ITERATIVE BLOW-UPS FOR MAPS WITH BOUNDED $\mathcal{A}$-VARIATION: A REFINEMENT, WITH APPLICATION TO BD AND BV.

MARCO CAROCCIA AND NICOLAS VAN GOETHEM

Abstract. We refine the iterated blow-up techniques. This technique, combined with a rigidity result and a specific choice of the kernel projection in the Poincaré inequality, might be employed to completely linearize blow-ups along at least one sequence. We show how to implement such argument by applying it to derive affine blow-up limits for BD and BV functions around Cantor points. In doing so we identify a specific subset of points - called totally singular points having blow-ups with completely singular gradient measure $Dp = D^s p$, $\mathcal{E}p = \mathcal{E}^s p$ - at which such linearization fails.

Contents

1. Introduction

In this paper we consolidate the iterated blow-up approach that has been developed in [14] based on the work [18]. This approach consists of iteratively blowing up a function at a point, relying on the principle expressed in [31, Theorem 14.16], which can be summarized by Preiss' result that *tangent measures to tangent measures are tangent measures* (see also Theorem 3.5 below). In other words, if h is a blow-up of u at x, and g is a blow-up of h at y, then, by means of this principle, we can deduce that g must be a blow-up of u at x.

This idea has already been successfully implemented in [18] to obtain relaxation and in [14] to obtain integral representation results for variational functionals in the context of maps of bounded deformation in the spirit of [11]. For applications, see also [6, 10, 15, 16]. The aim of this paper is to establish a general framework in order to apply this technique to general first-order operators $\mathcal{A}$. Note that the original technique introduced in [18] deals with iterative blow-ups of the measure $\mathcal{A}u$ ($\mathcal{E}u$ in that case), slightly different from the approach proposed here, which considers blow-ups of the function u. The main difference lies in the following fact: by considering blow-ups of $\mathcal{A}u$ one can obtain relaxation and homogenization results for energies depending solely on $\mathcal{A}u$. By focusing on the blow-ups of u, instead, it is possible to obtain relaxation and integral representation for energies depending on $\mathcal{A}u$ **and** u, up to some extent (as done in [14]). To obtain this slightly more general result, some additional ingredients are, however, required: a rigidity result and the structure of the operator *projection onto the kernel* $\mathcal{R}: L^1_{loc}(\mathbb{R}^n;\mathbb{R}^m) \to \mathrm{Ker}(\mathcal{A})$, appearing in the Poincaré inequality (see Proposition 4.3 and Lemma 4.4 in the context of BD). By rigidity result, we mean some general structure property implied by having constant polar $\frac{\mathrm{d}\mathcal{A}u}{\mathrm{d}|\mathcal{A}u|}$ on the convex set K. In some sense, since we need to keep track of the pointwise value of u (to allow for energies depending on $\mathcal{A}u$ and u) it is no surprise that we must gain some information on $u(x)$, encoded as information on the kernel projection $\mathcal{R}$, and the rigidity of blow-up structure at x. Once these two ingredients are at hand heuristically, by iteratively and alternatively blowing up, using rigidity and the kernel structure, we can find at least one affine blow-up of u at $|\mathcal{A}u|$-a.e. point, out of a specific set identified as the totally singular points (cf. Definitions 4.1 and 4.5). We show how to implement this scheme at Cantor points, in the BV and BD case as a refinement of the technique developed in [14] in Section 4. In particular by relying upon Definition 3.3 and Proposition 3.6, which embed the idea of iterative blow-ups into a solid general framework, we prove the following two facts.

Theorem (Affine blow-ups for BV functions - Theorem 4.2)**.** *Let $n \geq 2$ and $u \in \mathrm{BV}(\Omega;\mathbb{R}^m)$. Let K be a center-symmetric convex set. Then for $|D^c u|$-a.e. $x \in \Omega \setminus \mathrm{TS}(u)$ there exists a vanishing sequence $\varepsilon_i \downarrow 0$ such that*

$$u_{K,\varepsilon_i,x}(y) \to \frac{\mathrm{d}Du}{\mathrm{d}|Du|}(x)y \quad \textit{strictly in } \mathrm{BV}(K;\mathbb{R}^n).$$

Theorem (Affine blow-up for BD functions - Theorem 4.6)**.** *Let $u \in \mathrm{BD}(\Omega)$. Let K be a center-symmetric convex set. Then for $|\mathcal{E}^c u|$-a.e. $x \in \Omega \setminus \mathrm{TS}(u)$ there exists a sequence $\varepsilon_i \downarrow 0$ such that*

$$u_{K,\varepsilon_i,x}(y) \to \frac{\mathrm{d}\mathcal{E}u}{\mathrm{d}|\mathcal{E}u|}(x)y \quad \textit{strictly in } \mathrm{BD}(K)$$

Here, u_{K,x,ε_i} is the typical blow-up sequence considered when dealing with homogenization problems

$$u_{K,\varepsilon,x}(y) := \frac{u(x+\varepsilon y) - \mathcal{R}_K[u(x+\varepsilon\cdot)](y)}{\frac{|\mathcal{A}u|(K_\varepsilon(x))}{|K|\varepsilon^{n-1}}},$$

where $\mathcal{A} = \mathcal{E}$ in the BD case, $\mathcal{A} = D$ in the BV case, and $\mathcal{R}_K$ is a specific linear bounded operator (related to K), mapping L^1 onto $\mathrm{Ker}(\mathcal{A})$. The set $\mathrm{TS}(u)$ is a specific set of points, called *totally singular points*, defined for BV (Definition 4.1) and for BD (Definition 4.5) as those points for which all the blow-ups are given by functions h having only the singular part in their gradient (i.e. $D^s h = Dh$ or $\mathcal{E}h = \mathcal{E}^s h$ for the symmetric gradient). The exclusion of these points is due to the following fact. Heuristically, we eliminate the singular part of $\mathcal{E}u$ (Du in the BV case) in the final blow-up by iteratively blowing up at absolutely continuous points of each new blow-up, ending with an affine function. If we try to apply this idea at a totally singular point, after the first blow-up we can no longer find a point of absolute continuity where $\psi'(x) \neq 0$ to perform a second blow-up (since $D\psi = D^s\psi$). While a complete linearization is possible only outside of $\mathrm{TS}(u)$, we underline the general statements [18], [14, Proposition 4.6] for BD that allow to linearize at least one direction for $|\mathcal{E}^c u|$-a.e. $x \in \Omega$. Thus, at totally singular points, the process must stop, and we are left with one-dimensional blow-ups with a vanishing absolutely continuous part. Of course further blow-ups can follow, with a suitable application of Proposition 3.6 if, for the blow-up h, it holds $e(h)\mathcal{L}^n \neq 0$. We also point out that in [14] the linearization is achieved for specific convex sets P but with this refined technology it can be checked that [14, Proposition 4.6] can be proven for any general K.

As we briefly introduce at the beginning of Section 4, we consider the iterative blow-up strategy to be a fruitful and robust scheme for tackling homogenization and relaxation challenges.

Organization of the paper. In Section 2 we introduce the main notations and we retrieve, in a survey-like presentation, the results in literature on general first-order differential operator $\mathcal{A}$ and the space of bounded $\mathcal{A}$-variation. In Section 3, we define and elaborate on the set of *strict blow-up limit* (Definition 3.3) $\mathrm{bu}_K(u,x)$, we prove that is never empty and that *strict blow-ups of blow-ups are strict blow-ups* (i.e. Proposition 3.6). Finally in Section 4, we apply the iterative blow-up technique to obtain affine blow-ups for BD and BV functions.

2. Preliminaries

We collect here some preliminary results from literature that will be used in the sequel, together with the setting of the notation used in rest of the paper.

2.1. **General notations.** The letter n will always denote the ambient Euclidean space dimension. We will denote by $B_r(x)$ the ball of radius r and centered at x. Whenever $x=0$ we just write B_r, as well as in the case $r=1$ when we simply write $B(x)$. More in general, given a convex body K we denote by $K_r(x) := x + rK$. We denote by $\mathbb{M}^{m\times n}$ the set of $n\times n$ matrices. The notation e_i stands for the i-th vector of the canonical basis of $\mathbb{R}^n$, Id denotes the $n\times n$ identity matrix. The notation $\mathcal{L}^n$, $\mathcal{H}^{n-1}$ stand for the n-dimensional Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$ while $\mathcal{M}(E;V)$ is the space of all finite V-valued Radon measures on E. The space $\mathrm{Lin}(X;Y)$ denotes the family of all linear maps between the two vector spaces X and Y.

Let then $\{A_j\}_{j=1}^n \subset \mathrm{Lin}(\mathbb{R}^m, V)$, for some Euclidean space V, and define the *first order linear operator with constant coefficient* $\mathcal{A} : C^1(\mathbb{R}^n;\mathbb{R}^m) \to C^0(\mathbb{R}^n;V)$ to be:

$$\mathcal{A}u(x) := \sum_{j=1}^n A_j \partial_j u(x).$$

To such an operator, and for each $\xi \in \mathbb{R}^n$, we associate the *symbol* $\mathbb{A}[\xi] : \mathbb{R}^m \to V$ defined by

$$\mathbb{A}[\xi]\eta := \sum_{j=1}^n \xi_j A_j \eta.$$

We may use the intuitive notation introduced in [12]:

$$\mathbb{A}[\xi]\eta = \eta \otimes_{\mathbb{A}} \xi,$$

where we have defined the bi-linear map $\otimes_{\mathbb{A}}(\eta,\xi) = \eta \otimes_{\mathbb{A}} \xi = \mathbb{A}[\xi]\eta$. Observe that, formally for $u \in C^1(\mathbb{R}^n;\mathbb{R}^m)$ and since $D = (\partial_1,\dots,\partial_n)$,

$$\mathcal{A}u(x) = \mathbb{A}[D]u.$$

2.2. **Maps of bounded $\mathcal{A}$-variation.** Now we define the set of functions with bounded $\mathcal{A}$-variation as the functional space (still adopting the notation in [12])

$$\mathrm{BV}^{\mathcal{A}}(\Omega;\mathbb{R}^m) := \left\{ u \in L^1(\Omega;\mathbb{R}^m) \;\middle|\; \mathcal{A}u \in \mathcal{M}(\Omega;V) \right\},$$

where $\mathcal{M}(\Omega;V)$ is the set of V-valued finite Radon measures (see [4]), and with the distributional $\mathcal{A}$-gradient defined as the following measure:

$$\int_\Omega \varphi \cdot \mathrm{d}\mathcal{A}u := \int_\Omega \mathcal{A}^*\varphi \cdot u \,\mathrm{d}x, \;\forall\, \varphi \in C_c^\infty(\Omega;V),$$

