

FUNCTIONS OF BOUNDED VARIATION AND LIPSCHITZ ALGEBRAS IN METRIC MEASURE SPACES

ENRICO PASQUALETTO AND GIACOMO ENRICO SODINI

ABSTRACT. Given a unital algebra \mathcal{A} of locally Lipschitz functions defined over a metric measure space (X, d, m) , we study two associated notions of function of bounded variation and their relations: the space $BV_H(X; \mathcal{A})$, obtained by approximating in energy with elements of \mathcal{A} , and the space $BV_W(X; \mathcal{A})$, defined through an integration-by-parts formula that involves derivations acting in duality with \mathcal{A} . Our main result provides a sufficient condition on the algebra \mathcal{A} under which $BV_H(X; \mathcal{A})$ coincides with the standard metric BV space $BV_H(X)$, which corresponds to taking as \mathcal{A} the collection of all locally Lipschitz functions. Our result applies to several cases of interest, for example to Euclidean spaces and Riemannian manifolds equipped with the algebra of smooth functions, or to Banach and Wasserstein spaces equipped with the algebra of cylinder functions. Analogous results for metric Sobolev spaces $H^{1,p}$ of exponent $p \in (1, \infty)$ were previously obtained by several different authors.

CONTENTS

1. Introduction	1
1.1. General overview	1
1.2. Contents of the paper	2
1.3. Motivations and comparison with the previous literature	4
Acknowledgements	4
2. Preliminaries	5
2.1. Metric/measure spaces	5
2.2. L^p -Banach L^∞ -modules	7
2.3. Lipschitz derivations	8
3. Metric BV and $H^{1,1}$ spaces	9
3.1. The space BV_H	9
3.2. The space $H^{1,1}$	11
3.3. The space BV_W	12
4. Equivalence results	15
Appendix A. The total variation measure	20
References	24

1. INTRODUCTION

1.1. General overview. For more than two decades, functions of bounded variation and sets of finite perimeter have been studied in the general setting of metric measure spaces, starting from the paper [32]. Rather refined results are available on those metric measure spaces (often called PI

Date: March 28, 2025.

2020 Mathematics Subject Classification. 53C23, 26A45, 49J52, 46E35, 46N10.

Key words and phrases. Functions of bounded variation; Lipschitz algebras; metric measure spaces; derivations.

spaces) that are doubling and support a weak Poincaré inequality (see e.g. [32, 1, 2, 25, 26, 27]), and even deeper structural properties have been proven when also a lower synthetic Ricci curvature bound is imposed, in the class of the so-called $\text{RCD}(K, N)$ spaces [3, 13, 12, 11, 8]. However, a fully consistent theory of BV functions does not require any additional assumptions on the metric measure space under consideration, and that will be our object of study in the present paper.

The first notion of metric BV space, introduced by Miranda Jr. in [32], is formulated in terms of an energy approximation by locally Lipschitz functions. However, the algebra of locally Lipschitz functions depends solely on the metric structure of the ambient space, and as such it is not capable of detecting some additional features that the underlying metric space may have. Due to this reason, in several cases of interest – which we shall briefly mention in Section 1.3 – it is desirable to know that a smaller, specific algebra of approximating functions can be used to define the BV space. This is exactly the primary goal of the present paper: to give a sufficient (and effective) condition on some algebra of functions in order that the corresponding BV space (obtained via approximation) coincides with the original one that was defined in terms of locally Lipschitz functions.

More than ten years after the BV space via approximation was developed, Ambrosio and Di Marino introduced in [4] another notion of metric BV space and showed its equivalence with the former. Their definition is expressed in terms of the behaviour of functions along suitably-selected curves, where the exceptional curve families are detected using the so-called test plans. A third approach was then proposed by Di Marino in [17, 16], by means of an integration-by-parts formula involving an appropriate notion of derivation with divergence. Also the latter approach turned out to be equivalent to the first two. Generalised versions of the BV space via derivations will have a key role in this paper, as we shall discuss more in details later. Finally, we mention that a fourth notion of metric ‘Newtonian-type’ BV space (that we will not employ in this paper) was introduced more recently by Martio in [30, 31]. His notion is based upon the behaviour of functions along AM -almost every rectifiable curve, where AM denotes the so-called approximation modulus. The full equivalence of this latter approach with the other ones we discussed above was proved in [33].

1.2. Contents of the paper. Let (X, d, \mathbf{m}) be a metric measure space (as in Definition 2.1 below). For any locally Lipschitz function $f: X \rightarrow \mathbb{R}$, we denote by $\text{lip}_a(f): X \rightarrow [0, +\infty)$ its asymptotic slope function (see (2.1)), which assigns to each point $x \in X$ the ‘infinitesimal Lipschitz constant’ of f at x . The algebra of test functions we will consider in the various definitions of BV space is that of bounded locally Lipschitz functions whose asymptotic slope is integrable, which we denote by $\text{LIP}_*(X)$; see Definition 2.2. Given any unital separating subalgebra \mathcal{A} of $\text{LIP}_*(X)$, we define:

- The space $\text{BV}_H(X; \mathcal{A})$ of those functions $f \in L^1(\mathbf{m})$ that can be approximated in energy by elements of \mathcal{A} ; see Definition 3.1. Any given $f \in \text{BV}_H(X; \mathcal{A})$ is associated with a quantity $\|\mathbf{D}f\|_{*, \mathcal{A}} \in [0, +\infty)$, which we call its total variation (see (3.1)). In the distinguished case $\mathcal{A} = \text{LIP}_*(X)$, where we use the shorter notations $\text{BV}_H(X)$ and $\|\mathbf{D}f\|_*$, our definition is consistent with the one of [32, 4]. Whereas each function $f \in \text{BV}_H(X)$ is also naturally associated with a total variation measure $|\mathbf{D}f|_*$ satisfying $|\mathbf{D}f|_*(X) = \|\mathbf{D}f\|_*$ (see Definition 3.2), we do not know whether it is possible to define some total variation measure associated to an arbitrary function $f \in \text{BV}_H(X; \mathcal{A})$ when $\mathcal{A} \neq \text{LIP}_*(X)$. Indeed, the fact that the set-valued function $|\mathbf{D}f|_*$ as in Definition 3.2 is actually a σ -additive measure is due to the fact that $\text{LIP}_*(X)$ consists of a sufficiently vast class of locally Lipschitz functions, for which the results we will present in Appendix A are satisfied.

- The space $BV_W(X; \mathcal{A})$ of those functions $f \in L^1(\mathfrak{m})$ that satisfy an integration-by-parts formula in duality with the space $\text{Der}_\infty^\infty(X; \mathcal{A})$ of Lipschitz \mathcal{A} -derivations $b: \mathcal{A} \rightarrow L^1(\mathfrak{m})$ having divergence $\text{div}(b) \in L^\infty(\mathfrak{m})$; see Definitions 2.6, 2.7 and 3.10. Any $f \in BV_W(X; \mathcal{A})$ comes with a corresponding total \mathcal{A} -variation measure $|\mathbf{D}f|_{\mathcal{A}}$, as it is shown in Proposition 3.12. In the special case $\mathcal{A} = \text{LIP}_*(X)$, we just write $BV_W(X)$ and $|\mathbf{D}f|$ for the sake of brevity. The original definition in [17, 16] corresponds to choosing as \mathcal{A} the space of all boundedly-supported Lipschitz functions.

It is worth pointing out that – differently from most of the previous literature on the topic – in this paper we are not assuming completeness nor separability of the ambient metric space (X, d) . The possible lack of separability does not cause major issues, since we still require the reference measure \mathfrak{m} to be Radon (recall Definition 2.1). On the other hand, allowing for non-complete metric spaces leads to additional difficulties, as we will see in the sequel. We prefer to consider possibly non-complete spaces in order to include e.g. open domains in a given ambient space. Non-separable spaces are also interesting e.g. in view of the potential generalisation of the theories we consider here to the setting of extended metric-topological measure spaces [5, 36]. Let us also highlight that here we assume \mathfrak{m} to be a finite measure. This choice was made mostly for convenience, as it allows us to work with unital subalgebras of $\text{LIP}_*(X)$; this would not be possible for infinite reference measures, because constant functions are integrable only with respect to a finite measure. Extending our definitions and results to metric spaces equipped with a (possibly infinite) boundedly-finite Radon measure would be interesting, but outside the scope of this paper.

The main results of the present paper can be briefly summarised as follows:

- For any metric measure space (X, d, \mathfrak{m}) and any unital separating subalgebra $\mathcal{A} \subseteq \text{LIP}_*(X)$, we prove in Theorem 4.8 that

$$BV_W(X; \mathcal{A}) \subseteq BV_H(X; \mathcal{A}),$$

as well as the inequality $\|\mathbf{D}f\|_{*, \mathcal{A}} \leq |\mathbf{D}f|_{\mathcal{A}}(X)$ for every $f \in BV_W(X; \mathcal{A})$. The verification of this statement is based on a rather general duality argument in Convex Analysis, which is inspired by the proof of [35, Theorem 5.4] (and [28, Theorem 3.3]), where metric Sobolev spaces are involved instead.

- Assuming either that (X, d) is complete or that its associated topological space is a Radon space, we obtain in Theorem 4.9 the identification

$$BV_H(X) = BV_W(X)$$

together with the identity of measures $|\mathbf{D}f| = |\mathbf{D}f|_*$ for every $f \in BV_H(X)$. At this level of generality, this result seems to be new. In the case where (X, d) is complete, we can also deduce from the results of [4] that every function $f \in BV_H(X)$ can be approximated in energy by a sequence of bounded *globally* Lipschitz functions; see Proposition 3.5. Here, the completeness assumption cannot be dropped, as we will showcase in Example 3.6.

- Assuming that (X, d) is complete and \mathcal{A} is a *good algebra*, we prove in Theorem 4.10 that

$$BV_H(X; \mathcal{A}) = BV_H(X), \quad (1.1)$$

with $\|\mathbf{D}f\|_{*, \mathcal{A}} = \|\mathbf{D}f\|_*$ for every $f \in BV_H(X)$. In other words, being a good algebra is a sufficient condition for \mathcal{A} to be dense in energy in $BV_H(X)$. By a ‘good algebra’ we mean a unital separating subalgebra \mathcal{A} of $\text{LIP}_*(X)$ consisting of bounded globally Lipschitz functions with the property that truncated distance functions from a point can be ‘approximated well in $H^{1,1}$ ’ by elements of \mathcal{A} , where the Sobolev space $H^{1,1}(X; \mathcal{A})$ we consider is the one discussed in Definition 3.7; for details, we refer to Definition 4.1,

where we will introduce the concept of good algebra. The proof of (1.1) relies on the key approximation result Proposition 4.4, where the assumption of \mathcal{A} being a good algebra is used in an essential way. Along the way, we also obtain that $BV_H(X; \mathcal{A}) = BV_W(X; \mathcal{A})$.

1.3. Motivations and comparison with the previous literature. In studying functions of bounded variation defined using different algebras of locally Lipschitz functions, we have been strongly inspired by the works [36, 20], where unital separating subalgebras \mathcal{A} of bounded Lipschitz functions were used to define metric Sobolev spaces $H^{1,p}(X; \mathcal{A})$ of exponent $p \in (1, \infty)$. It is shown in [36] that – in the more general framework of extended metric-topological measure spaces – if \mathcal{A} is a *compatible algebra* (cf. with Example 4.2), then $H^{1,p}(X; \mathcal{A})$ coincides with the ‘standard’ Sobolev space $H^{1,p}(X)$ defined via approximation with bounded Lipschitz functions. Later on, a characterisation of those Lipschitz algebras \mathcal{A} for which $H^{1,p}(X; \mathcal{A}) = H^{1,p}(X)$ has been provided in [20] (in the case where the ambient space is a metric measure space). Our main Theorem 4.10 can be regarded as a version of the above result for metric BV spaces, but with some differences:

- We give only a sufficient condition for the identification $BV_H(X; \mathcal{A}) = BV_H(X)$, namely the fact that \mathcal{A} is a good algebra. Conversely, one can easily realise (see Proposition 4.11) that a necessary condition for $BV_H(X; \mathcal{A}) = BV_H(X)$ to hold is that \mathcal{A} is a *weakly good subalgebra*, which we define in a similar way as good algebras, but where the approximation of truncated distance functions is done in BV (as opposed to $H^{1,1}$). A natural guess is that \mathcal{A} being a weakly good subalgebra is in fact *equivalent* to $BV_H(X; \mathcal{A}) = BV_H(X)$. However, we were not able to prove it, and thus we leave it as an open problem. Another guess would be that \mathcal{A} being a good algebra is equivalent to $H^{1,1}(X; \mathcal{A}) = H^{1,1}(X)$, but this problem is – arguably – even more difficult, due to the functional-analytic complications one encounters when studying metric Sobolev spaces of exponent $p = 1$ (cf. with [7]).
- Our proof strategy for Theorem 4.10 is different from the one of [36, 20]. In the latter works, the identification results are obtained directly at the level of the spaces $H^{1,p}(X; \mathcal{A})$, making use of (generalised) Hopf–Lax flow techniques. In this paper, instead, we first prove equivalence results for metric BV spaces defined using derivations, and then we show their equivalence with the BV spaces defined via approximation. As a theory of derivations on extended metric-topological measure spaces has been recently developed in [35], we expect that it is possible to extend many of the notions and results of this paper to that setting.

We conclude the introduction by discussing some motivations behind our interest in good algebras. As we will observe in Example 4.2, the family of good algebras includes all the compatible algebras considered by Savaré in [36], thus in particular the algebra of smooth functions in the Euclidean space (or in any Riemannian manifold), as well as the algebra of *cylinder functions* in a Banach space or in Wasserstein/Hellinger/Hellinger–Kantorovich spaces. As it is evident e.g. from the works [18, 36, 20, 37, 15], whenever the ambient space possesses ‘nice’ structural features, in order to investigate fine analytic properties it is of fundamental importance to detect a family of regular functions, which are associated with a ‘geometric’ notion of differential. We point out that Theorem 4.10 was already known to be true in two cases: for $X = \mathbb{R}^n$ with the Euclidean norm, \mathbf{m} a Radon measure on \mathbb{R}^n and $\mathcal{A} = C_c^\infty(\mathbb{R}^n)$ the algebra of compactly-supported smooth functions, see [21, Theorem 5.7]; for $X = \mathbb{B}$ a separable Banach space, \mathbf{m} a finite Borel measure on \mathbb{B} and $\mathcal{A} = \text{Cyl}(\mathbb{B})$ the algebra of smooth cylindrical functions, see [34, Theorem 4.1].

Acknowledgements. The first named author was supported by the Research Council of Finland grant 362898.

2. PRELIMINARIES

Let us begin by fixing some terminology and conventions. Throughout this paper, by an **algebra** we mean a commutative, associative algebra over the field \mathbb{R} of real numbers. We say that an algebra \mathcal{A} is **unital** if it has a multiplicative unit. Whenever we say that \mathcal{A}' is a unital subalgebra of \mathcal{A} , we implicitly make the assumption that \mathcal{A} and \mathcal{A}' share the same multiplicative unit.

Given any real number $a \in \mathbb{R}$, we denote by $a^+ := a \vee 0$ and $a^- := a \wedge 0$ its **positive part** and its **negative part**, respectively. Given a set X , we indicate by $\mathbb{1}_S$ the **characteristic function** of a subset S of X , i.e. we define $\mathbb{1}_S(x) := 1$ for every $x \in S$ and $\mathbb{1}_S(x) := 0$ for every $x \in X \setminus S$.

For any two Banach spaces \mathbb{B}_1 and \mathbb{B}_2 , we denote by $\mathcal{L}(\mathbb{B}_1; \mathbb{B}_2)$ the vector space of all bounded linear operators between \mathbb{B}_1 and \mathbb{B}_2 . Recall that $\mathcal{L}(\mathbb{B}_1; \mathbb{B}_2)$ is a Banach space with respect to the operator norm, given by $\|T\|_{\mathcal{L}(\mathbb{B}_1; \mathbb{B}_2)} := \sup\{\|T(v)\|_{\mathbb{B}_2} : v \in \mathbb{B}_1, \|v\|_{\mathbb{B}_1} \leq 1\}$ for all $T \in \mathcal{L}(\mathbb{B}_1; \mathbb{B}_2)$. We denote by $\mathbb{B}' := \mathcal{L}(\mathbb{B}; \mathbb{R})$ the topological dual of a Banach space \mathbb{B} .

