ENERGY-DISSIPATION BALANCE FOR DYNAMIC VISCOELASTIC PROBLEMS WITH MEMORY IN DOMAINS WITH GROWING CRACKS

GIANNI DAL MASO AND RODICA TOADER

Abstract. We consider a dynamic viscoelastic problem with memory in a domain with a crack growing with constant velocity. Through a careful analysis of the singularity of the solutions around the crack tip we show that for suitable values of the material constants there exist solutions that satisfy the energy-dissipation balance. It is known that this is not possible for the Kelvin-Voigt model.

2020 Mathematics Subject Classification: 49J45, 74D05, 74H35, 74R10.

1. Introduction

We study a dynamic viscoelastic problem with memory in the isothermal antiplane case in a domain with a prescribed time-dependent crack growing along the x-axis.

Let Ω be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary. We assume that $(0,0)\in\Omega$. For every $a\in\mathbb{R}$ let

$$\Gamma_a := (-\infty, a]\times\{0\}. \tag{1.1}$$

In our problem the crack at time t is given by $\Gamma_{\ell(t)}\cap\Omega$, where $\ell(t)$ is a prescribed non-decreasing function of time. Given a time-dependent Dirichlet boundary condition $w(t)$, in our viscoelastic model with memory the boundary value problem for the displacement $u(t)$ is formally written as

$$\ddot u(t)-\operatorname{div}\sigma(t)=0 \text{ in } \Omega\setminus\Gamma_{\ell(t)}\,,\quad \sigma(t)e_2=0 \text{ on } \Gamma_{\ell(t)}\cap\Omega\,,\quad u(t)=w(t) \text{ on } \partial\Omega\,. \tag{1.2}$$

Here and henceforth dots denote time derivatives, div is the space divergence, and $\sigma(t)$ denotes the stress, which is the sum of the elastic stress $\sigma_e(t)$ and the viscous stress $\sigma_v(t)$, while e_2 is the second vector of the canonical basis in $\mathbb{R}^2$, which is normal to $\Gamma_{\ell(t)}$. The constitutive relations for these components of the stress are

$$\sigma_e(t)=c_e\nabla u(t)\,,$$
$$\sigma_v(t)=c_v\nabla u(t)-c_v\int_{-\infty}^t e^{s-t}\nabla u(s)ds\,,$$

for suitable constants $c_e>0$ and $c_v>0$, where ∇ denotes the space gradient. By a convenient choice of the units of u and t we can assume that $c_e+c_v=1$, so that

$$\sigma(t)=\sigma_e(t)+\sigma_v(t)=\nabla u(t)-c_v\int_{-\infty}^t e^{s-t}\nabla u(s)ds\,.$$

Preprint SISSA 02/2025/MATE.

Therefore the boundary value problem for $u(t)$ can be formally written in the more explicit form

$$\ddot{u}(t) - \Delta u(t) = -c_v \int_{-\infty}^{t} e^{s-t} \Delta u(s) ds \qquad \text{in } \Omega \setminus \Gamma_{\ell(t)}\,, \tag{1.3}$$

$$\partial_y u(t) = c_v \int_{-\infty}^{t} e^{s-t} \partial_y u(s) ds \qquad \text{on } \Gamma_{\ell(t)} \cap \Omega\,, \tag{1.4}$$

$$u(t) = w(t) \qquad \text{on } \partial\Omega\,, \tag{1.5}$$

where Δ is the space Laplacian and ∂_y denotes the partial derivative with respect to y.

In the model of crack growth we are considering, the functions $\ell(t)$ and $u(t)$ are related by the dynamic energy-dissipation balance (see, e.g., [10] for the case without viscosity). This condition, introduced by Mott [15], extends to the dynamic regime the classical Griffith's criterion for the quasistatic problem (see [11]). In our case the relevant energy terms are:

• the sum of the kinetic and the elastic energies

$$\mathcal{E}(t) := \frac{1}{2} \int_{\Omega\setminus\Gamma_{\ell(t)}} |\dot{u}(t)|^2 dxdy + \frac{c_e}{2} \int_{\Omega\setminus\Gamma_{\ell(t)}} |\nabla u(t)|^2 dxdy\,;$$

• the energy dissipated by viscosity in the time interval $[t_1, t_2]$, which can be written as

$$\mathcal{D}(t_1,t_2) := \frac{c_v}{2} \int_{\Omega\setminus\Gamma_{\ell(t_2)}} |\nabla u(t_2)|^2 dxdy - \frac{c_v}{2} \int_{\Omega\setminus\Gamma_{\ell(t_1)}} |\nabla u(t_1)|^2 dxdy - \int_{t_1}^{t_2} \Big(\int_{\Omega\setminus\Gamma_{\ell(t)}} F_u(t) \nabla \dot{u}(t) dxdy \Big) dt\,,$$

where

$$F_u(t) := c_v \int_{-\infty}^{t} e^{s-t} \nabla u(s) ds\,; \tag{1.6}$$

• the work of the forces acting on $\partial\Omega$ in the time interval $[t_1, t_2]$

$$\mathcal{W}(t_1,t_2) := \int_{t_1}^{t_2} \Big(\int_{\partial\Omega} \sigma(t) \nu \dot{w}(t) d\mathcal{H}^1 \Big) dt\,,$$

where ν is the outer unit normal to $\partial\Omega$;

• the energy dissipated by the crack growth in the time interval $[t_1, t_2]$, which, according to Griffith's theory, is assumed to be proportional to the added length, i.e.,

$$\mathcal{K}(t_1,t_2) := \beta(\ell(t_2) - \ell(t_1))\,,$$

for some constant $\beta > 0$ that represents the fracture toughness of the material.

The energy-dissipation balance is given by

$$\mathcal{E}(t_2) - \mathcal{E}(t_1) + \mathcal{D}(t_1,t_2) + \mathcal{K}(t_1,t_2) = \mathcal{W}(t_1,t_2) \tag{1.7}$$

for almost every $t_1 < t_2$.

In the Kelvin-Voigt model, where the viscous stress is given by $\sigma_v(t) = c_v \nabla \dot{u}(t)$ and the energy dissipated by viscosity is given by

$$\mathcal{D}(t_1,t_2) := c_v \int_{t_1}^{t_2} \Big(\int_{\Omega\setminus\Gamma_{\ell(t)}} |\nabla \dot{u}(t)|^2 dxdy \Big) dt\,,$$

we can prove that

$$\mathcal{E}(t_2) - \mathcal{E}(t_1) + \mathcal{D}(t_1,t_2) = \mathcal{W}(t_1,t_2)$$

for every $t_1 < t_2$, see [6]. This implies that the energy-dissipation balance (1.7) is satisfied if and only if $\mathcal{K}(t_1,t_2) = 0$, hence $\ell(t_1) = \ell(t_2)$ for every $t_1 < t_2$. Therefore the crack cannot grow in the Kelvin-Voigt model when the energy-dissipation balance is satisfied. This is known as the viscoelastic paradox for crack growth, see, e.g., [17].

In this paper, when c_v is sufficiently small and $\beta > 0$ is arbitrary, we provide an example of solution to (1.3)-(1.5), with $\ell(t) = ct$ for some constant $0 < c < 1$ and a suitable Dirichlet boundary condition $w(t)$, such that the energy-dissipation balance (1.7) is satisfied (see Theorem 6.1). Hence the viscoelastic paradox does not occur in this model with memory.

In particular, this shows that the model considered in [3], based on the energy-dissipation balance and on a maximal dissipation condition, leads to a growing crack for suitable values of the data.

Our result is obtained as a consequence of the precise form of the singular behaviour of the gradient of the solution $u(t)$ near the crack tip (see Section 6). We analyse this singularity only when the solution has the special form $u(t,x,y) = v(x-ct,y)$, which requires a special form of the boundary datum $w(t)$. By (1.3) the function v must satisfy an elliptic equation with a non-local term in a domain with the fixed crack Γ_0 (see Section 3). The singularity of the gradient of the solution of a local elliptic equation in these domains has been studied in detail by Grisvard. The main difficulty in our analysis is to extend these results to our non-local equation (see Section 4). Our approach is perturbative, therefore it requires a restriction on the viscosity constant c_v and the crack tip velocity c.

2. Preliminaries

2.1. **Basic notation.** In this paper we study an evolutionary boundary value problem in a suitable time interval and in a domain contained in the plane $\mathbb{R}^2$ with coordinates x and y. Given a time-dependent function ψ defined on a subset of $\mathbb{R}^2$, its time derivative is denoted by $\dot\psi$, the partial derivatives with respect to the spatial coordinates x and y are denoted by ∂_x and ∂_y, respectively, while ∇, div, and Δ denote the gradient, the divergence, and the Laplacian with respect to (x,y).

2.2. **A model of viscoelastic material with memory.** We shall consider a specific model for a viscoelastic material with fading memory in the isothermal antiplane case, with reference configuration $\Omega \subset \mathbb{R}^2$. In detail, the evolution of the (scalar) displacement $u(t)$ is governed by the partial differential equation

$$\ddot u(t) - \operatorname{div}\sigma(t) = 0\,, \tag{2.1}$$

where the stress $\sigma(t) \in \mathbb{R}^2$ is decomposed as $\sigma(t) := \sigma_e(t) + \sigma_v(t)$, with the elastic stress σ_e and the viscous stress σ_v given by the constitutive equations

$$\sigma_e(t) := c_e \nabla u(t) \quad \text{and} \quad \sigma_v(t) := c_v \nabla u(t) - c_v \int_{-\infty}^t e^{s-t} \nabla u(s) ds\,. \tag{2.2}$$

Here and henceforth $c_e > 0$ is the elasticity constant and $c_v > 0$ is the viscosity constant.

The study of models for viscoelastic materials with memory goes back to Maxwell [14], Boltzmann [2], and Volterra [20]-[22]. There is now a huge literature on these models for which we refer to the books [18, 8, 19, 9, 1] and the references therein. We refer to Dafermos [5] for the precise formulation of (2.1) and (2.2) in suitable function spaces and for the proof of an existence result with prescribed time dependent Dirichlet boundary conditions.

For this problem the following energy-dissipation balance holds:

$$\mathcal{E}(t_2) - \mathcal{E}(t_1) + \mathcal{D}(t_1,t_2) = \mathcal{W}(t_1,t_2)\,, \quad \text{for } t_1 < t_2\,, \tag{2.3}$$

where

$$\mathcal{E}(t) := \frac{1}{2}\int_\Omega |\dot u(t)|^2 dxdy + \frac{c_e}{2}\int_\Omega |\nabla u(t)|^2 dxdy$$

is the sum of the kinetic and elastic energies, $\mathcal{W}(t_1,t_2)$ is the work, in the time interval $[t_1,t_2]$, of the external forces due to the imposed time-dependent Dirichlet boundary condition, while

$$\mathcal{D}(t_1,t_2) := \frac{c_v}{2}\int_\Omega |\nabla u(t_2)|^2 dxdy - \frac{c_v}{2}\int_\Omega |\nabla u(t_1)|^2 dxdy - \int_{t_1}^{t_2}\Big(\int_\Omega F_u(t)\nabla \dot u(t) dxdy\Big) dt \tag{2.4}$$

with $F_u(t)$ defined in (1.6). In view of (2.3) $\mathcal{D}(t_1,t_2)$ has to be interpreted as the energy dissipated by viscosity in the interval $[t_1,t_2]$ (see, e.g., [17, Chapter 7]).

2.3. **The evolution problem with a prescribed growing crack.** We now introduce a model for the dynamic evolution of a viscoelastic material with memory in a domain with a prescribed growing crack. The reference configuration is a bounded open domain $\Omega\subset\mathbb{R}^2$ with Lipschitz boundary. For simplicity we assume that $(0,0)\in\Omega$ and that the crack grows along the x-axis. More precisely, we fix $T_0<T_1$ and a non decreasing function $\ell\colon[T_0,T_1]\to\mathbb{R}$. The crack at time $t\in[T_0,T_1]$ is given by $\Gamma_{\ell(t)}\cap\Omega$, where Γ is defined in (1.1).

In this case equation (2.1) is replaced by

$$\ddot u(t)-\operatorname{div}\sigma(t)=0\quad\text{in }\Omega\setminus\Gamma_{\ell(t)}\,,\qquad \sigma(t)e_2=0\quad\text{on }\Gamma_{\ell(t)}\cap\Omega\,,\tag{2.5}$$

where $\sigma(t)=\sigma_e(t)+\sigma_v(t)$ and the constitutive equations for the elastic and viscous stresses are still given by (2.2), while e_2 is the second vector of the canonical basis in $\mathbb{R}^2$, which is normal to $\Gamma_{\ell(t)}$. Of course, in this non-local in time formulation the usual initial conditions at $t=T_0$ are replaced by the condition $u(t)=u_0(t)$ for a.e. $t\in(-\infty,T_0)$, where u_0 is a prescribed function.

By a suitable choice of the units of u and t we can assume that $c_e+c_v=1$, so that, setting $\alpha:=c_v\in(0,1)$, we have

$$\sigma(t)=\nabla u(t)-\alpha\int_{-\infty}^t e^{s-t}\nabla u(s)ds\tag{2.6}$$

and the boundary value problem can be formally written in the more explicit form:

$$\ddot u(t)-\Delta u(t)=-\alpha\int_{-\infty}^t e^{s-t}\Delta u(s)ds\qquad\text{in }\Omega\setminus\Gamma_{\ell(t)}\ \text{ for a.e. } t\in(T_0,T_1)\,,\tag{2.7}$$

$$\partial_y u(t)=\alpha\int_{-\infty}^t e^{s-t}\partial_y u(s)ds\qquad\text{on }\Gamma_{\ell(t)}\cap\Omega\ \text{ for a.e. } t\in(T_0,T_1)\,,\tag{2.8}$$

$$u(t)=u_0(t)\qquad\text{for a.e. } t\in(-\infty,T_0)\,.\tag{2.9}$$

We assume that $u_0\in L^\infty((-\infty,T_0);H^1(\Omega\setminus\Gamma_{\ell(T_0)}))$. Let us introduce the function spaces which are used to study (2.7)-(2.8) in the time interval (T_0,T_1). For every $t\in[T_0,T_1]$ let $V_t:=H^1(\Omega\setminus\Gamma_{\ell(t)})$, let $H:=L^2(\Omega)$, let

$$\mathcal{V}:=\{u\in L^\infty((T_0,T_1);V_{T_1})\cap H^1((T_0,T_1);H): u(t)\in V_t\ \text{ for a.e. } t\in(T_0,T_1)\}\,,$$

and let

$$\mathcal{V}_0:=\{u\in\mathcal{V}: u(t)=0\text{ on }\partial\Omega\text{ for a.e. } t\in(T_0,T_1)\}\,.$$

Given

$$u\in L^\infty((T_0,T_1);H^1(\Omega\setminus\Gamma_{\ell(T_1)})),\tag{2.10}$$

to write in a precise way the weak form of (2.7)-(2.9) it is convenient to introduce, for every $t\in(T_0,T_1)$, the function

$$F_u(t):=\alpha\int_{-\infty}^{T_0}e^{-(t-s)}\nabla u_0(s)ds+\alpha\int_{T_0}^t e^{-(t-s)}\nabla u(s)ds\,,\tag{2.11}$$

defined as a Bochner integral in the space $L^2(\Omega;\mathbb{R}^2)$. Of course, if u is extended to $(-\infty,T_0)$ by setting $u(t)=u_0(t)$ for $t\in(-\infty,T_0)$, the function F_u satisfies (1.6). Note that $F_u\in L^\infty((T_0,T_1);L^2(\Omega;\mathbb{R}^2))$. To fulfill the formal requirement (2.6) we set

$$\sigma(t):=\nabla u(t)-F_u(t)\tag{2.12}$$

for a.e. $t\in(T_0,T_1)$.

The following definition is inspired by [7, Definition 2.7].

Definition 2.1. A weak solution of (2.7)-(2.9) is a function $u \in \mathcal{V}$ satisfying (2.9) and the equality

$$-\int_{T_0}^{T_1} \Big(\int_{\Omega\setminus\Gamma_{\ell(t)}} \dot u(t)\dot\varphi(t) dxdy \Big) dt + \int_{T_0}^{T_1} \Big(\int_{\Omega\setminus\Gamma_{\ell(t)}} \nabla u(t)\nabla\varphi(t) dxdy \Big) dt = \int_{T_0}^{T_1} \Big(\int_{\Omega\setminus\Gamma_{\ell(t)}} F_u(t)\nabla\varphi(t) dxdy \Big) dt \tag{2.13}$$

for every $\varphi \in \mathcal{V}_0$ with $\varphi(T_0) = \varphi(T_1) = 0$ a.e. in Ω.

Definition 2.2. Given $w \in \mathcal{V}$ we say that u is a weak solution of (2.7)-(2.9) with Dirichlet boundary condition $u = w$ on $\partial\Omega$ if it is a weak solution in the sense of Definition 2.1 and $u(t) = w(t)$ on $\partial\Omega$, in the sense of traces, for a.e. $t \in (T_0, T_1)$.

Remark 2.3. The following uniqueness result is proved in [4]: for every $w \in \mathcal{V}$ there exists at most one weak solution of (2.7)-(2.9) in (T_0, T_1) with Dirichlet boundary condition $u = w$ on $\partial\Omega$.

Remark 2.4. So far the existence result has been proved in [16] under stronger assumptions on the Dirichlet boundary data: $w \in H^2((T_0,T_1); L^2(\Omega)) \cap H^1((T_0,T_1); H^1(\Omega\setminus\Gamma_{\ell(T_0)}))$.

2.4. **Energy-dissipation balance in the presence of cracks.** We now analyse the energetic terms associated to a weak solution u of (2.7)-(2.9) in (T_0, T_1) in the sense of Definition 2.1. For a.e. $t \in (T_0, T_1)$ let $\mathcal{E}(t)$ be the sum of the kinetic and elastic energy at time t, that is

$$\mathcal{E}(t) := \frac{1}{2}\int_{\Omega\setminus\Gamma_{\ell(t)}} |\dot u(t)|^2 dxdy + \frac{c_e}{2}\int_{\Omega\setminus\Gamma_{\ell(t)}} |\nabla u(t)|^2 dxdy\,. \tag{2.14}$$

To write the viscous dissipation we assume in addition that

$$u(t) \in W^{2,p}(\Omega\setminus\Gamma_{\ell(t)}) \quad \text{and} \quad \dot u(t) \in W^{1,p}(\Omega\setminus\Gamma_{\ell(t)}) \quad \text{for a.e. } t \in (T_0,T_1)\,, \tag{2.15}$$

for some $1 < p < 4/3$, and

$$F_u(t) \in L^q(\Omega\setminus\Gamma_{\ell(t)}; \mathbb{R}^2) \quad \text{for a.e. } t \in (T_0,T_1)\,, \tag{2.16}$$

where q is the exponent conjugate to p. Moreover, we assume that

$$\operatorname*{ess\,sup}_{t\in(T_0,T_1)} \big(\|u(t)\|_{W^{2,p}(\Omega\setminus\Gamma_{\ell(t)})} + \|\dot u(t)\|_{W^{1,p}(\Omega\setminus\Gamma_{\ell(t)})} \big) < +\infty\,, \tag{2.17}$$

$$\operatorname*{ess\,sup}_{t\in(T_0,T_1)} \|F_u(t)\|_{L^q(\Omega\setminus\Gamma_{\ell(t)})} < +\infty\,. \tag{2.18}$$

According to (2.4) for a.e. $T_0 < t_1 < t_2 < T_1$ the viscous dissipation between t_1 and t_2 is given by

$$\mathcal{D}(t_1,t_2) := \frac{c_v}{2}\int_{\Omega\setminus\Gamma_{\ell(t_2)}} |\nabla u(t_2)|^2 dxdy - \frac{c_v}{2}\int_{\Omega\setminus\Gamma_{\ell(t_1)}} |\nabla u(t_1)|^2 dxdy - \int_{t_1}^{t_2}\Big(\int_{\Omega\setminus\Gamma_{\ell(t)}} F_u(t)\nabla\dot u(t) dxdy \Big) dt\,. \tag{2.19}$$

We now analyse the work done to produce the boundary displacement $u(t) = w(t)$. Besides (2.15) and (2.17) we assume that

$$F_u(t) \in W^{1,p}(\Omega\setminus\Gamma_{\ell(t)}; \mathbb{R}^2) \quad \text{for a.e. } t \in (T_0,T_1)\,, \tag{2.20}$$

$$\operatorname*{ess\,sup}_{t\in(T_0,T_1)} \|F_u(t)\|_{W^{1,p}(\Omega\setminus\Gamma_{\ell(t)})} < +\infty\,, \tag{2.21}$$

for the same $1 < p < 4/3$. About w we assume that

$$w(t) \in H^1(\Omega\setminus\Gamma_{\ell(T_0)}) \quad \text{and} \quad \dot w(t) \in W^{1,r}(\Omega\setminus\Gamma_{\ell(T_0)}) \quad \text{for a.e. } t \in (T_0,T_1)\,, \tag{2.22}$$

$$\operatorname*{ess\,sup}_{t\in(T_0,T_1)} \big(\|w(t)\|_{H^1(\Omega\setminus\Gamma_{\ell(T_0)})} + \|\dot w(t)\|_{W^{1,r}(\Omega\setminus\Gamma_{\ell(T_0)})} \big) < +\infty\,, \tag{2.23}$$

for some $1 < r < 2$ with $1/r + 1/p \leq 3/2$. The force acting on the boundary of Ω has density $\sigma(t)\nu$, where σ is given in (2.12) and ν is the outer unit normal to $\partial\Omega$. Then (2.15) and (2.20) give

$$\sigma(t) \in W^{1,p}(\Omega \setminus \Gamma_{\ell(t)}) \quad \text{for a.e. } t \in (T_0, T_1)\,, \tag{2.24}$$

while (2.17) and (2.21) give

$$\operatorname*{ess\,sup}_{t\in(T_0,T_1)} \|\sigma(t)\|_{W^{1,p}(\Omega\setminus\Gamma_{\ell(t)})} < +\infty\,.$$

Therefore the trace of $\sigma(t)$ on $\partial\Omega$ is well defined, it belongs to $L^{p/(2-p)}(\partial\Omega)$ (see [13, Theorem 18.24]), and

$$\operatorname*{ess\,sup}_{t\in(T_0,T_1)} \|\sigma(t)\|_{L^{p/(2-p)}(\partial\Omega)} < +\infty\,. \tag{2.25}$$

On the other hand by (2.22) and [13, Theorem 18.24] the trace of $\dot w(t)$ on $\partial\Omega$ belongs to $L^{r/(2-r)}(\partial\Omega)$ and by (2.23)

$$\operatorname*{ess\,sup}_{t\in(T_0,T_1)} \|\dot w(t)\|_{L^{r/(2-r)}(\partial\Omega)} < +\infty\,. \tag{2.26}$$

Since our assumption $1/r + 1/p \leq 3/2$ implies that $(2-r)/r + (2-p)/p \leq 1$, the integral

$$\int_{\partial\Omega} \sigma(t)\nu\dot w(t) d\mathcal{H}^1 \tag{2.27}$$

is well defined for a.e. $t \in (T_0, T_1)$. It represents the power at time t of the force acting on $\partial\Omega$. Therefore the work done by this force in the time interval $[t_1, t_2] \subset (T_0, T_1)$ is given by

$$\mathcal{W}(t_1, t_2) = \int_{t_1}^{t_2} \Big(\int_{\partial\Omega} \sigma(t)\nu\dot w(t) d\mathcal{H}^1 \Big) dt\,. \tag{2.28}$$

Note that the integral in time is well defined thanks to (2.25) and (2.26).