where the L^2-adjoint operator $\mathcal{A}^* : C^1(\mathbb{R}^n;V) \to C^0(\mathbb{R}^n;\mathbb{R}^m)$ is defined, starting from $\{A_j^*\}_{j=1}^n \subset \mathrm{Lin}(V;\mathbb{R}^m)$, and with $A_j z \cdot \eta = z \cdot A_j^*\eta$, for all $z \in \mathbb{R}^m, \eta \in V$, as

$$\mathcal{A}^* v = -\sum_{j=1}^n A_j^* \partial_j v.$$

As classical examples, the spaces of function of bounded variations $\mathrm{BV}(\mathbb{R}^n;\mathbb{R}^m) = \mathrm{BV}^D(\mathbb{R}^n;\mathbb{R}^m)$ (with $\mathcal{A}u = Du = \mathbb{A}[D]u = u \otimes D$; $\mathcal{A}^*\varphi = -\mathrm{div}\varphi$ and $V = \mathbb{R}^{m\times n}$), the spaces of function of *bounded deformations* $\mathrm{BD}(\mathbb{R}^n) = \mathrm{BV}^{\mathcal{E}}(\mathbb{R}^n;\mathbb{R}^n)$ with

$$\mathcal{E}u = \frac{1}{2}\left(Du + D^T u\right)$$

(with $\mathcal{E}u = \mathbb{E}[D]u := u \otimes_{\mathcal{E}} D$; $\mathcal{E}^*\varphi = -\mathrm{div}\varphi$ and $V = \mathbb{R}^{n\times n}_{\mathrm{sym}}$) and the space of *bounded deviatoric deformations* $\mathrm{BD}_{\mathrm{dev}}(\mathbb{R}^n) = \mathrm{BV}^{\mathcal{E}_d}(\mathbb{R}^n;\mathbb{R}^n)$ with

$$\mathcal{E}_d u = \mathcal{E}u - \frac{\mathrm{div}(u)}{n}\mathrm{Id}$$

(with $\mathcal{E}_d u = \mathbb{E}_d[D]u := u \otimes_{\mathcal{E}_d} D$; $\mathcal{E}_d^*\varphi = -\mathrm{div}\varphi$ and $V = \mathbb{R}^{n\times n}_{\mathrm{sym}_0}$). For references about these spaces we refer to [3, 4, 9, 22, 28, 34, 36].

Let us now introduce some crucial concepts in order to deepen the properties of these operators.

2.2.1. *Ellipticity and Cancelling properties.*

Definition 2.1 (Elliptic)**.** We say that the operator $\mathcal{A}$ is *elliptic* (or $\mathbb{R}$-elliptic) if for any $\xi \in \mathbb{R}^n \setminus \{0\}$ the map $\mathbb{A}[\xi] : \mathbb{R}^m \to V$ is one-to-one. Equivalently, if and only if for all non-zero ξ there exists a constant $c = c(n, \mathcal{A}) > 0$ such that

$$|\mathbb{A}[\xi]\eta| \geq c|\xi||\eta|, \quad \forall \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^m.$$

Intuitively ellipticity means that the equation $\mathcal{A}u = f$ is left-invertible (i.e., has a unique solution) in Fourier space [24]. As a consequence of the celebrated work by Calderón and Zygmund [13] we have that $\mathcal{A}$ is ($\mathbb{R}$-)elliptic if and only if for each $1 < p < \infty$ there exists $c = c(p, n, \mathcal{A})$ such that coercivity is in force:

$$\|Du\|_{L^p(\mathbb{R}^n;\mathbb{M}^{m\times n})} \leq c\|\mathcal{A}u\|_{L^p(\mathbb{R}^n;V)}, \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^m). \tag{2.1}$$

Inequality (2.1) is well-known to be false for $p = 1$ (cf. [32]) but under a stronger assumptions on $\mathcal{A}$ we can infer a Poincaré-type inequality and a Sobolev-type inequality.

Definition 2.2 (Cancelling)**.** We say that $\mathcal{A}$ is cancelling if

$$\bigcap_{\xi\in\mathbb{R}^n\setminus\{0\}} \mathrm{Im}(\mathbb{A}[\xi]) = \{0\}.$$

To the best of the authors' knowledge, the first contribution on the L^p-differentiability results of $BV^{\mathcal{A}}$-maps under ellipticity and cancellation (not necessarily $\mathbb{C}$-elliptic) properties can be found in [33].

For an elliptic and canceling first order linear operator we have the following Theorem.

Theorem 2.3 (Gagliardo-Nirenberg-Sobolev, see [37])**.** *If $\mathcal{A}$ is an elliptic and canceling first-order linear operator then there exists a $c > 0$, depending on n and $\mathbb{A}$ only, such that*

$$\|u\|_{L^{\frac{n}{n-1}}} \leq c\|\mathcal{A}u\|_{L^1} \quad \textit{for all } u \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^m).$$

It is not easy to localize Theorem 2.3 to general bounded open set. In this sense the best result available in literature (see Proposition 2.5) requires the introduction of the $\mathbb{C}$-ellipticity, a strong property that allows to develop all main variational tools on $\mathrm{BV}^{\mathcal{A}}$.

Definition 2.4 ($\mathbb{C}$-Elliptic)**.** We say that the operator $\mathcal{A}$ is $\mathbb{C}$-*elliptic* if for any $\xi \in \mathbb{C}^n \setminus \{0\}$ the complexification of its symbol $\mathbb{A}[\xi] : \mathbb{C}^m \to V + iV$ (being i the imaginary unit) is one-to-one.

Obviously a $\mathbb{C}$-elliptic operator is also elliptic. In [12, Theorem 2.6] is shown that $\mathbb{C}$-ellipticity is equivalent to the property of finite-dimensionality of the kernel of $\mathcal{A}$, i.e. $\dim\{v \in \mathcal{D}'(\Omega;V) : \mathcal{A}v = 0\} < \infty$. This makes easy to verify, for instance, that $D, \mathcal{E}$ are $\mathbb{C}$-elliptic as well as to verify that $\mathcal{E}_d$ is **not** $\mathbb{C}$-elliptic in $n = 2$ (whereas it is in bigger dimension).

As a matter of fact, many important operators are not $\mathbb{C}$-elliptic, as for instance the curl (of order 1) or the incompatibility operator (of order 2), since their kernels, consisting of all gradients of scalar function (for the curl), and of all symmetric gradients of vector functions (for the incompatibility [30]), have not finite dimension. However note that operators curl and inc may verify elliptic-like properties, in the sense that (2.1) holds for some specific Sobolev spaces V (cf. [35, Lemma 7] for the curl and [5, Theorem 3.9] for the incompatibility).

Finally, $\mathbb{C}$-ellipticity allows one to localize Theorem 2.3 to obtain the following Poincaré-Sobolev inequality. In the following, $\Pi^U_{\mathrm{Ker}} : L^1(U;\mathbb{R}^m) \to \mathrm{Ker}(\mathcal{A})$ stands for a bounded linear projection operator onto the kernel of $\mathcal{A}$, denoted as $\mathrm{Ker}(\mathcal{A})$.

Proposition 2.5 (Poincaré-Sobolev inequality)**.** *Let $\mathcal{A}$ be a $\mathbb{C}$-elliptic first order differential operator with constant coefficients. Let K be a center-symmetric convex set. Then there exists a constant c depending on n and K only such that*

$$\|u - \Pi^{K_r(x)}_{\mathrm{Ker}} u\|_{L^{\frac{n}{n-1}}(K_r(x);\mathbb{R}^m)} \leq c|\mathcal{A}u|(\overline{K_r(x)})$$

for all $x \in \mathbb{R}^d$, $r > 0$ and $u \in \mathrm{BV}^{\mathcal{A}}_{loc}(\mathbb{R}^n;\mathbb{R}^m)$.

Remark 2.6. We underline that the result of Proposition 2.5 is present in literature, in [24, Proposition 2.5] in the case $K = B$. By carefully looking at the proof, the argument relies upon the extension operator

$$E : W^{\mathcal{A},1}(B_r(x)) \to W^{\mathcal{A},1}(\mathbb{R}^d)$$

built in [25]. However, the argument for building the extension operator in [25] is not sensible to the shape of the ball and thus it can be applied verbatim to produce an extension operator

$$E : W^{\mathcal{A},1}(K_r(x)) \to W^{\mathcal{A},1}(\mathbb{R}^d)$$

for generic convex sets K (in fact on every so-called Jones domain). The same proof as in [24, Proposition 2.5] yields the result of Proposition 2.5 on generic convex sets K. We are deeply grateful to F. Gmeineder for the fruitful clarification about this subject.

The space $\mathrm{BV}^{\mathcal{A}}(\Omega)$, endowed with the norm $\|u\|_{\mathrm{BV}^{\mathcal{A}}} := |\mathcal{A}u|(\Omega) + \|u\|_{L^1}$, is a Banach space. The Poincaré-Sobolev inequality in (2.5) provide a standard argument, by following for instance the ideas in [28], to prove the following compactness Theorem.

Theorem 2.7 (Compactness Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary and let $\mathcal{A}$ be a $\mathbb{C}$-elliptic first-order linear operator. Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathrm{BV}^{\mathcal{A}}(\Omega;\mathbb{R}^m)$. Suppose that*

$$\sup_{k\in\mathbb{N}}\{\|u_k\|_{\mathrm{BV}^{\mathcal{A}}}\} < +\infty.$$

Then there exists $u \in \mathrm{BV}^{\mathcal{A}}(\Omega)$ and a subsequence $h(k)$ such that $u_{h(k)} \to u$ in L^1 and $\mathcal{A}u_{h(k)} \rightharpoonup^* \mathcal{A}u$.

The notation $\mathcal{A}u_{h(k)} \rightharpoonup^* \mathcal{A}u$ stands for the standard *weak* convergence* of Radon measures (see [29] or [23]).

2.2.2. *Poincaré inequality for $\mathbb{C}$-elliptic operators.* Let $\mathcal{A}$ be a linear, first order, homogeneous differential operator with constant coefficients on $\mathbb{R}^n$ which is $\mathbb{C}$-elliptic. Let K be a convex set of $\mathbb{R}^n$. Then it is known, [21, Theorem 3.7], that for a uniform constant $c = c(K,n) > 0$ the following Poincaré-type of inequality holds:

$$\|u - \Pi^K_{\mathrm{Ker}} u\|_{L^1(K;\mathbb{R}^m)} \leq c|\mathcal{A}u|(K). \tag{2.2}$$

Since we need to apply Poincaré inequality (for compactness purposes) on specific projection operators the following Proposition might be of some use. The proof is exactly as in [12], retrieved here for the sake of completeness.

