^2(\Omega_{1,\rho};\mathbb{R}^{3\times3})\,, \quad \nabla\widetilde{R}_h \rightharpoonup \nabla R_\rho \text{ weakly in } L^2(\Omega_{1,\rho};\mathbb{R}^{3\times3\times3})\,. \end{aligned} \tag{4.13}$$

We now verify that

$$\begin{aligned} &\text{(i)} \quad \partial_3 r_\rho = 0 \ \ \mathcal{L}^3\text{-a.e. in } \Omega_{1,\rho}\,, \quad R_\rho \in SO(3) \text{ with } \partial_3 R_\rho = 0 \ \ \mathcal{L}^3\text{-a.e. in } \Omega_{1,\rho}\,. \\ &\text{(ii)} \quad \nabla' r_\rho = R'_\rho\,, \text{ where } R'_\rho := \big(R_\rho e_1 | R_\rho e_2\big) \in SBV^2(\Omega_{1,\rho};\mathbb{R}^{3\times2}) \end{aligned} \tag{4.14}$$

where we recall the notation $\nabla' := (\partial_1,\partial_2)$. Indeed, by the lower semicontinuity of the L^2-norm under weak convergence, (4.10), and (4.12) we obtain

$$\begin{aligned} \|\partial_3 r_\rho\|_{L^2(\Omega_{1,\rho})} + \|\partial_3 R_\rho\|_{L^2(\Omega_{1,\rho})} &\le \liminf_{h\to0} \|\partial_3\widetilde{r}_h\|_{L^2(\Omega_{1,\rho})} + \liminf_{h\to0} \|\partial_3\widetilde{R}_h\|_{L^2(\Omega_{1,\rho})} \\ &\le \liminf_{h\to0} \big(h\|\nabla_h\widetilde{r}_h\|_{L^2(\Omega_{1,\rho})}\big) + \liminf_{h\to0} \big(h\|\nabla_h\widetilde{R}_h\|_{L^2(\Omega_{1,\rho})}\big) \\ &\le C \liminf_{h\to0} h = 0\,. \end{aligned}$$

Moreover, by (4.9), (3.37), (3.34)–(3.35), (3.40), (3.31), and (3.26),

$$\begin{aligned} \int_{\Omega_{1,\rho}} \mathrm{dist}^2(\widetilde{R}_h, SO(3))\,\mathrm{d}x &= h^{-1}\int_{\Omega_{h,\rho}} \mathrm{dist}^2(R_h, SO(3))\,\mathrm{d}x \le h^{-1}\sum_{i\in I^h_{\mathrm{int}}}\int_{Q_h(i)\cap\Omega_{h,\rho}} |R_h - R_{i,h}|^2\,\mathrm{d}x \\ &\le h^{-1}\sum_{i\in I^h_{\mathrm{int}}}\int_{Q_h(i)\cap\Omega_{h,\rho}} \Big|\sum_{j\in\mathcal{N}(i)} \psi^j_h(x)(R_{j,h}-R_{i,h})\Big|^2\,\mathrm{d}x \\ &\le Ch^{-1}h^3\sum_{i\in I^h_{\mathrm{int}}}\sum_{j\in\mathcal{N}(i)} |R_{j,h}-R_{i,h}|^2 \le Ch^{-1}\sum_{i\in I^h_{\mathrm{int}}}\sum_{j\in\mathcal{N}(i)}(\varepsilon_{i,h}+\varepsilon_{j,h}) \\ &\le Ch^{-1}\sum_{i\in I^h_{\mathrm{g}}}\varepsilon_{i,h} \le Ch^2\,. \end{aligned} \tag{4.15}$$

By passing to the limit as $h\to0$ in (4.15), and using (4.13)(ii), we obtain

$$\int_{\Omega_{1,\rho}} \mathrm{dist}^2(R_\rho, SO(3))\,\mathrm{d}x = \lim_{h\to0}\int_{\Omega_{1,\rho}} \mathrm{dist}^2(\widetilde{R}_h, SO(3))\,\mathrm{d}x = 0 \Rightarrow \mathrm{dist}(R_\rho, SO(3)) = 0 \ \text{ a.e. in } \Omega_{1,\rho}\,.$$

This concludes the proof of (4.14)(i). We now get that

$$y_h \to r_\rho \text{ in measure on } \Omega_{1,\rho}\,, \quad \nabla_h y_h \to R_\rho \text{ in measure on } \Omega_{1,\rho}\,, \tag{4.16}$$

which follows easily from (4.13)(i),(ii), (4.9), (2.7), and (3.7)(iv) via a change of variables. In particular, using (2.10), (2.4)(iv), (2.9), and (2.19), we obtain

$$\sup_{h>0}\big(\|\nabla y_h\|_{L^2(\Omega_{1,\rho})} + \mathcal{H}^2(J_{y_h}) + \|y_h\|_{L^\infty(\Omega_{1,\rho})}\big) \le C\,,$$

so that (4.16) and *Ambrosio's closure theorem* [4, Theorems 4.7] lead to (4.14)(ii).

Recalling (3.1), the convergence in (4.16), together with a monotonicity argument as $\rho\to0$, allows us to define fields $r\in SBV^2(\Omega;\mathbb{R}^3)$ and $R\in SBV^2(\Omega;\mathbb{R}^{3\times3})$, such that

$$\begin{aligned} &\text{(i)} \quad \partial_3 r = 0 \ \ \mathcal{L}^3\text{-a.e. in } \Omega\,, \quad R\in SO(3) \text{ with } \partial_3 R = 0 \ \ \mathcal{L}^3\text{-a.e. in } \Omega\,, \\ &\text{(ii)} \quad \nabla' r = R'\,, \text{ where } R' := \big(Re_1|Re_2\big) \in SBV^2(\Omega;\mathbb{R}^{3\times2})\,, \end{aligned} \tag{4.17}$$

so that in particular $Re_3 = \partial_1 r \wedge \partial_2 r$, and

$$y_h \to r \text{ in measure on } \Omega\,, \quad \nabla_h y_h \to R \text{ in measure on } \Omega\,. \tag{4.18}$$

We next prove that

$$\nabla_h y_h \to R \text{ strongly in } L^2(\Omega;\mathbb{R}^{3\times 3})\,. \tag{4.19}$$

Indeed, defining $L_h := \{x \in \Omega : |\nabla_h y_h(x)| > 2\sqrt{3}\}$, by (4.1) and (2.4)(iv) we get

$$3\mathcal{L}^3(L_h) \le \frac{1}{4}\int_{L_h} |\nabla_h y_h|^2 \,\mathrm{d}x \le \int_\Omega \mathrm{dist}^2(\nabla_h y_h, SO(3))\,\mathrm{d}x \le Ch^2\,. \tag{4.20}$$

This shows that $\chi_{\Omega\setminus L_h} \to 1$ boundedly in measure on Ω and thus, in view of (4.18), we get $\chi_{\Omega\setminus L_h}\nabla_h y_h \to R$ strongly in $L^2(\Omega;\mathbb{R}^{3\times 3})$. By (4.20) we further get $\|\chi_{L_h}\nabla_h y_h\|_{L^2(\Omega)} \le Ch$, which concludes the proof of (4.19).

Hence, setting $\widetilde{y} := r$, collecting (4.17)–(4.19) and recalling the uniform bound on deformations assumed in (2.2), we derive (2.20)(ii)-(iii). This concludes the proof of compactness. □

Corollary 4.1. *In the setting of Proposition 3.1, let $(\widetilde{w}_h)_{h>0} \subset SBV^2(\Omega_{1,\rho};\mathbb{R}^3)$ be defined by*

$$\widetilde{w}_h(x) := w_h(T_h x)\,, \tag{4.21}$$

where $w_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^3)$ is as in (3.9) and T_h as in (1.5). Then, for $(y,V) \in \mathcal{A}$ given in Theorem 2.1, we also have, up to subsequences (not relabeled),

$$\begin{aligned} &\text{(i)} \quad \widetilde{w}_h \longrightarrow \widetilde{y} \textit{ in } L^1(\Omega_{1,\rho};\mathbb{R}^3)\,,\\ &\text{(ii)} \quad \nabla_h \widetilde{w}_h \to \big(\nabla'\widetilde{y}, \partial_1\widetilde{y}\wedge\partial_2\widetilde{y}\big) \textit{ strongly in } L^2(\Omega_{1,\rho};\mathbb{R}^{3\times 3})\,. \end{aligned} \tag{4.22}$$

Proof. Using (4.10)–(4.11) with $\widetilde{w}_h$ in place of $\widetilde{r}_h$ (cf. (3.9)(iii)) and once again *Ambrosio's SBV compactness theorem*, the corollary follows from (3.9)(i),(ii) and (3.7)(iv), after a simple change of variables, together with (4.9) and (4.13)(ii). □

5. Proof of Theorem 2.3(i)

In this section we give the proof of the lower bound of Theorem 2.3, for which we prove separately the lower bound for the bulk and the surface part of the energy. Recalling Definition 2.2, we consider $(y_h, V_h)_{h>0}$ and $(y,V) \in \mathcal{A}$ such that $(y_h,V_h) \xrightarrow{\tau} (y,V)$, i.e., (2.20)(i)–(iii) hold true. We start with the lower bound of the elastic energy.

Proposition 5.1. *Suppose that $(y_h,V_h) \xrightarrow{\tau} (y,V)$ for some $(y,V) \in \mathcal{A}$, cf. (2.13). Then,*

$$\liminf_{h\to 0}\Big(h^{-2}\int_{\Omega\setminus\overline{V_h}} W(\nabla_h y_h)\,\mathrm{d}x\Big) \ge \frac{1}{24}\int_{S\setminus V} \mathcal{Q}_2(\mathrm{II}_y(x'))\,\mathrm{d}x'\,, \tag{5.1}$$

where we also recall the definition of II_y in (2.17).

Proof. Since it is not restrictive to assume that the sequence of total energies $(\mathcal{E}^h(y_h,V_h))_{h>0}$ is bounded, i.e., that (3.6) holds, we can apply Proposition 3.1 for $\rho > 0$ small and the sequence $(v_h,E_h)_{h>0}$ related to $(y_h,V_h)_{h>0}$ via (2.6)–(2.7).

Recalling the definition of $I^h_{\rm int}$ in (3.32), we introduce the sequence of piecewise constant rotation fields $(\bar{R}_h)_{h>0} \subset SBV^2(\Omega_{1,\rho};\mathbb{R}^{3\times 3})$, defined by

$$\bar{R}_h(x) := \begin{cases} \mathrm{Id} & \text{if } x \in T_{1/h}(Q_h(i)) \cap \Omega_{1,\rho} \ \text{ for some } i \notin I^h_{\rm int}\,,\\ R_{i,h} & \text{if } x \in T_{1/h}(Q_h(i)) \cap \Omega_{1,\rho} \ \text{ for some } i \in I^h_{\rm int}\,. \end{cases}$$

With a similar argument as in (3.49), recalling (4.9), (3.37), (3.34)–(3.35), (3.40), (3.31), and (3.26), we estimate

$$
\begin{aligned}
h^{-2}\int_{\Omega_{1,\rho}} |\widetilde{R}_h - \bar{R}_h|^2 \,\mathrm{d}x = h^{-3}\int_{\Omega_{h,\rho}} |R_h - \bar{R}_h|^2 &= h^{-3}\sum_{i\in I^h_{\mathrm{int}}}\int_{\Omega_{h,\rho}\cap Q_h(i)} \Big|\Big(\sum_{j\in\mathcal{N}(i)} \psi^j_h R_{j,h}\Big) - R_{i,h}\Big|^2 \\
&\le h^{-3}\sum_{i\in I^h_{\mathrm{int}}}\int_{Q_h(i)} \Big|\sum_{j\in\mathcal{N}(i)} \psi^j_h (R_{j,h} - R_{i,h})\Big|^2 \le C\sum_{i\in I^h_{\mathrm{int}}}\sum_{j\in\mathcal{N}(i)} |R_{j,h} - R_{i,h}|^2 \\
&\le Ch^{-3}\sum_{i\in I^h_{\mathrm{int}}}\sum_{j\in\mathcal{N}(i)} (\varepsilon_{j,h} + \varepsilon_{i,h}) \le Ch^{-3}\sum_{i\in I^h_{\mathrm{g}}} \varepsilon_{i,h} \le C\,. \qquad (5.2)
\end{aligned}
$$

Thus, combining (5.2) with (3.9)(ii) and using a change of variables, we deduce

$$
\sup_{h>0} \|h^{-1}(\bar{R}^T_h \nabla_h \widetilde{w}_h - \mathrm{Id})\|_{L^2(\Omega_{1,\rho})} \le C\,, \qquad (5.3)
$$

where $\widetilde{w}_h$ is defined as in (4.21). In particular, we deduce that there exists $G \in L^2(\Omega_{1,\rho};\mathbb{R}^{3\times3})$ such that, up to a subsequence (not relabeled),

$$
G_h := \frac{\bar{R}^T_h \nabla_h \widetilde{w}_h - \mathrm{Id}}{h} \rightharpoonup G \quad \text{weakly in } L^2(\Omega_{1,\rho};\mathbb{R}^{3\times3})\,. \qquad (5.4)
$$

We can then proceed with a classical linearization argument, cf. [36, 57], which we nevertheless detail in Appendix B for the reader's convenience. This leads to

$$
\liminf_{h\to0} \Big(h^{-2}\int_{\Omega\setminus\overline{V_h}} W(\nabla_h y_h)\,\mathrm{d}x\Big) \ge \frac{1}{2}\int_{\Omega_{1,\rho}} \mathcal{Q}_3(G)\,\mathrm{d}x \ge \frac{1}{2}\int_{\Omega_{1,\rho}} \mathcal{Q}_2(G')\,\mathrm{d}x\,, \qquad (5.5)
$$

where we recall (2.12), and where for a matrix $F \in \mathbb{R}^{3\times3}$ we use the notation $F' \in \mathbb{R}^{2\times2}$ for its upper-leftmost (2×2)-block.

Hence, as in the purely elastic setting, we are confronted with identifying the upper-left block G' of the weak limit G in (5.4). Although the sequence $(\widetilde{w}_h)_{h>0}$ is not Sobolev in the entire domain, due to its construction, cf. (3.9) and (4.21), it does not exhibit jumps in the transversal x_3-direction, so that the strategy originally devised in [36, Proof of Theorem 6.1(i)] is applicable. We next give the details of the proof.

