Multiplicity of critical points in presence of a linking: application to a superlinear boundary value problem

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Abstract

We consider a general nonlinear elliptic problem of the second order whose associated functional presents two linking structures and we prove the existence of three nontrivial solutions to the problem.

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1 Introduction

In this paper we consider the following problem

$$(P) \qquad \begin{cases} -\Delta u - \lambda u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 3$, $\lambda \in \mathbb{R}$ and $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$. Multiplicity results for solutions of nonlinear boundary value problems have been faced by a large number of authors, in different situations: sublinear, asymptotically linear, superlinear, subcritical, critical, supercritical... We are interested in a *superlinear* and *subcritical* problem (see Section 2). We will make the standard superlinear and subcritical assumptions on g ([4], [14], [22]) and we will show that for some values of λ, problem (P) has at least three nontrivial solutions (the trivial solution being a solution of (P) as well). Such a result seems to be new and, under quite general assumptions, it improves some previous results for analogous problems and it also parallels many results for different equations (see below).

There are many multiplicity results, in particular for the autonomous case, that is $g(x,u) = g(u)$, or when $\lambda = 0$ ([10], [26]). The literature concerning such results is quite extensive (especially considering the peculiar features that g has in different cases) and we just refer to [5], [6], [8], [11], [12], the other papers cited in this introduction and the references quoted therein.

A common way to face this problem is to look for solutions having one sign in their domain ([4]) and then possibly sign-changing solutions ([9]). We will follow a different approach, instead. Namely, if a C^1 functional f defined on a Hilbert space has a linking structure, then it has a nontrivial critical point (see [4]); nevertheless it may happen that ∇f has finer properties, and in this case f has two nontrivial critical points which may be at the same level (see [16]). We will apply such an abstract result to problem (P) and we will get the existence of three nontrivial solutions, the third one being given by an additional linking structure.

Concerning the nonautonomous case, we refer, as inspiring results and papers to which compare our result, to [1], [2], [4], [13], [21], [24] and [25], although the nonlinearity g has, in some cases, a different behaviour.

In particular in [4], in a substantially similar situation, the existence of a positive and a negative solution is proved for any $\lambda < \lambda_1$, while in [24] a larger number of solution is provided, but for a different nonlinearity: in fact, while the equation in problem (P) generalizes the equation $-\Delta u - \lambda u - |u|^{s-2}u = 0$, the equation studied in [24] (as well as in the one in [2] and [7]) generalizes the equation $-\Delta u - \lambda u + |u|^{s-2}u = 0$ and a minimization approach can be used. On the other hand in [25] a problem which is a perturbation of a symmetric one is considered, providing the existence of infinitely many solutions and infinitely many solutions are also provided when some symmetries on the related functional are assumed ([1], [4], [13]). It is also worth noting that our results is related to the one found in [3], although under different assumptions on g.

More precisely, our result is in the spirit of the result of [21], where the author studies the problem

$$\begin{cases} -\Delta u - \lambda u + t((u+\alpha e_1)^-)^p = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $t, \alpha > 0$ and e_1 is the first (positive) eigenfunction of $-\Delta$ on $H_0^1(\Omega)$ and she proves the analogous of Theorem 1. But, of course, the function $-((t+\alpha e_1)^-)^p$ doesn't satisfy (g_4) for every t (see Section 2).

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2 Assumptions, variational setting and main theorem

In this section we precise the hypotheses about g and we prepare the variational setting for the main result (Theorem 1).

First of all we assume the standard assumptions for a superlinear and subcritical nonlinearity, although with some restrictions (see Remark 2):

(g_1) g is a Carathéodory's function;

(g_2) there exits constants $a_1, a_2 > 0$ and $s \in (1, \frac{N+2}{N-2})$ such that $\forall\, t$ in $\mathbb{R}$ and for a.e. x in Ω

$$|g(x,t)| \le a_1 + a_2|t|^s;$$

(g_3) $g(x,t) = o(|t|)$ as $t \to 0$ uniformly in Ω;

(g_4) $\forall\, t \neq 0$ and for a.e. x in Ω

$$0 < \mu G(x,t) \le g(x,t)t,$$

where $\mu = s+1$ and $G(x,t) = \int_0^t g(x,\sigma)\, d\sigma$.

Note that (g_3) implies that $G(x,|t|) = o(|t^2|)$ as $t \to 0$ uniformly in Ω and (g_2) implies that $G(x,t) \le a_1|t| + a_2|t|^{s+1}$ $\forall\, t$ and for a.e. x in Ω.

We remark that such assumptions are quite natural and appear quite often while studying nonlinear subcritical problems (see [22], [14]).

