LIPSCHITZ REGULARITY RESULTS FOR A CLASS OF OBSTACLE PROBLEMS WITH NEARLY LINEAR GROWTH

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Abstract. This paper deals with the Lipschitz continuity of solutions to variational obstacle problems with nearly linear growth. The main tool used here is a new higher differentiability result which reveals to be crucial because it allows us to perform the linearization procedure to transform the constrained problem in an unconstrained one and it permits us to deduce the equivalence between our minimization problem and its corresponding variational formulation. Our results hold true for a large class of example for which the Lavrentiev phenomenon does not occurr, not necessarily for lagrangians dependent on the modulus of the gradient. We assume the same Sobolev regularity both for the gradient of the obstacle and for the coefficients.

1. Introduction

In this paper we study the Lipschitz continuity of the solutions to variational obstacle problems of the form

$$\min\left\{\int_\Omega f(x,Dw) : w \in \mathcal{K}_\psi(\Omega)\right\}, \tag{1.1}$$

in the case of nearly linear growth condition, where Ω, f and $\mathcal{K}_\psi(\Omega)$ will be specified below; the aim of our work is to complement the results cointained in the paper [13] where authors assume that the integrand $f(x,Dw)$ satisfies $(p,q)-$growth conditions and, as a function of the $x-$variable, belongs to a suitable Sobolev class; they then prove the Lipschitz continuity of the solutions to variational obstacle problems under the above-mentioned conditions.
A model functional that can be considered the following

$$w \mapsto \int_\Omega |Dw|\log(1+|Dw|) + a(x)(1+|Dw|^2)^{q/2}\,dx$$

with $q>1$ and $a(\cdot)$ a Lipschitz or bounded Sobolev coefficient.

There are few results in literature dealing with the case of nearly linear growth condition, see [36], [56] in the case of equations of functionals, see [35], [18] in the case of obstacle problems, and more in general [2], [6], [21], [40], [41], [45], [47], [55] in the case with the subquadratic growth.

The key tool we use here is a new higher differentiability result for solutions to obstacle problems with nearly linear growth obtained in the paper [39] in the spirit of the recent contributions [26], [27], [37], [38]; this new result allows us to adapt here the linearization

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technique already successfully employed in [4] is order to trasform the constrained problem in an unconstrained one. Higher differentiability and higher integrability of second order derivatives has been recently widely investigate both in the context of unconstrained and constrained problems, see for instance [9], [10], [14], [42], [43], [57], [58], [59].

The main feature of the paper is that we consider the same Sobolev dependence either for the gradient of the obstacle and for the partial map $x \mapsto D_\xi f(x,\xi)$, which turns to be a quite natural assumption. The idea of replacing the Lipschitz dependence on the coefficients in the non autonomous case with a Sobolev one has been intensively used in the last years, see for instance the survey [53] together with the references therein for a general overview. To better clarify the main technicalities, we will anyway state and prove first the case with a Lipschitz dependence.

The relationship between the ellipticity and the growth exponent we impose, namely (1.9), is the one considered for the first time in the series of papers [22], [23], [24], [25] and it is sharp (in view of the well known counterexamples, see for instance [49]) also to obtain the Lipschitz continuity of solutions to elliptic equations and systems and minimizers of related functionals with $p, q-$growth; therefore our results can be framed into the research concerning regularity results under non standard growth conditions, that, after the pioneering papers by Marcellini [50]–[52] has attracted growing attention, see among the others [3], [5], [15], [16], [19], [46], [53], [54], [60].

More in details, here Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 2$, the function $\psi : \Omega \to [-\infty, +\infty)$, called *obstacle*, belongs to the Sobolev space $W^{1,1}(\Omega)$ and the class $\mathcal{K}_\psi(\Omega)$ is defined as follows

$$\mathcal{K}_\psi(\Omega) := \left\{ w \in u_0 + W_0^{1,1}(\Omega) : w \geq \psi \text{ a.e. in } \Omega \right\}, \tag{1.2}$$

where $u_0 \in W^{1,1}(\Omega)$ is a fixed boundary value.
To avoid trivialities, in what follows we shall assume that $\mathcal{K}_\psi$ is not empty. We also assume that a solution to (1.1) is such that $f(x, Du) \in L^1_{\mathrm{loc}}(\Omega)$. As it has been shown in [28], in case of non-standard growth condition (at least in the autonomous case), this turns to be the right class of competitors.

Remark 1.1. Let us notice that, by replacing u_0 by $\tilde{u}_0 = \max\{u_0, \psi\}$, we may assume that the boundary value function u_0 satisfies $u_0 \geq \psi$ in Ω. Indeed $\tilde{u}_0 = (\psi - u_0)^+ + u_0$ and since

$$0 \leq (\psi - u_0)^+ \leq (u - u_0)^+ \in W_0^{1,\mu}(\Omega),$$

the function $(\psi - u_0)^+$, and hence $u - \tilde{u}_0$, belongs to $W_0^{1,\mu}(\Omega)$. Moreover assumptions $f(x, Du) \in L^1_{\mathrm{loc}}(\Omega)$ and $f(x, Du_0) \in L^1_{\mathrm{loc}}(\Omega)$ imply $f(x, D\tilde{u}_0) \in L^1_{\mathrm{loc}}(\Omega)$. Indeed we have

$$\begin{aligned} \int_\Omega f(x, D\tilde{u}_0)\,dx &= \int_{\Omega \cap \{u_0 \geq \psi\}} f(x, Du_0)\,dx + \int_{\Omega \cap \{u_0 < \psi\}} f(x, D\psi)\,dx \\ &\leq \int_\Omega [f(x, Du_0) + f(x, D\psi)]\,dx < +\infty \end{aligned}$$

where we used that $f(x,\xi) \geq 0$, by virtue of the left inequality in (1.5).

As we are dealing with non standard growth conditions, the Lavrentiev phenomenon may occurr (for more details see for instance [61], [29], [30], [7], [8]). On the other hand in the paper [17] a wide discussion about obstacle problems and a related suitable notion of relaxation has been introduced, and this setting can be extended to the case of nearly linear growth in view of Proposition 1.1 of [18]. Therefore we will assume to be in a situation where the Lavrentiev phenomenon does not appear. In the sequel we will denote with $\mathcal{L}(u,\tilde{\Omega})$ the gap functional over the set $\tilde{\Omega}$, in the spirit of [1], [29], [48].
As we already remarked, in order to better clarify the details of the techniques employed, we first state and prove the result with Lipschitz dependence on both the obstacle and the partial map $x \mapsto D_\xi f(x,\xi)$ and we concentrate in a second step on the Sobolev dependence. More precisely, we will deal with variational integral

$$\mathcal{F}(u) := \int_\Omega f(x, Du)\, dx, \tag{1.3}$$

where $f : \Omega \times \mathbb{R}^n \to [0,+\infty)$ is a Carathéodory function which is convex and of class $\mathcal{C}^2$ with respect to the second variable. We consider the exponents $q > 1$ and $\mu < 2$ such that the following bound holds

$$1 \le \frac{q}{2-\mu} < 1 + \frac{1}{n}. \tag{1.4}$$

We suppose that there exist two positive constants ν, L and a function $\bar{F} : [0,+\infty) \to [0,+\infty)$ such as $\lim_{t\to+\infty} \frac{\bar{F}(t)}{t} = +\infty$ such that

$$\nu \bar{F}(|\xi|) + \nu\,(1+|\xi|^2)^{\frac{2-\mu}{2}} \le f(x,\xi) \le L\,(1+|\xi|^2)^{\frac{q}{2}}, \tag{1.5}$$

$$\nu\,(1+|\xi|^2)^{-\frac{\mu}{2}}|\lambda|^2 \le \sum_{i,j} f_{\xi_i\xi_j}(x,\xi)\lambda_i\lambda_j \le L\,(1+|\xi|^2)^{\frac{q-2}{2}}|\lambda|^2, \tag{1.6}$$

$$|f_{x\xi}(x,\xi)| \le \; L\,(1+|\xi|^2)^{\frac{q-1}{2}}, \tag{1.7}$$

for all $\lambda, \xi \in \mathbb{R}^n$, $\lambda = \lambda_i$, $\xi = \xi_i$, $i = 1, 2, \ldots, n$, a.e. in Ω. Our main result with these hypothesis reads as follows

Theorem 1.2. *Let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to the obstacle problem* (1.1) *such that $\mathcal{L}(u, B_R) = 0$ for all $B_R \Subset \Omega$, under the assumptions* (1.5)-(1.7) *and* (1.4). *If $\psi \in W^{2,\infty}_{\mathrm{loc}}(\Omega)$, then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and the following estimate*

$$\|Du\|_{L^\infty(B_\rho)} \le \; C \left(\int_{B_R} (1 + f(x, Du))\, dx \right)^\beta \tag{1.8}$$

holds for every $0 < \rho < R$ and with positive constants C and β depending on $n, q, \mu, \nu, L, R, \rho$ and the local bounds for $\|D\psi\|_{W^{1,\infty}}$.

In the second part of the paper, we consider instead the exponents q and μ, where again $\mu < 2$ and $q > 1$, bounded by

$$1 \le \frac{q}{2-\mu} < 1 + \frac{r-n}{rn} = 1 + \frac{1}{n} - \frac{1}{r} \tag{1.9}$$

where $r > n$, so $\frac{1}{r} < \frac{1}{n}$. Moreover, we suppose that there exist two positive constants ν, L, a function $\bar{F} : [0, +\infty) \to [0, +\infty)$ such as $\lim_{t \to +\infty} \frac{\bar{F}(t)}{t} = +\infty$ and a function $h : \Omega \to [0, +\infty)$ such as $h(x) \in L^r_{\mathrm{loc}}(\Omega)$, such that (1.5) and (1.6) keep being valid and (1.7) is modified in

$$|f_{x\xi}(x, \xi)| \leq \, h(x) \, (1 + |\xi|^2)^{\frac{q-1}{2}} \tag{1.10}$$

for all $\lambda, \xi \in \mathbb{R}^n$, $\lambda = \lambda_i$, $\xi = \xi_i$, $i = 1, 2, \ldots, n$, a.e. in Ω.

Our main result with these modified hypothesis reads as follows

Theorem 1.3. *Let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to the obstacle problem* (1.1) *such that $\mathcal{L}(u, B_R) = 0$ for all $B_R \Subset \Omega$, under the assumptions* (1.5), (1.6), (1.10) *and* (1.9). *If $\psi \in W^{2,r}_{\mathrm{loc}}(\Omega)$, then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and the following estimate*

$$\|Du\|_{L^\infty(B_\rho)} \leq \, C \, \left(\int_{B_R} (1 + f(x, Du)) \, dx \right)^{\beta} \tag{1.11}$$

holds for every $0 < \rho < R$ and with positive constants C and β depending on $n, q, \mu, \nu, L, R, \rho$, on the local bounds for $\|D\psi\|_{W^{1,r}}$ and $\|h\|_{L^r}$.

The paper is organized as follows. Section 2 contains the notations and some preliminary results that will be needed in the sequel. Section 3 is devoted to the proof of Theorem 1.2 and Section 4 is dedicated to the proof of the main result, Theorem 1.3. Finally in Section 5 we provide some comments about the approximation and the conclusion.

2. Notations and preliminary results

Throughout the paper we will denote by B_ρ and B_R balls of radii respectively ρ and R (with $\rho < R$) compactly contained in Ω and with the same center, let us say $x_0 \in \Omega$. Moreover in the sequel constants will be denoted by C, regardless their actual value. Only the relevant dependencies will be highlighted.

First of all we state the following lemma which has important application in the so called hole-filling method. Its proof can be found for example in [44, Lemma 6.1].

