

# VARIATIONAL CONVERGENCES UNDER MOVING ANISOTROPIES

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ABSTRACT. We study the asymptotic behaviour of sequences of integral functionals depending on moving anisotropies. We introduce and describe the relevant functional setting, establishing uniform Meyers-Serrin type approximations, Poincaré inequalities and compactness properties. We prove several  $\Gamma$ -convergence results, and apply the latter to the study of  $H$ -convergence of anisotropic linear differential operators.

## 1. INTRODUCTION

In this paper we study the asymptotic behaviour of sequences of functionals of the form

$$\int_{\Omega} f_h(x, X^h u(x)) \, dx, \quad u \in W_{X^h}^{1,p}(\Omega),$$

for a sequence of anisotropies  $(X^h)_h = (X_1^h, \dots, X_m^h)_h$ , where  $\Omega$  is a bounded domain of  $\mathbf{R}^n$  and  $m \leq n$ . As  $h$  tends to  $\infty$ , we describe the convergence of these functionals, their minima, minimizers and momenta, to the corresponding limit quantities, associated with a limit functional

$$(1.1) \quad \int_{\Omega} f(x, Xu(x)) \, dx, \quad u \in W_X^{1,p}(\Omega),$$

driven by a limit anisotropy  $X = (X_1, \dots, X_m)$ . The natural variational convergence to take into account is  $\Gamma$ -convergence (see e.g. [3, 13]). This technique, which is extremely useful both for theoretical purposes and practical applications, remains a crucial tool in the study of various problems in mechanics, such as homogenization, phase transitions and the asymptotic analysis of PDEs. In addition to an interest in  $\Gamma$ -convergence itself, we explore its applications to the study of anisotropic, linear, symmetric PDEs, referring back to the so-called  $G$ -convergence or  $H$ -convergence (see e.g. [34, 39]).

The investigation of anisotropic variational functionals as in (1.1), as well as their related functional frameworks, is originally motivated by L. Hörmander's seminal work on *hypoelliptic operators* [27]. The latter constitutes a milestone in the study of differential operators with underlying *sub-Riemannian* type structures. Important evidences are the works of G.B. Folland [18] and of L. Rothschild and E.M. Stein [38] in the context of *stratified* and *nilpotent Lie groups*. We refer to [2] for further accounts on analysis on Lie groups, and to [1, 26, 28] for thorough introductions to sub-Riemannian geometry.

As a prototypical example, consider the smooth anisotropy  $X = (X_1, X_2)$  on  $\mathbf{R}^3$ , where

$$(1.2) \quad X_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} \quad \text{and} \quad X_2 = \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3}.$$

The latter generate, via Lie-brackets, the Lie algebra of the so-called *first Heisenberg group*, and induce on it a *sub-Riemannian structure*. The relevant hypoelliptic operator associated

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2020 *Mathematics Subject Classification*. 49J45, 49Q20, 53C17.

*Key words and phrases*. Integral representation;  $\Gamma$ -convergence;  $H$ -convergence; Local functionals; Anisotropic functionals; Moving anisotropies; Vector fields.

*Acknowledgements*. The authors thank Fares Essebei, Andrea Pinamonti, Francesco Serra Cassano and Giacomo Vianello for interesting and valuable conversations on the topic of the paper.

with  $X$  is the so-called *sub-Laplacian*

$$(1.3) \quad (X_1)^2 u + (X_2)^2 u,$$

and its associated *Dirichlet energy* reads as

$$(1.4) \quad \int_{\Omega} (|X_1 u|^2 + |X_2 u|^2) \, dx.$$

As (1.3) is not elliptic, (1.4) is not coercive. Moreover, the Euclidean Sobolev spaces  $W^{1,2}(\Omega)$  are not the correct finiteness domains for functionals as in (1.3). These issues clearly obstruct the use of classical techniques to study minimization properties of (1.4), preventing for instance the  $L^2$ -lower semicontinuity of its Euclidean extension to  $L^2(\Omega)$ , namely

$$\begin{cases} \int_{\Omega} (|X_1 u|^2 + |X_2 u|^2) \, dx & \text{if } u \in W^{1,2}(\Omega), \\ \infty & \text{if } u \in L^2(\Omega) \setminus W^{1,2}(\Omega). \end{cases}$$

To overcome these obstacles, building on the foundational work of G.B. Folland and E.M. Stein [19], the correct functional framework has been developed by B. Franchi, R. Serapioni and F. Serra Cassano [20, 21] and by N. Garofalo and D.M. Nhieu [22, 23]. Precisely, when  $1 \leq p < \infty$ , the study of anisotropic functionals in the greater generality of (1.1) can be carried out via the introduction of suitable anisotropic Sobolev and  $BV$  spaces, say  $W_X^{1,p}(\Omega)$  and  $BV_X(\Omega)$ , emerging as the natural domains of functionals as in (1.1). Although this construction is possible for arbitrary anisotropies, the specific structure of  $X$  may lead to good approximation results *à la* Meyers-Serrin, as well as to Poincaré inequalities and Rellich-Kondrachov compactness properties. We refer to Section 2 for more specific insights. When instead  $p = \infty$ , the reader is referred to [36, 41, 42] for some anisotropic  $L^\infty$ -variational problems.

Asymptotic results for functionals in the *static* case as in (1.1), i.e. when the anisotropy  $X$  is fixed, started in [31, 32, 33] to generalize classical integral representation and  $\Gamma$ -convergence results due to G. Buttazzo and G. Dal Maso [6]. These results were later extended in [15, 16] to complete integral functionals of the form

$$\int_{\Omega} f(x, u(x), Xu(x)) \, dx, \quad u \in W_X^{1,p}(\Omega),$$

generalizing the corresponding Euclidean counterparts [5, 7]. In all the above-mentioned contributions, the authors assume a full-rank condition on the family  $X$ , the so-called *Linear Independence Condition (LIC)*. This technical assumption has been definitely removed by the third-named author in [40]. Thanks to this set of results, variational properties for static anisotropic functionals are now well-understood, and available in great generality.

In this paper we extend the results obtained in the aforementioned literature by considering *moving anisotropies*. Namely, a static anisotropy is replaced by a sequence  $(X^h)_h$  converging to a limit anisotropy  $X$ . Although our setting will be fairly more general, it models, as a particular case, the regularization of a sub-Riemannian structure via Riemannian approximants. As an instance, consider the anisotropy  $X = (X_1, X_2)$  introduced in (1.2) to model the first sub-Riemannian Heisenberg group. Its natural sub-Riemannian structure, say  $g$ , can be approximated by a sequence of Riemannian metrics, say  $g_h$ , each induced by the family of vector fields  $X^h = (X_1, X_2, X_3^h)$ , where

$$X_3^h = \frac{1}{h} \frac{\partial}{\partial x_3}.$$

The benefits of such an approximation are manifold. For instance, we stress that the natural approximating operators related to (1.3), namely

$$(X_1)^2 u + (X_2)^2 u + (X_3^h)^2 u,$$

are precisely the *Laplace-Beltrami* operators associated with the Riemannian structures  $g_h$ . In turn, they are uniformly elliptic operators, their Dirichlet energies are coercive, and their

associated functional frameworks boil down to the Euclidean ones. We refer e.g. to [9, 14] for further insights, and to [8, 12, 24, 35, 37] for a non-exhaustive list of applications of this approximation technique.

The study of asymptotic properties of functionals of the form

$$(1.5) \quad F_h(u) = \begin{cases} \int_{\Omega} f_h(x, X^h u(x)) dx & \text{if } u \in W_{X^h}^{1,p}(\Omega), \\ \infty & \text{if } u \in L^p(\Omega) \setminus W_{X^h}^{1,p}(\Omega), \end{cases}$$

when, say,  $1 < p < \infty$ , requires some preliminary precautions. Indeed, contrarily to the static case, there are no *a priori* relations between the finiteness domains of each functional in (1.5). Indeed, the latter may naturally vary according to the associated anisotropy. To this aim, it is important to understand how the approximating Sobolev spaces relate to the limit one, as discussed thoroughly in Section 3.

Prescribing Dirichlet boundary conditions is also a delicate matter. Indeed, just as any finiteness domain in (1.5) depends on the associated anisotropy, it is natural in this setting to prescribe, for each fixed anisotropy  $X^h$ , a different boundary condition, say  $\varphi_h \in W_{X^h}^{1,p}(\Omega)$ , and to consider sequences of functionals of the form

$$(1.6) \quad F_h^{\varphi_h}(u) = \begin{cases} \int_{\Omega} f_h(x, X^h u(x)) dx & \text{if } u \in W_{X^h, \varphi_h}^{1,p}(\Omega), \\ \infty & \text{if } u \in L^p(\Omega) \setminus W_{X^h, \varphi_h}^{1,p}(\Omega), \end{cases}$$

where  $W_{X^h, \varphi_h}^{1,p}(\Omega)$  is the correct affine Sobolev space associated with the couple  $(X^h, \varphi_h)$ . We stress that, although moving boundary conditions may also be considered in the static case, in the moving one this choice is forced by the very structure of the problem. In this regard, we will discuss the right convergence assumptions on  $(\varphi_h)_h$  to the limit boundary condition, say  $\varphi$ , in order to suitably relate the approximating and the limit affine Sobolev spaces (cf. Section 3).

A third important novelty in considering sequences of functionals as in (1.5) and (1.6) consists in the fact that, unlike in the static case, possible coercivity and boundedness conditions typically depend on the anisotropy, and are thus not uniform along the sequence. Indeed, we will deal with growth conditions of the form

$$(1.7) \quad d_1 \int_{\Omega} |X^h u(x)|^p dx \leq \int_{\Omega} f_h(x, X^h u(x)) dx \leq \int_{\Omega} a(x) dx + d_2 \int_{\Omega} |X^h u(x)|^p dx,$$

where  $a \in L^1(\Omega)$  and  $0 < d_1 \leq d_2$  are fixed. This lack of uniformity, as discussed thoroughly in Section 6, prevents the use of some classical tools to establish  $\Gamma$ -compactness properties (cf. Remark 6.6), and requires a careful analysis of the asymptotic behaviour of the partial norms  $\|X^h u\|_{L^p(\Omega; \mathbf{R}^m)}$ .

A first fundamental step in handling these issues, as discussed in Section 4, consists in establishing Meyers-Serrin type approximation results that keep track of the approximating sequence of anisotropies (cf. Theorem 4.2). Roughly speaking, under uniform convergence of the anisotropies, functions in  $W_X^{1,p}(\Omega)$  can be well-approximated by a sequence of smooth functions  $(u_h)_h$ , where each  $u_h \in W_{X^h}^{1,p}(\Omega)$ , in the sense that

$$u_h \rightarrow u \text{ strongly in } L^p(\Omega) \quad \text{and} \quad X^h u_h \rightarrow Xu \text{ strongly in } L^p(\Omega; \mathbf{R}^m).$$

This approximation property, as explained in Section 6, is crucial to deal with moving growth conditions as in (1.7). In the same spirit, it is possible to provide a uniform approximation result for affine Sobolev spaces (cf. Theorem 4.3) under suitable convergence assumptions on the approximating sequence of boundary data. We stress that, in the static case, this result would follow *by definition*, as functions in  $W_{X, \varphi}^{1,p}(\Omega)$  are, up to a translation by  $\varphi$ , in the closure of  $\mathbf{C}_c^\infty(\Omega)$ . In order to show the validity of the aforementioned approximations, we introduce a non-standard mollification technique (cf. (4.1) and (4.2)), whose convergence rate is appropriately related to the convergence rate of the anisotropies.

The same mollification procedure also plays a key role in [Section 5](#), where we establish Rellich-Kondrachov compactness properties and Poincaré inequalities that, again, keep into account the entire sequence  $(X^h)_h$  (cf. [Theorem 5.2](#) and [Theorem 5.3](#), respectively). Precisely, we show that, as soon as  $X$  is strong enough to guarantee the validity of these properties in the limit, then such properties extend uniformly along  $(X^h)_h$ . We emphasise that such uniformity is fundamental in bypassing the non-uniformity of the coercivity assumptions as in [\(1.7\)](#). Indeed, a careful combination of the previous results will still provide, as in the static setting, boundedness of minimizing sequences and convergence of minima and minimizers (cf. [Theorem 6.10](#)).

The core of this work is [Section 6](#), where we provide several results related to  $\Gamma$ -convergence. Owing to [Section 4](#), we first prove  $\Gamma$ -convergence of the partial norms  $\|X^h u\|_{L^p(\Omega, \mathbf{R}^m)}$  to the limit one  $\|Xu\|_{L^p(\Omega, \mathbf{R}^m)}$  (cf. [Theorem 6.3](#)). As already mentioned, this first feature is crucial in the establishment of a general  $\Gamma$ -compactness theorem for functionals as in [\(1.5\)](#) (cf. [Theorem 6.5](#)). Next, we keep into account Dirichlet boundary data, obtaining  $\Gamma$ -compactness for sequences of functionals as in [\(1.6\)](#) (cf. [Theorem 6.8](#)). As pointed out in [Remark 6.9](#), the fact that the boundary data necessarily evolves with the anisotropies causes some challenges in the identification of the appropriate recovery sequences, as the latter need to keep into account the above-mentioned evolution. In this regard, a key tool is provided by the aforementioned uniform approximation result for affine Sobolev spaces, [Theorem 4.3](#). As already highlighted, the results of [Section 5](#) allow us to prove the convergence of minima and minimizers for  $\Gamma$ -converging sequences of functionals, as in [\(1.6\)](#), as stated in [Theorem 6.10](#).

We conclude [Section 6](#) by specializing our main  $\Gamma$ -compactness result, [Theorem 6.5](#), to the class of anisotropic *quadratic forms*

$$F_h(u) = \begin{cases} \int_{\Omega} \langle A^h(x) X^h u(x), X^h u(x) \rangle_{\mathbf{R}^m} dx & \text{if } u \in W_{X^h}^{1,2}(\Omega), \\ \infty & \text{if } u \in L^2(\Omega) \setminus W_{X^h}^{1,2}(\Omega), \end{cases}$$

whence establishing  $\Gamma$ -compactness in this particular subclass (cf. [Theorem 6.13](#)). The study of the latter is particularly relevant in connection with anisotropic sequences of symmetric, linear differential problems of the form

$$(1.8) \quad \begin{cases} \mu u + \operatorname{div}_{X^h}(A^h(x) X^h u) = f & \text{in } \Omega, \\ u_h \in W_{X^h, \varphi_h}^{1,2}(\Omega), \end{cases}$$

as well as with their asymptotic behaviour. We stress that the latter are clear generalisations of the prototypical differential problems associated with [\(1.3\)](#). In [Section 7](#), we adapt the classical notion of  $H$ -convergence for keeping into account the moving anisotropic framework (cf. [Definition 7.1](#)), and we establish a related  $H$ -compactness property (cf. [Theorem 7.2](#)). This will follow as a corollary of the previous results, as well as of some additional convergence properties established in [Theorem 6.15](#) to deal with the so-called *momenta* associated with [\(1.8\)](#).

The paper is structured as follows. In [Section 2](#), we recall definitions and properties of the static anisotropic functional setting. In [Section 3](#), we introduce the relevant classes of moving anisotropies we are interested in, and we discuss some preliminary functional results. Later, in [Section 4](#), we establish uniform Meyers-Serrin type approximations for anisotropic Sobolev spaces ([Theorem 4.2](#)) and for affine anisotropic Sobolev spaces ([Theorem 4.3](#)). In [Section 5](#), we state and prove a uniform Rellich-Kondrachov compactness result ([Theorem 5.2](#)), as well as a uniform Poincaré inequality ([Theorem 5.3](#)). In [Section 6](#), we first show the main  $\Gamma$ -compactness result, [Theorem 6.5](#), as well as its extension to functionals including Dirichlet boundary conditions ([Theorem 6.8](#)) and quadratic forms ([Theorem 6.13](#)). Moreover, we provide convergence of minima, minimizers ([Theorem 6.10](#)) and momenta ([Theorem 6.15](#)). As a corollary, in [Section 7](#) we obtain the  $H$ -compactness result, [Theorem 7.2](#).