Proposition 2.8 (Poincaré inequality)**.** *Let K be a fixed convex set. Let $\mathcal{A}$ be $\mathbb{C}$-elliptic and let $\mathcal{R} : L^1(K;\mathbb{R}^m) \to \mathrm{Ker}(\mathcal{A})$ be a linear, bounded operator such that $\mathcal{R}(L) = L$ for all $L \in \mathrm{Ker}(\mathcal{A})$. Then there exists a uniform constant $c = c(\mathcal{R}, K, n)$ depending on $n, \mathcal{R}$ and K such that*

$$\|u - \mathcal{R}[u]\|_{L^1(K;\mathbb{R}^m)} \leq c|\mathcal{A}u|(K). \tag{2.3}$$

Proof. Since $\mathcal{A}$ is $\mathbb{C}$-elliptic, inequality (2.2) holds. Then

$$\begin{aligned}\|u - \mathcal{R}[u]\|_{L^1(K;\mathbb{R}^m)} &\leq \|u - \Pi^K_{\mathrm{Ker}} u\|_{L^1(K;\mathbb{R}^m)} + \|\mathcal{R}[u] - \Pi^K_{\mathrm{Ker}} u\|_{L^1(K;\mathbb{R}^m)} \\ &\leq c|\mathcal{A}u|(K) + \|\mathcal{R}[u - \Pi^K_{\mathrm{Ker}} u]\|_{L^1(K;\mathbb{R}^m)}\end{aligned}$$

Since $\mathcal{R}$ is continuous we also have

$$\|\mathcal{R}(u - \Pi^K_{\mathrm{Ker}} u)\|_{L^1(K;\mathbb{R}^m)} \leq c(\mathcal{R}, K, n)\|u - \Pi^K_{\mathrm{Ker}} u\|_{L^1(K;\mathbb{R}^m)} \leq c|\mathcal{A}u|(K)$$

thus proving (2.3). ◻

2.3. **Structure of maps of bounded $\mathcal{A}$-variation maps.** We here collect the results related to the structure of $\mathcal{A}u$ under different assumptions on $\mathcal{A}$. The majority of these results heavily rely upon the pioneering works [8],[12] and [17].

2.3.1. *Trace and Gauss-Green theorems.* Under $\mathbb{C}$-ellipticity, in [12] is shown that a function $u \in \mathrm{BV}^{\mathcal{A}}(\Omega;\mathbb{R}^m)$ possesses the trace on $\partial\Omega$ and the Gauss-Green formula

$$\int_\Omega \varphi(x)\cdot \mathrm{d}\mathcal{A}u(x) = \int_\Omega \mathcal{A}^*\varphi(x)\cdot u(x)\,\mathrm{d}x + \int_{\partial\Omega} (u(y)\otimes_{\mathbb{A}} \nu_\Omega(y))\cdot\varphi(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)$$

holds true for all $\varphi \in C^\infty(\Omega;V)$. The trace is continuous under *strict convergence*: namely if $u_k \to u$ in L^1 and $|\mathcal{A}u_k|(\Omega)\to|\mathcal{A}u|(\Omega)$ then the trace $u_k\Big|_{\partial\Omega} \to u\Big|_{\partial\Omega}$ in L^1. To the knowledge of the authors, the first trace inequalities on $s\in(n-1,n)$-dimensional sets can be found in [26].

Definition 2.9 (Approximate jump)**.** Let $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$. We say that a point x is an *approximate jump point* of u if there exist *distinct* vectors $u^+,u^-\in\mathbb{R}^m$ and a direction $\nu\in\mathbb{S}^{n-1}$ satisfying

$$\begin{cases} \lim\limits_{r\to 0} \fint_{B_r^+(x,\nu)} |u(y)-u^+|\,\mathrm{d}y = 0,\\ \lim\limits_{r\to 0} \fint_{B_r^-(x,\nu)} |u(y)-u^-|\,\mathrm{d}y = 0. \end{cases}$$

Here, we use the notation

$$B_r^+(x,\nu) := \{y\in B_r(x)\ \nu\cdot(x-y)>0\}, \qquad B_r^-(x,\nu) := \{y\in B_r(x)\ \nu\cdot(x-y)<0\}$$

for the *ν-oriented half-balls* centred at x. We refer to u^+,u^- as the *one-sided limits* of u at x with respect to ν. The triplet is uniquely idendified up to a change of sign for ν and a permutation of u^+,u^-. The collection of all points of approximate jump for u is denoted as J_u.

We underline the remarkable result in [19] proving that, for a function $u\in L^1_{loc}(\Omega;\mathbb{R}^m)$ (without any additional information on the derivatives of u), J_u is $\mathcal{H}^{n-1}$-rectifiable.

To the knowledge of the authors, the first characterisation of continuity points, and first contribution on the fine properties of $BV^{\mathcal{A}}$-maps can be found in [20]. Moreover, the first systematic understanding of the algebraic properties of symbols to yield trace theorems and hereafter fine properties on lower dimensional sets can be found in [27].

2.3.2. *Radon-Nikodým decomposition of $\mathcal{A}u$.* Let $u\in\mathrm{BV}^{\mathcal{A}}(\Omega;\mathbb{R}^m)$ and let us denote by S_u the complement of the set of points of approximate continuity of u (those points where $u^+=u^-$). Then Radon-Nikodým writes as

$$\mathcal{A}u = \mathcal{A}^a u + \mathcal{A}^s u = \mathcal{A}^a u + \underbrace{\mathcal{A}^s u \llcorner (\Omega\setminus S_u)}_{=\mathcal{A}^c} + \underbrace{\mathcal{A}^s u \llcorner (S_u\setminus J_u)}_{=\mathcal{A}^{dd}} + \underbrace{\mathcal{A}^s u \llcorner J_u}_{=\mathcal{A}^j}.$$

The term $\mathcal{A}^c$ is the *diffuse continuous or Cantor-singular part.* The term $\mathcal{A}^{dd}$ is the *diffuse discontinuous singular part.*

It is proved in [24, Lemma 3.1] that, under $\mathbb{R}$-ellipticity of $\mathcal{A}$, $u\in\mathrm{BV}^{\mathcal{A}}(\Omega;\mathbb{R}^m)$ is L^p-differentiable (and hence approximately-differentiable) $\mathcal{L}^n$-almost everywhere for any $1\le p<$

$\frac{n}{n-1}$. For $p = \frac{n}{n-1}$ the same holds but $\mathbb{C}$-ellipticity is needed for $\mathcal{A}$. Finally, for $\mathbb{R}$-elliptic operator (see [2, Theorem 3.4.]) we have $\mathcal{L}^n$-almost everywhere that

$$\frac{d\mathcal{A}u}{d\mathcal{L}^n}(x)\mathcal{L}^n = \mathbb{A}[\nabla]u(x), \tag{2.4}$$

yielding $\mathcal{A}^a u = \mathbb{A}[\nabla]u\mathcal{L}^n$.

For $\mathbb{C}$-elliptic operators it is known [9, Theorem 1.2], that $(S_u \setminus J_u)$ is purely $\mathcal{H}^{n-1}$-unrectifiable and that the jump parts writes as [12]

$$\mathcal{A}^j = (u^+ - u^-) \otimes_{\mathbb{A}} \nu \mathcal{H}^{n-1} \llcorner J_u.$$

Finally, for first order linear operators $\mathcal{A}$ satisfying a specific hypothesis called *rank*$_{\mathcal{A}}$*-one* property, in [7] it is proven that $|\mathcal{A}u|(S_u \setminus J_u) = 0$ implying $\mathcal{A}^{dd} = 0$.
Moreover, according to [7], the rank-one property allows one to perform one-dimensional slicing of such maps, i.e., there exist directions a, b such that $\forall v \in V$ it holds

$$v \cdot \mathcal{A}u = \int_{\pi_a} Du^b_{y,a} d\mathcal{H}^{n-1}(y),$$

where $\cdot$ is by Riesz intended for the duality (V, V^*), with the slice of u defined as

$$u^b_{y,a} : \Omega^a_y := \{s \in \mathbb{R} : y + sa \in \Omega\} \to \mathbb{R},$$

where $u^b_{y,\eta}(t) := b \cdot u(y + ta)$ and with the hyper-plane orthogonal to a:

$$\pi_a := \{\zeta \in \mathbb{R}^n : a \cdot \zeta = 0\}.$$

Moreover it holds that $u^b_{y,a} \in BV(\Omega^a_y; \mathbb{R})$ for $\mathcal{H}^{n-1}$-almost all $y \in \pi_a$. These results in BV and BD can be found in [3, 4, 34].

2.3.3. *Annihilators and structure property of the polar of* $\mu = \mathcal{A}u$. Given the Radon measure $\sigma \in \mathcal{M}(\Omega; W)$, let the Radon measure $\mu \in \mathcal{M}(\Omega; V)$ satisfy

$$\mathcal{B}\mu = \sum_{j=1}^n B_j \partial_j \mu = \sigma \quad \text{ in } \mathcal{D}'(\Omega; W),$$

with tensor coefficients $B_j \in \mathrm{Lin}(V, W) \sim W \times V^*$, with its *symbol* $\mathbb{B}[\xi] : V \to W$

$$\mathbb{B}[\xi] := \sum_{j=1}^n \xi_j B_j.$$

The wave cone of $\mathcal{B}$ is defined as

$$\Lambda_{\mathcal{B}} := \bigcup_{\xi \in S^{n-1}} \mathrm{Ker}\mathbb{B}[\xi] \subset V.$$

Under these conditions De Philippis and Rindler in [17] have proved that the polar of μ has the following particular structure

$$\frac{d\mu}{|d\mu|} \in \Lambda_{\mathcal{B}} \quad |\mu|^s - \text{a.e. in } \Omega.$$

In $\mathrm{BV}(\Omega; \mathbb{R}^m)$ and for $\mathcal{B}$ the *curl operator*, one recovers Alberti's rank-one theorem [1],

$$\frac{dD^s u}{d|D^s u|}(x) = a(x) \otimes b(x),$$

for $D^s u$-almost all $x \in \Omega$ with $a(x), b(x) \in \mathbb{R}^n$. In $\mathrm{BD}(\Omega;\mathbb{R}^n)$ and for $\mathcal{B}$ the *incompatibility operator* [5], one finds

$$\frac{d\mathcal{E}^s u}{d|\mathcal{E}^s u|}(x) = a(x) \odot b(x),$$

for $\mathcal{E}^s u$-almost all $x \in \Omega$, and with $a(x), b(x) \in \mathbb{R}^n$.