Finally, according to Griffith's theory, the energy dissipated by the crack growth in the time interval $[t_1, t_2]$ is assumed to be proportional to the added length, i.e.,

$$\mathcal{K}(t_1, t_2) := \beta(\ell(t_2) - \ell(t_1))\,, \tag{2.29}$$

for some constant $\beta > 0$ that represents the fracture toughness of the material.

Definition 2.5. Let w be a function satisfying (2.22) and (2.23) and let u be a weak solution of (2.7)-(2.9) in the sense of Definition 2.1 with Dirichlet boundary condition $u = w$ on $\partial\Omega$. Assume that (2.15)-(2.18), (2.20), and (2.21) hold. We say that u satisfies the energy-dissipation balance if

$$\mathcal{E}(t_2) - \mathcal{E}(t_1) + \mathcal{D}(t_1, t_2) + \mathcal{K}(t_1, t_2) = \mathcal{W}(t_1, t_2)$$

for a.e. $t_1 < t_2$ in (T_0, T_1).

3. A particular solution

In the rest of the paper, under suitable assumptions on the crack tip velocity and on the viscosity constant $\alpha > 0$, for an arbitrary fracture toughness $\beta > 0$ we construct a particular Dirichlet boundary condition w such that the corresponding solution u satisfies the energy-dissipation balance.

For every $a > 0$ let $R_a := (-a, a) \times (-1, 1)$ and $S_a := (-a, +\infty) \times (-1, 1)$. Throughout the rest of the paper we choose $T_0 = -1$ and $T_1 = 1$. Moreover, we fix a constant $0 < c < 1$, which represents the constant velocity of the crack tip and we consider only the function $\ell(t) = ct$ for $t \in (-1, 1)$. We study our evolutionary problem in the time interval $(-1, 1)$ with reference configuration $\Omega = R_1$. We want to find a solution u to (2.7)-(2.9) of the form

$$u(t, x, y) = v(x - ct, y)\,, \quad -\infty < t \leq 1\,, \quad (x, y) \in R_1 \setminus \Gamma_{ct}\,, \tag{3.1}$$

for some function $v\colon S_\ell \setminus \Gamma_0 \to \mathbb{R}$, with

$$\ell := 1 + c\,, \tag{3.2}$$

taking

$$u_0(t,x,y) := v(x-ct,y) \quad \text{for every } t < -1 \text{ and } (x,y) \in R_1 \setminus \Gamma_{-c}\,. \tag{3.3}$$

By (2.7) and (2.8) for $-1 \le t \le 1$ the formal boundary value problem that v must satisfy for $(x,y) \in R_1 \backslash \Gamma_{ct}$ is

$$(1-c^2)\partial_x^2 v(x-ct,y) + \partial_y^2 v(x-ct,y) = \alpha \int_{-\infty}^t e^{s-t}\Delta v(x-cs,y)ds,$$

$$\partial_y v(x-ct,0) = \alpha \int_{-\infty}^t e^{s-t}\partial_y v(x-cs,0)ds \quad \text{for } -1 \le x \le ct\,.$$

This happens in particular if v is a solution of the formal boundary value problem

$$(1-c^2)\partial_x^2 v(x,y) + \partial_y^2 v(x,y) = \alpha \int_{-\infty}^t e^{s-t}\Delta v(x+c(t-s),y)ds \quad \text{in } R_\ell \setminus \Gamma_0,$$

$$\partial_y v(x,0) = \alpha \int_{-\infty}^t e^{s-t}\partial_y v(x+c(t-s),0)ds \quad \text{for } -\ell \le x \le 0\,,$$

which can be written in the form

$$(1-c^2)\partial_x^2 v(x,y) + \partial_y^2 v(x,y) = \alpha \int_0^{+\infty} e^{-s}\Delta v(x+cs,y)ds \quad \text{in } R_\ell \setminus \Gamma_0\,, \tag{3.4}$$

$$\partial_y v(x,0) = \alpha \int_0^{+\infty} e^{-s}\partial_y v(x+cs,0)ds \quad \text{for } -\ell \le x \le 0\,. \tag{3.5}$$

The arguments used so far are only formal. We shall introduce a weak formulation of the boundary value problem (3.4)-(3.5) and we shall prove, in a rigorous way, that for every weak solution v of this problem the function u given by (3.1) is a weak solution of (2.7)-(2.9) according to Definition 2.1 with u_0 given by (3.3).

We begin with a lemma that allows us to give a precise meaning to the integrals from 0 to $+\infty$ which appear in the weak formulation.

Lemma 3.1. *Let $1 \le p < +\infty$, $a > 0$, $\gamma > 0$, and $z \in L^p(S_a \setminus \Gamma_0)$. Then*

$$\int_{-1}^1 \Big(\int_{-a}^{+\infty} \Big(\int_0^{+\infty} e^{-s}|z(x+\gamma s,y)|ds\Big)^p dx\Big)dy \le \int_{-1}^1 \Big(\int_{-a}^{+\infty} |z(x,y)|^p dx\Big)dy\,. \tag{3.6}$$

Proof. Let us fix $y \in (-1,1)$. By a change of variables for every $x > -a$ we have

$$\int_0^{+\infty} e^{-s}|z(x+\gamma s,y)|ds = \int_{-\infty}^{+\infty} f(\sigma)g(x+\sigma)d\sigma\,,$$

where $f(\sigma) := \frac{1}{\gamma}e^{-\frac{\sigma}{\gamma}}$ for $\sigma \ge 0$ and $f(\sigma) := 0$ for $\sigma < 0$, while $g(x) = |z(x,y)|$ for $x > -a$ and $g(x) = 0$ for $x \le -a$. Since $\int_{\mathbb{R}} f(\sigma)d\sigma = 1$, by the Young Inequality for convolutions we have

$$\int_{-a}^{+\infty} \Big(\int_0^{+\infty} e^{-s}|z(x+\gamma s,y)|ds\Big)^p dx \le \int_{-a}^{+\infty} |z(x,y)|^p dx\,.$$

Integrating with respect to y we get (3.6). ☐

Definition 3.2. Given $a > 0$ we say that v is a weak solution of the boundary value problem

$$(1-c^2)\partial_x^2 v(x,y) + \partial_y^2 v(x,y) = \alpha \int_0^{+\infty} e^{-s}\Delta v(x+cs,y)ds \quad \text{in } R_a \setminus \Gamma_0\,, \tag{3.7}$$

$$\partial_y v(x,0) = \alpha \int_0^{+\infty} e^{-s}\partial_y v(x+cs,0)ds \quad \text{for } -a \le x \le 0\,. \tag{3.8}$$

if $v \in H^1(S_a \setminus \Gamma_0)$ and the equality

$$(1-c^2)\int_{R_a\setminus\Gamma_0} \partial_x v(x,y)\partial_x\varphi(x,y)dxdy + \int_{R_a\setminus\Gamma_0} \partial_y v(x,y)\partial_y\varphi(x,y)dxdy$$
$$= \alpha\int_{R_a\setminus\Gamma_0} \Big(\int_0^{+\infty} e^{-s}\nabla v(x+cs,y)ds\Big)\nabla\varphi(x,y)dxdy \tag{3.9}$$

holds for every $\varphi \in H^1(R_a\setminus\Gamma_0)$ with $\varphi = 0$ on ∂R_a.

We now prove in a rigorous way that a weak solution of (3.4)-(3.5) generates a weak solution of (2.7)-(2.9) with u_0 given by (3.3).

Lemma 3.3. *Let $v \in H^1(S_\ell \setminus \Gamma_0)$ be a weak solution of* (3.4)-(3.5) *according to Definition 3.2 and let u_0 be defined by* (3.3). *Then u defined by* (3.1) *is a weak solution of* (2.7)-(2.9) *in $\Omega = R_1$ according to Definition 2.1.*

Proof. Let us check that $u_0 \in L^\infty((-\infty,-1); H^1(R_1\setminus\Gamma_{-c}))$ and $u \in \mathcal{V}$. It follows immediately from the definitions that $u_0(s) \in H^1(R_1\setminus\Gamma_{-c})$ for a.e. $s \in (-\infty,-1)$ and $u(t) \in H^1(R_1\setminus\Gamma_{ct})$ for a.e. $t\in(-1,1)$. Moreover,

$$\int_{R_1\setminus\Gamma_{-c}} \big(|u_0(s,x,y)|^2 + |\nabla u_0(s,x,y)|^2\big)dxdy \le \int_{S_\ell\setminus\Gamma_0} \big(|v(x,y)|^2 + |\nabla v(x,y)|^2\big)dxdy\,,$$
$$\int_{R_1\setminus\Gamma_{ct}} \big(|u(t,x,y)|^2 + |\nabla u(t,x,y)|^2\big)dxdy \le \int_{S_\ell\setminus\Gamma_0} \big(|v(x,y)|^2 + |\nabla v(x,y)|^2\big)dxdy\,,$$

for a.e. $s\in(-\infty,-1)$ and a.e. $t\in(-1,1)$. Hence $u_0 \in L^\infty((-\infty,-1); H^1(R_1\setminus\Gamma_{-c}))$ and $u \in L^\infty((-1,1);V_1)$. Using the equality $\dot u(t,x,y) = -c\partial_x v(x-ct,y)$, it can be easily shown that $u\in H^1((-1,1);H)$.

Let $\varphi \in \mathcal{V}_0$ with $\varphi(-1)=\varphi(1)=0$ on R_1. Then for a.e. $t\in(-1,1)$ we have $\varphi(t)\in H^1(R_1\setminus\Gamma_{ct})$ and $\varphi(t)=0$ on ∂R_1, hence we can extend it to a function $\varphi(t)\in H^1(\mathbb{R}^2\setminus\Gamma_{ct})$ by setting $\varphi=0$ on $\mathbb{R}^2\setminus R_1$. For a.e. $t\in(-1,1)$ and for every $(x,y)\in R_\ell\setminus\Gamma_0$ we set

$$\varphi_c(t,x,y) := \varphi(t,x+ct,y)\,. \tag{3.10}$$

Let $R_1^{ct} := (-1-ct, 1-ct)\times(-1,1)$. Since

$$\varphi_c(t,\cdot,\cdot) = 0 \quad \text{on } \mathbb{R}^2\setminus R_1^{ct} \tag{3.11}$$

and $R_1^{ct}\subset R_\ell$, for a.e. $t\in(-1,1)$ we have $\varphi_c(t,\cdot,\cdot)\in H^1(R_\ell\setminus\Gamma_0)$ and $\varphi_c(t,\cdot,\cdot)=0$ on ∂R_ℓ. Using $\varphi_c(t)$ as test function in (3.9) for a.e. $t\in(-1,1)$ we obtain

$$(1-c^2)\int_{R_1^{ct}\setminus\Gamma_0} \partial_x v(x,y)\partial_x\varphi_c(t,x,y)dxdy + \int_{R_1^{ct}\setminus\Gamma_0} \partial_y v(x,y)\partial_y\varphi_c(t,x,y)dxdy$$
$$= \alpha\int_{R_1^{ct}\setminus\Gamma_0} \Big(\int_0^{+\infty} e^{-s}\nabla v(x+cs,y)ds\Big)\nabla\varphi_c(t,x,y)dxdy\,. \tag{3.12}$$

Taking into account (3.1), by a change of variable for every $(x,y)\in R_1^{ct}\setminus\Gamma_0$ we obtain

$$\int_0^{+\infty} e^{-s}\nabla v(x+cs,y)ds = \int_{-\infty}^t e^{-(t-s)}\nabla v(x+c(t-s),y)ds$$
$$= \int_{-\infty}^t e^{-(t-s)}\nabla u(s,x+ct,y)ds\,, \tag{3.13}$$

hence (3.12) can be written as

$$\begin{aligned}(1-c^2)\int_{R_1^{ct}\setminus\Gamma_0}&\partial_x u(t,x+ct,y)\partial_x\varphi(t,x+ct,y)dxdy\\&+\int_{R_1^{ct}\setminus\Gamma_0}\partial_y u(t,x+ct,y)\partial_y\varphi(t,x+ct,y)dxdy\\=\alpha\int_{R_1^{ct}\setminus\Gamma_0}&\Big(\int_{-\infty}^t e^{-(t-s)}\nabla u(s,x+ct,y)ds\Big)\nabla\varphi(t,x+ct,y)dxdy\,.\end{aligned}\tag{3.14}$$

By a change of variables we obtain

$$\begin{aligned}(1-c^2)\int_{R_1\setminus\Gamma_{ct}}&\partial_x u(t,x,y)\partial_x\varphi(t,x,y)dxdy\\&+\int_{R_1\setminus\Gamma_{ct}}\partial_y u(t,x,y)\partial_y\varphi(t,x,y)dxdy\\=\alpha\int_{R_1\setminus\Gamma_{ct}}&\Big(\int_{-\infty}^t e^{-(t-s)}\nabla u(s,x,y)ds\Big)\nabla\varphi(t,x,y)dxdy\,.\end{aligned}\tag{3.15}$$

This equality can be written in the form

$$\begin{aligned}-c^2\int_{R_1\setminus\Gamma_{ct}}\partial_x u(t,x,y)\partial_x\varphi(t,x,y)dxdy+\int_{R_1\setminus\Gamma_{ct}}\nabla u(t,x,y)\nabla\varphi(t,x,y)dxdy\\=\alpha\int_{R_1\setminus\Gamma_{ct}}\Big(\int_{-\infty}^t e^{-(t-s)}\nabla u(s,x,y)ds\Big)\nabla\varphi(t,x,y)dxdy\,.\end{aligned}\tag{3.16}$$

Recalling the definition (2.11) of $F_u(t)$ and the definition (3.3) of u_0, to conclude the proof of (2.13) it remains to show that

$$\int_{-1}^1\Big(\int_{R_1\setminus\Gamma_{ct}}\dot u(t,x,y)\dot\varphi(t,x,y)dxdy\Big)dt=c^2\int_{-1}^1\Big(\int_{R_1\setminus\Gamma_{ct}}\partial_x u(t,x,y)\partial_x\varphi(t,x,y)dxdy\Big)dt\,.\tag{3.17}$$

By (3.1) and (3.10) we have

$$\dot u(t,x,y)=-c\partial_x v(x-ct,y)\quad\text{and}\quad\dot\varphi(t,x,y)=\dot\varphi_c(t,x-ct,y)-c\partial_x\varphi_c(t,x-ct,y)\,,$$

hence

$$\begin{aligned}\int_{R_1\setminus\Gamma_{ct}}\dot u(t,x,y)\dot\varphi(t,x,y)dxdy=-c\int_{R_1\setminus\Gamma_{ct}}\partial_x v(x-ct,y)\dot\varphi_c(t,x-ct,y)dxdy\\+c^2\int_{R_1\setminus\Gamma_{ct}}\partial_x v(x-ct,y)\partial_x\varphi_c(t,x-ct,y)dxdy\,.\end{aligned}$$

Therefore, to prove (3.17) it is enough to show that

$$\int_{-1}^1\Big(\int_{R_1\setminus\Gamma_{ct}}\partial_x v(x-ct,y)\dot\varphi_c(t,x-ct,y)dxdy\Big)dt=0\,.$$

Changing variables again, by (3.11) the left-hand side can be written as

$$\int_{-1}^1\Big(\int_{R_\ell\setminus\Gamma_0}\partial_x v(x,y)\dot\varphi_c(t,x,y)dxdy\Big)dt=\int_{R_\ell\setminus\Gamma_0}\partial_x v(x,y)(\varphi_c(1,x,y)-\varphi_c(-1,x,y))dxdy\,,$$

and the last integral is equal to 0 because $\varphi_c(-1)=\varphi_c(1)=0$ as a consequence of the fact that $\varphi(-1)=\varphi(1)=0$. This concludes the proof. □

It is convenient to consider a change of variables which transforms (3.7) and (3.8) into the problem

$$\Delta\hat{v}(x,y)=\alpha\int_0^{+\infty}e^{-s}\Delta\hat{v}(x+\gamma s,y)ds+\alpha\gamma^2\int_0^{+\infty}e^{-s}\partial_x^2\hat{v}(x+\gamma s,y)ds \qquad \text{in } R_{\hat{a}}\setminus\Gamma_0\,, \quad (3.18)$$

$$\partial_y\hat{v}(x,0)=\alpha\int_0^{+\infty}e^{-s}\partial_y\hat{v}(x+\gamma s,0)ds \qquad \text{for } -\hat{a}\le x\le 0\,, \quad (3.19)$$

for suitable constants $\gamma>0$ and $\hat{a}>0$.

Definition 3.4. Given $\gamma>0$ and $\hat{a}>0$, we say that $\hat{v}$ is a weak solution of (3.18)-(3.19) if $\hat{v}\in H^1(S_{\hat{a}}\setminus\Gamma_0)$ and

$$\int_{R_{\hat{a}}\setminus\Gamma_0}\nabla\hat{v}(x,y)\nabla\varphi(x,y)dxdy=\alpha\int_{R_{\hat{a}}\setminus\Gamma_0}\Big(\int_0^{+\infty}e^{-s}\nabla\hat{v}(x+\gamma s,y)ds\Big)\nabla\varphi(x,y)dxdy$$
$$+\alpha\gamma^2\int_{R_{\hat{a}}\setminus\Gamma_0}\Big(\int_0^{+\infty}e^{-s}\partial_x\hat{v}(x+\gamma s,y)ds\Big)\partial_x\varphi(x,y)dxdy \quad (3.20)$$

for every $\varphi\in H^1(R_{\hat{a}}\setminus\Gamma_0)$ with $\varphi=0$ on $\partial R_{\hat{a}}$.

We now show the equivalence between (3.7)-(3.8) and (3.18)-(3.19).

Lemma 3.5. *Let $a>0$ and let $\hat{a}:=a/\lambda$, with $\lambda:=\sqrt{1-c^2}$. Let $v\in H^1(S_a\setminus\Gamma_0)$ and let $\hat{v}\in H^1(S_{\hat{a}}\setminus\Gamma_0)$ be defined by $\hat{v}(x,y):=v(\lambda x,y)$ for every $(x,y)\in S_{\hat{a}}\setminus\Gamma_0$. Then v is a weak solution of (3.7)-(3.8) according to Definition 3.2 if and only if $\hat{v}$ is a weak solution of (3.18)-(3.19), with $\gamma:=c/\sqrt{1-c^2}$, in the sense of Definition 3.4.*

Proof. Let $\varphi\in H^1(R_{\hat{a}}\setminus\Gamma_0)$ with $\varphi=0$ on $\partial R_{\hat{a}}$ and $\check{\varphi}\in H^1(R_a\setminus\Gamma_0)$ with $\check{\varphi}=0$ on ∂R_a be such that $\check{\varphi}(x,y)=\varphi(\frac{x}{\lambda},y)$ for every $(x,y)\in R_a\setminus\Gamma_0$.

Since $\lambda=\sqrt{1-c^2}$, using the relations between v and $\hat{v}$ and φ and $\check{\varphi}$, respectively, we see that

$$(1-c^2)\int_{R_a\setminus\Gamma_0}\partial_xv(x,y)\partial_x\check{\varphi}(x,y)dxdy+\int_{R_a\setminus\Gamma_0}\partial_yv(x,y)\partial_y\check{\varphi}(x,y)dxdy$$
$$=\alpha\int_{R_a\setminus\Gamma_0}\Big(\int_0^{+\infty}e^{-s}\nabla v(x+cs,y)ds\Big)\nabla\check{\varphi}(x,y)dxdy$$

is equivalent to

$$\int_{R_a\setminus\Gamma_0}\partial_x\hat{v}(\frac{x}{\lambda},y)\partial_x\varphi(\frac{x}{\lambda},y)dxdy+\int_{R_a\setminus\Gamma_0}\partial_y\hat{v}(\frac{x}{\lambda},y)\partial_y\varphi(\frac{x}{\lambda},y)dxdy$$
$$=\alpha\int_{R_a\setminus\Gamma_0}\Big(\int_0^{+\infty}e^{-s}\frac{1}{\lambda^2}\partial_x\hat{v}(\frac{x+cs}{\lambda},y)ds\Big)\partial_x\varphi(\frac{x}{\lambda},y)dxdy$$
$$+\alpha\int_{R_a\setminus\Gamma_0}\Big(\int_0^{+\infty}e^{-s}\partial_y\hat{v}(\frac{x+cs}{\lambda},y)ds\Big)\partial_y\varphi(\frac{x}{\lambda},y)dxdy\,,$$

which in turn, by a change of variables, is equivalent to

$$\int_{R_{\hat{a}}\setminus\Gamma_0}\partial_x\hat{v}(x,y)\partial_x\varphi(x,y)dxdy+\int_{R_{\hat{a}}\setminus\Gamma_0}\partial_y\hat{v}(x,y)\partial_y\varphi(x,y)dxdy$$
$$=\frac{\alpha}{\lambda^2}\int_{R_{\hat{a}}\setminus\Gamma_0}\Big(\int_0^{+\infty}e^{-s}\partial_x\hat{v}(x+\gamma s,y)ds\Big)\partial_x\varphi(x,y)dxdy$$
$$+\alpha\int_{R_{\hat{a}}\setminus\Gamma_0}\Big(\int_0^{+\infty}e^{-s}\partial_y\hat{v}(x+\gamma s,y)ds\Big)\partial_y\varphi(x,y)dxdy\,.$$

Hence (3.9) is equivalent to (3.20), which concludes the proof. □

We conclude this section by proving the existence and uniqueness of a solution $\hat{v}$ of (3.18)-(3.19), which in view of the previous lemma provides the existence of a unique solution v of (3.7)-(3.8).

Theorem 3.6. *Let $\gamma > 0$, $\hat{a} > 0$, and $w \in H^1(S_{\hat{a}} \setminus \Gamma_0)$. Assume that $\alpha(1+\gamma^2) < 1$. Then there exists a unique solution $\hat{v}$ of (3.18)-(3.19), according to Definition 3.4, with $\hat{v} = w$ on $\partial R_{\hat{a}}$ in the sense of traces and $\hat{v} = w$ a.e. on $S_{\hat{a}} \setminus R_{\hat{a}}$.*

Proof. We consider the space $H^1_{\partial R_{\hat{a}}}(R_{\hat{a}} \setminus \Gamma_0) := \{z \in H^1(R_{\hat{a}} \setminus \Gamma_0) : z = 0 \text{ on } \partial R_{\hat{a}} \setminus \Gamma_0\}$. When needed every $z \in H^1_{\partial R_{\hat{a}}}(R_{\hat{a}} \setminus \Gamma_0)$ is extended to a function $z \in H^1(S_{\hat{a}} \setminus \Gamma_0)$ by setting $z = 0$ on $S_{\hat{a}} \setminus R_{\hat{a}}$.

Let $\hat{v} \in H^1(S_{\hat{a}} \setminus \Gamma_0)$ with $\hat{v} = w$ on $\partial R_{\hat{a}}$ in the sense of traces and $\hat{v} = w$ a.e. on $S_{\hat{a}} \setminus R_{\hat{a}}$. It is convenient to write $\hat{v} = w + \zeta$, with $\zeta \in H^1_{\partial R_{\hat{a}}}(R_{\hat{a}} \setminus \Gamma_0)$. By (3.20) the function $\hat{v}$ is a solution of our problem if and only if ζ satisfies

$$\begin{aligned}&\int_{R_{\hat{a}} \setminus \Gamma_0} \nabla\zeta(x,y)\nabla\varphi(x,y)dxdy + \int_{R_{\hat{a}} \setminus \Gamma_0} \nabla w(x,y)\nabla\varphi(x,y)dxdy \\ &= \alpha \int_{R_{\hat{a}} \setminus \Gamma_0} \Big(\int_0^{+\infty} e^{-s}\big((1+\gamma^2)\partial_x\zeta(x+\gamma s, y), \partial_y\zeta(x+\gamma s, y)\big)ds\Big)\nabla\varphi(x,y)dxdy \qquad (3.21)\\ &+\alpha \int_{R_{\hat{a}} \setminus \Gamma_0} \Big(\int_0^{+\infty} e^{-s}\big((1+\gamma^2)\partial_x w(x+\gamma s, y), \partial_y w(x+\gamma s, y)\big)ds\Big)\nabla\varphi(x,y)dxdy\end{aligned}$$

for every $\varphi \in H^1_{\partial R_{\hat{a}}}(R_{\hat{a}} \setminus \Gamma_0)$. Thanks to Lemma 3.1 the above integrals are well-defined.