Remark 1. Integrating (g_4) we get that there exists $c_1 > 0$ such that $\forall\, t$ in $\mathbb{R}$ and for a.e. x in Ω, $G(x,t) \ge c_1|t|^\mu$.

Remark 2. **1a)** It is worth noting that in the general case in (g_4) one asks the existence of $\mu > 2$ such that the inequality holds. But by (g_2) and Remark 1, one immediately gets $\mu \le s+1$ (and so $s > 1$). So we require a stronger assumption on μ, which is however satisfied whenever $g(x,t)$ "behaves" like $|t|^{s-2}t$.

b) Contrary to [4], (g_4) is assumed globally.

It is well known that problem (P) has a nontrivial solution (see [4]), and of course (g_3) implies that $u = 0$ is a solution as well. We want to show that, under some restrictions on λ, other nontrivial solutions can be found.

Let $0 < \lambda_1 < \lambda_2 \le \ldots$ be the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$, if $i \ge 1$ set $H_i = \mathrm{Span}(e_1, \ldots, e_i)$, where e_i is the eigenfunction corresponding to λ_i and define $H_i^\perp$ as the orthogonal complement of H_i in $H_0^1(\Omega)$.

We can now state our main theorem.

Theorem 1. *Assume* (g_1) $-$ (g_4)*. Then* $\forall\, i \ge 2$ *there exists* $\delta_i > 0$ *such that* $\forall\, \lambda \in (\lambda_i - \delta_i, \lambda_i)$*, problem* (P) *has at least 3 nontrivial solutions.*

3 Inequalities and technical Lemmas

The proof of Theorem 1 will be done in several steps. First of all let us recall that $H_0^1(\Omega)$, as usual, is endowed with the scalar product $\langle u, v\rangle = \int Du \cdot Dv$, which induces the usual norm $\|u\|^2 = \int |Du|^2$. Now consider the functional $f_\lambda : H_0^1(\Omega) \longrightarrow \mathbb{R}$ defined as

$$f_\lambda(u) = \frac{1}{2}\int_\Omega |Du|^2\,dx - \frac{\lambda}{2}\int_\Omega u^2\,dx - \int_\Omega G(x,u)\,dx.$$

It is well known that f is a C^1 functional on $H_0^1(\Omega)$ and that it satisfies $(PS)_c$ $\forall\, c$ in $\mathbb{R}$ (see [4] and also Theorem 2.15 and Proposition B.35 in [22]), that is

$\forall\,(u_n)_n$ in $H_0^1(\Omega)$ such that $f_\lambda(u_n) \to c$ and $f_\lambda'(u_n) \to 0$, there exists a converging subsequence.

We want to show that if there exist i and j in $\mathbb{N}$ such that $\lambda_{i-1} < \lambda < \lambda_i = \ldots = \lambda_j < \lambda_{j+1}$, and λ is sufficiently near to λ_i, then the topological situation described in Theorem 3 (see Appendix) holds (with $X_1 = H_{i-1}, X_2 = \mathrm{Span}(e_i,\ldots,e_j)$ and $X_3 = H_j^\perp$) and in particular that $(\nabla)(f, H_{i-1}\oplus H_j^\perp, a, b)$ (see Appendix) holds for suitable a and b.

To do that we start from the following notations. If $j < k$ in $\mathbb{N}$ we set:

$$S_k^+(\rho) = \left\{u \in H_k^\perp \;\middle|\; \|u\| = \rho\right\},$$

$$\mathcal{T}_{j,k}(R) = \Big\{u \in H_k \;\Big|\; \|u\| = R\Big\} \bigcup \Big\{u \in H_j \;\Big|\; \|u\| \le R\Big\}.$$

We can now state our first Lemma.

Lemma 1. *Assume $\lambda_{i-1} < \lambda_i = \ldots = \lambda_j < \lambda_{j+1}$ and $\lambda \in (\lambda_{i-1}, \lambda_j)$. Then there exist R and $\rho > 0$, $R > \rho > 0$, such that*

$$\sup f_\lambda\Big(\mathcal{T}_{i-1,j}(R)\Big) < \inf f_\lambda\Big(S_{i-1}^+(\rho)\Big).$$