## 2. ANISOTROPIC FUNCTIONAL FRAMEWORKS

**Main notations.** If no ambiguity arises, we let  $\infty = +\infty$  and  $\mathbf{N} = \mathbf{N} \setminus \{0\}$ . If  $1 \leq p \leq \infty$ , we denote by  $p'$  the Hölder-conjugate exponent of  $p$ . We write  $(a_h)_h \subseteq (A_h)_h$  meaning that  $a_h \in A_h$  for any  $h \in \mathbf{N}$ . We fix  $m, n \in \mathbf{N}$  such that  $0 < m \leq n$ . If  $A, B \subseteq \mathbf{R}^n$ , we let  $A + B = \{a + b : a \in A, b \in B\}$ . For  $\alpha, \beta \in \mathbf{N}$ , we denote by  $M(\alpha, \beta)$  the set of matrices with  $\alpha$  rows and  $\beta$  columns. Vectors in  $\mathbf{R}^\alpha$  are seen as matrices in  $M(\alpha, 1)$ . We denote by  $\langle \cdot, \cdot \rangle_{\mathbf{R}^\alpha}$  the scalar product between vectors in  $\mathbf{R}^\alpha$ . Fixed an open and bounded set  $\Omega \subseteq \mathbf{R}^n$ , we denote by  $\mathcal{A}$  the class of all the open subsets of  $\Omega$ , and for a function  $u$  defined in  $\Omega$ , we may tacitly assume that it is defined on  $\mathbf{R}^n$  by extending it to be zero outside  $\Omega$ . Finally, we denote by  $Du$  the (distributional) Euclidean gradient of  $u$ .

**2.1. Anisotropies.** Throughout the paper,  $\Omega$  is an open and bounded subset of  $\mathbf{R}^n$ . Given a family  $X := (X_1, \dots, X_m)$  of Lipschitz continuous vector fields on  $\Omega$ , such that

$$X_j = \sum_{i=1}^n c_{j,i} \frac{\partial}{\partial x_i}, \quad \text{with } c_{j,i} \in \text{Lip}(\Omega)$$

for any  $j = 1, \dots, m$  and any  $i = 1, \dots, n$ , we denote by  $\mathcal{C}(x) := [c_{j,i}(x)]_{\substack{i=1, \dots, n \\ j=1, \dots, m}}$  the  $m \times n$  coefficient matrix associated with  $X$ . We may refer to  $X$  as (static) anisotropy, or  $X$ -gradient. This notation is motivated by the following definition.

**Definition 2.1.** Let  $u$  and  $V$  be a 1-dimensional and an  $m$ -dimensional distribution.

(1) The  $X$ -gradient of  $u$  is the  $m$ -dimensional distribution defined by

$$Xu(\varphi) = -u(\text{div}(\mathcal{C}^T \varphi)) \quad \text{for any } \varphi \in \mathbf{C}_c^\infty(\Omega; \mathbf{R}^m).$$

(2) The  $X$ -divergence of  $V$  is the 1-dimensional distribution defined by

$$\text{div}_X(V)(\varphi) = -V(\mathcal{C}D\varphi) \quad \text{for any } \varphi \in \mathbf{C}_c^\infty(\Omega).$$

Clearly, in the isotropic case, the above definition returns the usual notions of gradient and divergence. In what follows, we will sometimes consider the following classes of  $X$ -gradients.

**Definition 2.2.** We let  $\mathcal{M} = \mathcal{M}(\Omega_0, C_d, c, C)$  be the class of  $X$ -gradients that are defined and Lipschitz continuous on an open set  $\Omega_0 \supseteq \bar{\Omega}$ , and that satisfy the following three conditions.

- (H.1) Let  $d$  be the Carnot-Carathéodory distance induced by  $X$  on  $\Omega_0$  (cf. e.g. [26]). Then  $d$  is finite and continuous with respect to the usual topology of  $\mathbf{R}^n$ .
- (H.2) There exist positive constants  $R, C_d$  such that

$$|B_d(x, 2r)| \leq C_d |B_d(x, r)|$$

for any  $x \in \bar{\Omega}$  and  $r \leq R$ . Here,  $B_d(x, r)$  denotes the open metric ball with respect to  $d$ .

- (H.3) There exist positive constants  $c, C$  such that for every open ball  $B := B_d(\bar{x}, r)$ , with  $\bar{x} \in \Omega$  such that  $cB := B_d(\bar{x}, cr) \subseteq \Omega_0$ , and for every  $u \in \text{Lip}(c\bar{B})$  and  $x \in \bar{B}$ ,

$$\left| u(x) - \frac{1}{|B|} \int_B u(y) \, dy \right| \leq C \int_{cB} |Xu(y)| \frac{d(x, y)}{|B_d(x, d(x, y))|} \, dy.$$

Several important families of  $X$ -gradients belong to the class  $\mathcal{M}(\Omega_0, C_d, c, C)$ . In particular, we recall that all Hörmander vector fields, i.e. smooth vector fields for which the rank of the Lie algebra generated by  $X_1, \dots, X_m$  equals  $n$  at any point of  $\Omega_0$ , belong to the class  $\mathcal{M}(\Omega_0, C_d, c, C)$ . Among Hörmander vector fields, we recall the following instances.

- (i) *Euclidean vector fields:*  $X = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$ , defined on  $\mathbf{R}^n$ .
- (ii) *Grushin vector fields:*  $X = \left( \frac{\partial}{\partial x_1}, x_1 \frac{\partial}{\partial x_2} \right)$ , defined on  $\mathbf{R}^2$ .
- (iii) *Heisenberg vector fields:*  $X = \left( \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} \right)$ , defined on  $\mathbf{R}^3$ .

In particular, Heisenberg vector fields serve as a prototype of *Carnot groups vector fields* (cf. e.g. [2]). It is worth noting that all Carnot groups vector fields are, by definition, Hörmander vector fields, and hence belong to the class  $\mathcal{M}(\Omega_0, C_d, c, C)$ . For further insights on this topic and for examples of families of vector fields in the class  $\mathcal{M}(\Omega_0, C_d, c, C)$  that do not satisfy the Hörmander condition, the reader is referred to [32].

**2.2. Sobolev spaces.** Owing to Definition 2.1, we introduce the main functional framework.

**Definition 2.3.** *The anisotropic Sobolev spaces associated to an  $X$ -gradient are defined by*

$$\begin{aligned} W_X^{1,p}(\Omega) &:= \{u \in L^p(\Omega) : Xu \in L^p(\Omega; \mathbf{R}^m)\}, \\ H_X^{1,p}(\Omega) &:= \overline{\mathbf{C}^\infty(\Omega) \cap W_X^{1,p}(\Omega)}^{\|\cdot\|_{W_X^{1,p}(\Omega)}}, \\ W_{X,0}^{1,p}(\Omega) &:= \overline{\mathbf{C}_c^\infty(\Omega) \cap W_X^{1,p}(\Omega)}^{\|\cdot\|_{W_X^{1,p}(\Omega)}}, \\ W_{X,\varphi}^{1,p}(\Omega) &:= \{u \in W_X^{1,p}(\Omega) : u - \varphi \in W_{X,0}^{1,p}(\Omega)\} \quad \text{for any } \varphi \in W_X^{1,p}(\Omega). \end{aligned}$$

It is well-known (cf. [19]) that the vector space  $W_X^{1,p}(\Omega)$ , endowed with the norm

$$\|u\|_{W_X^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega; \mathbf{R}^m)},$$

is a Banach space for any  $1 \leq p < +\infty$ , reflexive when  $1 < p < +\infty$ . The Lipschitz continuity assumption on the family  $X$  ensures that  $W^{1,p}(\Omega)$  embeds continuously into  $W_X^{1,p}(\Omega)$  (cf. [15, 31]), where  $W^{1,p}(\Omega)$  denotes the Euclidean Sobolev space. Precisely, for any  $u \in W^{1,p}(\Omega)$ , the  $X$ -gradient admits the Euclidean representation

$$(2.1) \quad Xu(x) = \mathcal{C}(x)Du(x)$$

for a.e.  $x \in \Omega$ . We also recall that, as for classical Sobolev spaces,  $H_X^{1,p}(\Omega) \subseteq W_X^{1,p}(\Omega)$ . The classical result of Meyers and Serrin is still valid in this anisotropic setting and it was proved, independently, in [20] and [23]. We state the result in what follows, and refer again the interested reader to [32] for further discussions on this topic.

**Theorem 2.4.** *For any  $1 \leq p < \infty$ , it holds that  $H_X^{1,p}(\Omega) = W_X^{1,p}(\Omega)$ .*

If, in addition, the family  $X$  belongs to the class  $\mathcal{M}(\Omega_0, C_d, c, C)$ , then the following Poincaré inequality and Rellich-Kondrachov theorem hold. We refer the interested reader to [32, Proposition 2.16] and [21, Theorem 3.4], respectively.

**Proposition 2.5** (Poincaré inequality). *Let  $\Omega \subseteq \mathbf{R}^n$  be connected, let  $1 \leq p < \infty$  and let  $X \in \mathcal{M}(\Omega_0, C_d, c, C)$ . Then, there exists a positive constant  $c_{p,\Omega} = c_{p,\Omega}(p, \Omega)$  such that*

$$c_{p,\Omega} \int_{\Omega} |u|^p dx \leq \int_{\Omega} |Xu|^p dx \quad \text{for any } u \in W_{X,0}^{1,p}(\Omega).$$

**Theorem 2.6** (Rellich-Kondrachov). *For  $1 \leq p < \infty$ ,  $W_{X,0}^{1,p}(\Omega)$  compactly embeds in  $L^p(\Omega)$ .*

### 3. MOVING ANISOTROPIES

We fix a sequence of  $X$ -gradients  $(X^h)_h$  with  $m$  components  $X^h = (X_1^h, \dots, X_m^h)$ , identified by a sequence of matrices  $(\mathcal{C}^h)_h$  as in Section 2.1. Throughout the paper, we assume that there exists a limit family of vector fields  $X = (X_1, \dots, X_m)$ , identified by a Lipschitz continuous coefficient matrix  $\mathcal{C}$ , such that

$$(A) \quad \mathcal{C}^h \rightarrow \mathcal{C} \text{ uniformly on } \Omega.$$

We may sometimes use the index  $\infty$  to refer to  $X$ . As already mentioned, (A) will be sufficient to infer good approximation properties of anisotropic Sobolev functions. Instead, most of the  $\Gamma$ -compactness properties will be achieved for the following two classes of  $X$ -gradients.

**Definition 3.1.** We say that  $(X^h)_h \in \mathcal{S}_1$  if (A) holds and

$$(S.1) \quad \mathcal{C}^h = \begin{bmatrix} \mathcal{C} \\ \mathcal{D}^h \end{bmatrix} \quad \text{and} \quad \mathcal{C} = \begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix},$$

with a natural abuse of notation in the second property.

**Definition 3.2.** We say that  $(X^h)_h \in \mathcal{S}_2$  if (A) holds and if

$$(S.2) \quad (X^h)_h \text{ is uniformly bounded in } \text{Lip}(\Omega, M(m, n)).$$

**Remark 3.3.** As pointed out in the introduction,  $\mathcal{S}_1$  models those frameworks where a sub-Riemannian structure induced by  $X$  is approximated by a sequence of converging Riemannian structures. Instead,  $\mathcal{S}_2$  allows anisotropies with arbitrary shapes, provided the uniform bound (S.2) is satisfied.

**Remark 3.4.** If  $(X^h)_h \in \mathcal{S}_2$ , by (S.2) and up to a subsequence, we may always assume that

$$\frac{\partial \mathcal{C}_{j,i}^h}{\partial x_k} \rightarrow \frac{\partial \mathcal{C}_{j,i}}{\partial x_k} \quad \text{weakly}^* \text{ in } L^\infty(\Omega)$$

for any  $j = 1, \dots, m$  and any  $i, k = 1, \dots, n$ .

**Remark 3.5.** We stress that  $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$ . Nevertheless, neither  $\mathcal{S}_1 \subseteq \mathcal{S}_2$  nor  $\mathcal{S}_2 \subseteq \mathcal{S}_1$ . First, it is clear that  $\mathcal{S}_2 \not\subseteq \mathcal{S}_1$ . To see that  $\mathcal{S}_1 \not\subseteq \mathcal{S}_2$ , consider the following instance. Let  $\Omega = (-1, 1)^2 \subseteq \mathbf{R}_{(x_1, x_2)}^2$ . For any  $h \in \mathbf{N}$ , let  $X^h = (X_1, X_2^h)$ , where

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = f_h(x_2) \frac{\partial}{\partial x_2} \quad \text{and} \quad f_h(x_2) = \begin{cases} -\frac{1}{h} & \text{if } x_2 \in (-1, -\frac{1}{h^2}), \\ hx_2 & \text{if } x_2 \in (-\frac{1}{h^2}, \frac{1}{h^2}), \\ \frac{1}{h} & \text{if } x_2 \in (\frac{1}{h^2}, 1). \end{cases}$$

A simple computation reveals that  $(X^h)_h \in \mathcal{S}_1 \setminus \mathcal{S}_2$ .

In order to address compactness properties, we will need to further refine our assumptions. Indeed, we will show that, when

$$(B) \quad X \in \mathcal{M} = \mathcal{M}(\Omega_0, C_d, c, C),$$

it is possible to derive uniform compactness properties and Poincaré inequalities. To this aim, we introduce the following subclasses of  $\mathcal{S}_1$  and  $\mathcal{S}_2$ .

**Definition 3.6.** For  $i \in \{1, 2\}$ , we say that  $(X^h)_h \in \mathcal{T}_i$  if  $(X^h)_h \in \mathcal{S}_i$  and (B) holds.

In the rest of this section, we investigate some preliminary properties of moving anisotropies. We introduce the *anisotropic Rayleigh quotients*. If  $(X^h)_h$  satisfies (A), we let

$$\mathcal{R}_h := \inf \left\{ \frac{\int_\Omega |X^h u|^p \, dx}{\int_\Omega |u|^p \, dx} : u \in W_{X^h, 0}^{1,p}(\Omega), u \neq 0 \right\} = \inf \left\{ \frac{\int_\Omega |X^h u|^p \, dx}{\int_\Omega |u|^p \, dx} : u \in \mathbf{C}_c^\infty(\Omega), u \neq 0 \right\}$$

for any  $h \in \mathbf{N}$ , and we let  $\mathcal{R}$  be the Rayleigh quotient associated with  $X$ . Above, the second equality follows by definition of  $W_{X^h, 0}^{1,p}(\Omega)$ .

**3.1. Preliminary functional properties.** The following two propositions describes the relations between the approximating Sobolev spaces and the limit one in  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , respectively.

**Proposition 3.7.** Let  $(X^h)_h \in \mathcal{S}_1$ . Let  $1 < p < \infty$ . Then  $W_{X^h}^{1,p}(\Omega) \subseteq W_X^{1,p}(\Omega)$ . Precisely,

$$(3.1) \quad X^h u = (X_1 u, \dots, X_m u, X_{m+1}^h u, \dots, X_n^h u) \quad \text{and} \quad X u = (X_1 u, \dots, X_m u, 0, \dots, 0).$$

for any  $h \in \mathbf{N}$  and  $u \in W_{X^h}^{1,p}(\Omega)$ . Moreover, if  $(u_h)_h \subseteq W_{X^h}^{1,p}(\Omega)$  satisfies  $u_h \rightarrow u$  weakly in  $L^p(\Omega)$ , then

$$\|X u\|_{L^p(\Omega; \mathbf{R}^m)} \leq \liminf_{h \rightarrow \infty} \|X^h u_h\|_{L^p(\Omega; \mathbf{R}^m)}.$$

*Proof.* Let  $\varphi = (\varphi_1, \varphi_2) \in \mathbf{C}_c^\infty(\Omega; \mathbf{R}^m \times \mathbf{R}^{n-m})$ . Then

$$- \int_{\Omega} u \operatorname{div}(\mathcal{C}^T \varphi) \, dx = - \int_{\Omega} u \operatorname{div}((\mathcal{C}^h)^T(\varphi_1, 0)) \, dx = \int_{\Omega} \langle (X_1^h u, \dots, X_m^h u, 0, \dots, 0), \varphi \rangle_{\mathbf{R}^m} \, dx,$$

whence the first claims follow. Let  $(u_h)_h \subseteq W_{X^h}^{1,p}(\Omega)$  be such that  $u_h \rightarrow u$  weakly in  $L^p(\Omega)$ . Assume that  $\liminf_{h \rightarrow \infty} \|X^h u_h\|_{L^p(\Omega; \mathbf{R}^m)} < \infty$ . By reflexivity and (3.1), there exists  $v \in L^p(\Omega; \mathbf{R}^m)$  such that, up to a subsequence,  $X u_h \rightarrow v$  weakly in  $L^p(\Omega; \mathbf{R}^m)$ . Then  $u \in W_X^{1,p}(\Omega)$  and  $v = Xu$ . The thesis follows by the weak lower semicontinuity of the  $L^p$ -norm.  $\square$

**Proposition 3.8.** *Let  $(X^h)_h \in \mathcal{S}_2$ . Let  $1 < p < \infty$ . Let  $(u_h)_h \subseteq W_{X^h}^{1,p}(\Omega)$ .*

- (1) *If  $u_h \rightarrow u$  weakly in  $L^p(\Omega)$  and  $X^h u_h \rightarrow v$  weakly in  $L^p(\Omega; \mathbf{R}^m)$ , then  $u \in W_X^{1,p}(\Omega)$ .*
- (2) *If in addition  $u_h \rightarrow u$  strongly in  $L^p(\Omega)$ , then  $v = Xu$  and*

$$\|Xu\|_{L^p(\Omega; \mathbf{R}^m)} \leq \liminf_{h \rightarrow \infty} \|X^h u_h\|_{L^p(\Omega; \mathbf{R}^m)}.$$