3. Iterated strict blow-ups

We consider $\mathcal{A}$ to be a generic $\mathbb{C}$-elliptic operator and we consider $\mathrm{BV}^{\mathcal{A}}(\Omega;\mathbb{R}^m)$ to be the space of all the functions with bounded $\mathcal{A}$ variation from an open bounded regular set Ω into $\mathbb{R}^m$. For any fixed convex set K, $u \in \mathrm{BV}^{\mathcal{A}}(\Omega;\mathbb{R}^m)$, $x \in \Omega$ and $\varepsilon < \mathrm{dist}(x,\partial\Omega)$ we define

$$u_{K,\varepsilon,x}(y) := \frac{u(x+\varepsilon y) - \mathcal{R}_K[u(x+\varepsilon\cdot)](y)}{\frac{|\mathcal{A}u|(K_\varepsilon(x))}{|K|\varepsilon^{n-1}}}.$$

being $\mathcal{R}_K : L^1(K;\mathbb{R}^m) \to \mathrm{Ker}(\mathcal{A})$ any linear and bounded operator fixing $\mathrm{Ker}(\mathcal{A}) \cap L^1(K;\mathbb{R}^m)$. Note that we have $\mathcal{R}_K[u_{K,\varepsilon,x}] = 0$, by the very Definition of the blow-up sequence.

We use standard notations for the push-forward of measures, and in particular, given $\mu \in \mathcal{M}_{loc}(\mathbb{R}^n;V)$, $x \in \mathbb{R}^n$ and $\varepsilon > 0$, we will consider the push forward with the map $F^{x,\varepsilon}(y) := \frac{y-x}{\varepsilon}$ defined as

$$F^{x,\varepsilon}_{\#}\mu(A) := \mu(x+\varepsilon A).$$

Note that Preiss' tangent space $\mathrm{Tan}(\mu,x)$ at a given point $x \in \mathbb{R}^n$, is defined as the subset of non zero measures $\nu \in \mathcal{M}_{loc}(\mathbb{R}^n;V)$ such that ν is the local weak* limit of $1/c_i F^{x,\varepsilon_i}_{\#}\mu$, for some sequence $\varepsilon_i \downarrow 0$ as $i \uparrow +\infty$ and for some positive sequence $\{c_i\}_{i\in\mathbb{N}}$ (see [4], [31], [34]).

Remark 3.1 (Derivative of blow-ups)**.** Note that, with the given notations, an easy computation shows that

$$\mathcal{A}u_{K,\varepsilon,x} = |K| \frac{F^{x,\varepsilon}_{\#}\mathcal{A}u}{F^{x,\varepsilon}_{\#}|\mathcal{A}u|(K)}.$$

Morever, any L^1-limit point v of $\{u_{K,\varepsilon,x}\}_{\varepsilon>0}$ satisfies $\mathcal{A}u_{K,\varepsilon,x} \rightharpoonup^* \mathcal{A}v$, and hence has constant polar on K, i.e. satisfies

$$\mathcal{A}v = \left(\frac{\mathrm{d}\mathcal{A}u}{\mathrm{d}|\mathcal{A}u|}(x)\right)|\mathcal{A}v| \quad \text{on } K.$$

To ensure that the total variation is preserved along the blow-up limit procedure we recall the ensuing result.

Lemma 3.2 (Tangent measure with unit mass, Lemma 10.6 [34])**.** *Let $\mu \in \mathcal{M}_{loc}(\mathbb{R}^n;V)$. Then, for $|\mu|$-a.e. $x \in \mathbb{R}^n$ and for every bounded, open, convex set K the following assertions hold*

(a) *There exists a tangent measure $\gamma \in \mathrm{Tan}(\mu,x)$ such that $|\gamma|(K) = 1$, $|\gamma|(\partial K) = 0$;*
(b) *There exists $\varepsilon_i \downarrow 0$ as $i \uparrow +\infty$ such that*

$$\frac{F^{x,\varepsilon_i}_{\#}\mu}{F^{x,\varepsilon_i}_{\#}|\mu|(K)} \rightharpoonup^* \gamma \quad \textit{in} \quad \mathcal{M}(\overline{K};V).$$

Up to the light of the previous Proposition it is convenient to introduce the following notion limit point for blow-ups (subordinated to the proof of Proposition 3.6).

Definition 3.3 (Strict blow-up limit of u)**.** Let K be a convex set. We say that $(v,\gamma) \in \mathrm{BV}^{\mathcal{A}}(K;\mathbb{R}^m) \times \mathcal{M}(\overline{K};V)$ is a *strict blow-up limit* for $u \in \mathrm{BV}^{\mathcal{A}}(\Omega;\mathbb{R}^n)$ at a point x, and with respect to the convex set K (and we write $(v,\gamma) \in \mathrm{bu}_K(u;x)$), if there is a vanishing sequence $\{\varepsilon_i\}_{i\in\mathbb{N}}$ such that

1) $u_{K,\varepsilon_i,x}$ converges strictly to v in $\mathrm{BV}^{\mathcal{A}}(K;\mathbb{R}^m)$:
$$u_{K,\varepsilon_i,x} \to v \text{ in } L^1 \text{ and } |\mathcal{A}u_{K,\varepsilon_i,x}|(K) \to |\mathcal{A}v|(K);$$
2) $\gamma \in \mathrm{Tan}(\mathcal{A}u,x)$ is such that
$$\begin{aligned}
&\frac{1}{\mathcal{L}^n(K)}\mathcal{A}u_{K,\varepsilon_i,x} \overset{*}{\rightharpoonup} \gamma \text{ in } \mathcal{M}(\bar{K};V)\\
&|\gamma|(K) = |\gamma(K)| = 1, \quad |\gamma|(\partial K) = 0,\\
&\gamma = \frac{\mathrm{d}\mathcal{A}u}{\mathrm{d}|\mathcal{A}u|}(x)|\gamma| \text{ for } |\gamma| \text{ -a.e. } x \text{ in } K\\
&\mathcal{A}v = \mathcal{L}^n(K)\gamma \llcorner K.
\end{aligned}$$

The following ensures that $\mathrm{bu}_K(u;x)$ is never empty.

Lemma 3.4. *Let K be a center symmetric convex set. Let $u \in \mathrm{BV}^{\mathcal{A}}(\Omega,\mathbb{R}^m)$. Then, for $|\mathcal{A}u|$-a.e. $x \in \Omega$ the set $\mathrm{bu}_K(u;x)$ is not empty.*

Proof. We apply Lemma 3.2 to find $\gamma \in \mathrm{Tan}(\mathcal{A}u,x)$ and $\varepsilon_i \to 0$ such that
$$\frac{F^{x,\varepsilon_i}_{\#}\mathcal{A}u}{F^{x,\varepsilon_i}_{\#}|\mathcal{A}u|(K)} \overset{*}{\rightharpoonup} \gamma \quad \text{in } \mathcal{M}(\overline{K};\mathbb{R}^m).$$
Set $u_i := u_{K,\varepsilon_i,x}$. Since $\mathcal{R}_K[u_{K,\varepsilon_i,x}] = 0$, by (2.3)
$$\|u_i\|_{L^1(K;V)} = \|u_i - \mathcal{R}_K[u_i]\|_{L^1(K;V)} \le c|\mathcal{A}u_i|(K) = c\mathcal{L}^n(K)$$
for some c depending on $K, \mathcal{R}_K$ and n only. This implies (see Theorem 2.7) that, up to a further subsequence, $u_i \to v$ in $L^1(K;\mathbb{R}^m)$ with $v \in \mathrm{BV}^{\mathcal{A}}(K;\mathbb{R}^m)$. In particular
$$\mathcal{A}u_i \overset{*}{\rightharpoonup} \mathcal{A}v.$$
Now by the properties of γ and v follows that $(v,\gamma) \in \mathrm{bu}_K(u;x)$. □

Finally, with the help of the next result, we will be able to implement an iterated blow-up Lemma.

Theorem 3.5 (Tangent measures to tangent measures are tangent measures, cf. [31, Theorem 14.16])**.** *Let $\mu \in \mathcal{M}_{loc}(\mathbb{R}^n;V)$ be a Radon measure. Then for $|\mu|$-a.e. $x \in \mathbb{R}^n$ any $\nu \in \mathrm{Tan}(\mu,x)$ satisfies the following properties*

(a) *For any convex set K,*
$$\frac{F^{y,\rho}_{\#}\nu}{F^{y,\rho}_{\#}|\nu|(K)} \in \mathrm{Tan}(\mu,x) \quad \textit{for all} y \in \mathrm{spt}\,\nu, \rho > 0;$$
(b) $\mathrm{Tan}(\nu,y) \subseteq \mathrm{Tan}(\mu,x)$ *for all $y \in \mathrm{spt}\,\nu$;*

Note that the original result in [31, Theorem 14.16] is proven for $k=1$. However, the good properties of tangent space of measures (i.e. [4, Theorem 2.44] or [34, Lemma 10.4]) allow us to immediately extend its validity for generic k.

We now prove the central result, repeatedly used in our argument, that will allow us to take advantage of the notion of strict blow-up limit.

Proposition 3.6 (Blow-ups of blow-ups are blow-ups)**.** *Let $\mathcal{A}$ be a $\mathbb{C}$-elliptic operator. Fix K_1, K_2 two open bounded convex sets and pick $u \in \mathrm{BV}^{\mathcal{A}}(\Omega;\mathbb{R}^m)$. For $|\mathcal{A}u|$-a.e. $x \in \Omega$, any $h \in \mathrm{BV}^{\mathcal{A}}(K_1;\mathbb{R}^m)$ that is an L^1-limit point of $\{u_{K_1,\varepsilon,x}\}_{\varepsilon>0}$ satisfies the following property:*

- *for $|\mathcal{A}h|$−a.e. $y \in K_1$ if $(g,\gamma_g) \in \mathrm{bu}_{K_2}(h;y)$ then $(g - \mathcal{R}_{K_2}[g], \gamma_g) \in \mathrm{bu}_{K_2}(u;x)$.*

Remark 3.7. Let us remark the role of K_1 and K_2. In principle h is an L^1 blow-up of u, $\{u_{K_1,\varepsilon,x}\}_{\varepsilon>0}$ along the convex K_1 and g is a strict blow-up of h along the convex set K_2, $\{h_{K_2,\delta,y}\}_{\delta>0}$. The above Proposition in particular says that not only g is a blow-up of u but can also be reached as a sequence along K_2: $\{u_{K_2,\varepsilon_i,x}\}_{i\in\mathbb{N}}$. In particular we have the freedom to choose the convex set in doing the linearization procedure.

