Quantization and motion law for Ginzburg-Landau vortices

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Abstract

We study the vortex trajectories for the two-dimensional complex parabolic Ginzburg-Landau equation without well-preparedness assumption. We prove that the trajectory set is rectifiable, satisfies a weak motion law. In the case of degree ± 1 vortices, the motion law is satisfied in the classical sense. Moreover, dissipation occurs only at a finite number of times. Away from these times, possible collisions and splittings of vortices are constrained by algebraic equations involving their topological degrees.

Quantization properties of the energy and potential densities play a central role in the proofs.

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1 Introduction

This paper is a sequel of our previous paper [3] on the two-dimensional dissipative Ginzburg-Landau equation

$$(\mathrm{PGL})_\varepsilon \quad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \text{on } \mathbb{R}^2 \times \mathbb{R}^+_*, \\ u_\varepsilon(z, 0) = u^0_\varepsilon(z) & \text{for } z = x + iy \equiv (x, y) \in \mathbb{R}^2 . \end{cases}$$

Of special interest is the asymptotic limit $\varepsilon \to 0$. In particular, it has been recognized that for this asymptotics the energy bound

$$(\mathrm{H}_0) \qquad \mathcal{E}_\varepsilon(u^0_\varepsilon) = \int_{\mathbb{R}^N} e_\varepsilon(u^0_\varepsilon) = \int_{\mathbb{R}^N} \frac{|\nabla u^0_\varepsilon|^2}{2} + \frac{1}{4\varepsilon^2}(1 - |u^0_\varepsilon|^2)^2 \leq M_0 |\log \varepsilon|$$

for the initial datum u^0_ε, with some constant $M_0 > 0$ independent of ε, allows the formation of interesting topological defects, called vortices, which will be described below. It has also been recognized that in the two-dimensional case that we study here, accelerating time, e.g. considering the time scale $s =$

$\frac{t}{|\log \varepsilon|}$, is appropriate to investigate the dynamics of these vortices. Therefore we consider the map $\mathfrak{u}_\varepsilon$, defined on $\mathbb{R}^2 \times \mathbb{R}^+$ by

$$\mathfrak{u}_\varepsilon(z,s) = u_\varepsilon(z, s|\log \varepsilon|).$$

The first result of [3] (which extended in particular earlier works [11, 12]) stated a compactness and rigidity result for limiting maps of $(\mathfrak{u}_\varepsilon)_{\varepsilon>0}$.

Theorem 1 ([3]). *For a subsequence $\varepsilon_n \to 0$ we have*

$$\mathfrak{u}_{\varepsilon_n}(z,s) \to \mathfrak{u}_*(z,s) = \prod_{i=1}^{\ell(s)} \left(\frac{z - a_i(s)}{|z - a_i(s)|} \right)^{d_i(s)} \exp[i(\langle \vec{c}(s), z\rangle + b(s)], \qquad (1)$$

where, for $i = 1, ..., \ell(s)$, $a_i(s) \in \mathbb{R}^2$, $d_i(s) \in \mathbb{Z}$ and where $b(s) \in [0, 2\pi)$ and $\vec{c} : \mathbb{R}^+ \to \mathbb{R}^2$ is a lipschitz function. The convergence [1] in (1) is uniform on every compact subset of $\mathbb{R}^2 \times \mathbb{R}^+ \setminus \Sigma_{\mathfrak{v}}$, where

$$\Sigma_{\mathfrak{v}} = \cup_{s>0} \cup_{i=1}^{\ell(s)} \{a_i(s)\}.$$

We proved moreover that the numbers $\ell(s)$ and $d_i(s)$ are uniformly bounded, i.e. there exists a constant $C(M_0)$ such that

$$\ell(s) \leq C(M_0), \qquad |d_i(s)| \leq C(M_0), \qquad (2)$$

and that, except for a finite number of times,

$$d_i(s) \neq 0. \qquad (3)$$

We also showed that $\Sigma_{\mathfrak{v}}$ is closed, and of locally finite two-dimensional parabolic Hausdorff measure.