Lemma 2.1. *Let $h : [\rho_0, R_0] \to \mathbb{R}$ be a non-negative bounded function and $0 < \vartheta < 1$, $A, B \geq 0$ and $\beta > 0$. Assume that*

$$h(s) \leq \vartheta h(t) + \frac{A}{(t-s)^\beta} + B,$$

for all $\rho_0 \leq s < t \leq R_0$. Then

$$h(r) \leq \frac{cA}{(R_0 - \rho_0)^\beta} + cB,$$

where $c = c(\vartheta, \beta) > 0$.

We now present the higher differentiability result we need in the sequel. The proof can be found in [39].

Theorem 2.2. *Let* $u \in \mathcal{K}_\psi(\Omega)$ *be a solution to the obstacle problem* (1.1)*, such that* $\mathcal{L}(u, B_R) = 0$ *for all* $B_R \Subset \Omega$*, under the assumptions* (1.5)*,* (1.6)*,* (1.10) *and* (1.9)*. Then*

$$D\psi \in W^{1,r}_{\mathrm{loc}}(\Omega) \Rightarrow V_{2-\mu}(Du) := (1 + |Du|^2)^{-\mu/4} Du \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

This result allows us to obtain the validity of the variational inequality for our minimization problem. This is a relevant point, already in the autonomous case, see for instance [11], [12], [28].

Proposition 2.3. *Let* $u \in \mathcal{K}_\psi(\Omega)$ *be a solution to the obstacle problem* (1.1)*, under the assumptions* (1.5)*,* (1.6)*,* (1.10) *and* (1.9)*. Then, if* $D\psi \in W^{1,r}_{\mathrm{loc}}(\Omega)$*, then the following variational inequality holds*

$$\int_\Omega D_\xi f(x, Du) \cdot D(\varphi - u)\, dx \geq 0 \tag{2.1}$$

for all $\varphi \in W^{1,q}_{\mathrm{loc}}(\Omega)$*,* $\varphi \geq \psi$*.*

The proof of this result can be achieved arguing as in [13], observing that, by virtue of Theorem 2.2 and (1.9), we have the right higher integrability to pass to the limit in the variational inequality, namely the fact that

$$V_{2-\mu}(Du) \in L^{\frac{2n}{n-2}}_{\mathrm{loc}}(\Omega) \Rightarrow Du \in L^q_{\mathrm{loc}}(\Omega),$$

by virtue of (1.9), which yields

$$\frac{q}{2-\mu} < 1 + \frac{1}{r} - \frac{1}{n} < \frac{n}{n-2}.$$

For more details see [39].

3. A priori estimate: the Lipschitz case

3.1. **The linearization procedure.** The linearization trocedure is a process which goes back to [31] and later was refined in [20], see also [32], [33], [34]. We will follow the lines of [4].

We consider a smooth function $h_\varepsilon : (0, \infty) \to [0, 1]$ such that $h'_\varepsilon(s) \leq 0$ for all $s \in (0, \infty)$ and

$$h_\varepsilon(s) = \begin{cases} 1 & \text{for} \quad s \leq \varepsilon \\ 0 & \text{for} \quad s \geq 2\varepsilon \end{cases}$$

Consider the function

$$\varphi = u + t \cdot \eta \cdot h_\varepsilon(u - \psi)$$

with $\eta \in C^1_0(\Omega)$, $\eta \geq 0$ and $0 < t << 1$ as test function, in the variational inequality (2.1). We have

$$\int_\Omega D_\xi f(x, Du) \cdot D(\eta h_\varepsilon(u - \psi))\, dx \geq 0 \qquad \forall\, \eta \in C^1_0(\Omega)\,.$$

Since

$$\eta \mapsto L(\eta) = \int_\Omega D_\xi f(x, Du) \cdot D(\eta h_\varepsilon(u - \psi))\, dx$$

is a bounded positive linear functional, by the Riesz representation theorem there exists a nonnegative measure λ_ε such that

$$\int_\Omega D_\xi f(x,Du)\cdot D(\eta h_\varepsilon(u-\psi))\,dx = \int_\Omega \eta d\lambda_\varepsilon \qquad \forall\,\eta\in C^1_0(\Omega)\,.$$

It is not difficult to prove that the measure λ_ε is independent to ε. Therefore we can write

$$\int_\Omega D_\xi f(x,Du)\cdot D(\eta h_\varepsilon(u-\psi))\,dx = \int_\Omega \eta\, d\lambda \qquad \forall\,\eta\in C^1_0(\Omega)\,.$$

By Theorem 2.2 we have that

$$V_{2-\mu}(Du) := (1+|Du|^2)^{-\frac{\mu}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega), \tag{3.1}$$

Now, in order to identify the measure λ, we may pass to the limit as $\varepsilon\downarrow 0$

$$\int_\Omega -\mathrm{div}(D_\xi f(x,Du))\chi_{[u=\psi]}\eta\,dx = \int_\Omega \eta\,d\lambda \qquad \forall\,\eta\in C^1_0(\Omega)\,. \tag{3.2}$$

By introducing

$$g := -\mathrm{div}(D_\xi f(x,Du))\chi_{[u=\psi]}; \tag{3.3}$$

and combining our results we obtain

$$\int_\Omega D_\xi f(x,Du)\cdot D\eta\,dx = \int_\Omega g\eta\,dx \qquad \forall\,\eta\in C^1_0(\Omega)\,. \tag{3.4}$$

We are left to obtain an L^∞ estimate for g: since $Du = D\psi$ a.e. on the contact set, by (1.6) and (1.7) and the assumption $D\psi\in W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$, we have

$$\begin{aligned}|g| &= \left|\mathrm{div}(D_\xi f(x,Du))\chi_{[u=\psi]}\right| = |\mathrm{div}(D_\xi f(x,D\psi))|\\ &\le \sum_{k=1}^n |f_{\xi_k x_k}(x,D\psi)| + \sum_{k,i=1}^n |f_{\xi_k\xi_i}(x,D\psi)\psi_{x_k x_i}|\\ &\le L(1+|D\psi|^2)^{\frac{q-1}{2}} + L(1+|D\psi|^2)^{\frac{q-2}{2}}|D^2\psi|\end{aligned}$$

that is $g\in L^\infty_{\mathrm{loc}}(\Omega)$.

3.2. **A priori estimate and conclusion.** Our starting point is now (3.4). We make use of the supplementary assumption $u\in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, which is needed in order to let (3.4) to be satisfied; this assumption will be removed by means of the approximation procedure. By this further requirement and Theorem 2.2, the "second variation" system holds

$$\int_\Omega \left(\sum_{i,j=1}^n f_{\xi_i\xi_j}(x,Du)u_{x_j x_s}D_{x_i}\varphi + \sum_{i=1}^n f_{\xi_i x_s}(x,Du)D_{x_i}\varphi\right)dx = \int_\Omega g\,D_{x_s}\varphi\,dx, \tag{3.5}$$

for all $s=1,\dots,n$ and for all $\varphi\in W^{1,2}_0(\Omega)$. We fix $0<\rho<R$ with B_R compactly contained in Ω and we choose $\eta\in\mathcal{C}^1_0(\Omega)$ such that $0\le\eta\le 1$, $\eta\equiv 1$ on B_ρ, $\eta\equiv 0$ outside B_R and $|D\eta|\le\frac{C}{(R-\rho)}$. We test (3.5) with $\varphi=\eta^2(1+|Du|^2)^\gamma u_{x_s}$, for some $\gamma\ge 0$ so that

$$D_{x_i}\varphi = 2\eta\eta_{x_i}(1+|Du|^2)^\gamma u_{x_s} + 2\eta^2\gamma(1+|Du|^2)^{\gamma-1}|Du|D_{x_i}(|Du|)u_{x_s} + \eta^2(1+|Du|^2)^\gamma u_{x_s x_i}$$

Inserting in (3.5) we get:

$$
\begin{aligned}
0 \;=\; & \int_\Omega \sum_{i,j=1}^n f_{\xi_i\xi_j}(x,Du)u_{x_jx_s}2\eta\eta_{x_i}(1+|Du|^2)^\gamma u_{x_s}\,dx \\
& + \int_\Omega \sum_{i,j=1}^n f_{\xi_i\xi_j}(x,Du)u_{x_jx_s}\eta^2(1+|Du|^2)^\gamma u_{x_sx_i}\,dx \\
& + \int_\Omega \sum_{i,j=1}^n f_{\xi_i\xi_j}(x,Du)u_{x_jx_s}2\eta^2\gamma(1+|Du|^2)^{\gamma-1}|Du|D_{x_i}(|Du|)u_{x_s}\,dx \\
& + \int_\Omega \sum_{i=1}^n f_{\xi_ix_s}(x,Du)2\eta\eta_{x_i}(1+|Du|^2)^\gamma u_{x_s}\,dx \\
& + \int_\Omega \sum_{i=1}^n f_{\xi_ix_s}(x,Du)\eta^2(1+|Du|^2)^\gamma u_{x_sx_i}\,dx \\
& + \int_\Omega \sum_{i=1}^n f_{\xi_ix_s}(x,Du)2\eta^2\gamma(1+|Du|^2)^{\gamma-1}|Du|D_{x_i}(|Du|)u_{x_s}\,dx \\
& - \int_\Omega g2\eta\eta_{x_s}\,(1+|Du|^2)^\gamma\,u_{x_s}\,dx \\
& - \int_\Omega g2\eta^2\gamma\,(1+|Du|^2)^{\gamma-1}|Du|\,D_{x_s}(|Du|)\,u_{x_s}\,dx \\
& - \int_\Omega g\,\eta^2(1+|Du|^2)^\gamma u_{x_sx_s}\,dx \\
& =: I_{1,s}+I_{2,s}+I_{3,s}+I_{4,s}+I_{5,s}+I_{6,s}+I_{7,s}+I_{8,s}+I_{9,s}.
\end{aligned}
$$

We sum in the previous equation all terms with respect to s from 1 to n, and we denote by $I_1 - I_9$ the corresponding integrals.
By the Cauchy-Schwarz inequality, the Young inequality and (1.6), we have

$$
\begin{aligned}
|I_1| \;=\; & \left|\int_\Omega 2\eta\,(1+|Du|^2)^\gamma \sum_{i,j,s=1}^n f_{\xi_i\xi_j}(x,Du)\,u_{x_jx_s}\eta_{x_i}u_{x_s}\,dx\right| \\
\le\; & \int_\Omega 2\eta\,(1+|Du|^2)^\gamma \\
& \times \sum_{s=1}^n \left|\sum_{i,j=1}^n f_{\xi_i\xi_j}(x,Du)\,\eta_{x_i}\eta_{x_j}u_{x_s}^2\right|^{\frac12} \left|\sum_{i,j=1}^n f_{\xi_i\xi_j}(x,Du)\,u_{x_sx_i}\,u_{x_sx_j}\right|^{\frac12}\,dx \\
\le\; & \int_\Omega 2\eta\,(1+|Du|^2)^\gamma \\
& \times \left|\sum_{i,j,s=1}^n f_{\xi_i\xi_j}(x,Du)\,\eta_{x_i}\eta_{x_j}u_{x_s}^2\right|^{\frac12} \left|\sum_{i,j,s=1}^n f_{\xi_i\xi_j}(x,Du)\,u_{x_sx_i}\,u_{x_sx_j}\right|^{\frac12}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\int_\Omega (1+|Du|^2)^\gamma \left|\sum_{i,j,s=1}^n f_{\xi_i\xi_j}(x,Du)\,\eta_{x_i}\eta_{x_j}u_{x_s}^2\right| dx\\
&\quad +\frac{1}{4}\int_\Omega \eta^2\,(1+|Du|^2)^\gamma \sum_{i,j,s=1}^n f_{\xi_i\xi_j}(x,Du)\,u_{x_sx_i}\,u_{x_sx_j}\,dx\\
&\leq C\,L\int_\Omega (1+|Du|^2)^{\frac{q-2}{2}+\gamma}\left|\sum_{i,j,s=1}^n \eta_{x_i}\eta_{x_j}u_{x_s}^2\right| dx\\
&\quad +\frac{1}{4}\int_\Omega \eta^2\,(1+|Du|^2)^\gamma \sum_{i,j,s=1}^n f_{\xi_i\xi_j}(x,Du)\,u_{x_sx_i}\,u_{x_sx_j}\,dx\\
&\leq C\int_\Omega |D\eta|^2\,(1+|Du|^2)^{\frac{q-2}{2}+\gamma}\left|\sum_{s=1}^n u_{x_s}^2\right| dx\\
&\quad +\frac{1}{4}\int_\Omega \eta^2(1+|Du|^2)^\gamma \sum_{i,j,s=1}^n f_{\xi_i\xi_j}(x,Du)\,u_{x_sx_i}\,u_{x_sx_j}\,dx\\
&\leq C\int_\Omega |D\eta|^2\,(1+|Du|^2)^{\frac{q}{2}+\gamma}\,dx\\
&\quad +\frac{1}{4}\int_\Omega \eta^2\,(1+|Du|^2)^\gamma \sum_{i,j,s=1}^n f_{\xi_i\xi_j}(x,Du)\,u_{x_jx_s}u_{x_ix_s}\,dx
\end{aligned}
$$