Proof. By (g_2) and (g_3) we get in a standard way that, given $\epsilon > 0$, there exists $\delta > 0$ such that $G(x,u) \le \epsilon|u|^2 + C(\delta)|u|^{s+1}$. Moreover, by the fact that $\int |Dz|^2 \ge \lambda_i \int z^2$ $\forall\, z$ in $H_{i-1}^\perp$, we get the existence of $\rho > 0$ such that

$$\inf f_\lambda\Big(S_{i-1}^+(\rho)\Big) > 0.$$

Moreover $f_\lambda(H_{i-1}) \le 0$. To conclude the proof it is enough to show that

$$\lim_{\substack{\|u\|\to\infty,\\ u\in H_j}} f_\lambda(u) = -\infty.$$

Such a result easily follows from Remark 1. In fact if $u \in H_i$, then

$$f_\lambda(u) \le \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_\Omega u^2\,dx - c_1\int_\Omega |u|^\mu\,dx,$$

and since all norms in H_i are equivalent, the thesis follows. ∎

Now take $a \in \big(\sup f_\lambda(\mathcal{T}_{i-1,j}(R), \inf f_\lambda(S^+_{i-1}(\rho))\big)$ and $b > \sup f_\lambda(\overline{B_j(R)})$, where $B_j(R)$ is the ball in H_j with radius R. If one shows that $(\nabla)(f_\lambda, H_{i-1} \oplus H_j^\perp, a, b)$ holds, then Theorem 3 can be applied and Theorem 2 (see Section 4) can be proved.

Let us start with two Lemmas which will be useful to prove the (∇)–condition.

Lemma 2. *Assume $\lambda_{i-1} < \lambda_i = \ldots = \lambda_j < \lambda_{j+1}$ for some $i \leq j$ in $\mathbb{N}$. Then $\forall\, \delta > 0\ \exists\, \varepsilon_0 > 0$ such that $\forall\, \lambda$ in $[\lambda_{i-1}+\delta, \lambda_{j+1}-\delta]$, the unique critical point u of f_λ constrained on $H_{i-1} \oplus H_j^\perp$ such that $f_\lambda(u) \in [-\varepsilon_0, \varepsilon_0]$, is the trivial one.*

Proof. Assume by contradiction that there exist $\delta > 0$, λ_n in $[\lambda_{i-1}+\delta, \lambda_{j+1}-\delta]$ and u_n in $H_{i-1} \oplus H_j^\perp \setminus \{0\}$ such that

$$f_{\lambda_n}(u_n) = \frac{1}{2}\int_\Omega |Du_n|^2\, dx - \frac{\lambda_n}{2}\int_\Omega u_n^2\, dx - \int_\Omega G(x, u_n)\, dx \longrightarrow 0$$

and such that $\forall\, z$ in $H_{i-1} \oplus H_j^\perp$

$$\int_\Omega Du_n \cdot Dz\, dx - \lambda_n \int_\Omega u_n z\, dx - \int_\Omega g(x, u_n) z\, dx = 0. \tag{1}$$

Of course, up to a subsequence, we can assume $\lambda_n \to \lambda$ in $[\lambda_{i-1}+\delta, \lambda_{j+1}-\delta]$. Choose $z = u_n$ in (1). Then

$$0 = \int_\Omega |Du_n|^2\, dx - \lambda_n \int_\Omega u_n^2\, dx - \int_\Omega g(x, u_n) u_n\, dx =$$

$$2f_{\lambda_n}(u_n) + \int_\Omega [2G(x, u_n) - g(x, u_n)u_n]\, dx \leq 2f_{\lambda_n}(u_n) + (2-\mu)\int_\Omega G(x, u_n)\, dx$$

by (g_4). In particular

$$\lim_{n\to\infty} \int_\Omega G(x, u_n)\, dx = 0. \tag{2}$$

Now take v_n in H_{i-1} and w_n in $H_j^\perp$ such that $u_n = v_n + w_n\ \forall\, n$ and choose $z = v_n - w_n$ in (1). Then

$$\int_\Omega |Dv_n|^2\, dx - \lambda_n \int_\Omega v_n^2\, dx - \left(\int_\Omega |Dw_n|^2\, dx - \lambda_n \int_\Omega w_n^2\, dx\right)$$

$$= \int_\Omega g(x, u_n)(v_n - w_n)\, dx.$$

Since $\int |Dz|^2 \leq \lambda_{i-1} \int z^2\ \forall\, z$ in H_{i-1} and $\int |Dz|^2 \geq \lambda_{j+1} \int z^2\ \forall\, z$ in $H_j^\perp$, we get that there exists $c > 0$ independent of n ($c = \delta/\lambda_{j+1}$) such that

$$c\|u_n\|^2 \leq \int_\Omega g(x, u_n)(w_n - v_n)\, dx \qquad \forall\, n. \tag{3}$$

Here we used the fact that $\|v_n\|^2+\|w_n\|^2 = \|u_n\|^2$, since v_n and w_n are orthogonal.