Let $\widetilde{G}_h := (G_h e_1|G_h e_2) \in \mathbb{R}^{3\times2}$ denote the matrix of the first two columns of G_h, and similarly $\widetilde{G}$ for the weak L^2-limit G. In the slightly smaller domain $\Omega_{1,2\rho}$, we consider the finite difference quotients in x_3-direction, $(H_h)_{h>0} \subset L^2(\Omega_{1,2\rho};\mathbb{R}^{3\times2})$, defined by

$$
H_h(x',x_3) := \frac{\widetilde{G}_h(x',x_3+z) - \widetilde{G}_h(x',x_3)}{z} = (\bar{R}_h)^T \frac{\frac{1}{h}\nabla'\widetilde{w}_h(x',x_3+z) - \frac{1}{h}\nabla'\widetilde{w}_h(x',x_3)}{z}\,, \qquad (5.6)
$$

where $|z| \le \rho$ $(z \neq 0)$. In view of (5.4), we have

$$
H_h \rightharpoonup H := \frac{\widetilde{G}(x',x_3+z) - \widetilde{G}(x',x_3)}{z} \quad \text{weakly in } L^2(\Omega_{1,2\rho};\mathbb{R}^{3\times2})\,. \qquad (5.7)
$$

By (5.3) and (4.22)(ii), $(\bar{R}_h)_{h>0}$ converges boundedly in measure to $(\nabla'\widetilde{y}|b_{\widetilde{y}}) \in SBV^2(\Omega_{1,2\rho};\mathbb{R}^{3\times3})$, where

$$
b_{\widetilde{y}} := \partial_1\widetilde{y} \wedge \partial_2\widetilde{y}\,, \qquad (5.8)
$$

so that by (5.6)–(5.8) we obtain

$$
\frac{\frac{1}{h}\nabla'\widetilde{w}_h(x',x_3+z) - \frac{1}{h}\nabla'\widetilde{w}_h(x',x_3)}{z} \rightharpoonup (\nabla'\widetilde{y}|b_{\widetilde{y}})H \quad \text{weakly in } L^2(\Omega_{1,2\rho};\mathbb{R}^{3\times2})\,. \qquad (5.9)
$$

In order to identify the weak limit H, we can then argue as in the end of the proof of the lower bound for the elastic energy in [57, Section 5]. Setting

$$f_h(x', x_3) := \frac{\widetilde{w}_h(x', x_3 + z) - \widetilde{w}_h(x', x_3)}{hz} \in SBV^2(\Omega_{1,2\rho}; \mathbb{R}^3), \tag{5.10}$$

we observe that, by (3.9)(iv) and a slicing argument,

$$f_h(x', x_3) = \int_0^1 \frac{1}{h} \partial_3 \widetilde{w}_h(x', x_3 + tz)\, \mathrm{d}t\,.$$

By Corollary 4.1, see (4.22) and (5.8), we have that $\frac{1}{h}\partial_3 \widetilde{w}_h \to b_{\widetilde{y}}$ strongly in $L^2(\Omega_{1,\rho}; \mathbb{R}^3)$, and therefore

$$f_h \to \int_0^1 b_{\widetilde{y}}(\cdot, \cdot + tz)\, \mathrm{d}t = b_{\widetilde{y}} \ \text{ strongly in } L^2(\Omega_{1,2\rho}; \mathbb{R}^3)\,,$$

where the last equality follows from the fact that $b_{\widetilde{y}}$ is independent of x_3. By (5.9), (4.22)(ii), and (3.9)(iii), we have

$$\sup_{h>0} \|\nabla f_h\|_{L^2(\Omega_{1,2\rho})} < +\infty \quad \text{and} \quad \sup_{h>0} \mathcal{H}^2(J_{f_h}) < +\infty\,,$$

so that, by the basic closure theorem in SBV, cf. [4, Theorem 4.7], we deduce that

$$\nabla' f_h \rightharpoonup \nabla' b_{\widetilde{y}} \ \text{ weakly in } L^2(\Omega_{1,2\rho}; \mathbb{R}^{3\times3})\,. \tag{5.11}$$

Combining (5.9), (5.11), the fact that $(y, V) \in \mathcal{A}$ (see (2.13)), and recalling the identification (2.15), and (2.17), we obtain

$$H = (\nabla' \widetilde{y} | b_{\widetilde{y}})^T \nabla' b_{\widetilde{y}} \in SBV^2(\Omega_{1,2\rho}; \mathbb{R}^{3\times2})\,.$$

Thus, (5.7) implies

$$\widetilde{G}(x', x_3) = \widetilde{G}(x', 0) + x_3 H(x') \ \text{ for } (x', x_3) \in \Omega_{1,2\rho}\,. \tag{5.12}$$

Using the bilinearity of $\mathcal{Q}_2$ in (2.12), (5.12), that $\int_{-1/2+2\rho}^{1/2-2\rho} x_3 \, \mathrm{d}x_3 = 0$, and the definition of the second fundamental form (2.17), we obtain

$$\begin{aligned} \frac{1}{2}\int_{\Omega_{1,\rho}} \mathcal{Q}_2(G')\, \mathrm{d}x &= \frac{1}{2}\int_{\Omega_{1,2\rho}} \mathcal{Q}_2(G'(x', 0))\, \mathrm{d}x + \frac{1}{2}\int_{\Omega_{1,2\rho}} x_3^2 \mathcal{Q}_2(\mathrm{II}_y(x'))\, \mathrm{d}x\,, \\ &\geq \frac{1}{2}\int_{\Omega_{1,2\rho}} x_3^2 \mathcal{Q}_2(\mathrm{II}_y(x'))\, \mathrm{d}x\,. \end{aligned} \tag{5.13}$$

Then, (5.1) follows from (5.5) and (5.13), after letting $\rho \to 0$. □

We now proceed with the lower bound for the surface part of the energy, namely

$$\mathcal{E}^h_{\mathrm{surf}}(V_h) := \mathcal{E}^h(y_h, V_h) - h^{-2} \int_{\Omega \setminus \overline{V_h}} W(\nabla_h y_h)\, \mathrm{d}x = h^{-1} \mathcal{G}^{\kappa_h}_{\mathrm{surf}}(E_h; \Omega_h)\,,$$

where we refer to (2.8) and (3.4). Our approach deviates significantly from the proof of lower bounds in relaxation results for energies defined on pairs of deformations and sets, cf. [13], [20], [57], the main reason being that our piecewise nonlinear geometric rigidity result allows for a control only in a large part of $\Omega \setminus \overline{V_h}$.

While the justification for the middle term on the right-hand side of (2.18) is standard, for the last one therein the argument is based on a fine blow-up analysis around jump points of $J_{(y,\nabla' y)} \setminus \partial^* V$. In particular, a delicate contradiction argument is employed to obtain the desired density lower bound of the surface energy. For technical reasons, the latter is augmented with a vanishing contribution of the elastic energy, see (5.15) and (5.16) below. A suitable (two-dimensional in nature) blow-up of the deformations, their derivatives, as well as the void sets (cf. (5.19)), together with a De-Giorgi type argument, will allow us to identify, up to translations, an appropriate three-dimensional thin rod on which the sequence $(v_h, E_h)_{h>0}$ enjoys uniform energy bounds, related to the bending energy for thin rods with voids, see [34, Equations (2.3) and (2.8)]. Finally, our compactness result from [34, Theorem 2.1] and the particular structure of the limiting pair will allow us to conclude the contradictory argument.

Proposition 5.2. *Suppose that* $(y_h, V_h) \xrightarrow{\tau} (y, V)$ *for some* $(y, V) \in \mathcal{A}$. *Then,*

$$\liminf_{h\to 0} \mathcal{E}^h_{\mathrm{surf}}(V_h) \geq \mathcal{H}^1(\partial^* V \cap S) + 2\mathcal{H}^1\big(J_{(y,\nabla' y)} \setminus \partial^* V\big)\,. \tag{5.14}$$

Proof. Let $(E_h)_{h>0}$ be the void sets associated to $(V_h)_{h>0}$ according to (2.6). Let $(\widetilde{y}, \widetilde{V})$ be the pair associated to (y, V) as in (2.15) and (2.16) respectively.

Step (1): Blow-up argument. In order to prove (5.14), we perform a blow-up argument. Let $h > 0$ and $\eta > 0$ (η will be eventually sent to 0 after we send $h \to 0^+$). We introduce the family of Radon measures $\mu_{\eta,h} : \mathfrak{M}(S) \to \mathbb{R}_+$, defined by

$$\mu_{\eta,h}(K) := \eta h^{-3} \int_{\left(K\times\left(-\frac{h}{2},\frac{h}{2}\right)\right)\setminus \overline{E}_h} W(\nabla v_h)\,\mathrm{d}x + h^{-1}\mathcal{G}^{\kappa_h}_{\mathrm{surf}}\left(E_h; K \times \left(-\frac{h}{2}, \frac{h}{2}\right)\right)\,, \tag{5.15}$$

where we recall (2.7) and (3.4). By the assumption that the sequence $(\mathcal{E}^h(y_h, V_h))_{h>0}$ is bounded (cf. also the proof of Lemma 5.1), and after passing to a subsequence (not relabeled), we may suppose that $(\mu_{\eta,h})_{h>0}$ converges weakly* to some Radon measure μ_η. Let also

$$\lambda := \mathcal{H}^1 \llcorner_{(\partial^* V \cup J_{(y,\nabla' y)})\cap S}\,,$$

and $\mathrm{d}\mu_\eta/\mathrm{d}\lambda$ be the corresponding Radon-Nikodym derivative. In view of the lower semicontinuity of the mass under weak*-convergence, the equi-boundedness of the total energy, and the arbitrariness of $\eta > 0$, the estimate in (5.14) will follow by proving that for every Lebesgue point of μ_η with respect to λ there holds

$$\frac{\mathrm{d}\mu_\eta}{\mathrm{d}\lambda}(x_0) \geq \begin{cases} 1\,, & \text{if } x_0 \in \partial^* V \cap S\,,\\ 2\,, & \text{if } x_0 \in J_{(y,\nabla' y)} \setminus \partial^* V\,.\end{cases} \tag{5.16}$$

Now, fix $x_0 \in (\partial^* V \cup J_{(y,\nabla' y)}) \cap S$ such that a generalized unit normal at the point x_0 exists, which we denote by $\nu(x_0)$. Since this property holds for $\mathcal{H}^1$-a.e. point, it suffices to prove (5.16) in this case. Without loss of generality we assume that $x_0 = 0$ and $\nu(x_0) = e_1$. For $r < 1$ we let $Q_r := (-\frac{r}{2}, \frac{r}{2})^2$. Noting that $\lambda(Q_r) = r + o(r)$ as $r \to 0$, in order to prove (5.16), it suffices to show that

$$\liminf_{r\to 0}\liminf_{h\to 0} \frac{\mu_{\eta,h}(Q_r)}{r} \geq \begin{cases} 1\,, & \text{if } x_0 \in \partial^* V \cap S\,,\\ 2\,, & \text{if } x_0 \in J_{(y,\nabla' y)} \setminus \partial^* V\,.\end{cases} \tag{5.17}$$

Step (2): Boundary of voids. Regarding the first case in (5.17), for $0 \in \partial^* V \cap S$, with a change of variables, cf. (2.9), we can estimate from below

$$\begin{aligned}\mu_{\eta,h}(Q_r) &\geq h^{-1}\mathcal{H}^2\left(\partial E_h \cap \left(Q_r \times \left(-\frac{h}{2}, \frac{h}{2}\right)\right)\right)\\ &= \int_{\partial V_h \cap \left(Q_r \times \left(-\frac{1}{2},\frac{1}{2}\right)\right)} \left|\left(\nu^1_{V_h}, \nu^2_{V_h}, h^{-1}\nu^3_{V_h}\right)\right| \mathrm{d}\mathcal{H}^2 \geq \mathcal{H}^2\left(\partial V_h \cap \left(Q_r \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)\right)\,.\end{aligned} \tag{5.18}$$

Since $V_h \to \widetilde{V}$ in $L^1(\Omega)$, the L^1-lower semicontinuity of the perimeter, (5.18), (2.16), and the fact that $0 \in \partial^* V \cap S$ imply

$$\liminf_{r\to 0}\liminf_{h\to 0}\frac{\mu_{\eta,h}(Q_r)}{r} \geq \liminf_{r\to 0}\liminf_{h\to 0}\frac{\mathcal{H}^2\left(\partial V_h \cap \left(Q_r \times \left(-\frac{1}{2},\frac{1}{2}\right)\right)\right)}{r}$$
$$\geq \liminf_{r\to 0}\frac{\mathcal{H}^2\big(\partial^*\widetilde{V} \cap \left(Q_r \times \left(-\frac{1}{2},\frac{1}{2}\right)\right)\big)}{r} = \liminf_{r\to 0}\frac{\mathcal{H}^1\left(\partial^* V \cap Q_r\right)}{r} = 1\,.$$

Step (3): Jump points. Regarding the second case in (5.17), let $0 \in J_{(y,\nabla' y)} \setminus \partial^* V$, in particular we have $0 \in V^0$, where by V^0 we denote the set of points with two-dimensional density 0 with respect to V. We now define the auxiliary fields $Y_h\colon \Omega_h \to \mathbb{R}^3 \times \mathbb{R}^{3\times 3}$ and $Y\colon S \to \mathbb{R}^3 \times \mathbb{R}^{3\times 3}$ as

$$Y_h := (v_h, \nabla v_h) \ \text{ and } Y := \big(y, (\nabla' y, \partial_1 y \wedge \partial_2 y)\big)\,,$$

respectively, and observe that

$$J_Y = J_{(y,\nabla' y)}\,.$$

The assumption $0 \in V^0 \cap J_{(y,\nabla' y)} = V^0 \cap J_Y$ together with (2.20)(ii),(iii) and a scaling argument implies

$$\begin{aligned}
&\text{(i)} \quad \lim_{r\to 0}\lim_{h\to 0}\frac{\mathcal{L}^3\left(E_h \cap \left(Q_r \times \left(-\frac{h}{2},\frac{h}{2}\right)\right)\right)}{r^2 h} = 0\,,\\
&\text{(ii)} \quad \lim_{r\to 0}\lim_{h\to 0} \fint_{Q_r\times\left(-\frac{h}{2},\frac{h}{2}\right)} |Y_h - Y^{\pm}|\,\mathrm{d}x = 0\,,
\end{aligned} \tag{5.19}$$

where

$$Y^{\pm} := \begin{cases} Y^+(0)\,, & \text{if } x_1 > 0\\ Y^-(0)\,, & \text{if } x_1 \leq 0\,, \end{cases} \tag{5.20}$$

with $Y^+(0) = (y^+, R^+)$ and $Y^-(0) = (y^-, R^-)$ being the one-sided traces of Y at $0 \in J_Y$. Suppose by contradiction that the desired assertion was false, i.e., there exists $0 < \delta < 1$ such that

$$\liminf_{r\to 0}\liminf_{h\to 0}\frac{\mu_{\eta,h}(Q_r)}{r} < 2 - \delta\,.$$

We will proceed to show a contradiction, namely we will show that

$$\text{(a)} \quad y^+ = y^- \qquad \text{and} \qquad \text{(b)} \quad R^+ = R^-\,. \tag{5.21}$$

This implies that $Y^+(0) = Y^-(0)$ and thus $0 \notin J_Y$: a contradiction. Up to passing to subsequences in r and h, for all $0 < r \leq r_0(\delta)$ and $0 < h \leq h_0(r)$ small enough,

$$\mu_{\eta,h}(Q_r) \leq (2-\delta) r\,. \tag{5.22}$$

Step (4): Preparations for the proof of (5.21). The following arguments will be performed for fixed $r > 0$, which will be chosen sufficiently small along the proof. For $j \in \mathbb{Z}$ we define the (pairwise disjoint) stripes

$$S_h(j) := hje_2 + \Big(-\frac{r}{2},\frac{r}{2}\Big) \times \Big(-\frac{h}{2},\frac{h}{2}\Big)\,,$$

and note that $S_h(j) \subseteq Q_r$ for all $|j| \leq \frac{r}{2h} - \frac{1}{2}$. We set $N_h := \lfloor \frac{r}{2h} - \frac{1}{2} \rfloor$ and define

$$\mathcal{S}_h^{\text{good}} := \left\{ j \in \mathbb{Z}\colon |j| \leq N_h\,,\ \ \mu_{\eta,h}(S_h(j)) < \left(2 - \frac{\delta}{2}\right) h \right\}\,, \tag{5.23}$$

and $\mathcal{S}_h^{\rm bad} := \{j \in \mathbb{Z}\colon |j| \le N_h\} \setminus \mathcal{S}_h^{\rm good}$. In view of (5.22) and (5.23), we can estimate

$$\#\mathcal{S}_h^{\rm bad} \le \frac{4-2\delta}{(4-\delta)}\frac{r}{h}\,.$$

Therefore, for $h \in (0, h_0(r)]$ small enough, we obtain

$$\#\mathcal{S}_h^{\rm good} = 2N_h + 1 - \#\mathcal{S}_h^{\rm bad} \ge 2\left(\frac{r}{2h} - \frac{1}{2}\right) - 1 - \frac{4-2\delta}{(4-\delta)}\frac{r}{h} \ge \frac{\delta}{4}\frac{r}{h}\,. \tag{5.24}$$

We note that (5.24) and (5.19)(i),(ii) imply that there exists $j_h \in \mathcal{S}_h^{\rm good}$ such that

$$\mathcal{L}^3\left(E_h \cap \left(S_h(j_h) \times \left(-\frac{h}{2}, \frac{h}{2}\right)\right)\right) + \int_{S_h(j_h)\times\left(-\frac{h}{2},\frac{h}{2}\right)} |Y_h - Y^{\pm}| \le \frac{4}{\delta}\sigma_r r h^2\,, \tag{5.25}$$

for all $0 < r \le r_0$ and $0 < h \le h_0$, for a modulus of continuity $\sigma_r \to 0^+$ as $r \to 0^+$. Since $j_h \in \mathcal{S}_h^{\rm good}$, by the definition in (5.23) and (5.15) we immediately have

$$\eta h^{-4}\int_{\left(S_h(j_h)\times\left(-\frac{h}{2},\frac{h}{2}\right)\right)\setminus \overline{E}_h} W(\nabla v_h)\,\mathrm{d}x + h^{-2}\mathcal{G}_{\rm surf}^{\kappa_h}\left(E_h; S_h(j_h)\times\left(-\frac{h}{2},\frac{h}{2}\right)\right) < 2 - \frac{\delta}{2}\,. \tag{5.26}$$

Introducing the notation

$$S_{r,h,h} = \Big(-\frac{r}{2},\frac{r}{2}\Big) \times \Big(-\frac{h}{2},\frac{h}{2}\Big)^2\,,$$

up to translation of E_h and the domain of v_h by $-hj_he_2$ (not relabeled), (5.26) is equivalent to

$$h^{-4}\int_{S_{r,h,h}\setminus\overline{E}_h} \eta W(\nabla v_h)\,\mathrm{d}x + h^{-2}\mathcal{G}_{\rm surf}^{\kappa_h}(E_h; S_{r,h,h}) < 2 - \frac{\delta}{2}\,. \tag{5.27}$$