Notice that for fixed $s > 0$, the limiting map $\mathfrak{u}_*$ is completely determined by a finite number of parameters, namely the set of points $\{a_i(s)\}_{1\leq i \leq \ell(s)}$, which are usually referred to as **vortices**, the integers $d_i(s) \in \mathbb{Z}$, which are the degrees of the vortices, the number $b(s) \in \mathbb{R}$, which is a constant

[1] In [3] the convergence in (1) was stated a little differently, namely $\mathfrak{u}_{\varepsilon_n} \times \nabla \mathfrak{u}_{\varepsilon_n} \to w_* \times \nabla w_* + \vec{c}$, and $|\mathfrak{u}_{\varepsilon_n}| \to 1$. Here $w_*(z,s) = \prod_{i=1}^{\ell(s)} \left(\frac{z-a_i(s)}{|z-a_i(s)|} \right)^{d_i(s)}$. Moreover, the convergence (1) holds in $\mathcal{C}^k$ for the space variables, uniformly in the time variable, away from $\Sigma_{\mathfrak{v}}$.

phase shift, and the vector $\vec{c}(s) \in \mathbb{R}^2$, reminiscent of a wavenumber.[2] This reduction in the limit to a finite dimensional situation motivated us to use the term "rigidity" to describe the limiting map.

The set

$$\Sigma_{\mathfrak{v}} = \cup_{s>0} \Sigma_{\mathfrak{v}}^s = \cup_{s>0} \cup_{i=1}^{\ell(s)} \{a_i(s)\}$$

represents the trajectories of the vortices: we will therefore refer to it in some places as the **trajectory set**. The next important step in order to understand the limiting dynamics is to describe carefully $\Sigma_{\mathfrak{v}}$: this is one of the main goals of this paper. In [3] we derived a first, rather weak regularity property of $\Sigma_{\mathfrak{v}}$, restated here in Lemma 2.3 and 2.5. Our first result in this paper is

Theorem 2. *The trajectory set $\Sigma_{\mathfrak{v}}$ is a closed, one-dimensional countably rectifiable subset of $\mathbb{R}^2 \times \mathbb{R}^+$.*

Recall that a set $\Sigma \subset \mathbb{R}^N$ is said to be 1-dimensional countably rectifiable if it is contained in a countable union of lipschitz curves, except possibly for a subset of zero 1-dimensional Hausdorff measure. Since $\Sigma_{\mathfrak{v}}$ describes the dynamics of vortices, we will actually show that $\Sigma_{\mathfrak{v}}$ is contained in a countable union of graphs of Lipschitz functions defined from time intervals of $\mathbb{R}^+$ into $\mathbb{R}^2$.

If moreover the one-dimensional Hausdorff measure of $\Sigma_{\mathfrak{v}}$ were known to be locally finite, then one would be able to define a (one-dimensional) tangent space to $\Sigma_{\mathfrak{v}}$ in a weak sense at almost every point[3] $(a_i(s), s)$, in view of general results in geometric measure theory. This would give a natural meaning to the speed of the vortices. Since we do not have this information, we adopt the next definition.

Definition 1. *Let $(a_i(s_0), s_0) \in \Sigma_{\mathfrak{v}}$. The vector $\vec{v} \in \mathbb{R}^2$ is said to be an approximate speed of the vortex $a_i(s_0)$ at time s_0 if there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^2$ such that $f(s_0) = a_i(s_0)$, $f(s) \in \Sigma_{\mathfrak{v}}^s$ $\forall\, s > 0$ and f has an approximate derivative [4] at s_0 equal to $\vec{v}$.*

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[2]Notice that functions $\vec{c}$ and b depend only on the time variable s, but **not** on the space variable z. The function $\vec{c}$ can be directly deduced from the initial value, for instance by Fourier transform. It accounts for persistence of low frequency oscillations in the phase over the diverging time period considered, namely $t = s|\log \varepsilon|$. The possible presence of low frequencies is of course related to the fact that the domain $\mathbb{R}^2$ is unbounded. In case of bounded domains, the function $\vec{c}$ would vanish. Whereas we have a good control for $\vec{c}$, we do not know if similar properties hold for b. This would require refined estimates for the time derivative $\partial_t u_\varepsilon$ at the ε level.

[3]with respect to the one-dimensional Hausdorff measure.

[4]For the notion of approximate derivative, see [9], 3.1.2.