2.1. Metric/measure spaces. In this section, we discuss several concepts and results concerning spaces that are equipped with a distance and/or a measure. Given a measure space (X, Σ, μ) and an exponent $p \in [1, \infty]$, we denote by $(L^p(\mu), \|\cdot\|_{L^p(\mu)})$ the **Lebesgue space** of exponent p over (X, Σ, μ) . We write $L^p(\mu)^+$ to indicate the set of all $f \in L^p(\mu)$ satisfying $f \geq 0$ μ -a.e. on X . We also recall that $L^p(\mu)$ is a Riesz space if endowed with the usual pointwise μ -a.e. order. Under suitable assumptions (for example, if μ is σ -finite), it holds that $L^p(\mu)$ is a Dedekind complete lattice, which means that every order-bounded subset of $L^p(\mu)$ has a supremum and an infimum. For any set M of non-negative measures on (X, Σ) , we denote its **supremum measure** by $\bigvee M$:

$$\bigvee M(E) := \sup \left\{ \sum_{n \in \mathbb{N}} \mu_n(E_n) \mid (\mu_n)_{n \in \mathbb{N}} \subseteq M, (E_n)_{n \in \mathbb{N}} \subseteq \Sigma \text{ partition of } E \right\} \quad \text{for every } E \in \Sigma.$$

We recall that $\bigvee M$ is the least measure on (X, Σ) satisfying $\bigvee M \geq \mu$ for every $\mu \in M$, where the partial order \leq on the set of all non-negative measures on (X, Σ) is defined in the following way: we declare that $\mu \leq \nu$ if and only if $\mu(E) \leq \nu(E)$ for every $E \in \Sigma$.

Let (X, d) be a given metric space. We denote by $C(X)$ the unital algebra (with unit $\mathbb{1}_X$) of all real-valued continuous functions on X . We also consider its unital subalgebra $C_b(X)$ consisting of all bounded continuous functions. If $(\mu_n)_{n \in \mathbb{N}}$ and μ are finite non-negative Borel measures on X , then we say that μ_n **weakly converges** to μ if

$$\int f d\mu = \lim_{n \rightarrow \infty} \int f d\mu_n \quad \text{for every } f \in C_b(X).$$

We denote by $\mathcal{M}(X)$ the vector space of all finite, signed **Radon measures** on X . Then $\mathcal{M}(X)$ is a Banach space if endowed with the total variation norm. We denote by $\mathcal{M}_+(X)$ the set of all those measures $\mu \in \mathcal{M}(X)$ that are non-negative. To any measure $\mu \in \mathcal{M}(X)$, we associate its **total variation measure** $|\mu| \in \mathcal{M}_+(X)$. The **support** of a measure $\mu \in \mathcal{M}(X)$ is the closed subset $\text{spt}(\mu)$ of X given by

$$\text{spt}(\mu) := \{x \in X \mid |\mu|(B_r(x)) > 0 \text{ for every } r > 0\},$$

where $B_r(x) := \{y \in X : d(x, y) < r\}$ denotes the open ball in X of center x and radius r . Since the inner regularity of μ ensures that μ is concentrated on a σ -compact set, it can be readily checked that $\text{spt}(\mu)$ is separable and μ is concentrated on $\text{spt}(\mu)$. With a slight abuse of terminology, we say that a metric space (X, d) is **Radon** if its induced topological space is a Radon space, i.e. every finite Borel measure defined on it is a Radon measure. Since all Souslin spaces are Radon spaces, we have that every (Borel subset of a) complete and separable metric space is Radon. See

e.g. [9, Theorem 7.4.3, Section 7.14(vii)] and the references therein for a more detailed discussion.

A distinguished subalgebra of $C(X)$ is the space of all real-valued **locally Lipschitz** functions on X , which is given by

$$\text{LIP}_{loc}(X) := \left\{ f \in C(X) \mid \inf_{r>0} \text{Lip}(f; B_r(x)) < +\infty \text{ for every } x \in X \right\},$$

where the **Lipschitz constant** $\text{Lip}(f; E)$ of the function f on a set $E \subseteq X$ is defined as

$$\text{Lip}(f; E) := \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} \mid x, y \in E, x \neq y \right\} \in [0, +\infty],$$

where we adopt the convention that $\sup(\emptyset) := 0$. The **asymptotic slope** $\text{lip}_a(f): X \rightarrow [0, +\infty)$ of a function $f \in \text{LIP}_{loc}(X)$ is given by

$$\text{lip}_a(f)(x) := \inf_{r>0} \text{Lip}(f; B_r(x)) \quad \text{for every } x \in X. \quad (2.1)$$

It is easy to check that $\text{lip}_a(f)$ is upper semicontinuous (thus, Borel measurable). The space of all (globally) Lipschitz functions, i.e. of those $f \in \text{LIP}_{loc}(X)$ with $\text{Lip}(f) := \text{Lip}(f; X) < +\infty$, is denoted by $\text{LIP}(X)$. Note that $\text{LIP}_b(X) := C_b(X) \cap \text{LIP}(X)$ is a unital subalgebra of $C_b(X)$.

In this paper, we focus on the following class of metric measure spaces:

Definition 2.1 (Metric measure space). *We say that a triple (X, d, m) is a **metric measure space** if (X, d) is a metric space and $m \in \mathcal{M}_+(X)$ (i.e. $m \geq 0$ is a finite Radon measure on X).*

We point out that this notion of metric measure space is different from other approaches considered in the literature. In this paper, the following subalgebra of $C_b(X)$ will have a key role.

Definition 2.2 (The algebra $\text{LIP}_\star(X)$). *Let (X, d, m) be a metric measure space. Then we define*

$$\text{LIP}_\star(X) := \{ f \in C_b(X) \cap \text{LIP}_{loc}(X) \mid \text{lip}_a(f) \in L^1(m) \}.$$

Note that $\text{LIP}_\star(X)$ depends both on the distance d and the measure m , even though we do not indicate it for brevity. Since $\text{LIP}_\star(X)$ is a vector subspace of $C_b(X)$ and $fg \in \text{LIP}_\star(X)$ for every $f, g \in \text{LIP}_\star(X)$ (as $\text{lip}_a(fg) \leq |f|\text{lip}_a(g) + |g|\text{lip}_a(f) \in L^1(m)$), we have that $\text{LIP}_\star(X)$ is a unital subalgebra of $C_b(X)$. We will work also with unital **separating** subalgebras \mathcal{A} of $\text{LIP}_\star(X)$, i.e.

$$\sup_{f \in \mathcal{A}} |f(x) - f(y)| > 0 \quad \text{for every } x, y \in X \text{ with } x \neq y.$$

By the Stone–Weierstrass theorem, if \mathcal{A} is a unital separating subalgebra of $\text{LIP}_\star(X)$, then

$$\mathcal{A} \text{ is dense in } L^p(m) \text{ for every } p \in [1, \infty), \quad (2.2)$$

see e.g. [36, Lemma 2.1.27] (notice that, even if there the algebra \mathcal{A} is assumed to be *compatible*, one can easily check that the simple proof of the mentioned Lemma does not require additional assumptions on \mathcal{A}). It can be readily checked that $\text{LIP}_\star(X)$ itself is a separating algebra.

In the sequel, we will need a partition-of-unity-type result involving functions of a given unital separating subalgebra of $\text{LIP}_\star(X)$, see Lemma 2.4 below. To prove it, we will use the following simple consequence of Weierstrass' approximation theorem:

Lemma 2.3. *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f \in C([a, b])$ be such that $f(t) > 0$ for every $t \in [a, b]$. Then there exists a sequence $(p_n)_n$ of polynomials such that $0 \leq p_n(t) \leq f(t)$ for every $t \in [a, b]$ and every $n \in \mathbb{N}$, and p_n converges uniformly on $[a, b]$ to f as $n \rightarrow \infty$.*

Proof. Let $\alpha := \min_{[a,b]} f > 0$. For every $n \in \mathbb{N}$ with $n > n_0 := \lceil 2/\alpha \rceil$, by the Weierstrass approximation theorem, there is a polynomial \tilde{p}_n such that $\|f - \tilde{p}_n\|_{C_b([a,b])} < 1/n$. It is then enough to set $p_n := \tilde{p}_{n_0+n} - \frac{1}{n+n_0}$ for every $n \in \mathbb{N}$. \square

Lemma 2.4. *Let (X, d, \mathbf{m}) be a metric measure space, \mathcal{A} a unital separating subalgebra of $\text{LIP}_*(X)$ and E_1, \dots, E_n a Borel partition of X . Fix any $\varepsilon > 0$. Then there exist $\eta_1, \dots, \eta_n \in \mathcal{A}$ such that the following properties hold:*

- i) $0 \leq \eta_i \leq 1$ for every $i = 1, \dots, n$.
- ii) $\sum_{i=1}^n \eta_i(x) \leq 1$ for every $x \in X$.
- iii) $\|\eta_i - \mathbb{1}_{E_i}\|_{L^1(\mathbf{m})} \leq \varepsilon$ for every $i = 1, \dots, n$.

Proof. Fix $\delta > 0$. Since \mathcal{A} is dense in $L^1(\mathbf{m})$, for any $i = 1, \dots, k$ we can find $(\psi_i^k)_k \subseteq \mathcal{A}$ such that $\mathbb{1}_{E_i}(x) = \lim_k \psi_i^k(x)$ for \mathbf{m} -a.e. $x \in X$. Thanks to [36, Corollary 2.1.24], we can also assume that $0 \leq \psi_i^k \leq 1$. By Lemma 2.3, there is a polynomial p such that

$$0 \leq p(t) \leq \frac{1}{t}, \quad \left| p(t) - \frac{1}{t} \right| \leq \delta \quad \text{for every } t \in [\delta, n + \delta].$$

Since $p(\sum_{j=1}^n \psi_j^k(x) + \delta) \rightarrow p(1 + \delta)$ as $k \rightarrow \infty$ for \mathbf{m} -a.e. $x \in X$ and $p \circ (\sum_{j=1}^n \psi_j^k + \delta) \leq \frac{1}{\delta}$ holds \mathbf{m} -a.e. on X for every $k \in \mathbb{N}$, by the dominated convergence theorem we can find $k_0 \in \mathbb{N}$ such that, letting $\psi_i := \psi_i^{k_0}$, it holds that $\|p(\sum_{j=1}^n \psi_j + \delta) - p(1 + \delta)\|_{L^1(\mathbf{m})} \leq \delta$ and $\|\psi_i - \mathbb{1}_{E_i}\|_{L^1(\mathbf{m})} \leq \delta$ for every $i = 1, \dots, n$. Now, let us define

$$\eta_i := p\left(\sum_{j=1}^n \psi_j + \delta\right) \psi_i \in \mathcal{A} \quad \text{for every } i = 1, \dots, n.$$

Notice that $\eta_i \geq 0$ for every $i = 1, \dots, n$ and $\sum_{i=1}^n \eta_i \leq (\sum_{j=1}^n \psi_j + \delta)^{-1} (\sum_{i=1}^n \psi_i) \leq 1$, which proves i) and ii). Finally, for any given $i = 1, \dots, n$ we can estimate

$$\begin{aligned} \|\eta_i - \mathbb{1}_{E_i}\|_{L^1(\mathbf{m})} &\leq \left\| p\left(\sum_{j=1}^n \psi_j + \delta\right) \psi_i - \psi_i \right\|_{L^1(\mathbf{m})} + \|\psi_i - \mathbb{1}_{E_i}\|_{L^1(\mathbf{m})} \\ &\leq \left\| p\left(\sum_{j=1}^n \psi_j + \delta\right) - \frac{1}{1 + \delta} \right\|_{L^1(\mathbf{m})} + \left(1 - \frac{1}{1 + \delta}\right) \mathbf{m}(X) + \delta \\ &\leq \left\| p\left(\sum_{j=1}^n \psi_j + \delta\right) - p(1 + \delta) \right\|_{L^1(\mathbf{m})} + \left| p(1 + \delta) - \frac{1}{1 + \delta} \right| \mathbf{m}(X) + (\mathbf{m}(X) + 1)\delta \\ &\leq 2(\mathbf{m}(X) + 1)\delta, \end{aligned}$$

which proves the validity of iii) if we choose $\delta > 0$ so that $2(\mathbf{m}(X) + 1)\delta \leq \varepsilon$. \square

2.2. L^p -Banach L^∞ -modules. Let us recall the language of L^p -Banach L^∞ -modules that was introduced in [23] (see also [22]). Following [23, Definition 1.2.10] (and with a slight change of terminology), we give the following definition:

Definition 2.5 (L^p -Banach L^∞ -module). *Let (X, Σ, μ) be a σ -finite measure space and $p \in [1, \infty]$. Let \mathcal{M} be a module over the commutative ring $L^\infty(\mu)$. Then we say that \mathcal{M} is an $L^p(\mu)$ -Banach $L^\infty(\mu)$ -module if it is endowed with a map $|\cdot|: \mathcal{M} \rightarrow L^p(\mu)^+$, called **pointwise norm**, such that the following conditions are satisfied:*

- i) *Given any $v, w \in \mathcal{M}$ and $f \in L^\infty(\mu)$, it holds that*

$$|v| = 0 \implies v = 0,$$

$$|v + w| \leq |v| + |w|,$$

$$|fv| = |f||v|,$$

- where all equalities and inequalities involving the pointwise norm are in the μ -a.e. sense.
- ii) The **glueing property** holds, i.e. if $(E_n)_{n \in \mathbb{N}} \subseteq \Sigma$ is a partition of the set X and a sequence $(v_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}$ satisfies $\sum_{n \in \mathbb{N}} \|\mathbb{1}_{E_n} v_n\|_{L^p(\mu)}^p < +\infty$, then there exists a (necessarily unique) element $v = \sum_{n \in \mathbb{N}} \mathbb{1}_{E_n} v_n \in \mathcal{M}$ such that $\mathbb{1}_{E_n} v = \mathbb{1}_{E_n} v_n$ for every $n \in \mathbb{N}$.
 - iii) The norm $\|v\|_{\mathcal{M}} := \|v\|_{L^p(\mu)}$ on \mathcal{M} is complete.

Observe that each $L^p(\mu)$ -Banach $L^\infty(\mu)$ -module is in particular a Banach space. We say that a subset S of \mathcal{M} **generates** \mathcal{M} provided the $L^\infty(\mu)$ -linear span of S is dense in \mathcal{M} . An **isomorphism** $\Phi: \mathcal{M} \rightarrow \mathcal{N}$ between two $L^p(\mu)$ -Banach $L^\infty(\mu)$ -modules \mathcal{M} and \mathcal{N} is an $L^\infty(\mu)$ -linear bijection satisfying $|\Phi(v)| = |v|$ for every $v \in \mathcal{M}$.

Since any given $L^1(\mu)$ -Banach $L^\infty(\mu)$ -module \mathcal{M} is a Banach space, we can consider its dual Banach space \mathcal{M}' . Alternatively, we can consider its dual in the sense of Banach modules: following [23, Definition 1.2.6], we define \mathcal{M}^* as the set of all $L^\infty(\mu)$ -linear maps $\omega: \mathcal{M} \rightarrow L^1(\mu)$ for which there exists a function $g \in L^\infty(\mu)^+$ such that

$$|\omega(v)| \leq g|v| \text{ holds } \mu\text{-a.e. on } X \text{ for every } v \in \mathcal{M}. \quad (2.3)$$

Then \mathcal{M}^* is an $L^\infty(\mu)$ -Banach $L^\infty(\mu)$ -module if endowed with the pointwise operations and with

$$|\omega| := \bigwedge \{g \in L^\infty(\mu)^+ \mid g \text{ satisfies (2.3)}\} \in L^\infty(\mu)^+ \quad \text{for every } \omega \in \mathcal{M}^*.$$

We say that \mathcal{M}^* is the **module dual** of \mathcal{M} . The relation between \mathcal{M}' and \mathcal{M}^* is clarified by the following result: letting $\text{INT}_{\mathcal{M}}: \mathcal{M}^* \rightarrow \mathcal{M}'$ be the operator given by

$$\text{INT}_{\mathcal{M}}(\omega)(v) := \int \omega(v) \, d\mu \quad \text{for every } \omega \in \mathcal{M}^* \text{ and } v \in \mathcal{M},$$

we know from [23, Proposition 1.2.13] that

$$\text{INT}_{\mathcal{M}} \text{ is an isometric isomorphism of Banach spaces.} \quad (2.4)$$

2.3. Lipschitz derivations. In this paper, we consider *Lipschitz derivations* in the sense of [17, 16]. Borrowing from [17, Section 1.1] and [7, Section 4.1], we give the ensuing definitions:

Definition 2.6 (Derivation). *Let (X, d, \mathfrak{m}) be a metric measure space. Let \mathcal{A} be a unital separating subalgebra of $\text{LIP}_*(X)$. Then a map $b: \mathcal{A} \rightarrow L^1(\mathfrak{m})$ is called a **Lipschitz \mathcal{A} -derivation** if:*

- i) *There exists $G \in L^\infty(\mathfrak{m})^+$ such that $|b(f)| \leq G \text{lip}_a(f)$ holds \mathfrak{m} -a.e. for every $f \in \mathcal{A}$.*
- ii) *The map b is a linear operator that satisfies the **Leibniz rule**, i.e.*

$$b(fg) = f b(g) + g b(f) \quad \text{for every } f, g \in \mathcal{A}.$$

We denote by $\text{Der}^\infty(X; \mathcal{A})$ the space of all Lipschitz \mathcal{A} -derivations on (X, d, \mathfrak{m}) .