Moreover

$$\left|\int_\Omega g(x,u_n)(w_n - v_n)\,dx\right|$$

$$\leq \left(\int_\Omega |g(x,u_n)|^{1+1/s}\,dx\right)^{s/(s+1)} \left(\int_\Omega |w_n - v_n|^{s+1}\,dx\right)^{1/(s+1)}.$$

By the Sobolev's embedding theorem there exists a universal constant $\gamma_{s+1} > 0$ such that $\|v_n - w_n\|_{L^{s+1}} \leq \gamma_{s+1}\|v_n - w_n\| = \gamma_{s+1}\|u_n\|$. In this way, since $u_n \neq 0$, (3) implies that there exists $c' > 0$ such that

$$\|u_n\| \leq c' \left(\int_\Omega |g(x,u_n)|^{1+1/s}\,dx\right)^{s/(s+1)} \qquad \forall\, n. \tag{4}$$

Up to subsequences, there are two cases: $\|u_n\| \to \infty$ or $\|u_n\|$ is bounded.

First case: $\|u_n\|$ *is unbounded.* Then we can suppose that there exists u in $H_{i-1} \oplus H_j^\perp$ such that $u_n/\|u_n\| \rightharpoonup u$.

First of all (2) implies

$$2\frac{f_{\lambda_n}(u_n)}{\|u_n\|} \longrightarrow 1 - \lambda \int_\Omega u^2\,dx = 0,$$

so that $u \not\equiv 0$. Moreover (g_2), Remark 1 and the fact that $s+1=\mu$ imply

$$\int_\Omega |g(x,u_n)|^{1+1/s}\,dx \leq a_1' + a_2' \int_\Omega G(x,u_n)\,dx.$$

But the last quantity is bounded (by (2)) and (4) leads to a contradiction.

Second case: $\|u_n\|$ *is bounded.* We can suppose there exists u in $H_{i-1} \oplus H_j^\perp$ such that $u_n \rightharpoonup u$. Moreover (2) implies $u = 0$.

If $u_n \to 0$, then (4) would give

$$1 \leq \lim_{n\to\infty} c' \frac{\left(\int_\Omega |g(x,u_n)|^{1+1/s}\,dx\right)^{s/(s+1)}}{\|u_n\|} = 0,$$

the limit being 0 by (g_2) and (g_3).

So there should exist $\sigma > 0$ such that $\|u_n\| \geq \sigma\ \forall\, n$. But also in this case (g_3) and (4) would give

$$\sigma \leq \lim_{n\to\infty} c' \left(\int_\Omega |g(x,u_n)|^{1+1/s}\,dx\right)^{s/(s+1)} = 0.$$

The Lemma is thus completely proved. $\square$

For the following Lemma we denote by $P : H_0^1(\Omega) \longrightarrow \mathrm{Span}(e_i, \dots, e_j)$ and $Q : H_0^1(\Omega) \longrightarrow H_{i-1} \oplus H_i^\perp$ the orthogonal projections.

Lemma 3. *Suppose $\lambda_{i-1} < \lambda_i = \dots = \lambda_j < \lambda_{j+1}$, $\lambda \in \mathbb{R}$ and $(u_n)_n$ in $H_0^1(\Omega)$ is such that $(f_\lambda(u_n))_n$ is bounded, $Pu_n \to 0$ and $Q\nabla f_\lambda(u_n) \to 0$. Then $(u_n)_n$ is bounded.*

Proof. Assume by contradiction that $(u_n)_n$ is unbounded. Then we can suppose that there exists u in $H_0^1(\Omega)$ such that $u_n/\|u_n\| \rightharpoonup u$.

Note that $u_n = Pu_n + Qu_n$, $Pu_n \to 0$ and $Q\nabla f_\lambda(u_n) \to 0$, where $\nabla f_\lambda(u_n) = u_n + \Delta^{-1}(\lambda u_n + g(x, u_n))$. In particular

$$\langle Q\nabla f_\lambda(u_n), u_n\rangle = \langle \nabla f_\lambda(u_n), u_n\rangle - \langle P\nabla f_\lambda(u_n), u_n\rangle = \|u_n\|^2 - \lambda \int_\Omega u_n^2\, dx$$

$$- \int_\Omega g(x, u_n)u_n\, dx - \int_\Omega D\Big(P\big(u_n + \Delta^{-1}(\lambda u_n + g(x, u_n))\big)\Big) \cdot Du_n\, dx.$$

But for every z in $H_0^1(\Omega)$, Pz is a smooth function and $Pz \perp Qz$, so that the last integral in the previous equation is equal to

$$\int_\Omega |DPu_n|^2\, dx - \lambda \int_\Omega |Pu_n|^2\, dx - \int_\Omega g(x, u_n)Pu_n\, dx.$$