On the other hand, using (1.6) and the fact that $D_{x_j}(|Du|)|Du| = \sum_{k=1}^n u_{x_jx_k}u_{x_k}$, we can estimate the term I_3 as follows:

$$
\begin{aligned}
I_3 &= \int_\Omega \sum_{i,j,s=1}^n f_{\xi_i\xi_j}(x,Du)u_{x_jx_s}\left[2\eta^2\gamma(1+|Du|^2)^{\gamma-1}D_{x_i}(|Du|)|Du|\right]u_{x_s}\,dx\\
&\geq 2\gamma\int_\Omega \eta^2|Du|^{2\gamma-1}\sum_{i,j,s=1}^n f_{\xi_i\xi_j}(x,Du)D_{x_i}(|Du|)u_{x_jx_s}u_{x_s}\,dx\\
&= 2\gamma\int_\Omega \eta^2|Du|^{2\gamma-1}\sum_{i,j=1}^n f_{\xi_i\xi_j}(x,Du)D_{x_i}(|Du|)\left(\sum_{s=1}^n u_{x_jx_s}u_{x_s}\right) dx\\
&= 2\gamma\int_\Omega \eta^2|Du|^{2\gamma}\sum_{i,j=1}^n f_{\xi_i\xi_j}(x,Du)D_{x_i}(|Du|)D_{x_j}(|Du|)\,dx\\
&\geq 2\gamma\,\nu\int_\Omega \eta^2|Du|^{2\gamma}|D(|Du|)|^2(1+|Du|^2)^{-\frac{\mu}{2}}\,dx\\
&\geq 0
\end{aligned}
$$

We can estimate the fourth term by the Cauchy-Schwarz and the Young inequalities, together with (1.7), as follows

$$\begin{aligned}
|I_4| &= \left|2\int_\Omega \eta(1+|Du|^2)^\gamma \sum_{i,s=1}^n f_{\xi_i x_s}(x,Du)\eta_{x_i}u_{x_s}\,dx\right| \\
&\overset{(1.7)}{\leq} 2L\int_\Omega \eta\,(1+|Du|^2)^{\gamma+\frac{q-1}{2}} \sum_{i,s=1}^n |\eta_{x_i}u_{x_s}|\,dx \\
&\leq C\int_\Omega \eta|D\eta||Du|\,(1+|Du|^2)^{\gamma+\frac{q-1}{2}}\,dx \\
&\leq C\int_\Omega (\eta^2+|D\eta|^2)(1+|Du|^2)^{\gamma+\frac{q}{2}}\,dx
\end{aligned}$$

We can estimate the fifth term observing that

$$\frac{q-1}{2}=\frac{2q-2}{4}=\frac{2q-2-\mu+\mu}{4}=-\frac{\mu}{4}+\frac{2q-2+\mu}{4}=-\frac{\mu}{4}+\frac{q}{2}-\frac{2-\mu}{4} \tag{3.6}$$

and by the Cauchy-Schwarz and the Young inequalities, together with (1.7), as follows

$$\begin{aligned}
|I_5| &= \left|\int_\Omega \eta^2(1+|Du|^2)^\gamma \sum_{i,s=1}^n f_{\xi_i x_s}(x,Du)u_{x_s x_i}\,dx\right| \\
&\overset{(1.7)}{\leq} L\int_\Omega \eta^2(1+|Du|^2)^{\gamma+\frac{q-1}{2}} \left|\sum_{i,s=1}^n u_{x_s x_i}\right| dx \\
&= L\int_\Omega \eta^2(1+|Du|^2)^{\gamma+\frac{q-1}{2}}|D^2u|\,dx \\
&= L\int_\Omega \left[\eta^2(1+|Du|^2)^{\gamma-\frac{\mu}{2}}|D^2u|^2\right]^{\frac{1}{2}} \left[\eta^2(1+|Du|^2)^{\gamma+q-\frac{2-\mu}{2}}\right]^{\frac{1}{2}} dx \\
&\leq \frac{L}{4}\int_\Omega \eta^2(1+|Du|^2)^{\gamma-\frac{\mu}{2}}|D^2u|^2\,dx + C\int_\Omega \eta^2(1+|Du|^2)^{\gamma+q-\frac{2-\mu}{2}}\,dx
\end{aligned}$$

Finally, by (3.6), $|D(|Du|)|\leq|D^2u|$, the Cauchy-Schwarz and the Young inequalities and (1.7) we have that

$$\begin{aligned}
|I_6| &= \left|2\gamma\int_\Omega \sum_{i,s=1}^n f_{\xi_i x_s}(x,Du)\eta^2(1+|Du|^2)^{\gamma-1}|Du|D_{x_i}(|Du|)u_{x_s}\,dx\right| \\
&= \left|2\gamma\int_\Omega \eta^2(1+|Du|^2)^{\gamma-1}|Du|\sum_{i,s=1}^n f_{\xi_i x_s}(x,Du)D_{x_i}(|Du|)u_{x_s}\,dx\right| \\
&\leq 2\,\gamma\int_\Omega \eta^2(1+|Du|^2)^{\gamma-\frac{1}{2}}\left|\sum_{i,s=1}^n f_{\xi_i x_s}(x,Du)D_{x_i}(|Du|)u_{x_s}\right| dx \\
&\leq 2\,\gamma\,L\int_\Omega \eta^2(1+|Du|^2)^{\gamma-\frac{1}{2}+\frac{q-1}{2}}\left|\sum_{i,s=1}^n D_{x_i}(|Du|)u_{x_s}\right| dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \; 2\,\gamma\,L \int_\Omega \eta^2(1+|Du|^2)^{\gamma-\frac{1}{2}+\frac{q-1}{2}}|D(|Du|)||Du|\,dx \\
&\leq \; 2\,\gamma\,L \int_\Omega \eta^2(1+|Du|^2)^{\gamma+\frac{q-1}{2}}|D^2u|\,dx \\
&\overset{(3.6)}{=} \; L \int_\Omega \left[\eta^2(1+|Du|^2)^{\gamma-\frac{\mu}{2}}|D^2u|^2\right]^{\frac{1}{2}} \left[4\,\gamma^2\,\eta^2(1+|Du|^2)^{\gamma+q-\frac{2-\mu}{2}}\right]^{\frac{1}{2}}\,dx \\
&\leq \; \frac{L}{4}\int_\Omega \eta^2|D^2u|^2(1+|Du|^2)^{\gamma-\frac{\mu}{2}}\,dx + C\,\gamma^2 \int_\Omega \eta^2(1+|Du|^2)^{\gamma+q-\frac{2-\mu}{2}}\,dx
\end{aligned}
$$

where the constant C depends only on q, μ, L but it is independent of γ.
Let us now deal with the terms containing the function g. We use the bound

$$\|g\|_{L^\infty_{\text{loc}}(\Omega)} \leq C,$$

established in Section 3.1; even in this case the constant C is independent of γ.
We first have

$$
\begin{aligned}
|I_7| &= \left|\int_\Omega 2\,\eta\,(1+|Du|^2)^\gamma \sum_{s=1}^n g\,\eta_{x_s}\,u_{x_s}\,dx\right| \\
&\leq \int_\Omega 2\,\eta\,(1+|Du|^2)^\gamma\,\|g\|_{L^\infty(B_R)}\,|D\eta|\,|Du|\,dx \\
&\leq C\,\|g\|_{L^\infty(B_R)} \int_\Omega |D\eta|^2\,(1+|Du|^2)^{\gamma+\frac{1}{2}}\,dx \\
&\leq C\,\|g\|_{L^\infty(B_R)} \int_\Omega |D\eta|^2\,(1+|Du|^2)^{\gamma+\frac{q}{2}}\,dx \\
&\leq C\,\|g\|_{L^\infty(B_R)} \int_\Omega |D\eta|^2\,(1+|Du|^2)^{\gamma+q-\frac{2-\mu}{2}}\,dx
\end{aligned}
$$

because we know that

$$
\begin{gathered}
q \geq 2-\mu \\
2q \geq q+2-\mu \\
2q-2+\mu \geq q
\end{gathered}
$$

so we have the inequality

$$q-\frac{2-\mu}{2} = \frac{2q-2+\mu}{2} \geq \frac{q}{2} \tag{3.7}$$

We know that

$$
\begin{aligned}
|I_8| &\leq 2\gamma\|g\|_{L^\infty(B_R)} \int_\Omega \eta^2(1+|Du|^2)^{\gamma-1}\,|D(|Du|)|\,|Du|^2\,dx \\
&\leq 2\gamma\|g\|_{L^\infty(B_R)} \int_\Omega \eta^2(1+|Du|^2)^{\gamma-1}\,|D^2u|\,|Du|^2\,dx
\end{aligned}
$$

and that

$$|I_9| \leq \|g\|_{L^\infty(B_R)} \int_\Omega \eta^2\,(1+|Du|^2)^\gamma\,|D^2u|\,dx$$

so we can estimate them together and, going on as we did in I_6, we have

$$\begin{aligned}
|I_8|+|I_9| &\leq 2\gamma\,\|g\|_{L^\infty(B_R)}\int_\Omega \eta^2(1+|Du|^2)^{\gamma-1}\,|D^2u|\,|Du|^2\,dx \\
&\quad +\|g\|_{L^\infty(B_R)}\int_\Omega \eta^2\,(1+|Du|^2)^{\gamma}\,|D^2u|\,dx \\
&\leq 2(\gamma+1)\,\|g\|_{L^\infty(B_R)}\int_\Omega \eta^2(1+|Du|^2)^{\gamma}\,|D^2u|\,dx \\
&\leq \frac{L}{4}\int_\Omega \eta^2|D^2u|^2(1+|Du|^2)^{\gamma-\frac{\mu}{2}}\,dx \\
&\quad +C\,\|g\|^2_{L^\infty(B_R)}\,(1+\gamma^2)\int_\Omega \eta^2(1+|Du|^2)^{\gamma+q-\frac{2-\mu}{2}}\,dx
\end{aligned}$$