Proof. We pick an $x \in \Omega$ for which Theorem 3.5 is in force with respect to the measure $\mathcal{A}u$ (which is $|\mathcal{A}u|$ almost everywhere in Ω). Consider $h \in \mathrm{BV}^{\mathcal{A}}(K_1;\mathbb{R}^m)$ an L^1-limit point of $\{u_{K_1,\varepsilon,x}\}_{\varepsilon>0}$ and note that we can find a vanishing sequence $\{\varepsilon_i\}_{i\in\mathbb{N}}$ such that (in L^1)

$$u_{K_1,\varepsilon_i,x} \to h.$$

Notice that this means

$$\mathcal{A}u_{K_1,\varepsilon_i,x} \overset{*}{\rightharpoonup} \mathcal{A}h \quad \text{in } \mathcal{M}(K_1;V)$$

and it also means

$$\mathcal{L}^n(K_1)\frac{F^{x,\varepsilon_i}_{\#}\mathcal{A}u}{F^{x,\varepsilon_i}_{\#}|\mathcal{A}u|(K_1)} = \mathcal{A}u_{K_1,\varepsilon_i,x} \overset{*}{\rightharpoonup} \mathcal{A}h.$$

But then we have $\mathcal{A}h \in \mathrm{Tan}(\mathcal{A}u,x)$. In particular, because of Theorem 3.5, for all $y \in \mathrm{spt}(\mathcal{A}h)$, we can infer $\mathrm{Tan}(\mathcal{A}h,y) \subset \mathrm{Tan}(\mathcal{A}u,x)$. Fix one of such y for which also Lemma 3.4 holds (which are $|\mathcal{A}h|$-a.e. $y \in K_1$) and notice that, for $(g,\gamma_g) \in \mathrm{bu}_{K_2}(h,y)$ we can find a vanishing sequence $\{\delta_i\}_{i\in\mathbb{N}}$ such that

$$h_{K_2,\delta_i,y} \to g$$

strictly in $\mathrm{BV}^{\mathcal{A}}(K_2;\mathbb{R}^n)$ and

$$\frac{1}{\mathcal{L}^n(K_2)}\mathcal{A}h_{K_2,\delta_i,y} \overset{*}{\rightharpoonup} \gamma_g \quad \text{in } \mathcal{M}(\bar{K}_2;V)$$

and thus

$$\mathcal{L}^n(K_2)\frac{F^{y,\delta_i}_{\#}\mathcal{A}h}{F^{y,\delta_i}_{\#}|\mathcal{A}h|(K_2)} = \mathcal{A}h_{K_2,\delta_i,y} \overset{*}{\rightharpoonup} \mathcal{L}^n(K_2)\gamma_g \quad \text{in } \mathcal{M}(\bar{K}_2;V)$$

implying $\mathcal{L}^n(K_2)\gamma_g \in \mathrm{Tan}(\mathcal{A}h,y)$. We now know that the choice of y ensures $\mathcal{L}^n(K_2)\gamma_g \in \mathrm{Tan}(\mathcal{A}h,y) \subset \mathrm{Tan}(\mathcal{A}u,x)$ and therefore it must exists a sequence $\{c_i\}_{i\in\mathbb{N}}$ and a vanishing sequence $\{\rho_i\}_{i\in\mathbb{N}}$ such that

$$c_i F^{x,\rho_i}_{\#}\mathcal{A}u \overset{*}{\rightharpoonup} \mathcal{L}^n(K_2)\gamma_g.$$

Since $|\gamma_g|(\partial K_2) = 0$ we have

$$c_i F^{x,\rho_i}_{\#}|\mathcal{A}u|(K_2) \to \mathcal{L}^n(K_2)|\gamma_g|(K_2) = \mathcal{L}^n(K_2)$$

implying that

$$\mathcal{L}^n(K_2)\frac{F^{x,\rho_i}_{\#}\mathcal{A}u}{F^{x,\rho_i}_{\#}|\mathcal{A}u|(K_2)} \overset{*}{\rightharpoonup} \mathcal{L}^n(K_2)\gamma_g \quad \text{in } \mathcal{M}(\bar{K}_2;V). \tag{3.1}$$

Since $\mathcal{A}g = \mathcal{L}^n(K_2)\gamma_g$ on K_2 we obtain

$$\mathcal{L}^n(K_2)\frac{F_{\#}^{x,\rho_i}\mathcal{A}u}{F_{\#}^{x,\rho_i}|\mathcal{A}u|(K_2)} \rightharpoonup^* \mathcal{A}g \quad \text{on } \mathcal{M}(K_2;V).$$

Consider now $u_{K_2,\rho_i,x}$ and notice that

$$\|u_{K_2,\rho_i,x}\|_{L^1(K_2;\mathbb{R}^m)} \leq c|\mathcal{A}u_{K_2,\rho_i,x}|(K_2) = c\mathcal{L}^n(K_2)$$

and thus, up to extract a subsequence (not relabeled) we can infer $u_{K_2,\rho_i,x} \to \bar{g} \in \mathrm{BV}^{\mathcal{A}}(K_2;\mathbb{R}^m)$. In particular we also can deduce by this convergence that

$$\mathcal{A}g = \mathcal{A}\bar{g} \quad \Rightarrow \quad g = \bar{g} + L \ \text{ for some } L \in \mathrm{Ker}(\mathcal{A}).$$

This, combined with (3.1) implies that $(g - L, \gamma_g) = (\bar{g}, \gamma_g) \in \mathrm{bu}_{K_2}(u;x)$. To identify L we just use the fact that $\mathcal{R}_{K_2}$ is linear, continuous, keeps $\mathrm{Ker}(\mathcal{A})$ fixed and $\mathcal{R}_{K_2}[u_{K_2,\rho_i,x}] = 0$ for all $i \in \mathbb{N}$ yielding $\mathcal{R}_{K_2}[g - L] = \mathcal{R}_{K_2}[\bar{g}] = 0$. Thus

$$\mathcal{R}_{K_2}[g] = L.$$

□

4. Application

For a function u differentiable at x the following holds

$$\frac{u(x+\varepsilon y) - u(x)}{\varepsilon|\nabla u(x)|} \xrightarrow{\varepsilon\to 0} \frac{\nabla u(x)}{|\nabla u(x)|}y.$$

This relation also holds at points of approximate differentiability of a BV function, where the gradient is understood as the approximate gradient [4]. In this case, since $\frac{|Du|(K_\varepsilon(x))}{|K_\varepsilon(x)|} \xrightarrow{\varepsilon\to 0} |\nabla u(x)|$, we can write the equivalent relation

$$u_{K,\varepsilon,x}(y) = \frac{u(x+\varepsilon y) - (u)_K}{\frac{|Du|(K_\varepsilon(x))}{|K|\varepsilon^{n-1}}} \xrightarrow{\varepsilon\to 0} \frac{dD^a u}{d|D^a u|}(x)y = \frac{\nabla u(x)}{|\nabla u(x)|}y,$$

where $(\cdot)_K$ denotes the average over the convex set K. The same holds $\mathcal{L}^d$-a.e. for function with bounded deformation (cf. [3]).

In general, for a function with bounded $\mathcal{A}$-variation, approximate differentiability implies (see for instance [24]) that

$$u_{K,\varepsilon,x}(y) = \frac{u(x+\varepsilon y) - \mathcal{R}_K[u](y)}{\frac{|\mathcal{A}u|(K_\varepsilon(x))}{|K|\varepsilon^{n-1}}} \xrightarrow{\varepsilon\to 0} \frac{d\mathcal{A}^a u}{d|\mathcal{A}^a u|}(x)y,$$

at $\mathcal{L}^n$-a.e. point. Additionally, using the definition of *jump points*, it is quite a standard argument to verify that, for $\mathcal{H}^{n-1}$-a.e. $x \in J_u$, we have $u_{K,\varepsilon,x}(y) \xrightarrow{\varepsilon\to 0} H_{u^+(x),u^-(x),\nu(x)}(y)$ where

$$H_{u^+,u^-,\nu}(y) := \begin{cases} u^+ & \text{if } y\cdot\nu > 0 \\ u^- & \text{if } y\cdot\nu < 0. \end{cases}$$

By means of iterative blow-ups we will show that also at Cantor points a sequence of ε can be selected to converge to an affine function capturing the local behavior of the polar vector of the gradient measure, in the case $\mathcal{A} = D$ (BV-functions) and $\mathcal{A} = \mathcal{E}$ (BD-functions). Notice that, unlike the case of approximate differentiable points and jump points, where the blow-ups converge to a unique limit, at Cantor points we can only select a good sequence along which the blow-ups converge to a good limit. This is due to the highly irregular structure that the blow-ups can have around Cantor-type points. However, in typical homogenization and relaxation results, having one good blow-up is already enough for the applications.

The iterated blow-ups strategy can be described as a repeated application of Proposition 3.6, a rigidity structure results (as the one stated in Proposition 4.3), and the structure of the operator $\mathcal{R}$ appearing in the Poincaré inequality. For instance, considering the case $\mathcal{A} = \mathcal{E}$ the general strategy (following the milestones in [14] and [18]) can be illustrated like this:

1) If x is a point of approximate differentiability or a jump point, then blow-ups are given by the approximate differential $e(u)(x)y$ or by the jump function H;
2) If x is neither a jump point nor an approximate differentiability point then, by [17], we have that $\frac{\mathrm{d}\mathcal{E}^s u}{\mathrm{d}|\mathcal{E}^s u|}$ must have a precise shape (i.e., it belongs to the wave cone of the annihilator of $\mathcal{E}$);
   2.1) Then we consider a blow-up v, L^1-limit of $u_{K,\varepsilon_i,x}$. This blow-up will have the property of satisfying
$$\mathcal{E}v = \frac{\eta \odot \xi}{|\eta \odot \xi|}|\mathcal{E}v| \text{ on } K;$$
   2.2) The rigidity result in Proposition 4.3 will now yield information on the shape of v. In particular (in the BD case) we have
$$v(x) = \eta h_1(x \cdot \xi) + \xi h_2(x \cdot \eta) + L(x);$$
   2.3) If $x \notin \mathrm{TS}(u)$, by selecting a specific point on the domain we perform a second blow-up on v, which linearizes both directions, resulting in one blow-up of the form
$$w(y) = (\kappa_1 \xi \otimes \eta + \kappa_2 \eta \otimes \xi)y;$$
   2.4) By employing the information on $\mathcal{R}$ (with a further application of the rigidity result if η and ξ are parallel) the constant κ_i is proven to be $\kappa_i = \frac{1}{2|\eta \odot \xi|}$ yielding
$$g(y) = \frac{\eta \odot \xi}{|\eta \odot \xi|}y = \frac{\mathrm{d}\mathcal{E}u}{\mathrm{d}|\mathcal{E}u|}(x)y;$$
   2.5) By employing Proposition 3.6 we conclude that g can be obtained as the strict limit of $u_{K,\tilde{\varepsilon}_i,x}$ for some $\tilde{\varepsilon}_i \downarrow 0$.