*Proof.* Let  $p'$  the Hölder-conjugate exponent of  $p$ . Let  $(u_h)_h \subseteq W_{X^h}^{1,p}(\Omega)$  be such that  $u_h \rightarrow u$  weakly in  $L^p(\Omega)$ . By (S.2),

$$(3.2) \quad u_h c_{j,i}^h \rightarrow u c_{j,i} \text{ weakly in } L^p(\Omega)$$

for any  $i = 1, \dots, n$  and any  $j = 1, \dots, m$ . Fix  $\varphi \in \mathbf{C}_c^\infty(\Omega; \mathbf{R}^m)$ . Then

$$\begin{aligned} - \int_{\Omega} u \operatorname{div}(\mathcal{C}^T \varphi) \, dx &= - \int_{\Omega} u \sum_{i=1}^n \sum_{j=1}^m c_{j,i} \frac{\partial \varphi_j}{\partial x_i} \, dx - \int_{\Omega} u \sum_{i=1}^n \sum_{j=1}^m \frac{\partial c_{j,i}}{\partial x_i} \varphi_j \, dx \\ &\stackrel{(3.2)}{=} - \lim_{h \rightarrow \infty} \int_{\Omega} u_h \sum_{i=1}^n \sum_{j=1}^m c_{j,i}^h \frac{\partial \varphi_j}{\partial x_i} \, dx - \int_{\Omega} u \sum_{i=1}^n \sum_{j=1}^m \frac{\partial c_{j,i}}{\partial x_i} \varphi_j \, dx \\ &= \lim_{h \rightarrow \infty} \left( - \int_{\Omega} u_h \operatorname{div}((\mathcal{C}^h)^T \varphi) \, dx + \int_{\Omega} u_h \sum_{i=1}^n \sum_{j=1}^m \frac{\partial c_{j,i}^h}{\partial x_i} \varphi_j \, dx \right) - \int_{\Omega} u \sum_{i=1}^n \sum_{j=1}^m \frac{\partial c_{j,i}}{\partial x_i} \varphi_j \, dx \\ &= \lim_{h \rightarrow \infty} \left( \int_{\Omega} \langle X^h u_h, \varphi \rangle_{\mathbf{R}^m} \, dx + \int_{\Omega} u_h \sum_{i=1}^n \sum_{j=1}^m \frac{\partial c_{j,i}^h}{\partial x_i} \varphi_j \, dx \right) - \int_{\Omega} u \sum_{i=1}^n \sum_{j=1}^m \frac{\partial c_{j,i}}{\partial x_i} \varphi_j \, dx \\ &= \int_{\Omega} \langle v, \varphi \rangle_{\mathbf{R}^m} \, dx + \lim_{h \rightarrow \infty} \int_{\Omega} u_h \sum_{i=1}^n \sum_{j=1}^m \frac{\partial c_{j,i}^h}{\partial x_i} \varphi_j \, dx - \int_{\Omega} u \sum_{i=1}^n \sum_{j=1}^m \frac{\partial c_{j,i}}{\partial x_i} \varphi_j \, dx. \end{aligned}$$

By (S.2) and Hölder's inequality, if  $\|\varphi\|_{L^{p'}(\Omega; \mathbf{R}^m)} \leq 1$  the last line is bounded above independently of  $\varphi$ . In particular,  $u \in W_X^{1,p}(\Omega)$ . If in addition  $u_h \rightarrow u$  strongly in  $L^p(\Omega)$ , then the last two terms in the last line simplify by (S.2) and Remark 3.4. Hence  $v = Xu$  by definition. The thesis follows by the lower semicontinuity of the  $L^p$ -norm with respect to weak convergence.  $\square$

Our next result states the convergence of Rayleigh quotients when  $(X^h)_h \in \mathcal{S}_1$ . As we will show in Theorem 5.3, the same result holds in  $\mathcal{S}_2$  under the additional assumption (B).

**Proposition 3.9.** *Let  $(X^h)_h \in \mathcal{S}_1$ . Let  $1 < p < \infty$ . Then*

$$(3.3) \quad \mathcal{R} \leq \mathcal{R}_h \text{ for any } h \in \mathbf{N} \quad \text{and} \quad \lim_{h \rightarrow \infty} \mathcal{R}_h = \mathcal{R}.$$

*Proof.* Fix  $u \in \mathbf{C}_c^\infty(\Omega)$  such that  $u \neq 0$ . Then Proposition 3.7 implies that

$$\mathcal{R} \leq \frac{\int_{\Omega} |Xu|^p \, dx}{\int_{\Omega} |u|^p \, dx} \leq \frac{\int_{\Omega} |X^h u|^p \, dx}{\int_{\Omega} |u|^p \, dx},$$

whence the first property of (3.3) follows. On the other hand, fix  $u \in \mathbf{C}_c^\infty(\Omega)$  such that  $u \neq 0$ . By (A),  $X^h u \rightarrow Xu$  uniformly on  $\Omega$ , whence

$$\limsup_{h \rightarrow \infty} \mathcal{R}_h \leq \limsup_{h \rightarrow \infty} \frac{\int_{\Omega} |X^h u|^p dx}{\int_{\Omega} |u|^p dx} = \frac{\int_{\Omega} |Xu|^p dx}{\int_{\Omega} |u|^p dx}.$$

Being  $u \neq 0$  arbitrarily chosen in  $\mathbf{C}_c^\infty(\Omega)$ , (3.3) follows.  $\square$

Finally, we describe the relation between the approximating affine Sobolev spaces and the limit one under the sole assumption (A).

**Proposition 3.10.** *Let  $(X^h)_h$  satisfy (A). Let  $1 \leq p < \infty$ . Let  $(\varphi_h)_h \subseteq W_{X^h}^{1,p}(\Omega)$ , and let  $\varphi \in W_X^{1,p}(\Omega)$ . Assume that*

$$(3.4) \quad \varphi_h \rightarrow \varphi \text{ weakly in } L^p(\Omega) \quad \text{and} \quad X^h \varphi_h \rightarrow X\varphi \text{ weakly in } L^p(\Omega; \mathbf{R}^m).$$

Let  $(u_h)_h \subseteq W_{X^h, \varphi_h}^{1,p}(\Omega)$ , and let  $u \in W_X^{1,p}(\Omega)$ . Assume that

$$(3.5) \quad u_h \rightarrow u \text{ weakly in } L^p(\Omega) \quad \text{and} \quad X^h u_h \rightarrow Xu \text{ weakly in } L^p(\Omega; \mathbf{R}^m).$$

Then  $u \in W_{X, \varphi}^{1,p}(\Omega)$ .

*Proof.* We postpone the proof until the end of Section 4.2.  $\square$

#### 4. UNIFORM APPROXIMATION BY SMOOTH FUNCTIONS

In this section we establish the counterpart of Theorem 2.4, replacing a fixed  $X$ -gradient with a sequence  $(X^h)_h$  satisfying the sole assumption (A). In particular, our results will hold both in  $\mathcal{S}_1$  and in  $\mathcal{S}_2$ . Moreover, we provide a similar characterization for affine Sobolev spaces, and finally we extend the previous considerations to the anisotropic  $BV$  setting. Fix a sequence  $(X^h)_h$  satisfying (A). In particular, there exists a function  $\sigma : [0, \infty) \rightarrow (0, \infty)$  such that

$$(4.1) \quad \sup\{ |(\mathcal{C}^h(x) - \mathcal{C}(x))\xi| : x \in \Omega, \xi \in \mathbf{R}^n, |\xi| = 1 \} \leq \sigma(h)^{n+2}$$

for any  $h \in \mathbf{N}$  and  $\lim_{h \rightarrow \infty} \sigma(h) = 0$ . Exploiting (4.1), we introduce a particular sequence of mollifiers which is tailored to our setting. Precisely, Let  $J \in \mathbf{C}_c^\infty(\mathbf{R}^n)$  be a standard spherically symmetric mollifier. We define the family of mollifiers  $(J_h)_h$  by letting

$$(4.2) \quad J_h(x) = \frac{1}{\sigma(h)^n} J\left(\frac{x}{\sigma(h)}\right)$$

for any  $x \in \mathbf{R}^n$  and any  $h \in \mathbf{N}$ . Clearly, there is a constant  $C(J, n) > 0$ , depending only on  $J$  and  $n$ , such that

$$(4.3) \quad |DJ_h(x)| \leq \frac{C(J, n)}{\sigma(h)^{n+1}}$$

for any  $x \in \mathbf{R}^n$  and any  $h \in \mathbf{N}$ .

**4.1. A uniform Meyers-Serrin type approximation.** The following lemma constitute the technical core of this section.

**Lemma 4.1.** *Let  $(X^h)_h$  satisfy (A). Let  $1 \leq p < \infty$ ,  $u \in L^p(\Omega)$  and  $(J_h)_h$  be as in (4.2). Then*

$$\lim_{h \rightarrow \infty} \|X^h(J_h * u) - X(J_h * u)\|_{L^p(\Omega; \mathbf{R}^m)} = 0.$$

*Proof.* Notice that, in view of (4.1), (4.3) and Jensen's inequality,

$$\begin{aligned}
\|X^h(J_h * u) - X(J_h * u)\|_{L^p(\Omega; \mathbf{R}^m)}^p &= \int_{\Omega} |X^h(J_h * u) - X(J_h * u)|^p dx \\
&= \int_{\Omega} |(\mathcal{C}^h - \mathcal{C})D(J_h * u)|^p dx \\
&\stackrel{(4.1)}{\leq} \sigma(h)^{p(n+2)} \int_{\Omega} |D(J_h * u)|^p dx \\
&= \sigma(h)^{p(n+2)} \int_{\Omega} |(DJ_h * u)|^p dx \\
&= \sigma(h)^{p(n+2)} \int_{\Omega} \left| \int_{B(x, \sigma(h))} DJ_h(x-y)u(y) dy \right|^p dx \\
&= (\omega_n)^p \sigma(h)^{p(2n+2)} \int_{\Omega} \left| \int_{B(x, \sigma(h))} DJ_h(x-y)u(y) dy \right|^p dx \\
&\leq (\omega_n)^{p-1} \sigma(h)^{p(2n+2)-n} \int_{\Omega} \int_{B(x, \sigma(h))} |DJ_h(x-y)u(y)|^p dy dx \\
&\stackrel{(4.3)}{\leq} C(J, n)^p (\omega_n)^{p-1} \sigma(h)^{p(2n+2)-n-p(n+1)} \int_{\Omega} \int_{B(x, \sigma(h))} |u(y)|^p dy dx \\
&\leq |\Omega| C(J, n)^p (\omega_n)^{p-1} \sigma(h)^{p(2n+2)-n-p(n+1)} \|u\|_{L^p(\Omega)}^p,
\end{aligned}$$

where  $\omega_n$  is the Lebesgue measure of the Euclidean unit ball of  $\mathbf{R}^n$ . Since

$$(4.4) \quad p(2n+2) - n - p(n+1) = n(p-1) + p \geq p > 0,$$

the thesis follows.  $\square$

In view of Lemma 4.1, we prove the following Meyers-Serrin type approximation result.

**Theorem 4.2.** *Let  $(X^h)_h$  satisfy (A). Let  $1 \leq p < \infty$ . Let  $u \in W_X^{1,p}(\Omega)$ . Then there exists a sequence  $(u_h)_h \subseteq \mathbf{C}^\infty(\Omega) \cap W_{X^h}^{1,p}(\Omega)$  such that*

$$u_h \rightarrow u \text{ strongly in } L^p(\Omega) \quad \text{and} \quad X^h u_h \rightarrow Xu \text{ strongly in } L^p(\Omega; \mathbf{R}^m).$$

*Proof.* Let  $u \in W_X^{1,p}(\Omega)$ , and fix an open set  $\tilde{\Omega} \Subset \Omega$ . For any  $h \in \mathbf{N}$ , define  $u_h = J_h * u$ , being  $(J_h)_h$  as in (4.2). First, [20, Proposition 1.2.2] implies that  $u_h \rightarrow u$  strongly in  $L^p(\tilde{\Omega})$  and  $Xu_h \rightarrow Xu$  strongly in  $L^p(\tilde{\Omega}, \mathbf{R}^m)$ . Exploiting Lemma 4.1, we infer that  $X^h u_h \rightarrow Xu$  strongly in  $L^p(\tilde{\Omega}, \mathbf{R}^m)$ . The global statement follows *verbatim* as in the proof of [22, Theorem A.2].  $\square$

**4.2. A uniform characterization of affine Sobolev spaces.** In this section we provide an affine counterpart of Theorem 4.2. To this aim, for  $1 \leq p < \infty$  and  $\varphi \in W_X^{1,p}(\Omega)$ , set

$$\begin{aligned}
W_{X,\varphi}^{1,p}(\Omega, (X^h)_h) &:= \{u \in W_X^{1,p}(\Omega) : \text{there exists } (v_h)_h \subseteq \mathbf{C}_c^\infty(\Omega) \text{ such that} \\
&\quad v_h \rightarrow u - \varphi \text{ strongly in } L^p(\Omega) \text{ and } X^h v_h \rightarrow X(u - \varphi) \text{ strongly in } L^p(\Omega; \mathbf{R}^m)\}.
\end{aligned}$$

This affine space depends *a priori* on  $(X^h)_h$ . However, the following characterization holds.

**Theorem 4.3.** *Let  $(X^h)_h$  satisfy (A). Let  $1 \leq p < \infty$ . Let  $\varphi \in W_X^{1,p}(\Omega)$ . Then*

$$W_{X,\varphi}^{1,p}(\Omega) = W_{X,\varphi}^{1,p}(\Omega, (X^h)_h).$$

*Proof.* It suffices to consider the case  $\varphi = 0$ . Fix  $u \in W_{X,0}^{1,p}(\Omega, (X^h)_h)$ . By definition, there exists a sequence  $(u_h)_h \subseteq \mathbf{C}_c^\infty(\Omega)$  such that

$$(4.5) \quad u_h \rightarrow u \text{ strongly in } L^p(\Omega) \quad \text{and} \quad X^h u_h \rightarrow Xu \text{ strongly in } L^p(\Omega; \mathbf{R}^m).$$

Let  $(J_h)_h$  be the sequence of mollifiers defined by (4.2). Since  $\text{supp } u_h \Subset \Omega$  for any  $h \in \mathbf{N}$ , up to consider a subsequence of  $(J_h)_h$  we may assume that  $\text{supp } J_h \Subset \Omega$  and  $\sigma(h) \leq 1$  for any  $h \in \mathbf{N}$ , where  $\sigma$  is the modulus of continuity introduced in (4.1). For any  $h \in \mathbf{N}$ , set  $v_h = J_h * u_h$ . Then  $(v_h)_h \subseteq \mathbf{C}_c^\infty(\Omega)$ . Owing to [20, Proposition 1.2.2], we can select a further subsequence of  $(J_h)_h$  such that

$$(4.6) \quad \|u_h - v_h\|_{L^p(\Omega)} \leq \frac{1}{h} \quad \text{and} \quad \|X^h u_h - X^h v_h\|_{L^p(\Omega; \mathbf{R}^m)} \leq \frac{1}{h}$$

for any  $h \in \mathbf{N}$ . Moreover, arguing as in the proof of Lemma 4.1,

$$(4.7) \quad \|X^h v_h - X v_h\|_{L^p(\Omega; \mathbf{R}^m)}^p \leq |\Omega| C(J, n)^p (\omega_n)^{p-1} \sigma(h)^{p(2n+2)-n-p(n+1)} \|u_h\|_{L^p(\Omega)}^p.$$

By (4.4), (4.5), (4.6) and (4.7),  $v_h \rightarrow u$  strongly in  $L^p(\Omega)$  and  $X v_h \rightarrow X u$  strongly in  $L^p(\Omega; \mathbf{R}^m)$ , that is  $u \in W_{X,0}^{1,p}(\Omega)$ . Since the converse implication follows *verbatim*, the thesis follows.  $\square$

With a similar argument, we provide the proof of Proposition 3.10.