Summing up and using (1.6) we obtain

$$\int_\Omega \eta^2\,(1+|Du|^2)^{-\frac{\mu}{2}+\gamma}\,|D^2u|^2\,dx \leq C\,(1+\gamma^2)\int_\Omega (\eta^2+|D\eta|^2)\,(1+|Du|^2)^{q-\frac{2-\mu}{2}+\gamma}\,dx \tag{3.8}$$

where the constant C depends on ν, L, n, q, μ and on the local bounds on function g but is independent of γ. By Sobolev embedding theorem, recalling that $\mu \leq 2$, we have

$$\begin{aligned}
&\left(\int_\Omega \eta^{2^*}(1+|Du|^2)^{\left(\gamma+\frac{2-\mu}{2}\right)\frac{2^*}{2}}\,dx\right)^{\frac{2}{2^*}} \\
&= \left(\int_\Omega \eta^{2^*}(1+|Du|^2)^{\left(\frac{\gamma}{2}+\frac{2-\mu}{4}\right)2^*}\,dx\right)^{\frac{2}{2^*}} \\
&\leq C\int_\Omega \left|D\left[\eta\,(1+|Du|^2)^{\frac{\gamma}{2}+\frac{2-\mu}{4}}\right]\right|^2 dx \\
&\leq C\int_\Omega |D\eta|^2\,(1+|Du|^2)^{\gamma+\frac{2-\mu}{2}}\,dx \\
&\quad +C\,(1+\gamma^2)\int_\Omega \eta^2\,\left[(1+|Du|^2)^{\frac{\gamma}{2}+\frac{2-\mu}{4}-1}\,|Du|\,|D^2u|\right]^2 dx \\
&= C\int_\Omega |D\eta|^2\,(1+|Du|^2)^{\gamma+\frac{2-\mu}{2}}\,dx \\
&\quad +C\,(1+\gamma^2)\int_\Omega \eta^2\,(1+|Du|^2)^{\gamma+\frac{2-\mu}{2}-2}\,|Du|^2\,|D^2u|^2\,dx \\
&= C\int_\Omega |D\eta|^2\,(1+|Du|^2)^{\gamma+\frac{2-\mu}{2}}\,dx \\
&\quad +C\,(1+\gamma^2)\int_\Omega \eta^2\,(1+|Du|^2)^{\gamma-\frac{\mu}{2}-1}\,|Du|^2\,|D^2u|^2\,dx \\
&\leq C\int_\Omega |D\eta|^2\,(1+|Du|^2)^{\gamma+\frac{2-\mu}{2}}\,dx \\
&\quad +C\,(1+\gamma^2)\int_\Omega \eta^2\,(1+|Du|^2)^{\gamma-\frac{\mu}{2}}\,|D^2u|^2\,dx,
\end{aligned}$$

where we set

$$2^* = \begin{cases} \dfrac{2n}{n-2} & \text{if } n \geq 3 \\ \\ \text{any finite exponent} & \text{if } n = 2 \end{cases}$$

and we can observe that $2^* > 2$. Thanks to the left hand side of (1.4), we know that

$$1 \leq \frac{q}{2-\mu}$$

$$2 - \mu \leq q$$

$$\frac{2-\mu}{2} \leq q - \frac{2-\mu}{2}$$

which allow us to say that

$$\gamma + \frac{2-\mu}{2} \leq q - \frac{2-\mu}{2} + \gamma,$$

so we can write

$$\begin{aligned} \left(\int_\Omega \eta^{2^*} (1 + |Du|^2)^{\left(\gamma + \frac{2-\mu}{2}\right)\frac{2^*}{2}} \, dx \right)^{\frac{2}{2^*}} \quad &\leq \quad C \int_\Omega |D\eta|^2 \, (1 + |Du|^2)^{q - \frac{2-\mu}{2} + \gamma} \, dx \\ &\quad + C\,(1+\gamma^2) \int_\Omega \eta^2 \, (1 + |Du|^2)^{\gamma - \frac{\mu}{2}} \, |D^2 u|^2 \, dx. \end{aligned}$$

Thanks to (3.8), we finally get

$$\left(\int_\Omega \eta^{2^*} (1 + |Du|^2)^{\left(\gamma + \frac{2-\mu}{2}\right)\frac{2^*}{2}} \, dx \right)^{\frac{2}{2^*}} \quad \leq \quad C(1+\gamma^2) \int_\Omega (\eta^2 + |D\eta|^2)(1 + |Du|^2)^{\gamma + q - \frac{2-\mu}{2}} \, dx$$

from which we deduce

$$\left(\int_{B_\rho} (1 + |Du|^2)^{\left(\gamma + \frac{2-\mu}{2}\right)\frac{2^*}{2}} \, dx \right)^{\frac{2}{2^*}} \leq C \, \frac{(1+\gamma^2)}{(R-\rho)^2} \int_{B_R} (1 + |Du|^2)^{\gamma + q - \frac{2-\mu}{2}} \, dx$$

for any $0 < \rho < R$.
We introduce now the quantity σ such that

$$q - \frac{2-\mu}{2} = \frac{2-\mu}{2} + \sigma; \tag{3.9}$$

we observe that $\sigma > 0$ due to left hand side of assumption (1.4). Therefore

$$\begin{aligned} \left(\int_{B_\rho} (1 + |Du|^2)^{\left(\gamma + \frac{2-\mu}{2}\right)\frac{2^*}{2}} \, dx \right)^{\frac{2}{2^*}} \quad &\leq \quad C \, \frac{(1+\gamma^2)}{(R-\rho)^2} \int_{B_R} (1 + |Du|^2)^{\gamma + q - \frac{2-\mu}{2}} \, dx \\ &= \quad C \, \frac{(1+\gamma^2)}{(R-\rho)^2} \int_{B_R} (1 + |Du|^2)^{\gamma + \frac{2-\mu}{2} + \sigma} \, dx \end{aligned}$$

which allows us to say that

$$\left(\int_{B_\rho}(1+|Du|^2)^{\left(\gamma+\frac{2-\mu}{2}\right)\frac{2^*}{2}}\,dx\right)^{\frac{2}{2^*}} \le C\,\frac{(1+\gamma^2)}{(R-\rho)^2}\,\|(1+|Du|^2)\|^{\sigma}_{L^\infty(B_R)}\int_{B_R}(1+|Du|^2)^{\gamma+\frac{2-\mu}{2}}\,dx \tag{3.10}$$

At this point we inductively define the exponents

$$\gamma_1:=0,\qquad \gamma_{k+1}:=\left[\left(\gamma_k+\frac{2-\mu}{2}\right)\frac{2^*}{2}-\frac{2-\mu}{2}\right],\qquad \alpha_k:=\gamma_k+\frac{2-\mu}{2}, \tag{3.11}$$

for every integer $k\ge 1$. It follows that

$$\alpha_{k+1}=\left(\gamma_k+\frac{2-\mu}{2}\right)\frac{2^*}{2}=\chi\,\alpha_k\qquad\text{with }\chi:=\frac{2^*}{2}$$

By induction we can prove that

$$\gamma_k:=\frac{2-\mu}{2}\left[\left(\frac{2^*}{2}\right)^{k-1}-1\right]$$

Now we consider $0<\rho_0<R_0$ and set

$$R_k=\rho_0+\frac{(R_0-\rho_0)}{2^k}\qquad\forall k\ge 1$$

so that $R_{k+1}\le R_k$ for all $k\ge 1$ and

$$R_k-R_{k+1}=\frac{(R_0-\rho_0)}{2^{k+1}}.$$

We rewrite (3.10) with $\rho=R_{k+1}$ and $R=R_k$. We obtain

$$\begin{aligned}&\left(\int_{B_{R_{k+1}}}(1+|Du|^2)^{\left(\gamma_k+\frac{2-\mu}{2}\right)\frac{2^*}{2}}\,dx\right)^{\frac{2}{2^*}}\\ \le\;& C\,\frac{(1+\gamma_k^2)}{(R_k-R_{k+1})^2}\,\|(1+|Du|^2)\|^{\sigma}_{L^\infty(B_{R_k})}\int_{B_{R_k}}(1+|Du|^2)^{\gamma_k+\frac{2-\mu}{2}}\,dx\end{aligned} \tag{3.12}$$

from which we deduce

$$\begin{aligned}&\left(\int_{B_{R_{k+1}}}(1+|Du|^2)^{\left(\gamma_k+\frac{2-\mu}{2}\right)\frac{2^*}{2}}\,dx\right)^{\frac{2}{2^*}}\\ \le\;& C\,\frac{4^{k+1}\,(1+\gamma_k^2)}{(R_0-\rho_0)^2}\,\|(1+|Du|^2)\|^{\sigma}_{L^\infty(B_{R_k})}\int_{B_{R_k}}(1+|Du|^2)^{\gamma_k+\frac{2-\mu}{2}}\,dx\end{aligned}$$

We set

$$A_k:=\left(\int_{B_{R_k}}(1+|Du|^2)^{(\gamma_k+\frac{2-\mu}{2})}\,dx\right)^{\frac{1}{\gamma_k+\frac{2-\mu}{2}}}$$

so that

$$\begin{aligned} A_{k+1} &:= \left(\int_{B_{R_{k+1}}} (1+|Du|^2)^{(\gamma_{k+1}+\frac{2-\mu}{2})}\, dx \right)^{\frac{1}{\gamma_{k+1}+\frac{2-\mu}{2}}} \\ &= \left(\int_{B_{R_{k+1}}} (1+|Du|^2)^{(\gamma_k+\frac{2-\mu}{2})\frac{2^*}{2}}\, dx \right)^{\frac{1}{(\gamma_k+\frac{2-\mu}{2})\frac{2^*}{2}}} \end{aligned}$$

In this way, (3.12) becomes

$$A_{k+1} \le \left[C\, \frac{4^{k+1}\,(1+\gamma_k^2)}{(R_0-\rho_0)^2}\, \|(1+|Du|^2)\|^{\sigma}_{L^\infty(B_{R_k})} \right]^{\frac{1}{\gamma_k+\frac{2-\mu}{2}}} A_k$$

which can be also rewritten, in view of (3.11), as

$$\begin{aligned} A_{k+1} &\le \left[C\, \frac{4^{k+1}\,(1+\gamma_k^2)}{(R_0-\rho_0)^2}\, \|(1+|Du|^2)\|^{\sigma}_{L^\infty(B_{R_k})} \right]^{\frac{1}{\alpha_k}} A_k \\ &\le \prod_{i=1}^{k} \left[C\, \frac{4^{i+1}\,(1+\gamma_i^2)}{(R_0-\rho_0)^2}\, \|(1+|Du|^2)\|^{\sigma}_{L^\infty(B_{R_i})} \right]^{\frac{1}{\alpha_i}} A_1. \end{aligned}$$

We estimate now the term in the right hand side multiplying A_1 to show it is finite. First of all, once more inductively by (3.11), we have

$$\alpha_{k+1} = \frac{2-\mu}{2} \left(\frac{2^*}{2} \right)^k$$

so we can say that

$$\sum_{i=1}^{\infty} \frac{1}{\alpha_i} = \sum_{i=1}^{\infty} \frac{2}{2-\mu} \left(\frac{2}{2^*} \right)^{i-1} = \frac{2}{2-\mu} \sum_{i=1}^{\infty} \left(\frac{2}{2^*} \right)^{i-1}$$