The space $\text{Der}^\infty(X; \mathcal{A})$ is a module over the ring $L^\infty(\mathfrak{m})$, thus in particular it is a vector space. Given any $b \in \text{Der}^\infty(X; \mathcal{A})$, we denote by $|b|_{\mathcal{A}} \in L^\infty(\mathfrak{m})^+$ the \mathfrak{m} -a.e. minimal function G as in Definition 2.6 i), whose existence is guaranteed by the Dedekind completeness of $L^\infty(\mathfrak{m})$. The space $(\text{Der}^\infty(X; \mathcal{A}), |\cdot|_{\mathcal{A}})$ is an $L^\infty(\mathfrak{m})$ -Banach $L^\infty(\mathfrak{m})$ -module. When $\mathcal{A} = \text{LIP}_*(X)$, we just write $\text{Der}^\infty(X)$ and $|b|$ for brevity.

Definition 2.7 (Divergence). *Let (X, d, \mathfrak{m}) be a metric measure space. Let \mathcal{A} be a unital separating subalgebra of $\text{LIP}_*(X)$. Then we say that $b \in \text{Der}^\infty(X; \mathcal{A})$ has **divergence** $\text{div}(b) \in L^\infty(\mathfrak{m})$ if*

$$\int b(f) \, d\mu = - \int f \, \text{div}(b) \, d\mu \quad \text{for every } f \in \mathcal{A}.$$

We denote by $\text{Der}_\infty^\infty(X; \mathcal{A})$ the space of all derivations $b \in \text{Der}^\infty(X; \mathcal{A})$ having divergence.

Remark 2.8 (Independence of $\operatorname{div}(b)$ from \mathcal{A}). As \mathcal{A} is dense in $L^1(\mathbf{m})$ by (2.2), we have that $\operatorname{div}(b)$ is uniquely determined. In principle, $\operatorname{div}(b)$ may depend on the choice of the algebra \mathcal{A} . However, it is easy to check that if $\mathcal{A} \subseteq \tilde{\mathcal{A}}$ are unital separating subalgebras of $\operatorname{LIP}_*(X)$ and $b \in \operatorname{Der}_\infty^\infty(X; \tilde{\mathcal{A}})$, then $b \in \operatorname{Der}_\infty^\infty(X; \mathcal{A})$ and the two notions of divergence (in duality with \mathcal{A} and $\tilde{\mathcal{A}}$, respectively) coincide. Thus, with a slight abuse of notation, we will write only $\operatorname{div}(b)$ without stressing the dependence on the chosen algebra, since it will be clear from the context. ■

Note also that the space $\operatorname{Der}_\infty^\infty(X; \mathcal{A})$ is a module over the ring \mathcal{A} , thus in particular it is a vector subspace of $\operatorname{Der}^\infty(X; \mathcal{A})$. We denote by

$$\overline{\operatorname{Der}_\infty^\infty(X; \mathcal{A})}$$

the closure of $\operatorname{Der}_\infty^\infty(X; \mathcal{A})$ in $\operatorname{Der}^\infty(X; \mathcal{A})$. Note that $\overline{\operatorname{Der}_\infty^\infty(X; \mathcal{A})}$ is a Banach space with respect to the norm induced by $\operatorname{Der}^\infty(X; \mathcal{A})$.

3. METRIC BV AND $H^{1,1}$ SPACES

3.1. The space BV_H . A metric BV space defined in terms of approximating (locally) Lipschitz functions was introduced in [32] and revisited in [4]. Here, we consider its following variant:

Definition 3.1 (BV space via relaxation). *Let (X, d, \mathbf{m}) be a metric measure space. Let \mathcal{A} be a unital separating subalgebra of $\operatorname{LIP}_*(X)$. Then we define $\operatorname{BV}_H(X; \mathcal{A})$ as the space of all $f \in L^1(\mathbf{m})$ for which there exists $(f_n)_n \subseteq \mathcal{A}$ such that $f_n \rightarrow f$ in $L^1(\mathbf{m})$ and $\sup_{n \in \mathbb{N}} \int \operatorname{lip}_a(f_n) d\mathbf{m} < +\infty$.*

The **total variation** of a function $f \in \operatorname{BV}_H(X; \mathcal{A})$ is given by the quantity

$$\|\mathbf{D}f\|_{*, \mathcal{A}} := \inf \left\{ \liminf_{n \rightarrow \infty} \int \operatorname{lip}_a(f_n) d\mathbf{m} \mid (f_n)_n \subseteq \mathcal{A}, f_n \rightarrow f \text{ in } L^1(\mathbf{m}) \right\} < +\infty. \quad (3.1)$$

Clearly, there is a sequence $(f_n)_n \subseteq \mathcal{A}$ such that $f_n \rightarrow f$ in $L^1(\mathbf{m})$ and $\int \operatorname{lip}_a(f_n) d\mathbf{m} \rightarrow \|\mathbf{D}f\|_{*, \mathcal{A}}$. The **Cheeger \mathcal{A} -energy** functional $\mathcal{E}_\mathcal{A}: L^1(\mathbf{m}) \rightarrow [0, +\infty]$ is then defined as

$$\mathcal{E}_\mathcal{A}(f) := \begin{cases} \|\mathbf{D}f\|_{*, \mathcal{A}} & \text{if } f \in \operatorname{BV}_H(X; \mathcal{A}), \\ +\infty & \text{otherwise.} \end{cases}$$

It is a direct consequence of the definition that

$$\operatorname{BV}_H(X; \mathcal{A}) \subseteq \operatorname{BV}_H(X; \tilde{\mathcal{A}}), \quad \mathcal{E}_{\tilde{\mathcal{A}}} \leq \mathcal{E}_\mathcal{A} \quad \text{whenever } \mathcal{A} \subseteq \tilde{\mathcal{A}}. \quad (3.2)$$

In the case $\mathcal{A} = \operatorname{LIP}_*(X)$, we use the shorthand notations $\operatorname{BV}_H(X)$ and $\|\mathbf{D}f\|_*$. To any given function in $\operatorname{BV}_H(X)$, one can associate also a total variation measure, as follows:

Definition 3.2 (Total variation measure $|\mathbf{D}f|_*$). *Let (X, d, \mathbf{m}) be a metric measure space. Let $f \in \operatorname{BV}_H(X)$ be given. Then for any open set $U \subseteq X$ we define*

$$|\mathbf{D}f|_*(U) := \inf \left\{ \liminf_{n \rightarrow \infty} \int_U \operatorname{lip}_a(f_n) d\mathbf{m} \mid \operatorname{LIP}_*(U) \ni f_n \rightarrow f|_U \text{ in } L^1(\mathbf{m}|_U) \right\} \in [0, \|\mathbf{D}f\|_*].$$

Observe that $|\mathbf{D}f|_*(X) = \|\mathbf{D}f\|_*$. Furthermore, by arguing as in [4, Lemma 5.2] one can show that the set-function introduced in Definition 3.2 induces via Carathéodory's construction a finite Borel measure, still denoted by $|\mathbf{D}f|_*$. Videlicet, the Borel measure $|\mathbf{D}f|_*$ is characterised by

$$|\mathbf{D}f|_*(E) := \inf \{ |\mathbf{D}f|_*(U) \mid U \subseteq X \text{ open, } E \subseteq U \} \quad \text{for every Borel set } E \subseteq X.$$

This is shown in Appendix A, reporting a detailed version of the arguments in the proofs of [4, Lemmata 5.2 and 5.4] showing that they do not make use of the completeness nor separability of the ambient metric space. Moreover, even though in [4, Eq. (5.1)] the approximating sequence in the definition of $|\mathbf{D}f|_*(U)$ is not required to consist of bounded functions, it still coincides with our definition thanks to a standard truncation argument.

Remark 3.3. Assume $f \in \text{BV}_H(X)$ and $(f_n)_n \subseteq \text{LIP}_*(X)$ are chosen so that $f_n \rightarrow f$ in $L^1(\mathbf{m})$ and $\int \text{lip}_a(f_n) d\mathbf{m} \rightarrow \|\mathbf{D}f\|_*$. Then it holds that

$$\text{lip}_a(f_n)\mathbf{m} \rightharpoonup |\mathbf{D}f|_* \quad \text{weakly.}$$

Indeed, for any $U \subseteq X$ open we have $|\mathbf{D}f|_*(U) \leq \liminf_n \int_U \text{lip}_a(f_n) d\mathbf{m} = \liminf_n (\text{lip}_a(f_n)\mathbf{m})(U)$. Taking into account also that $(\text{lip}_a(f_n)\mathbf{m})(X) \rightarrow \|\mathbf{D}f\|_*(X)$, we conclude that $\text{lip}_a(f_n)\mathbf{m} \rightharpoonup |\mathbf{D}f|_*$ weakly, thanks to the Portmanteau theorem. \blacksquare

Remark 3.4. It follows from Remark 3.3 that

$$|\mathbf{D}f|_* \text{ is concentrated on } \text{spt}(\mathbf{m}) \text{ for every } f \in \text{BV}_H(X).$$

Indeed, as $X \setminus \text{spt}(\mathbf{m})$ is open, we have that $|\mathbf{D}f|_*(X \setminus \text{spt}(\mathbf{m})) \leq \liminf_n \int_{X \setminus \text{spt}(\mathbf{m})} \text{lip}_a(f_n) d\mathbf{m} = 0$. However, even though $\text{spt}(\mathbf{m})$ is separable, it could happen that $|\mathbf{D}f|_*$ is not a Radon measure, but under the additional assumption that (X, d) is complete it holds that $|\mathbf{D}f|_*$ is Radon (since in this case $\text{spt}(\mathbf{m})$ is complete and separable, thus it is a Radon space). \blacksquare

It is highly non-trivial to prove that, assuming the ambient metric space is complete, every function in $\text{BV}_H(X)$ can be approximated (on the whole X) by a sequence of bounded Lipschitz functions. The validity of this claim follows from the results of [16], as we are going to see:

Proposition 3.5. *Let (X, d, \mathbf{m}) be a metric measure space such that (X, d) is complete. Then*

$$\text{BV}_H(X) = \text{BV}_H(X; \text{LIP}_b(X))$$

and it holds that $\|\mathbf{D}f\|_{, \text{LIP}_b(X)} = \|\mathbf{D}f\|_*$ for every $f \in \text{BV}_H(X)$.*

Proof. Under the additional assumption that (X, d) is separable, the statement is a direct consequence of [16, Theorem 4.5.3]. Let us now show how to drop the separability assumption. For brevity, let us denote by S the support of \mathbf{m} . Defining the so-called *weak-BV spaces* BV_w as it is done in [16, Section 4.4.3] (or [4, Section 5.3]), by arguing as in [16, Section 4.4] one can check that

$$\text{BV}_H(X; \text{LIP}_b(X)) \subseteq \text{BV}_H(X) \subseteq \text{BV}_w(X),$$

as well as $|\mathbf{D}f|_w(X) \leq \|\mathbf{D}f\|_* \leq \|\mathbf{D}f\|_{*, \text{LIP}_b(X)}$ for every $f \in \text{BV}_H(X; \text{LIP}_b(X))$, where $|\mathbf{D}f|_w$ denotes the total variation measure corresponding to the weak-BV space $\text{BV}_w(X)$. Observe that $\text{BV}_H(X; \text{LIP}_b(X))$ can be identified with $\text{BV}_H(S; \text{LIP}_b(S))$ (thanks to [19]), and that $\text{BV}_w(X)$ can be identified with $\text{BV}_w(S)$ (since ∞ -test plans are concentrated on curves that lie in the support of \mathbf{m}); in both cases, also the total variations coincide. Since $\text{BV}_H(S; \text{LIP}_b(S)) = \text{BV}_w(S)$ and $|\mathbf{D}f|_w(S) = \|\mathbf{D}f\|_{*, \text{LIP}_b(S)}$ for every $f \in \text{BV}_w(S)$ by [16, Theorem 4.5.3] (together with the fact that S is complete and separable), we can finally conclude that the statement holds. \square

We point out that the completeness assumption in Proposition 3.5 cannot be dropped, as it is shown by the following example:

Example 3.6. Let us consider the space $X \subseteq \mathbb{R}$, which we define as

$$X := \bigcup_{n=1}^{\infty} (n - 2^{-n}, n + 2^{-n}) \setminus \{n\}.$$

We equip X with the restriction d of the Euclidean distance and the restriction \mathbf{m} of the one-dimensional Lebesgue measure. Note that (X, d, \mathbf{m}) is a metric measure space with (X, d) separable but non-complete. Let us now consider the function

$$f := \mathbb{1}_{\bigcup_{n=1}^{\infty} (n, n+2^{-n})} \in \text{LIP}_*(X).$$

Since $\text{lip}_a(f) \equiv 0$ we see that $f \in \text{BV}_H(X)$ with $\|\mathbf{D}f\|_* = 0$. However, we claim that $f \notin \text{BV}_H(X; \text{LIP}_b(X))$. To prove it, fix an arbitrary sequence $(f_j)_j \subseteq \text{LIP}_b(X)$ such that $f_j \rightarrow f$ in $L^1(\mathbf{m})$. Note that each f_j can be uniquely extended to a bounded Lipschitz function on $\bigcup_{n=1}^\infty [n - 2^{-n}, n + 2^{-n}]$, which we still denote by f_j . Up to a passing to a non-relabelled subsequence, we can assume that $\liminf_j \int \text{lip}_a(f_j) d\mathbf{m}$ is in fact a limit. Up to a further subsequence, we also have that $f_j \rightarrow f$ in the pointwise \mathbf{m} -a.e. sense. In particular, for any given $k \in \mathbb{N}$ we can find $j_k \in \mathbb{N}$ and $(t_\ell^k)_{\ell=1}^k, (s_\ell^k)_{\ell=1}^k \subseteq X$, with $\ell - 2^{-\ell} < t_\ell^k < \ell < s_\ell^k < \ell + 2^{-\ell}$ for all $\ell = 1, \dots, k$, such that $f_j(t_\ell^k) \leq \frac{1}{3}$ and $f_j(s_\ell^k) \geq \frac{2}{3}$ for every $\ell = 1, \dots, k$ and $j \geq j_k$. Then

$$\frac{1}{3} \leq f_j(s_\ell^k) - f_j(t_\ell^k) = \int_{t_\ell^k}^{s_\ell^k} f_j'(r) dr \leq \int_{\ell-2^{-\ell}}^{\ell+2^{-\ell}} \text{lip}_a(f_j)(r) dr \quad \text{for all } \ell = 1, \dots, k \text{ and } j \geq j_k.$$

Summing over $\ell = 1, \dots, k$ and letting $j \rightarrow \infty$, we deduce that

$$\lim_{j \rightarrow \infty} \int \text{lip}_a(f_j) d\mathbf{m} \geq \limsup_{j \rightarrow \infty} \sum_{\ell=1}^k \int_{\ell-2^{-\ell}}^{\ell+2^{-\ell}} \text{lip}_a(f_j)(r) dr \geq \frac{k}{3}.$$

Due to the arbitrariness of $k \in \mathbb{N}$, we conclude that $\lim_j \int \text{lip}_a(f_j) d\mathbf{m} = +\infty$ and thus accordingly $f \notin \text{BV}_H(X; \text{LIP}_b(X))$. \blacksquare

3.2. The space $H^{1,1}$. A metric Sobolev space defined in terms of approximating Lipschitz functions was introduced in [14] and revisited in [6]. Here, we consider its following variant:

Definition 3.7 (The space $H^{1,1}$). *Let (X, d, \mathbf{m}) be a metric measure space. Let \mathcal{A} be a unital separating subalgebra of $\text{LIP}_*(X)$. Then we define $H^{1,1}(X; \mathcal{A})$ as the space of all $f \in L^1(\mathbf{m})$ for which there exist a sequence $(f_n)_n \subseteq \mathcal{A}$ and a function $G \in L^1(\mathbf{m})^+$ such that $f_n \rightarrow f$ strongly in $L^1(\mathbf{m})$ and $\text{lip}_a(f_n) \rightharpoonup G$ weakly in $L^1(\mathbf{m})$.*

Given any $f \in H^{1,1}(X; \mathcal{A})$, we say that $\tilde{G} \in L^1(\mathbf{m})^+$ is a **relaxed \mathcal{A} -slope** of f if $\tilde{G} \geq G$ for some $G \in L^1(\mathbf{m})^+$ as in Definition 3.7. The set of all relaxed \mathcal{A} -slopes of f is a closed convex sublattice of $L^1(\mathbf{m})$, thus it admits a unique minimal element $|\mathbf{D}f|_{*, \mathcal{A}}$, which we call the **minimal relaxed \mathcal{A} -slope** of f . The space $H^{1,1}(X; \mathcal{A})$ is a Banach space if endowed with the norm

$$\|f\|_{H^{1,1}(X; \mathcal{A})} := \|f\|_{L^1(\mathbf{m})} + \| |\mathbf{D}f|_{*, \mathcal{A}} \|_{L^1(\mathbf{m})} \quad \text{for every } f \in H^{1,1}(X; \mathcal{A}).$$

Observe that $\mathcal{A} \subseteq H^{1,1}(X; \mathcal{A})$ and

$$|\mathbf{D}f|_{*, \mathcal{A}} \leq \text{lip}_a(f) \quad \text{for every } f \in \mathcal{A}. \quad (3.3)$$

Minimal relaxed \mathcal{A} -slopes satisfy also the following locality property: for any $f, g \in H^{1,1}(X; \mathcal{A})$,

$$|\mathbf{D}f|_{*, \mathcal{A}} = |\mathbf{D}g|_{*, \mathcal{A}} \quad \text{holds } \mathbf{m}\text{-a.e. on } \{f = g\}. \quad (3.4)$$

The interested reader can find a proof of the existence of the minimal relaxed \mathcal{A} -slope and of the completeness of $H^{1,p}(X; \mathcal{A})$ in [7, Section 5.2] for \mathcal{A} being the algebra of Lipschitz functions with bounded support, or in [36, Section 3.1] in case $p > 1$. The locality property of the relaxed \mathcal{A} -slope can be proven exactly as in [36, Theorem 3.1.12(b)], where it is proven for $p > 1$.