For $r > 0$ fixed, we now apply [34, Theorem 2.1] to obtain

$$((y^{\rm rod}|d_2|d_3), I) \in SBV^2_{\rm isom}\left(-\frac{r}{2},\frac{r}{2}\right) \times \mathcal{P}\left(-\frac{r}{2},\frac{r}{2}\right)$$

(cf. the definition in [34, (2.13)]) such that, up to a subsequence in h (not relabeled),

$$\begin{aligned}
&\chi_{V_h^{\rm rod}} \longrightarrow \chi_{V^{\rm rod}} \quad \text{in } L^1(S_{r,1,1})\,,\\
&y_h^{\rm rod} \longrightarrow \bar{y}^{\rm rod} \text{ in } L^1(S_{r,1,1};\mathbb{R}^3)\,,
\end{aligned} \tag{5.28}$$

where $y_h^{\rm rod}(x) = v_h(x_1, hx_2, hx_3)$ and $\bar{y}^{\rm rod}(x) = y^{\rm rod}(x_1)$ for $x \in S_{r,1,1}$, as well as

$$V_h^{\rm rod} = \{x \in S_{r,1,1}\colon (x_1, hx_2, hx_3) \in E_h\}\,,$$

and $V^{\rm rod} = I \times (-\frac{1}{2},\frac{1}{2})^2$ (see also [34, Equations (2.6) and (2.14)] for the notations). By definition of $SBV^2_{\rm isom}\left(-\frac{r}{2},\frac{r}{2}\right)$, we particularly have that

$$R^{\rm rod}(x_1) := (\partial_1 y^{\rm rod}|d_2|d_3)(x_1) \in SO(3) \text{ for } \mathcal{L}^1\text{-a.e. } x_1 \in \left(-\frac{r}{2},\frac{r}{2}\right)\,. \tag{5.29}$$

Moreover, by [34, Theorem 2.1] we get

$$\chi_{S_{r,1,1}\setminus V_h^{\rm rod}}\big(\partial_1 y_h^{\rm rod}, \tfrac{1}{h}\partial_2 y_h^{\rm rod}, \tfrac{1}{h}\partial_3 y_h^{\rm rod}\big) \rightharpoonup \chi_{S_{r,1,1}\setminus V^{\rm rod}}\bar{R}^{\rm rod} \ \text{ weakly in } L^2(S_{r,1,1};\mathbb{R}^{3\times3})\,, \tag{5.30}$$

where $\bar{R}^{\rm rod}(x) = R^{\rm rod}(x_1)$ for $x \in S_{r,1,1}$ (see also [34, Equations (2.9) and (2.15)] for the notations).

By the lower semicontinuity result in [34, Lemmata 5.2 and 5.3] and (5.27), we then get

$$c_*\eta\int_{(-\frac{r}{2},\frac{r}{2})} |(R^{\rm rod})^T\partial_1 R^{\rm rod}|^2\,\mathrm{d}x_1 + \mathcal{H}^0\big(\partial I \cap (-\tfrac{r}{2},\tfrac{r}{2})\big) + 2\mathcal{H}^0\big((J_{y^{\rm rod}} \cup J_{R^{\rm rod}}) \setminus \partial I\big) < 2 - \frac{\delta}{2} \tag{5.31}$$

for some $c_* > 0$. Here, we observe that the quadratic form on the right-hand side of [34, Equation (5.3)] is coercive, which can be seen by comparison to its form in the isotropic case addressed in [52, Remark 3.5].

Now, (5.31) implies

$$(J_{y^{\rm rod}} \cup J_{R^{\rm rod}}) \setminus \partial I = \emptyset \quad \text{and} \quad \mathcal{H}^0\left(\partial I \cap \left(-\tfrac{r}{2}, \tfrac{r}{2}\right)\right) \le 1\,. \tag{5.32}$$

From (5.25) and a change of variables we get $\mathcal{L}^1(I) \le \frac{r}{3}$, provided that $r > 0$ is small enough such that $\sigma_r < \frac{\delta}{4}\frac{1}{3}$. This and (5.32) imply that $I \subset \left(-\frac{r}{2}, -\frac{r}{6}\right)$ or $I \subset \left(\frac{r}{6}, \frac{r}{2}\right)$. Furthermore, for $Y^{\rm rod} := (y^{\rm rod}, R^{\rm rod})$, by (5.25), and (5.28)–(5.30), the lower semicontinuity of the L^1-norm under weak convergence, and a change of variables, we have

$$\int_{(-\frac{r}{6},\frac{r}{6})} |Y^{\rm rod} - Y^{\pm}|\,{\rm d}x_1 \le \frac{4}{\delta} r\sigma_r\,, \tag{5.33}$$

where we recall (5.20).

Step (5): Proof of (5.21). First, we show (5.21)(a), i.e., $y^+ = y^-$. Assume by contradiction that $|y^+ - y^-| > 0$, and choose $r > 0$ such that

$$r < |y^+ - y^-| \quad \text{and} \quad \sigma_r < \frac{\delta}{48}|y^+ - y^-|\,. \tag{5.34}$$

Let $x^- \in (-\frac{r}{6}, 0)$, $x^+ \in (0, \frac{r}{6})$ be such that

$$|y^{\rm rod}(x^+) - y^+| \le \frac{6}{r}\int_{(0,\frac{r}{6})} |Y^{\rm rod} - Y^+|\,{\rm d}x_1 \quad \text{and} \quad |y^{\rm rod}(x^-) - y^-| \le \frac{6}{r}\int_{(-\frac{r}{6},0)} |Y^{\rm rod} - Y^-|\,{\rm d}x_1\,.$$

Then, by (5.33) and (5.34) we obtain

$$\begin{aligned} |y^{\rm rod}(x^+) - y^{\rm rod}(x^-)| &\ge |y^+ - y^-| - |y^{\rm rod}(x^+) - y^+| - |y^{\rm rod}(x^-) - y^-| \\ &\ge |y^+ - y^-| - \frac{6}{r}\int_{(-\frac{r}{6},\frac{r}{6})} |Y^{\rm rod} - Y^{\pm}|\,{\rm d}x_1 \ge \frac{1}{2}|y^+ - y^-| > r/2\,, \end{aligned} \tag{5.35}$$

where the last step follows from the choice of r in (5.34). On the other hand, as $J_{y^{\rm rod}} \cap (-\frac{r}{6}, \frac{r}{6}) = \emptyset$, cf. (5.32), by the Fundamental Theorem of Calculus and (5.29), we obtain

$$|y^{\rm rod}(x^+) - y^{\rm rod}(x^-)| \le \int_{(x^-,x^+)} |\partial_1 y^{\rm rod}|\,{\rm d}x_1 = |x^+ - x^-| \le r/3\,. \tag{5.36}$$

Now, (5.35) and (5.36) contradict each other, which shows that $y^+ = y^-$.

Now, in a similar manner, we show (5.21)(b), i.e., $R^+ = R^-$. Assume by contradiction that $|R^+ - R^-| > 0$ and choose $r > 0$ small enough so that

$$r^{1/4} < |R^+ - R^-| \quad \text{and } \sigma_r < \frac{\delta}{48}|R^+ - R^-|\,. \tag{5.37}$$

Let $x^- \in (-\frac{r}{6}, 0)$, $x^+ \in (0, \frac{r}{6})$ be such that

$$|R^{\rm rod}(x^+) - R^+| \le \frac{6}{r}\int_{(0,\frac{r}{6})} |Y^{\rm rod} - Y^+|\,{\rm d}x_1 \quad \text{and} \quad |R(x^-) - R^-| \le \frac{6}{r}\int_{(-\frac{r}{6},0)} |Y^{\rm rod} - Y^-|\,{\rm d}x_1\,.$$

This, along with (5.33) and the choice of r in (5.37), shows

$$\begin{aligned} |R^{\rm rod}(x^+) - R^{\rm rod}(x^-)| &\ge |R^+ - R^-| - |R^{\rm rod}(x^+) - R^+| - |R^{\rm rod}(x^-) - R^-| \\ &\ge |R^+ - R^-| - \frac{6}{r}\int_{(-\frac{r}{6},\frac{r}{6})} |Y^{\rm rod} - Y^{\pm}|\,{\rm d}x_1 \ge \frac{1}{2}|R^+ - R^-| > \frac{1}{2}r^{1/4}\,. \end{aligned} \tag{5.38}$$

On the other hand, using (5.31) we get that $\|\partial_1 R^{\rm rod}\|_{L^2((-\frac{r}{6},\frac{r}{6}))} \le \bar{C}$ for a constant $\bar{C} > 0$ depending on η, but independent of r. Using this L^2-bound, together with the Fundamental Theorem of Calculus along with the fact that $J_{R^{\rm rod}} \cap (-\frac{r}{6},\frac{r}{6}) = \emptyset$, cf. again (5.32), shows

$$|R^{\rm rod}(x^+) - R^{\rm rod}(x^-)| \le \bar{C} r^{1/2} \,. \tag{5.39}$$

For $r > 0$ sufficiently small (depending on $\eta > 0$), (5.38) and (5.39) contradict each other, which shows $R^+ = R^-$. This concludes the proof. □

6. Proof of Theorem 2.3(ii)

In this last section we construct recovery sequences for admissible limits $(y, V) \in \mathcal{A}$, see (2.13), for which we proceed in several steps.

Step (1): Preparations. The first step is devoted to the smoothening of the void set V and covering most of the jump set $J_{(y,\nabla' y)}$ by a suitable smooth void set. We fix an arbitrary error parameter $\eta \in (0,1)$, which we will send to zero only at the end of the proof by means of a diagonal argument. We choose a smooth set $Z_\eta \subset \mathbb{R}^2$ such that

$$\mathcal{L}^2((V \triangle Z_\eta) \cap S) \le \eta, \qquad \mathcal{H}^1(\partial Z_\eta \cap S) \le \mathcal{H}^1(\partial^* V \cap S) + \eta, \qquad \mathcal{H}^1(\partial^* V \setminus \overline{Z_\eta}) \le \eta. \tag{6.1}$$

Indeed, we first apply [54, Theorem 3.1, Remark 3.2(i)] to find a relatively open set $Z'_\eta \in \mathcal{P}(S)$ such that $\partial Z'_\eta \cap S$ is a 1-dimensional C^1-submanifold, with

$$\mathcal{L}^2(V \triangle Z'_\eta) \le \frac{\eta}{2} \quad \text{and} \quad \mathcal{H}^1(\partial^* V \triangle \partial Z'_\eta) \le \frac{\eta}{2} \,.$$

Then, for (6.1), it suffices to choose a smooth set $Z_\eta \supset Z'_\eta$ with

$$\mathcal{L}^2(Z_\eta \setminus Z'_\eta) \le \frac{\eta}{2} \quad \text{and} \quad \mathcal{H}^1(\partial Z_\eta \cap S) \le \mathcal{H}^1(\partial Z'_\eta \cap S) + \frac{\eta}{2} \,.$$

Let

$$J_\eta := (J_{(y,\nabla' y)} \cup \partial^* V) \setminus \overline{Z_\eta} \,.$$

By a standard Besicovitch covering argument (see, e.g., [29, Equations (2.3), (2.6)] for details), we can find a finite number of pairwise disjoint closed rectangles $(R_i)_{i=1}^N$, so that for every $i = 1, \ldots, N$, $R_i \subset\subset S \setminus Z_\eta$, with length l_i and height ηl_i, and

$$\mathcal{H}^1\left(J_\eta \setminus \bigcup_{i=1}^N R_i\right) \le \eta, \qquad \sum_{i=1}^N l_i \le (1+\eta)\mathcal{H}^1(J_\eta) \,. \tag{6.2}$$

We can also pick pairwise disjoint smooth sets $T_i \subset\subset S \setminus Z_\eta$, $i = 1, \ldots, N$, so that

$$T_i \supset R_i \,, \quad \mathcal{L}^2(T_i) \le (1+\eta)\mathcal{L}^2(R_i) \,, \quad \text{and} \quad \mathcal{H}^1(\partial T_i) \le (1+\eta)\mathcal{H}^1(\partial R_i) \,. \tag{6.3}$$

We define $V_\eta := Z_\eta \cup \bigcup_{i=1}^N T_i$ and $y_\eta \in SBV^{2,2}_{\rm isom}(S;\mathbb{R}^3)$ by

$$y_\eta(x') := \begin{cases} y(x') & \text{for } x' \in S \setminus V_\eta \,, \\ x' & \text{for } x' \in V_\eta \,. \end{cases} \tag{6.4}$$

We also denote by $\widetilde{y}_\eta \colon \Omega \to \mathbb{R}^3$ the corresponding deformation indicated by the identification (2.15). By the fact that the jump set of $(y_\eta, \nabla' y_\eta)$ is contained in $J_\eta \cup \partial V_\eta$, (6.2) and (6.3) yield

$$\mathcal{H}^1\big(J_{(y_\eta,\nabla' y_\eta)} \cap (S \setminus \overline{V_\eta})\big) \le \eta \,. \tag{6.5}$$

Moreover, (6.1)–(6.3) also imply that

$$\begin{aligned}\mathcal{H}^1(\partial V_\eta \cap S) &\leq \mathcal{H}^1(\partial Z_\eta \cap S) + \sum_{i=1}^N \mathcal{H}^1(\partial T_i) \leq \mathcal{H}^1(\partial^* V \cap S) + \eta + \sum_{i=1}^N (1+\eta)(2+2\eta) l_i \\ &\leq \mathcal{H}^1(\partial^* V \cap S) + 2\mathcal{H}^1(J_\eta) + C\eta \leq \mathcal{H}^1(\partial^* V \cap S) + 2\mathcal{H}(J_\eta \setminus \partial^* V) + C\eta \\ &\leq \mathcal{H}^1(\partial^* V \cap S) + 2\mathcal{H}^1\big(J_{(y,\nabla' y)} \setminus \partial^* V\big) + C\eta\,, \end{aligned} \tag{6.6}$$

where $C > 0$ depends only on $\mathcal{H}^1(J_\eta)$ and thus only on (y, V). Using again (6.1)–(6.3), we also find

$$\begin{aligned}\mathcal{L}^2\big((V \triangle V_\eta) \cap S\big) &\leq \mathcal{L}^2((V \triangle Z_\eta) \cap S) + \sum_{i=1}^N \mathcal{L}^2(T_i) \\ &\leq \eta + (1+\eta)\eta \sum_{i=1}^N l_i^2 \leq \eta + C\eta \mathcal{H}^1(J_\eta) \leq C\eta\,, \end{aligned} \tag{6.7}$$

where we employed that $l_i \leq (1+\eta)\mathcal{H}^1(J_\eta) \leq C$ for every $i = 1, \ldots, N$.