In this way

$$\begin{aligned} \langle Q\nabla f_\lambda(u_n), u_n\rangle = 2f_\lambda(u_n) + 2\int_\Omega G(x, u_n)\, dx \\ - \int_\Omega g(x, u_n)u_n\, dx - \|Pu_n\|^2 + \lambda \int |Pu_n|^2\, dx + \int_\Omega g(x, u_n)Pu_n\, dx. \end{aligned} \tag{5}$$

Now observe that (g_2) implies that

$$\lim_{n\to\infty} \frac{\displaystyle\int_\Omega |g(x, u_n)Pu_n|\, dx}{\|u_n\|^s} = 0,$$

since $|g(x, u_n)Pu_n| \le \|Pu_n\|_\infty\big(a_1 + a_2|u_n|^s\big)$ and $\|Pu_n\|_\infty \to 0$. In this way, starting from (5), using (g_4) and dividing by $\|u_n\|^s$, we get $(s > 1)$

$$\lim_{n\to\infty} \frac{\displaystyle\int_\Omega G(x, u_n)\, dx}{\|u_n\|^s} = 0.$$

Thus Remark 1 implies

$$\lim_{n\to\infty} \frac{\displaystyle\int_\Omega |u_n|^\mu\, dx}{\|u_n\|^s} = 0,$$

and so $u \equiv 0$. In this way, dividing $2f_\lambda(u_n)$ by $\|u_n\|^2$, we get

$$\lim_{n\to\infty} \frac{\displaystyle\int_\Omega G(x,u_n)\,dx}{\|u_n\|^2} = 1, \tag{6}$$

and so there exists a constant $C > 0$ such that

$$\int_\Omega |u_n|^\mu \,dx \le C\|u_n\|^2. \tag{7}$$

Now let us show that

$$\lim_{n\to\infty} \frac{\displaystyle\int_\Omega g(x,u_n)Pu_n\,dx}{\|u_n\|^2} = 0, \tag{8}$$

which is obvious if $s \le 2$ (this is the case if $N \ge 6$). In fact

$$\lim_{n\to\infty} \frac{\displaystyle\int_\Omega |g(x,u_n)Pu_n|\,dx}{\|u_n\|^2} \le \frac{\|Pu_n\|_\infty}{\|u_n\|^2}\left(a_1 + a_2\int_\Omega |u_n|^s\,dx\right)$$

$$\le \|Pu_n\|_\infty \left[\frac{a_1}{\|u_n\|^2} + \frac{a_2'}{\|u_n\|^{2-2s/\mu}}\left(\frac{\int_\Omega |u_n|^\mu\,dx}{\|u_n\|^2}\right)^{s/\mu}\right],$$

and the thesis follows from (7), since $2 - 2s/\mu > 0$.

In this way (5), (g_4) and (8) imply

$$\lim_{n\to\infty} \frac{\displaystyle\int_\Omega G(x,u_n)\,dx}{\|u_n\|^2} = 0,$$

which contradicts (6). □

Now, by Lemma 2 and Lemma 3 we can prove the following fundamental result.

Proposition 1. *Assume $\lambda_{i-1} < \lambda_i = \ldots = \lambda_j < \lambda_{j+1}$ for some $i \le j$ in $\mathbb{N}$. Then $\forall\,\delta > 0\ \exists\,\varepsilon_0 > 0$ such that $\forall\,\lambda \in [\lambda_{i-1}+\delta, \lambda_{j+1}-\delta]$ and $\forall\,\varepsilon', \varepsilon'' \in (0,\varepsilon_0)$, $\varepsilon' < \varepsilon''$, the condition $(\nabla)(f_\lambda, H_{i-1}\oplus H_j^\perp, \varepsilon', \varepsilon'')$ holds.*

Proof. Assume by contradiction that there exists $\delta > 0$ such that $\forall\,\varepsilon_0 > 0$ there exist λ in $[\lambda_{i-1}+\delta, \lambda_{j+1}-\delta]$ and $\varepsilon', \varepsilon''$ in $(0,\varepsilon_0)$, such that $(\nabla)(f, H_{i-1}\oplus H_j^\perp, \varepsilon', \varepsilon'')$ does not hold.

Take $\varepsilon_0 > 0$ as given by Lemma 2. Then there exists $(u_n)_n$ in $H_0^1(\Omega)$ such that $d(u_n, H_{i-1}\oplus H_j^\perp) \to 0$, $f_\lambda(u_n) \in [\varepsilon', \varepsilon'']$ and $Q\nabla f_\lambda(u_n) \to 0$. Then by Lemma 3 $(u_n)_n$ is bounded. Assume $u_n \rightharpoonup u$.