We have for $n \ge 3$

$$\begin{aligned} & \lim_{k\to+\infty} \prod_{i=1}^{k} \left[\|1+|Du|^2\|^{\sigma}_{L^\infty(B_{R_i})} \right]^{\frac{1}{\alpha_i}} \\ \le\ & \lim_{k\to+\infty} \exp\left(\log\left(\prod_{i=1}^{k} \left[\|1+|Du|^2\|^{\sigma}_{L^\infty(B_{R_0})} \right]^{\frac{1}{\alpha_i}} \right)\right) \\ =\ & \lim_{k\to+\infty} \exp\left(\sum_{i=1}^{k} \left(\frac{\sigma\, \log \|1+|Du|^2\|_{L^\infty(B_{R_0})}}{\alpha_i} \right)\right) \\ =\ & \lim_{k\to+\infty} \|1+|Du|^2\|^{\sigma \sum_{i=1}^{k}\left(\frac{1}{\alpha_i}\right)}_{L^\infty(B_{R_0})} \\ =\ & \|1+|Du|^2\|^{\sigma \sum_{i=1}^{\infty}\left(\frac{1}{\alpha_i}\right)}_{L^\infty(B_{R_0})} \\ =\ & \|1+|Du|^2\|^{\sigma \frac{2}{2-\mu} \sum_{i=1}^{\infty}(\frac{2}{2^*})^{i-1}}_{L^\infty(B_{R_0})} \end{aligned}$$

$$
\begin{aligned}
&\overset{(3.9)}{=} \quad \|1+|Du|^2\|_{L^\infty(B_{R_0})}^{(q-2+\mu)\,\frac{2}{2-\mu}\,\sum_{i=1}^{\infty}\left(\frac{2}{\frac{2n}{n-2}}\right)^{i-1}} \\
&= \quad \|1+|Du|^2\|_{L^\infty(B_{R_0})}^{(q-2+\mu)\,\frac{2}{2-\mu}\,\sum_{i=1}^{\infty}(\frac{n-2}{n})^{i-1}} \\
&= \quad \|1+|Du|^2\|_{L^\infty(B_{R_0})}^{(q-2+\mu)\,\frac{2}{2-\mu}\,\frac{1}{1-\frac{n-2}{n}}} \\
&= \quad \|1+|Du|^2\|_{L^\infty(B_{R_0})}^{\frac{q-2+\mu}{2-\mu}\,n}
\end{aligned}
$$

On the other hand, let us define

$$
M_k = \prod_{i=1}^{k} \left[C\, 4^{i+1}\,(1+\gamma_i^2)\right]^{\frac{1}{\alpha_i}} = \exp\left(\sum_{i=1}^{k} \frac{\log(C\,(1+\gamma_i^2)\,4^{i+1})}{\alpha_i}\right)
$$

and we have

$$
\begin{aligned}
& \lim_{k\to+\infty} \sum_{i=1}^{k} \frac{\log(C\,(1+\gamma_i^2)\,4^{i+1})}{\alpha_i} \\
=\ & \lim_{k\to+\infty} \sum_{i=1}^{k} \frac{\log(C)+(i+1)\,\log(4)+\log(1+\gamma_i^2)}{\alpha_i} \\
=\ & \sum_{i=1}^{\infty} \frac{\log(C)+(i+1)\,\log(4)+\log(1+\gamma_i^2)}{\alpha_i} \\
=\ & \frac{2}{2-\mu} \sum_{i=1}^{\infty} \frac{\log(C)+(i+1)\,\log(4)+\log(1+\gamma_i^2)}{\chi^{i-1}} \\
\le\ & \frac{2}{2-\mu} \sum_{i=1}^{\infty} \frac{\log(C)+(i+1)\,\log(4)+2\,\log(\gamma_i)}{\chi^{i-1}} \\
\le\ & \frac{2}{2-\mu} \sum_{i=1}^{\infty} \frac{\log(C)+(i+1)\,\log(4)+2\,\log(\chi)\,(i-1)}{\chi^{i-1}} \\
=\ & \frac{2}{2-\mu} \sum_{i=1}^{\infty} \frac{Ai+B}{\chi^{i-1}} \\
=\ & \frac{2}{2-\mu} \sum_{i=1}^{\infty} (Ai+B)\left(\frac{2}{2^*}\right)^{i-1}
\end{aligned}
$$

Since $\chi = \frac{2^*}{2} \geq 1$ and A, B are constants, we have that

$$
\lim_{k\to+\infty} M_k \leq \exp\left(\frac{2}{2-\mu} \sum_{i=1}^{\infty} (Ai+B)\left(\frac{2}{2^*}\right)^{i-1}\right) = M < \infty.
$$

Thanks to that and the fact that $R_0 > R_1$, together with the fact that $A_1 < \infty$, letting $k \to +\infty$, we can write that

$$\|1+|Du|^2\|_{L^\infty(B_{\rho_0})} \leq \frac{M}{(R_0-\rho_0)^{\frac{2n}{2-\mu}}} \|1+|Du|^2\|^{\frac{q-2+\mu}{2-\mu}n} \left(\int_{B_{R_0}} (1+|Du|^2)^{\frac{2-\mu}{2}} dx\right)^{\frac{2}{2-\mu}}$$

Assumption (1.4) implies:

$$\frac{q-2+\mu}{2-\mu}\, n = \left(\frac{q}{2-\mu} - 1\right) n < 1,$$

so we can use the Young's inequality with exponents $\frac{2-\mu}{(q-2+\mu)\,n}$ and $\frac{2-\mu}{(2-\mu)-(q-2+\mu)\,n}$, to get:

$$\|1+|Du|^2\|_{L^\infty(B_{\rho_0})} \leq \frac{1}{2}\|1+|Du|^2\|_{L^\infty(B_{R_0})} + \left(\frac{M}{(R_0-\rho_0)^{\frac{2n}{2-\mu}}}\right)^\theta \left(\int_{B_{R_0}} (1+|Du|^2)^{\frac{2-\mu}{2}} dx\right)^{\frac{2\theta}{2-\mu}}$$

for an exponent $\theta = \theta\,(n, r, q, \mu) > 0$. Since previous estimate holds true for $\rho < \rho_0 < R_0 < R$ by Lemma 2.1 we get the desired a priori estimate

$$\|1+|Du|^2\|_{L^\infty(B_\rho)} \leq \left(\frac{M}{(R-\rho)^{\frac{2n}{2-\mu}}}\right)^\theta \left(\int_{B_R} (1+|Du|^2)^{\frac{2-\mu}{2}} dx\right)^{\frac{2\theta}{2-\mu}}$$

The case $n = 2$ can be treated in an analogous way, only we need to observe that in this case we would have

$$\prod_{i=1}^{k} \left[\|1+|Du|^2\|^{\sigma}_{L^\infty(B_{R_i})}\right]^{\frac{1}{\alpha_i}} \leq \|1+|Du|^2\|^{\frac{\sigma\, 2}{2-\mu}\frac{2^*}{2^*-2}}_{L^\infty(B_{R_0})}$$

and passing to the limit in the exponent of the term in the right hand side as $2^* \to \infty$ we would obtain

$$\frac{2\sigma}{2-\mu} < 1 \Leftrightarrow q < \frac{3}{2}(2-\mu),$$

which is exactly (1.4) in the case $n = 2$. The rest of the proof follows similarly.

4. A priori estimate: the Sobolev case

By proceeding along the same lines as in Section 3.1 to obtain

$$\int_\Omega D_\xi f(x, Du)\cdot D\eta\, dx = \int_\Omega g\eta\, dx \qquad \forall\, \eta \in C^1_0(\Omega)\,. \tag{4.1}$$

We are left to obtain an L^r estimate for g: since $Du = D\psi$ a.e. on the contact set, by (1.6) and (1.10) and the assumption $D\psi \in W^{1,r}_{\rm loc}(\Omega;\mathbb{R}^n)$, we have

$$\begin{aligned} |g| &= \left|\operatorname{div}(D_\xi f(x,Du))\chi_{[u=\psi]}\right| = |\operatorname{div}(D_\xi f(x,D\psi))| \\ &\leq \sum_{k=1}^n |f_{\xi_k x_k}(x,D\psi)| + \sum_{k,i=1}^n |f_{\xi_k\xi_i}(x,D\psi)\psi_{x_k x_i}| \\ &\leq h(x)\,(1+|D\psi|^2)^{\frac{q-1}{2}} + L(1+|D\psi|^2)^{\frac{q-2}{2}}|D^2\psi| \end{aligned}$$

that is $g \in L^r_{\rm loc}(\Omega)$.

By requiring the same additional a priori assumptions as in Section 3.2, also in this case we can assume that the "second variation" system holds

$$\int_\Omega \left(\sum_{i,j=1}^n f_{\xi_i\xi_j}(x,Du)u_{x_jx_s}D_{x_i}\varphi + \sum_{i=1}^n f_{\xi_i x_s}(x,Du)D_{x_i}\varphi \right) dx = \int_\Omega g\, D_{x_s}\varphi\, dx, \tag{4.2}$$

for all $s = 1, \dots, n$ and for all $\varphi \in W_0^{1,2}(\Omega)$. We fix $0 < \rho < R$ with B_R compactly contained in Ω and we choose $\eta \in \mathcal{C}_0^1(\Omega)$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on B_ρ, $\eta \equiv 0$ outside B_R and $|D\eta| \le \frac{C}{(R-\rho)}$. We test (3.5) with $\varphi = \eta^2(1+|Du|^2)^\gamma u_{x_s}$, for some $\gamma \ge 0$ so that

$$D_{x_i}\varphi = 2\eta\eta_{x_i}(1+|Du|^2)^\gamma u_{x_s} + 2\eta^2\gamma(1+|Du|^2)^{\gamma-1}|Du|D_{x_i}(|Du|)u_{x_s} + \eta^2(1+|Du|^2)^\gamma u_{x_sx_i}$$

Inserting in (4.2) we get:

$$\begin{aligned} 0 \;=\;& \int_\Omega \sum_{i,j=1}^n f_{\xi_i\xi_j}(x,Du)u_{x_jx_s}2\eta\eta_{x_i}(1+|Du|^2)^\gamma u_{x_s}\, dx \\ &+ \int_\Omega \sum_{i,j=1}^n f_{\xi_i\xi_j}(x,Du)u_{x_jx_s}\eta^2(1+|Du|^2)^\gamma u_{x_sx_i}\, dx \\ &+ \int_\Omega \sum_{i,j=1}^n f_{\xi_i\xi_j}(x,Du)u_{x_jx_s}2\eta^2\gamma(1+|Du|^2)^{\gamma-1}|Du|D_{x_i}(|Du|)u_{x_s}\, dx \\ &+ \int_\Omega \sum_{i=1}^n f_{\xi_i x_s}(x,Du)2\eta\eta_{x_i}(1+|Du|^2)^\gamma u_{x_s}\, dx \\ &+ \int_\Omega \sum_{i=1}^n f_{\xi_i x_s}(x,Du)\eta^2(1+|Du|^2)^\gamma u_{x_sx_i}\, dx \\ &+ \int_\Omega \sum_{i=1}^n f_{\xi_i x_s}(x,Du)2\eta^2\gamma(1+|Du|^2)^{\gamma-1}|Du|D_{x_i}(|Du|)u_{x_s}\, dx \\ &- \int_\Omega g2\eta\eta_{x_s}\,(1+|Du|^2)^\gamma\, u_{x_s}\, dx \\ &- \int_\Omega g2\eta^2\gamma\,(1+|Du|^2)^{\gamma-1}|Du|\, D_{x_s}(|Du|)\, u_{x_s}\, dx \\ &- \int_\Omega g\,\eta^2(1+|Du|^2)^\gamma u_{x_sx_s}\, dx \\ &=: I_{1,s} + I_{2,s} + I_{3,s} + I_{4,s} + I_{5,s} + I_{6,s} + I_{7,s} + I_{8,s} + I_{9,s}. \end{aligned}$$

We sum in the previous equation all terms with respect to s from 1 to n, and we denote by $I_1 - I_9$ the corresponding integrals.

By the fact that $D_{x_j}(|Du|)|Du| = \sum_{k=1}^n u_{x_jx_k}u_{x_k}$, the Cauchy-Schwarz inequality, the Young inequality and (1.6), we can estimate the terms I_1 and I_3 as in Section 3.2.