Baring this construction in mind, our goal now is to construct sets $(\widetilde{V}_h)_{h>0}$ and functions $(y_h)_{h>0}$ as follows (for the sake of not overburdening the notation in the following, subscripts h will indicate that the objects depend on both h and η): we need to find smooth sets $(W_h^{\rm void})_{h>0} \subset \mathcal{A}_{\rm reg}(\mathbb{R}^2)$ with

$$\lim_{h\to 0} \mathcal{L}^2(W_h^{\rm void}) = 0\,, \qquad \mathcal{H}^1(\partial W_h^{\rm void}) \leq C\eta\,, \tag{6.8}$$

such that also the set $V_h := \mathrm{int}(\overline{V_\eta \cup W_h^{\rm void}})$ is smooth, and $\widetilde{V}_h := V_h \times (-\frac{1}{2}, \frac{1}{2}) \subset \Omega$ satisfies

$$\begin{aligned} &\text{(i)} \quad \chi_{\widetilde{V}_h} \to \chi_{\widetilde{V}_\eta} \quad \text{in } L^1(\Omega) \quad \text{as } h \to 0 \\ &\text{(ii)} \quad \kappa_h \int_{\partial \widetilde{V}_h \cap \Omega} |\boldsymbol{A}|^2 \,\mathrm{d}\mathcal{H}^2 \to 0 \quad \text{as } h \to 0\,, \end{aligned} \tag{6.9}$$

where $\widetilde{V}_\eta := V_\eta \times (-\frac{1}{2}, \frac{1}{2})$. Moreover, we need to find a sequence $(y_h)_{h>0}$ with $y_h \in W^{1,2}(\Omega \setminus \widetilde{V}_h; \mathbb{R}^3)$ such that

$$\begin{aligned} &\text{(i)} \quad y_h \to \widetilde{y}_\eta \quad \text{strongly in } L^1(\Omega;\mathbb{R}^3) \quad \text{as } h \to 0\,, \\ &\text{(ii)} \quad \nabla_h y_h \longrightarrow \big(\nabla' \widetilde{y}_\eta, \partial_1 \widetilde{y}_\eta \wedge \partial_2 \widetilde{y}_\eta\big) \quad \text{strongly in } L^2(\Omega;\mathbb{R}^{3\times 3}) \quad \text{as } h \to 0\,, \\ &\text{(iii)} \quad \limsup_{h\to 0} \left(h^{-2} \int_{\Omega \setminus \widetilde{V}_h} W(\nabla_h y_h(x))\,\mathrm{d}x \right) \leq \frac{1}{24} \int_{S \setminus V_\eta} \mathcal{Q}_2(\mathrm{II}_{y_\eta}(x'))\,\mathrm{d}x' + \eta\,, \\ &\text{(iv)} \quad \|y_h\|_{L^\infty(\Omega)} \leq M\,. \end{aligned} \tag{6.10}$$

Then, recalling (2.3), (2.8), (2.9), by (6.10)(iii), (6.8),(6.9), and the fact that $\partial V_h \setminus \partial V_\eta \subset \partial W_h^{\rm void}$, we find

$$\limsup_{h\to 0} \mathcal{E}^h(y_h, \widetilde{V}_h) \leq \Big(\frac{1}{24} \int_{S \setminus V_\eta} \mathcal{Q}_2(\mathrm{II}_{y_\eta}(x'))\,\mathrm{d}x' + \mathcal{H}^1(\partial V_\eta \cap S) \Big) + C\eta$$

where we used that (6.9)(ii) is equivalent to

$$h^{-1} \kappa_h \int_{\partial(T_h(\widetilde{V}_h)) \cap \Omega_h} |\boldsymbol{A}|^2 \,\mathrm{d}\mathcal{H}^2 \to 0 \quad \text{as } h \to 0\,,$$

since $\partial \widetilde{V}_h \cap \Omega$ is cylindrical over $\partial V_h \cap S$. Recalling (2.18), we also observe that

$$\limsup_{\eta \to 0} \Big(\frac{1}{24} \int_{S \setminus V_\eta} \mathcal{Q}_2(\mathrm{II}_{y_\eta}(x'))\,\mathrm{d}x' + \mathcal{H}^1(\partial V_\eta \cap S) \Big) \leq \mathcal{E}^0(y, V) \quad \text{as } \eta \to 0\,,$$

by (6.4) and (6.6),(6.7). As $\mathcal{L}^2(\{y_\eta \neq y\}) \le C\eta$ by (6.4) and (6.7), using a diagonal argument in the theory of Γ-convergence, we obtain the desired recovery sequence. Here, we also use (6.9)(i) and (6.10)(i),(ii) to see that the convergence τ defined in Definition 2.2 is satisfied, and we use (6.10)(iv) to guarantee that y_h (extended by $T_h(\mathrm{id})$ inside $\widetilde{V}_h$) is admissible, cf. (2.10).

Summarizing, in the sequel it suffices to construct the sets $(W_h^{\mathrm{void}})_{h>0}$ and the functions $(y_h)_{h>0}$ such that (6.8)–(6.10) hold.

Step (2): Meshes and auxiliary regularization of the jump set. In view of (6.5), the jump set $J_{(y_\eta,\nabla' y_\eta)}$ inside $S\setminus\overline{V_\eta}$ has small $\mathcal{H}^1$-measure. To be in the position to repeat the arguments from the elastic case [36, Section 6], it would be necessary to replace y_η by a Sobolev function on $S\setminus\overline{V_\eta}$. A first idea could be to apply Corollary 3.6 (or the corresponding Sobolev replacement in Theorem 3.4) on $S\setminus\overline{V_\eta}$. Yet, this approximation is not compatible with the nonlinear rotationally invariant elastic energy, and would provide inadequate estimates. Better approximations can be obtained by applying Corollary 3.6 on meshes of scales smaller or equal to h. To this end, for $n\in\mathbb{N}_0$ and some $\Lambda\ge 1$, we partition $\mathbb{R}^2$ up to a set of negligible measure into the squares

$$\mathcal{Q}_h^n := \big\{Q_h^n(p) := p + \Lambda 2^{-n}h(-\tfrac{1}{2},\tfrac{1}{2})^2,\ p\in\Lambda 2^{-n}h\mathbb{Z}^2\big\}\,. \tag{6.11}$$

The parameter Λ will be chosen eventually in (6.70) (depending only on η) and plays a role in an extension procedure, which will become clear in Steps 7–8 below. We write

$$U_\eta := S\setminus\overline{V_\eta} \tag{6.12}$$

for notational convenience. Our strategy consists in defining a Whitney-type covering related to U_η such that the jump set is covered by squares with small area, see Proposition 6.1 below for the precise statement and particularly (6.27). For technical reasons in this construction, it is convenient to regularize the jump set of y_η. For notational convenience, we write

$$(F_\eta,b_\eta) := (\nabla' y_\eta, \partial_1 y_\eta\wedge\partial_2 y_\eta)\,. \tag{6.13}$$

By the density result [19, Theorem 3.1] we can find functions $(z_h)_{h>0}\subset SBV^2(U_\eta;\mathbb{R}^3)$ and $(F_h,b_h)_{h>0}\subset SBV^2(U_\eta;\mathbb{R}^{3\times3})$ such that J_{z_h} and $J_{(F_h,b_h)}$ consist of a finite number of segments, and

$$\begin{aligned}
&\text{(i)}\ \ \|z_h - y_\eta\|_{L^1(U_\eta)} + \|\nabla' z_h - \nabla' y_\eta\|_{L^2(U_\eta)} \le h^2\,,\\
&\text{(ii)}\ \ \|(F_h,b_h)-(F_\eta,b_\eta)\|_{L^2(U_\eta)} + \|\nabla'(F_h,b_h)-\nabla'(F_\eta,b_\eta)\|_{L^2(U_\eta)} \le h^2\,,\\
&\text{(iii)}\ \ \mathcal{H}^1(\Gamma_h)\le 2\eta, \quad \text{for } \Gamma_h := J_{z_h}\cup J_{(F_h,b_h)}\,,\\
&\text{(iv)}\ \ \|z_h\|_{L^\infty(U_\eta)} \le \|y_\eta\|_{L^\infty(U_\eta)}\,, \quad \|b_h\|_{L^\infty(U_\eta)}\le\|b_\eta\|_{L^\infty(U_\eta)}\,,
\end{aligned} \tag{6.14}$$

where for (6.14)(iii) we used (6.5) and (6.12). Note that the jump sets of z_h and (F_h,b_h) depend on h and therefore this regularization does not appear to be helpful yet, as it does not allow for uniform estimates. The only reason for this approximation is that it guarantees that the Whitney-type covering in Proposition 6.1 terminates at some finite scale $\mathcal{Q}_h^{K_h}$ for $K_h\in\mathbb{N}$ depending on h, see the discussion below (6.33).

Step (3): Construction of W_h^{void}. Recalling (6.11), we denote generic squares in $\mathcal{Q}_h^n$, $n\in\mathbb{N}_0$, by q. By $\ell(q)$ we indicate the sidelength of the square, i.e., $\ell(q) := \Lambda 2^{-n}h$ for some $n\in\mathbb{N}_0$. Moreover, by q' and q'' we denote squares with the same center as q and

$$\ell(q') = \frac{3}{2}\ell(q), \qquad \ell(q'') = 21\ell(q)\,. \tag{6.15}$$

(The value 21 is chosen for definiteness only and could also be any odd number sufficiently large.) Since we consider squares of size $\sim h$, the jump set Γ_h, see (6.14)(iii), is not necessarily small compared to $\ell(q)$ in each square q, which might prevent the application of Corollary 3.6. To this end, we define

$$\mathcal{Q}_h^U := \{q \in \mathcal{Q}_h^0 \colon q \subset U_\eta\}\,, \tag{6.16}$$

and given some universal $\theta \in (0, \frac{1}{16})$ small to be specified later (see below (6.44)), we introduce the collection of *bad squares* defined by

$$\mathcal{Q}_h^{\rm bad} := \big\{q \in \mathcal{Q}_h^U \colon \mathcal{H}^1(\Gamma_h \cap q') \geq \theta\Lambda h\big\}\,. \tag{6.17}$$

We define the sets

$$U_h := \bigcup_{q \in \mathcal{Q}_h^U} \overline{q}\,, \qquad U_h^{\rm bad} := \bigcup_{q \in \mathcal{Q}_h^{\rm bad}} \overline{q''}\,. \tag{6.18}$$

From the definition of $\mathcal{Q}_h^{\rm bad}$ in (6.17) and (6.14)(iii) as well as (6.15), we get

$$\begin{aligned} \mathcal{L}^2(U_h^{\rm bad}) &\leq C(\Lambda h)^2 \#\mathcal{Q}_h^{\rm bad} \leq C\Lambda h \mathcal{H}^1(\Gamma_\eta) \leq C\Lambda h\,, \\ \mathcal{H}^1(\partial U_h^{\rm bad}) &\leq C(\Lambda h)\#\mathcal{Q}_h^{\rm bad} \leq C\mathcal{H}^1(\Gamma_\eta) \leq C\eta \end{aligned} \tag{6.19}$$

for a constant $C > 0$ depending only on θ, where we used that each $x \in \mathbb{R}^2$ is only contained in a (universally) bounded number of squares q', for $q \in \mathcal{Q}_h^0$.

Then, recalling the notation in (1.4), we can choose a smooth $W_h^{\rm void} \supset (U_h^{\rm bad})_{\Lambda h}$, satisfying (6.8). More precisely, since $U_h^{\rm bad}$ consists of at most $C\eta(\Lambda h)^{-1}$-many squares of sidelength Λh, this can be done in such a way that

$$\|\boldsymbol{A}\|_{L^\infty(\partial W_h^{\rm void})} \leq C(\Lambda h)^{-1}\,,$$

cf. also [33, Lemma 3.5] for a similar construction. Therefore, by a careful choice of the sets $W_h^{\rm void}$ so that also $V_h := {\rm int}(\overline{V_\eta \cup W_h^{\rm void}})$ is smooth, in view of (6.8), the set $\widetilde{V}_h = V_h \times (-\frac{1}{2}, \frac{1}{2})$ satisfies

$$\int_{\partial \widetilde{V}_h \cap \Omega} |\boldsymbol{A}|^2 \, {\rm d}\mathcal{H}^2 \leq \int_{(\partial V_\eta \cap S)\times(-\frac{1}{2},\frac{1}{2})} |\boldsymbol{A}|^2 \, {\rm d}\mathcal{H}^2 + C\mathcal{H}^1(\partial W_h^{\rm void})h^{-2} \leq C_\eta + C\eta h^{-2},$$

where $C_\eta > 0$ depends on V_η and thus on η. By (2.5) this shows (6.9)(ii). Clearly, (6.8) implies (6.9)(i).

For convenience, we define $U_h^{\rm good} := U_h \setminus U_h^{\rm bad}$. For later purposes, we extend the set $U_h^{\rm good}$ by adding two extra layers around it. More precisely, we define

$$U_h^{\rm ext} := U_h^{\rm good} \cup \bigcup_{q \in \mathcal{Q}_h^{\rm ext}} \overline{q}\,, \tag{6.20}$$

where

$$\mathcal{Q}_h^{\rm ext} := \Big\{q \in \mathcal{Q}_h^0 \colon q \notin U_h^{\rm good}\,,\ {\rm dist}_\infty(q, U_h^{\rm good}) \in \{0, \Lambda h\}\Big\}\,. \tag{6.21}$$

Using the definition of $U_h^{\rm bad}$ in (6.18), and the fact that $W_h^{\rm void} \supset (U_h^{\rm bad})_{\Lambda h}$, it is elementary to check that

$$U_h^{\rm ext} \cup W_h^{\rm void} \supset (U_\eta)_{(2-\sqrt{2})\Lambda h}\,. \tag{6.22}$$

Step (4): Construction of a Whitney-type covering. We now construct a covering of $U_h^{\rm good}$. The main point is that Corollary 3.6 (for $d = 2$) is then applicable in all squares and that the entire jump set Γ_h, cf. (6.14)(iii), can be covered by a set with small area, see (6.25)–(6.27). In the covering we also ensure that the squares at the boundary of $U_h^{\rm good}$ are in $\mathcal{Q}_h^0$, which later will allow us to easily extend the covering to the set $U_h^{\rm ext}$ defined in (6.20). For the next statement we recall the notations q, q', q'' introduced before (6.15), and refer to Figure 3 for an illustration of the covering the next Proposition describes.

Proposition 6.1. *There exists a covering of Whitney-type* $\mathcal{W}_h := (q_i)_{i\in\mathcal{I}} \subset \bigcup_{n=0}^{K_h} \mathcal{Q}_h^n$ *for some* $K_h \in \mathbb{N}$ *such that the squares* $(q_i)_{i\in\mathcal{I}}$ *are pairwise disjoint, and satisfy*

$$\begin{aligned} &\text{(i)} \quad \bigcup_{i\in\mathcal{I}} \overline{q_i} = \overline{U_h^{\rm good}}\,, \\ &\text{(ii)} \quad q_i' \cap q_j' \neq \emptyset \quad \implies \quad \frac{1}{2}\ell(q_i) \le \ell(q_j) \le 2\ell(q_i)\,, \\ &\text{(iii)} \quad \#\{j \in \mathcal{I}\colon\ q_i' \cap q_j' \neq \emptyset\} \le 12 \quad \textit{for all } i \in \mathcal{I}\,. \end{aligned} \tag{6.23}$$

Moreover, defining

$$\begin{aligned} \mathcal{W}_h^{\rm bdy} &:= \{q_i\colon \partial q_i \cap \partial U_h^{\rm good} \neq \emptyset\}\,, \\ \mathcal{W}_h^{\rm jump} &:= \{q_i \notin \mathcal{W}_h^{\rm bdy}\colon \theta^2\ell(q_i) \le \mathcal{H}^1(\Gamma_h \cap q_i')\}\,, \\ \mathcal{W}_h^{\rm empt} &:= \{q_i \notin \mathcal{W}_h^{\rm bdy}\colon \mathcal{H}^1(\Gamma_h \cap q_i') = 0\}\,, \\ \mathcal{W}_h^{\rm neigh} &:= \mathcal{W}_h \setminus \big(\mathcal{W}_h^{\rm bdy} \cup \mathcal{W}_h^{\rm jump} \cup \mathcal{W}_h^{\rm empt}\big)\,, \end{aligned} \tag{6.24}$$

it holds that

$$\begin{aligned} &\text{(i)} \quad q_i \in \mathcal{Q}_h^0 \quad \textit{for all } q_i \in \mathcal{W}_h^{\rm bdy}\,, \\ &\text{(ii)} \quad \mathcal{H}^1(\Gamma_h \cap q_i') \le \theta\ell(q_i) \quad \textit{for all } q_i \in \mathcal{W}_h^{\rm bdy} \cup \mathcal{W}_h^{\rm jump} \cup \mathcal{W}_h^{\rm neigh}\,, \\ &\text{(iii)} \quad q_i'' \subset W_h^{\rm cov} \quad \textit{for all } q_i \in \mathcal{W}_h^{\rm neigh}, \end{aligned} \tag{6.25}$$

where

$$W_h^{\rm cov} := \bigcup_{q\in\mathcal{W}_h^{\rm bdy}\cup\mathcal{W}_h^{\rm jump}} \overline{q''}\,. \tag{6.26}$$

In particular, the set $W_h^{\rm cov}$ *satisfies*

$$\mathcal{L}^2(W_h^{\rm cov}) \le Ch\,, \tag{6.27}$$

for $C > 0$ *only depending on* Λ, θ, *and* η.

Proof of Proposition 6.1. First, all $q \in \mathcal{Q}_h^0$ with $q \subset U_h^{\rm good}$ and $\partial q \cap \partial U_h^{\rm good} \neq \emptyset$ are collected in $\mathcal{W}_h^{\rm bdy}$. Note that (6.25)(i) holds and by the definition of $\mathcal{Q}_h^{\rm bad}$ in (6.17), property (6.25)(ii) is satisfied for each $q \in \mathcal{W}_h^{\rm bdy}$.

Step 1: Induction. The rest of the covering is constructed inductively. We first construct the collections $\mathcal{W}_h^{\rm jump}$ and $\mathcal{W}_h^{\rm neigh}$. All $q \in \mathcal{Q}_h^0$, $q \subset U_h^{\rm good}$, with

$$\mathcal{H}^1(\Gamma_h \cap q') \ge \theta^2\ell(q) = \theta^2\Lambda h\,, \tag{6.28}$$

or $q \in \mathcal{W}_h^{\rm bdy}$ are collected in $\mathcal{Y}_0^{\rm jump}$, and we define $Y_0 := \bigcup_{q\in\mathcal{Y}_0^{\rm jump}} \overline{q}$. (Note that for later purposes it is convenient to also add the squares at the boundary to this set.) For definiteness, we also define $\mathcal{Y}_0^{\rm neigh} = \emptyset$.