On $H^1_{\partial R_{\hat{a}}}(R_{\hat{a}} \setminus \Gamma_0)$ we consider the norm

$$\|\varphi\|_{H^1_{\partial R_{\hat{a}}}(R_{\hat{a}} \setminus \Gamma_0)} := \|\nabla\varphi\|_{L^2(R_{\hat{a}} \setminus \Gamma_0)}. \qquad (3.22)$$

Let $H^{-1}_{\partial R_{\hat{a}}}(R_{\hat{a}} \setminus \Gamma_0)$ be the dual space of $H^1_{\partial R_{\hat{a}}}(R_{\hat{a}} \setminus \Gamma_0)$ and let $\Phi \colon H^1(R_{\hat{a}} \setminus \Gamma_0) \to H^{-1}_{\partial R_{\hat{a}}}(R_{\hat{a}} \setminus \Gamma_0)$ be the linear operator defined by

$$\langle \Phi z, \varphi \rangle := \int_{R_{\hat{a}} \setminus \Gamma_0} \nabla z(x,y)\nabla\varphi(x,y)dxdy \qquad (3.23)$$

for every $z \in H^1(R_{\hat{a}} \setminus \Gamma_0)$ and $\varphi \in H^1_{\partial R_{\hat{a}}}(R_{\hat{a}} \setminus \Gamma_0)$. The restriction of Φ to $H^1_{\partial R_{\hat{a}}}(R_{\hat{a}} \setminus \Gamma_0)$ is denoted by Φ_0. It is well-known that Φ_0 is a bijective isometry.

Let $\Psi \colon H^1(S_{\hat{a}} \setminus \Gamma_0) \to H^{-1}_{\partial R_{\hat{a}}}(R_{\hat{a}} \setminus \Gamma_0)$ be the operator defined by

$$\langle \Psi z, \varphi \rangle := \int_{R_{\hat{a}} \setminus \Gamma_0} \Big(\int_0^{+\infty} e^{-s}\big((1+\gamma^2)\partial_x z(x+\gamma s, y), \partial_y z(x+\gamma s)\big)ds\Big)\nabla\varphi(x,y)dxdy \quad (3.24)$$

for every $z \in H^1(S_{\hat{a}} \setminus \Gamma_0)$ and every $\varphi \in H^1_{\partial R_{\hat{a}}}(R_{\hat{a}} \setminus \Gamma_0)$. By Lemma 3.1 we have that $\|\Psi\| \le 1 + \gamma^2$.

With this notation problem (3.21) becomes: find $\zeta \in H^1_{\partial R_{\hat{a}}}(R_{\hat{a}} \setminus \Gamma_0)$ such that

$$\Phi_0\zeta = \alpha\Psi\zeta + \alpha\Psi w - \Phi w, \qquad (3.25)$$

which is equivalent to

$$\zeta = \alpha\Phi_0^{-1}\Psi\zeta + \alpha\Phi_0^{-1}\Psi w - \Phi_0^{-1}\Phi w. \qquad (3.26)$$

By hypothesis $\alpha(1+\gamma^2) < 1$, hence the function $\zeta \mapsto \alpha\Phi_0^{-1}\Psi\zeta + \alpha\Phi_0^{-1}\Psi w - \Phi_0^{-1}\Phi w$ is a contraction in $H^1_{\partial R_{\hat{a}}}(R_{\hat{a}} \setminus \Gamma_0)$. Since ζ is a solution of problem (3.21) if and only if ζ is a fixed point of this function, we have existence and uniqueness of the solution ζ of problem (3.21). This concludes the proof. □

4. Singularity at the origin

The aim of this section is to show that, for a suitable Dirichlet boundary condition on ∂R_ℓ, the gradient of the solution v of (3.4)-(3.5) according to Definition 3.2 is singular, at least when α is sufficiently small. The detailed study of this singularity will be crucial for the developments of the next section.

Since we want the smallness condition for α to be independent of c, at least for c sufficiently far from 0 and 1, we fix two constants $0 < c_0 < c_1 < 1$ and in the rest of the paper we always assume

$$c_0 \le c \le c_1. \tag{4.1}$$

We set

$$\ell_1 := 1 + c_1\,, \quad \lambda_1 := \sqrt{1-c_1^2}\,, \quad \text{and} \quad \hat{\ell}_1 := \ell_1/\lambda_1 = (1+c_1)/\sqrt{1-c_1^2}\,. \tag{4.2}$$

To simplify the notation, in this section R denotes the rectangle $R_{\hat{\ell}_1}$ and S denotes the strip $S_{\hat{\ell}_1}$. We consider the space $H^1_{\partial R}(R \setminus \Gamma_0) := \{z \in H^1(R \setminus \Gamma_0) : z = 0 \text{ on } \partial R \setminus \Gamma_0\}$. When needed every $z \in H^1_{\partial R}(R \setminus \Gamma_0)$ is extended to a function $z \in H^1(S \setminus \Gamma_0)$ by setting $z = 0$ on $S \setminus R$. Hence the space $H^1_{\partial R}(R\setminus\Gamma_0)$ can be considered as a subspace of $H^1(S\setminus\Gamma_0)$.

In view of Lemma 3.5, we shall first describe the behaviour near the origin of the weak solution $\hat{v}$ of (3.18)-(3.19) with

$$\gamma := c/\sqrt{1-c^2}\,. \tag{4.3}$$

To this end we introduce the function $\psi\colon \mathbb{R}^2 \setminus \Gamma_0 \to \mathbb{R}$ defined by

$$\psi(x,y) := \sqrt{\rho(x,y)}\sin\frac{\theta(x,y)}{2} \qquad \text{for every } (x,y) \in \mathbb{R}^2 \setminus \Gamma_0\,, \tag{4.4}$$

where $\rho(x,y) := \sqrt{x^2+y^2}$ and $\theta(x,y) \in (-\pi,\pi)$ is the oriented angle between the positive x axis and the vector (x,y), so that $x = \rho\cos\theta$ and $y = \rho\sin\theta$. We observe that

$$\psi \in W^{1,s}(R \setminus \Gamma_0) \cap W^{2,p}(R \setminus \Gamma_0) \quad \text{for every } 1 \le s < 4 \text{ and every } 1 \le p < 4/3\,. \tag{4.5}$$

Since ψ is harmonic in $\mathbb{R}^2 \setminus \Gamma_0$ and $\partial_y\psi = 0$ on Γ_0, we have

$$\int_{R\setminus\Gamma_0} \nabla\psi(x,y)\nabla\varphi(x,y)dxdy = 0 \qquad \text{for every } \varphi \in H^1_{\partial R}(R \setminus \Gamma_0)\,. \tag{4.6}$$

We shall prove in Theorem 4.3 that, given $4/3 < r < 1$, under suitable assumptions on α depending on r, the weak solution $\hat{v}$ of (3.18)-(3.19) in the sense of Definition 3.4 with the Dirichlet boundary condition $\hat{v} = \psi$ on ∂R can be written as

$$\hat{v} = \kappa\psi + \hat{v}^{reg} \quad \text{in } R \setminus \Gamma_0\,,$$

with $\kappa > 0$ and $\hat{v}^{reg} \in W^{2,r}(R\setminus\Gamma_0)$. This result is based on the following classical estimates, due to Grisvard, concerning the weak solutions of the boundary value problems

$$\begin{cases} -\Delta z = f & \text{in } R \setminus \Gamma_0\,,\\ \partial_y z = 0 & \text{on } R \cap \Gamma_0\,,\\ z = 0 & \text{on } \partial R\,. \end{cases} \tag{4.7}$$

The main difficulty of this section is to extend these estimates to the case of (3.18)-(3.19) where non-local terms are present.

The structure of the solution of (4.7) near the origin is described by the following result.

Theorem 4.1. *Let $4/3 < r < 2$, let $f \in L^r(R\setminus\Gamma_0)$, and let $z \in H^1_{\partial R}(R\setminus\Gamma_0)$ be the unique solution of the problem*

$$\int_{R\setminus\Gamma_0} \nabla z(x,y)\nabla\varphi(x,y)dxdy = \int_{R\setminus\Gamma_0} f(x,y)\varphi(x,y)dxdy \quad \textit{for every } \varphi \in H^1_{\partial R}(R \setminus \Gamma_0)\,, \tag{4.8}$$

which is the weak formulation of (4.7)*. Then* z *can be written in a unique way as*

$$z = \kappa\psi + z^{reg} \quad \text{in } R \setminus \Gamma_0 \,, \tag{4.9}$$

with $\kappa \in \mathbb{R}$ *and* $z^{reg} \in W^{2,r}(R\setminus\Gamma_0)$*. Moreover, there exists a constant* $A_r > 0$*, independent of* f*, such that the estimate*

$$|\kappa| + \|z^{reg}\|_{W^{2,r}(R\setminus\Gamma_0)} \le A_r \|f\|_{L^r(R\setminus\Gamma_0)} \tag{4.10}$$

holds.

Proof. Using the results in [12, Theorem 4.4.3.7] we obtain that there exist $\kappa \in \mathbb{R}$ and $z^{reg} \in W^{2,r}(R \setminus \Gamma_0)$ such that (4.9) holds. Since the function ψ does not belong to $W^{2,r}(R \setminus \Gamma_0)$ we deduce that κ and z^{reg} are uniquely determined.

Let us now prove (4.10). Let X be the closed linear subspace of $Y := \mathbb{R} \times W^{2,r}(R\setminus\Gamma_0;\mathbb{R}^2)$ defined by

$$X := \{(\kappa, v) \in Y : \kappa\psi + z^{reg} = 0 \text{ on } \partial R \text{ and } \partial_y z^{reg} = 0 \text{ on } R \cap \Gamma_0\},$$

and let $\Lambda\colon X \to L^r(R \setminus \Gamma_0)$ be the continuous linear operator defined by

$$\Lambda(\kappa, z^{reg}) := -\Delta z^{reg} \,.$$

Given $f \in L^r(R\setminus\Gamma_0)$ we consider the unique solution z of (4.8), which can be represented as in (4.9) for some $\kappa \in \mathbb{R}$ and $z^{reg} \in W^{2,r}(R \setminus \Gamma_0)$, uniquely determined by z. Since $\partial_y z = 0$ and $\partial_y \psi = 0$ on $R \cap \Gamma_0$, we deduce that $\partial_y z^{reg} = 0$ on $R \cap \Gamma_0$. Moreover, since $z = 0$ on ∂R we have $\kappa\psi + z^{reg} = 0$ on ∂R. Finally, since $-\Delta z = f$ and $-\Delta\psi = 0$ in $R \setminus \Gamma_0$, we have also $-\Delta z^{reg} = f$ in $R \setminus \Gamma_0$. This implies that there exists a unique $(\kappa, z^{reg}) \in X$ such that $\Lambda(\kappa, z^{reg}) = f$, proving that $\Lambda\colon X \to L^r(R \setminus \Gamma_0)$ is bijective. Since Λ is continuous, by the Closed Graph Theorem its inverse is continuous, which implies (4.10). ☐

We consider now the case in which the homogeneous condition $z = 0$ on ∂R is replaced by the non-homogeneous boundary condition $z = w$ on ∂R.

Corollary 4.2. *Let* $4/3 < r < 2$*,* $f \in L^r(R \setminus \Gamma_0)$*,* $w \in W^{2,r}(R \setminus \Gamma_0)$*, and let* $z \in H^1_{\partial R}(R \setminus \Gamma_0) + w$ *be the unique weak solution of problem* (4.8) *in this space. Then* z *can be written in a unique way as*

$$z = \kappa\psi + z^{reg} \quad \text{in } R \setminus \Gamma_0 \,,$$

with $\kappa \in \mathbb{R}$ *and* $z^{reg} \in W^{2,r}(R\setminus\Gamma_0)$*. Moreover, there exists a constant* $B_r > 0$*, independent of* f *and* w*, such that the estimate*

$$|\kappa| + \|z^{reg}\|_{W^{2,r}(R\setminus\Gamma_0)} \le B_r(\|f\|_{L^r(R\setminus\Gamma_0)} + \|w\|_{W^{2,r}(R\setminus\Gamma_0)})$$

holds.

Proof. Since $4/3 < r < 2$ and hence $1 - 1/r < 1/r$, by [12, Theorem 1.5.2.8 and Remark 1.7.4] there exists a function $\zeta \in W^{2,r}(R \setminus \Gamma_0)$ such that

$$\partial_y \zeta = 0 \quad \text{on } R \cap \Gamma_0 \,, \quad \partial_\nu \zeta = \partial_\nu w \quad \text{on } \partial R \,,$$
$$\zeta = w \quad \text{on } \partial R \cup \Gamma_0 \,.$$

We observe that ζ is not uniquely determined. However, we can choose ζ so that

$$\|\zeta\|_{W^{2,r}(R\setminus\Gamma_0)} \le c_r \|w\|_{W^{2,r}(R\setminus\Gamma_0)}$$

for a suitable constant $c_r > 0$ independent of w. To prove this estimate, it is enough to consider the surjective continuous linear map Λ introduced in [12, Theorem 1.5.2.8] and to apply the Closed Graph Theorem to its quotient defined on the Banach space $W^{2,r}(R \setminus \Gamma_0)/\mathrm{Ker}\,\Lambda$.

The conclusion now follows by applying the previous theorem to the function $z - \zeta$. ☐

Since $\psi \notin H^1(S \setminus \Gamma_0)$, we fix a cut-off function $\omega \in C_c^\infty(\mathbb{R}^2)$ with $\omega = 1$ in R and we set

$$\psi_0 := \omega\psi\,, \tag{4.11}$$

observing that $\psi_0 = \psi$ in $R \setminus \Gamma_0$ and $\psi_0 \in H^1(S \setminus \Gamma_0) \cap W^{2,p}(S \setminus \Gamma_0)$ for $1 \le p < 4/3$. Moreover, we have $\psi_0 \in C^\infty(S \setminus R)$.

We are now ready to state the main result of this section. It concerns the structure of the weak solution $\hat v$ of (3.18)-(3.19) in $R \setminus \Gamma_0$ with boundary condition $\hat v = \psi_0$ on ∂R, whose existence and uniqueness are guaranteed by Theorem 3.6.

Theorem 4.3. *Given $4/3 < r < 2$, there exists a constant $M_r > 1$ such that, if $\alpha M_r \le 1 - c_1^2$ and $\gamma = c/\sqrt{1-c^2}$, then the unique solution $\hat v$ of (3.18)-(3.19) in $R \setminus \Gamma_0$ with boundary condition $\hat v = \psi$ on ∂R, satisfying $\hat v = \psi_0$ in $S \setminus R$, can be written in a unique way in the form*

$$\hat v = \kappa_0 \psi + \hat v^{reg} \quad in\ R \setminus \Gamma_0\,, \tag{4.12}$$

with $\kappa_0 \in \mathbb{R}$, $\kappa_0 > 0$, and $\hat v^{reg} \in W^{2,r}(R \setminus \Gamma_0)$.

Proof. Assume that $\alpha < 1 - c_1^2$. Recalling the definition of γ and the inequalities $c_0 \le c \le c_1$, this implies that $\alpha(1+\gamma^2) < 1$. Hence Theorem 3.6 with $\hat a = \hat\ell_1$ and $w = \psi_0$ guarantees existence and uniqueness of the solution $\hat v$ mentioned in the statement. Let Φ, Φ_0, and Ψ be the operators defined in the proof of that theorem. Therefore we can write

$$\hat v = \psi_0 + \zeta\,, \tag{4.13}$$

where $\zeta \in H^1_{\partial R}(R \setminus \Gamma_0)$ is the unique fixed point of the map $\zeta \mapsto \alpha\Phi_0^{-1}\Psi\zeta + \alpha\Phi_0^{-1}\Psi\psi_0 - \Phi_0^{-1}\Phi\psi_0 = \alpha\Phi_0^{-1}\Psi\zeta + \alpha\Phi_0^{-1}\Psi\psi_0$, where the equality is due to the fact that $\Phi\psi_0 = 0$ in $R \setminus \Gamma_0$ by (4.6). Since this map is a contraction in the space $H^1_{\partial R}(R \setminus \Gamma_0)$ with norm (3.22), its fixed point ζ can be obtained as limit of the sequence (ζ_n) defined inductively in the following way: $\zeta_0 = 0$ and $\zeta_{n+1} = \alpha\Phi_0^{-1}\Psi\zeta_n + \alpha z_0$ for $n \ge 0$, where

$$z_0 := \Phi_0^{-1}\Psi\psi_0 \in H^1_{\partial R}(R \setminus \Gamma_0). \tag{4.14}$$

Let $U\colon H^1(S \setminus \Gamma_0) \to H^1_{\partial R}(R \setminus \Gamma_0) \subset H^1(S \setminus \Gamma_0)$ be the operator defined by

$$U := \Phi_0^{-1}\Psi\,. \tag{4.15}$$

By construction we have $\zeta_{n+1} = \alpha U \zeta_n + \alpha z_0$. Therefore we can write ζ_{n+1} as

$$\zeta_{n+1} = \sum_{j=0}^{n} \alpha^{j+1} U^j z_0\,,$$

and consequently the fixed point ζ satisfies

$$\zeta = \sum_{j=0}^{\infty} \alpha^{j+1} U^j z_0 = \sum_{j=0}^{\infty} \alpha^{j+1} z_j\,, \tag{4.16}$$

where the series converges strongly in $H^1_{\partial R}(R \setminus \Gamma_0)$ and

$$z_j := U^j z_0\,. \tag{4.17}$$

In order to prove (4.12) we study the singularity of the gradients of the functions z_j.

Given $1 \le p < +\infty$, it is convenient to introduce the operators $V\colon L^p(S \setminus \Gamma_0) \to L^p(S \setminus \Gamma_0)$ defined by

$$(Vz)(x,y) := \int_0^{+\infty} e^{-s} z(x+\gamma s, y)\,ds \quad \text{for every } z \in L^p(S \setminus \Gamma_0) \tag{4.18}$$

and $V_R\colon L^p(R \setminus \Gamma_0) \to L^p(R \setminus \Gamma_0)$ defined by

$$(V_R z)(x,y) := (V z_R)(x,y) = \int_0^{\frac{\hat\ell_1 - x}{\gamma}} e^{-s} z(x+\gamma s, y)\,ds \quad \text{for every } z \in L^p(R \setminus \Gamma_0)\,, \tag{4.19}$$

where $\hat{\ell}_1$ is defined in (4.2) and z_R is the extension of z obtained by setting $z_R := 0$ in $S \setminus R$. The fact that V maps $L^p(S\setminus\Gamma_0)$ into $L^p(S\setminus\Gamma_0)$ follows from Lemma 3.1. Moreover, we observe that V maps $W^{1,p}(S\setminus\Gamma_0)$ into $W^{1,p}(S\setminus\Gamma_0)$ and that

$$\partial_x(Vz) = V(\partial_x z) \quad \text{and} \quad \partial_y(Vz) = V(\partial_y z). \tag{4.20}$$

The analogous result for V_R is given by the following lemma.

Lemma 4.4. *Let $1 \le p < +\infty$ and let $z \in W^{1,p}(R\setminus\Gamma_0)$. Then $V_R z \in W^{1,p}(R\setminus\Gamma_0)$ and*

$$\partial_x(V_R z)(x,y) = \int_0^{\frac{\hat{\ell}_1 - x}{\gamma}} e^{-s}\partial_x z(x+\gamma s, y)ds - \frac{1}{\gamma}e^{\frac{x-\hat{\ell}_1}{\gamma}} z(\hat{\ell}_1, y)\,,$$

$$\partial_y(V_R z)(x,y) = \int_0^{\frac{\hat{\ell}_1 - x}{\gamma}} e^{-s}\partial_y z(x+\gamma s, y)ds\,,$$

where $z(\hat{\ell}_1,\cdot)$ denotes the trace of z on the segment $\{\hat{\ell}_1\}\times(-1,1)$. Moreover, there exists a constant $D_p > 0$, independent of z and of $c\in[c_0,c_1]$, such that

$$\|V_R z\|_{W^{1,p}(R\setminus\Gamma_0)} \le D_p \|z\|_{W^{1,p}(R\setminus\Gamma_0)}\,. \tag{4.21}$$

Proof. The proof is standard if z is of class C^1 in a suitable neighbourhood of $\partial R\setminus\Gamma_0$. The general case can be obtained by approximation. Estimate (4.21) follows from the trace inequality and from Lemma 3.1 applied to the extensions of z, $\partial_x z$, and $\partial_y z$ obtained by setting them equal to 0 in $S\setminus R$, taking into account the fact that $1/\gamma$ is bounded from above due to (4.1). □

For future use we state also a more general version of the previous result.

Lemma 4.5. *Let $1\le p<+\infty$, $a\ge 1/2$, $\delta>0$, and $z\in W^{1,p}(R_a\setminus\Gamma_0)$. Then the function $z_{a,\delta}\colon R_a\setminus\Gamma_0\to\mathbb{R}$ defined by*

$$z_{a,\delta}(x,y) := \int_0^{\frac{a-x}{\delta}} e^{-s} z(x+\delta s, y)ds$$

belongs to $W^{1,p}(R_a\setminus\Gamma_0)$ and

$$\partial_x z_{a,\delta}(x,y) = \int_0^{\frac{a-x}{\delta}} e^{-s}\partial_x z(x+\delta s,y)ds - \frac{1}{\delta}e^{\frac{x-a}{\delta}} z(a,y)\,,$$

$$\partial_y z_{a,\delta}(x,y) = \int_0^{\frac{a-x}{\delta}} e^{-s}\partial_y z(x+\delta s,y)ds\,,$$

where $z(a,\cdot)$ denotes the trace of z on the segment $\{a\}\times(-1,1)$. Moreover, there exists a constant $E_p>0$, independent of z, a, and δ, such that

$$\|z_{a,\delta}\|_{W^{1,p}(R_a\setminus\Gamma_0)} \le E_p(1+\frac{1}{\delta})\|z\|_{W^{1,p}(R_a\setminus\Gamma_0)}\,. \tag{4.22}$$

Proof. As in the previous lemma the proof is standard if z is of class C^1 in a suitable neighbourhood of $\partial R_a\setminus\Gamma_0$, and the general case can be obtained by approximation. For all terms appearing in $\|z_{a,\delta}\|_{W^{1,p}(R_a\setminus\Gamma_0)}$, except for the L^p-norm of $\frac{1}{\delta}e^{\frac{x-a}{\delta}}z(a,y)$, the estimate (4.22) follows from Lemma 3.1 applied to the extensions of z, $\partial_x z$, and $\partial_y z$ obtained by setting them equal to 0 in $S_a\setminus R_a$. To estimate the L^p-norm of $\frac{1}{\delta}e^{\frac{x-a}{\delta}}z(a,\cdot)$ we use the continuity of the trace operator from $W^{1,p}((a-1,a)\times(-1,1))$ into $L^p(\{a\}\times(-1,1))$, observing that the norm of this operator does not depend on a and that the rectangle $(a-1,a)\times(-1,1)$ is contained in R_a. □

We shall use the following integrability result. Note that, while $V\psi_0$ depends on c through γ, the final estimate of its norm is independent of $c\in[c_0,c_1]$.