*Proof of Proposition 3.10.* Fix  $h \in \mathbf{N}$ . Since  $u_h \in W_{X^h, \varphi_h}^{1,p}(\Omega)$ , there exists  $v_h \in \mathbf{C}_c^\infty(\Omega)$  such that

$$(4.8) \quad \|v_h - (u_h - \varphi_h)\|_{L^p(\Omega)} \leq \frac{1}{h} \quad \text{and} \quad \|X^h v_h - X^h(u_h - \varphi_h)\|_{L^p(\Omega; \mathbf{R}^m)} \leq \frac{1}{h}.$$

Let  $(J_h)_h$  be the sequence of mollifiers defined by (4.2). As in the previous proof, we may assume that  $\text{supp } v_h + \text{supp } J_h \Subset \Omega$  and  $\sigma(h) \leq 1$  for any  $h \in \mathbf{N}$ , where  $\sigma$  is the modulus of continuity introduced in (4.1). For any  $h \in \mathbf{N}$ , set  $\tilde{v}_h = J_h * v_h$ . Then  $(\tilde{v}_h)_h \subseteq \mathbf{C}_c^\infty(\Omega)$ . Again by [20, Proposition 1.2.2], up to a further subsequence of  $(J_h)_h$ ,

$$(4.9) \quad \|v_h - \tilde{v}_h\|_{L^p(\Omega)} \leq \frac{1}{h} \quad \text{and} \quad \|X^h v_h - X^h \tilde{v}_h\|_{L^p(\Omega; \mathbf{R}^m)} \leq \frac{1}{h}$$

for any  $h \in \mathbf{N}$ . Again, arguing as in Lemma 4.1,

$$(4.10) \quad \|X^h \tilde{v}_h - X \tilde{v}_h\|_{L^p(\Omega; \mathbf{R}^m)}^p \leq |\Omega| C(J, n)^p (\omega_n)^{p-1} \sigma(h)^{p(2n+2)-n-p(n+1)} \|v_h\|_{L^p(\Omega)}^p.$$

By (3.4) and (3.5),  $(u_h - \varphi_h)_h$  is bounded in  $L^p(\Omega)$ . Combining this fact with (4.4), (4.8) and (4.10), we conclude that

$$(4.11) \quad \lim_{h \rightarrow \infty} \|X^h \tilde{v}_h - X \tilde{v}_h\|_{L^p(\Omega; \mathbf{R}^m)} = 0.$$

Let  $p'$  be the Hölder-conjugate exponent of  $p$ . Let  $\psi \in L^{p'}(\Omega)$ . Then

$$\begin{aligned} \left| \int_{\Omega} (\tilde{v}_h - (u - \varphi)) \psi \, dx \right| &\leq \int_{\Omega} |\tilde{v}_h - v_h| |\psi| \, dx + \int_{\Omega} |v_h - (u_h - \varphi_h)| |\psi| \, dx \\ &\quad + \left| \int_{\Omega} (u - u_h) \psi \, dx \right| + \left| \int_{\Omega} (\varphi - \varphi_h) \psi \, dx \right| \\ &\stackrel{(4.8), (4.9)}{\leq} \frac{2}{h} \|\psi\|_{L^{p'}(\Omega)} + \left| \int_{\Omega} (u - u_h) \psi \, dx \right| + \left| \int_{\Omega} (\varphi - \varphi_h) \psi \, dx \right|. \end{aligned}$$

Letting  $h \rightarrow \infty$ , recalling (3.4) and (3.5), and being  $\psi$  arbitrary in  $L^{p'}(\Omega)$ ,  $\tilde{v}_h \rightarrow (u - \varphi)$  weakly in  $L^p(\Omega)$ . Similarly, if  $\psi \in L^{p'}(\Omega, \mathbf{R}^m)$ ,

$$\begin{aligned} \left| \int_{\Omega} \langle X\tilde{v}_h - X(u - \varphi), \psi \rangle_{\mathbf{R}^m} dx \right| &\leq \int_{\Omega} |\langle X\tilde{v}_h - X^h\tilde{v}_h, \psi \rangle_{\mathbf{R}^m}| dx + \int_{\Omega} |\langle X^h\tilde{v}_h - X^h v_h, \psi \rangle_{\mathbf{R}^m}| dx \\ &\quad + \int_{\Omega} |\langle X^h v_h - X^h(u_h - \varphi_h), \psi \rangle_{\mathbf{R}^m}| dx \\ &\quad + \left| \int_{\Omega} \langle Xu - X^h u_h, \psi \rangle_{\mathbf{R}^m} dx \right| + \left| \int_{\Omega} \langle X\varphi - X^h \varphi_h, \psi \rangle_{\mathbf{R}^m} dx \right| \\ &\stackrel{(4.8), (4.9)}{\leq} \left( \|X^h\tilde{v}_h - X\tilde{v}_h\|_{L^p(\Omega; \mathbf{R}^m)} + \frac{2}{h} \right) \|\psi\|_{L^{p'}(\Omega, \mathbf{R}^m)} \\ &\quad + \left| \int_{\Omega} \langle Xu - X^h u_h, \psi \rangle_{\mathbf{R}^m} dx \right| + \left| \int_{\Omega} \langle X\varphi - X^h \varphi_h, \psi \rangle_{\mathbf{R}^m} dx \right|. \end{aligned}$$

Combining (3.4), (3.5) and (4.11), we conclude that  $X\tilde{v}_h \rightarrow X(u - \varphi)$  weakly in  $L^p(\Omega, \mathbf{R}^m)$ . Therefore, by [10, Proposition 2.2],  $\tilde{v}_h \rightarrow u - \varphi$  weakly in  $W_X^{1,p}(\Omega)$ . Arguing exactly as in the proof [10, Theorem 1.1], we conclude that  $u \in W_{X,\varphi}^{1,p}(\Omega)$ .  $\square$

**4.3. A uniform Anzellotti-Giaquinta type approximation.** Although beyond the scopes of this paper, for the sake of completeness we provide an Anzellotti-Giaquinta approximation result in the case of *functions of  $X$ -bounded variation*. We refer to [20] for their main definitions and properties. Again, the key of the proof is encoded in the following lemma.

**Lemma 4.4.** *Let  $(X^h)_h$  satisfy (A). Let  $u \in L^1(\Omega)$ . Let  $(J_h)_h$  be as in (4.2). Then*

$$\limsup_{h \rightarrow \infty} \left\{ \int_{\Omega} u \operatorname{div} \left( ((\mathcal{C}^h)^T \varphi) * J_h - \mathcal{C}^T(\varphi * J_h) \right) dx : \varphi \in \mathbf{C}_c^\infty(\Omega; \mathbf{R}^n), \|\varphi\|_\infty \leq 1 \right\} = 0.$$

*Proof.* Let  $\varphi$  be as in the statement, and set

$$S = \int_{\Omega} u \operatorname{div} \left( ((\mathcal{C}^h)^T \varphi) * J_h - \mathcal{C}^T(\varphi * J_h) \right) dx.$$

Then

$$\begin{aligned} S &= \sum_{i=1}^n \int_{\Omega} u \frac{\partial}{\partial x_i} \left[ ((\mathcal{C}^h)^T \varphi)_i * J_h - (\mathcal{C}^T(\varphi * J_h))_i \right] dx \\ &= \sum_{i=1}^n \sum_{j=1}^m \int_{\Omega} u \frac{\partial}{\partial x_i} \int_{\mathbf{R}^n} (c_{j,i}^h(y) - c_{j,i}(x)) \varphi_j(y) J_h(x - y) dy dx \\ &= \sum_{i=1}^n \sum_{j=1}^m \int_{\Omega} u \int_{\mathbf{R}^n} (c_{j,i}^h(y) - c_{j,i}(x)) \varphi_j(y) \frac{\partial J_h}{\partial x_i}(x - y) dy dx \\ &\quad - \sum_{i=1}^n \sum_{j=1}^m \int_{\Omega} u \int_{\mathbf{R}^n} \frac{\partial c_{j,i}^h}{\partial x_i}(x) \varphi_j(y) J_h(x - y) dy dx \\ &= \sum_{i=1}^n \sum_{j=1}^m \int_{\Omega} u \int_{\mathbf{R}^n} (c_{j,i}^h(y) - c_{j,i}(y)) \varphi_j(y) \frac{\partial J_h}{\partial x_i}(x - y) dy dx \\ &\quad + \sum_{i=1}^n \sum_{j=1}^m \int_{\Omega} u \int_{\mathbf{R}^n} (c_{j,i}(y) - c_{j,i}(x)) \varphi_j(y) \frac{\partial J_h}{\partial x_i}(x - y) dy dx \\ &\quad - \sum_{i=1}^n \sum_{j=1}^m \int_{\Omega} u \int_{\mathbf{R}^n} \frac{\partial c_{j,i}^h}{\partial x_i}(x) \varphi_j(y) J_h(x - y) dy dx. \end{aligned}$$

Let us denote respectively by I, II and III the terms in the last three lines. Arguing *verbatim* as in the proof of [20, Lemma 2.1.1], we infer that  $|\text{II}| + |\text{III}| \rightarrow 0$  as  $h \rightarrow +\infty$  uniformly with respect to  $\varphi$ . Therefore we are left to estimate I. In view of (4.1) and (4.3), we get that

$$\begin{aligned} |\text{I}| &\stackrel{(4.1)}{\leq} \sigma(h)^{n+2} \sum_{i=1}^n \sum_{j=1}^m \int_{\Omega} |u| \int_{\mathbf{R}^n} \left| \varphi_j(y) \frac{\partial J_h}{\partial x_i}(x-y) \right| dy dx \\ &\stackrel{(4.3)}{\leq} C(J, n) \sigma(h) \sum_{i=1}^n \sum_{j=1}^m \int_{\Omega} |u| \int_{\mathbf{R}^n} |\varphi_j(y)| dy dx \\ &\leq m n C(J, n) \sigma(h) \int_{\Omega} |u| dx. \end{aligned}$$

Therefore I goes to 0 as  $h \rightarrow +\infty$  uniformly with respect to  $\varphi$ , whence the thesis follows.  $\square$

The anisotropic Anzellotti-Giaquinta approximation result reads as follows.

**Theorem 4.5.** *Let  $(X^h)_h$  satisfy (A). Let  $u \in BV_X(\Omega)$ . There exists  $(u_h)_h \subseteq \mathbf{C}^\infty(\Omega) \cap W_{X^h}^{1,1}(\Omega)$  such that*

$$u_h \rightarrow u \text{ strongly in } L^p(\Omega) \quad \text{and} \quad \lim_{h \rightarrow \infty} \int_{\Omega} |X^h u_h| dx = \int_{\Omega} d|Xu|.$$

*Proof.* Owing to Lemma 4.4, the proof follows *verbatim* as in [20].  $\square$

## 5. UNIFORM COMPACTNESS AND POINCARÉ INEQUALITIES

In this section we prove the counterparts of Theorem 2.6 and Proposition 2.5 replacing a fixed anisotropy with a sequence  $(X^h)_h$  satisfying (A) and (B). In particular, our results will hold both in  $\mathcal{T}_1$  and in  $\mathcal{T}_2$ . Before going on, we point out that neither the assumptions of  $\mathcal{S}_1$  nor those of  $\mathcal{S}_2$  are strong enough to guarantee compactness, as the next example shows.

**Example 5.1.** For any  $h \in \mathbf{N}$ , consider the family  $X^h = (X_1, X_2^h)$  defined on  $\mathbf{R}_{(x_1, x_2)}^2$  by

$$X_1 = \frac{\partial}{\partial x_1} \quad \text{and} \quad X_2^h = \frac{1}{h} \frac{\partial}{\partial x_2}.$$

For any  $h \in \mathbf{N}$ ,  $X_1$  and  $X_2^h$  are smooth and Lipschitz continuous. Moreover,  $(X^h)_h$  converges uniformly, with all its derivatives, to  $X = (X_1, 0)$ , whence  $(X^h)_h \in \mathcal{S}_1 \cap \mathcal{S}_2$ . Let  $\Omega = (-\pi, \pi)^2$ . For any  $h \in \mathbf{N}$ , set  $u_h(x_1, x_2) := \sin(hx_2)$  for any  $(x_1, x_2) \in \mathbf{R}^2$ . Then  $(u_h)_h \subseteq \mathbf{C}^\infty(\bar{\Omega})$  and  $X^h u_h(x_1, x_2) = (0, \cos(hx_2))$  for any  $(x_1, x_2) \in \mathbf{R}^2$ . In particular,  $\|u_h\|_{L^2(\Omega)} = \|X^h u_h\|_{L^2(\Omega; \mathbf{R}^m)} = \pi\sqrt{2}$  for any  $h \in \mathbf{N}$ . By Riemann-Lebesgue lemma,  $u_h \rightarrow 0$  weakly in  $L^2(\Omega)$ , but it does not admit any subsequence converging strongly in  $L^2(\Omega)$ .

Our first result is the following uniform Rellich-Kondrachov compactness property.

**Theorem 5.2** (Uniform Rellich-Kondrachov). *Let  $(X^h)_h$  satisfy (A) and (B). Let  $1 < p < \infty$ . Let  $(\varphi_h)_h \subseteq W_{X^h}^{1,p}(\Omega)$ , and let  $\varphi \in W_X^{1,p}(\Omega)$ . Assume that*

$$(5.1) \quad \varphi_h \rightarrow \varphi \text{ strongly in } L^p(\Omega) \quad \text{and} \quad X^h \varphi_h \rightarrow X\varphi \text{ weakly in } L^p(\Omega; \mathbf{R}^m).$$

*Let  $(u_h)_h \subseteq W_{X^h, \varphi_h}^{1,p}(\Omega)$ . Assume that there exists  $M > 0$  such that*

$$(5.2) \quad \|u_h\|_{L^p(\Omega)}, \|X^h u_h\|_{L^p(\Omega; \mathbf{R}^m)} \leq M \text{ for any } h \in \mathbf{N}.$$

*Then there exists  $u \in W_{X, \varphi}^{1,p}(\Omega)$  such that, up to a subsequence,  $u_h \rightarrow u$  strongly in  $L^p(\Omega)$  and  $X^h u_h \rightarrow Xu$  weakly in  $L^p(\Omega; \mathbf{R}^m)$ .*

*Proof.* Assume  $\varphi_h, \varphi = 0$  for any  $h \in \mathbf{N}$ . By definition, there exists  $(v_h)_h \subseteq \mathbf{C}_c^\infty(\Omega)$  such that

$$(5.3) \quad \|u_h - v_h\|_{L^p(\Omega)} \leq \frac{1}{h} \quad \text{and} \quad \|X^h u_h - X^h v_h\|_{L^p(\Omega; \mathbf{R}^m)} \leq \frac{1}{h}$$

for any  $h \in \mathbf{N}$ . Let  $(J_h)_h$  be the sequence of mollifiers defined by (4.2). Arguing as in the proof of Theorem 4.3, we may assume that  $\text{supp } v_h + \text{supp } J_h \Subset \Omega$  and  $\sigma(h) \leq 1$  for any  $h \in \mathbf{N}$ , where  $\sigma$  is defined in (4.1). For any  $h \in \mathbf{N}$ , set  $\tilde{v}_h = J_h * v_h$ . Then  $(\tilde{v}_h)_h \subseteq \mathbf{C}_c^\infty(\Omega)$ . Moreover, arguing again as in the proof of Theorem 4.3, we may assume that

$$(5.4) \quad \|v_h - \tilde{v}_h\|_{L^p(\Omega)} \leq \frac{1}{h} \quad \text{and} \quad \|X^h v_h - X^h \tilde{v}_h\|_{L^p(\Omega; \mathbf{R}^m)} \leq \frac{1}{h}$$

and

$$(5.5) \quad \|X^h \tilde{v}_h - X \tilde{v}_h\|_{L^p(\Omega; \mathbf{R}^m)}^p \leq |\Omega| C(J, n)^p (\omega_n)^{p-1} \sigma(h)^{p(2n+2)-n-p(n+1)} \|v_h\|_{L^p(\Omega)}^p.$$

Combining (4.4), (5.2), (5.3), (5.4) and (5.5),  $(\tilde{v}_h)_h$  is bounded in  $W_X^{1,p}(\Omega)$ . Therefore, by (B) and reflexivity, there exists  $u \in W_{X,0}^{1,p}(\Omega)$  such that, up to a subsequence,  $\tilde{v}_h \rightarrow u$  strongly in  $L^p(\Omega)$  and  $X \tilde{v}_h \rightarrow Xu$  weakly in  $L^p(\Omega; \mathbf{R}^m)$ . By (5.3) and (5.4),  $u_h \rightarrow u$  strongly in  $L^p(\Omega)$ . Moreover, again by (5.3), (5.4) and (5.5),  $X^h u_h \rightarrow Xu$  weakly in  $L^p(\Omega; \mathbf{R}^m)$ . Finally, we argue in the case of arbitrary  $(\varphi_h)_h$  and  $\varphi$  satisfying (5.1). Let  $(u_h)_h \subseteq W_{X^h, \varphi_h}^{1,p}(\Omega)$  satisfy (5.2). Then, by (5.1),  $(u_h - \varphi_h)_h \subseteq W_{X^h, 0}^{1,p}(\Omega)$  satisfies (5.2) with a possibly bigger  $M > 0$ . Therefore, there exists  $v \in W_{X,0}^{1,p}(\Omega)$  such that  $u_h - \varphi_h \rightarrow v$  strongly in  $L^p(\Omega)$  and  $X^h u_h - X^h \varphi_h \rightarrow Xv$  weakly in  $L^p(\Omega; \mathbf{R}^m)$ . Set  $u = v + \varphi$ . Then  $u \in W_{X, \varphi}^{1,p}(\Omega)$ ,  $u_h \rightarrow u$  strongly in  $L^p(\Omega)$  and  $X^h u_h \rightarrow Xv$  weakly in  $L^p(\Omega; \mathbf{R}^m)$  by (5.1). The thesis follows.  $\square$

As a corollary of Theorem 5.2, we recover the convergence of Rayleigh quotients, in analogy with Proposition 3.9, and a uniform Poincaré inequality.