We can estimate the fourth term by the Cauchy-Schwarz and the Young inequalities, together with (1.10), as follows

$$\begin{aligned} |I_4| &= \left| 2\int_\Omega \eta\,(1+|Du|^2)^\gamma \sum_{i,s=1}^n f_{\xi_i x_s}(x,Du)\,\eta_{x_i} u_{x_s}\,dx \right| \\ &\overset{(1.10)}{\le} 2\int_\Omega |h(x)|\,\eta\,(1+|Du|^2)^{\gamma+\frac{q-1}{2}} \sum_{i,s=1}^n |\eta_{x_i} u_{x_s}|\,dx \\ &\le C\int_\Omega |h(x)|\,\eta\,|D\eta||Du|\,(1+|Du|^2)^{\gamma+\frac{q-1}{2}}\,dx \\ &\le C\int_\Omega |h(x)|\,(\eta^2+|D\eta|^2)(1+|Du|^2)^{\gamma+\frac{q}{2}}, \end{aligned}$$

We can estimate the fifth term by (3.6) and by the Cauchy-Schwarz and the Young inequalities, together with (1.10), as follows

$$\begin{aligned} |I_5| &= \left| \int_\Omega \eta^2(1+|Du|^2)^\gamma \sum_{i,s=1}^n f_{\xi_i x_s}(x,Du) u_{x_s x_i}\,dx \right| \\ &\overset{(1.10)}{\le} \int_\Omega |h(x)|\,\eta^2\,(1+|Du|^2)^{\gamma+\frac{q-1}{2}} \left|\sum_{i,s=1}^n u_{x_s x_i}\right| dx \\ &= \int_\Omega |h(x)|\,\eta^2\,(1+|Du|^2)^{\gamma+\frac{q-1}{2}} |D^2u|\,dx \\ &= \int_\Omega \left[\eta^2(1+|Du|^2)^{\gamma-\frac{\mu}{2}}|D^2u|^2\right]^{\frac12} \left[h^2(x)\,\eta^2(1+|Du|^2)^{\gamma+q-\frac{2-\mu}{2}}\right]^{\frac12} dx \\ &\le \frac14\int_\Omega \eta^2(1+|Du|^2)^{\gamma-\frac{\mu}{2}}|D^2u|^2\,dx + \int_\Omega h^2(x)\,\eta^2(1+|Du|^2)^{\gamma+q-\frac{2-\mu}{2}}\,dx. \end{aligned}$$

Finally, by (3.6), $|D(|Du|)| \le |D^2u|$, the Cauchy-Schwarz and the Young inequalities and (1.10) we have that

$$\begin{aligned} |I_6| &= \left| 2\gamma \int_\Omega \sum_{i,s=1}^n f_{\xi_i x_s}(x,Du)\eta^2(1+|Du|^2)^{\gamma-1}|Du|D_{x_i}(|Du|)u_{x_s}\,dx \right| \\ &= \left| 2\gamma \int_\Omega \eta^2(1+|Du|^2)^{\gamma-1}|Du| \sum_{i,s=1}^n f_{\xi_i x_s}(x,Du)D_{x_i}(|Du|)u_{x_s}\,dx \right| \\ &\le 2\,\gamma \int_\Omega \eta^2(1+|Du|^2)^{\gamma-\frac12} \left|\sum_{i,s=1}^n f_{\xi_i x_s}(x,Du)D_{x_i}(|Du|)u_{x_s}\right| dx \\ &\le 2\,\gamma \int_\Omega \eta^2\,|h(x)|\,(1+|Du|^2)^{\gamma-\frac12+\frac{q-1}{2}} \left|\sum_{i,s=1}^n D_{x_i}(|Du|)u_{x_s}\right| dx \\ &\le 2\,\gamma \int_\Omega \eta^2\,|h(x)|\,(1+|Du|^2)^{\gamma-\frac12+\frac{q-1}{2}} |D(|Du|)||Du|\,dx \end{aligned}$$

$$
\begin{aligned}
&\leq 2\gamma \int_\Omega \eta^2\, |h(x)|\, (1+|Du|^2)^{\gamma+\frac{q-1}{2}} |D^2u|\, dx \\
&\overset{(3.6)}{=} \int_\Omega \left[\eta^2(1+|Du|^2)^{\gamma-\frac{\mu}{2}} |D^2u|^2\right]^{\frac{1}{2}} \left[4\,\gamma^2\,\eta^2\,h^2(x)\,(1+|Du|^2)^{\gamma+q-\frac{2-\mu}{2}}\right]^{\frac{1}{2}} dx \\
&\leq \frac{1}{4}\int_\Omega \eta^2|D^2u|^2(1+|Du|^2)^{\gamma-\frac{\mu}{2}}\, dx + C\,\gamma^2 \int_\Omega \eta^2\, h^2(x)\,(1+|Du|^2)^{\gamma+q-\frac{2-\mu}{2}}\, dx,
\end{aligned}
$$

where the constant C depends only on q,μ but it is independent of γ.
Let us now deal with the terms containing the function g. We use the bound

$$
\|g\|_{L^r_{\mathrm{loc}}(\Omega)} \leq C,
$$

established thanks to the new assumption on the gradient of the obstacle; also in this case the constant C is independent of γ.
By (3.7), we first have

$$
\begin{aligned}
|I_7| &= \left|\int_\Omega 2\,\eta\,(1+|Du|^2)^{\gamma} \sum_{s=1}^n g\,\eta_{x_s}\,u_{x_s}\, dx\right| \\
&\leq \int_\Omega 2\,\eta\,(1+|Du|^2)^{\gamma}\, |g|\, |D\eta|\, |Du|\, dx \\
&\leq C\int_\Omega |g|\, |D\eta|^2\,(1+|Du|^2)^{\gamma+\frac{1}{2}}\, dx \\
&\leq C\int_\Omega |g|\, |D\eta|^2\,(1+|Du|^2)^{\gamma+\frac{q}{2}}\, dx \\
&\leq C\int_\Omega |g|\, |D\eta|^2\,(1+|Du|^2)^{\gamma+q-\frac{2-\mu}{2}}\, dx
\end{aligned}
$$

We know that

$$
\begin{aligned}
|I_8| &\leq 2\gamma\int_\Omega |g|\,\eta^2(1+|Du|^2)^{\gamma-1}\,|D(|Du|)|\,|Du|^2\, dx \\
&\leq 2\gamma\int_\Omega |g|\,\eta^2(1+|Du|^2)^{\gamma-1}\,|D^2u|\,|Du|^2\, dx
\end{aligned}
$$

and that

$$
|I_9| \leq \int_\Omega |g|\,\eta^2\,(1+|Du|^2)^{\gamma}\,|D^2u|\, dx
$$

so we can estimate them together and, going on as we did in I_6, we have

$$
\begin{aligned}
|I_8|+|I_9| &\leq 2\,\gamma\int_\Omega |g|\,\eta^2\,(1+|Du|^2)^{\gamma-1}\,|D^2u|\,|Du|^2\, dx + \int_\Omega |g|\,\eta^2\,(1+|Du|^2)^{\gamma}\,|D^2u|\, dx \\
&\leq 2\,(1+\gamma)\int_\Omega |g|\,\eta^2(1+|Du|^2)^{\gamma}\,|D^2u|\, dx \\
&\leq \frac{1}{4}\int_\Omega \eta^2|D^2u|^2(1+|Du|^2)^{\gamma-\frac{\mu}{2}}\, dx
\end{aligned}
$$

$$+C\,(1+\gamma^2)\int_\Omega |g|^2\,\eta^2\,(1+|Du|^2)^{\gamma+q-\frac{2-\mu}{2}}\,dx$$

Summing up and using (1.6), we obtain

$$\begin{aligned}&\int_\Omega \eta^2\,(1+|Du|^2)^{-\frac{\mu}{2}+\gamma}\,|D^2u|^2\,dx\\ &\leq\ C\,\Theta\,(1+\gamma^2)\left[\int_\Omega(\eta^{2m}+|D\eta|^{2m})\,(1+|Du|^2)^{\left(q-\frac{2-\mu}{2}+\gamma\right)m}\,dx\right]^{\frac{1}{m}}\end{aligned}\tag{4.3}$$

where the constant C depends on ν, L, q, μ but is independent of γ and where we set

$$\Theta = 1+\|g\|^2_{L^r(\Omega)}+\|h\|^2_{L^r(\Omega)}$$

and

$$m=\frac{r}{r-2}\tag{4.4}$$

By Sobolev embedding theorem, by the left hand side of (1.9), recalling that $\mu\leq 2$ and proceding as we did in Section 3.2 we have

$$\begin{aligned}\left(\int_\Omega \eta^{2^*}(1+|Du|^2)^{\left(\gamma+\frac{2-\mu}{2}\right)\frac{2^*}{2}}\,dx\right)^{\frac{2}{2^*}}\ &\leq\ C\int_\Omega |D\eta|^2\,(1+|Du|^2)^{q-\frac{2-\mu}{2}+\gamma}\,dx\\ &\quad +C\,(1+\gamma^2)\int_\Omega \eta^2\,(1+|Du|^2)^{\gamma-\frac{\mu}{2}}\,|D^2u|^2\,dx\end{aligned}$$

where we set 2^* the same way we did before, a part from the case $n=2$ for which we assume $2^*>2m$.
Thanks to (4.3), we finally get

$$\begin{aligned}&\left(\int_\Omega \eta^{2^*}(1+|Du|^2)^{\left(\gamma+\frac{2-\mu}{2}\right)\frac{2^*}{2}}\,dx\right)^{\frac{2}{2^*}}\\ &\leq\ C\,\Theta\,(1+\gamma^2)\left[\int_\Omega(\eta^{2m}+|D\eta|^{2m})(1+|Du|^2)^{\left(q-\frac{2-\mu}{2}+\gamma\right)m}\,dx\right]^{\frac{1}{m}}\end{aligned}$$

from which we deduce

$$\left(\int_{B_\rho}(1+|Du|^2)^{\left(\gamma+\frac{2-\mu}{2}\right)\frac{2^*}{2}}\,dx\right)^{\frac{2}{2^*}}\leq C\,\frac{\Theta\,(1+\gamma^2)}{(R-\rho)^2}\left[\int_{B_R}(1+|Du|^2)^{\left(q-\frac{2-\mu}{2}+\gamma\right)m}\,dx\right]^{\frac{1}{m}}$$

for any $0<\rho<R$.
At this point we introduce the quantity σ defined as

$$\sigma = q-\frac{2-\mu}{2}-\frac{2-\mu}{2m}$$

where we observe that $\sigma>0$ due to left hand side inequality of assumption (1.9) and the fact that $m>1$. That allow us to say that

$$q-\frac{2-\mu}{2}=\sigma+\frac{2-\mu}{2m}$$

Therefore

$$\begin{aligned}\left(\int_{B_\rho}(1+|Du|^2)^{\left(\gamma+\frac{2-\mu}{2}\right)\frac{2^*}{2}}\,dx\right)^{\frac{2}{2^*}} &\leq C\,\frac{\Theta\,(1+\gamma^2)}{(R-\rho)^2}\left[\int_{B_R}(1+|Du|^2)^{\left(q-\frac{2-\mu}{2}+\gamma\right)m}\,dx\right]^{\frac{1}{m}}\\ &= C\,\frac{\Theta\,(1+\gamma^2)}{(R-\rho)^2}\left[\int_{B_R}(1+|Du|^2)^{\left(\sigma+\frac{2-\mu}{2m}+\gamma\right)m}\,dx\right]^{\frac{1}{m}}\end{aligned}$$

which allow us to say that

$$\begin{aligned}&\left(\int_{B_\rho}(1+|Du|^2)^{\left(\gamma+\frac{2-\mu}{2}\right)\frac{2^*}{2}}\,dx\right)^{\frac{2}{2^*}}\\ &\leq\; C\,\frac{\Theta\,(1+\gamma^2)}{(R-\rho)^2}\,\|(1+|Du|^2)\|^{\sigma}_{L^\infty(B_R)}\left[\int_{B_R}(1+|Du|^2)^{\gamma m+\frac{2-\mu}{2}}\,dx\right]^{\frac{1}{m}}\end{aligned}\tag{4.5}$$