Notice that, if not empty, the set of approximate speeds might well be not reduced to a singleton (for instance, in case of branching points of the trajectories). However, we have

Theorem 3. *For almost every $s > 0$ and for any $i = 1, ..., \ell(s)$, the vortex $a_i(s)$ has a unique approximate speed denoted $\dot{a}_i(s)$ and given by*

$$\dot{a}_i(s) = \frac{1}{d_i(s)} \left[\vec{c}(s)^{\perp} + 2 \sum_{i \neq j = 1}^{\ell(s)} d_j(s) \nabla_{a_i} (\log |a_i(s) - a_j(s)|) \right]. \qquad (4)$$

Theorem 3 shows that the motion law for the vortices is given by an ordinary differential equation, at least in a weak sense, outside an exceptional set of null measure. The next result shows that the ODE governing the dynamics is satisfied in a classical sense for degree ± 1 vortices, and also, for arbitrary degrees, outside a closed set with empty interior.

Theorem 4. *i) There exists an open dense set $\mathcal{O}$ in $\mathbb{R}^+$ such that for every subinterval $J \subset \mathcal{O}$, the number of vortices $\ell(s)$ and the degrees $d_i(s)$, $i = 1, ..., \ell(s)$, are constant for $s \in J$, and the set $\Sigma_{\mathfrak{v}} \cap \mathbb{R}^2 \times J$ is given by a disjoint union of smooth curves, which are integral curves of the ODE (4).*

ii) If $s_0 \in \mathbb{R}^+_*$ is such that $|d_i(s_0)| = 1$ for any $i = 1, ..., \ell(s_0)$, then the maximal interval of existence $I_{s_0} = (s_0, s_{max})$ of the ODE (4) with initial time s_0 is contained in $\mathcal{O}$.

iii) Assume s_{max} in statement ii) is such that $d_i(s_{max}) \in \{-1, 0, +1\}$ for any $i \in 1, ..., \ell(s_{max})$, and consider the the ODE (4) with initial time s_{max} and with points a_i such that $d_i(s_{max}) = 0$ dropped. Then its maximal interval of existence is contained in $\mathcal{O}$.

In other words, statement iii) is a unique continuation principle (governed by ODE's) as long as collisions do not lead to multiple degrees.

Statement ii) in Theorem 4 was already proved in [11, 12, 16, 18] in the case of "well-prepared" initial data, i.e. having l vortices of degree $+1$ and -1 and an energy $E_\varepsilon(u_\varepsilon^0) = \pi l |\log \varepsilon| + O(1)$: this is actually the minimal energy required for such a vortex configuration. In [17], this well-prepared assumption was somewhat relaxed to $E_\varepsilon(u_\varepsilon^0) \leq \pi l |\log \varepsilon| + \frac{|\log \varepsilon|}{(\log |\log \varepsilon|)^\beta}$ for some $\beta > 1$. In this case $\vec{c} \equiv 0$, and the ODE (4) is the gradient flow of the Kirchhoff function

$$W(a_1, ..., a_l) = 2 \sum_{i \neq j = 1}^{l} d_i d_j \log |a_i - a_j|.$$

We emphasize that equation (4) was shown there to be verified everywhere, in the classical sense, but only up to the **first** collision time s_{max}. Such **collisions** are however an unavoidable aspect of the complete dynamics, in particular when vortices of opposite degrees are present.[5]

In the general case, when the degrees are not assumed to be equal to ± 1, one of the main obstacles that we need to face is the fact that the total number of vortices is a priori not constant, even not locally constant. Besides collisions, the possible **splitting** of vortices of multiple degree (i.e. $|d_i(s)| > 1$) into several vortices of different degrees (of arbitrary sign), and their possible later **recombinations** represent both a mathematical and conceptual difficulty.

The language and tools of geometric measure theory are one way to circumvent these difficulties. However, we believe that the results in Theorem 2 and 3 might be improved, and in particular that set $\Sigma_{\mathfrak{v}}$ is a finite union of smooth disjoint curves with finitely many branching points.