Suppose that for some $k \in \mathbb{N}$ the collections $\mathcal{Y}_j^{\rm jump}, \mathcal{Y}_j^{\rm neigh} \subset \mathcal{Q}_h^j$, $0 \le j \le k$, and

$$Y_k := \bigcup_{j=0}^{k} \bigcup_{q\in\mathcal{Y}_j^{\rm jump}\cup\mathcal{Y}_j^{\rm neigh}} \overline{q} \tag{6.29}$$

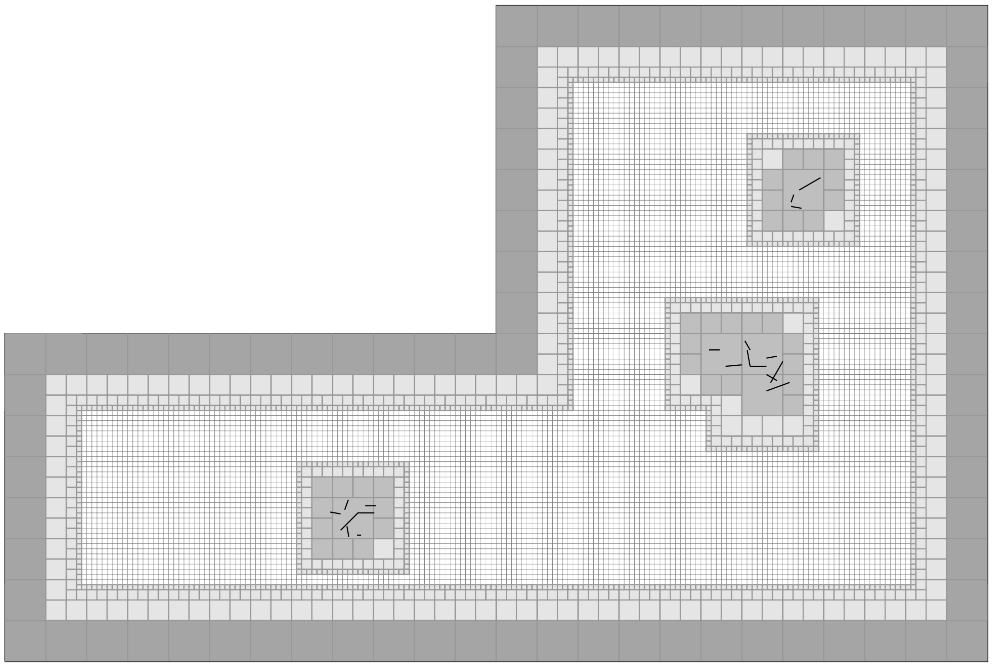

Figure 3. An illustration of the dyadic construction giving the covering $\mathcal{W}_h$. The jump set Γ_h is depicted by the black segments.

have already been constructed such that

$$\begin{aligned}
&\text{(i)} \quad q_1' \cap q_2' \neq \emptyset \quad \text{for } q_1, q_2 \in \bigcup_{j=0}^{k} \mathcal{Y}_j^{\rm jump} \cup \mathcal{Y}_j^{\rm neigh} \qquad \Longrightarrow \qquad \frac{1}{2}\ell(q_2) \leq \ell(q_1) \leq 2\ell(q_2)\,, \\
&\text{(ii)} \quad \theta^2 \ell(q) \leq \mathcal{H}^1(\Gamma_h \cap q') \quad \text{for all } q \in \bigcup_{j=0}^{k} \mathcal{Y}_j^{\rm jump} \setminus \mathcal{W}_h^{\rm bdy}\,, \\
&\text{(iii)} \quad \mathcal{H}^1(\Gamma_h \cap q') \leq \theta \ell(q) \quad \text{for all } q \in \bigcup_{j=0}^{k} \mathcal{Y}_j^{\rm jump} \cup \mathcal{Y}_j^{\rm neigh} \text{ and } q \in \mathcal{Y}_k^{\rm rest}\,, \\
&\text{(iv)} \quad \text{for each } q \in \mathcal{Y}_k^{\rm neigh} \ \exists j \in \{0, \ldots, k-1\},\, \widetilde{q} \in \mathcal{Y}_j^{\rm jump}, \text{ so that } \operatorname{dist}_\infty(q, \widetilde{q}) \leq \Lambda h \sum_{l=j}^{k-1} 2^{-l}\,, \\
&\text{(v)} \quad \text{for all } q \in \bigcup_{j=0}^{k} (\mathcal{Y}_j^{\rm jump} \cup \mathcal{Y}_j^{\rm neigh}) \setminus \mathcal{W}_h^{\rm bdy} \colon \partial q \cap \partial Y_k \neq \emptyset \implies q \in \mathcal{Q}_h^k\,,
\end{aligned} \tag{6.30}$$

where we have set

$$\mathcal{Y}_k^{\rm rest} := \{q \in \mathcal{Q}_h^k \colon q \subset U_h^{\rm good} \setminus Y_k\}\,.$$

Clearly, by construction all the above properties are satisfied for $k = 0$, see (6.17), (6.28), and (6.29).

Before we proceed with the induction step, let us briefly explain the relevance of these properties. First, (6.30)(i) will be needed for (6.23)(ii),(iii), and (6.30)(ii),(iii) are essential for (6.25)(ii) and for the definition of $\mathcal{W}_h^{\rm jump}$, respectively. Next, (6.30)(iv) will lead to (6.25)(iii), and finally (6.30)(v) is needed to obtain (6.30)(i) in the next iteration step.

We now come to the step $k+1$. We define $\mathcal{Y}_{k+1}^{\rm jump}$ and $\mathcal{Y}_{k+1}^{\rm neigh}$ as follows: recalling (6.29), we let

$$\mathcal{Y}_{k+1}^{\rm neigh} := \{q \in \mathcal{Q}_h^{k+1} \colon q \subset U_h^{\rm good} \setminus Y_k\,, \quad \partial q \cap \partial Y_k \neq \emptyset\}\,. \tag{6.31}$$

Then, we let

$$\mathcal{Y}_{k+1}^{\rm jump} := \{q \in \mathcal{Q}_h^{k+1} \setminus \mathcal{Y}_{k+1}^{\rm neigh} \colon q \subset U_h^{\rm good} \setminus Y_k\,, \quad \mathcal{H}^1(\Gamma_h \cap q') \geq \theta^2 \ell(q)\}\,. \tag{6.32}$$

We now confirm the properties (6.30) for the step $k+1$. First, (6.30)(i),(v) for step k guarantee (6.30)(i) for step $k+1$. Moreover, the construction in (6.31) and (6.32) directly shows (6.30)(ii),(v) in step $k+1$.

Next, we address (6.30)(iii). To this end, fix $q \in \mathcal{Y}_{k+1}^{\rm jump} \cup \mathcal{Y}_{k+1}^{\rm neigh} \cup \mathcal{Y}_{k+1}^{\rm rest}$, and choose the unique square $q_* \in \mathcal{Q}_h^k$ with $q \subset q_*$. Note that $\mathcal{H}^1(\Gamma_h \cap q'_*) < \theta^2 \ell(q_*)$, as otherwise q_* would have been added to $\mathcal{Y}_k^{\rm jump}$ in the previous iteration step. This shows

$$\mathcal{H}^1(\Gamma_h \cap q') \leq \mathcal{H}^1(\Gamma_h \cap q'_*) < \theta^2 \ell(q_*) = 2\theta^2 \ell(q) \leq \theta \ell(q)\,,$$

where we used that $0 < \theta \leq 1/2$, recalling the choice before (6.17).

It remains to show (6.30)(iv) in step $k+1$. For $q \in \mathcal{Y}_{k+1}^{\rm neigh}$, in view of the definitions (6.31) and (6.29), property (6.30)(v) in step k yields that there exists $\hat{q} \in \mathcal{Y}_k^{\rm jump} \cup \mathcal{Y}_k^{\rm neigh}$ such that $\mathrm{dist}_\infty(q, \hat{q}) = 0$. If $\hat{q} \in \mathcal{Y}_k^{\rm jump}$, the statement follows for $\widetilde{q} = \hat{q}$. Otherwise, if $\hat{q} \in \mathcal{Y}_k^{\rm neigh}$, by (6.30)(iv) in step k, we find $j \in \{0, \ldots, k-1\}$ and $\widetilde{q} \in \mathcal{Y}_j^{\rm jump}$ such that $\mathrm{dist}_\infty(\hat{q}, \widetilde{q}) \leq \Lambda h \sum_{l=j}^{k-1} 2^{-l}$. Since $\mathrm{dist}_\infty(q, \widetilde{q}) \leq \mathrm{dist}_\infty(\hat{q}, \widetilde{q}) + \Lambda h 2^{-k}$, the statement also follows in this case.

Step 2: Definition of the covering. We now show that we can terminate the iteration at some step K_h. To this end, we claim that there exists $K_h \in \mathbb{N}$ such that for all $q \in \mathcal{Q}_h^{K_h}$, $q \subset U_h^{\rm good} \setminus Y_{K_h}$, we have

$$\mathcal{H}^1(\Gamma_h \cap q') = 0\,. \tag{6.33}$$

In fact, choose K_h large enough such that $\Lambda 2^{-K_h} h$ is smaller than each of the length of the finite number of segments forming Γ_h, cf. (6.14)(iii). (This is the only point where we use the regularity and polygonal structure of the jump set.) Suppose by contradiction that there exists $q \in \mathcal{Q}_h^{K_h}$ such that $q \subset U_h^{\rm good} \setminus Y_{K_h}$ and $\mathcal{H}^1(\Gamma_h \cap q') \neq 0$. Choose $q_* \in \mathcal{Q}_h^{K_h - 1}$ with $q_* \supset q$, and observe that $q_* \in \mathcal{Y}_{K_h-1}^{\rm rest}$. As Γ_h consists of line segments whose length exceed $\Lambda 2^{-K_h} h$, it is elementary to verify that

$$\mathcal{H}^1(\Gamma_h \cap q'_*) \geq \frac{1}{4} \ell(q) = \frac{1}{8} \ell(q_*)\,,$$

which contradicts (6.30)(iii) as $0 < \theta < \frac{1}{16}$.

We now come to the definition of $\mathcal{W}_h$, cf. (6.24). First, $\mathcal{W}_h^{\rm bdy}$ has already been defined, and we let

$$\mathcal{W}_h^{\rm jump} := \bigcup_{j=0}^{K_h} \mathcal{Y}_j^{\rm jump} \setminus \mathcal{W}_h^{\rm bdy}\,. \tag{6.34}$$

Moreover, we introduce the auxiliary collection

$$\mathcal{W}_h^{\rm neigh,aux} := \bigcup_{j=0}^{K_h} \mathcal{Y}_j^{\rm neigh}\,.$$

In view of (6.30)(i),(v), we can cover of $U_h^{\rm good} \setminus Y_{K_h}$ with cubes in $\mathcal{Q}_h^{K_h}$, denoted by $\mathcal{W}_h^{\rm empt,aux}$ such that $\mathcal{W}_h^{\rm bdy} \cup \mathcal{W}_h^{\rm jump} \cup \mathcal{W}_h^{\rm neigh,aux} \cup \mathcal{W}_h^{\rm empt,aux}$ satisfy (6.23), where we use that (6.23)(iii) is a simple consequence of (6.23)(ii). Eventually, we define

$$\mathcal{W}_h^{\rm empt} := \mathcal{W}_h^{\rm empt,aux} \cup \big\{ q \in \mathcal{W}_h^{\rm neigh,aux} \colon \mathcal{H}^1(\Gamma_h \cap q') = 0 \big\}, \quad \mathcal{W}_h^{\rm neigh} := \mathcal{W}_h^{\rm neigh,aux} \setminus \mathcal{W}_h^{\rm empt}. \tag{6.35}$$

We note that the covering $\mathcal{W}_h$ consisting of the four families in (6.24) still satisfies (6.23). The collections satisfy the respective properties stated in (6.24) by (6.30)(ii) and (6.33)–(6.35).

It remains to show (6.25)–(6.27). Property (6.25)(i) holds by construction and (6.25)(ii) follows from (6.30)(iii). Next, (6.25)(iii) follows by an elementary computation using (6.29), (6.34)–(6.35), property (6.30)(iv), and the fact that $\ell(q'') = 21\ell(q)$. Finally, using the property of $\mathcal{W}_h^{\rm jump}$ in (6.24), as well as (6.17)–(6.19) we compute

$$\begin{aligned}\mathcal{L}^2(W_h^{\rm cov}) &= \sum_{q \in \mathcal{W}_h^{\rm bdy} \cup \mathcal{W}_h^{\rm jump}} \mathcal{L}^2(q'') \le Ch \sum_{q \in \mathcal{W}_h^{\rm bdy} \cup \mathcal{W}_h^{\rm jump}} \ell(q) \\ &\le C\theta^{-2} h \sum_{q \in \mathcal{W}_h^{\rm jump}} \mathcal{H}^1(\Gamma_h \cap q') + Ch\mathcal{H}^1(\partial U_h^{\rm good}) \le C\theta^{-2} h \mathcal{H}^1(\Gamma_h) + Ch \le Ch\,,\end{aligned}$$

for a constant $C > 0$ depending on Λ, θ and η, where in the penultimate step we have employed (6.23)(iii), the definition of $U_h^{\rm good}$ before (6.20), and the fact that (by the regularity of U_η)

$$\mathcal{H}^1(\partial U_h) \le C\mathcal{H}^1(\partial U_\eta) \le C_\eta\,.$$

The last step follows from (6.14)(iii). This concludes the proof of the proposition. □

Step (5): Auxiliary estimates on the Whitney-type covering. Before we can come to the definition of the deformations $(y_h)_{h>0}$, we need some preliminary estimates on the squares of the Whitney-type covering $\mathcal{W}_h$ defined in Proposition 6.1 which allow us to control the behavior on adjacent squares. Recalling the notation in (6.12)–(6.13), for notational convenience, we define

$$a_h := h^{-2}\big(|\nabla' z_h - \nabla' y_\eta| + |(F_h, b_h) - (F_\eta, b_\eta)|\big) \in L^2(U_\eta)\,, \tag{6.36}$$

which is bounded in $L^2(U_\eta)$, uniformly in $h > 0$, by (6.14). For every $i \in \mathcal{I}$ we apply Corollary 3.6 on q_i' for the function (F_h, b_h) to find a set of finite perimeter $\omega_i^1 \subset q_i'$ and a matrix field $(F_i, b_i) \in \mathbb{R}^{3\times 3}$ such that

$$\begin{aligned}&\text{(i)} \quad \mathcal{H}^1(\partial^* \omega_i^1) \le C\mathcal{H}^1(\Gamma_h \cap q_i')\,, \\ &\text{(ii)} \quad \|(F_h, b_h) - (F_i, b_i)\|_{L^2(q_i' \setminus \omega_i^1)} \le C\ell(q_i) \|\nabla'(F_h, b_h)\|_{L^2(q_i')}\,,\end{aligned} \tag{6.37}$$

where we recall the definition of Γ_h in (6.14)(iii). (Clearly, the objects $\omega_i^1, (F_i, b_i)$ also depend on h which we do not include in the notation for simplicity.) In view of (6.36), (6.37), and the fact that $\nabla' y_\eta = F_\eta$, see (6.13), we also get

$$\|\nabla' z_h - F_i\|_{L^2(q_i' \setminus \omega_i^1)} \le C\ell(q_i) \|\nabla'(F_h, b_h)\|_{L^2(q_i')} + Ch^2 \|a_h\|_{L^2(q_i')}\,. \tag{6.38}$$

Then, we define

$$u_i(x') := \chi_{q_i' \setminus \omega_i^1}(x') \big(z_h(x') - F_i x'\big) \quad \text{for } x' \in q_i'\,. \tag{6.39}$$

We note that $u_i \in SBV^2(q_i'; \mathbb{R}^3)$ with

$$\|\nabla' u_i\|_{L^2(q_i')} = \|\nabla' z_h - F_i\|_{L^2(q_i' \setminus \omega_i^1)} \le Ch \|\nabla'(F_h, b_h)\|_{L^2(q_i')} + Ch^2 \|a_h\|_{L^2(q_i')}\,,$$

and

$$\mathcal{H}^1(J_{u_i}) \le C\mathcal{H}^1(\Gamma_h \cap q_i')\,, \tag{6.40}$$

where we also used that $\ell(q_i) \le Ch$ for a constant $C > 0$ depending on Λ, together with (6.37)(i) and the fact that

$$J_{u_i} \subset (J_{z_h} \cap q_i') \cup \partial^* \omega_i^1 \subset (\Gamma_h \cap q_i') \cup \partial^* \omega_i^1 \, .$$