We now inductively define the exponents

$$\gamma_1 := 0,\qquad \gamma_{k+1} := \frac{1}{m}\left[\left(\gamma_k+\frac{2-\mu}{2}\right)\frac{2^*}{2}-\frac{2-\mu}{2}\right],\qquad \alpha_k := m\,\gamma_k+\frac{2-\mu}{2},$$

for every integer $k\geq 1$. It follows that

$$\alpha_{k+1}=\left(\gamma_k+\frac{2-\mu}{2}\right)\frac{2^*}{2}=\chi\,\alpha_k+\tau$$

where we have set

$$\chi := \frac{2^*}{2m}\qquad\text{and}\qquad \tau := \frac{2^*\alpha_1}{r}=\frac{2^*(2-\mu)}{2r}$$

By induction we can prove that

$$\alpha_{k+1}=\alpha_1\chi^k+\tau\sum_{i=0}^{k-1}\chi^i=\frac{2-\mu}{2}\,\chi^k+\tau\sum_{i=0}^{k-1}\chi^i\tag{4.6}$$

and

$$\gamma_{k+1}=\frac{\alpha_1}{m}(\chi^k-1)+\frac{\tau}{m}\sum_{i=0}^{k-1}\chi^i=\frac{2-\mu}{2m}(\chi^k-1)+\frac{\tau}{m}\sum_{i=0}^{k-1}\chi^i\tag{4.7}$$

For a later use, we record the elementary estimate

$$\gamma_{k+1}\leq\frac{2\,\alpha_1}{\chi-1}\,\chi^{k+1}=\frac{2-\mu}{\chi-1}\,\chi^{k+1}\tag{4.8}$$

Now we consider $0<\rho_0<R_0$ and set

$$R_k=\rho_0+\frac{(R_0-\rho_0)}{2^k}\qquad\forall k\geq 1$$

so that $R_{k+1} \leq R_k$ for all $k \geq 1$ and

$$R_k - R_{k+1} = \frac{(R_0 - \rho_0)}{2^{k+1}}.$$

We rewrite (4.5) with $\rho = R_{k+1}$ and $R = R_k$. We obtain

$$\begin{aligned} &\left(\int_{B_{R_{k+1}}} (1+|Du|^2)^{\left(\gamma_k + \frac{2-\mu}{2}\right)\frac{2^*}{2}}\, dx\right)^{\frac{2}{2^*}} \\ &\leq \; C\, \frac{\Theta\,(1+\gamma_k^2)}{(R_k - R_{k+1})^2}\, \|(1+|Du|^2)\|^{\sigma}_{L^\infty(B_{R_k})} \left[\int_{B_{R_k}} (1+|Du|^2)^{m\gamma_k + \frac{2-\mu}{2}}\, dx\right]^{\frac{1}{m}} \end{aligned} \tag{4.9}$$

from which we deduce

$$\begin{aligned} &\left(\int_{B_{R_{k+1}}} (1+|Du|^2)^{\left(\gamma_k + \frac{2-\mu}{2}\right)\frac{2^*}{2}}\, dx\right)^{\frac{2}{2^*}} \\ &\leq \; C\, \frac{4^{k+1}\,\Theta\,(1+\gamma_k^2)}{(R_0 - \rho_0)^2}\, \|(1+|Du|^2)\|^{\sigma}_{L^\infty(B_{R_k})} \left[\int_{B_{R_k}} (1+|Du|^2)^{m\gamma_k + \frac{2-\mu}{2}}\, dx\right]^{\frac{1}{m}} \end{aligned}$$

and we can write

$$\begin{aligned} &\int_{B_{R_{k+1}}} (1+|Du|^2)^{\left(\gamma_k + \frac{2-\mu}{2}\right)\frac{2^*}{2}}\, dx \\ &\leq \; \left[C\, \frac{4^{k+1}\,\Theta\,(1+\gamma_k^2)}{(R_0 - \rho_0)^2}\right]^{\frac{2^*}{2}} \|(1+|Du|^2)\|^{\frac{2^*\sigma}{2}}_{L^\infty(B_{R_k})} \left[\int_{B_{R_k}} (1+|Du|^2)^{m\gamma_k + \frac{2-\mu}{2}}\, dx\right]^{\chi} \end{aligned} \tag{4.10}$$

For each $k \in \mathbb{N}$, we define:

$$A_k := \left(\int_{B_{R_k}} (1+|Du|^2)^{\alpha_k} dx\right)^{\frac{1}{\alpha_k}}$$

where $\alpha_k = m\gamma_k + \frac{2-\mu}{2}$ and $\alpha_{k+1} = \left(\gamma_k + \frac{2-\mu}{2}\right)\frac{2^*}{2}$. So we can rewrite (4.8) as:

$$A_{k+1} \leq \left[C\, \frac{4^{k+1}\,\Theta\,(1+\gamma_k^2)}{(R_0 - \rho_0)^2}\right]^{\frac{2^*}{2\alpha_{k+1}}} \left(\|(1+|Du|^2)\|^{\frac{2^*\sigma}{2}}_{L^\infty(B_{R_k})}\right)^{\frac{1}{\alpha_{k+1}}} A_k^{\frac{\alpha_k \chi}{\alpha_{k+1}}}$$

Iterating this inequality we obtain:

$$A_{k+1} \leq \prod_{i=1}^{k} \left[C\, \frac{4^{i+1}\,\Theta\,(1+\gamma_i^2)}{(R_0 - \rho_0)^2}\right]^{\frac{2^*\chi^{k-i}}{2\alpha_{k+1}}} \left(\|(1+|Du|^2)\|^{\frac{2^*\sigma}{2}}_{L^\infty(B_R)}\right)^{\frac{1}{\alpha_{k+1}}\sum_{i=0}^{k-1}\chi^i} A_1^{\frac{\chi^k \alpha_1}{\alpha_{k+1}}} \tag{4.11}$$

We can notice that

$$\lim_{k\to+\infty} \frac{1}{\alpha_{k+1}} \sum_{i=0}^{k-1} \chi^i \;=\; \lim_{k\to+\infty} \frac{\sum_{i=0}^{k-1}\chi^i}{\alpha_1 \chi^k + \tau \sum_{i=0}^{k-1}\chi^i}$$

$$\begin{aligned} &= \lim_{k\to+\infty} \frac{\chi^k - 1}{\alpha_1(\chi-1)\chi^k + \tau(\chi^k - 1)} \\ &= \frac{1}{\alpha_1(\chi-1)+\tau} \\ &\leq \frac{2}{(\chi-1)\,\alpha_1} \end{aligned}$$

and that

$$\lim_{k\to+\infty} \frac{\chi^k \alpha_1}{\alpha_{k+1}} = \frac{\alpha_1(\chi-1)}{\alpha_1(\chi-1)+\tau} = \delta \tag{4.12}$$

We define:

$$M_k := \prod_{i=1}^{k} \left[C\, 4^{i+1}\,(1+\gamma_i^2)\right]^{\frac{2^*\chi^{k-i}}{2\alpha_{k+1}}} = \exp\left[\frac{2^*}{2\,\alpha_{k+1}} \sum_{i=1}^{k} \chi^{k-i} \log\left(C\,4^{i+1}(1+\gamma_i^2)\right)\right]$$

and we have, thanks to (4.6):

$$\begin{aligned} M_k &= \exp\left[\frac{2^*}{2\,\alpha_{k+1}} \sum_{i=1}^{k} \chi^{k-i}\log(C) + \chi^{k-i}(i+1)\log(4) + \chi^{k-i}\log(1+\gamma_i^2)\right] \\ &= \exp\left[\frac{2^*}{2}\sum_{i=1}^{k} \frac{\log(C)\,(\chi-1)}{[\alpha_1\,(\chi-1)+\tau]\,\chi^i - \tau\,\chi^{i-k}}\right] \exp\left[\frac{2^*}{2}\sum_{i=1}^{k} \frac{(i+1)\,\log(4)\,(\chi-1)}{[\alpha_1\,(\chi-1)+\tau]\,\chi^i - \tau\,\chi^{i-k}}\right] \\ &\times \exp\left[\frac{2^*}{2}\sum_{i=1}^{k} \frac{\log(1+\gamma_i^2)\,(\chi-1)}{[\alpha_1\,(\chi-1)+\tau]\,\chi^i - \tau\,\chi^{i-k}}\right] \\ &= \exp(L_{1k}) + \exp(L_{2k}) + \exp(L_{3k}) \end{aligned}$$

Now we show that those three quantities are bounded by some constants $C_i, i = 1, \dots, 6$ depending only on the data n, r, μ, q.
Let us start with the estimate of L_1 as follows:

$$\begin{aligned} L_1 &= \lim_{k\to+\infty} \frac{2^*}{2} \sum_{i=1}^{k} \frac{\log(C)\,(\chi-1)}{[\alpha_1\,(\chi-1)+\tau]\,\chi^i - \tau\,\chi^{i-k}} \\ &= \lim_{k\to+\infty} \frac{2^*}{2} \sum_{i=1}^{k} \frac{\log(C)}{\chi^i\,\left[\frac{2-\mu}{2}\,(\chi-1)+\tau-\frac{\tau}{\chi^k}\right]} \\ &\leq \lim_{k\to+\infty} \frac{2^*}{2} \sum_{i=1}^{k} \frac{\log(C)}{\chi^i\,(\chi-1)\,\frac{2-\mu}{2}} \\ &= \frac{2^*\,\log(C)}{(2-\mu)(\chi-1)} \sum_{i=1}^{\infty} \left(\frac{1}{\chi}\right)^i \\ &\leq C_1 < +\infty \end{aligned}$$

Then, for L_2 we can say that:

$$
\begin{aligned}
L_2 &= \lim_{k\to+\infty} \frac{2^*}{2} \sum_{i=1}^{k} \frac{(i+1)\,\log(4)\,(\chi-1)}{[\alpha_1\,(\chi-1)+\tau]\,\chi^i - \tau\,\chi^{i-k}} \\
&= \lim_{k\to+\infty} \frac{2^*}{2} \sum_{i=1}^{k} \frac{(i+1)\,\log(C)}{\chi^i\,\left[\frac{2-\mu}{2}\,(\chi-1)+\tau-\frac{\tau}{\chi^k}\right]} \\
&\leq \lim_{k\to+\infty} \frac{2^*}{2} \sum_{i=1}^{k} \frac{(i+1)\,\log(C)}{\chi^i\,(\chi-1)\,\frac{2-\mu}{2}} \\
&= \frac{2^*\,\log(C)}{(2-\mu)(\chi-1)} \sum_{i=1}^{\infty} \frac{i+1}{\chi^i} \\
&= \frac{2^*\,\log(C)}{(2-\mu)(\chi-1)} \left[\sum_{i=1}^{\infty} i\,\left(\frac{1}{\chi}\right)^i + \sum_{i=1}^{\infty}\left(\frac{1}{\chi}\right)^i\right] \\
&\leq C_2 < +\infty
\end{aligned}
$$