The Radon measures $\mathfrak{v}_\varepsilon^s$ defined for $s \geq 0$ on $\mathbb{R}^2 \times \{s\}$ by

$$\mathfrak{v}_\varepsilon^s(x) = \frac{e_\varepsilon(\mathfrak{u}_\varepsilon(x,s))}{|\log \varepsilon|}\, dx,$$

as well as the measures

$$W_\varepsilon^s ds \equiv V_\varepsilon(\mathfrak{u}_\varepsilon)\, dx\, ds = \frac{(1-|\mathfrak{u}_\varepsilon|^2)^2}{4\varepsilon^2}\, dx\, ds,$$

are central in the proofs. These quantities possess remarkable properties inherited from the equation $(\mathrm{PGL})_\varepsilon$. It was shown in [3] (Theorem 4 and 5 there) that there exists a subsequence $\varepsilon_n \to 0$ such that, for each $s > 0$,

$$\mathfrak{v}_{\varepsilon_n}^s \rightharpoonup \mathfrak{v}_*^s = \sum_{i=1}^{\ell(s)} \theta_i(s)\delta_{a_i(s)} \qquad \text{as } n \to \infty, \tag{5}$$

for some non negative densities $\theta_i(s)$. Here we prove that these densities are actually quantized, and related to the degrees of the vortices. A similar property holds for the potential V_ε as well.

[5] In particular, if u_ε^0 has two vortices of degree 1 and -1 located at the points $a_{-1}(0) = -1$ and $a_1(0) = 1$, then in view of (4), the limiting map $\mathfrak{u}_*$ has two vortices given by $a_i(s) = (-1)^i\sqrt{1-2s}$, $i = -1, 1$. These two vortices will collide at time $s = \frac{1}{2}$. This is a special case of collision of vortices with total degree zero. Such a situation is analyzed in [3], Theorem 3 (see also for bounded domains the more recent paper [17] which gives in particular a detailed analysis of a single dipole ± 1 annihilation). In the example given above the two vortices disappear after collision time and the map $\mathfrak{u}_*$ is then constant.

Theorem 5. *For all but finitely many $s > 0$ we have*

$$\theta_i(s) = \pi d_i^2(s). \tag{6}$$

Moreover, as $n \to +\infty$,

$$V_{\varepsilon_n}(\mathfrak{u}_{\varepsilon_n})dxds \rightharpoonup W_* = W_*^s ds = \frac{\pi}{2}\sum_{i=1}^{\ell(s)} d_i^2(s)\delta_{a_i(s)} \otimes ds \tag{7}$$

in the sense of measures on $\mathbb{R}^2 \times \mathbb{R}^+$.

In particular, the measures $\frac{1}{\pi}\mathfrak{v}_*^s$ and $\frac{2}{\pi}W_*^s$ coincide and are equal to sums of Dirac masses located at the points $a_i(s)$, with integer valued weights. They are therefore **quantized**.

Notice that (6) and (7) were already derived in the elliptic case in [2] for minimizers, and in [7] for critical points: as a matter of fact, a substantial part of our proof of (6) and (7) relies heavily on the method presented in [7]. This part, mainly relying on elliptic PDE techniques (the focus is put on the perturbed stationary Ginzburg-Landau equation), is presented in the Appendix. Independently, the perturbed stationary Ginzburg-Landau equation is considered in [17] with quite similar results.

At this stage, it is also worthwhile to notice that as a consequence of Theorem 5 ii) of [3], the quantity $A(s) \equiv \sum_i d_i(s)^2$ is non-increasing: since it is an integer it is also piecewise constant. More precisely, we have

Lemma 1. *There exists a finite set $\{0 = \tau_0 < \tau_1 < \cdots < \tau_q < \tau_{q+1} = +\infty\}$ such that for every $s \in (\tau_k, \tau_{k+1})$*

$$A(s) \equiv \sum_{i=1}^{\ell(s)} d_i^2(s) = \tfrac{1}{\pi}\mathfrak{v}_*^s(\mathbb{R}^2) = \tfrac{2}{\pi}W_*^s(\mathbb{R}^2) = n_k = \mathrm{Cste},$$

where $n_k \in \mathbb{N}$ depends only on k and $n_{k+1} < n_k$.

An important consequence of Lemma 1 is that the dissipation rate $|\partial_t u_\varepsilon|^2$ vanishes asymptotically on $\mathbb{R}^2 \times (\tau_k, \tau_{k+1})$ (see Corollary 3.1), yielding a flavour of reversibility to the evolution equation. This allows us to establish the "straight cone property", stated in Proposition 4.1 ii), which is a major ingredient in our proofs of Theorem 2 and Theorem 3.

The following theorem is a substantial extension to the κ-confinement result presented in [3] Theorem 3.