Lemma 4.6. *Let $1 \le r < 2$. Then the function $V\psi_0$ belongs to $W^{2,r}(R \setminus \Gamma_0)$ and there exists a constant $F_r > 0$, independent of $c \in [c_0, c_1]$, such that $\|V\psi_0\|_{W^{2,r}(R\setminus\Gamma_0)} \le F_r$.*

Proof. Since $\psi_0 \in H^1(S \setminus \Gamma_0)$, by Lemma 3.1 applied to ψ_0, $\partial_x\psi_0$, and $\partial_y\psi_0$ we deduce from (4.20) that $V\psi_0 \in W^{1,r}(R \setminus \Gamma_0)$ and that

$$\|V\psi_0\|_{W^{1,r}(R\setminus\Gamma_0)} \le F_{r,0} \tag{4.23}$$

for a suitable positive constant $F_{r,0}$ independent of $c \in [c_0, c_1]$.

Since, by (4.20), $\partial_{xx}(V\psi_0) = V(\partial_{xx}\psi_0)$, $\partial_{xy}(V\psi_0) = V(\partial_{xy}\psi_0)$, and $\partial_{yy}(V\psi_0) = V(\partial_{yy}\psi_0)$, to deal with the second derivatives of $V\psi_0$ we consider the functions

$$(x,y) \mapsto \int_0^{+\infty} e^{-s}|\partial_{xx}\psi_0(x+\gamma s, y)|ds\,, \tag{4.24}$$

$$(x,y) \mapsto \int_0^{+\infty} e^{-s}|\partial_{xy}\psi_0(x+\gamma s, y)|ds\,, \tag{4.25}$$

$$(x,y) \mapsto \int_0^{+\infty} e^{-s}|\partial_{yy}\psi_0(x+\gamma s, y)|ds\,. \tag{4.26}$$

We claim that they belong to $L^r(R \setminus \Gamma_0)$ for every $1 \le r < 2$ and that their L^r-norms are bounded by a constant depending only on r.

We prove the claim only for (4.24), the proof for the other ones being analogous. By direct computation we see that there exists a constant $A > 0$ such that

$$|\psi(x,y)| \le (|x|+|y|)^{1/2}\,, \quad |\partial_x\psi(x,y)| \le \frac{A}{(|x|+|y|)^{1/2}}\,, \quad |\partial_{xx}\psi(x,y)| \le \frac{A}{(|x|+|y|)^{3/2}}$$

for every $(x,y) \in \mathbb{R}^2 \setminus \Gamma_0$. Since $\partial_{xx}\psi_0 = \partial_{xx}\psi\,\omega + 2\partial_x\psi\,\partial_x\omega + \psi\,\partial_{xx}\omega$, there exists a constant $B > 0$ such that for every $(x,y) \in \mathbb{R}^2 \setminus \Gamma_0$ we have

$$|\partial_{xx}\psi_0(x,y)| \le \frac{B}{(|x|+|y|)^{3/2}}\chi(x,y) + \frac{B}{(|x|+|y|)^{1/2}}\chi(x,y) + B(|x|+|y|)^{1/2}\chi(x,y)\,,$$

where $\chi \in C_c^\infty(\mathbb{R}^2)$, $0 \le \chi \le 1$ on $\mathbb{R}^2$, and $\chi = 1$ on $\operatorname{supp}\omega$.

Then for every $(x,y) \in \mathbb{R}^2 \setminus \Gamma_0$ we have

$$\int_0^{+\infty} e^{-s}|\partial_{xx}\psi_0(x+\gamma s, y)|ds \le B\,f_1(x,y) + B\,f_2(x,y) + B\,f_3(x,y)\,,$$

where

$$f_1(x,y) := \int_0^{+\infty} e^{-s}\frac{1}{(|x+\gamma s|+|y|)^{3/2}}ds\,,$$

$$f_2(x,y) := \int_0^{+\infty} e^{-s}\frac{\chi(x+\gamma s, y)}{(|x+\gamma s|+|y|)^{1/2}}ds\,,$$

$$f_3(x,y) := \int_0^{+\infty} e^{-s}(|x+\gamma s|+|y|)^{1/2}ds\,.$$

By Lemma 3.1 for every $r < 4$ we have

$$\|f_2\|_{L^r(R\setminus\Gamma_0)} \le \Big(\int_{\operatorname{supp}\chi}\frac{dxdy}{(|x|+|y|)^{r/2}}\Big)^{1/r} =: F_{r,2}\,.$$

Since

$$f_3(x,y) \le (|x|+|y|)^{1/2} + \gamma^{1/2}\int_0^{+\infty} e^{-s}s^{1/2}ds\,,$$

the function f_3 belongs to $L^\infty(R\setminus\Gamma_0)$. Recalling that the function $c \mapsto \gamma$ is increasing and that $c_0 \le c \le c_1$, we deduce that

$$\|f_3\|_{L^r(R\setminus\Gamma_0)} \le F_{r,3}$$

for a suitable positive constant $F_{r,3}$ independent of c.

As for f_1, in order to integrate by parts we notice that

$$\frac{d}{ds}\frac{1}{(|x+\gamma s|+|y|)^{1/2}}=-\frac{1}{2}\frac{\gamma\,\mathrm{sign}(x+\gamma s)}{(|x+\gamma s|+|y|)^{3/2}}\,.$$

Hence for $x>0$ we have

$$f_1(x,y)=-\frac{2}{\gamma}\frac{e^{-s}}{(|x+\gamma s|+|y|)^{1/2}}\Big|_{s=0}^{s=+\infty}-\frac{2}{\gamma}\int_0^{+\infty}\frac{e^{-s}}{(|x+\gamma s|+|y|)^{1/2}}ds\leq\frac{2}{\gamma(|x|+|y|)^{1/2}}\,.$$

To estimate f_1 for $x<0$ we write

$$\begin{aligned}f_1(x,y)&=\int_0^{-x/\gamma}\frac{e^{-s}ds}{(|x+\gamma s|+|y|)^{3/2}}+\int_{-x/\gamma}^{+\infty}\frac{e^{-s}ds}{(|x+\gamma s|+|y|)^{3/2}}\\&=\frac{2}{\gamma}\frac{e^{-s}}{(|x+\gamma s|+|y|)^{1/2}}\Big|_{s=0}^{s=-x/\gamma}+\int_0^{-x/\gamma}\frac{2}{\gamma}\frac{e^{-s}ds}{(|x+\gamma s|+|y|)^{1/2}}\\&\quad-\frac{2}{\gamma}\frac{e^{-s}}{(|x+\gamma s|+|y|)^{1/2}}\Big|_{s=-x/\gamma}^{s=+\infty}-\int_{-x/\gamma}^{+\infty}\frac{2}{\gamma}\frac{e^{-s}ds}{(|x+\gamma s|+|y|)^{1/2}}\leq\frac{C}{\gamma|y|^{1/2}}\end{aligned}$$

for a suitable constant $C>0$ independent of x and y. Recalling that the function $c\mapsto\gamma$ is increasing and that $c_0\leq c\leq c_1$, we obtain that $f_1\in L^r(R\setminus\Gamma_0)$ for every $r<2$ and

$$\|f_1\|_{L^r(R\setminus\Gamma_0)}\leq F_{r,1}$$

for some positive constant $F_{r,1}$ independent of c. Together with the results for f_2 and f_3 this concludes the proof of the claim for (4.24).

The conclusion of the lemma follows from (4.23) and from the estimates obtained in the claim. □

In the following lemma the estimates for the first derivatives proved in Lemma 4.4 are extended, under more restrictive assumptions, to second order derivatives.

Lemma 4.7. *Let $1<r<2$ and let $z\in H^1(R\setminus\Gamma_0)$ with $z(\hat\ell_1,y)=0$ in the sense of traces on the segment $\{\hat\ell_1\}\times(-1,1)$. Assume that there exist $\kappa\in\mathbb{R}$ and $z^{reg}\in W^{2,r}(R\setminus\Gamma_0)$ such that*

$$z=\kappa\psi+z^{reg}\quad\text{in } R\setminus\Gamma_0\,. \tag{4.27}$$

Then $V_Rz\in W^{2,r}(R\setminus\Gamma_0)$ and there exists a constant $G_r>0$, independent of z, κ, z^{reg}, and $c\in[c_0,c_1]$, such that

$$\|V_Rz\|_{W^{2,r}(R\setminus\Gamma_0)}\leq G_r(|\kappa|+\|z^{reg}\|_{W^{2,r}(R\setminus\Gamma_0)})\,. \tag{4.28}$$

Proof. By Lemma 4.4 applied to z we have that $V_Rz\in H^1(R\setminus\Gamma_0)$ and that for a.e. $(x,y)\in R\setminus\Gamma_0$

$$\partial_x(V_Rz)(x,y)=\int_0^{\frac{\hat\ell_1-x}{\gamma}}e^{-s}\partial_xz(x+\gamma s,y)ds\,, \tag{4.29}$$

$$\partial_y(V_Rz)(x,y)=\int_0^{\frac{\hat\ell_1-x}{\gamma}}e^{-s}\partial_yz(x+\gamma s,y)ds\,, \tag{4.30}$$

which implies

$$\partial_x(V_Rz)=V_R(\partial_xz)\quad\text{and}\quad\partial_y(V_Rz)=V_R(\partial_yz)\,. \tag{4.31}$$

By applying Lemma 3.1 to suitable extensions of V_Rz, $\partial_x(V_Rz)$, and $\partial_y(V_Rz)$, we obtain that

$$\|V_Rz\|_{W^{1,r}(R\setminus\Gamma_0)}\leq\|z\|_{W^{1,r}(R\setminus\Gamma_0)}\,. \tag{4.32}$$

To prove that $V_Rz\in W^{2,r}(R\setminus\Gamma_0)$ we fix $1<\hat r<4/3$ with $\hat r\leq r$, and begin by proving that $V_Rz\in W^{2,\hat r}(R\setminus\Gamma_0)$. By (4.5) and (4.27) we have that $z\in W^{2,\hat r}(R\setminus\Gamma_0)$. Recalling

(4.31), by Lemma 4.4 applied to $\partial_x z$ and $\partial_y z$, with $p=\hat{r}$, we obtain that $V_R z \in W^{2,\hat{r}}(R\backslash\Gamma_0)$ and for a.e. $(x,y)\in R\setminus\Gamma_0$

$$\partial_{xx}(V_R z)(x,y) = \int_0^{\frac{\hat{\ell}_1-x}{\gamma}} e^{-s}\partial_{xx}z(x+\gamma s,y)ds - \frac{1}{\gamma}e^{\frac{x-\hat{\ell}_1}{\gamma}}\partial_x z(\hat{\ell}_1,y)\,, \tag{4.33}$$

$$\partial_{xy}(V_R z)(x,y) = \int_0^{\frac{\hat{\ell}_1-x}{\gamma}} e^{-s}\partial_{xy}z(x+\gamma s,y)ds\,, \tag{4.34}$$

$$\partial_{yy}(V_R z)(x,y) = \int_0^{\frac{\hat{\ell}_1-x}{\gamma}} e^{-s}\partial_{yy}z(x+\gamma s,y)ds\,, \tag{4.35}$$

where in the last equalities we used the fact that the trace of z on the segment $\{\hat{\ell}_1\}\times(-1,1)$ satisfies $z(\hat{\ell}_1,y)=0$ for a.e. $y\in(-1,1)$, and hence $\partial_y z(\hat{\ell}_1,y)=0$ for a.e. $y\in(-1,1)$.

These equalities allow us to improve the regularity of $V_R z$ and to obtain that $V_R z\in W^{2,r}(R\setminus\Gamma_0)$. Indeed, by (4.27) we obtain

$$\begin{aligned}\partial_{xx}(V_R z)(x,y) = \kappa\int_0^{\frac{\hat{\ell}_1-x}{\gamma}} e^{-s}\partial_{xx}\psi(x+\gamma s,y)ds - \frac{\kappa}{\gamma}e^{\frac{x-\hat{\ell}_1}{\gamma}}\partial_x\psi(\hat{\ell}_1,y)\\ +\int_0^{\frac{\hat{\ell}_1-x}{\gamma}} e^{-s}\partial_{xx}z^{reg}(x+\gamma s,y)ds - \frac{1}{\gamma}e^{\frac{x-\hat{\ell}_1}{\gamma}}\partial_x z^{reg}(\hat{\ell}_1,y)\,.\end{aligned} \tag{4.36}$$

From the proof of Lemma 4.6 we know that the function

$$(x,y)\mapsto \int_0^{\frac{\hat{\ell}_1-x}{\gamma}} e^{-s}\partial_{xx}\psi(x+\gamma s,y)ds$$

belongs to $L^r(R\setminus\Gamma_0)$ and that its L^r-norm is bounded by F_r. Moreover, the function

$$(x,y)\mapsto \frac{1}{\gamma}e^{\frac{x-\hat{\ell}_1}{\gamma}}\partial_x\psi(\hat{\ell}_1,y) \tag{4.37}$$

belongs to $L^r(R\setminus\Gamma_0)$ because ψ is of class C^∞ in a neighbourhood of the segment $\{\hat{\ell}_1\}\times[-1,1]$. Recalling that the function $c\mapsto\gamma$ is increasing and that $c_0\le c\le c_1$, we deduce that the L^r-norm of (4.37) is bounded by some positive constant H_r.

By Lemma 3.1 the function

$$(x,y)\mapsto \int_0^{\frac{\hat{\ell}_1-x}{\gamma}} e^{-s}\partial_{xx}z^{reg}(x+\gamma s,y)ds$$

belongs to $L^r(R\setminus\Gamma_0)$ and its norm is bounded by $\|z^{reg}\|_{W^{2,r}(R\backslash\Gamma_0)}$. Moreover, using the monotonicity of $c\mapsto\gamma$, the inequalities $c_0\le c\le c_1$ and the continuity of the trace operator imply that there exists a constant $K_r>0$ such that

$$\Big(\int_{R\backslash\Gamma_0}|\frac{1}{\gamma}e^{-\frac{\hat{\ell}_1-x}{\gamma}}\partial_x\varphi(\hat{\ell}_1,y)|^r dxdy\Big)^{1/r}\le K_r\|\varphi\|_{W^{2,r}(R\backslash\Gamma_0)}\,, \tag{4.38}$$

for every $\varphi\in W^{2,r}(R\setminus\Gamma_0)$. Applying this inequality to $\varphi=z^{reg}$, from (4.36) and the previous inequalities we conclude that

$$\|\partial_{xx}(V_R z)\|_{L^r(R\backslash\Gamma_0)}\le\kappa(F_r+H_r)+(1+K_r)\|z^{reg}\|_{W^{2,r}(R\backslash\Gamma_0)}. \tag{4.39}$$

Similarly, we estimate the other two terms $\partial_{xy}(V_R z)$ and $\partial_{yy}(V_R z)$ in $L^r(R\setminus\Gamma_0)$. Together with the estimate (4.32) of the $W^{1,r}$-norm, this concludes the proof. □

Proof of Theorem 4.3 (continuation). We now introduce two operators Ψ_1 and Ψ_2 from $H^1(S\setminus\Gamma_0)$ into $H^{-1}_{\partial R}(R\setminus\Gamma_0)$ defined by

$$\langle\Psi_1 z,\varphi\rangle := \int_{R\setminus\Gamma_0}\Big(\int_0^{+\infty} e^{-s}\nabla z(x+\gamma s,y)ds\Big)\nabla\varphi(x,y)dxdy\,, \tag{4.40}$$

$$\langle\Psi_2 z,\varphi\rangle := \int_{R\setminus\Gamma_0}\Big(\int_0^{+\infty} e^{-s}\partial_x z(x+\gamma s,y)ds\Big)\partial_x\varphi(x,y)dxdy\,, \tag{4.41}$$

for every $z\in H^1(S\setminus\Gamma_0)$ and every $\varphi\in H^1_{\partial R}(R\setminus\Gamma_0)$, where $\gamma=c/\sqrt{1-c^2}$ as in (4.3).

Recalling the definition (3.24) of Ψ we have

$$\Psi=\Psi_1+\gamma^2\Psi_2 \quad \text{in } H^1(S\setminus\Gamma_0)\,. \tag{4.42}$$

We also note that by (4.20) we have

$$\langle\Psi_1 z,\varphi\rangle = \int_{R\setminus\Gamma_0}\nabla(Vz)(x,y)\nabla\varphi(x,y)dxdy\,,$$

for every $z\in H^1(S\setminus\Gamma_0)$ and every $\varphi\in H^1_{\partial R}(R\setminus\Gamma_0)$, hence

$$\Psi_1=\Phi V \quad \text{in } H^1(S\setminus\Gamma_0)\,, \tag{4.43}$$

where Φ is defined in (3.23).

In the following lemma we use these operators to describe in detail the structure of the functions z_j introduced in (4.17).

Lemma 4.8. *Let $4/3<r<2$. Then there exists a constant $\mu_r\ge 1$, independent of α, such that for every $\mu\ge\mu_r$ and every $j\ge 0$ there exist $\kappa_j\in\mathbb{R}$ and $z_j^{reg}\in W^{2,r}(R\setminus\Gamma_0)$ satisfying*

$$z_j=\kappa_j\psi+z_j^{reg} \quad \textit{in } R\setminus\Gamma_0 \quad \textit{and} \quad |\kappa_j|+\|z_j^{reg}\|_{W^{2,r}(R\setminus\Gamma_0)}\le\mu^{j+1}\,. \tag{4.44}$$

Proof. We proceed by induction. Taking into account (4.14), (4.42), and (4.43) we have

$$z_0=\Phi_0^{-1}\Psi\psi_0=\Phi_0^{-1}\Phi V\psi_0+\gamma^2\Phi_0^{-1}\Psi_2\psi_0\,. \tag{4.45}$$

Let us consider the term $z_{0,1}:=\Phi_0^{-1}\Phi V\psi_0$. We have $z_{0,1}\in H^1_{\partial R}(R\setminus\Gamma_0)$ and $\Phi z_{0,1}=\Phi V\psi_0$. Hence $z_{0,1}=V\psi_0+v_{0,1}$ with $v_{0,1}\in H^1(R\setminus\Gamma_0)$ satisfying $\Phi v_{0,1}=0$, i.e.,

$$\int_{R\setminus\Gamma_0}\nabla v_{0,1}(x,y)\nabla\varphi(x,y)dxdy=0$$

for every $\varphi\in H^1_{\partial R}(R\setminus\Gamma_0)$. Since $z_{0,1}=0$ on ∂R we deduce that $v_{0,1}=-V\psi_0$ on ∂R. By Corollary 4.2 there exist $\kappa_{0,1}\in\mathbb{R}$ and $v_{0,1}^{reg}\in W^{2,r}(R\setminus\Gamma_0)$ such that

$$v_{0,1}=\kappa_{0,1}\psi+v_{0,1}^{reg} \quad \text{in } R\setminus\Gamma_0\,, \tag{4.46}$$

$$|\kappa_{0,1}|+\|v_{0,1}^{reg}\|_{W^{2,r}(R\setminus\Gamma_0)}\le B_r\|V\psi_0\|_{W^{2,r}(R\setminus\Gamma_0)}\le B_rF_r\,, \tag{4.47}$$

where in the last inequality we used Lemma 4.6.

To study the term $z_{0,2}:=\Phi_0^{-1}\Psi_2\psi_0$, we set

$$f_0(x,y):=\int_0^{+\infty}e^{-s}\partial_{xx}\psi_0(x+\gamma s,y)ds=(V\partial_{xx}\psi_0)(x,y)\,.$$

By Lemma 4.6 we have $f_0\in L^r(R\setminus\Gamma_0)$ and $\|f_0\|_{L^r(R\setminus\Gamma_0)}\le F_r$.

We claim that $\Psi_2\psi_0=-f_0$ in $H^{-1}_{\partial R}(R\setminus\Gamma_0)$, which means that

$$\int_{R\setminus\Gamma_0}\Big(\int_0^{+\infty}e^{-s}\partial_x\psi_0(x+\gamma s,y)ds\Big)\partial_x\varphi(x,y)dxdy=-\int_{R\setminus\Gamma_0}f_0(x,y)\varphi(x,y)dxdy$$

for every $\varphi\in H^1_{\partial R}(R\setminus\Gamma_0)$. This can be written in the form

$$\int_{R\setminus\Gamma_0}(V\partial_x\psi_0)(x,y)\,\partial_x\varphi(x,y)dxdy=-\int_{R\setminus\Gamma_0}f_0(x,y)\varphi(x,y)dxdy. \tag{4.48}$$

By (4.20) we have $\partial_x(V\partial_x\psi_0) = V\partial_{xx}\psi_0 = f_0$, and $\partial_y(V\partial_x\psi_0) = V\partial_{xy}\psi_0$. By Lemma 4.6 the functions $V\partial_{xx}\psi_0$ and $V\partial_{xy}\psi_0$ belong to $L^r(R\setminus\Gamma_0)$, hence $V\partial_x\psi_0 \in W^{1,r}(R\setminus\Gamma_0)$. Since $\varphi \in H^1_{\partial R}(R\setminus\Gamma_0)$, and hence $\varphi \in L^q(R\setminus\Gamma_0)$ by the Sobolev Embedding Theorem for every $1 \le q < +\infty$, we can integrate by parts in the left-hand side of (4.48). Since the x-component of the normal to Γ_0 is zero, no boundary term appears and, using the equality $\partial_x(V\partial_x\psi_0) = f_0$, from the integration by parts we obtain (4.48), concluding the proof of the claim.