**Theorem 5.3** (Uniform Poincaré inequality). *Let  $(X^h)_h$  satisfy (A) and (B). Let  $1 < p < \infty$ . Then*

$$(5.6) \quad \lim_{h \rightarrow \infty} \mathcal{R}_h = \mathcal{R}.$$

If  $\Omega$  is connected, then  $\mathcal{R} > 0$ . In particular, up to a subsequence,

$$(5.7) \quad \int_{\Omega} |u|^p dx \leq \frac{2}{\mathcal{R}} \int_{\Omega} |X^h u|^p dx$$

for any  $h \in \mathbf{N}$  and any  $u \in W_{X^h, 0}^{1,p}(\Omega)$ . Finally, if  $(\varphi_h)_h \subseteq W_{X^h}^{1,p}(\Omega)$ , then

$$(5.8) \quad \int_{\Omega} |u|^p dx \leq \frac{2^{2p-1}}{\mathcal{R}} \int_{\Omega} |X^h u|^p dx + \frac{2^{2p-1}}{\mathcal{R}} \int_{\Omega} |X^h \varphi_h|^p dx + 2^{p-1} \int_{\Omega} |\varphi_h|^p dx$$

for any  $h \in \mathbf{N}$  and any  $u \in W_{X^h, \varphi_h}^{1,p}(\Omega)$ .

*Proof.* Arguing *verbatim* as in the proof of Proposition 3.9,

$$(5.9) \quad \limsup_{h \rightarrow \infty} \mathcal{R}_h \leq \mathcal{R}.$$

Let us prove the converse inequality. By definition, for any  $h \in \mathbf{N}$  there exists  $u_h \in \mathbf{C}_c^\infty(\Omega)$  such that  $u_h \neq 0$  and

$$(5.10) \quad \frac{\int_{\Omega} |X^h u_h|^p dx}{\int_{\Omega} |u_h|^p dx} \leq \mathcal{R}_h + \frac{1}{h}.$$

For any  $h \in \mathbf{N}$ , define  $v_h = (\|u_h\|_{L^p(\Omega)})^{-1} u_h$ . Notice that, by (5.10),

$$(5.11) \quad \frac{\int_{\Omega} |X^h v_h|^p dx}{\int_{\Omega} |v_h|^p dx} = \frac{\int_{\Omega} |X^h u_h|^p dx}{\int_{\Omega} |u_h|^p dx} \leq \mathcal{R}_h + \frac{1}{h}.$$

Moreover, by definition,  $(v_h)_h \subseteq \mathbf{C}_c^\infty(\Omega)$ ,  $\|v_h\|_{L^p(\Omega)} = 1$  for any  $h \in \mathbf{N}$  and

$$\limsup_{h \rightarrow \infty} \|X^h v_h\|_p^p = \limsup_{h \rightarrow \infty} \frac{\int_{\Omega} |X^h u_h|^p dx}{\int_{\Omega} |u_h|^p dx} \leq \limsup_{h \rightarrow \infty} \left( \mathcal{R}_h + \frac{1}{h} \right) \leq \mathcal{R}$$

by (5.9). Therefore, by Theorem 5.2, up to a subsequence there exists  $v \in W_{X,0}^{1,p}(\Omega)$  such that  $v_h \rightarrow v$  strongly in  $L^p(\Omega)$  and  $X^h v_h \rightarrow Xv$  weakly in  $L^p(\Omega; \mathbf{R}^m)$ . Moreover,  $\|v\|_{L^p(\Omega)} = 1$ , so that  $v \neq 0$ . Then, we conclude that

$$\mathcal{R} \leq \frac{\int_{\Omega} |Xv|^p dx}{\int_{\Omega} |v|^p dx} \leq \liminf_{h \rightarrow \infty} \frac{\int_{\Omega} |X^h v_h|^p dx}{\int_{\Omega} |v_h|^p dx} \leq \liminf_{h \rightarrow \infty} \left( \mathcal{R}_h + \frac{1}{h} \right) = \liminf_{h \rightarrow \infty} \mathcal{R}_h,$$

whence (5.6). The fact that  $\mathcal{R} > 0$  and (5.7) follow by Proposition 2.5 and (5.6). Finally, (5.8) is straightforward (cf. e.g. [10, Corollary 2.6]).  $\square$

## 6. $\Gamma$ -CONVERGENCE FOR FUNCTIONALS DEPENDING ON MOVING ANISOTROPIES

In this section we establish several  $\Gamma$ -convergence properties for sequences of integral functionals depending on a sequence  $(X^h)_h$  belonging to either  $\mathcal{S}_1$  or  $\mathcal{S}_2$ . Let us recall some basic preliminaries. For a complete account to  $\Gamma$ -convergence, as well as for the vocabulary of local functionals, we refer the reader to [3, 13]. We just recall that if  $(\mathcal{X}, \tau)$  is a first-countable topological space, a sequence of functionals  $(F_h)_h : \mathcal{X} \rightarrow [0, \infty]$  is said to  $\Gamma(\tau)$ -converge to a functional  $F : \mathcal{X} \rightarrow [0, \infty]$  if the following two conditions hold.

- For any  $u \in \mathcal{X}$  and any sequence  $(u_h)_h$  converging to  $u$  in  $\tau$ , then

$$(\text{liminf}) \quad F(u) \leq \liminf_{h \rightarrow \infty} F_h(u_h).$$

- For any  $u \in \mathcal{X}$ , there exists a sequence  $(u_h)_h$  converging to  $u$  in  $\tau$  such that

$$(\text{limsup}) \quad F(u) \geq \limsup_{h \rightarrow \infty} F_h(u_h).$$

Sequences for which (limsup) holds are known as *recovery sequences*. Equivalently, defining the  $\Gamma$ -lower limit and  $\Gamma$ -upper limit respectively by

$$F'(u) := \Gamma - \liminf_{h \rightarrow \infty} F_h(u) := \inf \left\{ \liminf_{h \rightarrow \infty} F_h(u_h) : u_h \rightarrow u \text{ in } \tau \right\}$$

and

$$F''(u) := \Gamma - \limsup_{h \rightarrow \infty} F_h(u) := \inf \left\{ \limsup_{h \rightarrow \infty} F_h(u_h) : u_h \rightarrow u \text{ in } \tau \right\},$$

$(F_h)_h$   $\Gamma$ -converges to  $F : (X, \tau) \rightarrow [0, \infty]$  if and only if

$$\Gamma - \liminf_{h \rightarrow \infty} F_h(u) = \Gamma - \limsup_{h \rightarrow \infty} F_h(u) = F(u)$$

for any  $u \in X$ . We say that  $F$  is the  $\Gamma$ -limit of  $(F_h)_h$  and we write  $F = \Gamma - \lim_{h \rightarrow \infty} F_h$ . Throughout this section, we fix a sequence  $(X^h)_h \subseteq \mathcal{S}_1 \cup \mathcal{S}_2$ , an open and bounded set  $\Omega \subseteq \mathbf{R}^n$  and  $1 < p < \infty$ . The wider class of functionals we are interested in is made of (local) integral functionals of the form

$$F_h(u, A) := \begin{cases} \int_{\Omega} f_h(x, X^h u(x)) dx & \text{if } A \in \mathcal{A}, u \in L^p(\Omega) \cap W_{X^h}^{1,p}(A), \\ \infty & \text{otherwise in } L^p(\Omega), \end{cases}$$

where each integrand  $f_h$  belongs to the following class.

**Definition 6.1.** Fix  $h \in \mathbf{N}$ .  $\mathcal{I}_p^h(a, d_1, d_2)$  is the class of functions  $f_h : \Omega \times \mathbf{R}^m \rightarrow \mathbf{R}$  such that:

- (I<sub>1</sub>)  $f_h : \Omega \times \mathbf{R}^m \rightarrow \mathbf{R}$  is a Carathéodory function;
- (I<sub>2</sub>) for a.e.  $x \in \Omega$ , the function  $f_h(x, \cdot) : \mathbf{R}^m \rightarrow \mathbf{R}$  is convex;

(I<sub>3</sub>) there exist two constants  $0 < d_1 \leq d_2$  and a non-negative function  $a \in L^1(\Omega)$  such that

$$(6.1) \quad d_1 |\mathcal{C}^h(x)\xi|^p \leq f_h(x, \mathcal{C}^h(x)\xi) \leq a(x) + d_2 |\mathcal{C}^h(x)\xi|^p$$

for a.e.  $x \in \Omega$  and for every  $\xi \in \mathbf{R}^n$ .

**Remark 6.2.** If  $F_h$  is the (local) integral functional associated with some  $f_h \in \mathcal{I}_p^h(a, d_1, d_2)$ , then [40, Theorem 4.2] implies that

$$(6.2) \quad d_1 \|X^h u\|_{L^p(A)}^p \leq F_h(u, A) \leq \int_A a(x) dx + d_2 \|X^h u\|_{L^p(A)}^p$$

for any  $A \in \mathcal{A}$  and any  $u \in W_{X^h}^{1,p}(A)$ .

**6.1.  $\Gamma$ -convergence of norms.** Our first result focuses on the special case of norms. [Theorem 6.3](#) will be propaedeutic to the proof of the main compactness theorem, [Theorem 6.5](#).

**Theorem 6.3.** Let  $(X^h)_h \in \mathcal{S}_1 \cup \mathcal{S}_2$ . Let  $1 < p < \infty$ . For  $h \in \mathbf{N}$ , set  $\Psi_p^h : L^p(\Omega) \times \mathcal{A} \rightarrow [0, \infty]$  and  $\Psi_p : L^p(\Omega) \times \mathcal{A} \rightarrow [0, \infty]$  respectively by

$$\Psi_p^h(u, A) = \begin{cases} \int_A |X^h u|^p dx & \text{if } A \in \mathcal{A}, u \in W_{X^h}^{1,p}(A), \\ \infty & \text{otherwise,} \end{cases}$$

and

$$\Psi_p(u, A) = \begin{cases} \int_A |Xu|^p dx & \text{if } A \in \mathcal{A}, u \in W_X^{1,p}(A), \\ \infty & \text{otherwise.} \end{cases}$$

Then,

$$\Psi_p(\cdot, A) = \Gamma(L^p) - \lim_{h \rightarrow \infty} \Psi_p^h(\cdot, A)$$

for any  $A \in \mathcal{A}$ .

*Proof.* Fix  $A \in \mathcal{A}$  and  $u \in L^p(\Omega)$ . Assume first that  $u \notin W_X^{1,p}(A)$ . Then,  $\Psi_p(u, A) = \infty$ , so that (limsup) follows with  $u_h = u$ . On the other hand, if there exists  $(u_h)_h \in L^p(\Omega)$  such that  $u_h \rightarrow u$  in  $L^p(\Omega)$  and  $\liminf_{h \rightarrow \infty} \Psi_p^h(u_h, A) < \infty$ , then either [Proposition 3.7](#) or [Proposition 3.8](#) would imply that  $u \in W_X^{1,p}(A)$ , a contradiction. Instead, assume  $u \in W_X^{1,p}(A)$ . The existence of a sequence as in (limsup) follows at once from [Theorem 4.2](#), while (liminf) follows as above in view of either [Proposition 3.7](#) or [Proposition 3.8](#).  $\square$

**Remark 6.4.** Arguing as in the proof of [Theorem 6.3](#), we may show  $\Gamma$ -convergence of complete moving anisotropic Sobolev norms.

**6.2.  $\Gamma$ -compactness in the general case.** Owing to [Theorem 6.3](#), we can prove compactness for arbitrary classes of integral functionals.

**Theorem 6.5.** Let  $(X^h)_h \in \mathcal{S}_1 \cup \mathcal{S}_2$ . Let  $1 < p < \infty$ . Let  $a \in L^1(\Omega)$ ,  $a \geq 0$ , and  $0 < d_1 \leq d_2$ . Let  $(f^h)_h \subseteq \mathcal{I}_p^h(a, d_1, d_2)$ . Denote by  $(F_h)_h : L^p(\Omega) \times \mathcal{A} \rightarrow [0, \infty]$  the corresponding sequence of integral functionals, each represented by

$$(6.3) \quad F_h(u, A) = \begin{cases} \int_A f_h(x, X^h u(x)) dx & \text{if } A \in \mathcal{A}, u \in W_{X^h}^{1,p}(A), \\ \infty & \text{otherwise.} \end{cases}$$

Then there exist  $F : L^p(\Omega) \times \mathcal{A} \rightarrow [0, \infty]$  and  $f \in \mathcal{I}_p^\infty(a, d_1, d_2)$  such that, up to a subsequence,

$$(6.4) \quad F(\cdot, A) = \Gamma(L^p) - \lim_{h \rightarrow \infty} F_h(\cdot, A)$$

for any  $A \in \mathcal{A}$ , and  $F$  is represented by

$$(6.5) \quad F(u, A) = \begin{cases} \int_A f(x, Xu(x)) dx & \text{if } A \in \mathcal{A}, u \in W_X^{1,p}(A), \\ \infty & \text{otherwise.} \end{cases}$$

Moreover,  $f$  can be uniquely chosen in  $\mathcal{I}_p^\infty(a, d_1, d_2)$  in such a way that

$$(6.6) \quad f(x, \eta) = f(x, \mathcal{C}(x)\xi_\eta)$$

for a.e.  $x \in \Omega$  and any  $\eta \in \mathbf{R}^m$ , where  $\xi_\eta$  is any vector in  $\mathbf{R}^n$  such that

$$(6.7) \quad \eta = \mathcal{C}(x)\xi_\eta + \tilde{\eta} \quad \text{and} \quad \langle \mathcal{C}(x)\xi_\eta, \tilde{\eta} \rangle_{\mathbf{R}^m} = 0 \quad \text{for some } \tilde{\eta} \in \mathbf{R}^m.$$

**Remark 6.6.** Although reminiscent of the approach presented in [13], ours differs from the former. Indeed, the classical approach to  $\Gamma$ -compactness consists of three main steps.

- (1) Given a sequence of integral functionals  $(F_h)_h$ , extract a subsequence, say again  $(F_h)_h$  which  $\bar{\Gamma}$ -converges to a local functional  $F$ , where  $\bar{\Gamma}$ -convergence (cf. [13, Chapter 16]) is a variational convergence tailored to local functionals.
- (2) Upgrade  $\bar{\Gamma}$ -convergence to  $\Gamma$ -convergence.
- (3) Show that  $F$  is an integral functional.

The proof of (2) (cf. e.g. [13, Proposition 18.6]) relies on uniform growth conditions like

$$F_h(u, A) \leq \tilde{G}(u, A)$$

for a suitable lower-semicontinuous measure  $G$ . On the other hand, we can only dispose of non-uniform growth conditions of the form

$$F_h(u, A) \leq \tilde{G}_h(u, A) := \int_A a(x) dx + d_2 \|X^h u\|_{L^p(A)}^p.$$

Nevertheless, [Theorem 6.3](#) ensures that  $(\tilde{G}_h)_h$   $\Gamma$ -converges. We will show that this weaker framework will be still sufficient to promote  $\bar{\Gamma}$ -convergence to  $\Gamma$ -convergence. In addition, once  $\Gamma$ -convergence is achieved, in order to conclude we will need to apply the anisotropic integral representation result [40, Theorem 4.1].