In particular we show that (excluding the small family of *totally singular* points that we identify in step 2.3) almost all Cantor points must have at least a blow-up which is $\frac{\mathrm{d}\mathcal{E}u}{\mathrm{d}|\mathcal{E}u|}(x)y$. If a point $x \in \mathrm{TS}(u) \setminus J_u$ - still by applying Proposition 3.6 and given the rigidity result - we can always provide a blow-up affine at least in one direction as done in [14]. With this information at hand, it is now easy to proceed to relaxation and integral representation. Note that the final affine shape is not a surprise since the blow-ups are defined by removing the lower order terms $\mathcal{R}[u(x+\varepsilon\cdot)]$ (and this is crucial in giving the limiting affine shape; otherwise some contribution from $\mathcal{R}[u(x+\varepsilon\cdot)]$ might arise, requiring much more knowledge of the pointwise behavior of u around x).

The excluded set $\mathrm{TS}(u)$ is made by those point x for which all the L^1 blow-ups h are singular: $\mathcal{E}h = \mathcal{E}^s h$. This is a very specific property and brings to a natural question: how does a function u look if all its blow-ups have zero absolutely continuous part and can be expressed as sums of one-dimensional functions?

Generally speaking, the above strategy works also for a generic $\mathbb{C}$-elliptic operator $\mathcal{A}$. Indeed the crucial ingredients are: a rigidity result describing the structure of functions with constant polar $\mathcal{A}u = P|\mathcal{A}u|$, and a specific choice of the application $\mathcal{R}$ appearing in the Poincaré inequality 2.3. While the choice of $\mathcal{R}$ can be done in rather general contexts, the rigidity structure seems, instead, specific to each $\mathcal{A}$ and that would be where the main difficulties we believe to rely. Nonetheless, we see in this strategy a fruitful starting point for pursuing homogenization and integral representation results at least for some specific operators, such as the deviatoric strain $\mathcal{E}_{\rm dev}u := \mathcal{E}u - \frac{\operatorname{div}(u)}{d}\mathrm{Id}$, whose rigidity result will be published in a subsequent work by the authors.

4.1. **Affine blow-ups in** BV**.** We conclude by showing a similar statement in the context of BV maps.

Definition 4.1 (Totally singular points for BV maps)**.** For $u \in \mathrm{BV}(\Omega;\mathbb{R}^n)$, consider the blow-up sequences

$$u_{K,\varepsilon,x}(y) := \frac{u(x+\varepsilon y) - (u)_{K_\varepsilon(x)}}{\frac{|Du|(K_\varepsilon(x))}{|K|\varepsilon^{n-1}}}.$$

A point $x \in \Omega$ is said BV-totally singular for $u \in \mathrm{BV}(\Omega;\mathbb{R}^m)$ if for any L^1 limit points $p \in \mathrm{BV}(K;\mathbb{R}^m)$ of $\{u_{K,x,\varepsilon}\}_{\varepsilon>0}$ it holds $Dp = D^s p$. We denote by $\mathrm{TS}(u)$ the set of points which are BV-totally singular for u.

Theorem 4.2. *Let $n \geq 2$ and $u \in \mathrm{BV}(\Omega;\mathbb{R}^m)$. Then, for any convex set with $K = -K$ and for $|D^c u|$-a.e. $x \in \Omega \setminus \mathrm{TS}(u)$ there exists a vanishing sequence $\varepsilon_i \downarrow 0$ such that*

$$u_{K,\varepsilon_i,x}(y) \to \frac{\mathrm{d}Du}{\mathrm{d}|Du|}(x)y$$

strictly on $\mathrm{BV}(K;\mathbb{R}^m)$.

The proof of Theorem 4.2 it is an easy application of Proposition 3.6.

Proof of Theorem 4.2. Having chosen a Cantor point x such that

$$\frac{\mathrm{d}Du}{\mathrm{d}|Du|}(x) = \frac{\eta\otimes\xi}{|\eta\otimes\xi|},$$

(which are $|D^c u|$ almost all in Ω) thanks to the rigidity of BV blow-ups (see for instance [4]) we can infer that any BV limit point p of $\{u_{K,\varepsilon,x}\}_{\varepsilon>0}$ has the shape

$$p(y) = \psi(y\cdot\eta)\xi + c$$

for some $c,\xi \in \mathbb{R}^m$, $\eta \in \mathbb{R}^n$ and $\psi \in \mathrm{BV}_{loc}(\mathbb{R})$. Since $x \notin \mathrm{TS}(u)$ we can find at least one BV limit point with a ψ such that $\psi' \neq 0$ on a set of positive Lebesgue measure. Then, on setting $\Psi(z) := \psi(z\cdot\eta)\xi$ we have $\Psi' \neq 0$ on a set with positive Lebesgue measure. We thus can select a point $y \in K$ such that

a) $y \in \mathrm{spt}(Dp)$ with $y\cdot\eta$ a point of approximate differentiability for ψ and a Lebesgue point for $y \mapsto \frac{\mathrm{d}Dp}{\mathrm{d}|Dp|}(y)$;

b) we have

$$\frac{\Psi(y+\rho z)-(\Psi)_{K_\rho(y)}}{\rho} \to \beta_1(\eta\otimes\xi)z + c \text{ in } L^1(K)$$

for some $\beta_1 \in \mathbb{R}$, $c \in \mathbb{R}^m$;

c) it holds

$$\frac{|Dp|(K_\rho(y))}{\mathcal{L}^n(K_\rho(y))} \to \beta_3 > 0$$

.

We now pick $(g, \gamma_g) \in \mathrm{bu}_K(p; y)$ (which is not empty thanks to Lemma 3.4). Notice that by applying Proposition 3.6 we have $(g - (g)_K, \gamma) \in \mathrm{bu}_K(u; x)$. Recall that for BV:

$$\mathcal{R}_K(w) = (w)_K := \fint_K w \,\mathrm{d}x.$$

In particular $g - (g)_K$ can be reached as a BV-strict limit point of $\{u_{K,\varepsilon,x}\}_{\varepsilon>0}$. It is now enough to characterize $g - (g)_K$ as a blow-up of p.

Characterization of the blow-up g: notice that

$$p_{K,\rho_i,y}(z) = \frac{\Psi(y+\rho_i z) - (\Psi(y+\rho_i\cdot))_K}{\frac{|Dp|(K_{\rho_i}(y))}{|K|\rho_i^{n-1}}}.$$

Since

$$(\Psi(y+\rho_i\cdot))_K = (\Psi)_{K_{\varrho_i}(y)}$$

we just conclude that

$$p_{K,\rho_i,y}(z) = \frac{\Psi(y+\rho_i z) - (\Psi(y+\rho_i\cdot))_K}{\frac{|Dp|(K_{\rho_i}(y))}{|K|\rho_i^{n-1}}} = \frac{\Psi(y+\rho_i z) - (\Psi)_{K_{\varrho_i}(y)}}{\varrho_i\frac{|Dp|(K_{\rho_i}(y))}{|K|\rho_i^{n}}} \to \frac{\beta_1}{\beta_3}(\eta\otimes\xi)z + c$$

in $L^1(K;\mathbb{R}^m)$. Since $g(z) - (g)_K \in \mathrm{bu}_K(u;x)$ and $(g)_K = c$ we get

$$g(z) - (g)_K = \frac{\beta_1}{\beta_3}(\eta\otimes\xi)z.$$

Also we have strict convergence of the blow-ups and this implies that $|Dg|(K) = |Dp_{K,\varrho_i,x}|(K) = \mathcal{L}^n(K)$ implying

$$\frac{\beta_1}{\beta_3} = \frac{1}{|\eta\otimes\xi|}$$

and thus

$$g(z) - (g)_K = \left(\frac{\eta\otimes\xi}{|\eta\otimes\xi|}\right)z = \frac{\mathrm{d}Du}{\mathrm{d}|Du|}(x)z \in \mathrm{bu}_K(u;x).$$

□

4.2. **Affine blow-ups in** BD**.** Beyond the structure Theorems given in Subsection 2.3, we can refer - for $\mathcal{E}$ - to the fine properties obtained in [3]. For $u \in \mathrm{BD}(\Omega)$ the following well-known splitting is in force

$$\mathcal{E}u = e(u)\mathcal{L}^n + [u] \odot \nu_u \mathcal{H}^{n-1} \llcorner_{J_u} + \mathcal{E}^c u$$

where $e(u)(x) := \frac{d\mathcal{E}u}{\mathrm{d}\mathcal{L}^n}(x)$ (which can be computed as $e(u)(x) = \frac{\nabla u(x)+\nabla u(x)^t}{2}$, being $\nabla u(x)$ the approximate differential of u at x, existing $\mathcal{L}^n$-a.e. on Ω - cf. (2.4)), J_u the *jump set of* u, ν_u a unitary vector field normal to J_u, $[u] := u^+ - u^-$ the jump set, $a \odot b := \frac{a\otimes b + b \otimes a}{2}$ and $\mathcal{E}^c u$ stands for the cantor part of the measure $\mathcal{E}u$, supported on C_u:

$$C_u := \left\{ x \in \Omega \setminus S_u : \lim_{r\downarrow 0} \frac{|Eu|(B_r(x))}{r^n} = +\infty, \lim_{r\downarrow 0} \frac{|Eu|(B_r(x))}{r^{n-1}} = 0 \right\}.$$

Recall also that ([3, Theorem 6.1]) it holds

$$|\mathcal{E}u|(S_u \setminus J_u) = 0.$$

As a consequence of the more general result in [17], described in Subsection 2.3.3 for $|\mathcal{E}^c u|$−a.e. $x \in \Omega$ we have

$$\frac{\mathrm{d}\mathcal{E}^c u}{\mathrm{d}|\mathcal{E}^c u|}(x) = \frac{\eta(x) \odot \xi(x)}{|\eta(x) \odot \xi(x)|}$$

for some Borel $|\mathcal{E}^c u|$-measurable vector fields $\eta, \xi : \Omega \to \mathbb{R}^n$.

We report a rigidity result for BD maps with constant polar vector. The result can be found in [18, Theorem 2.10 (i)-(ii)] (see also [14, Proposition 3.9]).