By Theorem 4.1 there exist $\kappa_{0,2} \in \mathbb{R}$ and $z^{reg}_{0,2} \in W^{2,r}(R\setminus\Gamma_0)$ such that

$$z_{0,2} = \Phi_0^{-1}\Psi_2\psi_0 = -\Phi_0^{-1}f_0 = \kappa_{0,2}\psi + z^{reg}_{0,2} \quad \text{in } R\setminus\Gamma_0\,, \tag{4.49}$$

$$|\kappa_{0,2}| + \|z^{reg}_{0,2}\|_{W^{2,r}(R\setminus\Gamma_0)} \le A_r\|f_0\|_{L^r(R\setminus\Gamma_0)} \le A_rF_r\,. \tag{4.50}$$

Therefore (4.45)-(4.47), together with the monotonicity of $c \mapsto \gamma$ and the inequalities $c_0 \le c \le c_1$, imply that

$$z_0 = z_{0,1} + \gamma^2 z_{0,2} = (\kappa_{0,1} + \gamma^2\kappa_{0,2})\psi + v^{reg}_{0,1} + V\psi_0 + \gamma^2 z^{reg}_{0,2} \quad \text{in } R\setminus\Gamma_0 \tag{4.51}$$

$$|\kappa_{0,1} + \gamma^2\kappa_{0,2}| + \|v^{reg}_{0,1} + V\psi_0 + \gamma^2 z^{reg}_{0,2}\|_{W^{2,r}(R\setminus\Gamma_0)} \le (B_r + 1 + \gamma_1^2A_r)F_r\,, \tag{4.52}$$

where $\gamma_1 := c_1/\sqrt{1-c_1^2}$. This implies that (4.44) is satisfied for $j = 0$ with $\kappa_0 := \kappa_{0,1} + \gamma^2\kappa_{0,2}$ and $z^{reg}_0 := v^{reg}_{0,1} + V\psi_0 + \gamma^2 z^{reg}_{0,2}$ provided

$$\mu \ge \mu_{r,0} := (B_r + 1 + \gamma_1^2A_r)F_r. \tag{4.53}$$

Let now $j \ge 1$ and assume that (4.44) holds for $j-1$, that is to say

$$z_{j-1} = \kappa_{j-1}\psi + z^{reg}_{j-1} \text{ in } R\setminus\Gamma_0 \quad \text{and} \quad |\kappa_{j-1}| + \|z^{reg}_{j-1}\|_{W^{2,r}(R\setminus\Gamma_0)} \le \mu^j \tag{4.54}$$

for suitable $\kappa_{j-1} \in \mathbb{R}$, $z^{reg}_{j-1} \in W^{2,r}(R\setminus\Gamma_0)$, and $\mu > 0$. By (4.15) and (4.17) we have $z_j = \Phi_0^{-1}\Psi z_{j-1}$. Therefore (4.42) gives

$$z_j = z_{j,1} + \gamma^2 z_{j,2}\,, \tag{4.55}$$

where, taking (4.43) into account, we set

$$z_{j,1} := \Phi_0^{-1}\Psi_1 z_{j-1} = \Phi_0^{-1}\Phi V z_{j-1}\,, \tag{4.56}$$

$$z_{j,2} := \Phi_0^{-1}\Psi_2 z_{j-1}\,. \tag{4.57}$$

Let us consider first the term $z_{j,1}$. We have $z_{j,1} \in H^1_{\partial R}(R\setminus\Gamma_0)$ and $\Phi z_{j,1} = \Phi V z_{j-1}$. Hence

$$z_{j,1} = V z_{j-1} + v_{j,1}\,, \tag{4.58}$$

where $v_{j,1} \in H^1(R\setminus\Gamma_0)$ satisfies $\Phi v_{j,1} = 0$, i.e.,

$$\int_{R\setminus\Gamma_0} \nabla v_{j,1}(x,y)\nabla\varphi(x,y)\,dxdy = 0$$

for every $\varphi \in H^1_{\partial R}(R\setminus\Gamma_0)$. Since $z_{j,1} = 0$ on ∂R, we deduce that $v_{j,1} = -Vz_{j-1}$ on ∂R. By Corollary 4.2 there exist $\kappa_{j,1} \in \mathbb{R}$ and $v^{reg}_{j,1} \in W^{2,r}(R\setminus\Gamma_0)$ such that

$$v_{j,1} = \kappa_{j,1}\psi + v^{reg}_{j,1} \quad \text{in } R\setminus\Gamma_0\,, \tag{4.59}$$

$$|\kappa_{j,1}| + \|v^{reg}_{j,1}\|_{W^{2,r}(R\setminus\Gamma_0)} \le B_r\|Vz_{j-1}\|_{W^{2,r}(R\setminus\Gamma_0)}\,. \tag{4.60}$$

Since $z_{j-1} \in H^1_{\partial R}(R\setminus\Gamma_0)$, we have $Vz_{j-1} = V_Rz_{j-1}$. By Lemma 4.7 and (4.54) we have $V_Rz_{j-1} \in W^{2,r}(R\setminus\Gamma_0)$ and

$$\|Vz_{j-1}\|_{W^{2,r}(R\setminus\Gamma_0)} = \|V_Rz_{j-1}\|_{W^{2,r}(R\setminus\Gamma_0)} \le G_r(|\kappa_{j-1}| + \|z^{reg}_{j-1}\|_{W^{2,r}(R\setminus\Gamma_0)})\,. \tag{4.61}$$

Together with (4.54) and (4.58)-(4.60) this gives

$$z_{j,1} = \kappa_{j,1}\psi + v^{reg}_{j,1} + Vz_{j-1} \quad \text{in } R\setminus\Gamma_0 \tag{4.62}$$

$$|\kappa_{j,1}| + \|Vz_{j-1} + v^{reg}_{j,1}\|_{W^{2,r}(R\setminus\Gamma_0)} \le (B_r+1)G_r\mu^j\,. \tag{4.63}$$

To deal with $z_{j,2}$ introduced in (4.57) we set

$$f_{j-1}(x,y) := \int_0^{\frac{\hat\ell_1 - x}{\gamma}} e^{-s}\partial_{xx} z_{j-1}(x+\gamma s, y)ds - \frac{1}{\gamma}e^{-\frac{\hat\ell_1 - x}{\gamma}}\partial_x z_{j-1}(\hat\ell_1, y)\,.$$

We claim that $f_{j-1} \in L^r(R\setminus\Gamma_0)$. By (4.54) we have that

$$\begin{aligned} f_{j-1}(x,y) = \kappa_{j-1}\int_0^{\frac{\hat\ell_1 - x}{\gamma}} e^{-s}\partial_{xx}\psi_0(x+\gamma s,y)ds + \int_0^{\frac{\hat\ell_1 - x}{\gamma}} e^{-s}\partial_{xx} z^{reg}_{j-1}(x+\gamma s,y)ds \\ -\frac{1}{\gamma}\kappa_{j-1}e^{-\frac{\hat\ell_1 - x}{\gamma}}\partial_x\psi_0(\hat\ell_1,y) - \frac{1}{\gamma}e^{-\frac{\hat\ell_1 - x}{\gamma}}\partial_x z^{reg}_{j-1}(\hat\ell_1,y)\,. \end{aligned} \tag{4.64}$$

From the proof of Lemma 4.6 we know that the function

$$(x,y)\mapsto \int_0^{\frac{\hat\ell_1 - x}{\gamma}} e^{-s}\partial_{xx}\psi_0(x+\gamma s,y)ds$$

belongs to $L^r(R\setminus\Gamma_0)$ and its norm is bounded by F_r. By Lemma 3.1 the function

$$(x,y)\mapsto \int_0^{\frac{\hat\ell_1 - x}{\gamma}} e^{-s}\partial_{xx} z^{reg}_{j-1}(x+\gamma s,y)ds$$

also belongs to $L^r(R\setminus\Gamma_0)$ and its norm is bounded by $\|z^{reg}_{j-1}\|_{W^{2,r}(R\setminus\Gamma_0)}$. Arguing as in (4.37) we obtain that the $L^r(R\setminus\Gamma_0)$-norm of the function $(x,y)\mapsto \frac{1}{\gamma}e^{-\frac{\hat\ell_1 - x}{\gamma}}\partial_x\psi_0(\hat\ell_1,y)$ is bounded by H_r. As for the last term in (4.64), using (4.38) with $\varphi = z^{reg}_{j-1}$ we obtain

$$\Big(\int_{R\setminus\Gamma_0}|\frac{1}{\gamma}e^{-\frac{\hat\ell_1 - x}{\gamma}}\partial_x z^{reg}_{j-1}(\hat\ell_1,y)|^r dxdy\Big)^{1/r} \le K_r\|z^{reg}_{j-1}\|_{W^{2,r}(R\setminus\Gamma_0)}\,.$$

From (4.64) and from these inequalities we obtain that $f_{j-1}\in L^r(R\setminus\Gamma_0)$ and

$$\|f_{j-1}\|_{L^r(R\setminus\Gamma_0)} \le (F_r+H_r)|\kappa_{j-1}| + (1+K_r)\|z^{reg}_{j-1}\|_{W^{2,r}(R\setminus\Gamma_0)}\,. \tag{4.65}$$

We claim that $\Psi_2 z_{j-1} = -f_{j-1}$ in $H^{-1}_{\partial R}(R\setminus\Gamma_0)$. Recalling that $z_{j-1}=0$ on $S\setminus R$ the claim is equivalent to

$$\int_{R\setminus\Gamma_0}\Big(\int_0^{\frac{\hat\ell_1 - x}{\gamma}} e^{-s}\partial_x z_{j-1}(x+\gamma s,y)ds\Big)\partial_x\varphi(x,y)dxdy = -\int_{R\setminus\Gamma_0} f_{j-1}(x,y)\varphi(x,y)dxdy \tag{4.66}$$

for every $\varphi\in H^1_{\partial R}(R\setminus\Gamma_0)$. We first want to prove that

$$\partial_x\Big(\int_0^{\frac{\hat\ell_1 - x}{\gamma}} e^{-s}\partial_x z_{j-1}(x+\gamma s,y)ds\Big) = -f_{j-1}(x,y)\,.$$

By (4.54) we have

$$\begin{aligned} \int_0^{\frac{\hat\ell_1 - x}{\gamma}} e^{-s}\partial_x z_{j-1}(x+\gamma s,y)ds = \kappa_{j-1}\int_0^{\frac{\hat\ell_1 - x}{\gamma}} e^{-s}\partial_x\psi_0(x+\gamma s,y)ds \\ + \int_0^{\frac{\hat\ell_1 - x}{\gamma}} e^{-s}\partial_x z^{reg}_{j-1}(x+\gamma s,y)ds\,. \end{aligned}$$

By Lemma 4.4 the function

$$(x,y)\mapsto \int_0^{\frac{\hat\ell_1 - x}{\gamma}} e^{-s}\partial_x z^{reg}_{j-1}(x+\gamma s,y)ds \tag{4.67}$$

belongs to $W^{1,r}(R\setminus\Gamma_0)$ and

$$\partial_x\Big(\int_0^{\frac{\hat\ell_1-x}{\gamma}} e^{-s}\partial_x z^{reg}_{j-1}(x+\gamma s,y)ds\Big)$$
$$=\int_0^{\frac{\hat\ell_1-x}{\gamma}} e^{-s}\partial_{xx} z^{reg}_{j-1}(x+\gamma s,y)ds-\frac{1}{\gamma}e^{\frac{\hat\ell_1-x}{\gamma}}\partial_x z^{reg}_{j-1}(\hat\ell_1,y)\,. \tag{4.68}$$

We now apply Lemma 4.4 with $p=1$ to the function $u=\partial_x\psi_0$ obtaining that the function

$$(x,y)\mapsto\int_0^{\frac{\hat\ell_1-x}{\gamma}} e^{-s}\partial_x\psi_0(x+\gamma s,y)ds \tag{4.69}$$

belongs to $W^{1,1}(R\setminus\Gamma_0)$ and

$$\partial_x\Big(\int_0^{\frac{\hat\ell_1-x}{\gamma}} e^{-s}\partial_x\psi_0(x+\gamma s,y)ds\Big)$$
$$=\int_0^{\frac{\hat\ell_1-x}{\gamma}} e^{-s}\partial_{xx}\psi_0(x+\gamma s,y)ds-\frac{1}{\gamma}e^{\frac{\hat\ell_1-x}{\gamma}}\partial_x\psi_0(\hat\ell_1,y)\,. \tag{4.70}$$

Since the term $\frac{1}{\gamma}e^{\frac{\hat\ell_1-x}{\gamma}}\partial_x\psi_0(\hat\ell_1,y)$ is bounded and the function

$$(x,y)\mapsto\int_0^{\frac{\hat\ell_1-x}{\gamma}} e^{-s}\partial_{xx}\psi_0(x+\gamma s,y)ds$$

belongs to $L^r(R\setminus\Gamma_0)$ (see the proof of Lemma 4.6), the function (4.69) belongs to $W^{1,r}(R\setminus\Gamma_0)$.

By (4.54), (4.67) and the previous remarks we deduce that

$$(x,y)\mapsto\int_0^{\frac{\hat\ell_1-x}{\gamma}} e^{-s}\partial_x z_{j-1}(x+\gamma s,y)ds \tag{4.71}$$

belongs to $W^{1,r}(R\setminus\Gamma_0)$, while (4.54), (4.68), and (4.70) give

$$\partial_x\Big(\int_0^{\frac{\hat\ell_1-x}{\gamma}} e^{-s}\partial_x z_{j-1}(x+\gamma s,y)ds\Big)=-f_{j-1}(x,y)\,. \tag{4.72}$$

On the other hand, since $z_{j-1}\in H^1(S\setminus\Gamma_0)$, by Lemma 3.1 we have also that the function (4.71) belongs to $L^2(R\setminus\Gamma_0)$. Since by the Sobolev Embedding Theorem, every $\varphi\in H^1(R\setminus\Gamma_0)$ belongs to $L^q(R\setminus\Gamma_0)$ for every $q<+\infty$ we can integrate by parts in the left-hand side of (4.66). Since the x-component of the normal to Γ_0 is zero no boundary term appears, and using (4.72) from the integration by parts we obtain (4.66), concluding the proof of the claim.

By Theorem 4.1 and by (4.57) there exist $\kappa_{j,2}\in\mathbb{R}$ and $z^{reg}_{j,2}\in W^{2,r}(R\setminus\Gamma_0)$ such that

$$z_{j,2}=\Phi_0^{-1}\Psi_2 z_{j-1}=-\Phi_0^{-1}f_{j-1}=\kappa_{j,2}\psi+z^{reg}_{j,2}\quad\text{in } R\setminus\Gamma_0\,, \tag{4.73}$$
$$|\kappa_{j,2}|+\|z^{reg}_{j,2}\|_{W^{2,r}(R\setminus\Gamma_0)}\le A_r\|f_{j-1}\|_{L^r(R\setminus\Gamma_0)}\,.$$

By (4.65) this inequality gives

$$\begin{aligned}|\kappa_{j,2}|+\|z^{reg}_{j,2}\|_{W^{2,r}(R\setminus\Gamma_0)}&\le A_r(F_r+H_r)|\kappa_{j-1}|+A_r(1+K_r)\|z^{reg}_{j-1}\|_{W^{2,r}(R\setminus\Gamma_0)}\\&\le A_r(F_r+H_r+1+K_r)\mu^j\,,\end{aligned} \tag{4.74}$$

where in the last inequality we used the inequality in (4.54).

Therefore, setting $\kappa_j := \kappa_{j,1} + \gamma^2 \kappa_{j,2}$ and $z_j^{reg} := v_{j,1}^{reg} + V z_{j-1} + \gamma^2 z_{j,2}^{reg}$, and using again the monotonicity of $c \mapsto \gamma$ and the inequalities $c_0 \le c \le c_1$, from (4.55), (4.62), (4.63), (4.73), and (4.74) we obtain

$$z_j = \kappa_j \psi + z_j^{reg} \quad \text{in } R \setminus \Gamma_0 \,,$$
$$|\kappa_j| + \|z_j^{reg}\|_{W^{2,r}(R\setminus\Gamma_0)} \le \big((B_r+1)G_r + \gamma_1^2 A_r(F_r + H_r + 1 + K_r)\big)\mu^j \,,$$

where $\gamma_1^2 = c_1^2/(1-c_1^2)$. This implies that (4.44) holds for $j \ge 1$ provided $\mu \ge \mu_{r,1} := (B_r+1)G_r + \gamma_1^2 A_r(F_r + H_r + 1 + K_r)$, thus concluding the proof of the inductive step.

Therefore (4.44) holds for every $j \ge 0$ if $\mu \ge \mu_r := \max\{1, \mu_{r,0}, \mu_{r,1}\}$. This concludes the proof. □

The following result provides the structure of the solution ζ of (3.21).

Lemma 4.9. *Let $4/3 < r < 2$. Assume that $2\alpha\mu_r \le 1 - c_1^2$, where $\mu_r \ge 1$ is the constant provided by Lemma 4.8. Let $\gamma = c/\sqrt{1-c^2}$ as in (4.3). Then the unique solution $\zeta \in H^1_{\partial R}(R \setminus \Gamma_0)$ of problem (3.21), with $\hat a = \hat\ell_1$, can be written in the form*

$$\zeta = \kappa\psi + \zeta^{reg} \quad \textit{in } R \setminus \Gamma_0 \,, \tag{4.75}$$

with $-1 < \kappa < 1$ and $\zeta^{reg} \in W^{2,r}(R \setminus \Gamma_0)$.

Proof. Since $\mu_r \ge 1$, the inequality $2\alpha\mu_r \le 1 - c_1^2 \le 1 - c^2$ and the definition of γ imply $\alpha(1+\gamma^2) < 1$. By Theorem 3.6 there is a unique solution ζ of problem (3.21) with $\hat a = \hat\ell_1$. By (4.16) ζ can be written as

$$\zeta = \sum_{j=0}^{\infty} \alpha^{j+1} z_j \,, \tag{4.76}$$

where z_j are defined in (4.17) and do not depend on α. Choosing $\mu := \mu_r$, by Lemma 4.8 for every j we have

$$z_j = \kappa_j\psi + z_j^{reg} \text{ in } R \setminus \Gamma_0 \quad \text{and} \quad |\kappa_j| + \|z_j^{reg}\|_{W^{2,r}(R\setminus\Gamma_0)} \le \mu^{j+1} \,. \tag{4.77}$$

Since $0 < \alpha\mu < 1/2$ the series $\sum_{j=0}^{\infty} \alpha^{j+1}\kappa_j$ and $\sum_{j=0}^{\infty} \alpha^{j+1} z_j^{reg}$ converge in $\mathbb{R}$ and $W^{2,r}(R \setminus \Gamma_0)$, respectively. Therefore, choosing

$$\kappa := \sum_{j=0}^{\infty} \alpha^{j+1}\kappa_j \qquad \text{and} \qquad \zeta^{reg} := \sum_{j=0}^{\infty} \alpha^{j+1} z_j^{reg} \tag{4.78}$$

from (4.76) we obtain (4.75). Since

$$\sum_{j=0}^{\infty} \alpha^{j+1}\mu^{j+1} = \frac{\alpha\mu}{1-\alpha\mu} < 1 \,,$$

from (4.77) and (4.78) we get $|\kappa| < 1$. □

Proof of Theorem 4.3 (conclusion). Let $M_r := 2\mu_r \ge 2$, where μ_r is the constant provided by Lemma 4.8. By (4.13) we have $\hat v = \psi_0 + \zeta$. By Lemma 4.9 this implies

$$\hat v = (1+\kappa)\psi + \zeta^{reg} \quad \text{in } R \setminus \Gamma_0 \,,$$

with $|\kappa| < 1$, which gives (4.12) with $\kappa_0 := 1 + \kappa > 0$ and $\hat v^{reg} := \zeta^{reg}$. □

The following result provides the structure of the solution v of problem (3.7)-(3.8). Let $\check\psi \colon \mathbb{R}^2 \setminus \Gamma_0 \to \mathbb{R}$ and $\check\psi_0 \colon \mathbb{R}^2 \setminus \Gamma_0 \to \mathbb{R}$ the functions defined by

$$\check\psi(x,y) := \psi(\tfrac{x}{\lambda}, y) \quad \text{and} \quad \check\psi_0(x,y) := \psi_0(\tfrac{x}{\lambda}, y) \,, \tag{4.79}$$

where ψ and ψ_0 are defined in (4.4) and (4.11), while $\lambda = \sqrt{1-c^2}$.

Corollary 4.10. *Let $4/3 < r < 2$, let $a > 0$, let $\alpha_1 := (1 - c_1^2)/M_r$, where $M_r > 1$ is the constant provided by Theorem 4.3, and let*

$$\check{\ell} := \hat{\ell}_1 \lambda = \ell_1 \frac{\lambda}{\lambda_1} = (1 + c_1) \frac{\sqrt{1 - c^2}}{\sqrt{1 - c_1^2}} . \tag{4.80}$$

Assume that $\alpha \le \alpha_1$. Then the unique solution v of problem (3.7)-(3.8) in $R_{\check{\ell}} \backslash \Gamma_0$ (according to Definition 3.2) with boundary condition $v = a\check{\psi}$ on $\partial R_{\check{\ell}} \setminus \Gamma_0$ and satisfying $v = a\check{\psi}_0$ in $S_{\check{\ell}} \setminus R_{\check{\ell}}$ can be written in a unique way in the form

$$v = \kappa \check{\psi} + v^{reg} \quad \text{in } R_{\check{\ell}} \setminus \Gamma_0 ,$$

where $\kappa = a\kappa_0$, with $\kappa_0 > 0$ the constant provided by Theorem 4.3, and $v^{reg} \in W^{2,r}(R_{\check{\ell}} \backslash \Gamma_0)$.

Proof. We consider first the case $a = 1$. Let $\gamma = c/\sqrt{1 - c^2}$ as in (4.3). Since the inequality $\alpha \le \alpha_1$ implies that $\alpha M_r \le 1 - c_1^2$, by Theorem 4.3 the unique solution $\hat{v}$ of (3.18)-(3.19) in $R \setminus \Gamma_0$ with boundary condition $\hat{v} = \psi$ on ∂R, and satisfying $\hat{v} = \psi_0$ in $S \setminus R$, can be written in a unique way in the form

$$\hat{v} = \kappa_0 \psi + \hat{v}^{reg} \quad \text{in } R \setminus \Gamma_0 ,$$

with $\kappa_0 > 0$ and $\hat{v}^{reg} \in W^{2,r}(R \setminus \Gamma_0)$. By Lemma 3.5 we have $v(x, y) = \hat{v}(\frac{x}{\lambda}, y)$, hence $v = \kappa_0 \check{\psi}_0 + v^{reg}$, with $v^{reg}(x, y) := \hat{v}^{reg}(\frac{x}{\lambda}, y)$.

In the general case $a > 0$ the solution v can be obtained multiplying by a the solution corresponding to $a = 1$. □

Remark 4.11. Since $\check{\psi} \in W^{2,p}(R_{\check{\ell}} \setminus \Gamma_0)$ for every $1 < p < \frac{4}{3}$, Corollary 4.10 implies that $v \in W^{2,p}(R_{\check{\ell}} \setminus \Gamma_0)$ for every $1 < p < \frac{4}{3}$.

5. Auxiliary results

The following lemmas will be used in the study of the energy-dissipation balance.

Let $\mathcal{H}^1$ be the 1-dimensional Hausdorff measure in $\mathbb{R}^2$. For every $\rho > 0$ let B_ρ be the open ball of centre $(0, 0)$ and radius ρ.

Lemma 5.1. *Assume that $u \in W^{1,r}(B_1)$ with $4/3 < r < 2$ and let $1 < q < 2$ be given by $\frac{1}{q} = \frac{2}{r} - \frac{1}{2}$. Then*

$$\int_{\partial B_1} |u|^2 d\mathcal{H}^1 < +\infty , \tag{5.1}$$

$$\int_{B_1} |u|^q |\nabla u|^q dxdy < +\infty , \tag{5.2}$$

$$\int_{\partial B_\rho} |u|^2 d\mathcal{H}^1 \le \rho \int_{\partial B_1} |u|^2 d\mathcal{H}^1 + 2C_q (\rho^2 - \rho^{\frac{q}{q-1}})^{1 - \frac{1}{q}} \Big(\int_{B_1} |u|^q |\nabla u|^q dxdy \Big)^{\frac{1}{q}} \tag{5.3}$$

for every $0 < \rho < 1$, where $C_q := (2\pi \frac{q-1}{2-q})^{1 - \frac{1}{q}}$. In particular, we have

$$\lim_{\rho \to 0+} \int_{\partial B_\rho} |u|^2 d\mathcal{H}^1 \to 0 . \tag{5.4}$$

Proof. Let $v := u^2$. Using the Sobolev Embedding Theorem we can prove that $v \in W^{1,q}(B_1)$ and $\nabla v = 2u\nabla u$, which gives (5.2). Inequality (5.1) follows from the fact that the trace of v belongs to $L^1(\partial B_1)$.

For $\theta \in [0, 2\pi]$ let $e(\theta) := (\cos\theta, \sin\theta)$. Assuming that $v \in C^1(\overline{B}_1)$ for every $0 < \rho < 1$ we can write

$$v(\rho e(\theta)) = v(e(\theta)) - \int_\rho^1 \nabla v(te(\theta)) e(\theta) dt .$$

Hence

$$v(\rho e(\theta)) \le v(e(\theta)) + \int_\rho^1 |\nabla v(te(\theta))| dt$$
$$\le v(e(\theta)) + \Big(\int_\rho^1 t|\nabla v(te(\theta))|^q dt\Big)^{\frac{1}{q}} \Big(\int_\rho^1 t^{-\frac{1}{q-1}} dt\Big)^{1-\frac{1}{q}}$$
$$= v(e(\theta)) + \Big(\int_\rho^1 t|\nabla v(te(\theta))|^q dt\Big)^{\frac{1}{q}} \Big(\tfrac{q-1}{q-2}(1-\rho^{\frac{q-2}{q-1}})\Big)^{1-\frac{1}{q}}.$$

Using the equalities

$$\int_{\partial B_\rho} v d\mathcal{H}^1 = \int_0^{2\pi} \rho v(\rho e(\theta)) d\theta\,,$$
$$\int_{B_1} |\nabla v|^q dxdy = \int_0^{2\pi} \Big(\int_0^1 t|\nabla v(te(\theta))|^q dt\Big) d\theta\,,$$

we obtain

$$\int_{\partial B_\rho} v d\mathcal{H}^1 \le \rho \int_0^{2\pi} v(e(\theta)) d\theta + \rho(2\pi)^{1-\frac{1}{q}} \Big(\int_{B_1} |\nabla v|^q dxdy\Big)^{\frac{1}{q}} \Big(\tfrac{q-1}{2-q}(\rho^{\frac{q-2}{q-1}} - 1)\Big)^{1-\frac{1}{q}}.$$

By density this inequality holds for every $v \in W^{1,q}(B_1)$. Recalling that $v = u^2$, and hence $\nabla v = 2u\nabla u$, from this formula we obtain (5.3).