In the case where $\mathcal{A} = \text{LIP}_*(X)$, we use the shorthand notations $H^{1,1}(X)$ and $|\mathbf{D}f|_*$.

Remark 3.8. It holds that $H^{1,1}(X; \mathcal{A}) \subseteq \text{BV}_H(X; \mathcal{A})$ and

$$\|\mathbf{D}f\|_{*, \mathcal{A}} \leq \int |\mathbf{D}f|_{*, \mathcal{A}} d\mathbf{m} \quad \text{for every } f \in H^{1,1}(X; \mathcal{A}).$$

The proof follows immediately from the definitions of the spaces and the weak lower semicontinuity of the $L^1(\mathbf{m})$ -norm. \blacksquare

In accordance with [23, Definition 2.2.1], the metric Sobolev space $H^{1,1}(X; \mathcal{A})$ induces a space of ‘integrable 1-forms’, which is an $L^1(\mathfrak{m})$ -Banach $L^\infty(\mathfrak{m})$ -module called the *cotangent module*, and a *differential* map that underlies the minimal relaxed slopes. Videlicet:

Theorem 3.9 (Cotangent module). *Let (X, d, \mathfrak{m}) be a metric measure space. Let \mathcal{A} be a unital separating subalgebra of $\text{LIP}_*(X)$. Then there exist an $L^1(\mathfrak{m})$ -Banach $L^\infty(\mathfrak{m})$ -module $L^1(T^*X; \mathcal{A})$, which we call the \mathcal{A} -**cotangent module**, and a linear operator $d_{\mathcal{A}}: \mathcal{A} \rightarrow L^1(T^*X; \mathcal{A})$ such that:*

- i) $|d_{\mathcal{A}}f| = |Df|_{*, \mathcal{A}}$ for every $f \in \mathcal{A}$.
- ii) $\{d_{\mathcal{A}}f : f \in \mathcal{A}\}$ generates $L^1(T^*X; \mathcal{A})$.

The pair $(L^1(T^*X; \mathcal{A}), d_{\mathcal{A}})$ is unique up to a unique isomorphism: given any pair (\mathcal{M}, d) with the same properties, there exists a unique isomorphism of $L^1(\mathfrak{m})$ -Banach $L^\infty(\mathfrak{m})$ -modules $\Phi: L^1(T^*X; \mathcal{A}) \rightarrow \mathcal{M}$ such that $d = \Phi \circ d_{\mathcal{A}}$. Moreover, $d_{\mathcal{A}}$ satisfies the **Leibniz rule**, i.e.

$$d_{\mathcal{A}}(fg) = f d_{\mathcal{A}}g + g d_{\mathcal{A}}f \quad \text{for every } f, g \in \mathcal{A}. \quad (3.5)$$

Proof. This kind of construction is due to Gigli [23]; see also [22] or [24]. Technically speaking, the stated existence and uniqueness are direct consequences of [29, Theorem 3.19]. The Leibniz rule (3.5) follows from the locality (3.4) of $|Df|_{*, \mathcal{A}}$, arguing as in [23, Theorem 2.2.6]. \square

In analogy with [23, Definition 2.3.1], we then define the **Sobolev \mathcal{A} -tangent module** as

$$L^\infty(TX; \mathcal{A}) := L^1(T^*X; \mathcal{A})^*.$$

Recall that $L^\infty(TX; \mathcal{A})$ is an $L^\infty(\mathfrak{m})$ -Banach $L^\infty(\mathfrak{m})$ -module. Recall also that the operator

$$\text{INT}_{\mathcal{A}} := \text{INT}_{L^1(T^*X; \mathcal{A})}: L^\infty(TX; \mathcal{A}) \rightarrow L^1(T^*X; \mathcal{A})'$$

is an isometric isomorphism of Banach spaces, see (2.4). To be consistent with [23], we will denote by $\omega(v) \in L^1(\mathfrak{m})$ (instead of $v(\omega) \in L^1(\mathfrak{m})$) the ‘pointwise duality pairing’ between two elements $\omega \in L^1(T^*X; \mathcal{A})$ and $v \in L^\infty(TX; \mathcal{A})$, even though $L^\infty(TX; \mathcal{A})$ is the module dual of $L^1(T^*X; \mathcal{A})$ (and, in general, not the other way round).

3.3. The space BV_W . A notion of metric BV space in terms of an integration-by-parts formula involving derivations was introduced in [17, Definition 3.1]; see also [16]. We generalise it as follows:

Definition 3.10 (BV space via derivations). *Let (X, d, \mathfrak{m}) be a metric measure space. Let \mathcal{A} be a unital separating subalgebra of $\text{LIP}_*(X)$. Then we define $\text{BV}_W(X; \mathcal{A})$ as the space of all functions $f \in L^1(\mathfrak{m})$ for which there exists a bounded linear operator $\mathbf{L}_f: \text{Der}_\infty^\infty(X; \mathcal{A}) \rightarrow \mathcal{M}(X)$ such that*

$$\mathbf{L}_f(b)(X) = - \int f \operatorname{div}(b) \, d\mathfrak{m} \quad \text{for every } b \in \text{Der}_\infty^\infty(X; \mathcal{A}), \quad (3.6a)$$

$$\mathbf{L}_f(hb) = h \mathbf{L}_f(b) \quad \text{for every } h \in \mathcal{A} \text{ and } b \in \text{Der}_\infty^\infty(X; \mathcal{A}), \quad (3.6b)$$

$$|\mathbf{L}_f(b)|(X \setminus \operatorname{spt}(\mathfrak{m})) = 0 \quad \text{for every } b \in \text{Der}_\infty^\infty(X; \mathcal{A}). \quad (3.6c)$$

Remark 3.11. Given any function $f \in \text{BV}_W(X; \mathcal{A})$, the operator \mathbf{L}_f is uniquely determined. Indeed, if another operator $\tilde{\mathbf{L}}_f$ satisfies the same properties, then (3.6b) and (3.6a) imply that

$$\int h d(\mathbf{L}_f(b) - \tilde{\mathbf{L}}_f(b)) = \mathbf{L}_f(hb)(X) - \tilde{\mathbf{L}}_f(hb)(X) = - \int f \operatorname{div}(hb) \, d\mathfrak{m} + \int f \operatorname{div}(hb) \, d\mathfrak{m} = 0$$

for every $h \in \mathcal{A}$ and $b \in \text{Der}_\infty^\infty(X; \mathcal{A})$, whence it follows that $\mathbf{L}_f = \tilde{\mathbf{L}}_f$ thanks to (2.2). \blacksquare

By Remark 3.11, it makes sense to define the **total \mathcal{A} -variation** of any $f \in \text{BV}_W(X; \mathcal{A})$ as

$$V_{\mathcal{A}}(f) := \|\mathbf{L}_f\|_{\mathcal{L}(\overline{\text{Der}}_{\infty}^{\infty}(X; \mathcal{A}); \mathcal{M}(X))},$$

where we keep the notation \mathbf{L}_f to indicate the unique bounded linear operator from $\overline{\text{Der}}_{\infty}^{\infty}(X; \mathcal{A})$ to $\mathcal{M}(X)$ that extends \mathbf{L}_f . One can readily check that $\text{BV}_W(X; \mathcal{A})$ is a Banach space if endowed with the norm

$$\|f\|_{\text{BV}_W(X; \mathcal{A})} := \|f\|_{L^1(\mathfrak{m})} + V_{\mathcal{A}}(f) \quad \text{for every } f \in \text{BV}_W(X; \mathcal{A}).$$

Proposition 3.12. *Let (X, d, \mathfrak{m}) be a metric measure space. Let \mathcal{A} be a unital separating subalgebra of $\text{LIP}_{\star}(X)$. For any $f \in \text{BV}_W(X; \mathcal{A})$, we define its **total \mathcal{A} -variation measure** as*

$$|\mathbf{D}f|_{\mathcal{A}} := \bigvee \{ |\mathbf{L}_f(b)| \mid b \in \text{Der}_{\infty}^{\infty}(X; \mathcal{A}), |b|_{\mathcal{A}} \leq 1 \}.$$

Then $|\mathbf{D}f|_{\mathcal{A}}$ is a finite non-negative Borel measure on X such that $|\mathbf{D}f|_{\mathcal{A}}(X) = V_{\mathcal{A}}(f)$.

Remark 3.13. Since $\mathbf{L}_f(0)$ is the null measure and $\mathbf{L}_f(-b) = -\mathbf{L}_f(b)$ for every $b \in \text{Der}_{\infty}^{\infty}(X; \mathcal{A})$, the definition of $|\mathbf{D}f|_{\mathcal{A}}$ can be equivalently rewritten as

$$|\mathbf{D}f|_{\mathcal{A}}(E) = \sup \sum_{i=1}^n \mathbf{L}_f(b_i)(E_i) \quad \text{for every Borel set } E \subseteq X,$$

where the supremum is taken among all finite Borel partitions $(E_i)_{i=1}^n$ of E and all collections $(b_i)_{i=1}^n \subseteq \text{Der}_{\infty}^{\infty}(X; \mathcal{A})$ such that $|b_i|_{\mathcal{A}} \leq 1$ and E_i is contained in the support of the positive part of $\mathbf{L}_f(b_i)$. This is because, given $b \in \text{Der}_{\infty}^{\infty}(X; \mathcal{A})$ and a Borel subset $F \subseteq X$, we have that

$$|\mathbf{L}_f(b)|(F) = \mathbf{L}_f(b)(F \cap P_b) + \mathbf{L}_f(-b)(F \setminus P_b),$$

where P_b is the support of the positive part of $\mathbf{L}_f(b)$. Thus one can always replace the partition $(E_i)_{i=1}^n$ with $(E_i \cap P_{b_i})_{i=1}^n \cup (E_i \setminus P_{b_i})_{i=1}^n$, and the collection $(b_i)_{i=1}^n$ with $(b_i)_{i=1}^n \cup (-b_i)_{i=1}^n$. ■

Proof of Proposition 3.12. Let $\lambda < |\mathbf{D}f|_{\mathcal{A}}(X)$ be fixed. Taking into account Remark 3.13, we can find a Borel partition E_1, \dots, E_n of X and $b_1, \dots, b_n \in \text{Der}_{\infty}^{\infty}(X; \mathcal{A})$ such that $|b_i|_{\mathcal{A}} \leq 1$ for all $i = 1, \dots, n$ and $\sum_{i=1}^n \mathbf{L}_f(b_i)(E_i) > \lambda$. Letting $\mu := \sum_{i=1}^n |\mathbf{L}_f(b_i)|$ and $\delta := \sum_{i=1}^n \mathbf{L}_f(b_i)(E_i) - \lambda$, we know from Lemma 2.4 that there exist non-negative functions $\eta_1, \dots, \eta_n \in \mathcal{A}$ with $\sum_{i=1}^n \eta_i \leq 1$ such that $\|\eta_i - \mathbb{1}_{E_i}\|_{L^1(\mu)} \leq \delta/n$ for every $i = 1, \dots, n$. Letting $b := \sum_{i=1}^n \eta_i b_i \in \text{Der}_{\infty}^{\infty}(X; \mathcal{A})$, we have that $|b|_{\mathcal{A}} \leq \sum_{i=1}^n \eta_i |b_i|_{\mathcal{A}} \leq 1$ and

$$\left| \sum_{i=1}^n \mathbf{L}_f(b_i)(E_i) - \mathbf{L}_f(b)(X) \right| \leq \sum_{i=1}^n \int |\mathbb{1}_{E_i} - \eta_i| d|\mathbf{L}_f(b_i)| \leq \delta,$$

whence it follows that

$$\lambda = \sum_{i=1}^n \mathbf{L}_f(b_i)(E_i) - \delta \leq \mathbf{L}_f(b)(X) \leq |\mathbf{L}_f(b)|(X) \leq V_{\mathcal{A}}(f).$$

Letting $\lambda \nearrow |\mathbf{D}f|_{\mathcal{A}}(X)$, we deduce that $|\mathbf{D}f|_{\mathcal{A}}(X) \leq V_{\mathcal{A}}(f)$, thus $|\mathbf{D}f|_{\mathcal{A}}$ is finite. Finally, we have $|\mathbf{L}_f(b)|(X) \leq |\mathbf{D}f|_{\mathcal{A}}(X)$ for all $b \in \text{Der}_{\infty}^{\infty}(X; \mathcal{A})$ with $|b|_{\mathcal{A}} \leq 1$, thus $V_{\mathcal{A}}(f) \leq |\mathbf{D}f|_{\mathcal{A}}(X)$. □

Remark 3.14. Assuming in addition that (X, d) is complete, it holds that

$$|\mathbf{D}f|_{\mathcal{A}} \text{ is a Radon measure for every } f \in \text{BV}_W(X; \mathcal{A}).$$

Indeed, $|\mathbf{D}f|_{\mathcal{A}}$ is concentrated on $\text{spt}(\mathfrak{m})$ as a consequence of (3.6c), thus it is a Radon measure because $\text{spt}(\mathfrak{m})$ is complete and separable. ■

When $\mathcal{A} = \text{LIP}_{\star}(X)$, we just write $\text{BV}_W(X)$ and $|\mathbf{D}f|$ for brevity.

Lemma 3.15. *Let (X, d, \mathbf{m}) be a metric measure space. Assume that (X, d) is either complete or a Radon space. Then it holds that $BV_H(X) \subseteq BV_W(X)$ and*

$$|\mathbf{D}f| \leq |\mathbf{D}f|_* \quad \text{for every } f \in BV_H(X).$$

Proof. Let $f \in BV_H(X)$ be given. Fix any $b \in \text{Der}_\infty^\infty(X)$. Take a sequence $(f_n)_n \subseteq \text{LIP}_*(X)$ such that $f_n \rightarrow f$ in $L^1(\mathbf{m})$ and $\int \text{lip}_a(f_n) d\mathbf{m} \rightarrow \|\mathbf{D}f\|_*$. Accounting for Remark 3.3, we have that $\text{lip}_a(f_n)\mathbf{m} \rightharpoonup |\mathbf{D}f|_*$ weakly. Since $|\mathbf{D}f|_*$ is a Radon measure by Remark 3.4, we deduce from [9, Theorem 8.6.4] that $(\text{lip}_a(f_n)\mathbf{m})_n$ is a tight sequence. As $|b(f_n)\mathbf{m}| = |b(f_n)|\mathbf{m} \leq |b|\text{lip}_a(f_n)\mathbf{m}$ for every $n \in \mathbb{N}$, we deduce that $(b(f_n)\mathbf{m})_n$ is a tight sequence as well. Since $(b(f_n)\mathbf{m})_n$ is also bounded in total variation norm, by [9, Theorem 8.6.7] we deduce that, up to a non-relabelled subsequence, $b(f_n)\mathbf{m} \rightharpoonup \mathbf{L}_f(b)$ weakly for some $\mathbf{L}_f(b) \in \mathcal{M}(X)$. For any $h \in \text{LIP}_*(X)$, we can compute

$$\int h d\mathbf{L}_f(b) = \lim_{n \rightarrow \infty} \int h b(f_n) d\mathbf{m} = - \lim_{n \rightarrow \infty} \int f_n (h \text{div}(b) + b(h)) d\mathbf{m} = - \int f (h \text{div}(b) + b(h)) d\mathbf{m}.$$