Proposition 4.3 (Rigidity)**.** *If* $w \in \mathrm{BD}_{loc}(\mathbb{R}^n)$ *is such that for some* $\eta, \xi \in \mathbb{R}^n$

$$\mathcal{E}w = \frac{\eta \odot \xi}{|\eta \odot \xi|} |\mathcal{E}w|,$$

then

(i) *if* $\eta \neq \pm\xi$

$$w(y) = \alpha_1(y \cdot \xi)\eta + \alpha_2(y \cdot \eta)\xi + \mathrm{L}y + \mathrm{v},$$

for some $\alpha_1, \alpha_2 \in \mathrm{BV}_{loc}(\mathbb{R})$, $\mathrm{L} \in \mathbb{M}^{n\times n}_{skew}$, $\mathrm{v} \in \mathbb{R}^n$*;*

(ii) *if* $\eta = \pm\xi$

$$w(y) = \alpha(y \cdot \eta)\eta + \mathrm{L}y + \mathrm{v},$$

for some $\alpha \in \mathrm{BV}_{loc}(\mathbb{R})$, $\mathrm{L} \in \mathbb{M}^{n\times n}_{skew}$, $\mathrm{v} \in \mathbb{R}^n$*.*

We then recall the following particular kernel projection (cf. [14, Lemma 3.5, Proposition 3.6], that is classical when $K = B$ is the ball (see [3, 28]). We recall that

$$\mathrm{Ker}(\mathcal{E} := \{z(y) := \mathrm{L}y + \mathrm{b} \mid \mathrm{L} \in \mathbb{M}^{n\times n}_{skew}, \ \in \mathbb{R}^n\},$$

Lemma 4.4. *Let* K *be a center-symmetric convex set (i.e.* $K = -K$*). For* $u \in \mathrm{BD}(K)$ *we define* $\mathcal{R}_K[u]$ *to be the affine map*

$$\mathcal{R}_K[u](y) := \mathrm{L}_K[u]y + \mathrm{b}_K[u]$$

with

$$\mathrm{L}_K[u] := \frac{1}{2\mathcal{L}^n(K)} \int_{\partial K} (u \otimes \nu_{\partial K} - \nu_{\partial K} \otimes u) \, \mathrm{d}\mathcal{H}^{n-1}$$

$$\mathrm{b}_K[u] := \frac{1}{\mathcal{H}^{n-1}(\partial K)} \int_{\partial K} u \, \mathrm{d}\mathcal{H}^{n-1}.$$

Then $\mathcal{R}_K : \mathrm{BD}(K) \to \mathrm{Ker}(\mathcal{E})$ *extends to a linear, bounded functional on* $L^1(K;\mathbb{R}^n)$ *with values in* $\mathrm{Ker}(\mathcal{E})$ *and* $\mathcal{R}_K[p] = p$ *for all* $p \in \mathrm{Ker}(\mathcal{E})$.

In particular $\mathcal{R}$ can be used to define the blow-ups, and in the Poincaré inequality (2.3). Note that for an affine function and for a center-symmetric convex set K, one has, by a simple integration by parts,

$$\mathcal{R}_K[Ax+b](y) = \frac{(A-A^T)}{2}y + b \tag{4.1}$$

With these notions retrieved we can now prove the blow-up selection principle Theorem. We first introduce the Definition of *totally singular points* that will make the statement more clean.

Definition 4.5 (Totally singular points for BD maps)**.** For $u \in \mathrm{BD}(\Omega)$, consider the blow-up sequences

$$u_{K,\varepsilon,x}(y) := \frac{u(x+\varepsilon y) - \mathcal{R}_K[u(x+\varepsilon\cdot)](y)}{\frac{|\mathcal{E}u|(K_\varepsilon(x))}{|K|\varepsilon^{n-1}}}.$$

A point $x \in \Omega$ is said to be a *totally singular point* for $u \in \mathrm{BD}(\Omega)$ if for any L^1-limit point $p \in \mathrm{BD}(K)$ of $\{u_{K,\varepsilon,x}\}_{\varepsilon>0}$ it holds $\mathcal{E}p = \mathcal{E}^s p$. We denote by $\mathrm{TS}(u)$ the set of points which are totally singular for u.

Theorem 4.6. *Let* $u \in \mathrm{BD}(\Omega)$. *Let* K *be a center-symmetric convex set. Then for* $|\mathcal{E}^c u|$*-a.e.* $x \in \Omega \setminus \mathrm{TS}(u)$ *there exists a sequence* $\varepsilon_i \downarrow 0$ *such that*

$$u_{K,\varepsilon_i,x}(y) \to \frac{\mathrm{d}\mathcal{E}u}{\mathrm{d}|\mathcal{E}u|}(x)y \quad \textit{strictly in } \mathrm{BD}(K)$$

Proof of Theorem 4.6. Since x is fixed for the rest of the proof, we simply denote by η, ξ the vectors of the polar. We prove the statement at any $x \in \Omega \setminus \mathrm{TS}(u)$ such that Proposition 3.6 is in force (which are $|\mathcal{E}^c u|$-almost all $x \in \Omega$). Since $x \notin \mathrm{TS}(u)$ then there exists an L^1 limit point h of $\{u_{K,\varepsilon_i,x}\}_{i\in\mathbb{N}}$ such that $e(h)\mathcal{L}^n \neq 0$. Because of Proposition 4.3 we have

$$h(y) = \alpha_1(y\cdot\eta)\xi + \alpha_2(y\cdot\xi)\eta + \mathrm{L}y + \mathrm{v}$$

for some $\alpha_i \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$, $\mathrm{L} \in \mathbb{M}^{n\times n}_{skew}$, $v \in \mathbb{R}^n$. Since $e(u) \neq 0$ it follows that $\alpha_1'(y\cdot\eta) + \alpha_2'(y\cdot\xi) \neq 0$. Set

$$\Psi_1(y) := \alpha_1(y\cdot\eta)\xi, \quad \Psi_2(y) := \alpha_2(y\cdot\xi)\eta$$

and let us now select $y \in K$ satisfying

a) $y \in \mathrm{spt}(\mathcal{E}h)$ with $y\cdot\eta$, $y\cdot\xi$ a point of approximate differentiability (respectively) for α_1, α_2 and a Lebesgue point for $y \mapsto \frac{\mathrm{d}\mathcal{E}h}{\mathrm{d}|\mathcal{E}h|}(y)$;

b) we have

$$\frac{\Psi_1(y+\rho\cdot) - (\Psi_1)_{\partial K_\rho(y)}}{\rho} \to \beta_1^{(1)}\xi\otimes\eta + c_1 \text{in } L^1(K;\mathbb{R}^n)$$

for some $\beta_1^{(1)} \in \mathbb{R}$, $c \in \mathbb{R}^n$;

c) it holds

$$\frac{\Psi_2(y+\rho\cdot) - (\Psi_2)_{\partial K_\rho(y)}}{\rho} \to \beta_1^{(2)}\eta\otimes\xi + c_2 \text{ in } L^1(K;\mathbb{R}^n)$$

for some $\beta_1^{(2)} \in \mathbb{R}$, $c \in \mathbb{R}^n$;

d) we have

$$\frac{D\Psi_1(K_\rho(y))}{\mathcal{L}^n(K)\rho^n} \to \beta_2^{(1)}\xi\otimes\eta;$$

e) it holds

$$\frac{D\Psi_2(K_\rho(y))}{\mathcal{L}^n(K)\rho^n} \to \beta_2^{(2)}\eta\otimes\xi;$$

d) we have

$$\frac{|\mathcal{E}h|(K_\rho(y))}{\mathcal{L}^n(K_\rho(y))} \to \beta_3 > 0.$$

Since $x \notin \mathrm{TS}(u)$ (and since $e(h) = (\Psi_1' + \Psi_2')\eta \odot \xi \neq 0$) we can guarantee that, for a set of positive $|\mathcal{E}h|$ measure we have $\beta_3 = \beta_2^{(1)} + \beta_2^{(2)} \neq 0$.

Consider now $(g,\gamma) \in \mathrm{bu}_K(h;y)$ (which is not empty due to Lemma 3.4) and notice that, as in the proof of 4.2, by applying Proposition 3.6 we have $(g - \mathcal{R}_K[g],\gamma) \in \mathrm{bu}_K(u;x)$. In particular $g-\mathcal{R}_K[g]$ can be reached as a BD-strict limit point of $\{u_{K,\varepsilon,x}\}_{\varepsilon>0}$. It is now enough to characterize $g - \mathcal{R}_K[g]$ as a blow-up of h.

Characterization of the blow-up g: Notice that

$$h_{K,\rho_i,y}(z) = \frac{\Psi_1(y+\rho_i z) + \Psi_2(y+\rho_i z) - \mathcal{R}_K[h(y+\rho_i\cdot)](z)}{\frac{|\mathcal{E}h|(K_{\rho_i}(y))}{|K|\rho_i^{n-1}}}$$

and that, by (4.1),

$$\mathcal{R}_K[h(y+\rho_i\cdot)](z) = \frac{|K|\rho_i^{n-1}}{|\mathcal{E}h|(K_{\rho_i}(y))}\left(\mathcal{R}_K[\Psi_2(y+\rho_i\cdot)](z) + \mathcal{R}_K\left[\Psi_1(y+\rho_i\cdot)\right](z)\right)$$

Moreover

$$\begin{aligned}
\mathrm{L}_K[\alpha_1((y+\rho_i\cdot)\cdot\eta)\xi] &= \frac{1}{2|K|}\int_{\partial K}\alpha_1((y+\rho_i z)\cdot\eta)(\xi\otimes\nu_K - \nu_K\otimes\xi)\,\mathrm{d}\mathcal{H}^{n-1}(z)\\
&= \frac{1}{2|K|\rho_i^{n-1}}\int_{\partial K_{\rho_i}(y)}\alpha_1(z\cdot\eta)(\xi\otimes\nu_{K_{\rho_i}(y)} - \nu_{K_{\rho_i}(y)}\otimes\xi)\,\mathrm{d}\mathcal{H}^{n-1}(z)\\
&= \frac{\rho_i}{2|K_{\rho_i}(y)|}\left[D\Psi_1(K_{\rho_i}(y)) - D\Psi_1(K_{\rho_i}(y))^t\right]
\end{aligned}$$

and analogously

$$\mathrm{L}_K[\alpha_2((y+\rho_i\cdot)\cdot\xi)\eta] = \frac{\rho_i}{2|K_{\rho_i}(y)|}\left[D\Psi_2(K_{\rho_i}(y)) - D\Psi_2(K_{\rho_i}(y))^t\right]$$

Thence

$$\begin{aligned}
h_{K,\rho_i,y}(z) =& \frac{|K|\rho_i^{n-1}}{|\mathcal{E}h|(K_{\rho_i}(y))}\Bigg[\Psi_1(y+\rho_i z) - (\Psi_1)_{\partial K_{\rho_i}(y)} - \frac{\rho_i}{2|K_{\rho_i}(y)|}\left[D\Psi_1(K_{\rho_i}(y)) - D\Psi_1(K_{\rho_i}(y))^t\right]z\\
&+\Psi_2(y+\rho_i z) - (\Psi_2)_{\partial K_{\rho_i}(y)} - \frac{\rho_i}{2|K_{\rho_i}(y)|}\left[D\Psi_2(K_{\rho_i}(y)) - D\Psi_2(K_{\rho_i}(y))^t\right]z\Bigg]
\end{aligned}$$