Finally, (5.4) is a consequence of (5.1)-(5.3). □

Lemma 5.2. *Assume $u \in W^{1,r}(B_1)$ with $4/3 < r < 2$. Then*

$$\frac{1}{\rho^{\frac{1}{2}}} \int_{\partial B_\rho} |u| d\mathcal{H}^1 \le \rho^{\frac{1}{2}} \int_{\partial B_1} |u| d\mathcal{H}^1 + C_r \Big(\int_B |\nabla u|^r dxdy\Big)^{\frac{1}{r}} \big(\rho^{\frac{3r-4}{2(r-1)}} - \rho^{\frac{r}{2(r-1)}}\big)^{1-\frac{1}{r}} \tag{5.5}$$

for every $0 < \rho < 1$, where $C_r := (2\pi\frac{r-1}{2-r})^{1-\frac{1}{r}}$. In particular,

$$\lim_{\rho \to 0+} \frac{1}{\rho^{\frac{1}{2}}} \int_{\partial B_\rho} |u| d\mathcal{H}^1 = 0\,. \tag{5.6}$$

Proof. For $\theta \in [0, 2\pi]$ let $e(\theta) := (\cos\theta, \sin\theta)$. Assuming that $u \in C^1(\overline{B}_1)$ we can write

$$u(\rho e(\theta)) = u(e(\theta)) - \int_\rho^1 \nabla u(te(\theta)) e(\theta) dt\,,$$

hence

$$|u(\rho e(\theta))| \le |u(e(\theta))| + \int_\rho^1 |\nabla u(te(\theta))| dt$$
$$\le |u(e(\theta))| + \Big(\int_\rho^1 t|\nabla u(te(\theta))|^r dt\Big)^{\frac{1}{r}} \Big(\int_\rho^1 t^{-\frac{1}{r-1}} dt\Big)^{1-\frac{1}{r}}$$
$$= |u(e(\theta))| + \Big(\int_\rho^1 t|\nabla u(te(\theta))|^r dt\Big)^{\frac{1}{r}} \Big(\tfrac{r-1}{r-2}(1-\rho^{\frac{r-2}{r-1}})\Big)^{1-\frac{1}{r}}.$$

Using the equalities

$$\int_{\partial B_\rho} |u| d\mathcal{H}^1 = \int_0^{2\pi} \rho |u(\rho e(\theta))| d\theta\,,$$
$$\int_{B_1} |\nabla u|^r dxdy = \int_0^{2\pi} \Big(\int_0^1 t|\nabla u(te(\theta))|^r dt\Big) d\theta\,,$$

we obtain

$$\frac{1}{\rho^{\frac{1}{2}}}\int_{\partial B_\rho}|u|d\mathcal{H}^1 \le \rho^{\frac{1}{2}}\int_{\partial B_1}|u|d\mathcal{H}^1 + \rho^{\frac{1}{2}}(2\pi)^{1-\frac{1}{r}}\Big(\int_{B_1}|\nabla u|^r dxdy\Big)^{\frac{1}{r}}\Big(\tfrac{r-1}{2-r}(\rho^{\frac{r-2}{r-1}}-1)\Big)^{1-\frac{1}{r}}.$$

By density the same inequality holds for every $u \in W^{1,r}(B_1)$ and this leads to (5.5), which implies (5.6). □

The following results deal with the behaviour of the singular part of the derivatives of the solution u to (2.7)-(2.9) of the form (3.1). Let us recall that $\check\psi(x,y) = \psi(\frac{x}{\lambda}, y)$, where ψ is defined in (4.4) and $\lambda = \sqrt{1-c^2}$.

Lemma 5.3. *Let* $-1 < t < 1$ *and* $0 < \rho < 1 - c|t|$. *Then*

$$\begin{aligned}\partial_t\Big(\frac{1}{2}\int_{R_1\setminus(\Gamma_{ct}\cup B_\rho(ct,0))}|\partial_x\check\psi(x-ct,y)|^2dxdy\Big)&\\ = -c\int_{R_1\setminus(\Gamma_{ct}\cup B_\rho(ct,0))}\partial_{xx}\check\psi(x-ct,y)\partial_x\check\psi(x-ct,y)dxdy + C_1^\lambda\,,&\end{aligned}\tag{5.7}$$

where

$$C_1^\lambda := \frac{\pi}{8\lambda}\frac{c}{1+\lambda}\,. \tag{5.8}$$

Proof. Let

$$R_1^{ct} := (-1-ct, 1-ct)\times(-1,1)\,. \tag{5.9}$$

By a change of variables we get

$$\int_{R_1\setminus(\Gamma_{ct}\cup B_\rho(ct,0))}|\partial_x\check\psi(x-ct,y)|^2dxdy = \int_{R_1^{ct}\setminus(\Gamma_0\cup B_\rho(0,0))}|\partial_x\check\psi(x,y)|^2dxdy\,,$$

which gives

$$\partial_t\Big(\frac{1}{2}\int_{R_1\setminus(\Gamma_{ct}\cup B_\rho(ct,0))}|\partial_x\check\psi(x-ct,y)|^2dxdy\Big) = \frac{c}{2}\Big(\int_{-1}^{1}|\partial_x\check\psi(-1-ct,y)|^2dy - \int_{-1}^{1}|\partial_x\check\psi(1-ct,y)|^2dy\Big).$$

On the other hand, by the same change of variables

$$-c\int_{R_1\setminus(\Gamma_{ct}\cup B_\rho(ct,0))}\partial_{xx}\check\psi(x-ct,y)\partial_x\check\psi(x-ct,y)dxdy = -c\int_{R_1^{ct}\setminus(\Gamma_0\cup B_\rho(0,0))}\partial_{xx}\check\psi(x,y)\partial_x\check\psi(x,y)dxdy\,.$$

Integrating by parts we obtain that the right-hand side is equal to

$$\begin{aligned}\frac{c}{2}\Big(\int_{-1}^{1}|\partial_x\check\psi(-1-ct,y)|^2dy - \int_{-1}^{1}|\partial_x\check\psi(1-ct,y)|^2dy\Big)&\\ +\frac{c}{2}\rho\int_{-\pi}^{\pi}|\partial_x\check\psi(\rho\cos\theta,\rho\sin\theta)|^2\cos\theta d\theta\,.&\end{aligned}$$

Since $\partial_x\check\psi$ is homogeneous of degree $-\frac{1}{2}$ we obtain that

$$\tfrac{1}{2}\rho|\partial_x\check\psi(\rho\cos\theta,\rho\sin\theta)|^2\cos\theta = \tfrac{1}{2}|\partial_x\check\psi(\cos\theta,\sin\theta)|^2\cos\theta\,.$$

Therefore (5.7) holds with

$$C_1^\lambda = -\frac{c}{2}\int_{-\pi}^{\pi}|\partial_x\check\psi(\cos\theta,\sin\theta)|^2\cos\theta d\theta\,. \tag{5.10}$$

In order to compute this integral it is convenient to introduce the polar coordinates $(\rho_\lambda, \theta_\lambda) = (\rho_\lambda(\theta), \theta_\lambda(\theta))$ of the point $(\frac{\cos\theta}{\lambda}, \sin\theta)$. Recalling the definitions of ψ and $\check\psi$ we have that $\partial_x\check\psi(\cos\theta,\sin\theta) = -\frac{1}{2\lambda}\rho_\lambda^{-1/2}\sin\frac{\theta_\lambda}{2}$, hence

$$C_1^\lambda = -\frac{c}{16\lambda^2}\int_{-\pi}^{\pi}\frac{1}{\rho_\lambda}(1-\cos\theta_\lambda)\cos\theta d\theta = \frac{c}{16\lambda^2}\int_{-\pi}^{\pi}\frac{\cos\theta_\lambda\cos\theta}{\rho_\lambda}d\theta\,, \tag{5.11}$$

where in the last equality we used the symmetry properties of $\rho_\lambda(\theta)$ and $\cos\theta$. Since $\rho_\lambda\cos\theta_\lambda = \frac{\cos\theta}{\lambda}$ and $\rho_\lambda^2 = \frac{\cos^2\theta}{\lambda^2} + \sin^2\theta$, from (5.11) we obtain

$$C_1^\lambda := \frac{c}{16\lambda}\int_{-\pi}^{\pi}\frac{\cos^2\theta}{\cos^2\theta + \lambda^2\sin^2\theta}d\theta\,, \tag{5.12}$$

which gives (5.8) by direct computation. □

Corollary 5.4. *For every $-1 < t_1 < t_2 < 1$ we have*

$$\begin{aligned}&-c\lim_{\rho\to 0+}\int_{t_1}^{t_2}\Big(\int_{R_1\setminus(\Gamma_{ct}\cup B_\rho(ct,0))}\partial_{xx}\check\psi(x-ct,y)\partial_x\check\psi(x-ct,y)dxdy\Big)dt\\ &= \frac{1}{2}\int_{R_1\setminus\Gamma_{ct_2}}|\partial_x\check\psi(x-ct_2,y)|^2dxdy - \frac{1}{2}\int_{R_1\setminus\Gamma_{ct_1}}|\partial_x\check\psi(x-ct_1,y)|^2dxdy - (t_2-t_1)C_1^\lambda\,,\end{aligned}$$

where C_1^λ is the constant introduced in (5.8).

Proof. Integrating (5.7) between t_1 and t_2 we obtain

$$\begin{aligned}&\frac{1}{2}\int_{R_1\setminus(\Gamma_{ct_2}\cup B_\rho(ct_2,0))}|\partial_x\check\psi(x-ct_2,y)|^2dxdy - \frac{1}{2}\int_{R_1\setminus(\Gamma_{ct_1}\cup B_\rho(ct_1,0))}|\partial_x\check\psi(x-ct_1,y)|^2dxdy\\ &= -c\int_{t_1}^{t_2}\Big(\int_{R_1\setminus(\Gamma_{ct}\cup B_\rho(ct,0))}\partial_{xx}\check\psi(x-ct,y)\partial_x\check\psi(x-ct,y)dxdy\Big)dt + (t_2-t_1)C_1^\lambda\,.\end{aligned}$$

Since $|\partial_x\check\psi|^2$ is integrable we can pass to the limit as $\rho\to 0+$ and conclude the proof. □

Lemma 5.5. *Let $-1 < t < 1$ and $0 < \rho < 1 - c|t|$. Then*

$$\begin{aligned}&\partial_t\Big(\frac{1}{2}\int_{R_1\setminus(\Gamma_{ct}\cup B_\rho(ct,0))}|\nabla\check\psi(x-ct,y)|^2dxdy\Big)\\ &= -c\int_{R_1\setminus(\Gamma_{ct}\cup B_\rho(ct,0))}\nabla\check\psi(x-ct,y)\nabla\partial_x\check\psi(x-ct,y)dxdy + c^2\,C_1^\lambda\,,\end{aligned} \tag{5.13}$$

where C_1^λ is defined in (5.8).

Proof. Let R_1^{ct} be defined as in the proof of Lemma 5.3. By a change of variables we get

$$\int_{R_1\setminus(\Gamma_{ct}\cup B_\rho(ct,0))}|\nabla\check\psi(x-ct,y)|^2dxdy = \int_{R_1^{ct}\setminus(\Gamma_0\cup B_\rho(0,0))}|\nabla\check\psi(x,y)|^2dxdy\,,$$

which gives

$$\partial_t\Big(\frac{1}{2}\int_{R_1\setminus(\Gamma_{ct}\cup B_\rho(ct,0))}|\nabla\check\psi(x-ct,y)|^2dxdy\Big) = \frac{c}{2}\Big(\int_{-1}^{1}|\nabla\check\psi(-1-ct,y)|^2dy - \int_{-1}^{1}|\nabla\check\psi(1-ct,y)|^2dy\Big).$$

On the other hand, by the same change of variables

$$-c\int_{R_1\setminus(\Gamma_{ct}\cup B_\rho(ct,0))}\nabla\check\psi(x-ct,y)\nabla\partial_x\check\psi(x-ct,y)dxdy = -c\int_{R_1^{ct}\setminus(\Gamma_0\cup B_\rho(0,0))}\nabla\check\psi(x,y)\nabla\partial_x\check\psi(x,y)dxdy\,.$$

Integrating by parts we obtain that the right-hand side is equal to

$$\begin{aligned}&\frac{c}{2}\Big(\int_{-1}^{1}|\nabla\check\psi(-1-ct,y)|^2dy - \int_{-1}^{1}|\nabla\check\psi(1-ct,y)|^2dy\Big)\\ &\quad+\frac{c}{2}\rho\int_{-\pi}^{\pi}|\nabla\check\psi(\rho\cos\theta,\rho\sin\theta)|^2\cos\theta d\theta\,.\end{aligned}$$

Since $\nabla\check\psi$ is positively homogeneous of degree $-\frac{1}{2}$ we obtain that

$$\tfrac{1}{2}\rho|\nabla\check\psi(\rho\cos\theta,\rho\sin\theta)|^2\cos\theta = \tfrac{1}{2}|\nabla\check\psi(\cos\theta,\sin\theta)|^2\cos\theta\,,$$

hence (5.13) holds with $c^2 C_1^\lambda$ replaced by

$$\hat{C}_1^\lambda = -\frac{c}{2}\int_{-\pi}^{\pi} |\nabla\check{\psi}(\cos\theta,\sin\theta)|^2 \cos\theta d\theta\,. \tag{5.14}$$

To conclude the proof it is thus enough to show that

$$\hat{C}_1^\lambda = c^2 C_1^\lambda\,. \tag{5.15}$$

From the proof of Lemma 5.3 we deduce that

$$\hat{C}_1^\lambda = C_1^\lambda - \frac{c}{2}\int_{-\pi}^{\pi} |\partial_y\check{\psi}(\cos\theta,\sin\theta)|^2 \cos\theta d\theta\,. \tag{5.16}$$

Using the notation introduced in that proof, from the definitions of ψ and $\check{\psi}$ we obtain that $\partial_y\check{\psi}(\cos\theta,\sin\theta) = \frac{1}{2}\rho_\lambda^{-1/2}\cos\frac{\theta_\lambda}{2}$ hence

$$\hat{C}_1^\lambda = C_1^\lambda - \frac{c}{16}\int_{-\pi}^{\pi} \frac{1}{\rho_\lambda}(1+\cos\theta_\lambda)\cos\theta d\theta = C_1^\lambda - \frac{c}{16}\int_{-\pi}^{\pi}\frac{\cos\theta_\lambda\cos\theta}{\rho_\lambda}d\theta\,, \tag{5.17}$$

where in the last equality we used the symmetry properties of $\rho_\lambda(\theta)$ and $\cos\theta$. By (5.11) we obtain that

$$\hat{C}_1^\lambda = (1-\lambda^2)C_1^\lambda = c^2 C_1^\lambda\,.$$

This gives (5.15), thus concluding the proof of the lemma. □

Corollary 5.6. *For every $-1 < t_1 < t_2 < 1$ we have*

$$\begin{aligned}&-c\lim_{\rho\to 0+}\int_{t_1}^{t_2}\Big(\int_{R_1\setminus(\Gamma_{ct}\cup B_\rho(ct,0))}\nabla\check{\psi}(x-ct,y)\nabla\partial_x\check{\psi}(x-ct,y)dxdy\Big)dt\\ &= \frac{1}{2}\int_{R_1\setminus\Gamma_{ct_2}}|\nabla\check{\psi}(x-ct_2,y)|^2dxdy - \frac{1}{2}\int_{R_1\setminus\Gamma_{ct_1}}|\nabla\check{\psi}(x-ct_1,y)|^2dxdy - (t_2-t_1)c^2C_1^\lambda\,.\end{aligned}$$

Proof. Integrating (5.13) between t_1 and t_2 we obtain

$$\begin{aligned}&\frac{1}{2}\int_{R_1\setminus(\Gamma_{ct_2}\cup B_\rho(ct_2,0))}|\nabla\check{\psi}(x-ct_2,y)|^2dxdy - \frac{1}{2}\int_{R_1\setminus(\Gamma_{ct_1}\cup B_\rho(ct_1,0))}|\nabla\check{\psi}(x-ct_1,y)|^2dxdy\\ &= -c\int_{t_1}^{t_2}\Big(\int_{R_1\setminus(\Gamma_{ct}\cup B_\rho(ct,0))}\nabla\check{\psi}(x-ct,y)\nabla\partial_x\check{\psi}(x-ct,y)dxdy\Big)dt + (t_2-t_1)c^2C_1^\lambda\,.\end{aligned}$$

Since $|\nabla\check{\psi}|^2$ is integrable we can pass to the limit as $\rho\to 0+$ and conclude the proof. □

The following remark will be used in some steps of the proof of the energy-dissipation balance.

Remark 5.7. Let I be a bounded open interval in $\mathbb{R}$, let Ω be a bounded open subset of $\mathbb{R}^n$, let $f\in W^{1,1}(I\times\Omega)$, and let $F\colon I\to\mathbb{R}$ be defined by

$$F(t) = \int_\Omega f(t,\xi)d\xi\,.$$

Then $F\in W^{1,1}(I)$ and

$$\dot{F}(t) = \int_\Omega \partial_t f(t,\xi)d\xi$$

for a.e. $t\in I$. The standard proof can be obtained by using the definition of derivatives in the sense of distributions and the Fubini Theorem.

6. Energy-dissipation balance

In this section we prove the following theorem, which is the main result of the paper.

Theorem 6.1. *There exists $0 < \alpha_1 < 1$ with the following property: if $0 < \alpha \leq \alpha_1$ and $c_0 \leq c \leq c_1$, then there exist $u_0 \in L^\infty((-\infty,-1);H^1(R_1 \setminus \Gamma_{-c}))$ and $w \in \mathcal{V}$, satisfying the regularity assumptions (2.22) and (2.23), such that the unique weak solution u of (2.7)-(2.9) with Dirichlet boundary condition $u = w$ on ∂R_1 (according to Definition 2.2) satisfies the energy-dissipation balance in the sense of Definition 2.5.*

Proof. We fix $4/3 < r_0 < 2$ and define $\alpha_1 := (1-c_1^2)/M_{r_0}$, where $M_{r_0} > 1$ is the constant provided by Theorem 4.3. Assume that

$$0 < \alpha \leq \alpha_1 \quad \text{and} \quad c_0 \leq c \leq c_1 \,. \tag{6.1}$$

As in Corollary 4.10, let $\check\psi(x,y) := \psi(\frac{x}{\lambda},y)$ and $\check\psi_0(x,y) := \psi_0(\frac{x}{\lambda},y)$, where ψ and ψ_0 are defined in (4.4) and (4.11), while $\lambda = \sqrt{1-c^2}$. We choose

$$a := \frac{2}{\sqrt{\pi}}\frac{\sqrt{\beta}}{\kappa_0}\,, \tag{6.2}$$

where β is the fracture toughness and $\kappa_0 > 0$ is the constant provided by Theorem 4.3. Recalling (3.2), we observe that (4.80) and (6.1) imply that

$$\check\ell := \hat\ell_1\lambda = \ell_1\frac{\lambda}{\lambda_1} = \ell_1\frac{\sqrt{1-c^2}}{\sqrt{1-c_1^2}} \geq \ell_1 = 1 + c_1 \geq 1 + c = \ell\,. \tag{6.3}$$

Let $v \in H^1(S_{\check\ell} \setminus \Gamma_0)$ be the unique solution of problem (3.7)-(3.8) in $R_{\check\ell} \setminus \Gamma_0$ (according to Definition 3.2) with boundary condition $v = a\check\psi$ on $\partial R_{\check\ell} \setminus \Gamma_0$ and satisfying $v = a\check\psi_0$ in $S_{\check\ell} \setminus R_{\check\ell}$. By Remark 4.11 we have that $v \in W^{2,p}(R_{\check\ell} \setminus \Gamma_0)$ for every $1 < p < 4/3$. By Corollary 4.10 we have

$$v = \kappa\check\psi + v^{reg} \quad \text{a.e. in } R_{\check\ell} \setminus \Gamma_0\,, \tag{6.4}$$

with

$$\kappa = a\kappa_0 = \frac{2}{\sqrt{\pi}}\sqrt{\beta} \quad \text{and} \quad v^{reg} \in W^{2,r_0}(R_{\check\ell} \setminus \Gamma_0)\,, \tag{6.5}$$

where in the second equality we used (6.2). Observing that $x - ct > -(1+c) = -\ell > -\check\ell$ for $-1 < x < 1$ and $t \leq 1$, we can consider the function u defined by

$$u(t,x,y) = v(x-ct,y) \quad \text{for } t \leq 1 \text{ and } (x,y) \in R_1 \setminus \Gamma_{ct}\,. \tag{6.6}$$

By (6.4) u can be written as

$$u(t,x,y) = \kappa\check\psi(x-ct,y) + v^{reg}(x-ct,y) \quad \text{for } t \in (-1,1) \text{ and } (x,y) \in R_1 \setminus \Gamma_{ct}\,. \tag{6.7}$$

We set

$$u_0(t) := u(t) \quad \text{for } t \leq -1\,.$$

Then $u_0 \in L^\infty((-\infty,-1);H^1(R_1 \setminus \Gamma_{-c}))$. By Lemma 3.3 u is a weak solution of (2.7)-(2.9) according to Definition 2.1. From the regularity of v and from (6.6) it follows that for every $1 < p < 4/3$ we have

$$u(t) \in W^{2,p}(R_1 \setminus \Gamma_{ct}) \quad \text{for every } t \in (-1,1)\,, \tag{6.8}$$

$$\sup_{t\in(-1,1)} \|u(t)\|_{W^{2,p}(R_1\setminus\Gamma_{ct}))} < +\infty. \tag{6.9}$$

Using the Sobolev Embedding Theorem we deduce that for every $1 < s < 4$

$$\nabla u(t) \in L^s(R_1 \setminus \Gamma_{ct};\mathbb{R}^2), \tag{6.10}$$

$$\sup_{t\in(-1,1)} \|\nabla u(t)\|_{L^s(R_1\setminus\Gamma_{ct})} < +\infty. \tag{6.11}$$

Since $\dot{u}(t,x,y) = -c\partial_x v(x-ct,y)$ and $\ddot{u}(t,x,y) = c^2\partial_{xx}v(x-ct,y)$, it follows also that for every $1<p<4/3$ we have

$$\dot{u}(t) \in W^{1,p}(R_1 \setminus \Gamma_{ct}) \quad \text{and} \quad \ddot{u}(t) \in L^p(R_1 \setminus \Gamma_{ct}), \tag{6.12}$$

$$\|\dot{u}(t)\|_{W^{1,p}(R_1\setminus\Gamma_{ct})} + \|\ddot{u}(t)\|_{L^p(R_1\setminus\Gamma_{ct})} \le C, \tag{6.13}$$

for a.e. $t \in (-1,1)$, where $C = c\|\partial_x v\|_{W^{1,p}(R_\ell\setminus\Gamma_0)} + c^2\|\partial_{xx}v\|_{L^p(R_\ell\setminus\Gamma_0)}$.