Note that $Q\nabla f_\lambda(u_n) = u_n - Pu_n + \Delta^{-1}(\lambda u_n + g(x,u_n))$, where $g(x,u_n) \to g(x,u)$ in $L^{1+1/s}(\Omega)$ by (g_2) and $\Delta^{-1} : H^{-1} \longrightarrow H_0^1(\Omega)$ is a compact operator. Then $u_n \to u$ and u is a critical point of f_λ constrained on $H_{i-1} \oplus H_j^\perp$.

By Lemma 2 $u = 0$, while $0 < \varepsilon' \le f_\lambda(u_n)\ \forall n$. The continuity of f_λ gives rise to a contradiction. □

In order to apply Theorem 3 we only have to show that $\sup f_\lambda(\overline{B_j(R)})$ is small enough. In order to do that, let us show the following Lemma.

Lemma 4.

$$\lim_{\lambda \to \lambda_j} \sup f_\lambda(H_j) = 0.$$

Proof. Assume by contradiction that there exist $\lambda_n \to \lambda_j$, $(u_n)_n$ in H_j and $\varepsilon > 0$ such that

$$\sup f_{\lambda_n}(H_j) = f_{\lambda_n}(u_n) \ge \varepsilon \qquad \forall n.$$

Note that f_λ attains a maximum in H_j by Remark 1.

If $(u_n)_n$ is bounded, we can assume that $u_n \to u$ in H_j. In this way

$$\varepsilon \le \frac{1}{2}\int_\Omega |Du|^2\,dx - \frac{\lambda_j}{2}\int_\Omega u^2\,dx - \int_\Omega G(x,u)\,dx \le 0.$$

So we can assume that $\|u_n\| \to \infty$. In this case (g_4) implies

$$0 < \varepsilon \le f_{\lambda_n}(u_n) \le \frac{1}{2}\|u_n\|^2 - \frac{\lambda_n}{2}\int_\Omega u_n^2\,dx - c_1\int_\Omega |u_n|^\mu\,dx,$$

and since all norms are equivalent in H_j the right hand side of the last inequality would tend to $-\infty$. □

4 Proof of Theorem 1

We can now prove the following preliminary result.

Theorem 2. *Assume $\lambda_{i-1} < \lambda_i = \ldots = \lambda_j < \lambda_{j+1}$ for some $i \le j$ in $\mathbb{N}$. Then there exists $\delta_1 > 0$ such that, $\forall\,\lambda$ in $(\lambda_j - \delta_1, \lambda_j)$, problem (P) has at least two nontrivial solution.*

Proof. Take $\delta' > 0$ and find ε_0 as in Proposition 1. Fix $\varepsilon' < \varepsilon'' < \varepsilon_0$. By Lemma 4 there exists $\delta_1 \le \delta'$ such that, if $\lambda \in (\lambda_j - \delta_1, \lambda_j)$, then $\sup f_\lambda(H_j) < \varepsilon''$ and by Proposition 1, $(\nabla)(f_\lambda, H_{i-1} \oplus H_j^\perp, \varepsilon', \varepsilon'')$ holds. Moreover, since $\lambda < \lambda_j$, the topological situation of Lemma 1 is satisfied. By Theorem 3 there exist two critical point u_1, u_2 of f_λ such that $f_\lambda(u_i) \in [\varepsilon', \varepsilon'']$, $i = 1, 2$. In particular u_1 and u_2 are nontrivial solutions of problem (P). □

In order to prove the existence of a third nontrivial solution, let us prove the following Lemma.

Lemma 5. *Suppose $\lambda_{i-1} < \lambda_i = \ldots = \lambda_j < \lambda_{j+1}$. Then there exist $\delta_i > 0$, $\rho_1 > 0$ and $R_1 > \rho_1$ such that $\forall\,\lambda$ in $(\lambda_i - \delta_i, \lambda_i)$*

$$\inf f_\lambda\Big(S_j^+(\rho_1)\Big) > \sup f_\lambda\Big(\mathcal{T}_{j,j+1}(R_1)\Big).$$

In particular there exists a critical point u of f_λ such that $f_\lambda(u) \geq \inf f_\lambda\big(S_j^+(\rho_1)\big)$.

Proof. Take λ in $[\lambda_{i-1}, \lambda_i)$. If $v \in H_j^\perp$, then $\int |Dv|^2 \geq \lambda_{j+1} \int u^2$, so that (g_2) and (g_3) imply that $\forall\,\tau > 0$ there exists $\rho_1 > 0$ such that, if $v \in H_j^\perp$ and $\|v\| = \rho_1$, then

$$f_\lambda(v) \geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{j+1}} - \tau\right)\|v\|^2.$$

Of course one take τ so small that the right hand side of the last inequality is greater than $C\rho_1^2$, where C is independent on λ and $C > 0$ (for example $C = 1 - \lambda_j/\lambda_{j+1} - \tau$).