Therefore, the measure $\mathbf{L}_f(b)$ is independent of the approximating sequence and $b(f_n)\mathbf{m} \rightharpoonup \mathbf{L}_f(b)$ holds for the original sequence $(f_n)_n$. In particular, we have that $\mathbf{L}_f(b)$ is concentrated on $\text{spt}(\mathbf{m})$, proving (3.6c). If $b, \tilde{b} \in \text{Der}_\infty^\infty(X)$ and $\lambda \in \mathbb{R}$, then $(\lambda b + \tilde{b})(f_n)\mathbf{m} = \lambda b(f_n)\mathbf{m} + \tilde{b}(f_n)\mathbf{m}$ weakly converges both to $\mathbf{L}_f(\lambda b + \tilde{b})$ and $\lambda \mathbf{L}_f(b) + \mathbf{L}_f(\tilde{b})$, so that \mathbf{L}_f is linear. Given $h, \tilde{h} \in \text{LIP}_*(X)$, we have that $\int \tilde{h} d\mathbf{L}_f(hb) = \lim_n \int \tilde{h}(hb)(f_n) d\mathbf{m} = \lim_n \int (\tilde{h}h)b(f_n) d\mathbf{m} = \int \tilde{h}h d\mathbf{L}_f(b) = \int \tilde{h} d(h \mathbf{L}_f(b))$, which yields (3.6b) thanks to (2.2). Moreover, for every $b \in \text{Der}_\infty^\infty(X)$ we can compute

$$\mathbf{L}_f(b)(X) = \lim_{n \rightarrow \infty} \int b(f_n) d\mathbf{m} = - \lim_{n \rightarrow \infty} \int f_n \text{div}(b) d\mathbf{m} = - \int f \text{div}(b) d\mathbf{m},$$

which gives (3.6a). Next, fix a derivation $b \in \text{Der}_\infty^\infty(X)$ with $|b| \leq 1$ and a compact set $K \subseteq X$. We claim that

$$\mathbf{L}_f(b)(K) \leq |\mathbf{D}f|_*(K), \quad (3.7)$$

whence (since $|\mathbf{L}_f(b)|$ is a Radon measure) it follows that $f \in BV_W(X)$ and $|\mathbf{D}f| \leq |\mathbf{D}f|_*$. To prove the claim, pick a sequence of open sets $(U_i)_i$ such that $K \subset U_i$ for every $i \in \mathbb{N}$ and satisfying $\mathbf{L}_f(b)(U_i) \rightarrow \mathbf{L}_f(b)(K)$, $|\mathbf{D}f|_*(U_i) \rightarrow |\mathbf{D}f|_*(K)$. Now fix $i \in \mathbb{N}$ and take a non-decreasing sequence $(\eta_{i,j})_j$ of Lipschitz cut-off functions $\eta_{i,j}: X \rightarrow [0, 1]$ such that $\eta_{i,j}(x) \nearrow \mathbb{1}_{U_i}(x)$ as $j \rightarrow \infty$ for every $x \in X$. Note that

$$\begin{aligned} \int \eta_{i,j} d\mathbf{L}_f(b) &= \lim_{n \rightarrow \infty} \int \eta_{i,j} b(f_n) d\mathbf{m} \leq \liminf_{n \rightarrow \infty} \int \eta_{i,j} |b| \text{lip}_a(f_n) d\mathbf{m} \\ &\leq \liminf_{n \rightarrow \infty} \int \eta_{i,j} \text{lip}_a(f_n) d\mathbf{m} = \int \eta_{i,j} d|\mathbf{D}f|_* \leq |\mathbf{D}f|_*(U_i) \end{aligned}$$

for every $i, j \in \mathbb{N}$. Applying the dominated convergence theorem, we deduce that

$$\mathbf{L}_f(b)(U_i) = \lim_{j \rightarrow \infty} \int \eta_{i,j} d\mathbf{L}_f(b) \leq |\mathbf{D}f|_*(U_i) \quad \text{for every } i \in \mathbb{N}.$$

Therefore, we can finally conclude that

$$\mathbf{L}_f(b)(K) = \lim_{i \rightarrow \infty} \mathbf{L}_f(b)(U_i) \leq \lim_{i \rightarrow \infty} |\mathbf{D}f|_*(U_i) = |\mathbf{D}f|_*(K),$$

which gives (3.7). Thus, the proof is complete. \square

Remark 3.16. In the case where (X, d) is complete, by slightly adapting the proof of Lemma 3.15 (and taking also Proposition 3.5 into account) one can prove that $BV_H(X) \subseteq BV_W(X; \text{LIP}_b(X))$ and $|\mathbf{D}f|_{\text{LIP}_b(X)} \leq |\mathbf{D}f|_*$ for every $f \in BV_H(X)$. \blacksquare

4. EQUIVALENCE RESULTS

Given a metric space (X, d) and $x \in X$, we define the 1-Lipschitz function $d_x : X \rightarrow [0, +\infty)$ as

$$d_x(y) := d(x, y) \quad \text{for every } y \in X.$$

Definition 4.1 (Good algebra). *Let (X, d, m) be a metric measure space. Let \mathcal{A} be a unital separating subalgebra of $LIP_b(X)$. Then we say that \mathcal{A} is a **good algebra** if, given any $x \in X$ and $r > 0$, there exist $(f_n)_n \subseteq \mathcal{A}$ and $\rho \in L^\infty(m)$ with $0 \leq \rho \leq 1$ such that $\|f_n - d_x \wedge r\|_{L^1(m)} \rightarrow 0$ and $\text{lip}_a(f_n) \rightharpoonup \rho$ weakly in $L^1(m)$.*

Note that a unital separating subalgebra \mathcal{A} of $LIP_b(X)$ is a good algebra precisely when

$$d_x \wedge r \in H^{1,1}(X; \mathcal{A}), \quad |D(d_x \wedge r)|_{*, \mathcal{A}} \leq 1 \quad \text{for every } x \in X \text{ and } r > 0.$$

Clearly, $LIP_b(X)$ itself is a good algebra.

Example 4.2 (Some good algebras). In [36] is introduced a notion of *compatible* algebra in the setting of *extended metric-topological measure spaces*. These are Hausdorff topological spaces (X, τ) endowed with a (possibly extended) metric d which has good compatibility properties with τ , together with a finite, non-negative, Radon measure m (on (X, τ)). Whenever τ coincides with the topology induced by d and d is finite, we recover our setting, i.e. (X, d, m) is a metric measure space in the sense of Definition 2.1. In this case, an algebra \mathcal{A} is said to be compatible if

$$d(x, y) = \sup \left\{ \frac{|f(x) - f(y)|}{\text{Lip}(f)} \mid f \in \mathcal{A}, \text{Lip}(f) \neq 0 \right\} \quad \text{for every } x, y \in X.$$

It is not difficult to check that this implies that the functions d_x , $x \in X$ can be approximated in the following sense: there exists a sequence $(f_n)_n \subseteq \mathcal{A}$ such that

$$f_n \rightarrow d_x \quad m\text{-a.e.}, \quad \text{lip}_a(f_n) \rightharpoonup \rho \text{ in } L^1(m) \quad \text{for some } 0 \leq \rho \leq 1.$$

This is precisely (even if there is considered only for $p > 1$) the density condition in [20, Theorem 2.13]. In turn this last condition implies that \mathcal{A} is a good algebra according to Definition 4.1, simply using a truncation argument of the distance. Therefore, a variety of algebras in many metric spaces are good algebras:

- (1) Smooth functions in Riemannian manifolds and Euclidean spaces [20, Remark 2.21].
- (2) Cylinder functions in Banach and Hilbert spaces [36, Example 2.1.19].
- (3) Algebras generated by approximating distances [36, Example 2.1.20].
- (4) Cylinder functions in the Wasserstein, Hellinger and Hellinger–Kantorovich metric spaces of (probability) measures [37, 20, 15]. ■

Remark 4.3. If \mathcal{A} is a good algebra, then the approximation $(f_n)_n$ of $d_x \wedge r$ can be also chosen so that $0 \leq f_n \leq r$ for every $n \in \mathbb{N}$. Indeed, letting $(f_n)_n$ be as in Definition 4.1, we have that $u_n := (f_n \wedge r) \vee 0 \rightarrow d_x \wedge r$ in $L^1(m)$. Thanks to [36, Lemma 2.1.26], we can find a sequence $(\tilde{g}_n)_n \subseteq \mathcal{A}$ such that $|\tilde{g}_n - u_n| \leq \frac{1}{n}$ and $\text{lip}_a(\tilde{g}_n) \leq \text{lip}_a(u_n)$ for all $n \in \mathbb{N}$. By the Dunford–Pettis theorem, we thus have (up to a non-relabelled subsequence) that $\text{lip}_a(\tilde{g}_n) \rightharpoonup \rho$ weakly in $L^1(m)$, for some function $\rho \in L^\infty(m)$ with $0 \leq \rho \leq 1$. Finally, letting $g_n := r(r + \frac{2}{n})^{-1}(\tilde{g}_n + \frac{1}{n})$ for every $n \in \mathbb{N}$, we obtain a sequence $(g_n)_n \subseteq \mathcal{A}$ with $0 \leq g_n \leq r$ such that $g_n \rightarrow d_x \wedge r$ strongly in $L^1(m)$ and $\text{lip}_a(g_n) \rightharpoonup \rho$ weakly in $L^1(m)$, as desired. ■

Proposition 4.4. *Let (X, d, m) be a metric measure space and \mathcal{A} a good subalgebra of $LIP_b(X)$. Let $f \in LIP_b(X)$ be given. Then there exists a sequence $(f_n)_n \subseteq \mathcal{A}$ such that the following hold:*

- i) $\inf_X f \leq f_n \leq \sup_X f$ for every $n \in \mathbb{N}$.
- ii) $f_n \rightarrow f$ in $L^1(m)$ as $n \rightarrow \infty$.

- iii) $(\text{lip}_a(f_n))_n$ is **dominated** in $L^1(\mathfrak{m})$, i.e. there exists $G \in L^1(\mathfrak{m})^+$ such that $\text{lip}_a(f_n) \leq G$ holds \mathfrak{m} -a.e. on X for every $n \in \mathbb{N}$.
- iv) $(\text{lip}_a(f_n) - \text{Lip}(f))^+ \rightarrow 0$ in $L^1(\mathfrak{m})$ as $n \rightarrow \infty$.

Proof. Without loss of generality, we can assume that $\text{Lip}(f) > 0$. Fix a sequence $(x_k)_k \subseteq X$ that is dense in $\text{spt}(\mathfrak{m})$. Define $L := \text{Lip}(f)$, $m := \inf_X f$, $M := \sup_X f$ and

$$\tilde{g}_k := (f(x_k) - Ld_{x_k}) \vee m = f(x_k) - L \left(d_{x_k} \wedge \frac{f(x_k) - m}{L} \right) \quad \text{for every } k \in \mathbb{N}.$$

Notice that $\tilde{g}_k \in \text{LIP}_b(X)$ and $\text{Lip}(\tilde{g}_k) \leq L$. Since \mathcal{A} is a good algebra, we can find $(f_k^j)_j \subseteq \mathcal{A}$ such that $f_k^j \rightarrow \tilde{g}_k$ in $L^1(\mathfrak{m})$ and $\text{lip}_a(f_k^j) \rightarrow |\text{D}\tilde{g}_k|_{*,\mathcal{A}} \leq L$ in $L^1(\mathfrak{m})$ as $j \rightarrow \infty$. By Remark 4.3, we can assume that $m \leq f_k^j \leq M$ for every $k, j \in \mathbb{N}$. Define

$$g_n := \tilde{g}_1 \vee \dots \vee \tilde{g}_n \in \text{LIP}_b(X) \quad \text{for every } n \in \mathbb{N}.$$

Given that $m \leq g_n \leq M$ for every $n \in \mathbb{N}$ and $g_n(x) \rightarrow f(x)$ for every $x \in \text{spt}(\mathfrak{m})$, we have that $g_n \rightarrow f$ in $L^1(\mathfrak{m})$ by the dominated convergence theorem. Now fix $n \in \mathbb{N}$. Since $f_1^j \vee \dots \vee f_n^j \rightarrow g_n$ in $L^1(\mathfrak{m})$ as $j \rightarrow \infty$, there exists $j(n) \in \mathbb{N}$ such that $\|f_1^{j(n)} \vee \dots \vee f_n^{j(n)} - g_n\|_{L^1(\mathfrak{m})} \leq 1/n$ and

$$\|h_n^k\|_{L^1(\mathfrak{m})} \leq \frac{1}{n2^n} \quad \text{for every } k = 1, \dots, n, \text{ where } h_n^k := |\text{lip}_a(f_k^{j(n)}) - |\text{D}\tilde{g}_k|_{*,\mathcal{A}}|.$$

By [36, Lemma 2.1.26], we can find a function $\tilde{f}_n \in \mathcal{A}$ such that $\sup_X |f_1^{j(n)} \vee \dots \vee f_n^{j(n)} - \tilde{f}_n| \leq 1/n$ and $\text{lip}_a(\tilde{f}_n) \leq \text{lip}_a(f_1^{j(n)} \vee \dots \vee f_n^{j(n)})$. In particular, $m - \frac{1}{n} \leq \tilde{f}_n \leq M + \frac{1}{n}$ for every $n \in \mathbb{N}$. It also follows that $\|\tilde{f}_n - g_n\|_{L^1(\mathfrak{m})} \leq (1 + \mathfrak{m}(X))/n$ and

$$\text{lip}_a(\tilde{f}_n) \leq (|\text{D}\tilde{g}_1|_{*,\mathcal{A}} + h_n^1) \vee \dots \vee (|\text{D}\tilde{g}_n|_{*,\mathcal{A}} + h_n^n) \leq L + h_n^1 \vee \dots \vee h_n^n \leq L + h_n^1 + \dots + h_n^n.$$

Note that $h_n := h_n^1 + \dots + h_n^n$ satisfies $\|h_n\|_{L^1(\mathfrak{m})} \leq 2^{-n}$, thus $h := \sum_{n=1}^{\infty} h_n$ belongs to $L^1(\mathfrak{m})$. Therefore, $\tilde{f}_n \rightarrow f$ in $L^1(\mathfrak{m})$ as $n \rightarrow \infty$ and $\text{lip}_a(\tilde{f}_n) \leq L + h \in L^1(\mathfrak{m})$ for every $n \in \mathbb{N}$. Moreover, we have that $(\text{lip}_a(\tilde{f}_n) - L)^+ \leq h_n \rightarrow 0$ in $L^1(\mathfrak{m})$ as $n \rightarrow \infty$. Finally, defining $(f_n)_n \subseteq \mathcal{A}$ as

$$f_n := c_n \left(\tilde{f}_n - m + \frac{1}{n} \right) + m, \quad \text{where we set } c_n := \frac{M - m}{M - m + \frac{2}{n}},$$

one can then easily check that $(f_n)_n$ satisfies i), ii), iii) and iv). The statement is achieved. \square

Theorem 4.5. *Let (X, d, \mathfrak{m}) be a metric measure space. Let \mathcal{A} be a good subalgebra of $\text{LIP}_b(X)$. Define the operator $\phi_{\mathcal{A}} : \text{Der}_{\infty}^{\infty}(X; \text{LIP}_b(X)) \rightarrow \text{Der}_{\infty}^{\infty}(X; \mathcal{A})$ as*

$$\phi_{\mathcal{A}}(b) := b|_{\mathcal{A}} \in \text{Der}_{\infty}^{\infty}(X; \mathcal{A}) \quad \text{for every } b \in \text{Der}_{\infty}^{\infty}(X; \text{LIP}_b(X)).$$

Then $\phi_{\mathcal{A}}$ is an isomorphism of \mathcal{A} -modules such that the identity $\text{div}(\phi_{\mathcal{A}}(b)) = \text{div}(b)$ holds for every $b \in \text{Der}_{\infty}^{\infty}(X; \text{LIP}_b(X))$. Moreover, it holds that

$$|\phi_{\mathcal{A}}(b)|_{\mathcal{A}} = |b| \quad \text{for every } b \in \text{Der}_{\infty}^{\infty}(X; \text{LIP}_b(X)).$$

Proof. Let us write $L_X := \text{LIP}_b(X)$ for brevity. First of all, notice that $\phi_{\mathcal{A}}(b) \in \text{Der}_{\infty}^{\infty}(X; \mathcal{A})$ and $|\phi_{\mathcal{A}}(b)|_{\mathcal{A}} \leq |b|$ for every $b \in \text{Der}_{\infty}^{\infty}(X; L_X)$. Moreover, the operator $\phi_{\mathcal{A}}$ is a homomorphism of \mathcal{A} -modules by construction.