*Proof of Theorem 6.5. Step 1.* According to [13, Theorem 19.4],  $F : L^p(\Omega) \times \mathcal{A} \rightarrow [0, \infty]$  belongs to  $\mathcal{M}_p(e_1, e_2, e_3, e_4, \mu)$  if  $F$  is a measure and if there exist  $e_1 \geq 1$ ,  $e_2, e_3, e_4 \geq 0$ , a finite measure  $\mu$ , independent of  $F$ , and a measure  $G : L^p(\Omega) \times \mathcal{A} \rightarrow [0, \infty]$ , which may depend on  $F$ , such that

$$(6.8) \quad G(u, A) \leq F(u, A) \leq e_1 G(u, A) + e_2 \|u\|_{L^p(A)}^p + \mu(A)$$

and

$$(6.9) \quad G(\varphi u + (1 - \varphi)v, A) \leq e_4(G(u, A) + G(v, A)) + e_3 e_4 (\max |D\varphi|^p) \|u - v\|_{L^p(A)} + \mu(A),$$

for any  $u, v \in L^p(\Omega)$ , any  $A \in \mathcal{A}$  and any  $\varphi \in \mathbf{C}_c^\infty(\Omega)$  such that  $0 \leq \varphi \leq 1$ . We claim that  $(F_h)_h \subseteq \mathcal{M}_p(e_1, e_2, e_3, e_4, \mu)$ . Fix  $h \in \mathbf{N}$ , and set  $\mu(B) := \int_B a(x) dx$ . Then  $\mu$  is a finite measure on  $\Omega$ . Moreover, in view of [31, Lemma 4.14], the non-negative local functional  $G_h := d_1 \Psi_p^h(u, A)$  is a measure. Hence, letting  $e_1 := \frac{d_2}{d_1}$  and  $e_2 := 0$ , (6.8) follows from (6.2). Moreover, recalling (A) and arguing *verbatim* as in the proof of [17, Proposition 3.3], (6.9) follows for suitable constants  $e_3, e_4 \geq 0$  independent of  $h \in \mathbf{N}$ .

**Step 2.** By [13, Theorem 19.5], there exists an  $L^p$ -lower semicontinuous functional  $F \in \mathcal{M}_p(e_1, e_2, e_3, e_4, \mu)$  such that, up to a subsequence,  $(F_h)_h$   $\bar{\Gamma}(L^p)$ -converges to  $F$ . By definition of  $\mathcal{M}_p(e_1, e_2, e_3, e_4, \mu)$ ,  $F$  is a measure. Moreover, by [13, Proposition 16.15],  $F$  is local.

**Step 3.** We claim that

$$(6.10) \quad d_1 \|Xu\|_{L^p(A)}^p \leq F(u, A) \leq \int_A a(x) dx + d_2 \|Xu\|_{L^p(A)}^p$$

for any  $A \in \mathcal{A}$  and any  $u \in W_X^{1,p}(A)$ . Indeed, let  $G := d_1 \Psi_p$ . Since  $(G_h)_h \subseteq \mathcal{M}_p(e_1, e_2, e_3, e_4, \mu)$ , we apply again [13, Theorem 19.5] to infer that, up to a further subsequence,  $(G_h)_h$   $\bar{\Gamma}(L^p)$ -converges to a suitable functional in  $\mathcal{M}_p(e_1, e_2, e_3, e_4, \mu)$ . On the other hand, we know from

**Theorem 6.3** that

$$(6.11) \quad G(\cdot, A) = \Gamma(L^p) - \lim_{h \rightarrow \infty} G_h(\cdot, A)$$

for any  $A \in \mathcal{A}$ . Then, [13, Proposition 16.4] implies that  $(G_h)_h \bar{\Gamma}(L^p)$ -converges to  $G$ . Therefore, passing to the  $\bar{\Gamma}(L^p)$ -limit in  $G_h(u, A) \leq F_h(u, A) \leq e_1 G_h(u, A) + \mu(A)$  allows to show (6.10).

**Step 4.** We claim that

$$(6.12) \quad F(\cdot, A) = \Gamma(L^p) - \lim_{h \rightarrow \infty} F_h(\cdot, A)$$

for any  $A \in \mathcal{A}$ . To this aim, fix  $A \in \mathcal{A}$  and  $u \in W_X^{1,p}(A) \cap L^p(\Omega)$ . We claim that

$$(6.13) \quad F(u, A) = F'(u, A) = F''(u, A).$$

By [13, Proposition 16.4], we already know that  $F(u, A) \leq F'(u, A) \leq F''(u, A)$ , so that we are left to show that  $F''(u, A) \leq F(u, A)$ . Following the proof of [13, Proposition 18.6], and noticing that  $\nu(B) := \int_B a \, dx + d_2 \Psi_p(u, B)$  is a finite measure on  $A$ , for any  $\varepsilon > 0$  we choose a compact set  $K_\varepsilon \Subset A$  such that  $\nu(A \setminus K_\varepsilon) \leq \varepsilon$ . Moreover, recalling (6.11) and by [13, Proposition 6.7], we infer that

$$(6.14) \quad F''(u, \tilde{A}) \leq \nu(\tilde{A})$$

for any  $\tilde{A} \in \mathcal{A}$ . Choose two open sets  $A', A'' \in \mathcal{A}$  such that  $K_\varepsilon \Subset A' \Subset A'' \Subset A$ . Since  $(F_h)_h \subseteq \mathcal{M}_p(e_1, e_2, e_3, e_4, \mu)$ , [13, Theorem 19.4] implies that  $(F_h)_h$  satisfies the *uniform fundamental estimate* (cf. [13, Definition 18.2]). Therefore, noticing that  $A = A' \cup (A \setminus K_\varepsilon)$ , we can apply [13, Proposition 18.3] to infer that

$$F''(u, A) \leq F''(u, A'') + F''(u, A \setminus K_\varepsilon) \leq F(u, A) + \nu(A \setminus K_\varepsilon) \leq F(u, A) + \varepsilon,$$

where in the semi-last inequality we exploited [13, Remark 16.1] together with (6.14). Letting  $\varepsilon \rightarrow 0$ , (6.13) follows. We are ready to prove (6.12). To this aim, fix  $A \in \mathcal{A}$  and  $u \in L^p(\Omega)$ . If  $u \in W_X^{1,p}(A)$ , the thesis follows from (6.13). Conversely, assume that  $u \notin W_X^{1,p}(A)$ , or equivalently that  $G(u, A) = \infty$ . Exploiting (6.11),  $G(u, A) \leq F'(u, A)$ . Then, by [13, Remark 14.5]  $G(u, A) \leq F(u, A)$ , so that  $F(u, A) = \infty$ . Since  $F(u, A) \leq F'(u, A) \leq F''(u, A)$ , we conclude that  $F(u, A) = F'(u, A) = F''(u, A)$ , whence (6.12) follows.

**Step 5.** By (6.12),  $F$  is  $L^p$ -lower semicontinuous. Moreover, arguing *verbatim* as in the proof of [40, Theorem 4.3],  $F(u + c, A) = F(u, A)$  for any  $A \in \mathcal{A}$ , any  $u \in C^\infty(A)$  and any  $c \in \mathbf{R}$ . In conclusion,  $F$  satisfies all the hypotheses of [40, Theorem 4.1], whence the thesis follows.  $\square$

**Remark 6.7.** The very same approach to **Theorem 6.5**, together with the integral representation results provided in [40], allows to establish  $\Gamma$ -compactness results for integral functionals associated with complete Lagrangians  $f(x, u, \eta)$  under suitable natural assumptions (cf. [40]).

**6.3.  $\Gamma$ -convergence for functionals with Dirichlet boundary conditions.** **Theorem 6.5** can be exploited to keep into account Dirichlet boundary conditions.

**Theorem 6.8.** *Let  $(X^h)_h \in \mathcal{S}_1 \cup \mathcal{S}_2$ . Let  $1 < p < \infty$ . Let  $(f^h)_h \subseteq \mathcal{I}_p^h(a, d_1, d_2)$  and  $f \in \mathcal{I}_p^\infty(a, d_1, d_2)$ , and let  $F_h, F : L^p(\Omega) \times \mathcal{A} \rightarrow [0, \infty]$  be the corresponding integral functionals, respectively defined in (6.3) and (6.5). Fix  $(\varphi_h)_h \subseteq W_{X^h}^{1,p}(\Omega)$  and  $\varphi \in W_X^{1,p}(\Omega)$  such that*

$$(6.15) \quad \varphi_h \rightarrow \varphi \text{ strongly in } L^p(\Omega) \quad \text{and} \quad X^h \varphi_h \rightarrow X \varphi \text{ strongly in } L^p(\Omega; \mathbf{R}^m).$$

Define the functionals  $F_h^{\varphi_h}, F^\varphi : L^p(\Omega) \rightarrow [0, \infty]$  by

$$(6.16) \quad F_h^{\varphi_h}(u) = \begin{cases} F_h(u, \Omega) & \text{if } u \in W_{X^h, \varphi_h}^{1,p}(\Omega), \\ \infty & \text{otherwise,} \end{cases} \quad F^\varphi(u) = \begin{cases} F(u, \Omega) & \text{if } u \in W_{X, \varphi}^{1,p}(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

Assume that

$$(6.17) \quad F(u, \Omega) = \Gamma(L^p) - \lim_{h \rightarrow \infty} F_h(u, \Omega).$$

Then

$$(6.18) \quad F^\varphi(u) = \Gamma(L^p) - \lim_{h \rightarrow \infty} F_h^{\varphi_h}(u).$$

*Proof.* First we prove that  $F^\varphi(u) \leq \Gamma(L^p) - \liminf_{h \rightarrow \infty} F_h^{\varphi_h}(u)$ . Let  $u \in L^p(\Omega)$ , and let  $(u_h)_h \subseteq L^p(\Omega)$  be such that  $u_h \rightarrow u$  strongly in  $L^p(\Omega)$ . If  $\liminf_{h \rightarrow \infty} F_h^{\varphi_h}(u_h) = \infty$ , ( $\liminf$ ) follows. Assume the contrary. In this way, we may suppose that  $u_h \in W_{X^h, \varphi_h}^{1,p}(\Omega)$  for any  $h \in \mathbf{N}$ . By (6.2),  $(X^h u_h)_h$  is bounded in  $L^p(\Omega; \mathbf{R}^m)$ . Therefore, up to subsequences and by either Proposition 3.7 or Proposition 3.8,  $X^h u_h \rightarrow Xu$  weakly in  $L^p(\Omega; \mathbf{R}^m)$ . Therefore, by Proposition 3.10,  $u \in W_{X, \varphi}^{1,p}(\Omega)$ . In this way,

$$F^\varphi(u) = F(u, \Omega) \stackrel{(6.17)}{\leq} \liminf_{h \rightarrow \infty} F_h(u_h, \Omega) = \liminf_{h \rightarrow \infty} F_h^{\varphi_h}(u_h),$$

whence the first claim follows. Now we prove that  $F^\varphi(u) \geq \Gamma(L^p) - \limsup_{h \rightarrow \infty} F_h^{\varphi_h}(u)$ . Let  $u \in L^p(\Omega)$ . Fix  $\varepsilon > 0$ . Again, we may assume that  $u \in W_{X, \varphi}^{1,p}(\Omega)$ . Hence, by Theorem 4.3 there exists a sequence  $(v_h)_h \subseteq \mathbf{C}_c^\infty(\Omega)$  such that  $v_h \rightarrow u - \varphi$  strongly in  $L^p(\Omega)$  and  $X^h v_h \rightarrow X(u - \varphi)$  strongly in  $L^p(\Omega; \mathbf{R}^m)$ . Therefore, recalling (6.2), up to a subsequence

$$\begin{aligned} F_h(v_h + \varphi_h, A) &\leq \int_A a \, dx + d_2 \int_A |X^h v_h + X^h \varphi_h|^p \, dx \\ &\leq \int_A a \, dx + 2^{p-1} d_2 \int_A |Xu|^p \, dx + 2^{p-1} d_2 \int_A (|X^h v_h - X(u - \varphi)|^p + |X^h \varphi_h - X\varphi|^p) \, dx \\ &\stackrel{(6.15)}{\leq} \int_A a \, dx + 2^{p-1} d_2 \int_A |Xu|^p \, dx + \frac{\varepsilon}{2} \end{aligned}$$

for any  $A \in \mathcal{A}$ . Therefore, there exists a compact set  $K_\varepsilon \subseteq \Omega$  such that

$$(6.19) \quad F_h(v_h + \varphi_h, \Omega \setminus K_\varepsilon) \leq \varepsilon.$$

By (6.17) there exists  $(u_h)_h \subseteq W_{X^h}^{1,p}(\Omega)$  such that  $u_h \rightarrow u$  strongly in  $L^p(\Omega)$  and  $F(u, \Omega) \geq \limsup_{h \rightarrow \infty} F_h(u_h, \Omega)$ . Let  $A', A'' \in \mathcal{A}$  be such that  $K_\varepsilon \subseteq A' \Subset A'' \Subset \Omega$ . By the proof of Theorem 6.5, we know that  $(F_h)_h$  satisfies the uniform fundamental estimate as in [13, Definition 18.2]. Therefore, there exists  $M > 0$  and a sequence of smooth cut-off functions between  $A'$  and  $A''$ , say  $(\psi_h)_h$ , such that, up to a subsequence,

$$\begin{aligned} (6.20) \quad F_h(\psi_h u_h + (1 - \psi_h)(v_h + \varphi_h), \Omega) &\leq (1 + \varepsilon) (F_h(u_h, \Omega) + F_h(v_h + \varphi_h, \Omega \setminus K)) \\ &\quad + \varepsilon \left( \|u_h\|_{L^p(\Omega)}^p + \|v_h + \varphi_h\|_{L^p(\Omega)}^p \right) + M \|u_h - (v_h + \varphi_h)\|_{L^p(\Omega)}^p \\ &\stackrel{(6.15), (6.19)}{\leq} (1 + \varepsilon) F_h(u_h, \Omega) + \varepsilon(1 + \varepsilon) + \varepsilon \left( 2\|u\|_{L^p(\Omega)}^p + 1 \right) + M\varepsilon. \end{aligned}$$

Set  $w_h := \psi_h u_h + (1 - \psi_h)(v_h + \varphi_h)$  for any  $h \in \mathbf{N}$ . Notice that  $w_h \rightarrow u$  strongly in  $L^p(\Omega)$ . Fix  $h \in \mathbf{N}$ . As  $u_h - \varphi_h \in W_{X^h}^{1,p}(\Omega)$ , by Theorem 2.4 there exists a sequence  $(u_k^h)_k \subseteq \mathbf{C}^\infty(\Omega) \cap W_{X^h}^{1,p}(\Omega)$  such that  $u_k^h \rightarrow u_h - \varphi_h$  strongly in  $L^p(\Omega)$  and  $X^h u_k^h \rightarrow X^h u_h - X^h \varphi_h$  strongly in  $L^p(\Omega; \mathbf{R}^m)$ . Notice that

$$w_h - \varphi_h = \psi_h (u_h - \varphi_h) + (1 - \psi_h) v_h.$$

For any  $k \in \mathbf{N}$ , set  $w_k^h = \psi_h u_k^h + (1 - \psi_h) v_h$ . As  $v_h \in \mathbf{C}_c^\infty(\Omega)$ ,  $(w_k^h)_k \subseteq \mathbf{C}_c^\infty(\Omega)$ . Since

$$w_k^h - (w_h - \varphi_h) = \psi_h (u_k^h - (u_h - \varphi_h)),$$

we conclude that  $w_k^h \rightarrow w_h - \varphi_h$  strongly in  $L^p(\Omega)$  and  $X^h w_k^h \rightarrow X^h(w_h - \varphi_h)$  strongly in  $L^p(\Omega; \mathbf{R}^m)$ . Therefore  $w_h \in W_{X^h, \varphi_h}^{1,p}(\Omega)$ , and in particular  $F_h(w_h, \Omega) = F_h^{\varphi_h}(w_h)$ . Combining this information with (6.20), we infer

$$(6.21) \quad F_h^{\varphi_h}(w_h) \leq (1 + \varepsilon)F_h(u_h, \Omega) + \varepsilon(1 + \varepsilon) + \varepsilon \left( 2\|u\|_{L^p(\Omega)}^p + 1 \right) + M\varepsilon,$$

whence

$$\begin{aligned} \Gamma(L^p) - \limsup_{h \rightarrow \infty} F_h^{\varphi_h}(u) &\leq \limsup_{h \rightarrow \infty} F_h^{\varphi_h}(w_h) \\ &\stackrel{(6.21)}{\leq} (1 + \varepsilon) \limsup_{h \rightarrow \infty} F_h(u_h, \Omega) + \varepsilon(1 + \varepsilon) + \varepsilon \left( 2\|u\|_{L^p(\Omega)}^p + 1 \right) + M\varepsilon \\ &= (1 + \varepsilon)F(u, \Omega) + \varepsilon(1 + \varepsilon) + \varepsilon \left( 2\|u\|_{L^p(\Omega)}^p + 1 \right) + M\varepsilon \\ &= (1 + \varepsilon)F^\varphi(u) + \varepsilon(1 + \varepsilon) + \varepsilon \left( 2\|u\|_{L^p(\Omega)}^p + 1 \right) + M\varepsilon. \end{aligned}$$

Being  $\varepsilon$  arbitrary, the thesis follows.  $\square$

**Remark 6.9.** Although reminiscent of the proof of [13, Theorem 21.1], our argument needs to differ from the former. Indeed, as the finiteness domain of  $F_h^{\varphi_h}$  depends on  $h$ , the limit function  $u$  in the proof of (limsup) cannot be used to construct a recovery sequence, but has to be properly well-approximated. On the other hand, our approach has allowed to consider a sequence of evolving boundary data instead of a fixed one.