By Remark 1 we get

$$f_\lambda(u) \leq \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)} - c_1\|u\|^\mu_{L^\mu(\Omega)},$$

and since all norms are equivalent in H_{j+1}, we get that $f_\lambda(u) \to -\infty$ if $u \in H_{j+1}$ and $\|u\| \to \infty$.

By Lemma 4 there exists $\delta_i > 0$ such that $\forall\,\lambda$ in $(\lambda_i - \delta_i, \lambda_i)$ it results

$$\sup f_\lambda(H_j) < C\rho_1^2.$$

Of course we can always assume that $\delta_i \leq \delta_1$, δ_1 being the one given in Theorem 2.

In this way, the classical Linking Theorem (see [22]) shows the existence of a critical point u of f_λ such that $f_\lambda(u) \geq C\rho_1^2$. □

Note that, although the topological structure found in Lemma 5 is equal to the one of Lemma 1, it is not possible to apply Theorem 3 again, since it is not clear if $(\nabla)(f_\lambda, H_j \oplus H_{j+1}^\perp, C\rho_1^2, \sup f_\lambda(\overline{B_{j+1}(R_1)}))$ holds.

Proof of Theorem 1. Take δ_i as given in Lemma 5. Then the critical point u found there is different from the critical points u_i found in Theorem 2, since

$$f_\lambda(u_i) \leq \sup f_\lambda(H_j) < C\rho_1^2 \leq f_\lambda(u).$$

5 Appendix

In this section we recall one theorem belonging to a class of some recent variational ones which provide the existence of several critical points under a "mixed type" assumption on the functional, in the sense that there are hypotheses both on the values of the functional on some suitable sets and on the values of its gradient. Theorems of this kind were first introduced in [16] and then developed in [17] (see also [15] for a sequential version of a theorem of this kind) and they were fruitfully applied in many cases to establish multiplicity results (see also [18], [19], [20], [21]).

For the definition and the proves we refer to [16].

Definition 1. Let X be a Hilbert space, $f : X \longrightarrow \mathbb{R}$ be a C^1 function, M a closed subspace of X, a, $b \in \mathbb{R} \cup \{-\infty, +\infty\}$. We say that condition $(\nabla)(f, M, a, b)$ holds if there exists $\gamma > 0$ such that

$$\inf \left\{ \|P_M \nabla f(u)\| \,\middle|\, a \leq f(u) \leq b, \operatorname{dist}(u, M) \leq \gamma \right\} > 0,$$

where $P_M : X \longrightarrow M$ is the orthogonal projection of X onto M.

This means that one requires that $f_{|M}$ has no critical points u with $a \leq f(u) \leq b$, with some uniformity.

Theorem 3 ((∇)–Theorem). *Let X be a Hilbert space and X_i, $i = 1, 2, 3$ three subspaces of X such that $X = X_1 \oplus X_2 \oplus X_3$ and $\dim X_i < \infty$ for $i = 1, 2$. Denote by P_i the orthogonal projection of X onto X_i. Let $f : X \longrightarrow \mathbb{R}$ be a $C^{1,1}$ function. Let $\rho, \rho', \rho'', \rho_1$ be such that $\rho_1 > 0$, $0 \leq \rho' < \rho < \rho''$ and define*

$$\Delta = \left\{ u \in X_1 \oplus X_2 \,\middle|\, \rho' \leq \|P_2 u\| \leq \rho'', \|P_1 u\| \leq \rho_1 \right\} \quad \textit{and} \quad T = \partial_{X_1 \oplus X_2} \Delta,$$

$$S_{23}(\rho) = \left\{ u \in X_2 \oplus X_3 \,\middle|\, \|u\| = R \right\} \textit{ and } B_{23}(\rho) = \left\{ u \in X_2 \oplus X_3 \,\middle|\, \|u\| \leq R \right\}.$$

Assume that

$$a' = \sup f(T) < \inf f(S_{23}(\rho)) = a''.$$

Let a and b be such that $a' < a < a''$ and $b > \sup f(\Delta)$. Assume $(\nabla)(f, X_1 \oplus X_3, a, b)$ holds and that $(PS)_c$ holds at any c in $[a, b]$. Then f has at least two critical points in $f^{-1}([a, b]))$. Moreover, if

$$\inf f(B_{23}(\rho)) > -\infty \qquad \textit{and} \qquad a_1 < \inf f(B_{23}(\rho))$$

and $(PS)_c$ holds at any c in $[a_1, b]$, then f has another critical level in $[a_1, a']$.