Theorem 6. *Let* $s_0 \in (\tau_k, \tau_{k+1})$, $a \in \mathbb{R}^2$, $r > 0$ *and* $0 < \kappa \leq \frac{1}{2}$ *be such that*

$$(H_\kappa(a, r, s_0)) \qquad\qquad \emptyset \neq \Sigma_{\mathfrak{v}}^{s_0} \cap B(a, r) \subset B(a, \kappa r).$$

There exist constants $0 < \kappa_1 \leq \frac{1}{2}$ *and* $\gamma_1 > 0$ *,depending only on* M_0*, such that if* $0 < \kappa \leq \kappa_1$ *and*

$$\mathrm{dist}(s_0, \{\tau_k, \tau_{k+1}\}) \geq \gamma_1 \kappa^2 r^2$$

then,

$$\Gamma \equiv \sum_{i \in J} d_i^2(s_0) - \left(\sum_{i \in J} d_i(s_0) \right)^2 = 0, \tag{8}$$

where we have set $J = \{i \in 1, ..., \ell(s_0) \mid a_i(s_0) \in B(a, r)\}$*. More precisely,*

if $\Gamma > 0$ *then* $\tau_{k+1} - s_0 < \gamma_1 \kappa^2 r^2$ *and if* $\Gamma < 0$ *then* $s_0 - \tau_k < \gamma_1 \kappa^2 r^2$*.*

Relation (8) was already proved in [7] for the stationary equation on a bounded domain. As a matter of fact, it is one of the key ingredients for proving quantization of the energy. A cluster of vortices $\{a_i(s_0)\}_{i \in J}$ for which $\Gamma \neq 0$ is called an unbalanced cluster in [17]: Theorem 6 shows that unbalanced confined clusters at small scale may only be found close to the collision times τ_k.

In particular, given $s_0 \in (\tau_k, \tau_{k+1})$ and $i \in \{1, ..., \ell(s_0)\}$, there exists $\Delta s_0 > 0$ and $r \equiv r(s_0) > 0$ such that

$$\Sigma_{\mathfrak{v}}^s \cap (B(a_i(s_0), r) \setminus B(a_i(s_0), r/2)) = \emptyset$$

and

$$\Gamma(s) \equiv \sum_{a_j(s) \in B(a_i(s_0), r)} d_j^2(s) - \left(\sum_{a_j(s) \in B(a_i(s_0), r)} d_j(s) \right)^2 = 0 \tag{9}$$

for every $s \in (s_0 - \Delta s_0, s_0 + \Delta s_0)$.

An important consequence is that vortex splittings or recombinations at times different from the τ_k's have to satisfy the **algebraic equilibrium equation** (9) for the degrees.

Combining Theorem 6 with scaling arguments and results for the perturbed elliptic Ginzburg-Landau equation presented in the Appendix, we may now improve the quantization result stated in (6) by

Theorem 7. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and $s_0 \notin \{\tau_0, ..., \tau_q\}$ be such that $\Sigma_{\mathfrak{v}}^{s_0} \cap \partial\Omega = \emptyset$. Then, if n is sufficiently large, we have*

$$|\int_\Omega e_{\varepsilon_n}(\mathfrak{u}_{\varepsilon_n}(x, s_0))dx - n_\Omega \pi |\log \varepsilon_n|| \\ \leq C(M_0, s_0, \Omega)(\|\partial_t u_{\varepsilon_n}(., s_0|\log \varepsilon_n|)\|_{L^2}^2 + 1), \quad (10)$$

where $n_\Omega = \sum_{a_i(s_0)\in\Omega} d_i^2(s_0)$.

In some sense, Theorem 7 is the parabolic generalization of the main result in [7]: in particular if $\|\partial_t u_\varepsilon(., s_0|\log \varepsilon|)\| = O(1)$, then $E_\varepsilon(\mathfrak{u}_\varepsilon(., s_0, \Omega)$ is up to an $O(1)$ error equal to an integer multiple of $\pi|\log \varepsilon|$, as in [7] for the elliptic case. Notice however that in general this integer is not equal to the square of the total degree as in [7]. In a forthcoming work, we will show, under mild compactness assumptions on the initial datum, that $E_\varepsilon(u_\varepsilon(., s_0|\log \varepsilon|), \Omega)$ is up to an $O(1)$ error equal to an integer multiple of $\pi|\log \varepsilon|$ for any $s_0 \notin \{\tau_0, ..., \tau_q\}$. This kind of result will provide an alternative proof of the rectifiability of $\Sigma_{\mathfrak{v}}$ and will moreover show that its $\mathcal{H}^1$ measure is locally finite.