To conclude, it remains to show that if $b \in \text{Der}_{\infty}^{\infty}(X; \mathcal{A})$ is given, there exists $\bar{b} \in \text{Der}_{\infty}^{\infty}(X; L_X)$ such that $\phi_{\mathcal{A}}(\bar{b}) = b$ and $|\bar{b}| \leq |b|_{\mathcal{A}}$. To prove it, fix a function $f \in L_X$ and take a sequence $(f_n)_n \subseteq \mathcal{A}$ that approximates f in the sense of Proposition 4.4. Since $|b(f_n)| \leq |b|_{\mathcal{A}} \text{lip}_a(f_n)$ for every $n \in \mathbb{N}$, we have that $(b(f_n))_n$ is dominated in $L^1(\mathfrak{m})$, thus the Dunford–Pettis theorem ensures that we can extract a subsequence $(f_{n_k})_k$ such that $b(f_{n_k}) \rightharpoonup \bar{b}(f)$ in $L^1(\mathfrak{m})$, for some

limit function $\bar{b}(f) \in L^1(\mathbf{m})$. Note that whenever a sequence $(\tilde{f}_i)_i \subseteq \mathcal{A}$ and a function $h \in L^1(\mathbf{m})$ satisfy $\tilde{f}_i \rightharpoonup f$ and $b(\tilde{f}_i) \rightharpoonup h$ weakly in $L^1(\mathbf{m})$, for any $g \in \mathcal{A}$ we have that

$$\begin{aligned} \int hg \, d\mathbf{m} &= \lim_{i \rightarrow \infty} \int g b(\tilde{f}_i) \, d\mathbf{m} = \lim_{i \rightarrow \infty} \int b(g\tilde{f}_i) - \tilde{f}_i b(g) \, d\mathbf{m} \\ &= - \lim_{i \rightarrow \infty} \int \tilde{f}_i (g \operatorname{div}(b) + b(g)) \, d\mathbf{m} = - \int f(g \operatorname{div}(b) + b(g)) \, d\mathbf{m}. \end{aligned} \quad (4.1)$$

This shows that $\bar{b}(f)$ is independent of the chosen subsequence $(f_{n_k})_k$ and that the following holds: if $(\tilde{f}_i)_i \subseteq \mathcal{A}$ satisfies $\tilde{f}_i \rightharpoonup f$ weakly in $L^1(\mathbf{m})$ and $(b(\tilde{f}_i))_i$ is dominated in $L^1(\mathbf{m})$, then $b(\tilde{f}_i) \rightharpoonup \bar{b}(f)$ weakly in $L^1(\mathbf{m})$. In particular, we have that $\bar{b}|_{\mathcal{A}} = b$. Let us now check that $\bar{b} \in \operatorname{Der}_{\infty}^{\infty}(X; L_X)$:

- Fix $f, g \in L_X$ and $\alpha, \beta \in \mathbb{R}$. Let $(f_n)_n \subseteq \mathcal{A}$ (resp. $(g_n)_n \subseteq \mathcal{A}$) be an approximating sequence for f (resp. g) as in Proposition 4.4. Letting $n \rightarrow \infty$ in the identity $b(\alpha f_n + \beta g_n) = \alpha b(f_n) + \beta b(g_n)$, we obtain that $\bar{b}(\alpha f + \beta g) = \alpha \bar{b}(f) + \beta \bar{b}(g)$, proving that \bar{b} is linear.
- Note that whenever $f \in L_X$ and $(f_n)_n \subseteq \mathcal{A}$ is an approximating sequence for f as in Proposition 4.4, we have that

$$\int \bar{b}(f) \, d\mathbf{m} = \lim_{n \rightarrow \infty} \int b(f_n) \, d\mathbf{m} = - \lim_{n \rightarrow \infty} \int f_n \operatorname{div}(b) \, d\mathbf{m} = - \int f \operatorname{div}(b) \, d\mathbf{m}. \quad (4.2)$$

- We claim that

$$\bar{b}(fg) = f b(g) + g \bar{b}(f) \quad \text{for every } f \in L_X \text{ and } g \in \mathcal{A}. \quad (4.3)$$

Indeed, letting $(f_n)_n \subseteq \mathcal{A}$ be an approximating sequence for f as in Proposition 4.4, we have that $b(f_n g) = f_n b(g) + g b(f_n)$ for every $n \in \mathbb{N}$, which gives $\bar{b}(fg) = f b(g) + g \bar{b}(f)$.

- Next, we claim that \bar{b} has the following property: if $f \in L_X$ and $(\tilde{f}_i)_i \subseteq L_X$ are such that $\tilde{f}_i \rightharpoonup f$ weakly in $L^1(\mathbf{m})$ and $(\bar{b}(\tilde{f}_i))_i$ is dominated in $L^1(\mathbf{m})$, then $\bar{b}(\tilde{f}_i) \rightharpoonup \bar{b}(f)$ weakly in $L^1(\mathbf{m})$. To prove it, one can argue as in (4.1), by using (4.2) and (4.3).
- Consequently, we can show that \bar{b} satisfies the Leibniz rule, i.e.

$$\bar{b}(fg) = f \bar{b}(g) + g \bar{b}(f) \quad \text{for every } f, g \in L_X. \quad (4.4)$$

Indeed, letting $(g_n)_n \subseteq \mathcal{A}$ be an approximating sequence for g as in Proposition 4.4, we know from (4.3) that $\bar{b}(fg_n) = f b(g_n) + g_n \bar{b}(f)$ for every $n \in \mathbb{N}$, whence (4.4) follows.

- We also claim that

$$|\bar{b}(f)| \leq \operatorname{Lip}(f) |b|_{\mathcal{A}} \quad \text{for every } f \in L_X. \quad (4.5)$$

To prove it, take an approximating sequence $(f_n)_n \subseteq \mathcal{A}$ for f as in Proposition 4.4 and set $h_n := (\operatorname{lip}_a(f_n) - \operatorname{Lip}(f))^+$. Letting $n \rightarrow \infty$ in $\bar{b}(f_n) \leq |b|_{\mathcal{A}} \operatorname{lip}_a(f_n) \leq |b|_{\mathcal{A}} (\operatorname{Lip}(f) + h_n)$, we get $\bar{b}(f) \leq \operatorname{Lip}(f) |b|_{\mathcal{A}}$. Doing the same for $-f$ instead, we obtain that $-\bar{b}(f) \leq \operatorname{Lip}(f) |b|_{\mathcal{A}}$, proving (4.5).

- Fix an equi-bounded, equi-Lipschitz sequence $(f_n)_n \subseteq L_X$ converging pointwise to $f \in L_X$. Since $|\bar{b}(f_n)| \leq \operatorname{Lip}(f_n) |b|_{\mathcal{A}}$ by (4.5), the sequence $(\bar{b}(f_n))_n$ is bounded in $L^{\infty}(\mathbf{m})$, thus it is dominated in $L^1(\mathbf{m})$. Given that $f_n \rightarrow f$ in $L^1(\mathbf{m})$ (by the dominated convergence theorem), we have that $\bar{b}(f_n) \rightharpoonup \bar{b}(f)$ weakly in $L^1(\mathbf{m})$ by the fourth point above. Since $L^1(\mathbf{m})$ is separable, by the Banach-Alaoglu theorem we can extract a subsequence $(f_{n_k})_k$ such that $(\bar{b}(f_{n_k}))_k$ has a weak* limit G in $L^{\infty}(\mathbf{m})$. In particular, $\bar{b}(f_{n_k}) \rightharpoonup G$ weakly in $L^1(\mathbf{m})$, thus $G = \bar{b}(f)$ and accordingly the original sequence $\bar{b}(f_n)$ weakly* converges to $\bar{b}(f)$ in $L^{\infty}(\mathbf{m})$. This proves that \bar{b} is weakly* continuous (in the sense of [7, Lemma 4.12 i]). By suitably adapting the proof of [7, Corollary 4.14] (exploiting the fact that $\operatorname{spt}(\mathbf{m})$ is separable), we can conclude that $|\bar{b}(f)| \leq |b|_{\mathcal{A}} \operatorname{lip}_a(f)$ for every $f \in L_X$.

All in all, we showed that $\bar{b} \in \operatorname{Der}_{\infty}^{\infty}(X; L_X)$, $\phi_{\mathcal{A}}(\bar{b}) = b$ and $|\bar{b}| \leq |b|_{\mathcal{A}}$. The proof is complete. \square

Corollary 4.6. *Let (X, d, m) be a metric measure space. Let \mathcal{A} be a good subalgebra of $\text{LIP}_b(X)$. Then $\text{BV}_W(X; \text{LIP}_b(X)) \subseteq \text{BV}_W(X; \mathcal{A})$ and $|\mathbf{D}f|_{\mathcal{A}} \leq |\mathbf{D}f|_{\text{LIP}_b(X)}$ for all $f \in \text{BV}_W(X; \text{LIP}_b(X))$.*

Proof. Let $f \in \text{BV}_W(X; \text{LIP}_b(X))$ and let $\mathbf{L}_f: \text{Der}_{\infty}^{\infty}(X; \text{LIP}_b(X)) \rightarrow \mathcal{M}(X)$ be as in Definition 3.10 for f and the algebra $\text{LIP}_b(X)$. We define the operator $\tilde{\mathbf{L}}_f: \text{Der}_{\infty}^{\infty}(X; \mathcal{A}) \rightarrow \mathcal{M}(X)$ as

$$\tilde{\mathbf{L}}_f(b) := \mathbf{L}_f(\phi_{\mathcal{A}}^{-1}(b)) \quad \text{for every } b \in \text{Der}_{\infty}^{\infty}(X; \mathcal{A}),$$

where $\phi_{\mathcal{A}}$ is as in Theorem 4.5. It is easy to check that $\tilde{\mathbf{L}}_f$ satisfies Definition 3.10 for f and the algebra \mathcal{A} . The inequality between the total variation measures follows immediately from the properties of $\phi_{\mathcal{A}}$. \square

Let us now fix a unital separating subalgebra \mathcal{A} of $\text{LIP}_{\star}(X)$. We can regard $d_{\mathcal{A}}$ as an unbounded linear operator $d_{\mathcal{A}}: L^1(m) \rightarrow L^1(T^*X; \mathcal{A})$ whose domain is $D(d_{\mathcal{A}}) = \mathcal{A}$. Given that \mathcal{A} is dense in $L^1(m)$, we have that $d_{\mathcal{A}}$ is densely defined, thus accordingly its adjoint operator

$$d_{\mathcal{A}}^*: L^1(T^*X; \mathcal{A})' \rightarrow L^{\infty}(m)$$

is well defined. Recall that the domain $D(d_{\mathcal{A}}^*)$ of $d_{\mathcal{A}}^*$ is a vector subspace of $L^1(T^*X; \mathcal{A})'$, and that $d_{\mathcal{A}}^*$ is characterised by $\int f d_{\mathcal{A}}^* V dm = V(d_{\mathcal{A}} f)$ for every $V \in D(d_{\mathcal{A}}^*)$ and $f \in \mathcal{A}$.

Lemma 4.7. *Let (X, d, m) be a metric measure space. Let \mathcal{A} be a unital separating subalgebra of $\text{LIP}_{\star}(X)$. Fix any $v \in L^{\infty}(TX; \mathcal{A})$ with $\text{INT}_{\mathcal{A}}(v) \in D(d_{\mathcal{A}}^*)$. Let us define $b_v: \mathcal{A} \rightarrow L^1(m)$ as*

$$b_v(f) := d_{\mathcal{A}} f(v) \in L^1(m) \quad \text{for every } f \in \mathcal{A}.$$

Then $b_v \in \text{Der}_{\infty}^{\infty}(X; \mathcal{A})$. Moreover, it holds that $|b_v|_{\mathcal{A}} \leq |v|$ and $\text{div}(b_v) = -d_{\mathcal{A}}^(\text{INT}_{\mathcal{A}}(v))$.*

Proof. First, observe that $d_{\mathcal{A}} f(v) \leq |v| |d_{\mathcal{A}} f| \leq |v| \text{lip}_a(f)$ for every $f \in \mathcal{A}$, so that $b_v(f) \in L^1(m)$ and b_v satisfies Definition 2.6 i) with $G = |v|$. The linearity of b_v follows from the linearity of $d_{\mathcal{A}}$. For any $f, g \in \mathcal{A}$, we deduce from (3.5) that

$$b_v(fg) = d_{\mathcal{A}}(fg)(v) = f d_{\mathcal{A}} g(v) + g d_{\mathcal{A}} f(v) = f b_v(g) + g b_v(f),$$

which proves that b_v satisfies Definition 2.6 ii), thus accordingly $b_v \in \text{Der}_{\infty}^{\infty}(X; \mathcal{A})$ and $|b_v|_{\mathcal{A}} \leq |v|$. Finally, recalling the definitions of $d_{\mathcal{A}}^*$ and $\text{INT}_{\mathcal{A}}$, for any $f \in \mathcal{A}$ we can compute

$$\int b_v(f) dm = \int d_{\mathcal{A}} f(v) dm = \text{INT}_{\mathcal{A}}(v)(d_{\mathcal{A}} f) = \int f d_{\mathcal{A}}^*(\text{INT}_{\mathcal{A}}(v)) dm,$$

which gives that $b_v \in \text{Der}_{\infty}^{\infty}(X; \mathcal{A})$ and $\text{div}(b_v) = -d_{\mathcal{A}}^*(\text{INT}_{\mathcal{A}}(v))$. The proof is thus complete. \square

We now define the **pre-Cheeger \mathcal{A} -energy** functional $\tilde{\mathcal{E}}_{\mathcal{A}}: L^1(m) \rightarrow [0, +\infty]$ as

$$\tilde{\mathcal{E}}_{\mathcal{A}}(f) := \begin{cases} \int |\mathbf{D}f|_{*, \mathcal{A}} dm & \text{if } f \in \mathcal{A}, \\ +\infty & \text{otherwise.} \end{cases}$$

It can happen that $\tilde{\mathcal{E}}_{\mathcal{A}}|_{\mathcal{A}} \neq \mathcal{E}_{\mathcal{A}}|_{\mathcal{A}}$, but $\mathcal{E}_{\mathcal{A}}$ is the $L^1(m)$ -lower semicontinuous envelope of $\tilde{\mathcal{E}}_{\mathcal{A}}$ (thanks to (3.3) and to Remark 3.8). Notice also that the functional $\tilde{\mathcal{E}}_{\mathcal{A}}$ can be expressed as

$$\tilde{\mathcal{E}}_{\mathcal{A}} = \|\cdot\|_{L^1(T^*X; \mathcal{A})} \circ d_{\mathcal{A}}. \quad (4.6)$$

We denote by $n_{\mathcal{A}}^*$ the Fenchel conjugate of $n_{\mathcal{A}} := \|\cdot\|_{L^1(T^*X; \mathcal{A})}: L^1(T^*X; \mathcal{A}) \rightarrow [0, +\infty)$, i.e.

$$n_{\mathcal{A}}^*(V) := \sup_{\omega \in L^1(T^*X; \mathcal{A})} (V(\omega) - \|\omega\|_{L^1(T^*X; \mathcal{A})}) \quad \text{for every } V \in L^1(T^*X; \mathcal{A})'.$$

Straightforward computations give that

$$n_{\mathcal{A}}^*(V) = \begin{cases} 0 & \text{for every } V \in L^1(T^*X; \mathcal{A})' \text{ with } \|V\|_{L^1(T^*X; \mathcal{A})'} \leq 1, \\ +\infty & \text{for every } V \in L^1(T^*X; \mathcal{A})' \text{ with } \|V\|_{L^1(T^*X; \mathcal{A})'} > 1. \end{cases} \quad (4.7)$$

Theorem 4.8. *Let (X, d, \mathbf{m}) be a metric measure space. Let \mathcal{A} be a unital separating subalgebra of $\text{LIP}_*(X)$. Then it holds that $\text{BV}_W(X; \mathcal{A}) \subseteq \text{BV}_H(X; \mathcal{A})$ and*

$$\|\mathbf{D}f\|_{*, \mathcal{A}} \leq |\mathbf{D}f|_{\mathcal{A}}(X) \quad \text{for every } f \in \text{BV}_W(X; \mathcal{A}).$$

Proof. Fix any $f \in \text{BV}_W(X; \mathcal{A})$. Since $\mathcal{E}_{\mathcal{A}}$ is convex and $L^1(\mathbf{m})$ -lower semicontinuous, it coincides with the double Fenchel conjugate $\tilde{\mathcal{E}}_{\mathcal{A}}^{**} := (\tilde{\mathcal{E}}_{\mathcal{A}}^*)^*$ of $\tilde{\mathcal{E}}_{\mathcal{A}}$. Therefore, using (4.6), [10, Theorem 5.1], (4.7) and Lemma 4.7, we obtain that

$$\begin{aligned} \mathcal{E}_{\mathcal{A}}(f) &= \tilde{\mathcal{E}}_{\mathcal{A}}^{**}(f) = \sup_{g \in L^\infty(\mathbf{m})} \left(\int f g \, d\mathbf{m} - \tilde{\mathcal{E}}_{\mathcal{A}}^*(g) \right) = \sup_{g \in L^\infty(\mathbf{m})} \left(\int f g \, d\mathbf{m} - (\mathbf{n}_{\mathcal{A}} \circ d_{\mathcal{A}})^*(g) \right) \\ &= \sup_{g \in L^\infty(\mathbf{m})} \left(\int f g \, d\mathbf{m} - \inf_{\substack{V \in D(d_{\mathcal{A}}^*) \\ d_{\mathcal{A}}^* V = g}} \mathbf{n}_{\mathcal{A}}^*(V) \right) = \sup_{V \in D(d_{\mathcal{A}}^*)} \left(\int f d_{\mathcal{A}}^* V \, d\mathbf{m} - \mathbf{n}_{\mathcal{A}}^*(V) \right) \\ &\leq \sup_{\substack{b \in \text{Der}_\infty^\infty(X; \mathcal{A}) \\ |b|_{\mathcal{A}} \leq 1}} \left(- \int f \, \text{div}(b) \, d\mathbf{m} \right) = \sup_{\substack{b \in \text{Der}_\infty^\infty(X; \mathcal{A}) \\ |b|_{\mathcal{A}} \leq 1}} \mathbf{L}_f(b)(X) = |\mathbf{D}f|_{\mathcal{A}}(X). \end{aligned}$$

This gives that $f \in \text{BV}_H(X; \mathcal{A})$ and $\|\mathbf{D}f\|_{*, \mathcal{A}} \leq |\mathbf{D}f|_{\mathcal{A}}(X)$, thus proving the statement. \square

Theorem 4.9. *Let (X, d, \mathbf{m}) be a metric measure space. Assume that (X, d) is either complete or a Radon space. Then it holds that*

$$\text{BV}_H(X) = \text{BV}_W(X)$$

and we have that $|\mathbf{D}f| = |\mathbf{D}f|_*$ for every $f \in \text{BV}_H(X)$.