By applying Corollary 3.6 on q_i' once more, this time for the function u_i, we obtain another set of finite perimeter $\omega_i^2 \subset q_i'$ with $\mathcal{H}^1(\partial^* \omega_i^2) \le C\mathcal{H}^1(\Gamma_h \cap q_i')$, see (6.40), and $c_i \in \mathbb{R}^3$ such that

$$\|u_i - c_i\|_{L^2(q_i' \setminus \omega_i^2)} \le C\ell(q_i) \|\nabla' u_i\|_{L^2(q_i')} \le C\ell(q_i)\big(h\|\nabla'(F_h, b_h)\|_{L^2(q_i')} + h^2 \|a_h\|_{L^2(q_i')}\big) \, . \tag{6.41}$$

For later reference, we note that the estimates (6.37)–(6.38) and (6.41) are true for $\omega_i^1 = \omega_i^2 = \emptyset$, whenever $q_i \in \mathcal{W}_h^{\rm empt}$, see (6.24). Now, we define the affine function

$$y_i(x') := F_i x' + c_i \, . \tag{6.42}$$

Recalling (6.39), we note that

$$z_h(x') = u_i(x') + y_i(x') - c_i \quad \text{for } x' \in q_i' \setminus \omega_i \, , \tag{6.43}$$

where we set $\omega_i := \omega_i^1 \cup \omega_i^2$. By the isoperimetric inequality and (6.25)(ii) we get

$$\mathcal{L}^2(\omega_i) \le C\big(\mathcal{H}^1(\partial^* \omega_i^1 \cup \partial^* \omega_i^2)\big)^2 \le C(\mathcal{H}^1(\Gamma_h \cap q_i'))^2 \le C\theta^2 \ell(q_i)^2 \le \frac{1}{100} \ell(q_i)^2 \, , \tag{6.44}$$

for $\theta \in (0, 1/16)$ sufficiently small. Given $i \in \mathcal{I}$, we define

$$\mathcal{N}_i := \{q_j \colon q_j' \cap q_i' \neq \emptyset\} \, , \qquad N(q_i) := \bigcup_{q_j \in \mathcal{N}_i} q_j' \, . \tag{6.45}$$

It is then easy to deduce that for each $i \in \mathcal{I}$

$$\begin{aligned} &\|y_i - y_j\|_{L^2(q_i')} + h\|(F_i, b_i) - (F_j, b_j)\|_{L^2(q_i')} \\ &\qquad \le C\ell(q_i)\Big(h\|\nabla'(F_h, b_h)\|_{L^2(N(q_i))} + h^2 \|a_h\|_{L^2(N(q_i))}\Big) \quad \text{for all } q_j \in \mathcal{N}_i \, . \end{aligned} \tag{6.46}$$

Indeed, since y_i is affine, by Lemma 3.7, (6.44), (6.43), (6.41),(6.23)(ii), and $q_i' \cap q_j' \subset N(q_i)$ for all $q_j \in \mathcal{N}_i$, we obtain

$$\begin{aligned} \|y_i - y_j\|_{L^2(q_i')} &\le C\|y_i - y_j\|_{L^2((q_i' \cap q_j') \setminus (\omega_i \cup \omega_j))} \le C\|y_i - z_h\|_{L^2(q_i' \setminus \omega_i)} + C\|z_h - y_j\|_{L^2(q_j' \setminus \omega_j)} \\ &\le C\|u_i - c_i\|_{L^2(q_i' \setminus \omega_i)} + C\|u_j - c_j\|_{L^2(q_j' \setminus \omega_j)} \\ &\le C\ell(q_i)\Big(h\|\nabla'(F_h, b_h)\|_{L^2(N(q_i))} + h^2 \|a_h\|_{L^2(N(q_i))}\Big) \, . \end{aligned} \tag{6.47}$$

Similarly, using that $(F_i, b_i), (F_j, b_j)$ are constant, (6.44), and (6.37)(ii), we get

$$\begin{aligned} h\|(F_i, b_i) - (F_j, b_j)\|_{L^2(q_i')} &\le Ch\|(F_i, b_i) - (F_j, b_j)\|_{L^2((q_i' \cap q_j') \setminus (\omega_i \cup \omega_j))} \\ &\le Ch\|(F_i, b_i) - (F_h, b_h)\|_{L^2(q_i' \setminus \omega_i)} + Ch\|(F_h, b_h) - (F_j, b_j)\|_{L^2(q_j' \setminus \omega_j)} \\ &\le Ch\ell(q_i)\|\nabla'(F_h, b_h)\|_{L^2(N(q_i))} \, . \end{aligned} \tag{6.48}$$

Combining (6.47) and (6.48), the estimate (6.46) follows.

By a similar argument, using (6.44), that $(F_\eta, b_\eta) \in SO(3)$ (see (6.13) and (6.4)), (6.37)(ii), and (6.36), we get

$$\begin{aligned} \|\operatorname{dist}((F_i, b_i), SO(3))\|_{L^2(q_i')} &\le C\|\operatorname{dist}((F_i, b_i), SO(3))\|_{L^2(q_i' \setminus \omega_i^1)} \le C\|(F_i, b_i) - (F_\eta, b_\eta)\|_{L^2(q_i' \setminus \omega_i^1)} \\ &\le C\|(F_i, b_i) - (F_h, b_h)\|_{L^2(q_i' \setminus \omega_i^1)} + C\|(F_h, b_h) - (F_\eta, b_\eta)\|_{L^2(q_i' \setminus \omega_i^1)} \\ &\le Ch\|\nabla'(F_h, b_h)\|_{L^2(q_i')} + Ch^2 \|a_h\|_{L^2(q_i')} \, . \end{aligned} \tag{6.49}$$

For notational convenience, recalling (6.14), we also introduce the functions

$$\begin{cases} \bar{y}_i(x') := z_h(x')\,, \quad \bar{b}_i := b_h(x') & \text{for } x' \in q_i'\,,\ q_i \in \mathcal{W}_h^{\rm empt}\,, \\ \bar{y}_i(x') := y_i(x'), \quad \bar{b}_i := b_i & \text{for } x' \in q_i'\,,\ q_i \notin \mathcal{W}_h^{\rm empt}\,. \end{cases} \tag{6.50}$$

Note that for every $q_i \notin \mathcal{W}_h^{\rm empt}$, the functions $\bar{y}_i, \bar{b}_i$ are affine or constant, respectively. The fact that $\omega_i^1 = \omega_i^2 = \emptyset$ for $q_i \in \mathcal{W}_h^{\rm empt}$, along with (6.37)–(6.38), (6.41)–(6.43), and (6.46) also implies, for all $q_j \in \mathcal{N}_i$,

$$\|\bar{y}_i - \bar{y}_j\|_{L^2(q_i')} + h\|(\nabla'\bar{y}_i, \bar{b}_i) - (\nabla'\bar{y}_j, \bar{b}_j)\|_{L^2(q_i')} \le C\ell(q_i)\big(h\|\nabla'(F_h, b_h)\|_{L^2(N(q_i))} + h^2\|a_h\|_{L^2(N(q_i))}\big)\,. \tag{6.51}$$

We are now ready to proceed to the next step, namely the definition of the approximating sequence $(y_h)_{h>0}$ satisfying (6.10). Recall the definition of $U_h^{\rm good}$ and $U_h^{\rm ext}$ before and in (6.20). Following the notation in (2.16), we also define

$$\widetilde{U}_h^{\rm good} := U_h^{\rm good} \times (-\tfrac{1}{2}, \tfrac{1}{2})\,, \qquad \widetilde{U}_h^{\rm ext} := U_h^{\rm ext} \times (-\tfrac{1}{2} - \Lambda h, \tfrac{1}{2} + \Lambda h)\,, \tag{6.52}$$

where for the second set it will turn out to be convenient to thicken slightly also in the x_3-direction. Our next steps (Steps 6 and 7) consist in defining y_h first on $\widetilde{U}_h^{\rm good}$ and then on $\widetilde{U}_h^{\rm ext}$. As during the extension we slightly change the function, we denote the functions in Step 6 by $\bar{y}_h$ and in Step 7 by y_h for a better distinction.

Step (6): Definition of $\bar{y}_h$ on $\widetilde{U}_h^{\rm good}$. Recalling the covering $\mathcal{W}_h := (q_i)_{i\in\mathcal{I}}$ provided by Proposition 6.1, we choose $(\varphi_i)_{i\in\mathcal{I}} \subset C_c^\infty(\mathbb{R}^2; [0,1])$ with

$$\text{(i)} \ \sum_{i\in\mathcal{I}} \varphi_i(x') = 1 \ \ \forall x' \in U_h^{\rm good}\,, \quad \text{(ii)} \ \operatorname{supp}(\varphi_i) \subset q_i'\,, \quad \text{(iii)} \ \|\nabla\varphi_i\|_\infty \le C\ell(q_i)^{-1} \ \ \forall i \in \mathcal{I}\,. \tag{6.53}$$

As the proof of the existence of such a partition is very similar to the construction of a partition of unity for Whitney coverings (see [62, Chapter VI.1]), we omit it here. We also refer to [30, Proof of Theorem 4.6] for similar arguments.

Fix $d \in C_0^1(S; \mathbb{R}^3)$ to be specified later, see the choice before (6.62) below. Recalling (6.50), we define $\bar{y}_h \in W^{1,2}(\widetilde{U}_h^{\rm good}; \mathbb{R}^3)$ by

$$\bar{y}_h(x) := \sum_{i\in\mathcal{I}} \varphi_i(x')\big(\bar{y}_i(x') + hx_3\bar{b}_i(x')\big) + h^2\frac{x_3^2}{2}d(x')\,. \tag{6.54}$$

We further introduce

$$W_h^{\rm bjn} := \bigcup_{q\in\mathcal{W}_h^{\rm bdy}\cup\mathcal{W}_h^{\rm jump}\cup\mathcal{W}_h^{\rm neigh}} q'\,, \qquad W_h^{\rm empt} := U_h^{\rm good} \setminus W_h^{\rm bjn}\,. \tag{6.55}$$

We again use the notation $\widetilde{W}_h^{\rm bjn}$ and $\widetilde{W}_h^{\rm empt}$ for the corresponding cylindrical sets in $\mathbb{R}^3$, see also (6.52). We also note that $W_h^{\rm bjn} \subset W_h^{\rm cov}$ by (6.25)(iii). Observe that the definition of $U_h^{\rm good}$ in Step 3, the definition of $\mathcal{Q}_h^U$ in (6.16), and (6.19) shows that $\mathcal{L}^2(U_\eta \setminus U_h^{\rm good}) \to 0$ as $h \to 0$. By the definition of $W_h^{\rm empt}$ in (6.55), together with (6.27) and the previous observation, we find

$$\mathcal{L}^2(U_\eta \setminus W_h^{\rm empt}) \to 0 \ \text{ as } h \to 0\,. \tag{6.56}$$

In view of (6.53) and (6.54), on $\widetilde{W}_h^{\rm empt}$, the definition of the deformation is similar to the ansatz for the recovery sequence in [36, Equation (6.24)], namely, using also (6.50), we have

$$\bar{y}_h(x) = z_h(x') + hx_3b_h(x') + h^2\frac{x_3^2}{2}d(x') \ \text{ for } x' \in \widetilde{W}_h^{\rm empt}\,, \tag{6.57}$$

with

$$\nabla_h \bar{y}_h(x) = \big(R_h(x') + hx_3(\nabla' b_h(x'), d(x'))\big) + h^2 \frac{x_3^2}{2}(\nabla' d(x'), 0)\,, \tag{6.58}$$

where $R_h(x') := (\nabla' z_h(x'), b_h(x'))$. Here, we can repeat the estimate from the purely elastic case [36, Proof of Theorem 6.1(ii)], which we perform here for the sake of completeness. In view of (6.58), we obtain

$$R_\eta^T \nabla_h \bar{y}_h - \mathrm{Id} = hx_3 R_\eta^T\big(\nabla' b_h, d\big) + h^2 \frac{x_3^2}{2} R_\eta^T (\nabla' d, 0) + R_\eta^T R_h - \mathrm{Id} =: A_h\,, \tag{6.59}$$

where we set $R_\eta := (F_\eta, b_\eta) \in SO(3)$, cf. (6.13). Define $\chi_h := \chi_{\widetilde{W}_h^{\rm empt}}$ for brevity. In view of (6.14)(i),(ii) and (6.56), we get that, up to subsequences in h (not relabeled),

$$\frac{1}{h}\chi_h(R_\eta^T R_h - \mathrm{Id}) \to 0 \quad \text{and} \quad \frac{1}{h}\chi_h A_h \to x_3 R_\eta^T(\nabla' b_\eta, d) \text{ pointwise } \mathcal{L}^3\text{-a.e. on } \Omega \setminus \widetilde{V}_\eta\,. \tag{6.60}$$

Moreover, (2.4)(i), (6.59), the growth from above on W in (2.4)(v), and (6.14)(ii) yield

$$h^{-2}\chi_h W(\nabla_h \bar{y}_h) \le C h^{-2}|A_h|^2 \le C(|\nabla' b_\eta|^2 + |d|^2 + |\nabla' d|^2) + \mathrm{O}(h) \quad \text{on } \widetilde{W}_h^{\rm empt}\,,$$

where $\mathrm{O}(h)$ has to be understood in the L^1-sense. Therefore, by the Dominated Convergence Theorem, a Taylor expansion of W, see (2.4) and (2.11), (6.56), (6.59), and (6.60), we obtain

$$\lim_{h\to 0} h^{-2}\int_{\widetilde{W}_h^{\rm empt}} W(\nabla_h \bar{y}_h(x))\,\mathrm{d}x = \frac{1}{2}\int_{\Omega\setminus\widetilde{V}_\eta} x_3^2 \mathcal{Q}_3(R_\eta^T(\nabla' b_\eta, d))\,\mathrm{d}x\,. \tag{6.61}$$

Now, recalling (2.12) and (2.17), we can choose a function $d \in C_0^1(S;\mathbb{R}^3)$ such that

$$\lim_{h\to 0} h^{-2}\int_{\widetilde{W}_h^{\rm empt}} W(\nabla_h \bar{y}_h(x))\,\mathrm{d}x \le \frac{1}{24}\int_{S\setminus V_\eta} \mathcal{Q}_2(\mathrm{II}_{y_\eta}(x'))\,\mathrm{d}x' + \eta\,. \tag{6.62}$$

We now come to the integral over $\widetilde{W}_h^{\rm bjn}$. On this set, the derivative of $\bar{y}_h$ reads as

$$\begin{aligned}\nabla_h \bar{y}_h(x) = &\sum_{j\in\mathcal{I}} \varphi_j(x')\big((\nabla'\bar{y}_j(x'), \bar{b}_j(x')) + hx_3(\nabla'\bar{b}_j(x'), d(x'))\big) + h^2\frac{x_3^2}{2}(\nabla' d(x'), 0)\\ &+ \sum_{j\in\mathcal{I}} \big(\bar{y}_j(x') + hx_3\bar{b}_j(x')\big)\otimes(\nabla'\varphi_j(x'), 0)\,.\end{aligned} \tag{6.63}$$

Fix $q_i \in \mathcal{W}_h^{\rm bdy} \cup \mathcal{W}_h^{\rm jump} \cup \mathcal{W}_h^{\rm neigh}$, and set $\widetilde{q}_i' := q_i' \times (-\frac{1}{2}, \frac{1}{2})$. Since $\nabla'\big(\sum_{j\in\mathcal{I}}\varphi_j\big) = 0$, see (6.53)(i), we get

$$\Big\|\sum_{j\in\mathcal{I}}\big(\bar{y}_j + hx_3\bar{b}_j\big)\otimes(\nabla'\varphi_j, 0)\Big\|_{L^2(\widetilde{q}_i')} = \Big\|\sum_{j\in\mathcal{I}}\big((\bar{y}_j - \bar{y}_i) + hx_3(\bar{b}_j - \bar{b}_i)\big)\otimes(\nabla'\varphi_j, 0)\Big\|_{L^2(\widetilde{q}_i')}\,.$$

By (6.51) and the fact that $\|\nabla'\varphi_j\|_\infty \le C\ell(q_i)^{-1}$ for all $j \in \mathcal{I}$ with $\mathrm{supp}(\varphi_j)\cap q_i' \neq \emptyset$, see (6.23)(ii) and (6.53)(ii),(iii), we thus find

$$\Big\|\sum_{j\in\mathcal{I}}\big(\bar{y}_j + hx_3\bar{b}_j\big)\otimes(\nabla'\varphi_j, 0)\Big\|_{L^2(\widetilde{q}_i')} \le Ch\|\nabla'(F_h, b_h)\|_{L^2(N(q_i))} + Ch^2\|a_h\|_{L^2(N(q_i))}\,.$$