References

[1] A. Ambrosetti, *On the existence of multiple solutions for a class of nonlinear boundary value problems*, Rend. Sem. Mat. Univ. Padova 49 (1973), 195–204.

[2] A. Ambrosetti - D. Lupo, *On a class of nonlinear Dirichlet problems with multiple solutions*, Nonlinear Anal. 8 (1984), no. 10, 1145–1150.

[3] A. Ambrosetti - G. Mancini, *Sharp nonuniqueness results for some non-linear problems*, Nonlinear Anal. 3 (1979), no. 5, 635–645.

[4] A. Ambrosetti - P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis 14 (1973), 349–381.

[5] H. Brezis - L. Nirenberg, *Remarks on finding critical points*, Comm. Pure Appl. Math. 44 (1991), no. 8-9, 939–963.

[6] T. Bartsch - Z. Q. Wang, *On the existence of sign changing solutions for semilinear Dirichlet problems*, Topol. Methods Nonlinear Anal. 7 (1996), no. 1, 115–131.

[7] N. P. Các - F. Schulz, *Multiple solutions for a class of semilinear Dirichlet problems*, Houston J. Math. 17 (1991), no. 1, 71–81.

[8] A. Castro - J. Cossio - J. M. Neuberger, *A minmax principle, index of the critical point, and existence of sign-changing solutions to elliptic boundary value problems*, Electron. J. Differential Equations 1998, No. 02, 1–18.

[9] A. Castro - J. Cossio - J. M. Neuberger, *A sign-changing solution for a superlinear Dirichlet problem*, Rocky Mountain J. Math. 27 (1997), no. 4, 1041–1053.

[10] A. Castro - J. Cossio - J. M. Neuberger, *On multiple solutions of a nonlinear Dirichlet problem*, Proceedings of the Second World Congress of Nonlinear Analysts, Part 6 (Athens, 1996). Nonlinear Anal. 30 (1997), no. 6, 3657–3662.

[11] E. N. Dancer, *Degenerate critical points, homotopy indices and Morse inequalities*, J. Reine Angew. Math. 350 (1984), 1–22.

[12] E. N. Dancer - Y. Du, *Existence of changing sign solutions for some semilinear problems with jumping nonlinearities at zero*, Proc. Roy. Soc. Edinburgh Sect. A 124 (1994), no. 6, 1165–1176.

[13] Z. Jin, *Multiple solutions for a class of semilinear elliptic equations*, Proc. Amer. Math. Soc. 125 (1997), no. 12, 3659–3667.

[14] S. J. Li - M. Willem, *Applications of local linking to critical point theory* J. Math. Anal. Appl. 189 (1995), no. 1, 6–32.

[15] A. Marino - D. Mugnai, *Asymptotical multiplicity and some reversed variational inequalities*, to appear on Topological Methods in Nonlinear Analysis.

[16] A. Marino - C. Saccon, *Some Variational Theorems of Mixed Type and Elliptic Problems with jumping nonlinearities*, Ann. Scuola Norm. Sup. Pisa Cl. Sc. (4), Vol. XXV, Fasc. 3-4 (1997), 661–665.

[17] A. Marino - C. Saccon, *Nabla Theorems and multiple solutions for some noncooperative elliptic systems*, Topological Methods in Nonlinear Analysis 17, no.2 (2001), 213–238.

[18] A. M. Micheletti - A. Pistoia, *On the number of solutions for a class of fourth order elliptic problems*, Comm. Appl. Nonlinear Anal. 6 (1999), no. 2, 49–69.

[19] A. M. Micheletti - A. Pistoia - C. Saccon, *Three solutions of a fourth order elliptic problem via variational theorems of mixed type*, Appl. Anal. 75 (2000), no. 1-2, 43–59.

[20] D. Mugnai, *On a "reversed" variational inequality*, Topological Methods in Nonlinear Analysis 17, no.2 (2001), 321–358.

[21] A. Pistoia, *An Application of a Variational Theorem of Mixed Type to a Superlinear Elliptic Problem*, J. Math. Anal. Appl. 237 (1999), 303–319.

[22] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, 65, American Mathematical Society, Providence, RI, 1986.

[23] E. A. B. Silva, *Critical point theorems and applications to a semilinear elliptic problem*, NoDEA Nonlinear Differential Equations Appl. 3 (1996), no. 2, 245–261.

[24] M. Struwe, *A note on a result of Ambrosetti and Mancini*, Ann. Mat. Pura Appl. (4) 131 (1982), 107–115.

[25] M. Struwe, *Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems*, Manuscripta Math. 32 (1980), no. 3-4, 335–364.

[26] Z. Q. Wang, *On a superlinear elliptic equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire 8 (1991), no. 1, 43–57.