Proof. It follows from Lemma 3.15 and Theorem 4.8. \square

Theorem 4.10. *Let (X, d, \mathbf{m}) be a metric measure space. Assume that (X, d) is complete. Let \mathcal{A} be a good subalgebra of $\text{LIP}_b(X)$. Then it holds that*

$$\text{BV}_H(X; \mathcal{A}) = \text{BV}_H(X) = \text{BV}_W(X; \mathcal{A}) = \text{BV}_W(X; \text{LIP}_b(X)).$$

Moreover, we have that $\|\mathbf{D}f\|_{*, \mathcal{A}} = \|\mathbf{D}f\|_* = |\mathbf{D}f|_{\mathcal{A}}(X) = |\mathbf{D}f|_{\text{LIP}_b(X)}(X)$ and $|\mathbf{D}f|_* = |\mathbf{D}f|_{\mathcal{A}} = |\mathbf{D}f|_{\text{LIP}_b(X)}$ for every $f \in \text{BV}_H(X)$.

Proof. First of all, observe that for any unital separating subalgebra $\mathcal{A} \subseteq \text{LIP}_b(X)$ it holds that

$$\begin{aligned} \text{BV}_W(X; \mathcal{A}) &\subseteq \text{BV}_H(X; \mathcal{A}) \subseteq \text{BV}_H(X) \subseteq \text{BV}_W(X; \text{LIP}_b(X)), \\ |\mathbf{D}f|_{\text{LIP}_b(X)}(X) &\leq \|\mathbf{D}f\|_* \leq \|\mathbf{D}f\|_{*, \mathcal{A}} \leq |\mathbf{D}f|_{\mathcal{A}}(X) \quad \text{for every } f \in \text{BV}_W(X; \mathcal{A}). \end{aligned} \tag{4.8}$$

The above inclusions (from left to right) and the above inequalities (from right to left) follow from Theorem 4.8, (3.2) and Remark 3.16. If \mathcal{A} is a good subalgebra, we know from Corollary 4.6 that $\text{BV}_W(X; \text{LIP}_b(X)) \subseteq \text{BV}_W(X; \mathcal{A})$ and $|\mathbf{D}f|_{\mathcal{A}} \leq |\mathbf{D}f|_{\text{LIP}_b(X)}$ for every $f \in \text{BV}_W(X; \text{LIP}_b(X))$. Therefore, the above inclusions and inequalities are all equalities. The statement follows. \square

We conclude by formulating a partial converse of Theorem 4.10:

Proposition 4.11. *Let (X, d, \mathbf{m}) be a metric measure space. Let \mathcal{A} be a unital separating subalgebra of $\text{LIP}_*(X)$. If $\text{BV}_H(X; \mathcal{A}) = \text{BV}_H(X)$ and $\|\mathbf{D}f\|_{*, \mathcal{A}} = \|\mathbf{D}f\|_*$ for every $f \in \text{BV}_H(X; \mathcal{A})$, then \mathcal{A} is a **weakly good subalgebra**: for every $x \in X$ and for every $r > 0$, there exist $(f_n)_n \subseteq \mathcal{A}$ and $\rho \in L^\infty(\mathbf{m})$ with $0 \leq \rho \leq 1$ such that $\|f_n - d_x \wedge r\|_{L^1(\mathbf{m})} \rightarrow 0$ and $\text{lip}_a(f_n)\mathbf{m} \rightharpoonup \rho\mathbf{m}$ weakly.*

Proof. Fix any $x \in X$ and $r > 0$. Given that $d_x \wedge r \in \text{LIP}_*(X) \subseteq \text{BV}_H(X) = \text{BV}_H(X; \mathcal{A})$ and $\|\mathbf{D}(d_x \wedge r)\|_{*, \mathcal{A}} = |\mathbf{D}(d_x \wedge r)|_*(X)$, we can find $(f_n)_n \subseteq \mathcal{A}$ such that $f_n \rightarrow d_x \wedge r$ in $L^1(\mathbf{m})$ and $\int \text{lip}_a(f_n) \, d\mathbf{m} \rightarrow |\mathbf{D}(d_x \wedge r)|_*(X)$. It follows from Remark 3.3 that $\text{lip}_a(f_n)\mathbf{m} \rightharpoonup |\mathbf{D}(d_x \wedge r)|_*$ weakly. Since $|\mathbf{D}(d_x \wedge r)|_* \leq \mathbf{m}$ as a direct consequence of the definition, we proved the statement. \square

APPENDIX A. THE TOTAL VARIATION MEASURE

We report here a detailed version of the arguments in the proofs of [4, Lemmata 5.2 and 5.4] showing that they do not make use of the completeness nor separability of the ambient metric space, and also that they still work with our notion of asymptotic slope and for the approximating class being $\text{LIP}_*(X)$ and not $\text{LIP}_{loc}(X)$.

Lemma A.1. *Let (X, d, m) be a metric measure space and let $M, N \subseteq X$ be open sets such that $\eta := d(M \setminus N, N \setminus M) > 0$. Set*

$$\phi(x) := \frac{3}{\eta} \min \left\{ (d(x, N \setminus M) - \eta/3)^+, \frac{\eta}{3} \right\} \quad \text{for every } x \in M \cup N,$$

and $H := \phi^{-1}((0, 1))$. Then H is an open subset of $M \cap N$ with $d(H, (M \setminus N) \cup (N \setminus M)) > 0$. Whenever $u \in \text{LIP}_*(M)$ and $v \in \text{LIP}_*(N)$, setting

$$w(x) := (1 - \phi(x))u(x) + \phi(x)v(x) \quad \text{for every } x \in M \cup N,$$

(where we extended trivially u and v to the whole set $M \cup N$), we have that $w \in \text{LIP}_*(M \cup N)$ and moreover it holds:

$$\begin{aligned} \int_{M \cup N} \text{lip}_a(w) \, dm &\leq \int_M \text{lip}_a(u) \, dm + \int_N \text{lip}_a(v) \, dm + \frac{3}{\eta} \int_{\overline{H}} |u - v| \, dm. \\ w &\equiv u \text{ on a neighbourhood of } M \setminus N, \quad w \equiv v \text{ on a neighbourhood of } N \setminus M. \\ \int_{M \cup N} |w - \sigma| \, dm &\leq \int_M |u - \sigma| \, dm + \int_N |v - \sigma| \, dm \quad \text{for every } \sigma \in L^1(M \cup N). \end{aligned}$$

Proof. It is clear that $\phi \in \text{LIP}_b(M \cup N)$, that $\phi \equiv 1$ on $\{x \in M \cup N : d(x, N \setminus M) > 2\eta/3\} \supseteq M \setminus N$, that $\phi \equiv 0$ on $\{x \in M \cup N : d(x, N \setminus M) < \eta/3\} \supseteq N \setminus M$, that $\phi(M \cup N) = [0, 1]$, and that ϕ is $3/\eta$ -Lipschitz continuous. The set $H := \phi^{-1}((0, 1))$ is therefore an open subset of $M \cap N$ and it satisfies $d(H, (M \setminus N) \cup (N \setminus M)) > 0$: indeed $x \in H$ if and only if $\eta/3 < d(x, N \setminus M) < 2\eta/3$, so that $d(x, N \setminus M) > \eta/3$ and $d(x, M \setminus N) > \eta - 2\eta/3 = \eta/3$. Given u and v as in the statement, let $w = \phi u + (1 - \phi)v$ in $M \cup N$ be as above. The convexity inequality for the asymptotic slope gives that $\text{lip}_a(w) \leq \phi \text{lip}_a(u) + (1 - \phi) \text{lip}_a(v) + \text{lip}_a(\phi)|u - v|$, so that $w \in \text{LIP}_*(M \cup N)$. By definition, it is clear that $w \equiv u$ in a neighbourhood of $M \setminus N$ and that $w \equiv v$ in a neighbourhood of $N \setminus M$. Writing $M \cup N = \text{int}(\{\phi = 0\}) \cup \text{int}(\{\phi = 1\}) \cup \overline{H}$, we have

$$\begin{aligned} \int_{M \cup N} \text{lip}_a(w) \, dm &\leq \int_{M \cup N} \phi \text{lip}_a(u) \, dm + \int_{M \cup N} (1 - \phi) \text{lip}_a(v) \, dm + \int_{M \cup N} \text{lip}_a(\phi)|u - v| \, dm \\ &\leq \int_M \text{lip}_a(u) \, dm + \int_N \text{lip}_a(v) \, dm + \frac{3}{\eta} \int_{\overline{H}} |u - v| \, dm, \end{aligned}$$

where we have also used that $\text{lip}_a(\phi) \equiv 0$ in the interior of sets of the form $\{\phi = c\}$, $c \in \{0, 1\}$, and that $\phi \leq \mathbb{1}_M$ and $1 - \phi \leq \mathbb{1}_N$. The last inequality in the statement follows from the latter fact and the triangle inequality, writing $\sigma = \phi\sigma + (1 - \phi)\sigma$. \square

Proposition A.2. *Let (X, d, m) be a metric measure space. Let τ_d be the topology induced by d . For $f \in L^1(m)$ we define the function $|\mathbf{D}f|_* : \tau_d \rightarrow [0, +\infty]$ as*

$$|\mathbf{D}f|_*(U) := \inf \left\{ \liminf_{n \rightarrow \infty} \int_U \text{lip}_a(f_n) \, dm \mid \text{LIP}_*(U) \ni f_n \rightarrow f|_U \text{ in } L^1(m|_U) \right\} \quad \text{for every } U \in \tau_d,$$

with the convention that $|\mathbf{D}f|_*(\emptyset) := 0$. Then $|\mathbf{D}f|_*$ satisfies the following properties:

- (1) If $A_1, A_2 \in \tau_d$ and $A_1 \subseteq A_2$, then $|\mathbf{D}f|_*(A_1) \leq |\mathbf{D}f|_*(A_2)$.
- (2) If $A_1, A_2 \in \tau_d$, then $|\mathbf{D}f|_*(A_1 \cup A_2) \leq |\mathbf{D}f|_*(A_1) + |\mathbf{D}f|_*(A_2)$, with equality if A_1 and A_2 are disjoint.

(3) If $(A_n)_n \subseteq \tau_d$ and $A_n \subseteq A_{n+1}$ for every $n \in \mathbb{N}$, then

$$\lim_{n \rightarrow \infty} |\mathbf{D}f|_*(A_n) = |\mathbf{D}f|_* \left(\bigcup_{n=1}^{\infty} A_n \right). \quad (\text{A.1})$$

In particular the formula

$$|\mathbf{D}f|_*(E) := \inf \{ |\mathbf{D}f|_*(U) \mid U \subseteq X \text{ open, } E \subseteq U \} \quad \text{for every } E \subseteq X, \quad (\text{A.2})$$

provides a σ -subadditive extension of $|\mathbf{D}f|_*$ whose collection of additive sets, in the sense of Carathéodory, contains $\mathcal{B}(X)$. Therefore $|\mathbf{D}f|_*: \mathcal{B}(X) \rightarrow [0, +\infty]$ is a Borel measure.

Proof. The monotonicity in (1) and the equality in (2) in case A_1 and A_2 are disjoint are obvious. We proceed to prove (3), setting $A := \bigcup_{n=1}^{\infty} A_n$.

Preparation to the proof of (3): by the monotonicity in (1), it is enough to show

$$\sup_n |\mathbf{D}f|_*(A_n) \geq |\mathbf{D}f|_*(A) \quad (\text{A.3})$$

so that we can assume w.l.o.g. that $\sup_n |\mathbf{D}f|_*(A_n) < +\infty$. We can restrict ourselves to the case

$$d(\overline{A_n}, X \setminus A_{n+1}) > 0 \quad \text{for every } n \in \mathbb{N}. \quad (\text{A.4})$$

To see this, it is enough to set

$$A'_n := \left\{ x \in X : d(x, X \setminus A_n) > \frac{1}{n} \right\} \in \tau_d, \quad \text{for every } n \in \mathbb{N},$$

so that $\bigcup_{n=1}^{\infty} A'_n = A$, condition (A.4) is satisfied for $(A'_n)_n$ and, since $A'_n \subseteq A_n$ for every $n \in \mathbb{N}$, by the monotonicity in (1), we also have $\sup_n |\mathbf{D}f|_*(A_n) \geq \sup_n |\mathbf{D}f|_*(A'_n)$.

We therefore assume (A.4) to hold for $(A_n)_n$ and we proceed to set

$$C_1 := A_1, \quad C_2 := A_2, \quad C_k := A_k \setminus \overline{A_{k-2}} \quad \text{for } k \geq 3.$$

It is clear that $(C_k)_k \subseteq \tau_d$, $d(C_{3k+i}, C_{3h+i}) > 0$ for every $k, h \geq 0$, $h \neq k$, and every $i \in \{1, 2, 3\}$. By construction we also have that $\sum_{j=0}^n C_{3j+i} \subseteq A_{3n+i}$ for every $n \in \mathbb{N}$ and $i \in \{1, 2, 3\}$, so that,

$$\sum_{k=1}^{\infty} |\mathbf{D}f|_*(C_k) = \sum_{i=1}^3 \sum_{j=0}^{\infty} |\mathbf{D}f|_*(C_{3j+i}) \leq 3 \sup_n |\mathbf{D}f|_*(A_n) < +\infty,$$

where we have also used the equality in (2) for disjoint sets. We fix $\varepsilon > 0$ and we find an integer $\bar{k} \in \mathbb{N}$ such that $\sum_{k=\bar{k}}^{\infty} |\mathbf{D}f|_*(C_k) \leq \varepsilon$. Point (3) will follow if we are able to construct a sequence $(f_m)_m \subseteq \text{LIP}_*(A)$ such that

$$|\mathbf{D}f|_*(A_{\bar{k}}) + 5\varepsilon \geq \liminf_{m \rightarrow \infty} \int_A \text{lip}_a(f_m) \, d\mathbf{m}, \quad f_m \rightarrow f|_A \text{ in } L^1(\mathbf{m}|_A), \quad (\text{A.5})$$

so that (A.3), and thus (3), follows. Notice that the sequence $(f_m)_m$ (and many of the objects defined below) depends also on ε but we do not stress this dependence. Let us set

$$B_{-2} := \emptyset, \quad B_{-1} := \emptyset, \quad B_0 := A_{\bar{k}}, \quad D_0 := A_{\bar{k}}, \quad B_h := A_{h+\bar{k}}, \quad D_h := C_{h+\bar{k}} \quad \text{for all } h \in \mathbb{N}.$$

Let $c_h > 0$ and H_h be the constant and the open set as in Lemma A.1 for the sets B_h and D_{h+1} , for every $h \geq 0$; we also set $c_{-1} := 0$ and $H_{-1} := \emptyset$. Note that $d(D_{h+1} \setminus B_h, B_h \setminus D_{h+1}) > 0$ for every $h \geq 0$ by construction. To construct f_m as above, we construct, for every $m \in \mathbb{N}$, sequences of functions $(\psi_h^m)_{h \geq 0}$, $(f_h^m)_{h \geq 0}$. Starting from those, we define a sequence $(\tilde{f}_m)_m \subset \text{LIP}_{loc}(A)$ satisfying (A.5) from which we will define the sought sequence $(f_m)_m \subset \text{LIP}_*(A)$ satisfying (A.5) by a simple truncation argument.

Construction of $(\psi_h^m)_h$: let $m \in \mathbb{N}$ and $h \geq 0$ be fixed and let us set

$$\gamma_h^m := \min \left\{ \frac{1}{m2^h}, \frac{\varepsilon}{2^{h+1}c_h}, \frac{\varepsilon}{2^h c_{h-1}} \right\} > 0.$$

By definition of $|\mathbf{D}f|_*(D_h)$, we can find a function $\psi_h^m \in \text{LIP}_*(D_h)$ such that

$$\int_{D_h} |\psi_h^m - f| \, d\mathbf{m} < \gamma_h^m, \quad \int_{D_h} \text{lip}_a(\psi_h^m) \, d\mathbf{m} < |\mathbf{D}f|_*(D_h) + \gamma_h^m.$$

It follows that

$$\int_{D_h} \text{lip}_a(\psi_h^m) \, d\mathbf{m} \leq |\mathbf{D}f|_*(D_h) + \frac{1}{m2^h}, \quad (\text{A.6})$$

$$\int_{D_h} |\psi_h^m - f| \, d\mathbf{m} \leq \frac{1}{m2^h}, \quad (\text{A.7})$$

$$c_h \int_{\overline{H_h}} |\psi_h^m - f| \, d\mathbf{m} \leq \frac{\varepsilon}{2^{h+1}}, \quad (\text{A.8})$$

$$c_{h-1} \int_{\overline{H_{h-1}}} |\psi_h^m - f| \, d\mathbf{m} \leq \frac{\varepsilon}{2^h}, \quad (\text{A.9})$$

where the latter two inequalities follow from the fact that $\overline{H_{h-1}} \subset B_{h-1} \cap D_h \subset D_h$ and $\overline{H_h} \subset B_h \cap D_{h+1} \subset D_h$.