This along with (6.63), (6.50), (6.51), the fact that $\nabla'\bar{b}_j \in \{0, \nabla' b_h\}$, and $\ell(q_i) \le Ch$, shows

$$\|\nabla_h\bar{y}_h - (\nabla'\bar{y}_i, \bar{b}_i)\|_{L^2(\widetilde{q}_i')} \le Ch\|\nabla'(F_h, b_h)\|_{L^2(N(q_i))} + Ch^2\|a_h\|_{L^2(N(q_i))} + Ch\|d\|_{W^{1,2}(q_i')}\,.$$

Then, by (6.36), (6.42), and (6.49), we get

$$\|\mathrm{dist}(\nabla_h\bar{y}_h, SO(3))\|_{L^2(\widetilde{q}_i')} \le Ch\|\nabla'(F_h, b_h)\|_{L^2(N(q_i))} + Ch^2\|a_h\|_{L^2(N(q_i))} + Ch\|d\|_{W^{1,2}(q_i')}\,.$$

Summing over all $q_i \in \mathcal{W}_h^{\rm bdy} \cup \mathcal{W}_h^{\rm jump} \cup \mathcal{W}_h^{\rm neigh}$ and using (6.55), (6.23)(iii), and (6.25)(iii), we deduce

$$\|{\rm dist}(\nabla_h \bar{y}_h, SO(3))\|^2_{L^2(\widetilde{W}_h^{\rm bjn})} \le Ch^2 \|\nabla'(F_h, b_h)\|^2_{L^2(W_h^{\rm cov})} + Ch^2 \|d\|^2_{W^{1,2}(W_h^{\rm cov})} + Ch^4 \|a_h\|^2_{L^2(W_h^{\rm cov})}\,, \tag{6.64}$$

where we also used $N(q_i) \subset q_i''$, see (6.45). Now, by (6.14)(ii), (6.27), and (6.36) we find that

$$h^{-2} \|{\rm dist}(\nabla_h \bar{y}_h, SO(3))\|^2_{L^2(\widetilde{W}_h^{\rm bjn})} \to 0 \quad \text{as } h \to 0\,.$$

This, along with (6.55), (6.62), and (2.4)(v) shows

$$\lim_{h\to 0} h^{-2} \int_{\widetilde{U}_h^{\rm good}} W(\nabla_h \bar{y}_h(x))\,{\rm d}x \le \frac{1}{24} \int_{S\setminus V_\eta} \mathcal{Q}_2({\rm II}_{y_\eta}(x'))\,{\rm d}x' + \eta\,.$$

Step (7): Definition of y_h on $\widetilde{U}_h^{\rm ext}$. We now come to the definition of y_h on the extended set $\widetilde{U}_h^{\rm ext}$ defined in (6.52). Recall that $\mathcal{W}_h^{\rm bdy} \subset \mathcal{Q}_h^0$, see (6.25)(i). We can thus extend the Whitney-type covering $\mathcal{W}_h$ given by Proposition 6.1 to a new covering, denoted by $\mathcal{W}_h^{\rm ext}$, by adding all squares of $\mathcal{Q}_h^{\rm ext}$, see (6.21). On each of these squares $q_i \in \mathcal{W}_h^{\rm ext} \setminus \mathcal{W}_h$, we pick one of the squares $q_j \in \mathcal{W}_h^{\rm bdy}$ which is closest to q_i and define $F_i := F_j$ and $b_i := b_j$, where F_j and b_j are given in (6.37). Accordingly, we also define the affine function y_i, see (6.42), and as in (6.50) we introduce the notation $\bar{y}_i(x') := y_i(x')$ and $\bar{b}_i := b_i$. Exploiting the fact that for neighboring squares the difference of these objects can be controlled, see (6.46) and its justification in (6.47) and (6.48), it is elementary to check that (6.51) still holds for the extended covering.

We let $(\varphi_i)_{i\in\mathcal{I}'}$ be a partition of unity related to $\mathcal{W}_h^{\rm ext}$ satisfying (6.53). Then, choosing a field $d \in C_0^1(S;\mathbb{R}^3)$ as in Step 6, we define $y_h \in W^{1,2}(\widetilde{U}_h^{\rm ext};\mathbb{R}^3)$ by

$$y_h(x) := \sum_{i\in\mathcal{I}'} \varphi_i\big(\bar{y}_i(x') + hx_3\bar{b}_i(x')\big) + h^2\frac{x_3^2}{2}d(x')\,. \tag{6.65}$$

The estimate (6.61) holds still true for y_h in place of $\bar{y}_h$ and $W_h^{\rm empt} \times (-\frac{1}{2} - \Lambda h, \frac{1}{2} + \Lambda h)$ in place of $\widetilde{W}_h^{\rm empt}$. In particular, the limit is not affected by the thickening in the x_3-direction. In a similar fashion, arguing as in Step 6, by replacing the estimate on $\widetilde{W}_h^{\rm bjn}$ in (6.64) accordingly by a calculation on $W_h^* := (W_h^{\rm bjn} \cup (U_h^{\rm ext} \setminus U_h^{\rm good})) \times (-\frac{1}{2} - \Lambda h, \frac{1}{2} + \Lambda h)$, we get

$$h^{-2}\|{\rm dist}(\nabla_h y_h, SO(3))\|^2_{L^2(W_h^*)} \le C\|\nabla'(F_h, b_h)\|^2_{L^2(W_h^{{\rm cov},*})} + C\|d\|^2_{W^{1,2}(W_h^{{\rm cov},*})} + Ch^2\|a_h\|^2_{L^2(W_h^{{\rm cov},*})}\,,$$

where we set $W_h^{{\rm cov},*} := W_h^{\rm cov} \cup (U_h^{\rm ext} \setminus U_h^{\rm good})$. Hence, repeating verbatim the argument after (6.64), we obtain

$$h^{-2}\|{\rm dist}(\nabla_h y_h, SO(3))\|^2_{L^2(W_h^*)} \to 0 \text{ as } h \to \infty\,. \tag{6.66}$$

A combination of these estimates as before, shows

$$\lim_{h\to 0} h^{-2} \int_{\widetilde{U}_h^{\rm ext}} W(\nabla_h y_h(x))\,{\rm d}x \le \frac{1}{24} \int_{S\setminus V_\eta} \mathcal{Q}_2({\rm II}_{y_\eta}(x'))\,{\rm d}x' + \eta\,. \tag{6.67}$$

Step (8): Conclusion. We define $y_h\colon \Omega \to \mathbb{R}^3$ by $y_h = T_h({\rm id})$ on $\widetilde{V}_h$ (recall its definition before (6.9)) and otherwise as the restriction of the function in (6.65) to $\Omega \setminus \widetilde{V}_h$. Here, we use (6.22), recall also (6.12), to ensure that $\Omega \setminus \widetilde{V}_h \subset \widetilde{U}_h^{\rm ext}$. We first treat the case that $\|y\|_{L^\infty(S)} < M$. In view of (6.4), (6.14)(iv), and (6.65), we find $\|y_h\|_{L^\infty(\Omega)} \le M$ for h sufficiently small, i.e., (6.10)(iv) holds. Here, recalling the estimates in Step 5 it is indeed not restrictive to assume that $\|\bar{y}_i\|_\infty \le \|z_h\|_\infty$

and $\|\bar{b}_i\|_\infty \le \|b_h\|_\infty$. From the representation of y_h and $\nabla_h y_h$ on $\widetilde{W}_h^{\rm empt}$, see (6.57)–(6.58), by using (6.9)(i), (6.14), and the fact that $\mathcal{L}^3((\Omega \setminus \widetilde{V}_h) \setminus \widetilde{W}_h^{\rm empt}) \to 0$, see (6.56), we find

$$\chi_{\widetilde{W}_h^{\rm empt}} y_h \to \widetilde{y}_\eta \text{ in } L^1(\Omega;\mathbb{R}^3)\,, \quad \text{and} \quad \chi_{\widetilde{W}_h^{\rm empt}} \nabla_h y_h \to (\nabla' \widetilde{y}_\eta, \partial_1 \widetilde{y}_\eta \wedge \partial_2 \widetilde{y}_\eta) \text{ strongly in } L^2(\Omega;\mathbb{R}^{3\times 3})\,.$$

This together with $\|y_h\|_{L^\infty(\Omega)} \le M$ and (6.66) shows (6.10)(i),(ii). Eventually, (6.10)(iii) follows from (6.67).

We close the proof by explaining the necessary adaptations in the case that $\|y\|_{L^\infty(S)} = M$. Note that the sequence $(y_h)_{h>0}$ as defined in Step 7 might not satisfy $\|y_h\|_{L^\infty} \le M$ in this case. Using (6.22) and recalling the definition of $\widetilde{U}_h^{\rm ext}$ in (6.52) we find

$$(\Omega \setminus \widetilde{V}_h)_{(2-\sqrt{2})\Lambda h} \subset \widetilde{U}_h^{\rm ext}.$$

Then, choosing a universal $C > 0$ large enough such that $\Omega \subset B_C(0) \subset \mathbb{R}^3$, and defining

$$\sigma_h := 1 + \frac{1}{2C}\Lambda h\,, \tag{6.68}$$

it is elementary to check that

$$\sigma_h x \in \widetilde{U}_h^{\rm ext} \quad \text{for all } x \in \Omega \setminus \widetilde{V}_h\,. \tag{6.69}$$

We define the sequence deformations $(\hat{y}_h)_{h>0}$ with $\hat{y}_h \in W^{1,2}(\Omega \setminus \widetilde{V}_h;\mathbb{R}^3)$, by $\hat{y}_h := T_h(\mathrm{id})$ on $\widetilde{V}_h$ and

$$\hat{y}_h(x) := \sigma_h^{-1} y_h(\sigma_h x) \qquad \text{on } \Omega \setminus \widetilde{V}_h,$$

where y_h is given in (6.65). By (6.69) this is well defined. From (6.65) and (6.14)(iv) we obtain $\|y_h\|_{L^\infty(\Omega)} \le M + Dh$, where $D := \|d\|_\infty + \|b_\eta\|_\infty$. Recalling the choice (6.68) and choosing further

$$\Lambda \ge \frac{2C \cdot D}{M}\,, \tag{6.70}$$

which clearly only depends on η, we find $\|\hat{y}_h\|_{L^\infty(\Omega)} \le M$. This shows (6.10)(iv). As $\sigma_h \to 1$ for $h \to 0$, we easily get that also (6.10)(i)–(iii) are satisfied. This concludes the proof.

Appendix A. Proofs of Proposition 3.8 and Corollary 3.9

Proof of Proposition 3.8. We recall once again that by $C > 0$ we denote generic constants which are independent of h, ρ. We fix $i \in I_{\rm g}^h$. Let

$$\mathcal{P}_{i,h} := \Big\{ (P_{i,h}^j)_j \text{ the connected components of } \hat{Q}_{h,\rho}(i) \setminus \partial E_h^* \Big\}\,,$$

with the enumeration being such that $\mathcal{L}^3(P_{i,h}^1)$ is always maximal. We can use the maximality of $P_{i,h}^1$ in terms of its volume, the relative isoperimetric inequality, and (3.23) to estimate

$$\begin{aligned} \mathcal{L}^3(\hat{Q}_{h,\rho}(i) \setminus P_{i,h}^1) &= \sum_{j\ge 2} \mathcal{L}^3(P_{i,h}^j) \le c_{\rm isop} \sum_{j \ge 2} [\mathcal{H}^2(\partial P_{i,h}^j \cap \hat{Q}_{h,\rho}(i))]^{3/2} \\ &\le c_{\rm isop} \alpha^{1/2} h \sum_{j\ge 2} \mathcal{H}^2(\partial P_{i,h}^j \cap \hat{Q}_{h,\rho}(i)) \le C h \mathcal{H}^2\big(\partial E_h^* \cap \hat{Q}_h(i)\big)\,, \end{aligned} \tag{A.1}$$

for $C := 2c_{\rm isop}\alpha^{1/2}$. Furthermore, by (3.2), (3.3), (3.23), (A.1), and the choice of α in (3.21) and ρ in (3.16),

$$\mathcal{L}^3(\hat{Q}_h(i) \setminus P_{i,h}^1) \le \mathcal{L}^3(\hat{Q}_h(i) \setminus \hat{Q}_{h,\rho}(i)) + 2c_{\rm isop}\alpha^{3/2}h^3 \le \frac{9h^3}{128} + \frac{9h^3}{128} = \frac{1}{64}\mathcal{L}^3(\hat{Q}_h(i))\,. \tag{A.2}$$

We now distinguish between the two cases, namely

$$\text{(a)} \;\; P_{i,h}^1 \subset E_h^*\,, \qquad \text{(b)} \;\; P_{i,h}^1 \cap E_h^* = \emptyset\,.$$

Case (a): If $P^1_{i,h} \subset E^*_h$, we just define $D_{i,h} := P^1_{i,h}$, $R_{i,h} := \mathrm{Id}$, $b_{i,h} := 0$, and $z_{i,h} \in W^{1,2}(\hat{Q}_{h,\rho}(i);\mathbb{R}^3)$ by $z_{i,h} := \mathrm{id}$. Then (3.27) holds by (A.1) and (A.2), while (3.28), (3.29), and (3.30) are trivially satisfied, recalling (3.19).

Case (b): Suppose now that $P^1_{i,h} \cap E^*_h = \emptyset$. Then, we apply Theorem 3.3 to the map v^*_h for $\rho > 0$, $\gamma := \kappa_h/h^2$, and $\eta_h \to 0$ satisfying (3.17). Property (3.11) provides a rotation $R^1_{i,h} \in SO(3)$ and $b^1_{i,h} \in \mathbb{R}^3$ such that

$$\text{(i)}\quad \int_{P^1_{i,h}} \big|\mathrm{sym}\big((R^1_{i,h})^T\nabla v^*_h - \mathrm{Id}\big)\big|^2\,\mathrm{d}x \le C\big(1 + C_{\eta_h}(h^{-2}\kappa_h)^{-15/2}h^{-3}\varepsilon_{i,h}\big)\varepsilon_{i,h}\,,$$
$$\text{(ii)}\quad h^{-2}\int_{P^1_{i,h}} |v^*_h - (R^1_{i,h}x + b^1_{i,h})|^2\,\mathrm{d}x + \int_{P^1_{i,h}} \big|(R^1_{i,h})^T\nabla v^*_h - \mathrm{Id}\big|^2\,\mathrm{d}x \le C_{\eta_h}(h^{-2}\kappa_h)^{-5}\varepsilon_{i,h}\,,$$

where we recall (3.22) and we have set $\gamma = \kappa_h/h^2$. Our choice of $(\eta_h)_{h>0}$ in (3.17), the definition in (3.23), and (2.5) ensure that, for $h > 0$ small enough depending on ρ,

$$\begin{aligned}&\text{(i)}\quad \int_{P^1_{i,h}} \big|\mathrm{sym}\big((R^1_{i,h})^T\nabla v^*_h - \mathrm{Id}\big)\big|^2\,\mathrm{d}x \le C_0\varepsilon_{i,h}\,,\\ &\text{(ii)}\quad h^{-2}\int_{P^1_{i,h}} |v^*_h - (R^1_{i,h}x + b^1_{i,h})|^2\,\mathrm{d}x + \int_{P^1_{i,h}} \big|(R^1_{i,h})^T\nabla v^*_h - \mathrm{Id}\big|^2\,\mathrm{d}x \le C_0 h^{-2/5}\varepsilon_{i,h}\,,\end{aligned} \tag{A.3}$$

for a universal constant $C_0 > 0$. It is also easy to verify that

$$|b^1_{i,h}| \le CM \tag{A.4}$$

for a universal constant $C > 0$ that is independent of $h > 0$. Indeed, since $\|v_h\|_{L^\infty(\Omega_h)} \le M$ for $M \ge 1$, using the triangle inequality we obtain

$$\mathcal{L}^3(P^1_{i,h})|b^1_{i,h}|^2 \le C\int_{P^1_{i,h}} |v^*_h(x) - (R^1_{i,h}x + b^1_{i,h})|^2 + C\mathcal{L}^3(P^1_{i,h})\big(\|v_h\|^2_{L^\infty(\Omega_h)} + (\mathrm{diam}(\Omega_h))^2\big)\,. \tag{A.5}$$

Using further (A.2), (A.3)(ii), and (3.23), we get

$$|b^1_{i,h}|^2 \le Ch^{-3}h^2h^{-2/5}\varepsilon_{i,h} + C(M^2 + C) \le C(M^2 + C)\,, \tag{A.6}$$

hence $|b^1_{i,h}| \le CM$.