Therefore, for a.e. $t\in(-1,1)$ the duality product $\langle \ddot{u}(t),\varphi\rangle_t$ considered in [7, Definition 2.15] reduces to

$$\langle \ddot{u}(t),\varphi\rangle_t = \int_{R_1\setminus\Gamma_{ct}} \ddot{u}(t)\varphi\, dxdy$$

for every $\varphi \in H^1(R_1\setminus\Gamma_{ct})$ with $\varphi = 0$ on ∂R_1. Moreover, by (6.13) we have $\ddot{u} \in L^\infty((-1,1);L^p(R_1))$ for every $1<p<4/3$ and by the definition of the weak derivative we have

$$-\int_{-1}^{1}\Big(\int_{R_1\setminus\Gamma_{ct}} \dot{u}(t)\dot{\varphi}(t)dxdy\Big)dt = \int_{-1}^{1}\Big(\int_{R_1\setminus\Gamma_{ct}} \ddot{u}(t)\varphi(t)dxdy\Big)dt$$

for every $\varphi \in \mathcal{V}_0$.

Substituting in the proof of [7, Theorem 2.17] the term $(f(t),\varphi(t))$ by $(F_u(t),\nabla\varphi(t))$, from (2.13) we deduce that for a.e. $t\in(-1,1)$ we have

$$\int_{R_1\setminus\Gamma_{ct}} \ddot{u}(t)\varphi dxdy + \int_{R_1\setminus\Gamma_{ct}} \nabla u(t)\nabla\varphi dxdy = \int_{R_1\setminus\Gamma_{ct}} F_u(t)\nabla\varphi dxdy \tag{6.14}$$

for every $\varphi\in H^1(R_1\setminus\Gamma_{ct})$ with $\varphi=0$ on ∂R_1.

Recalling (2.11), (6.6), and the equality $v=\check{\psi}_0$ in $S_{\check{\ell}}\setminus R_{\check{\ell}}$, for every $t\in(-1,1)$ and a.e. $(x,y)\in R_1$ by the change of variables $\tau := t-s$ we obtain

$$F_u(t,x,y) = \alpha\int_0^{+\infty} e^{-\tau}\nabla v(x-ct+c\tau,y)d\tau$$

$$= \alpha\int_0^{\frac{\check{\ell}-x}{c}+t} e^{-\tau}\nabla v(x-ct+c\tau,y)d\tau + \alpha\int_{\frac{\check{\ell}-x}{c}+t}^{+\infty} e^{-\tau}\nabla\check{\psi}_0(x-ct+c\tau,y)d\tau \tag{6.15}$$

$$= \kappa F^{sing}(t,x,y) + F_u^{reg}(t,x,y),$$

where

$$F^{sing}(t,x,y) := \alpha\int_0^{\frac{\check{\ell}-x}{c}+t} e^{-\tau}\nabla\check{\psi}(x-ct+c\tau,y)d\tau, \tag{6.16}$$

$$F_u^{reg}(t,x,y) := \alpha\int_0^{\frac{\check{\ell}-x}{c}+t} e^{-\tau}\nabla v^{reg}(x-ct+c\tau,y)d\tau \ + \alpha\int_{\frac{\check{\ell}-x}{c}+t}^{+\infty} e^{-\tau}\nabla\check{\psi}_0(x-ct+c\tau,y)d\tau. \tag{6.17}$$

We note that there exists $C_\lambda > 0$ such that

$$|\nabla\check{\psi}(x-ct+c\tau,y)| \le \frac{C_\lambda}{|x-ct+c\tau|^{1/2}+|y|^{1/2}}.$$

Using polar coordinates around $(ct,0)$ we write $x-ct=\rho\cos\theta$ and $y=\rho\sin\theta$. Therefore, from (6.16) and the last inequality we obtain that

$$|F^{sing}(t,x,y)| \le \alpha C_\lambda \int_0^{\frac{\check{\ell}-x}{c}+t} \frac{d\tau}{|\rho\cos\theta+c\tau|^{1/2}+|\rho\sin\theta|^{1/2}}, \tag{6.18}$$

and by the change of variables $\sigma = c\tau/\rho$ we get

$$|F^{sing}(t,x,y)| \le \rho^{1/2}\frac{\alpha C_\lambda}{c}\int_0^{\frac{\check{\ell}-(x-ct)}{\rho}} \frac{d\sigma}{|\cos\theta+\sigma|^{1/2}+|\sin\theta|^{1/2}}. \tag{6.19}$$

We have

$$\int_0^2 \frac{d\sigma}{|\cos\theta+\sigma|^{1/2}+|\sin\theta|^{1/2}} \le \int_{\cos\theta}^{\cos\theta+2} \frac{d\xi}{|\xi|^{1/2}} \le 2(3^{1/2}+1)\,.$$

Moreover, if $\frac{\check{\ell}-(x-ct)}{\rho} > 2$ we have also

$$\int_2^{\frac{\check{\ell}-(x-ct)}{\rho}} \frac{d\sigma}{|\cos\theta+\sigma|^{1/2}+|\sin\theta|^{1/2}} \le \int_2^{\frac{\check{\ell}-(x-ct)}{\rho}} \frac{d\sigma}{(\sigma-1)^{1/2}} \le 2\frac{(\check{\ell}-(x-ct))^{1/2}}{\rho^{1/2}}.$$

From these inequalities we obtain that

$$F^{sing}(t) \in L^\infty(R_1;\mathbb{R}^2) \quad \text{for every } t\in(-1,1)\,, \tag{6.20}$$

$$\sup_{t\in(-1,1)} \|F^{sing}(t)\|_{L^\infty(R_1)} < +\infty\,. \tag{6.21}$$

By (4.5), (6.16), and Lemma 4.5 with $a = \check{\ell}+ct > 1$, $\delta = c$, applied to $z(x,y) = \partial_x\check{\psi}(x-ct,y)$ and $z(x,y) = \partial_y\check{\psi}(x-ct,y)$, we obtain that for every $1<p<4/3$

$$F^{sing}(t) \in W^{1,p}(R_1\setminus\Gamma_{ct};\mathbb{R}^2) \quad \text{for every } t\in(-1,1)\,, \tag{6.22}$$

$$\sup_{t\in(-1,1)} \|F^{sing}(t)\|_{W^{1,p}(R_1\setminus\Gamma_{ct})} < +\infty\,. \tag{6.23}$$

We claim that

$$F_u^{reg}(t) \in W^{1,r_0}(R_1\setminus\Gamma_{ct};\mathbb{R}^2) \quad \text{for every } t\in(-1,1)\,, \tag{6.24}$$

$$\sup_{t\in(-1,1)} \|F_u^{reg}(t)\|_{W^{1,r_0}(R_1\setminus\Gamma_{ct})} < +\infty\,. \tag{6.25}$$

Since $\nabla v^{reg} \in W^{1,r_0}(R_{\check{\ell}}\setminus\Gamma_0;\mathbb{R}^2)$ by (6.5), by Lemma 4.5 with $p=r_0$, $a=\check{\ell}+ct>1$, and $\delta=c$, applied to $z(x,y)=\partial_x v^{reg}(x-ct,y)$ and $z(x,y)=\partial_y v^{reg}(x-ct,y)$, the function

$$(x,y)\mapsto \int_0^{\frac{\check{\ell}-x}{c}+t} e^{-\tau}\nabla v^{reg}(x-ct+c\tau,y)d\tau$$

belongs to $W^{1,r_0}(R_{\check{\ell}}\setminus\Gamma_{ct};\mathbb{R}^2)$ and its norm is estimated by the norm of ∇v^{reg}. Since the last term in (6.17) is a smooth function we deduce that (6.24) and (6.25) hold.

From (6.15) and from the estimates on $F^{sing}(t)$ and $F_u^{reg}(t)$ we obtain that

$$F_u(t) \in W^{1,p}(R_1\setminus\Gamma_{ct};\mathbb{R}^2) \tag{6.26}$$

for every $1<p<4/3$. Using the Sobolev Embedding Theorem for $F_u^{reg}(t)$, we deduce from the L^∞-estimate (6.21) for $F^{sing}(t)$ and from (6.24) and (6.25) that

$$F_u(t) \in L^{r_0^*}(R_1\setminus\Gamma_{ct};\mathbb{R}^2) \quad \text{for every } t\in(-1,1), \tag{6.27}$$

$$\sup_{t\in(-1,1)} \|F_u(t)\|_{L^{r_0^*}(R_1\setminus\Gamma_{ct})} < +\infty, \tag{6.28}$$

where

$$1/r_0^* = 1/r_0 - 1/2. \tag{6.29}$$

Using the L^∞-estimate of $F^{sing}(t)$ and the trace estimate for $F_u^{reg}(t)$ (see [13, Theorem 18.24]), we deduce that the traces $F_u(t)^+$ and $F_u(t)^-$ of $F_u(t)$ on Γ_{ct} from above and from below satisfy

$$F_u(t)^\pm \in L^{r_0/(2-r_0)}(\Gamma_{ct}\cap R_1;\mathbb{R}^2) \quad \text{for every } t\in(-1,1), \tag{6.30}$$

$$\sup_{t\in(-1,1)} \|F_u(t)^\pm\|_{L^{r_0/(2-r_0)}(\Gamma_{ct}\cap R_1)} < +\infty. \tag{6.31}$$

Since for every $1 < p < 4/3$ we have $\ddot{u}(t) \in L^p(R_1 \setminus \Gamma_{ct})$ by (6.12), while $\nabla u(t)$ and $F_u(t)$ belong to $W^{1,p}(R_1 \setminus \Gamma_{ct}; \mathbb{R}^2)$ by (6.8) and (6.26), equation (6.14) is equivalent to

$$\begin{cases} \ddot{u}(t) - \Delta u(t) = -\mathrm{div} F_u(t) & \text{in } R_1 \setminus \Gamma_{ct}\,, \\ \partial_y^{\pm} u(t) = F_u(t)^{\pm}_y & \text{on } \Gamma_{ct} \cap R_1\,, \end{cases} \tag{6.32}$$

for a.e. $t \in (-1,1)$, where the first line has to be considered as an equality between functions in $L^p(R_1 \setminus \Gamma_{ct})$, while the second one is to be intended in the sense of traces. Here and henceforth $\partial_y^{\pm} u(t)$ denote the traces of $\partial_y u(t)$ on $\Gamma_{ct} \cap R_1$ from above and from below and $F_u(t)^{\pm}_y$ denote the y-component of the vectors $F_u(t)^{\pm}$.

Let us fix $t \in (-1,1)$ such that (6.32) holds. Multiplying the first line in (6.32) by $\dot{u}(t)$ we obtain

$$\ddot{u}(t)\dot{u}(t) - \Delta u(t)\dot{u}(t) = -\mathrm{div} F_u(t)\dot{u}(t) \quad \text{a.e. in } R_1 \setminus \Gamma_{ct}\,. \tag{6.33}$$

Let us fix $0 < \rho < 1 - c|t|$. Using the regularity of $\check{\psi}$ far from $(0,0)$ and recalling that $\dot{u}(t,x,y) = -c\partial_x v(x - ct, y)$, from (6.5) and (6.7) we obtain

$$\dot{u}(t) \in W^{1,r_0}(R_1 \setminus (\Gamma_{ct} \cup \overline{B}_\rho(ct,0)))\,, \tag{6.34}$$

$$\nabla u(t) \in W^{1,r_0}(R_1 \setminus (\Gamma_{ct} \cup \overline{B}_\rho(ct,0)); \mathbb{R}^2)\,. \tag{6.35}$$

By (6.34) and by the Sobolev Embedding Theorem we have

$$\dot{u}(t) \in L^{r_0^*}(R_1 \setminus (\Gamma_{ct} \cup \overline{B}_\rho(ct,0))) \quad \text{with } 1/r_0^* = 1/r_0 - 1/2. \tag{6.36}$$

Let p_0 be the exponent conjugate to r_0^*, characterised by the equality $1/p_0 + 1/r_0^* = 1$. Since $r_0 > 4/3$ we have $1 < p_0 < 4/3$. By (6.8), (6.12), and (6.26) this implies that the functions $\ddot{u}(t)$, $\Delta u(t)$, and $\mathrm{div} F_u(t)$ belong to $L^{p_0}(R_1 \setminus \Gamma_{ct})$, hence all products in (6.33) are integrable on $R_1 \setminus (\Gamma_{ct} \cup \overline{B}_\rho(ct,0))$. Therefore we obtain

$$\int_{R_1 \setminus (\Gamma_{ct} \cup \overline{B}_\rho(ct,0))} \ddot{u}(t)\dot{u}(t)dxdy - \int_{R_1 \setminus (\Gamma_{ct} \cup \overline{B}_\rho(ct,0))} \Delta u(t)\dot{u}(t)dxdy = -\int_{R_1 \setminus (\Gamma_{ct} \cup \overline{B}_\rho(ct,0))} \mathrm{div} F_u(t)\dot{u}(t)dxdy\,. \tag{6.37}$$

Let $\hat{r} > 0$ be defined by $1/\hat{r} = 2/r_0 - 1/2$. Since $r_0 > 4/3$ we have $\hat{r} > 1$. From (6.34) and (6.35), using the Sobolev Embedding Theorem we obtain that $\dot{u}(t)\nabla u(t)$ belongs to $W^{1,\hat{r}}(R_1 \setminus (\Gamma_{ct} \cup \overline{B}_\rho(ct,0)); \mathbb{R}^2)$.

Since $4/3 < r_0 < 2$, there exists $1 < p_1 < 4/3$ such that

$$3/4 < 1/p_1 < 3/2 - 1/r_0\,. \tag{6.38}$$

Using again the Sobolev Embedding Theorem, from (6.26) (with $p = p_1$) and (6.34) we obtain that $\dot{u}(t)F_u(t) \in W^{1,\hat{p}}(R_1 \setminus (\Gamma_{ct} \cup \overline{B}_\rho(ct,0)); \mathbb{R}^2)$, with $\hat{p} > 1$ characterised by the equality $1/\hat{p} = 1/p_1 + 1/r_0 - 1/2$. Therefore we can integrate by parts the last two terms of (6.37) and we obtain

$$\begin{aligned} -\int_{R_1 \setminus (\Gamma_{ct} \cup \overline{B}_\rho(ct,0))} \Delta u(t)\dot{u}(t)dxdy &= \int_{R_1 \setminus (\Gamma_{ct} \cup \overline{B}_\rho(ct,0))} \nabla u(t)\nabla\dot{u}(t)dxdy + \int_{\partial B_\rho(ct,0)} \partial_\nu u(t)\dot{u}(t)d\mathcal{H}^1 - \int_{\partial R_1} \partial_\nu u(t)\dot{u}(t)d\mathcal{H}^1 \\ &\quad + \int_{\Gamma_{ct} \cap (R_1 \setminus \overline{B}_\rho(ct,0))} \partial_y^+ u(t)\dot{u}(t)^+ d\mathcal{H}^1 - \int_{\Gamma_{ct} \cap (R_1 \setminus \overline{B}_\rho(ct,0))} \partial_y^- u(t)\dot{u}(t)^- d\mathcal{H}^1\,, \\ -\int_{R_1 \setminus (\Gamma_{ct} \cup \overline{B}_\rho(ct,0))} \mathrm{div} F_u(t)\dot{u}(t)dxdy &= \int_{R_1 \setminus (\Gamma_{ct} \cup \overline{B}_\rho(ct,0))} F_u(t)\nabla\dot{u}(t)dxdy + \int_{\partial B_\rho(ct,0)} F_u(t)\nu\dot{u}(t)d\mathcal{H}^1 - \int_{\partial R_1} F_u(t)\nu\dot{u}(t)d\mathcal{H}^1 \\ &\quad + \int_{\Gamma_{ct} \cap (R_1 \setminus \overline{B}_\rho(ct,0))} F_u(t)^+_y \dot{u}(t)^+ d\mathcal{H}^1 - \int_{\Gamma_{ct} \cap (R_1 \setminus \overline{B}_\rho(ct,0))} F_u(t)^-_y \dot{u}(t)^- d\mathcal{H}^1\,. \end{aligned}$$

Here and henceforth $\dot u(t)^\pm$ denote the traces of $\dot u(t)$ on $\Gamma_{ct}\cap R_1$ from above and from below. From (6.37) and the previous equalities we obtain

$$\begin{aligned}&\int_{R_1\setminus(\Gamma_{ct}\cup\overline{B}_\rho(ct,0))}\ddot u(t)\dot u(t)dxdy+\int_{R_1\setminus(\Gamma_{ct}\cup\overline{B}_\rho(ct,0))}\nabla u(t)\nabla\dot u(t)dxdy+\int_{\partial B_\rho(ct,0)}\partial_\nu u(t)\dot u(t)d\mathcal{H}^1-\int_{\partial R_1}\partial_\nu u(t)\dot u(t)d\mathcal{H}^1\\&+\int_{\Gamma_{ct}\cap(R_1\setminus\overline{B}_\rho(ct,0))}\partial_y^+u(t)\dot u(t)^+d\mathcal{H}^1-\int_{\Gamma_{ct}\cap(R_1\setminus\overline{B}_\rho(ct,0))}\partial_y^-u(t)\dot u(t)^-d\mathcal{H}^1=\int_{R_1\setminus(\Gamma_{ct}\cup\overline{B}_\rho(ct,0))}F_u(t)\nabla\dot u(t)dxdy+\int_{\partial B_\rho(ct,0)}F_u(t)\nu\dot u(t)d\mathcal{H}^1\\&-\int_{\partial R_1}F_u(t)\nu\dot u(t)d\mathcal{H}^1+\int_{\Gamma_{ct}\cap(R_1\setminus\overline{B}_\rho(ct,0))}F_u(t)^+_y\dot u(t)^+d\mathcal{H}^1-\int_{\Gamma_{ct}\cap(R_1\setminus\overline{B}_\rho(ct,0))}F_u(t)^-_y\dot u(t)^-d\mathcal{H}^1\,.\end{aligned}\tag{6.39}$$

Let $\varphi\in C^\infty(\overline{R}_1)$ with $\varphi=1$ in a neighbourhood of ∂R_1 and $\varphi=0$ in a neighbourhood of $[-c,c]\times\{0\}$. For every $t\in(-1,1)$ and a.e. $(x,y)\in R_1\setminus\Gamma_{-c}$ we set

$$w(t,x,y):=u(t,x,y)\varphi(x,y)=v(x-ct,y)\varphi(x,y)\tag{6.40}$$

and we observe that $w\in\mathcal{V}$ and $u(t)=w(t)$ on ∂R_1 in the sense of traces for every $t\in(-1,1)$.

We study now the energy-dissipation balance for u. Note that by (6.8), (6.9), (6.12), and (6.13) the function u satisfies (2.15) and (2.17) for every $1<p<4/3$. Recalling (6.29) and (6.38), we see that the exponent q_1 conjugate to p_1 satisfies $q_1<r_0^*$. By (6.26), (6.27), and (6.28) this implies that F_u satisfies (2.16), (2.18), (2.20), and (2.21) with $p=p_1$ and $q=q_1$, while from the equality $\dot u(t,x,y)=-c\partial_x v(x-ct,y)$ and from (6.4) and (6.5) we obtain that w satisfies (2.22) and (2.23) for $p=p_1$ and $r=r_0$. Therefore the viscous dissipation is given by (2.19) and the work of the external forces acting on ∂R_1 is given by (2.28).

From (2.12), (2.28), and (6.39), for every $-1<t_1<t_2<1$ we obtain

$$\begin{aligned}&\int_{t_1}^{t_2}\Big(\int_{R_1\setminus(\Gamma_{ct}\cup\overline{B}_\rho(ct,0))}\ddot u(t)\dot u(t)dxdy\Big)dt+\int_{t_1}^{t_2}\Big(\int_{R_1\setminus(\Gamma_{ct}\cup\overline{B}_\rho(ct,0))}\nabla u(t)\nabla\dot u(t)dxdy\Big)dt\\&+\int_{t_1}^{t_2}\Big(\int_{\partial B_\rho(ct,0)}\partial_\nu u(t)\dot u(t)d\mathcal{H}^1\Big)dt+\int_{t_1}^{t_2}\Big(\int_{\Gamma_{ct}\cap(R_1\setminus\overline{B}_\rho(ct,0))}\partial_y^+u(t)\dot u(t)^+d\mathcal{H}^1\Big)dt-\int_{t_1}^{t_2}\Big(\int_{\Gamma_{ct}\cap(R_1\setminus\overline{B}_\rho(ct,0))}\partial_y^-u(t)\dot u(t)^-d\mathcal{H}^1\Big)dt\\&=\int_{t_1}^{t_2}\Big(\int_{R_1\setminus(\Gamma_{ct}\cup\overline{B}_\rho(ct,0))}F_u(t)\nabla\dot u(t)dxdy+\int_{\partial B_\rho(ct,0)}F_u(t)\nu\dot u(t)d\mathcal{H}^1\Big)dt+\mathcal{W}(t_1,t_2)\\&+\int_{t_1}^{t_2}\Big(\int_{\Gamma_{ct}\cap(R_1\setminus\overline{B}_\rho(ct,0))}F_u(t)^+_y\dot u(t)^+d\mathcal{H}^1\Big)dt-\int_{t_1}^{t_2}\Big(\int_{\Gamma_{ct}\cap(R_1\setminus\overline{B}_\rho(ct,0))}F_u(t)^-_y\dot u(t)^-d\mathcal{H}^1\Big)dt\,.\end{aligned}\tag{6.41}$$

We now study the limit as $\rho\to0+$ of all terms of (6.41). We begin with the integrals on $R_1\setminus(\Gamma_{ct}\cup\overline{B}_\rho(ct,0))$.

From (6.7) it follows that

$$\dot u(t)=-c\kappa\partial_x\check\psi(x-ct,y)-c\partial_x v^{reg}(x-ct,y)\,,\tag{6.42}$$

$$\ddot u(t)=c^2\kappa\partial_{xx}\check\psi(x-ct,y)+c^2\partial_{xx}v^{reg}(x-ct,y)\,,\tag{6.43}$$

hence

$$\begin{aligned}\ddot u(t)\dot u(t)=&-c^3\kappa^2\partial_x\check\psi(x-ct,y)\partial_{xx}\check\psi(x-ct,y)-c^3\kappa\partial_x\check\psi(x-ct,y)\partial_{xx}v^{reg}(x-ct,y)\\&-c^3\kappa\partial_x v^{reg}(x-ct,y)\partial_{xx}\check\psi(x-ct,y)-c^3\partial_x v^{reg}(x-ct,y)\partial_{xx}v^{reg}(x-ct,y)\,.\end{aligned}\tag{6.44}$$

Since $r_0>4/3$, using the Sobolev Embedding Theorem we deduce from (4.5) (with $p=p_1$), (6.5), and (6.38) that all terms but the first one are integrable on $R_1\setminus\Gamma_{ct}$ and that their integrals are bounded uniformly with respect to t. Therefore

$$\lim_{\rho\to 0+}\int_{R_1\setminus(\Gamma_{ct}\cup\overline{B}_\rho(ct,0))}\Big(-c^3\kappa\partial_x\check\psi(x-ct,y)\partial_{xx}v^{reg}(x-ct,y)$$
$$-c^3\kappa\partial_x v^{reg}(x-ct,y)\partial_{xx}\check\psi(x-ct,y)-c^3\partial_x v^{reg}(x-ct,y)\partial_{xx}v^{reg}(x-ct,y)\Big)dxdy$$
$$=\int_{R_1\setminus\Gamma_{ct}}\Big(-c^3\kappa\partial_x\check\psi(x-ct,y)\partial_{xx}v^{reg}(x-ct,y)$$
$$-c^3\kappa\partial_x v^{reg}(x-ct,y)\partial_{xx}\check\psi(x-ct,y)-c^3\partial_x v^{reg}(x-ct,y)\partial_{xx}v^{reg}(x-ct,y)\Big)dxdy$$
$$=\partial_t\Big(c^2\kappa\int_{R_1\setminus\Gamma_1}\partial_x\check\psi(x-ct,y)\partial_x v^{reg}(x-ct,y)dxdy\Big)$$
$$+\partial_t\Big(\frac{c^2}{2}\int_{R_1\setminus\Gamma_1}|\partial_x v^{reg}(x-ct,y)|^2dxdy\Big)\,,\tag{6.45}$$

where in the last equality we used Remark 5.7. Let $-1<t_1<t_2<1$. Integrating in time between t_1 and t_2 and using the Dominated Convergence Theorem, from Corollary 5.4 and (6.42) we obtain that

$$\lim_{\rho\to 0+}\int_{t_1}^{t_2}\Big(\int_{R_1\setminus(\Gamma_{ct}\cup\overline{B}_\rho(ct,0))}\ddot u(t)\dot u(t)dxdy\Big)dt$$
$$=\frac12\int_{R_1\setminus\Gamma_{ct_2}}|\dot u(t_2)|^2dxdy-\frac12\int_{R_1\setminus\Gamma_{ct_1}}|\dot u(t_1)|^2dxdy-c^2\kappa^2(t_2-t_1)C_1^\lambda\,,\tag{6.46}$$

where C_1^λ is defined in (5.8).