Construction of $(f_h^m)_h$: let $m \in \mathbb{N}$ be fixed; we construct by induction on h a sequence of functions $(f_h^m)_{h \geq -1}$ such that $f_{-1}^m := 0$, $f_0^m := \psi_0^m$ and satisfying, for every $h \geq 0$, the following conditions:

$$f_h^m \equiv f_{h-1}^m \quad \text{on } B_{h-2}, \quad f_h^m \equiv \psi_h^m \quad \text{on } B_h \setminus \overline{B_{h-1}}, \quad f_h^m \in \text{LIP}_*(B_h), \quad (\text{A.10})$$

$$\int_{B_h} |f_h^m - f| \, d\mathbf{m} \leq \frac{1}{m} \left(2 - \frac{1}{2^h} \right), \quad (\text{A.11})$$

$$\int_{B_h} \text{lip}_a(f_h^m) \, d\mathbf{m} \leq \int_{B_{h-1}} \text{lip}_a(f_{h-1}^m) \, d\mathbf{m} + \int_{D_h} \text{lip}_a(\psi_h^m) \, d\mathbf{m} + \frac{\varepsilon}{2^{h-1}}. \quad (\text{A.12})$$

Notice that the above conditions are satisfied for $h = 0$: the first condition in (A.10) is void, the second and the third one are true by construction; (A.11) follows by (A.7); (A.12) is trivially true.

We now show that the above three conditions being true for $h \geq 0$ implies they are true for $h + 1$. We apply Lemma A.1 to the open sets $M := B_h$ and $N := D_{h+1}$ and to the functions $u := f_h^m$ and $v := \psi_{h+1}^m$. We get the existence of a function $f_{h+1}^m \in \text{LIP}_*(B_{h+1})$ satisfying

$$\int_{B_{h+1}} \text{lip}_a(f_{h+1}^m) \, d\mathbf{m} \leq \int_{B_h} \text{lip}_a(f_h^m) \, d\mathbf{m} + \int_{D_{h+1}} \text{lip}_a(\psi_{h+1}^m) \, d\mathbf{m} + c_h \int_{\overline{H_h}} |f_h^m - \psi_{h+1}^m| \, d\mathbf{m}, \quad (\text{A.13})$$

$$f_{h+1}^m \equiv f_h^m \quad \text{on } B_h \setminus D_{h+1} \supseteq B_{h-1}, \quad f_{h+1}^m \equiv \psi_{h+1}^m \quad \text{on } D_{h+1} \setminus B_h \supseteq B_{h+1} \setminus \overline{B_h}, \quad (\text{A.14})$$

$$\int_{B_{h+1}} |f_{h+1}^m - \sigma| \, d\mathbf{m} \leq \int_{B_h} |f_h^m - \sigma| \, d\mathbf{m} + \int_{D_{h+1}} |\psi_{h+1}^m - \sigma| \, d\mathbf{m} \quad \text{for every } \sigma \in L^1(B_{h+1}). \quad (\text{A.15})$$

We see that (A.10) for $h + 1$ follows from (A.14); choosing $\sigma := f$ in (A.15) and using (A.11) and (A.7) we get (A.11) for $h + 1$; finally (A.12) for $h + 1$ follows from (A.13), (A.8) and (A.9) and the fact that by induction assumption $f_h^m \equiv \psi_h^m$ on $B_h \setminus \overline{B_{h-1}} \supseteq \overline{H_h}$.

Construction of $(f_m)_m$ and proof of (3): for every $m \in \mathbb{N}$ we set

$$\tilde{f}_m(x) := f_h^m(x) \quad \text{if } x \in B_{h-1}, \quad h \geq 1.$$

By the first equality in (A.10) each function \tilde{f}_m is well defined and belongs to $\text{LIP}_{loc}(A)$ for every $m \in \mathbb{N}$. Inequality (A.11) and the monotone convergence theorem give that $\int_A |\tilde{f}_m - f| \, d\mathbf{m} \leq 2/m$ for every $m \in \mathbb{N}$ so that $\tilde{f}_m \rightarrow f|_A$ in $L^1(\mathbf{m}|_A)$. Iterating (A.12) and using (A.6) we have that, for every $h \geq 1$, it holds

$$\begin{aligned} \int_{B_h} \text{lip}_a(f_h^m) \, d\mathbf{m} &\leq \int_{B_{h-1}} \text{lip}_a(f_{h-1}^m) \, d\mathbf{m} + \int_{D_h} \text{lip}_a(\psi_h^m) \, d\mathbf{m} + \frac{\varepsilon}{2^{h-1}} \\ &\leq \int_{B_{h-1}} \text{lip}_a(f_{h-1}^m) \, d\mathbf{m} + |\mathbf{D}f|_*(D_h) + \frac{1}{2^h} \left(\frac{1}{m} + 2\varepsilon \right) \\ &\leq \sum_{i=0}^h |\mathbf{D}f|_*(D_h) + \left(\frac{1}{m} + 2\varepsilon \right) \sum_{i=0}^h \frac{1}{2^h} \\ &\leq |\mathbf{D}f|_*(A_{\bar{k}}) + \sum_{k=\bar{k}}^{h+\bar{k}} |\mathbf{D}f|_*(C_k) + \left(\frac{1}{m} + 2\varepsilon \right) \sum_{i=0}^h \frac{1}{2^h} \\ &\leq |\mathbf{D}f|_*(A_{\bar{k}}) + \varepsilon + \left(\frac{1}{m} + 2\varepsilon \right) \sum_{i=0}^h \frac{1}{2^h}. \end{aligned}$$

Therefore, the monotone convergence theorem gives that

$$\int_A \text{lip}_a(\tilde{f}_m) \, d\mathbf{m} = \lim_{h \rightarrow \infty} \int_{B_h} \text{lip}_a(f_h^m) \, d\mathbf{m} \leq |\mathbf{D}f|_*(A_{\bar{k}}) + 5\varepsilon + \frac{2}{m},$$

which shows that $(\tilde{f}_m)_m \subset \text{LIP}_{loc}(A)$ satisfies (A.5). Finally we set

$$f_m := \tilde{f}_m \wedge m \in \text{LIP}_*(A) \quad \text{for every } m \in \mathbb{N}.$$

We see that $f_m \rightarrow f|_A$ in $L^1(\mathbf{m}|_A)$ and, since $\text{lip}_a(f_m) \leq \text{lip}_a(\tilde{f}_m)$ for every $m \in \mathbb{N}$, we also see that $(f_m)_m \subset \text{LIP}_*(A)$ satisfies the first part of (A.5). This concludes the proof of (3).

Proof of (2): it is enough to show

$$|\mathbf{D}f|_*(A'_1 \cup A'_2) \leq |\mathbf{D}f|_*(A_1) + |\mathbf{D}f|_*(A_2) \quad \text{if } A'_i \in \tau_d \text{ is s.t. } d(A'_i, X \setminus A_i) > 0, \quad i = 1, 2. \quad (\text{A.16})$$

Indeed, it is enough to define

$$A_i^n := \{x \in X : d(x, X \setminus A_i) > 1/n\} \in \tau_d, \quad \text{for every } i = 1, 2, \quad n \in \mathbb{N},$$

so that, if (A.16) is true for $A'_i := A_i^n$, $i = 1, 2$, $n \in \mathbb{N}$, then (3) gives that $\sup_n |\mathbf{D}f|_*(A_1^n \cup A_2^n) = |\mathbf{D}f|_*(A_1 \cup A_2)$ and we obtain (2). To show (A.16) we take sequences $(g_h^i)_h \subset \text{LIP}_*(A_i)$ satisfying

$$\lim_{h \rightarrow \infty} \int_{A_i} \text{lip}_a(g_h^i) \, d\mathbf{m} = |\mathbf{D}f|_*(A_i), \quad g_h^i \rightarrow f \text{ in } L^1(\mathbf{m}|_{A_i}), \quad i = 1, 2, \quad (\text{A.17})$$

and we apply Lemma A.1 with $M := (A'_1 \cup A'_2) \cap A_1$, $N := (A'_1 \cup A'_2) \cap A_2$, and, for every $h \in \mathbb{N}$, to the functions $u := g_h^1$, $v := g_h^2$. We obtain the existence of a positive constant $c > 0$ and, for every $h \in \mathbb{N}$, of a function $g_h \in \text{LIP}_*(A'_1 \cup A'_2)$ satisfying

$$\int_{A'_1 \cup A'_2} \text{lip}_a(g_h) \, d\mathbf{m} \leq \int_{A_1} \text{lip}_a(g_h^1) \, d\mathbf{m} + \int_{A_2} \text{lip}_a(g_h^2) \, d\mathbf{m} + c \int_{A_1 \cap A_2} |g_h^1 - g_h^2| \, d\mathbf{m}.$$

Passing to the limit as $h \rightarrow \infty$ and using (A.17), we get (A.16).

Finally, the given properties (1), (2), (3) imply that (A.2) defines an extension of $|\mathbf{D}f|_*$ to the family of all subsets of X which is an outer measure on X [9, Definition 1.11.1]; therefore its restriction to the σ -algebra of its Carathéodory-measurable subsets \mathcal{A} is a σ -additive measure [9, Theorem 1.11.4(iii)]. Finally, we observe that $\mathcal{B}(X) \subseteq \mathcal{A}$ as a consequence of [9, Theorem 1.11.10, Theorem 7.14.29] and (2). \square

REFERENCES

- [1] L. Ambrosio. Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces. *Advances in Mathematics*, 159:51–67, 2001.
- [2] L. Ambrosio. Fine properties of sets of finite perimeter in doubling metric measure spaces. *Set Valued Analysis*, 10:111–128, 2002.
- [3] L. Ambrosio, E. Bruè, and D. Semola. Rigidity of the 1-Bakry-Émery inequality and sets of finite perimeter in RCD spaces. *Geom. Funct. Anal.*, 29:949–1001, 2019.
- [4] L. Ambrosio and S. Di Marino. Equivalent definitions of BV space and of total variation on metric measure spaces. *J. Funct. Anal.*, 266(7):4150–4188, 2014.
- [5] L. Ambrosio, M. Erbar, and G. Savaré. Optimal transport, Cheeger energies and contractivity of dynamic transport distances in extended spaces. *Nonlinear Analysis*, 137:77–134, 2016.
- [6] L. Ambrosio, N. Gigli, and G. Savaré. Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces. *Rev. Mat. Iberoam.*, 29(3):969–996, 2013.
- [7] L. Ambrosio, T. Ikonen, D. Lučić, and E. Pasqualetto. Metric Sobolev spaces I: equivalence of definitions. *Milan Journal of Mathematics*, pages 1–93, 2024.
- [8] G. Antonelli, C. Brena, and E. Pasqualetto. The Rank-One Theorem on RCD spaces. *Analysis & PDE*, 17(8):2797–2840, 2024.
- [9] V. I. Bogachev. *Measure theory. Vol. I, II*. Springer-Verlag, Berlin, 2007.
- [10] G. Bouchitté, G. Buttazzo, and P. Seppecher. Energies with respect to a measure and applications to low dimensional structures. *Calc. Var. Partial Differential Equations*, 5(1):37–54, 1997.
- [11] C. Brena and N. Gigli. Calculus and fine properties of functions of bounded variation on RCD spaces. *The Journal of Geometric Analysis*, 34(1):11, 2023.
- [12] E. Bruè, E. Pasqualetto, and D. Semola. Constancy of the dimension in codimension one and locality of the unit normal on $\text{RCD}(K, N)$ spaces. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, XXIV:1765–1816, 2023.
- [13] E. Bruè, E. Pasqualetto, and D. Semola. Rectifiability of the reduced boundary for sets of finite perimeter over $\text{RCD}(K, N)$ spaces. *J. Eur. Math. Soc.*, 25(2):413–465, 2023.
- [14] J. Cheeger. Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.*, 9(3):428–517, 1999.
- [15] L. Dello Schiavo and G. E. Sodini. The Hellinger–Kantorovich metric measure geometry on spaces of measures. arXiv preprint, arXiv:2503.07802, 2025.
- [16] S. Di Marino. Recent advances on BV and Sobolev spaces in metric measure spaces, 2014. PhD thesis (cvgmt preprint).
- [17] S. Di Marino. Sobolev and BV spaces on metric measure spaces via derivations and integration by parts. arXiv preprint, arXiv:1409.5620, 2014.
- [18] S. Di Marino, N. Gigli, E. Pasqualetto, and E. Soultanis. Infinitesimal Hilbertianity of Locally $\text{CAT}(\kappa)$ -Spaces. *J. Geom. Anal.*, pages 1–65, Nov 2020.
- [19] S. Di Marino, N. Gigli, and A. Pratelli. Global Lipschitz extension preserving local constants. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 31(4):757–765, 2020.
- [20] M. Fornasier, G. Savaré, and G. E. Sodini. Density of subalgebras of Lipschitz functions in metric Sobolev spaces and applications to Wasserstein Sobolev spaces. *J. Funct. Anal.*, 285(11):110153, 2023.
- [21] M. S. Gelli and D. Lučić. A note on BV and 1-Sobolev functions on the weighted Euclidean space. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur.*, 33:757–794, 2023.
- [22] N. Gigli. Lecture notes on differential calculus on RCD spaces. *Publ. RIMS Kyoto Univ.*, 54, 2018.
- [23] N. Gigli. Nonsmooth differential geometry—an approach tailored for spaces with Ricci curvature bounded from below. *Mem. Amer. Math. Soc.*, 251(1196), 2018.
- [24] N. Gigli and E. Pasqualetto. Differential structure associated to axiomatic Sobolev spaces. *Expositiones Mathematicae*, 38(4):480–495, 2020.
- [25] J. Kinnunen, R. Korte, A. Lorent, and N. Shanmugalingam. Regularity of sets with quasiminimal boundary surfaces in metric spaces. *J. Geom. Anal.*, 23(4):1607–1640, 2013.

- [26] J. Kinnunen, R. Korte, N. Shanmugalingam, and H. Tuominen. Pointwise properties of functions of bounded variation in metric spaces. *Revista Matemática Complutense*, 27:41–67, 2014.
- [27] P. Lahti. A new Federer-type characterization of sets of finite perimeter. *Archive for Rational Mechanics and Analysis*, 236(2):801–838, 2020.
- [28] D. Lučić and E. Pasqualetto. Yet another proof of the density in energy of Lipschitz functions. *manuscripta mathematica*, 175:421–438, 2024.
- [29] D. Lučić and E. Pasqualetto. An axiomatic theory of normed modules via Riesz spaces. *The Quarterly Journal of Mathematics*, 75:1429–1479, 2024.
- [30] O. Martio. Functions of bounded variation and curves in metric measure spaces. *Advances in Calculus of Variations*, 9(4):305–322, 2016.
- [31] O. Martio. The space of functions of bounded variation on curves in metric measure spaces. *Conform. Geom. Dyn.*, 20:81–96, 2016.
- [32] M. Miranda Jr. Functions of bounded variation on “good” metric spaces. *Journal de Mathématiques Pures et Appliquées*, 82(8):975–1004, 2003.
- [33] F. Nobili, E. Pasqualetto, and T. Schultz. On master test plans for the space of BV functions. *Adv. Calc. Var.*, 2022.
- [34] E. Pasqualetto. Smooth approximations preserving asymptotic Lipschitz bounds. arXiv preprint, arXiv:2409.01772, 2024.
- [35] E. Pasqualetto and J. Taipalus. Derivations and Sobolev functions on extended metric-measure spaces. arXiv preprint, arXiv:2503.02596.
- [36] G. Savaré. Sobolev spaces in extended metric-measure spaces. In *New trends on analysis and geometry in metric spaces*, volume 2296 of *Lecture Notes in Math.*, pages 117–276. Springer, Cham, 2022.
- [37] G. E. Sodini. The general class of Wasserstein Sobolev spaces: density of cylinder functions, reflexivity, uniform convexity and Clarkson’s inequalities. *Calc. Var. Partial Differ. Equ.*, 62(7):212, 2023.

DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. Box 35 (MAD), FI-40014 UNIVERSITY OF JYVASKYLA
Email address: `enrico.e.pasqualetto@jyu.fi`

INSTITUT FÜR MATHEMATIK - FAKULTÄT FÜR MATHEMATIK - UNIVERSITÄT WIEN, OSKAR-MORGENSTERN-PLATZ
 1, 1090 WIEN (AUSTRIA)
Email address: `giacomo.sodini@univie.ac.at`