And for L_3, thanks to (4.7), we can conclude that

$$
\begin{aligned}
L_3 &= \lim_{k\to+\infty} \frac{2^*}{2} \sum_{i=1}^{k} \frac{\log(1+\gamma_i^2)\,(\chi-1)}{[\alpha_1\,(\chi-1)+\tau]\,\chi^i - \tau\,\chi^{i-k}} \\
&= \lim_{k\to+\infty} \frac{2^*}{2} \sum_{i=1}^{k} \frac{(\chi-1)\,\log(1+\gamma_i^2)}{\chi^i\,\left[\frac{2-\mu}{2}\,(\chi-1)+\tau-\frac{\tau}{\chi^k}\right]} \\
&= \lim_{k\to+\infty} \frac{2^*}{2} \sum_{i=2}^{k} \frac{(\chi-1)\,\log(1+\gamma_i^2)}{\chi^i\,\left[\frac{2-\mu}{2}\,(\chi-1)+\tau-\frac{\tau}{\chi^k}\right]} \\
&\leq \frac{2^*}{2-\mu} \sum_{i=2}^{\infty} \frac{\log(1+\gamma_i^2)}{\chi^i} \\
&\leq \frac{2^*}{2-\mu} \sum_{i=2}^{\infty} \frac{\log\left(\gamma_i^2\,\left(1+\frac{1}{\gamma_i^2}\right)\right)}{\chi^i} \\
&= \frac{2^*}{2-\mu} \sum_{i=2}^{\infty} \frac{\log(\gamma_i^2)+\log\left(1+\frac{1}{\gamma_i^2}\right)}{\chi^i} \\
&= \frac{2^*}{2-\mu} \sum_{i=2}^{\infty} \frac{\log(\gamma_i^2)}{\chi^i} + \frac{2^*}{2-\mu} \sum_{i=2}^{\infty} \frac{\log\left(1+\frac{1}{\gamma_i^2}\right)}{\chi^i} \\
&= L_4 + L_5
\end{aligned}
$$

where we can say that

$$
\begin{aligned}
L_4 &= \frac{2^*}{2-\mu}\sum_{i=2}^{\infty}\frac{\log(\gamma_i^2)}{\chi^i} \\
&= \frac{2\,2^*}{2-\mu}\sum_{i=2}^{\infty}\frac{\log(\gamma_i)}{\chi^i} \\
&= \frac{2^*}{2-\mu}\sum_{i=2}^{\infty}\frac{\log\left(\left(\frac{2-\mu}{2}+\frac{\tau}{\chi-1}\right)\frac{\chi^{i-1}-1}{m}\right)}{\chi^i} \\
&= C_3+\frac{2\,2^*}{2-\mu}\sum_{i=2}^{\infty}\frac{\log(\chi^{i-1}-1)}{\chi^i} \\
&\leq C_3+\frac{2\,2^*}{2-\mu}\sum_{i=2}^{\infty}\frac{(i-1)\log(\chi)}{\chi^i} \\
&\leq C_3+\frac{2\,2^*}{2-\mu}\sum_{i=2}^{\infty}\frac{i-1}{\chi^i} \\
&\leq C_3+C_4<+\infty
\end{aligned}
$$

$$
\begin{aligned}
L_5 &= \frac{2^*}{2-\mu}\sum_{i=2}^{\infty}\frac{\log\left(1+\frac{1}{\gamma_i^2}\right)}{\chi^i} \\
&\leq \frac{2\,2^*}{2-\mu}\sum_{i=2}^{\infty}\frac{\log\left(1+\frac{1}{\gamma_i}\right)}{\chi^i} \\
&\leq \frac{2\,2^*}{2-\mu}\sum_{i=2}^{\infty}\frac{1}{\chi^i\,\gamma_i} \\
&= \frac{2\,2^*}{2-\mu}\sum_{i=2}^{\infty}\frac{1}{\chi^i\,\left(\frac{2-\mu}{2}+\frac{\tau}{\chi-1}\right)\frac{\chi^{i-1}-1}{m}} \\
&= C_5\sum_{i=2}^{\infty}\frac{1}{\chi^i\,(\chi^{i-1}-1)} \\
&\leq C_5\sum_{i=2}^{\infty}\frac{1}{\chi^i\,(\chi-1)} \\
&= C_5\sum_{i=2}^{\infty}\frac{1}{\chi^i} \\
&\leq C_6<+\infty
\end{aligned}
$$

So we have that

$$
\lim_{k\to+\infty} M_k = \lim_{k\to+\infty}\exp(L_1)+\lim_{k\to+\infty}\exp(L_2)+\lim_{k\to+\infty}\exp(L_3)
$$

$$\begin{aligned} &\le \exp(C_1) + \exp(C_2) + \exp(C_3 + C_4 + C_6) \\ &\le M < +\infty \end{aligned}$$

Last but not least we have that

$$\begin{aligned} X &:= \lim_{k\to+\infty} \sum_{i=1}^{k} \frac{2^* \chi^{k-i}}{2\alpha_{k+1}} \\ &= \lim_{k\to+\infty} \frac{2^*}{2} \sum_{i=1}^{k} \frac{\chi^{k-i}}{\frac{2-\mu}{2}\,\chi^k + \tau \sum_{i=0}^{k-1} \chi^i} \\ &= \lim_{k\to+\infty} \frac{2^*}{2} \sum_{i=1}^{k} \frac{\chi^{k-i}}{\frac{2-\mu}{2}\,\chi^k + \tau \frac{\chi^k - 1}{\chi - 1}} \\ &= \lim_{k\to+\infty} \frac{2^*\,(\chi-1)}{2} \sum_{i=1}^{k} \frac{1}{\frac{2-\mu}{2}\,(\chi-1)\,\chi^i + \tau\chi^i - \tau\chi^{i-k}} \\ &= \lim_{k\to+\infty} \frac{2^*\,(\chi-1)}{2} \sum_{i=1}^{k} \frac{1}{\chi^i \left[\frac{2-\mu}{2}\,(\chi-1) + \tau - \frac{\tau}{\chi^k}\right]} \\ &\le \lim_{k\to+\infty} \frac{2^*\,(\chi-1)}{2} \sum_{i=1}^{k} \frac{1}{\chi^i\,(\chi-1)\frac{2-\mu}{2}} \\ &= \frac{2^*}{2-\mu} \sum_{i=1}^{\infty} \frac{1}{\chi^i} < \infty \end{aligned}$$

Thanks to (4.12) and letting $k \to +\infty$, noticing that $R_0 > R_1$, we can we can rewrite (4.11) as follows

$$\|1 + |Du|^2\|_{L^\infty(B_{\rho_0})} \le \frac{M\,\Theta^X}{(R_0 - \rho_0)^{2X}}\, \|1 + |Du|^2\|^{\frac{2^*\,\sigma}{2\,\alpha_1\,(\chi-1)+2\tau}} \left(\int_{B_{R_0}} (1 + |Du|^2)^{\frac{2-\mu}{2}}\, dx \right)^{\frac{2\,\delta}{2-\mu}}$$

Now we have that

$$\begin{aligned} E &:= \frac{2^*\,\sigma}{2\,\alpha_1\,(\chi-1) + 2\tau} \\ &= \frac{\frac{2^*\,\sigma}{2}}{\frac{2-\mu}{2}\,(\chi-1) + \frac{2^*\,\alpha_1}{r}} \\ &= \frac{\frac{2^*}{2}\,\left[q - (2-\mu)\,\left(\frac{1}{2} + \frac{1}{2m}\right)\right]}{\frac{2-\mu}{2}\,\left(\frac{2^*}{2m} - 1\right) + \frac{2^*\,(2-\mu)}{2r}} \\ &= \frac{2^*\,\left[q - (2-\mu)\,\left(\frac{1}{2} + \frac{1}{2m}\right)\right]}{(2-\mu)\,\left[\frac{2^*}{2m} - 1 + \frac{2^*}{r}\right]} \end{aligned}$$

and once more we have to prove that $E < 1$. Now, if $n \geq 3$ then we are done if and only if we have that

$$2^* \left[q - (2-\mu)\left(\frac{1}{2} + \frac{1}{2m}\right)\right] < (2-\mu)\left[\frac{2^*}{2m} - 1 + \frac{2^*}{r}\right]$$

$$2^* \left[q - (2-\mu)\left(\frac{1}{2} + \frac{1}{2m}\right)\right] < 2^*\,(2-\mu)\left[\frac{1}{2m} - \frac{1}{2^*} + \frac{1}{r}\right]$$

$$q < (2-\mu)\left[\frac{1}{2} + \frac{1}{2m} + \frac{1}{2m} - \frac{1}{2^*} + \frac{1}{r}\right]$$

but using the equality

$$\frac{1}{2m} - \frac{1}{2^*} = \frac{1}{n} - \frac{1}{r} \tag{4.13}$$

we know that

$$\begin{aligned}
\frac{1}{2} + \frac{1}{2m} + \frac{1}{2m} - \frac{1}{2^*} + \frac{1}{r} &= \frac{1}{2} + \frac{2}{2^*} + \frac{2}{n} - \frac{2}{r} - \frac{1}{2^*} + \frac{1}{r} \\
&= \left[\frac{1}{2} + \frac{1}{2^*} + \frac{2}{n}\right] - \frac{1}{r} \\
&= \left[\frac{1}{2} + \frac{n-2}{2n} + \frac{2}{n}\right] - \frac{1}{r} \\
&= \frac{n+n-2+4}{2n} - \frac{1}{r} \\
&= 1 + \frac{1}{n} - \frac{1}{r},
\end{aligned}$$

and the thesis is proved. On the other hand, if $n = 2$, then passing to the limit as $2^* \to \infty$ in the expression of E we deduce

$$\frac{\sigma}{(2-\mu)\left(\frac{1}{2m} + \frac{1}{r}\right)} < 1 \Leftrightarrow q < (2-\mu)\left[\frac{1}{2} + \frac{1}{m} + \frac{1}{r}\right] \overset{(4.4)}{=} (2-\mu)\left[\frac{3}{2} - \frac{1}{r}\right]$$

which is nothing but (1.9) with the choice $n = 2$. Thus we can use the Young's inequality with exponents $\frac{2\,\alpha_1\,(\chi-1)+2\tau}{2^*\,\sigma}$ and $\frac{2\,\alpha_1\,(\chi-1)+2\tau}{[2\,\alpha_1\,(\chi-1)+2\tau]-2^*\,\sigma}$, to get:

$$\|1+|Du|^2\|_{L^\infty(B_{\rho_0})} \leq \frac{1}{2}\,\|1+|Du|^2\|_{L^\infty(B_{R_0})} + \left(\frac{M\,\Theta^X}{(R_0-\rho_0)^{2X}}\right)^{\vartheta}\left(\int_{B_{R_0}}(1+|Du|^2)^{\frac{2-\mu}{2}}\,dx\right)^{\frac{2\delta\vartheta}{2-\mu}}$$

for an exponent $\vartheta = \vartheta\,(n,r,q,\mu) > 0$. Since previous estimate holds true for $\rho < \rho_0 < R_0 < R$, once more by Lemma 2.1 we finally get

$$\|1+|Du|^2\|_{L^\infty(B_\rho)} \leq \left(\frac{M\Theta^X}{(R-\rho)^{2X}}\right)^{\vartheta}\left(\int_{B_R}(1+|Du|^2)^{\frac{2-\mu}{2}}\,dx\right)^{\frac{2\delta\vartheta}{2-\mu}}$$

5. Conclusion

The Lipschitz regularity results usually are carried out over three steps: approximation, a priori estimates and passage to the limit. We note that our a priori estimates do not require any additional assumption on the structure of the Lagrangian f. This means that we can choose different methods in order to carry out our main results.
As long as we are in the situation where the Lavrentiev phenomenon does not occurr, we can approximate from above like it has been done in [39], to which we refer for the details (the only difference is obviously in the a priori estimate, but all the rest can be carried out in the same way). We remark that, differently from [25], we have enough regularity to pass to the limit without involving the relaxed functional.
In this respect, the presence of the function $\bar{F}$ with superlinear growth and the strict convexity of the functional reveal to be crucial in order to perform the passage to the limit.

Compliance with Ethical Standards

Conflict of interest: The authors declare that they have no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by the authors.

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