Similarly, from (6.7) it follows that

$$\nabla u(t)=\kappa\nabla\check\psi(x-ct,y)+\nabla v^{reg}(x-ct,y)\tag{6.47}$$
$$\nabla\dot u(t)=-c\kappa\nabla\partial_x\check\psi(x-ct,y)-c\nabla\partial_x v^{reg}(x-ct,y)\,,\tag{6.48}$$

hence

$$\nabla u(t)\nabla\dot u(t)=-c\kappa^2\nabla\check\psi(x-ct,y)\nabla\partial_x\check\psi(x-ct,y)-c\kappa\nabla\check\psi(x-ct,y)\nabla\partial_x v^{reg}(x-ct,y)$$
$$-c\kappa\nabla v^{reg}(x-ct,y)\nabla\partial_x\check\psi(x-ct,y)-c\nabla v^{reg}(x-ct,y)\nabla\partial_x v^{reg}(x-ct,y)\,.\tag{6.49}$$

Since $r_0>4/3$, using the Sobolev Embedding Theorem we deduce from (4.5) (with $p=p_1$), (6.5), and (6.38) that all terms but the first one are integrable on $R_1\setminus\Gamma_{ct}$ and that their integrals are bounded uniformly with respect to t. Therefore

$$\lim_{\rho\to 0+}\int_{R_1\setminus(\Gamma_{ct}\cup\overline{B}_\rho(ct,0))}\Big(-c\kappa\nabla\check\psi(x-ct,y)\nabla\partial_x v^{reg}(x-ct,y)$$
$$-c\kappa\nabla v^{reg}(x-ct,y)\nabla\partial_x\check\psi(x-ct,y)-c\nabla v^{reg}(x-ct,y)\nabla\partial_x v^{reg}(x-ct,y)\Big)dxdy$$
$$=\int_{R_1\setminus\Gamma_{ct}}\Big(-c\kappa\nabla\check\psi(x-ct,y)\nabla\partial_x v^{reg}(x-ct,y)$$
$$-c\kappa\nabla v^{reg}(x-ct,y)\nabla\partial_x\check\psi(x-ct,y)-c\nabla v^{reg}(x-ct,y)\nabla\partial_x v^{reg}(x-ct,y)\Big)dxdy$$
$$=\partial_t\Big(\kappa\int_{R_1\setminus\Gamma_1}\nabla\check\psi(x-ct,y)\nabla v^{reg}(x-ct,y)dxdy\Big)$$
$$+\partial_t\Big(\frac12\int_{R_1\setminus\Gamma_1}|\nabla v^{reg}(x-ct,y)|^2dxdy\Big)\,,\tag{6.50}$$

where in the last equality we used Remark 5.7. Integrating in time between t_1 and t_2 and using the Dominated Convergence Theorem and Corollary 5.6 we obtain from (6.47) that

$$\lim_{\rho\to 0}\int_{t_1}^{t_2}\Big(\int_{R_1\setminus(\Gamma_{ct}\cup\overline{B}_\rho(ct,0))}\nabla u(t)\nabla\dot u(t)dxdy\Big)dt$$
$$=\frac{1}{2}\int_{R_1\setminus\Gamma_{ct_2}}|\nabla u(t_2)|^2dxdy-\frac{1}{2}\int_{R_1\setminus\Gamma_{ct_1}}|\nabla u(t_1)|^2dxdy-c^2\kappa^2(t_2-t_1)C_1^\lambda\,. \tag{6.51}$$

By (6.48)

$$F_u(t)\nabla\dot u(t)=-c\kappa F_u(t)\nabla\partial_x\check\psi(x-ct,y)-cF_u(t)\nabla\partial_x v^{reg}(x-ct,y)\,.$$

Since $r_0>4/3$, by (4.5) (with $p=p_1$), (6.5), (6.27), and (6.38), the right-hand side in the above equality is integrable on $R_1\setminus\Gamma_{ct}$, hence

$$\lim_{\rho\to 0+}\int_{R_1\setminus(\Gamma_{ct}\cup\overline{B}_\rho(ct,0))}F_u(t)\nabla\dot u(t)dxdy=\int_{R_1\setminus\Gamma_{ct}}F_u(t)\nabla\dot u(t)dxdy\,. \tag{6.52}$$

We now consider the integrals on $\partial B_\rho(ct,0)$ that appear in (6.41). Using again (6.7) we obtain that

$$\begin{aligned}\partial_\nu u(t)\dot u(t)&=(\kappa\partial_\nu\check\psi(x-ct)+\partial_\nu v^{reg}(x-ct,y))(-c\kappa\partial_x\check\psi(x-ct,y)-c\partial_x v^{reg}(x-ct,y))\\&=-c\kappa^2\partial_\nu\check\psi(x-ct)\partial_x\check\psi(x-ct,y)-c\kappa\partial_\nu\check\psi(x-ct)\partial_x v^{reg}(x-ct,y)\\&\quad-c\kappa\partial_\nu v^{reg}(x-ct,y)\partial_x\check\psi(x-ct,y)-c\partial_\nu v^{reg}(x-ct,y)\partial_x v^{reg}(x-ct,y)\,.\end{aligned}$$

By Lemmas 5.1 and 5.2 the integrals on $\partial B_\rho(ct,0)$ of the last three terms tend to 0 as $\rho\to 0$ and are bounded uniformly with respect to t. Hence

$$\lim_{\rho\to 0+}\int_{\partial B_\rho(ct,0)}\partial_\nu u(t)\dot u(t)\,d\mathcal{H}^1=-c\kappa^2\lim_{\rho\to 0+}\int_{\partial B_\rho(0,0)}\partial_\nu\check\psi\partial_x\check\psi d\mathcal{H}^1=-\kappa^2C_2^\lambda\,,$$

where, using the fact that $\nabla\check\psi$ is positively homogeneous of degree $-\frac{1}{2}$, we have that

$$C_2^\lambda:=c\int_{-\pi}^{\pi}\partial_\nu\check\psi(\cos\theta,\sin\theta)\partial_x\check\psi(\cos\theta,\sin\theta)d\theta\,.$$

By the Dominated Convergence Theorem we conclude that

$$\lim_{\rho\to 0+}\int_{t_1}^{t_2}\Big(\int_{\partial B_\rho(ct,0)}\partial_\nu u(t)\dot u(t)\,d\mathcal{H}^1\Big)dt=-\kappa^2(t_2-t_1)C_2^\lambda\,, \tag{6.53}$$

for every $-1<t_1<t_2<1$.

We now compute C_2^λ. First of all, we note that $\partial_\nu\check\psi=\partial_x\check\psi\cos\theta+\partial_y\check\psi\sin\theta$, hence

$$C_2^\lambda=c\int_{-\pi}^{\pi}|\partial_x\check\psi(\cos\theta,\sin\theta)|^2\cos\theta d\theta+c\int_{-\pi}^{\pi}\partial_y\check\psi(\cos\theta,\sin\theta)\partial_x\check\psi(\cos\theta,\sin\theta)\sin\theta d\theta. \tag{6.54}$$

By (5.8) and (5.10) we have that

$$c\int_{-\pi}^{\pi}|\partial_x\check\psi(\cos\theta,\sin\theta)|^2\cos\theta d\theta=-2C_1^\lambda=-\frac{\pi}{4\lambda}\frac{c}{1+\lambda}\,. \tag{6.55}$$

To compute the second integral in (6.54) we introduce the polar coordinates $(\rho_\lambda,\theta_\lambda)=(\rho_\lambda(\theta),\theta_\lambda(\theta))$ of $(\frac{\cos\theta}{\lambda},\sin\theta)$. Using the definitions of ψ and $\check\psi$ we have that

$$\partial_y\check\psi(\cos\theta,\sin\theta)\partial_x\check\psi(\cos\theta,\sin\theta)=-\frac{1}{4\lambda}\frac{1}{\rho_\lambda}\sin\frac{\theta_\lambda}{2}\cos\frac{\theta_\lambda}{2}=-\frac{1}{8\lambda}\frac{1}{\rho_\lambda}\sin\theta_\lambda\,.$$

Since $\rho_\lambda\sin\theta_\lambda=\sin\theta$ and $\rho_\lambda^2=\frac{\cos^2\theta}{\lambda^2}+\sin^2\theta$, we obtain

$$c\int_{-\pi}^{\pi}\partial_y\check\psi(\cos\theta,\sin\theta)\partial_x\check\psi(\cos\theta,\sin\theta)\sin\theta d\theta=-\frac{c\lambda}{8}\int_{-\pi}^{\pi}\frac{\sin^2\theta}{\cos^2\theta+\lambda^2\sin^2\theta}d\theta\,.$$

By direct computation we get

$$\int_{-\pi}^{\pi} \frac{\sin^2\theta}{\cos^2\theta+\lambda^2\sin^2\theta}d\theta = \frac{2\pi}{\lambda}\frac{1}{1+\lambda}\,,$$

hence from (6.54) and (6.55) we obtain

$$C_2^\lambda = -\frac{\pi}{4\lambda}\frac{c}{1+\lambda} - \frac{c\lambda}{8}\frac{2\pi}{\lambda}\frac{1}{1+\lambda} = -\frac{\pi c}{4\lambda}\,. \tag{6.56}$$

By (6.15) and (6.42) we have

$$\begin{aligned} F_u(t)\nu\dot u(t) &= (\kappa F^{sing}(t)\nu + F_u^{reg}(t)\nu)(-c\kappa\partial_x\check\psi(x-ct,y) - c\partial_x v^{reg}(x-ct,y)) \\ &= -c\kappa^2F^{sing}(t)\nu\partial_x\check\psi(x-ct,y) - c\kappa F^{sing}(t)\nu\partial_x v^{reg}(x-ct,y) \\ &\quad -c\kappa F_u^{reg}(t)\nu\partial_x\check\psi(x-ct,y) - cF_u^{reg}(t)\nu\partial_x v^{reg}(x-ct,y)\,. \end{aligned}$$

Using Lemmas 5.1 and 5.2 we deduce from (6.20), (6.21), (6.24), and (6.25) that the integrals on $\partial B_\rho(ct,0)$ of all these terms tend to zero as $\rho\to 0$ and are bounded uniformly with respect to t. By the Dominated Convergence Theorem we conclude that

$$\lim_{\rho\to 0+}\int_{t_1}^{t_2}\Big(\int_{\partial B_\rho(ct,0)} F_u(t)\nu\dot u(t)\,d\mathcal{H}^1\Big)dt = 0\,, \tag{6.57}$$

for every $-1<t_1<t_2<1$.

Finally, we consider all integrals on $\Gamma_{ct}\cap(R_1\setminus\overline{B}_\rho(ct,0))$ that appear in (6.41). By (6.42) and (6.47) it follows that

$$\begin{aligned} \partial_y^+u(t)\dot u(t)^+ &= -c\kappa^2\partial_y^+\check\psi(x-ct,0)\partial_x^+\check\psi(x-ct,0) - c\kappa\partial_y^+\check\psi(x-ct,0)\partial_x^+v^{reg}(x-ct,0) \\ &\quad -c\kappa\partial_y^+v^{reg}(x-ct,0)\partial_x^+\check\psi(x-ct,0) - c\partial_y^+v^{reg}(x-ct,0)\partial_x^+v^{reg}(x-ct,0)\,, \end{aligned}$$

where $\partial_x^\pm$ denote the traces of the partial derivative ∂_x on (part of) Γ_{ct} from above and from below.

By (6.5) and by the trace estimate (see [13, Theorem 18.24]) the functions $x\mapsto\partial_x^+v^{reg}(x-ct,0)$ and $x\mapsto\partial_y^+v^{reg}(x-ct,0)$ belong to $L^{r_0/(2-r_0)}(\Gamma_{ct}\cap R_1)$. By (4.4) we have $x\mapsto\partial_y^+\check\psi(x-ct,0)=0$ on $\Gamma_{ct}\cap R_1$ and $x\mapsto\partial_x^+\check\psi(x-ct,0)\in L^r(\Gamma_{ct}\cap R_1)$ for every $1\le r<2$. Since $r_0>4/3$ we have $r_0/(2-r_0)>2$. Therefore, taking r equal to the conjugate exponent of $r_0/(2-r_0)$, from the expression for $\partial_y^+u(t)\dot u(t)^+$ we deduce that this function belongs to $L^1(\Gamma_0\cap R_1)$, hence

$$\lim_{\rho\to 0+}\int_{\Gamma_{ct}\cap(R_1\setminus\overline{B}_\rho(ct,0))}\partial_y^+u(t)\dot u(t)^+d\mathcal{H}^1 = \int_{\Gamma_{ct}\cap R_1}\partial_y^+u(t)\dot u(t)^+d\mathcal{H}^1. \tag{6.58}$$

In the same way we prove that $\partial_y^-u(t)\dot u(t)^-\in L^1(\Gamma_{ct}\cap R_1)$ and hence

$$\lim_{\rho\to 0+}\int_{\Gamma_{ct}\cap(R_1\setminus\overline{B}_\rho(ct,0))}\partial_y^-u(t)\dot u(t)^-d\mathcal{H}^1 = \int_{\Gamma_{ct}\cap R_1}\partial_y^-u(t)\dot u(t)^-d\mathcal{H}^1. \tag{6.59}$$

We now consider the integral of $F_u(t)_y^+\dot u(t)^+$. We already observed that the function $x\mapsto\partial_x^+v^{reg}(x-ct,0)$ belongs to $L^{r_0/(2-r_0)}(\Gamma_{ct}\cap R_\ell)$ and that $x\mapsto\partial_x^+\check\psi(x-ct,0)$ belongs to $L^r(\Gamma_{ct}\cap R_\ell)$ for every $1\le r<2$. Therefore (6.42) gives $\dot u(t)^+\in L^r(\Gamma_{ct}\cap R_1)$ for every $1\le r<2$. By (6.30) we have $F_u(t)^+\in L^{r_0/(2-r_0)}(\Gamma_{ct}\cap R_1;\mathbb{R}^2)$. Taking r equal to the exponent conjugate to $r_0/(2-r_0)$ we obtain that $F_u(t)_y^+\dot u(t)^+\in L^1(\Gamma_{ct}\cap R_1)$, which implies that

$$\lim_{\rho\to 0+}\int_{\Gamma_{ct}\cap(R_1\setminus\overline{B}_\rho(ct,0))}F_u(t)_y^+\dot u(t)^+d\mathcal{H}^1 = \int_{\Gamma_{ct}\cap R_1}F_u(t)_y^+\dot u(t)^+d\mathcal{H}^1\,. \tag{6.60}$$

In the same way we prove that

$$\lim_{\rho\to 0+}\int_{\Gamma_{ct}\cap(R_1\setminus\overline{B}_\rho(ct,0))}F_u(t)_y^-\dot u(t)^-d\mathcal{H}^1 = \int_{\Gamma_{ct}\cap R_1}F_u(t)_y^-\dot u(t)^-d\mathcal{H}^1. \tag{6.61}$$

By (6.41), (6.46), and (6.51)-(6.61) for a.e. $-1 < t_1 < t_2 < 1$ we obtain that

$$\frac{1}{2}\int_{R_1\setminus\Gamma_{ct_2}}|\dot u(t_2)|^2dxdy - \frac{1}{2}\int_{R_1\setminus\Gamma_{ct_1}}|\dot u(t_1)|^2dxdy + \frac{1}{2}\int_{R_1\setminus\Gamma_{ct_2}}|\nabla u(t_2)|^2dxdy - \frac{1}{2}\int_{R_1\setminus\Gamma_{ct_1}}|\nabla u(t_1)|^2dxdy$$

$$- c^2\kappa^2(t_2-t_1)C_1^\lambda - c^2\kappa^2(t_2-t_1)C_1^\lambda - \kappa^2(t_2-t_1)C_2^\lambda$$

$$= \int_{t_1}^{t_2}\Big(\int_{R_1\setminus\Gamma_{ct}}F_u(t)\nabla\dot u(t)dxdy\Big)dt - \int_{t_1}^{t_2}\Big(\int_{\Gamma_{ct}\cap R_1}\partial_y^+u(t)\dot u(t)^+d\mathcal{H}^1\Big)dt + \int_{t_1}^{t_2}\Big(\int_{\Gamma_{ct}\cap R_1}\partial_y^-u(t)\dot u(t)^-d\mathcal{H}^1\Big)dt$$

$$+\mathcal{W}(t_1,t_2) + \int_{t_1}^{t_2}\Big(\int_{\Gamma_{ct}\cap R_1}F_u(t)_y^+\dot u(t)^+d\mathcal{H}^1\Big)dt - \int_{t_1}^{t_2}\Big(\int_{\Gamma_{ct}\cap R_1}F_u(t)_y^-\dot u(t)^-d\mathcal{H}^1\Big)dt\,.$$

By (6.32) we have $\partial_y^+u(t) = F_u(t)_y^+$ and $\partial_y^-u(t) = F_u(t)_y^-$ $\mathcal{H}^1$-a.e. on $\Gamma_{ct}\cap R_1$, while by (5.8), (6.5), and (6.56) we have $-2c^2\kappa^2C_1^\lambda - \kappa^2C_2^\lambda = \frac{\pi}{4}c\kappa^2 = \beta c$. Therefore the previous equality reduces to

$$\frac{1}{2}\int_{R_1\setminus\Gamma_{ct_2}}|\dot u(t_2)|^2dxdy - \frac{1}{2}\int_{R_1\setminus\Gamma_{ct_1}}|\dot u(t_1)|^2dxdy + \frac{1}{2}\int_{R_1\setminus\Gamma_{ct_2}}|\nabla u(t_2)|^2dxdy - \frac{1}{2}\int_{R_1\setminus\Gamma_{ct_1}}|\nabla u(t_1)|^2dxdy$$

$$- \int_{t_1}^{t_2}\Big(\int_{R_1\setminus\Gamma_{ct}}F_u(t)\nabla\dot u(t)dxdy\Big)dt + \beta c(t_2-t_1) = \mathcal{W}(t_1,t_2)\,.$$

Recalling the equality $c_e + c_v = 1$ and the definitions of the energy $\mathcal{E}(t)$ (see (2.14)), of the viscous dissipation $\mathcal{D}(t_1,t_2)$ (see (2.19)), and of the energy $\mathcal{K}(t_1,t_2)$ dissipated by the crack growth (see (2.29)), the previous equality can be written in the form

$$\mathcal{E}(t_2) - \mathcal{E}(t_1) + \mathcal{D}(t_1,t_2) + \mathcal{K}(t_1,t_2) = \mathcal{W}(t_1,t_2)\,, \tag{6.62}$$

for a.e. $-1 < t_1 < t_2 < 1$, hence u satisfies the energy-dissipation balance according to Definition 2.5. □

Acknowledgements. This paper is based on work supported by the National Research Project (PRIN 2022) "Variational Methods for Stationary and Evolution Problems with Singularities and Interfaces", funded by the Italian Ministry of University, and Research. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] Amendola G., Fabrizio M., Golden J.M.: Thermodynamics of materials with memory. Theory and applications. *Springer, New York,* 2012.

[2] Boltzmann L.E.: Zur Theorie der Elastischen Nachwirkung. *Sitzungsberichte der mathematisch-naturwissenschaftlichen Classe der Kaiserlichen Akademie der Wissenschaften, Wien,* **70** (1874), 275–306.

[3] Cianci F.: Dynamic crack growth in viscoelastic materials with memory. *Milan J. Math.* **91** (2023), 331–351.

[4] Cianci F., Dal Maso G.: Uniqueness and continuous dependence for a viscoelastic problem with memory in domains with time dependent cracks. *Diff. Integr. Eqn.* **34** (2021), 595–620.

[5] Dafermos C.M.: An abstract Volterra equation with applications to linear viscoelasticity. *J. Differential Equations* **7** (1970), 554–569.

[6] Dal Maso G., Larsen C.: Existence for wave equations on domains with arbitrary growing cracks. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **22** (2011), 387–408.

[7] Dal Maso G., Toader R.: On the Cauchy problem for the wave equation on time-dependent domains. *J. Differential Equations* **266** (2019) 3209–3246.

[8] Day W.A.: The thermodynamics of simple materials with fading memory. *Springer Tracts in Natural Philosophy, Vol. 22. Springer-Verlag, New York-Heidelberg,* 1972.

[9] Fabrizio M., Morro A.: Mathematical problems in linear viscoelasticity. *SIAM Studies in Applied Mathematics, 12. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA,* 1992.

[10] Freund, L.B.: Dynamic fracture mechanics. *Cambridge University Press, New York,* 1990.

[11] Griffith A.: The phenomena of rupture and flow in solids. *Philos. Trans. Roy. Soc. London Ser. A* **221** (1920), 163–198.
[12] Grisvard P.: Elliptic problems in nonsmooth domains. *Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston, MA,* 1985.
[13] Leoni G.: A first course in Sobolev spaces. Second edition. *Graduate Studies in Mathematics, 181. American Mathematical Society, Providence, RI,* 2017.
[14] Maxwell J.C.: On the dynamical theory of gases, *Philos. Trans. Roy. Soc. London Ser. A* **157** (1867), 49–88.
[15] Mott, N.F.: Fracture of metals: theoretical considerations, *Engineering* **165** (1948), 16–18.
[16] Sapio F.: A dynamic model for viscoelasticity in domains with time-dependent cracks. *NoDEA Nonlinear Differential Equations Appl.* **28** (2021), Paper No. 67.
[17] Slepyan L.I.: Models and phenomena in fracture mechanics. *Foundations of Engineering Mechanics. Springer-Verlag, Berlin,* 2002.
[18] Truesdell C.: The elements of continuum mechanics. Springer-Verlag New York, Inc., New York, 1966.
[19] Tschoegl N.W.: The phenomenological theory of linear viscoelastic behavior. An introduction. *Springer-Verlag, Berlin,* 1989.
[20] Volterra V.: Sur la théorie mathématique des phénomènes héréditaires, J. Math. Pures Appl. 7 (1928), 249–298.
[21] Volterra V.: Sur les équations intégro-différentielles et leurs applications. *Acta Math.* **35**, (1912) 295–356.
[22] Volterra V.: Leçons sur les fonctions de lignes, *Gauthier-Villars, Paris,* 1913.

(Gianni Dal Maso) SISSA, Via Bonomea 265, 34136 Trieste, Italy
Email address, Gianni Dal Maso: `dalmaso@sissa.it`

(Rodica Toader) Università di Trieste, Trieste, Italy
Email address, Rodica Toader: `rodica.toader@units.it`