First we see that, thanks to our choice of y we have

$$\frac{\frac{\rho_i}{2|K_{\rho_i}(y)|}\left[D\Psi_j(K_{\rho_i}(y)) - D\Psi_j(K_{\rho_i}(y))^t\right]}{\frac{|\mathcal{E}h|(K_{\rho_i}(y))}{|K|\rho_i^{n-1}}} \to \frac{\beta_2^{(j)}}{2\beta_3}\left(\xi\otimes\eta - \eta\otimes\xi\right) \quad \text{(because of hypothesis d), e), f)).}$$

Also that, because of hypothesis b), c)

$$\frac{\Psi_j(y+\rho_i z) - (\Psi_j)_{\partial K_{\rho_i}(y)}}{\rho_i} \to \beta_1^{(j)}(\xi\otimes\eta)z + c$$

in L^1 and thus

$$\frac{\Psi_j(y+\rho_i z) - (\Psi_j)_{\partial K_{\rho_i}(y)}}{\frac{|\mathcal{E}h|(K_{\rho_i}(y))}{|K|\rho_i^{n-1}}} \to \frac{\beta_1^{(j)}}{\beta_3}(\xi\otimes\eta)z + \frac{c_j}{\beta_3}$$

in $L^1(K;\mathbb{R}^n)$. Thence we conclude that

$$h_{K,\rho_i,y} \to \kappa_1(\eta\otimes\xi)z + \kappa_2(\xi\otimes\eta)z + C$$

in $L^1(K;\mathbb{R}^n)$ and for some $\kappa_1,\kappa_2\in\mathbb{R}$, $C\in\mathbb{R}^n$. Since $h_{K,\rho_i,y}\to g$ strictly in $\mathrm{BD}(K)$ we conclude

$$g(z) = \kappa_1(\eta\otimes\xi)z + \kappa_2(\xi\otimes\eta)z + C.$$

We now know that $g - \mathcal{R}_K[g]$ is also a BD-strict limit point of $\{u_{K,\varepsilon,x}\}_{\varepsilon>0}$. In particular, setting $\mathcal{R}_K[g](z) = Lz + b$ for some $L\in\mathbb{M}^{n\times n}_{skew}$, by (4.1), we see that

$$\frac{\kappa_1-\kappa_2}{2}\left(\eta\otimes\xi - \xi\otimes\eta\right) - L = 0, \quad C - b = 0$$

which means $C = b$ and

$$L = \frac{\kappa_1-\kappa_2}{2}\left(\eta\otimes\xi - \xi\otimes\eta\right).$$

Thus

$$g(z) - \mathcal{R}_K[g](z) = \kappa(\eta\odot\xi)z$$

with $\kappa = \frac{\kappa_1+\kappa_2}{2}$. We now combine this information with the fact that $|\mathcal{E}g|(K) = \mathcal{L}^n(K)$, again implied by the strict convergence, to obtain

$$\mathcal{L}^n(K) = |\mathcal{E}g|(K) = \kappa|\eta\odot\xi|\mathcal{L}^n(K) \quad \Rightarrow \quad \kappa = \frac{1}{|\eta\odot\xi|}$$

achieving

$$g(z) - \mathcal{R}_K[g](z) = \left(\frac{\eta\odot\xi}{|\eta\odot\xi|}\right)z = \frac{\mathrm{d}\mathcal{E}u}{\mathrm{d}|\mathcal{E}u|}(x)z.$$

□

Remark 4.7. Notice that $x\notin \mathrm{TS}(u)$ is crucial in the above argument since a further linearization through an additional blow-up can be performed only if we can find a BD limit point h with $\mathcal{E}h \neq \mathcal{E}^s h$.

Acknowledgments: MC thanks the financial support of PRIN 2022R537CS "Nodal optimization, nonlinear elliptic equations, nonlocal geometric problems, with a focus on regularity" funded by the European Union under Next Generation EU. NVG was supported by the FCT project UIDB/04561/2020. The authors are deeply grateful to F. Gmeineder for the content of Proposition 2.5.

References

[1] G. Alberti. Rank one property for derivatives of functions with bounded variation. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 123(2):239–274, 1993.

[2] G. Alberti, S. Bianchini, and G. Crippa. On the L^p-differentiability of certain classes of functions. *Revista Matemática Iberoamericana*, 30(1):349–367, 2014.

[3] L. Ambrosio, A. Coscia, and G. Dal Maso. Fine properties of functions with bounded deformation. *Archive for Rational Mechanics and Analysis*, 139(3):201–238, 1997.

[4] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*, volume 254. Clarendon Press Oxford, 2000.

[5] S. Amstutz and N. Van Goethem. Analysis of the incompatibility operator and application in intrinsic elasticity with dislocations. *SIAM J. Math. Anal.*, 48(1):320–348, 2016.

[6] O. Anza Hafsa and J.-Ph. Mandallena. Stochastic homogenization of nonconvex integrals in the space of functions of bounded deformation. *Asymptotic Analysis*, 131(2):209 – 232, 2023.

[7] A. Arroyo-Rabasa. Slicing and fine properties for functions with bounded $\mathbb{A}$-variation. *arXiv preprint arXiv:2009.13513*, 4, 2020.

[8] A. Arroyo-Rabasa, G. De Philippis, J. Hirsch, and F. Rindler. Dimensional estimates and rectifiability for measures satisfying linear pde constraints. *Geometric and Functional Analysis*, 29(3):639–658, 2019.

[9] A. Arroyo-Rabasa and A. Skorobogatova. A look into some of the fine properties of functions with bounded $\mathbb{A}$-variation. 2019.

[10] A. C. Barroso, J. Matias, and E. Zappale. Relaxation for an optimal design problem in BD. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 153(3):721 – 763, 2023.

[11] G. Bouchitté, I. Fonseca, and L. Mascarenhas. A global method for relaxation. *Arch. Rational Mech. Anal.*, 145(1):51–98, 1998.

[12] D. Breit, L. Diening, and F. Gmeineder. On the trace operator for functions of bounded $\mathbb{A}$-variation. *Analysis e PDE*, 13(2):559 – 594, 2020.

[13] A. P. Calderon and A. Zygmund. On the existence of certain singular integrals. *Acta Mathematica*, 88(none):85 – 139, 1952.

[14] M. Caroccia, M. Focardi, and N. Van Goethem. On the integral representation of variational functionals on BD. *SIAM Journal on Mathematical Analysis*, 52(4):4022–4067, 2020.

[15] M. Cicalese, M. Focardi, and C. I. Zeppieri. Phase-field approximation of functionals defined on piecewise-rigid maps. *Journal of Nonlinear Science*, 31(5), 2021.

[16] V. Crismale, M. Friedrich, and F. Solombrino. Integral representation for energies in linear elasticity with surface discontinuities. *Advances in Calculus of Variations*, 15(4):705 – 733, 2022.

[17] G. De Philippis and F. Rindler. On the structure of $\mathcal{A}$-free measures and applications. *Annals of Mathematics*, pages 1017–1039, 2016.

[18] G. De Philippis and F. Rindler. Fine properties of functions of bounded deformation-an approach via linear pdes. *Mathematics in Engineering*, 2(3):386–422, 2020.

[19] G. Del Nin. Rectifiability of the jump set of locally integrable functions. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*, 22(3):1233–1240, 2021.

[20] L. Diening and F. Gmeineder. Continuity points via Riesz potentials for $\mathbb{C}$-elliptic operators. *Quarterly Journal of Mathematics*, 71(4):1201 – 1218, 2020.

[21] L. Diening and F. Gmeineder. Sharp trace and korn inequalities for differential operators. *Potential Analysis*, 2024.

[22] F. Ebobisse. Fine properties of functions with bounded deformation and applications in variational problems. *Ph.D. Thesis*, 1999.

[23] L. C. Evans. *Measure theory and fine properties of functions*. Routledge, 2018.

[24] F. Gmeineder and B. Raita. On critical L^p–differentiability of BD–maps. *Rev. Mat. Iberoam*, 5, 2017.

[25] F. Gmeineder and B. Raiţă. Embeddings for $\mathbb{A}$-weakly differentiable functions on domains. *J. Funct. Anal.*, 277(12):33, 2019. Id/No 108278.

[26] F. Gmeineder, B. Raiţă, and J. van Schaftingen. On limiting trace inequalities for vectorial differential operators. *Indiana University Mathematics Journal*, 70(5):2133 – 2176, 2021.

[27] F. Gmeineder, B. Raiţă, and J. Van Schaftingen. Boundary ellipticity and limiting L^1-estimates on halfspaces. *Advances in Mathematics*, 439, 2024.

[28] R. V. Kohn. New estimates for deformations in terms of their strains: I) estimates of Wirtinger type for nonlinear strains; II) functions whose linearized strains are measures. 1980.

[29] F. Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[30] G. B. Maggiani, R. Scala, and N. Van Goethem. A compatible-incompatible decomposition of symmetric tensors in L^p with application to elasticity. *Math. Methods Appl. Sci.*, 38(18):5217–5230, 2015.

[31] P. Mattila. *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*. Number 44. Cambridge university press, 1999.

[32] D. Ornstein. A non-inequality for differential operators in the L^1 norm. *Archive for Rational Mechanics and Analysis*, 11:40–49, 1962.

[33] B. Raiţă. Critical L^p-differentiability of $BV^{\mathbb{A}}$-maps and canceling operators. *Transactions of the American Mathematical Society*, 372(10):7297 – 7326, 2019.

[34] F. Rindler. *Calculus of Variations*. Springer, 2018.

[35] R. Scala and N. Van Goethem. Constraint reaction and the Peach-Koehler force for dislocation networks. *Math. Mech. Complex Syst.*, 4(2):105–138, 2016.

[36] R. Temam and G. Strang. Functions of bounded deformation. *Arch. Rational Mech. Anal.*, 75:7–21, 1980.

[37] J. Van Schaftingen. Limiting Sobolev inequalities for vector fields and canceling linear differential operators. *J. Eur. Math. Soc. (JEMS)*, 15(3):877–921, 2013.

Dipartimento di matematica, Politecnico di Milano
Email address: marco.caroccia@polimi.it

Centro de Matemática e Aplicações Fundamentais, UniversI.de de Lisboa
Email address: vangoeth@fc.ul.pt