**6.4. Convergence of minima and minimizers.** Once Theorem 5.2, Theorem 5.3 and Theorem 6.8 are established, we argue *verbatim* as in the Euclidean setting to infer convergence of minima and minimizers. In view of Section 7, we additionally take into account a continuous perturbation.

**Theorem 6.10.** *Let  $(X^h)_h \in \mathcal{T}_1 \cup \mathcal{T}_2$ . Let  $1 < p < \infty$ . Let  $(f^h)_h \subseteq \mathcal{I}_p^h(a, d_1, d_2)$  and  $f \in \mathcal{I}_p^\infty(a, d_1, d_2)$ . Let  $(\varphi_h)_h \subseteq W_{X^h}^{1,p}(\Omega)$  and  $\varphi \in W_X^{1,p}(\Omega)$  satisfy (6.15). Let  $F_h^\varphi, F^\varphi : L^2(\Omega) \rightarrow [0, \infty]$  be as in (6.16). Assume that (6.18) holds. Let  $G : L^p(\Omega) \rightarrow \mathbf{R}$  be continuous and such that, for any  $\varepsilon > 0$ , there are  $\delta_1 = \delta_1(\varepsilon) > 0$ ,  $\delta_2 = \delta_2(\varepsilon) > 0$  and  $\delta_3 = \delta_3(\varepsilon) > 0$  such that*

$$(6.22) \quad -\delta_1(\varepsilon) - \varepsilon \int_{\Omega} |u|^p dx \leq G(u) \leq \delta_2(\varepsilon) + \delta_3(\varepsilon) \int_{\Omega} |u|^p dx$$

for any  $u \in L^p(\Omega)$ . Up to a subsequence, the following properties hold.

(1)  $F^\varphi + G$  has a minimum, and

$$\inf_{L^p(\Omega)} (F_h^{\varphi_h} + G) \rightarrow \min_{L^p(\Omega)} (F^\varphi + G).$$

(2) Assume that there exists a sequence  $(u_h)_h \subseteq W_{X^h, \varphi_h}^{1,p}(\Omega)$  such that, for any  $h \in N$ ,  $u_h$  minimizes  $F_h^{\varphi_h} + G$ . Then there exist a minimum point  $u$  of  $F^\varphi + G$  such that

$$u_h \rightarrow u \text{ strongly in } L^p(\Omega).$$

*Proof.* As  $G$  is continuous, (6.18) and [13, Proposition 6.21] imply that

$$F^\varphi + G = \Gamma(L^p) - \lim_{h \rightarrow \infty} (F_h^{\varphi_h} + G).$$

For any  $h \in \mathbf{N}$ , set  $m_h = \inf_{L^p(\Omega)}(F_h^{\varphi_h} + G)$ . Notice that  $m_h \in \mathbf{R}$ . Let  $u_h \in L^p(\Omega)$  be such that  $m_h \leq F_h^{\varphi_h}(u_h) + G(u_h) \leq m_h + \frac{1}{h}$ . Then, for any  $\varepsilon > 0$ , there exists  $\hat{M} = \hat{M}(\varepsilon) > 0$  such that

$$\begin{aligned} F_h^{\varphi_h}(u_h) + G(u_h) &\leq m_h + 1 \\ &\leq F_h^{\varphi_h}(\varphi_h) + G(\varphi_h) + 1 \\ &\stackrel{(6.2),(6.22)}{\leq} \int_{\Omega} a \, dx + \delta_2(\varepsilon) + \delta_3(\varepsilon) \int_{\Omega} |\varphi_h|^p \, dx + d_2 \int_{\Omega} |X^h \varphi_h|^p \, dx + 1 \\ &\stackrel{(6.15)}{\leq} \hat{M}(\varepsilon). \end{aligned}$$

In particular, by [Theorem 5.3](#),

$$\begin{aligned} \hat{M}(\varepsilon) &\geq F_h^{\varphi_h}(u_h) + G(u_h) \\ &\stackrel{(6.2),(6.22)}{\geq} d_1 \int_{\Omega} |X^h u_h|^p \, dx - \varepsilon \int_{\Omega} |u_h|^p \, dx - \delta_1(\varepsilon) \\ &\stackrel{(5.8)}{\geq} \left( d_1 - \frac{\varepsilon 2^{2p-1}}{\mathcal{R}} \right) \int_{\Omega} |X^h u|^p \, dx - \frac{\varepsilon 2^{2p-1}}{\mathcal{R}} \int_{\Omega} |X^h \varphi_h|^p \, dx - \varepsilon 2^{p-1} \int_{\Omega} |\varphi_h|^p \, dx - \delta_1(\varepsilon) \end{aligned}$$

Combining the previous inequalities with [Theorem 5.3](#),  $(u_h)_h$  is bounded in  $L^p(\Omega)$  and  $(X^h u_h)_h$  is bounded in  $L^p(\Omega; \mathbf{R}^m)$ . By [Theorem 5.2](#) and up to a subsequence, there exists  $u \in W_{X,\varphi}^{1,p}(\Omega)$  such that  $u_h \rightarrow u$  strongly in  $L^p(\Omega)$ . The rest of the thesis follows as in [[13](#), [Theorem 7.2](#)].  $\square$

**Remark 6.11.** For any  $\mu \geq 0$  and any  $g \in L^q(\Omega)$ , set  $G : L^p(\Omega) \rightarrow \mathbf{R}$  by

$$G(u) = \frac{\mu}{p} \int_{\Omega} |u|^p \, dx - \int_{\Omega} g u \, dx$$

for any  $u \in L^p(\Omega)$ . It is easy to check that  $u$  is continuous and satisfies [\(6.22\)](#).

**6.5. Quadratic forms.** In this section we let  $p = 2$ , and we specialize [Theorem 6.5](#) to the class of quadratic forms. Moreover, as this will be important in the forthcoming [Section 7](#), we provide a convergence result of the so-called *momenta*, defining the latter to keep into account the moving anisotropic setting (cf. [Definition 7.1](#)).

**Definition 6.12.** Fix  $h \in \mathbf{N}$  and  $0 < \lambda \leq \Lambda$ . We denote by  $E^h(\Omega; \lambda, \Lambda)$  the class of symmetric, measurable matrix-valued functions  $A^h : \Omega \rightarrow \mathbf{R}^{m \times m}$  such that

$$(6.23) \quad \lambda |\mathcal{C}^h(x)\xi|^2 \leq \langle A^h(x)\mathcal{C}^h(x)\xi, \mathcal{C}^h(x)\xi \rangle_{\mathbf{R}^m} \leq \Lambda |\mathcal{C}^h(x)\xi|^2$$

for a.e.  $x \in \Omega$  and for every  $\xi \in \mathbf{R}^n$ .

Quadratic forms corresponding to elements of  $E^h(\Omega; \lambda, \Lambda)$  express as

$$(6.24) \quad F_h(u, A) = \begin{cases} \int_A \langle A^h(x)X^h u(x), X^h u(x) \rangle_{\mathbf{R}^m} \, dx & \text{if } u \in W_{X^h}^{1,2}(A), \\ \infty & \text{otherwise.} \end{cases}$$

**Theorem 6.13.** Let  $(X^h)_h \in \mathcal{S}_1 \cup \mathcal{S}_2$ . Let  $0 < \lambda \leq \Lambda$ . Let  $(A^h)_h \subseteq E^h(\Omega; \lambda, \Lambda)$ . For any  $h \in \mathbf{N}$ , denote by  $F_h$  the corresponding quadratic form in the sense of [\(6.24\)](#). There exists  $A \in E^\infty(\Omega; \lambda, \Lambda)$  and a corresponding quadratic form  $F : L^2(\Omega) \times \mathcal{A} \rightarrow [0, \infty]$  such that, up to subsequences, [\(6.4\)](#) holds.

*Proof.* Let  $f^h$  be the Lagrangian associated with  $F_h$ . Then  $(f^h)_h \subseteq \mathcal{I}_2^h(0, \lambda, \Lambda)$ . In virtue of [Theorem 6.5](#), there exist  $f \in \mathcal{I}_2(0, \lambda, \Lambda)$  and a corresponding integral functional  $F : L^2(\Omega) \times \mathcal{A} \rightarrow [0, \infty]$  such that, up to subsequences, [\(6.4\)](#) holds. Fix  $A \in \mathcal{A}$ . By [\(6.4\)](#) and [[13](#), [Theorem 11.10](#)], there is a symmetric bilinear form  $B : W_X^{1,2}(A) \times W_X^{1,2}(A) \rightarrow \mathbf{R}$  such that

$$\int_A f(x, Xu) \, dx = B(u, u)$$

for any  $u \in W_X^{1,p}(A)$ . Arguing *verbatim* as in the proof of [13, Theorem 22.1], there exists a  $n \times n$  symmetric, measurable matrix-valued function  $A_e$  such that

$$f(x, \mathcal{C}(x)\xi) = \langle A_e(x)\xi, \xi \rangle_{\mathbf{R}^n}$$

for a.e.  $x \in \Omega$  and for every  $\xi \in \mathbf{R}^n$ . Arguing as in [25, 40], there exists a unique, measurable,  $n \times m$  matrix-valued function  $\mathcal{C}_P(x)$  such that

$$\begin{aligned} \mathcal{C}_P(x)\mathcal{C}(x)\mathcal{C}_P(x) &= \mathcal{C}_P(x), & \mathcal{C}(x)\mathcal{C}_P(x)\mathcal{C}(x) &= \mathcal{C}(x), \\ \mathcal{C}_P(x)\mathcal{C}(x) &= \mathcal{C}(x)^T\mathcal{C}_P(x)^T, & \mathcal{C}(x)\mathcal{C}_P(x) &= \mathcal{C}_P(x)^T\mathcal{C}(x)^T \end{aligned}$$

for any  $x \in \Omega$ . Set  $A(x) = \mathcal{C}_P(x)^T A_e(x) \mathcal{C}_P(x)$ . Fix  $x \in \Omega$  and  $\eta \in \mathbf{R}^m$ , and let  $\xi_\eta$  be as in (6.7). Let

$$N_x = \ker(\mathcal{C}(x)) \quad \text{and} \quad V_x = \{\mathcal{C}(x)^T \eta : \eta \in \mathbf{R}^m\}.$$

From standard linear algebra, there are unique  $(\xi_\eta)_{N_x} \in N_x$  and  $(\xi_\eta)_{V_x} \in V_x$  such that

$$\xi_\eta = (\xi_\eta)_{N_x} + (\xi_\eta)_{V_x}.$$

By [31, Lemma 3.13],

$$\langle A_e(x)\xi_\eta, \xi_\eta \rangle_{\mathbf{R}^n} = \langle A_e(x)(\xi_\eta)_{V_x}, (\xi_\eta)_{V_x} \rangle_{\mathbf{R}^n}.$$

Moreover, by [40, Proposition 3.1],

$$(\xi_\eta)_{V_x} = \mathcal{C}_P(x)\mathcal{C}(x)\xi_\eta.$$

Therefore

$$\langle A_e(x)\xi_\eta, \xi_\eta \rangle_{\mathbf{R}^n} = \langle A_e(x)\mathcal{C}_P(x)\mathcal{C}(x)\xi_\eta, \mathcal{C}_P(x)\mathcal{C}(x)\xi_\eta \rangle_{\mathbf{R}^n} = \langle A(x)\mathcal{C}(x)\xi_\eta, \mathcal{C}(x)\xi_\eta \rangle_{\mathbf{R}^m},$$

whence

$$f(x, \eta) \stackrel{(6.6)}{=} f(x, \mathcal{C}(x)\xi_\eta) = \langle A_e(x)\xi_\eta, \xi_\eta \rangle_{\mathbf{R}^n} = \langle A(x)\mathcal{C}(x)\xi_\eta, \mathcal{C}(x)\xi_\eta \rangle_{\mathbf{R}^m}.$$

The thesis follows combining the previous equation with (2.1) and Theorem 2.4.  $\square$

The next crucial result is preliminary to the convergence of momenta as stated in the forthcoming Theorem 6.15.

**Theorem 6.14.** *Let  $(X^h)_h \in \mathcal{S}_1 \cup \mathcal{S}_2$ . Let  $(A^h)_h \subseteq E^h(\Omega; \lambda, \Lambda)$  and  $A \in E^\infty(\Omega; \lambda, \Lambda)$ . Let  $(\Phi_h)_h \subseteq L^2(\Omega; \mathbf{R}^n)$  and  $\Phi \in L^2(\Omega; \mathbf{R}^n)$  be such that  $\mathcal{C}^h \Phi_h \rightarrow \mathcal{C}\Phi$  strongly in  $L^2(\Omega, \mathbb{R}^m)$ . Define  $G_h^{\Phi_h}, G^\Phi : L^2(\Omega) \times \mathcal{A} \rightarrow [0, \infty]$  by*

$$G_h^{\Phi_h}(u, A) = \begin{cases} \int_A \langle A^h(X^h u + \mathcal{C}^h \Phi_h), X^h u + \mathcal{C}^h \Phi_h \rangle_{\mathbf{R}^m} dx & \text{if } u \in W_{X^h}^{1,2}(A), \\ \infty & \text{otherwise,} \end{cases}$$

and

$$G^\Phi(u, A) = \begin{cases} \int_A \langle A(Xu + \mathcal{C}\Phi), (Xu + \mathcal{C}\Phi) \rangle_{\mathbf{R}^m} dx & \text{if } u \in W_X^{1,2}(A), \\ \infty & \text{otherwise.} \end{cases}$$

If (6.4) holds, then

$$(6.25) \quad G^\Phi(\cdot, A) = \Gamma(L^2) - \lim_{h \rightarrow \infty} G_h^{\Phi_h}(\cdot, A).$$

for every  $A \in \mathcal{A}$ .

*Proof.* For any  $h \in \mathbf{N}$ , let  $F_h, F : L^2(\Omega) \times \mathcal{A} \rightarrow [0, \infty]$  be the quadratic forms associated with  $A^h$  and  $A$ . For each  $h \in \mathbf{N}$ , define

$$g_h^{\Phi_h}(x, \eta) := f_h(x, \eta + \mathcal{C}^h(x)\Phi_h(x)) = \langle A^h(x)(\eta + \mathcal{C}^h(x)\Phi_h(x)), \eta + \mathcal{C}^h(x)\Phi_h(x) \rangle_{\mathbf{R}^m}$$

and

$$\tilde{g}_h^{\Phi_h}(x, \eta) := g_h^{\Phi_h}(x, \eta) + \lambda |\mathcal{C}^h(x)\Phi_h(x)|^2$$

for a.e.  $x \in \Omega$  and for any  $\eta \in \mathbf{R}^m$ . Then, up to subsequences,

$$\begin{aligned}
 \tilde{g}_h^{\Phi_h}(x, \mathcal{C}^h(x)\xi) &= \langle A^h(x)\mathcal{C}^h(x)(\xi + \Phi_h(x)), \mathcal{C}^h(x)(\xi + \Phi_h(x)) \rangle_{\mathbf{R}^m} + \lambda |\mathcal{C}^h(x)\Phi_h(x)|^2 \\
 &\stackrel{(6.23)}{\leq} \Lambda |\mathcal{C}^h(x)(\xi + \Phi_h(x))|^2 + \lambda |\mathcal{C}^h(x)\Phi_h(x)|^2 \\
 &\leq (2\Lambda + \lambda) |\mathcal{C}^h(x)\Phi_h(x)|^2 + 2\Lambda |\mathcal{C}^h(x)\xi|^2 \\
 &\leq (2\Lambda + \lambda) |\tilde{k}(x)| + 2\Lambda |\mathcal{C}^h(x)\xi|^2
 \end{aligned}$$

for a suitable  $\tilde{k} \in L^1(\Omega)$ , where the last passage follows from [4, Theorem 4.9]. Similarly,

$$\tilde{g}_h^{\Phi_h}(x, \mathcal{C}^h(x)\xi) \stackrel{(6.23)}{\geq} \lambda |\mathcal{C}^h(x)\xi|^2 + 2\lambda |\mathcal{C}^h(x)\Phi_h(x)|^2 + 2\lambda \langle \mathcal{C}^h(x)\xi, \mathcal{C}^h(x)\Phi_h(x) \rangle_{\mathbf{R}^m} \geq \frac{\lambda}{2} |\mathcal{C}^h(x)\xi|^2$$

for a.e.  $x \in \Omega$  and for any  $\xi \in \mathbf{R}^n$ . Therefore

$$(6.26) \quad \tilde{g}_h^{\Phi_h} \in \mathcal{I}_2^h \left( (2\Lambda + \lambda) |\tilde{k}|, \frac{\lambda}{2}, 2\Lambda \right)$$

for every  $h \in \mathbf{N}$ . Since  $\mathcal{C}^h\Phi_h \rightarrow \mathcal{C}\Phi$  strongly in  $L^2(\Omega; \mathbf{R}^n)$ , Theorem 6.5 implies that there exists  $\hat{G}^\Phi : L^p(\Omega) \times \mathcal{A} \rightarrow [0, \infty]$  such that, up to subsequences,

$$\hat{G}^\Phi(\cdot, A) = \Gamma(L^2) - \lim_{h \rightarrow \infty} G_h^{\Phi_h}(\cdot, A)$$

for every  $A \in \mathcal{A}$ , and that there exists a function  $\hat{g}^\Phi$  such that

$$(6.27) \quad \hat{g}^\Phi + \lambda |\mathcal{C}(x)\Phi(x)|^2 \in \mathcal{I}_2 \left( (2\Lambda + \lambda) |\tilde{k}|, \frac{\lambda}{2}, 2\Lambda \right)$$

and such that  $\hat{G}^\Phi$  can be represented as

$$\hat{G}^\Phi(u, A) := \begin{cases} \int_A \hat{g}^\Phi(x, Xu(x)) dx & \text{if } A \in \mathcal{A}, u \in W_X^{1,2}(A), \\ \infty & \text{otherwise.} \end{cases}$$

To conclude, we show that

$$(6.28) \quad \hat{G}^\Phi(u, A) = G^\Phi(u, A)$$

for each  $A \in \mathcal{A}$  and  $u \in W_X^{1,2}(A)$ . We divide the proof of (6.28) into three steps.