We now show that we can use Theorem 3.4 to obtain a Sobolev function satisfying (3.30). Introducing the function $u_{i,h} \in SBV^2(\hat{Q}_{h,\rho}(i);\mathbb{R}^3)$ by

$$u_{i,h}(x) := \chi_{P^1_{i,h}}(x)\big[(R^1_{i,h})^T v^*_h(x) - x - (R^1_{i,h})^T b^1_{i,h}\big]\,, \tag{A.7}$$

we observe that $J_{u_{i,h}} \subset \partial E^*_h \cap \hat{Q}_{h,\rho}(i)$. Now, (A.7), (A.3), and (3.23) imply that

$$\begin{aligned}&\text{(i)}\quad \int_{\hat{Q}_{h,\rho}(i)} |\mathrm{sym}(\nabla u_{i,h})|^2\,\mathrm{d}x \le C\varepsilon_{i,h}\,,\\ &\text{(ii)}\quad h^{-2}\int_{\hat{Q}_{h,\rho}(i)} |u_{i,h}|^2\,\mathrm{d}x + \int_{\hat{Q}_{h,\rho}(i)} |\nabla u_{i,h}|^2\,\mathrm{d}x \le C\varepsilon_{i,h}^{9/10}\,.\end{aligned} \tag{A.8}$$

Applying Theorem 3.4 to the map $u_{i,h}$, in view of Remark 3.5 and (3.23), we obtain a set of finite perimeter $\omega_{i,h} \subset \hat{Q}_{h,\rho}(i)$ that satisfies

$$\mathcal{H}^2(\partial^*\omega_{i,h}) \le c_{\mathrm{KP}}\mathcal{H}^2(J_{u_{i,h}}) \le c_{\mathrm{KP}}\mathcal{H}^2(\partial E^*_h \cap \hat{Q}_{h,\rho}(i))\,, \tag{A.9}$$

and

$$\begin{aligned}\mathcal{L}^3(\omega_{i,h}) &\le c_{\mathrm{KP}}\big(\mathcal{H}^2(J_{u_{i,h}})\big)^{3/2} \le c_{\mathrm{KP}}\big(\mathcal{H}^2(\partial E_h^* \cap \hat{Q}_{h,\rho}(i))\big)^{3/2} \\ &\le c_{\mathrm{KP}}\alpha^{1/2} h \mathcal{H}^2(\partial E_h^* \cap \hat{Q}_{h,\rho}(i)) \le c_{\mathrm{KP}}\alpha^{3/2}h^3 \le \frac{9}{128}h^3 < \frac{1}{64}\mathcal{L}^3(\hat{Q}_h(i))\,,\end{aligned} \tag{A.10}$$

where we used again (3.21). Theorem 3.4 provides also a Sobolev function $\zeta_{i,h} \in W^{1,2}(\hat{Q}_{h,\rho}(i);\mathbb{R}^3)$ such that

$$\begin{aligned}&\text{(i)} \quad \zeta_{i,h} \equiv u_{i,h} \quad \text{on} \quad \hat{Q}_{h,\rho}(i) \setminus \omega_{i,h}\,,\\ &\text{(ii)} \quad \|\mathrm{sym}(\nabla \zeta_{i,h})\|_{L^2(\hat{Q}_{h,\rho}(i))} \le c_{\mathrm{KP}} \|\mathrm{sym}(\nabla u_{i,h})\|_{L^2(\hat{Q}_{h,\rho}(i))}\,,\\ &\text{(iii)} \quad \|\zeta_{i,h}\|_\infty \le \|u_{i,h}\|_\infty \le CM\,,\end{aligned} \tag{A.11}$$

where the final estimate in (A.11)(iii) follows from $\|v_h^*\|_\infty \le M$, (A.4), and the definition of $u_{i,h}$ in (A.7).

We then define as dominant component the set

$$D_{i,h} := P^1_{i,h} \setminus \omega_{i,h}\,, \tag{A.12}$$

so that by (A.1), (A.2), and (A.10) we indeed verify that (3.27) holds. The estimate (3.28) follows directly from the definition (A.12), the fact that $\partial P^1_{i,h} \cap \hat{Q}_{h,\rho}(i) \subset \partial E^*_h \cap \hat{Q}_{h,\rho}(i)$, and (A.9).

Applying the classical Korn's inequality in $W^{1,2}$, we find $A_{i,h} \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}$ such that

$$\int_{\hat{Q}_{h,\rho}(i)} |\nabla \zeta_{i,h} - A_{i,h}|^2 \,\mathrm{d}x \le C_{\mathrm{KP}} \int_{\hat{Q}_{h,\rho}(i)} |\mathrm{sym}(\nabla u_{i,h})|^2 \,\mathrm{d}x \le C C_{\mathrm{KP}} \varepsilon_{i,h} \tag{A.13}$$

for a universal $C_{\mathrm{KP}} > 0$, where we used (A.11)(ii) and (A.8)(i). We now set

$$z_{i,h} := R^1_{i,h}\zeta_{i,h} + R^1_{i,h}\mathrm{id} + b^1_{i,h} \in W^{1,2}(\hat{Q}_{h,\rho}(i);\mathbb{R}^3)\,, \tag{A.14}$$

and observe that by (A.7), (A.11)(i), and (A.12) it holds that $z_{i,h} \equiv v^*_h$ on $D_{i,h}$, which in particular implies (3.30)(i). Moreover, (3.30)(iii) follows from (A.11)(iii) and (A.4).

It remains to show (3.29) and (3.30)(ii). In view of (A.13) and (A.14), we have

$$\int_{\hat{Q}_{h,\rho}(i)} |\nabla z_{i,h} - R^1_{i,h}(\mathrm{Id} + A_{i,h})|^2 \,\mathrm{d}x \le C\varepsilon_{i,h}\,. \tag{A.15}$$

Next, we substitute $R^1_{i,h}(\mathrm{Id}+A_{i,h})$ in (A.15) by a suitable rotation. For this purpose, we prove that there exists $R_{i,h} \in SO(3)$ such that

$$\mathcal{L}^3(\hat{Q}_{h,\rho}(i))|R^1_{i,h}(\mathrm{Id} + A_{i,h}) - R_{i,h}|^2 \le C\varepsilon_{i,h}\,. \tag{A.16}$$

Indeed, by (3.27) together with (A.11)(i), (A.12), (A.8)(ii), and (A.13), we get

$$\begin{aligned}h^3|A_{i,h}|^2 \le \mathcal{L}^3(D_{i,h})|A_{i,h}|^2 &= \int_{D_{i,h}} \big|\nabla u_{i,h} + A_{i,h} - \nabla\zeta_{i,h}\big|^2 \,\mathrm{d}x \\ &\le 2\Big(\int_{D_{i,h}} |\nabla u_{i,h}|^2 \,\mathrm{d}x + \int_{D_{i,h}} |\nabla \zeta_{i,h} - A_{i,h}|^2 \,\mathrm{d}x\Big) \le C(\varepsilon_{i,h}^{9/10} + \varepsilon_{i,h})\,.\end{aligned}$$

Using that $\varepsilon_{i,h} \le h^4$, see (3.23), we obtain

$$|A_{i,h}|^2 \le Ch^{-3}\varepsilon_{i,h}^{9/10} \le Ch^{-7/5}\varepsilon_{i,h}^{1/2}\,.$$

A standard Taylor expansion, cf. [36, Equation (33)], gives

$$\mathrm{dist}(G, SO(3)) = |\mathrm{sym}(G) - \mathrm{Id}| + \mathrm{O}(|G - \mathrm{Id}|^2)\,,$$

which further yields

$$\mathrm{dist}^2\big(\mathrm{Id} + A_{i,h}, SO(3)\big) \le C|A_{i,h}|^4 \le Ch^{-14/5}\varepsilon_{i,h}\,,$$

i.e., there exists $R_{i,h} \in SO(3)$ such that

$$|R^1_{i,h}(\mathrm{Id} + A_{i,h}) - R_{i,h}|^2 \le Ch^{1/5}h^{-3}\varepsilon_{i,h} \le Ch^{-3}\varepsilon_{i,h} \le C(\mathcal{L}^3(\hat{Q}_{h,\rho}(i)))^{-1}\varepsilon_{i,h}\,.$$

This proves (A.16) which, combined with (A.15), gives

$$\int_{\hat{Q}_{h,\rho}(i)} |\nabla z_{i,h} - R_{i,h}|^2 \,\mathrm{d}x \le C\varepsilon_{i,h}\,.$$

This yields the second part of (3.30)(ii). Applying the Poincaré inequality on $W^{1,2}(\hat{Q}_{h,\rho}(i);\mathbb{R}^3)$ we obtain a vector $b_{i,h} \in \mathbb{R}^3$ such that the rigid motion $r_{i,h}(x) := R_{i,h}x + b_{i,h}$ satisfies

$$h^{-2}\int_{\hat{Q}_{h,\rho}(i)} |z_{i,h}(x) - r_{i,h}(x)|^2 \,\mathrm{d}x \le C\varepsilon_{i,h}\,,$$

concluding the proof of (3.30)(ii). Moreover, (3.29) is an immediate consequence of (3.30)(i),(ii). Finally, by repeating verbatim the argument in (A.5)–(A.6) with $b_{i,h}$ in place of $b^1_{i,h}$, we also obtain that $|b_{i,h}| \le CM$. This concludes the proof. □

Remark A.1. Note again that the indices considered in I^h_{g} are related to cuboids for which $\varepsilon_{i,h} \le h^4$. In [34, Equation (3.47)] this additional requirement was not necessary since the global elastic energy scaling was h^4. In contrast, in the present setting, the global elastic energy scaling is h^3, and an additional control is needed for the following reason: Since the proof of Proposition 3.8 relies on applying Theorem 3.3 on $\hat{Q}_{h,\rho}(i)$, in order to ensure that the constant in (3.11)(i) can be chosen independently of $h > 0$, it is essential that $\varepsilon_{i,h} \ll h^3$. Thus, using the global energy bound $\varepsilon_{i,h} \le h^3$ would not be sufficient for this purpose. Yet, considering any bound of the form $\varepsilon_{i,h} \le h^{3+\mu}$, for $\mu > 0$, would be sufficient, up to adjusting the curvature regularization parameter κ_h in (2.5).

The choice $\mu = 1$ is canonical, since the cardinality of indices i such that $\varepsilon_{i,h} > h^4$ is of the same order (namely h^{-1}) as the one of the indices i' for which $\mathcal{H}^2(\partial E^*_h \cap \hat{Q}_{h,\rho}(i')) > \alpha h^2$, i.e., for which the surface area condition in the definition (3.23) is violated.

Proof of Corollary 3.9. By (3.29) and the triangle inequality we can estimate

$$\begin{aligned}\int_{D_{i,h}\cap D_{i',h}} \big|r_{i,h} - r_{i',h}\big|^2 \,\mathrm{d}x &\le 2\int_{D_{i,h}} \big|v^*_h(x) - r_{i,h}(x)\big|^2 \,\mathrm{d}x + 2\int_{D_{i',h}} \big|v^*_h(x) - r_{i',h}(x)\big|^2 \,\mathrm{d}x \\ &\le Ch^2(\varepsilon_{i,h} + \varepsilon_{i',h})\,. \end{aligned} \tag{A.17}$$

Note that $\mathcal{L}^3(\hat{Q}_h(i) \cap \hat{Q}_h(i')) > \frac{9}{8}h^3$ and $\mathcal{L}^3(\hat{Q}_h(j) \setminus D_{j,h}) \le \frac{9}{32}h^3$ by (3.27) for $j = i, i'$. This yields

$$\mathcal{L}^3(D_{i,h} \cap D_{i',h}) \ge \mathcal{L}^3(\hat{Q}_h(i) \cap \hat{Q}_h(i')) - \mathcal{L}^3(\hat{Q}_h(i) \setminus D_{i,h}) - \mathcal{L}^3(Q_h(i') \setminus D_{i',h}) \ge \frac{9}{16}h^3\,.$$

Moreover, $\hat{Q}_h(i) \cup \hat{Q}_h(i')$ is contained in a ball of radius $r = ch$ for a universal constant $c > 0$. This along with (A.17) and Lemma 3.7 shows the first part of (3.31). The estimate for $|R_{i,h} - R_{i',h}|^2$ therein follows exactly in the same fashion, using again (3.29). □

Appendix B. A linearization argument for the elastic energy liminf.

We detail the by now classical linearization argument to obtain (5.5). Let $(\lambda_h)_{h>0} \subset (0,\infty)$ be such that

$$\lambda_h \to \infty\,,\ h\lambda_h \to 0 \ \text{ as } h \to 0\,, \tag{B.1}$$

and define

$$\Theta_{h,\rho} := \Omega_{1,\rho} \cap \{\widetilde{w}_h = y_h\} \cap \{|G_h| \le \lambda_h\}\,. \tag{B.2}$$

Note that $\mathcal{L}^3(\{\widetilde{w}_h \neq y_h\} \cap \Omega_{1,\rho}) \to 0$ by (4.21), (2.7), (3.9)(i), and a scaling argument. Combining this with the fact that $\sup_{h>0} \|G_h\|_{L^2(\Omega_{1,\rho})} \le C$, see (5.4), $\lambda_h \to +\infty$, and Chebyshev's inequality, we obtain

$$\mathcal{L}^3(\Omega_{1,\rho} \setminus \Theta_{h,\rho}) \to 0 \quad \text{as } h \to 0\,, \tag{B.3}$$

i.e., $\chi_{\Theta_{h,\rho}} \to 1$ boundedly in measure on $\Omega_{1,\rho}$ as $h \to 0$. By (2.10), $W(\mathrm{Id}) = 0$, $W \ge 0$, and the definition of $\Theta_{h,\rho}$, we get

$$\begin{aligned}\liminf_{h\to 0}\Big(h^{-2}\int_{\Omega\setminus\overline{V_h}} W(\nabla_h y_h)\,\mathrm{d}x\Big) &= \liminf_{h\to 0}\Big(h^{-2}\int_{\Omega} W(\nabla_h y_h)\,\mathrm{d}x\Big)\\ &\ge \liminf_{h\to 0}\Big(h^{-2}\int_{\Omega_{1,\rho}} \chi_{\Theta_{h,\rho}} W(\nabla_h \widetilde{w}_h)\,\mathrm{d}x\Big)\,.\end{aligned}$$

The regularity and the structural hypotheses on W, recall (2.4), imply that

$$W(\mathrm{Id}+F) = \tfrac{1}{2}\mathcal{Q}_3(F) + \Phi(F)\,,$$

where $\Phi\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfies

$$\sup\Big\{\tfrac{|\Phi(F)|}{|F|^2}\colon\ |F| \le \sigma\Big\} \to 0 \quad \text{as } \sigma \to 0\,. \tag{B.4}$$

Together with the definition of G_h in (5.4), we obtain

$$\begin{aligned}\liminf_{h\to 0}\Big(h^{-2}\int_{\Omega\setminus\overline{V_h}} W(\nabla_h y_h)\,\mathrm{d}x\Big) &\ge \liminf_{h\to 0}\Big(h^{-2}\int_{\Omega_{1,\rho}} \chi_{\Theta_{h,\rho}} W(\mathrm{Id}+hG_h)\,\mathrm{d}x\Big)\\ &\ge \liminf_{h\to 0}\int_{\Omega_{1,\rho}} \chi_{\Theta_{h,\rho}}\Big(\tfrac{1}{2}\mathcal{Q}_3(G_h) + h^{-2}\Phi(hG_h)\Big)\,\mathrm{d}x\\ &= \liminf_{h\to 0}\frac{1}{2}\int_{\Omega_{1,\rho}} \chi_{\Theta_{h,\rho}}\mathcal{Q}_3(G_h)\,\mathrm{d}x\,.\end{aligned} \tag{B.5}$$

In the passage to the last line above, we made use of the fact that

$$\limsup_{h\to 0}\int_{\Omega_{1,\rho}} \chi_{\Theta_{h,\rho}} h^{-2}|\Phi(hG_h)|\,\mathrm{d}x \le \limsup_{h\to 0}\left(\sup\Big\{\tfrac{|\Phi(hG_h)|}{|hG_h|^2}\colon\ |hG_h| \le h\lambda_h\Big\}\int_{\Omega_{1,\rho}} \chi_{\Theta_{h,\rho}}|G_h|^2\,\mathrm{d}x\right) = 0\,,$$

which follows from the fact that $(G_h)_{h>0}$ is bounded in $L^2(\Omega_{1,\rho};\mathbb{R}^{3\times 3})$, (B.2), (B.4), and $h\lambda_h \to 0$, cf. (B.1). Hence, (B.5), (5.4), the fact that $\chi_{\Theta_{h,\rho}} \to 1$ boundedly in measure in $\Omega_{1,\rho}$, see (B.3), and the convexity of $\mathcal{Q}_3$ imply that

$$\liminf_{h\to 0}\Big(h^{-2}\int_{\Omega\setminus\overline{V_h}} W(\nabla_h y_h)\,\mathrm{d}x\Big) \ge \frac{1}{2}\int_{\Omega_{1,\rho}} \mathcal{Q}_3(G)\,\mathrm{d}x \ge \frac{1}{2}\int_{\Omega_{1,\rho}} \mathcal{Q}_2\big(G'\big)\,\mathrm{d}x\,,$$

which is exactly (5.5).

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