**Step 1.** Let  $\xi \in \mathbf{R}^n$ . Set  $\Psi^h(x) = \Psi(x) = \xi$  for any  $h \in \mathbf{N}$  and any  $x \in \Omega$ . Let  $G^{\Psi_h}, G^\Psi$  and  $\hat{G}^\Psi$  be the corresponding functionals. If  $u_\xi(x) := \langle \xi, x \rangle_{\mathbf{R}^n}$  for any  $x \in \mathbb{R}^n$ , then

$$X^h u_\xi = \mathcal{C}^h \xi, \quad X u_\xi = \mathcal{C} \xi \quad \text{and} \quad G_h^{\Psi_h}(u, A) = F_h(u + u_\xi, A).$$

Let  $(u_h)_h$  be a recovery sequence for  $u$  relative to  $\hat{G}^\Psi$ . Then

$$G^\Psi(u, A) = F(u + u_\xi, A) \leq \liminf_{h \rightarrow \infty} F_h(u_h + u_\xi, A) = \liminf_{h \rightarrow \infty} G_h^{\Psi_h}(u_h, A) \leq \hat{G}^\Psi(u, A).$$

Conversely, let  $(v_h)_h$  be a recovery sequence for  $u + u_\xi$  relative to  $F$ . Then, as  $v_h - u_\xi \rightarrow u$  strongly in  $L^2(\Omega)$ ,

$$\hat{G}^\Psi(u, A) \leq \liminf_{g \rightarrow \infty} G_h^\Psi(v_h - u_\xi, A) = \liminf_{h \rightarrow \infty} F_h(v_h, A) \leq F(u + u_\xi, A) = G^\Psi(u, A).$$

Therefore, (6.28) holds with  $\Phi$  replaced by  $\Psi$ .

**Step 2.** Let  $\Psi_h = \Psi$  be *piecewise constant*, i.e., there exist  $\xi^1, \dots, \xi^N \in \mathbb{R}^n$  and  $A_1, \dots, A_N$  pairwise disjoint open sets such that  $|\Omega \setminus \cup_{i=1}^N A_i| = 0$  and

$$\Psi(x) := \sum_{i=1}^N \chi_{A_i}(x) \xi^i.$$

Owing to the previous case, (6.28) follows *verbatim* is in the proof of [32, Theorem 3.7], again replacing  $\Phi$  with  $\Psi$ .

**Step 3.** We are ready to prove (6.28) Let  $(\Psi_j)_j$  be a sequence of piecewise constant functions converging to  $\Phi$  strongly in  $L^2(\Omega; \mathbf{R}^n)$ . In particular,  $(\mathcal{C}\Psi_j)_j$  converges to  $\mathcal{C}\Phi$  strongly in  $L^2(\Omega; \mathbf{R}^m)$ . Arguing *verbatim* as in the proof of [32, Theorem 3.7], there exists a constant  $\tilde{c} = \tilde{c}(\lambda, \Lambda)$ , such that, up to subsequences,

$$(6.29) \quad |G^\Phi(u, A) - G^{\Psi_j}(u, A)| \leq \tilde{c} \|\mathcal{C}\Phi - \mathcal{C}\Psi_j\|_{L^2(\Omega; \mathbf{R}^m)} (\|Xu\|_{L^2(\Omega; \mathbf{R}^m)} + 2\|\mathcal{C}\Phi\|_{L^2(\Omega; \mathbf{R}^m)} + 1)$$

and, recalling (6.27),

$$(6.30) \quad |\hat{G}^\Phi(u, A) - \hat{G}^{\Psi_j}(u, A)| \leq \tilde{c} \|\mathcal{C}\Phi - \mathcal{C}\Psi_j\|_{L^2(\Omega; \mathbf{R}^m)} (\|Xu\|_{L^2(\Omega; \mathbf{R}^m)} + 2\|\mathcal{C}\Phi\|_{L^2(\Omega; \mathbf{R}^m)} + 1)$$

for any  $A \in \mathcal{A}$ , any  $u \in W_X^{1,2}(A)$  and any  $j \in \mathbf{N}$ . Moreover,

$$\begin{aligned} |\hat{G}^\Phi(u, A) - G^\Phi(u, A)| &\leq |\hat{G}^\Phi(u, A) - \hat{G}^{\Psi_j}(u, A)| + |\hat{G}^{\Psi_j}(u, A) - G^{\Psi_j}(u, A)| \\ &\quad + |G^{\Psi_j}(u, A) - G^\Phi(u, A)|. \end{aligned}$$

By (6.29), (6.30) and the previous steps, (6.28) follows.  $\square$

We are in position to infer convergence of momenta in the following moving anisotropic sense.

**Theorem 6.15.** *Let  $(X^h)_h \in \mathcal{S}_1 \cup \mathcal{S}_2$ . Let  $(A^h)_h \subseteq E^h(\Omega; \lambda, \Lambda)$ . Let  $A^\infty \in E^\infty(\Omega; \lambda, \Lambda)$ . Assume that (6.4) holds. Then, up to subsequences,*

$$\int_A \langle A^\infty Xu, \mathcal{C}\Phi \rangle_{\mathbf{R}^m} dx = \lim_{h \rightarrow \infty} \int_A \langle A^h X^h u_h, \mathcal{C}^h \Phi_h \rangle_{\mathbf{R}^m} dx$$

for any  $A \in \mathcal{A}$ , any  $u \in W_X^{1,2}(A)$ , any recovery sequence  $(u_h)_h$  relative to  $F$ , any  $\Phi \in L^2(\Omega; \mathbf{R}^n)$  and any  $(\Phi_h)_h \subseteq L^2(\Omega, \mathbb{R}^n)$  such that  $\mathcal{C}^h \Phi_h \rightarrow \mathcal{C}\Phi$  strongly in  $L^2(\Omega, \mathbb{R}^n)$ .

*Proof.* For any  $h \in \mathbf{N}$ , denote by  $(F_h)_h$  and  $F_\infty$  the quadratic forms associated with  $(A^h)_h$  and  $A^\infty$ . Fix  $t \in \mathbf{R}$ , and denote by  $G_h^{t\Phi_h}$  and  $G^{t\Phi}$  the functionals arising from Theorem 6.14. By the properties of  $(u_h)_h$  and Theorem 6.14,

$$\frac{G^{t\Phi}(u, A) - F_\infty(u, A)}{t} \leq \liminf_{h \rightarrow \infty} \frac{G_h^{t\Phi_h}(u_h, A) - F_h(u_h, A)}{t}.$$

Notice that

$$\frac{G^{t\Phi}(u, A) - F_\infty(u, A)}{t} = 2 \int_A \langle A^\infty Xu, \mathcal{C}\Phi \rangle_{\mathbf{R}^m} dx + t \int_A \langle A^\infty \mathcal{C}\Phi, \mathcal{C}\Phi \rangle_{\mathbf{R}^m} dx$$

and

$$\frac{G_h^{t\Phi_h}(u_h, A) - F_h(u_h, A)}{t} = 2 \int_A \langle A^h X^h u_h, \mathcal{C}^h \Phi_h \rangle_{\mathbf{R}^m} dx + t \int_A \langle A^h \mathcal{C}^h \Phi_h, \mathcal{C}^h \Phi_h \rangle_{\mathbf{R}^m} dx.$$

Since, up to subsequences,

$$\left| \int_A \langle A^h \mathcal{C}^h \Phi_h, \mathcal{C}^h \Phi_h \rangle_{\mathbf{R}^m} dx \right| \leq \Lambda \int_A |\mathcal{C}^h \Phi_h|^2 dx \leq \Lambda \left( |\Omega| + \int_A |\mathcal{C}\Phi|^2 dx \right),$$

and similarly for the second term, we let  $t \rightarrow 0$  and conclude that

$$(6.31) \quad \int_A \langle A^\infty Xu, \mathcal{C}\Phi \rangle_{\mathbf{R}^m} dx \leq \liminf_{h \rightarrow \infty} \int_A \langle A^h X^h u_h, \mathcal{C}^h \Phi_h \rangle_{\mathbf{R}^m} dx.$$

The thesis follows by evaluating (6.31) in  $\Psi = -\Phi$  and  $\Psi_h = -\Phi_h$ .  $\square$

7.  $H$ -CONVERGENCE FOR LINEAR OPERATORS UNDER MOVING ANISOTROPIES

In this section, owing to the results proved in [Section 6](#), we consider the asymptotic behaviour of suitable anisotropic, symmetric, linear differential operators, extending previous results in the static case (cf. [\[29, 30, 32\]](#), and cf. e.g. [\[11\]](#) for some recent non-local achievements). Let  $p = 2$ . Let  $\Omega$  be a bounded domain of  $\mathbf{R}^n$ . Let  $(X^h)_h \in \mathcal{T}_1 \cup \mathcal{T}_2$ . Let  $(A^h)_h \subseteq E^h(\Omega; \lambda, \Lambda)$ , for some fixed  $0 < \lambda \leq \Lambda$ . Let  $(\varphi_h)_h \subseteq W_{X^h}^{1,p}(\Omega)$  and  $\varphi \in W_X^{1,p}(\Omega)$  be such that [\(6.15\)](#) holds. For any  $h \in \mathbf{N}$ , let  $\mathcal{L}^h$  be the linear anisotropic differential operator defined by

$$\mathcal{L}^h := \operatorname{div}_{X^h}(A^h(x)X^h).$$

For any  $f \in L^2(\Omega)$  and any  $\mu \geq 0$ , we consider anisotropic elliptic problems of the form

$$(P_h) \quad \begin{cases} \mu u + \mathcal{L}^h(u) = f & \text{in } \Omega, \\ u_h \in W_{X^h, \varphi_h}^{1,2}(\Omega). \end{cases}$$

Accordingly, we say that  $u_h \in W_{X^h, \varphi_h}^{1,2}(\Omega)$  is a *weak solution* to  $(P_h)$  if

$$\mu \int_{\Omega} u_h v \, dx + \int_{\Omega} \langle A^h X^h u, X^h v \rangle_{\mathbf{R}^m} \, dx = \int_{\Omega} f v \, dx$$

for any  $v \in W_{X^h, 0}^{1,2}(\Omega)$ . If each  $X^h$  belongs to some  $\mathcal{M}_h = \mathcal{M}_h(\Omega_0^h, C_d^h, c_h, C_h)$ , then [Proposition 2.5](#), [Theorem 2.6](#), [\[10, Corollary 2.6\]](#), [Remark 6.11](#) and a standard application of the direct methods imply the existence of a unique minimizer  $u_h \in W_{X^h, \varphi_h}^{1,2}(\Omega)$  of the functional

$$u \mapsto \frac{\mu}{2} \int_{\Omega} |u|^2 \, dx + \frac{1}{2} \int_{\Omega} \langle A^h X^h u, X^h u \rangle_{\mathbf{R}^m} \, dx - \int_{\Omega} f u \, dx,$$

which in turn is the unique weak solution to  $(P_h)$ . We stress that, as we are assuming  $(X^h)_h \in \mathcal{T}_1 \cup \mathcal{T}_2$ , all our previous considerations hold replacing  $X^h$  with the limit  $X$ -gradient  $X$ . Under the above assumption, we provide the definition of  $H$ -convergence for sequences of operators  $(\mathcal{L}^h)_h \subseteq \mathcal{E}^h(\Omega; \lambda, \Lambda)$ .

**Definition 7.1.** *Let  $(\mathcal{L}^h)_h \subseteq \mathcal{E}^h(\Omega; \lambda, \Lambda)$ . We say that  $(\mathcal{L}^h)_h \subseteq \mathcal{E}^h(\Omega; \lambda, \Lambda)$   $H$ -converges to  $\mathcal{L}^\infty \in \mathcal{E}^\infty(\Omega; \lambda, \Lambda)$  if the following holds. For any  $f \in L^2(\Omega)$ , any  $\mu \geq 0$ , any  $(\varphi_h)_h \subseteq W_{X^h}^{1,p}(\Omega)$  and any  $\varphi \in W_X^{1,p}(\Omega)$  such that [\(6.15\)](#) holds, let  $(u_h)_h \subseteq W_{X^h, \varphi_h}^{1,2}(\Omega)$  and  $u_\infty \in W_{X, \varphi}^{1,2}(\Omega)$  be, respectively, the unique weak solutions to  $(P_h)$  and the corresponding limit problem. Then, up to a subsequence,*

$$u_h \rightarrow u_\infty \text{ strongly in } L^2(\Omega) \quad (\text{convergence of solutions})$$

and

$$\int_{\Omega} \langle A^h X^h u_h, C^h \Phi_h \rangle_{\mathbf{R}^m} \, dx \rightarrow \int_{\Omega} \langle A^\infty X u_\infty, C \Phi \rangle_{\mathbf{R}^m} \, dx \quad (\text{convergence of momenta})$$

for any  $\Phi \in L^2(\Omega; \mathbf{R}^n)$  and any  $(\Phi_h)_h \subseteq L^2(\Omega, \mathbb{R}^n)$  such that  $C^h \Phi_h \rightarrow C \Phi$  strongly in  $L^2(\Omega, \mathbb{R}^n)$ .

Owing to the results presented in [Section 6](#), the following  $H$ -compactness theorem holds.

**Theorem 7.2.** *Let  $\Omega \subseteq \mathbf{R}^n$  be a bounded domain. Let  $(X^h)_h \in \mathcal{T}_1 \cup \mathcal{T}_2$ . Assume that, for any  $h \in \mathbf{N}$ ,  $X^h$  belongs to some  $\mathcal{M}_h$  as in [Definition 2.2](#). Let  $(\mathcal{L}^h)_h \subseteq \mathcal{E}^h(\Omega; \lambda, \Lambda)$ . Then, up to a subsequence, there exists  $\mathcal{L}^\infty \in \mathcal{E}^\infty(\Omega; \lambda, \Lambda)$  such that  $\mathcal{L}_h$   $H$ -converges to  $\mathcal{L}^\infty$ .*

*Proof.* The proof follows combining [Theorem 6.8](#), [Theorem 6.10](#), [Remark 6.11](#), [Theorem 6.13](#) and [Theorem 6.15](#).  $\square$

## DECLARATIONS

**Funding information.** The authors are member of the Istituto Nazionale di Alta Matematica (INdAM), Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA). The authors have been supported by INdAM–GNAMPA 2023 Project *Equazioni differenziali alle derivate parziali di tipo misto o dipendenti da campi di vettori*, CUP E53C22001930001. S. V. is supported by MIUR-PRIN 2022 Project *Regularity problems in sub-Riemannian structures* (Grant Code: 2022F4F2LH), and by INdAM-GNAMPA 2025 Project *Structure of sub-Riemannian hypersurfaces in Heisenberg groups*, (Grant Code: CUP\_ES324001950001). A.M. is supported by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M), by MCIN/AEI/10.13039/501100011033 (PID2021-123903NB-I00), by Generalitat de Catalunya (2021-SGR-00087). A.M. has been supported by the INdAM-GNAMPA 2024 Project *Pairing and div-curl lemma: extensions to weakly differentiable vector fields and nonlocal differentiation* (Grant Code: CUP\_E53C23001670001). A.M. and F.P. are supported by the INdAM - GNAMPA 2025 Project *Metodi variazionali per problemi dipendenti da operatori frazionari isotropi e anisotropi* (Grant Code: CUP\_E5324001950001). This research was supported by the Centre de Recerca Matemàtica of Barcelona (CRM), under the International Programme for Research in Groups (IP4RG).

**Conflict of interest.** The authors declare that they have no financial interests or conflicts of interest related to the subject matter or materials discussed in this paper.

**Data availability statement.** Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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