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2 A brief account on some useful facts

In this section, we recall some formulas and results on $(\mathrm{PGL})_\varepsilon$, recast and complete them in a form suitable for our further analysis. We begin with the following well-known evolution formula for the localized energy density, from which the main results in this paper stem: for $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, we have

$$\frac{d}{ds}\int_{\mathbb{R}^2} \chi(x)\, d\mu_\varepsilon^t = -\int_{\mathbb{R}^2\times\{t\}} \chi(x)|\partial_t u_\varepsilon|^2\, dx \\ + \int_{\mathbb{R}^2\times\{t\}} \Big(D^2\chi \nabla u_\varepsilon \cdot \nabla u_\varepsilon - \Delta\chi e_\varepsilon(u_\varepsilon)\Big)\, dx, \quad (2.1)$$

where $t = s|\log \varepsilon|$. Using this formula and the PDE analysis of [3], we derive

Lemma 2.1. *Let $K \subset \mathbb{R}^2 \times \mathbb{R}^+ \setminus \Sigma_{\mathfrak{v}}$ be a compact set. Then*

$$e_\varepsilon(\mathfrak{u}_\varepsilon) \leq C(K), \qquad on\ K \quad (2.2)$$

$$||\mathfrak{u}_\varepsilon| - 1| \leq C(K)\varepsilon \qquad on\ K\,, \quad (2.3)$$

and

$$\int_K |\frac{\partial u_\varepsilon}{\partial t}|^2(x, s|\log\varepsilon|)\,dxds \leq \frac{C(K)}{|\log\varepsilon|}. \tag{2.4}$$

Proof. The first statement follows from (5.15) and (5.16) in [3]. For (2.3), we invoke Theorem 2.1 of [3], inequalities (2.23) and (2.24) there, which asserts that if $|u_\varepsilon| \geq 1 - \sigma_0$ on some parabolic cylinder $\Lambda \equiv B(x,r) \times [T - r^2, T]$, for some universal constant $0 < \sigma_0 < 1/2$, then

$$||1 - |u_\varepsilon|\,||_{L^\infty(\Lambda_{1/2})} \leq C(\Lambda, \sigma_0)\varepsilon^2|\log\varepsilon|,$$

where $\Lambda_{\frac{1}{2}} \equiv B(x, r/2) \times [T - r^2/4, T]$. Therefore $V_\varepsilon(u_\varepsilon) \to 0$ uniformly on $\Lambda_{1/2}$ and the conclusion follows.

Finally, formula (2.4) follows from identity (2.1) and (2.2). □

Next, we have, using the results from Appendix A.

Lemma 2.2. *For $0 < s_1 < s_2$ and sufficiently small ε, it holds*

$$\int_{\mathbb{R}^2\times[s_1,s_2]} |\partial_t u_\varepsilon|^2\,dxds \leq CM_0 \tag{2.5}$$

and

$$\int_{\mathbb{R}^2\times[s_1,s_2]} V_\varepsilon(u_\varepsilon)\,dxds \leq CM_0(1 + |s_2 - s_1|). \tag{2.6}$$

Proof. Inequality (2.5) is a direct consequence of the energy identity (2.1) for $\chi \equiv 1$,

$$\frac{d}{ds}\int_{\mathbb{R}^2} d\mu_\varepsilon^t = -\int_{\mathbb{R}^2\times\{t\}} |\partial_t u_\varepsilon|^2\,dx,$$

and the fact that $t = s|\log\varepsilon|$.

For (2.6) we invoke Proposition A.2 of the Appendix, with say $\beta = 1/2$. We write $(\text{PGL})_\varepsilon$ in the elliptic form

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2}u_\varepsilon(1 - |u_\varepsilon|^2) + f_\varepsilon, \tag{2.7}$$

where

$$f_\varepsilon = -\partial_t u_\varepsilon\,.$$

In view of (2.5), $\int_{[s_1,s_2]\times\mathbb{R}^2} |f_\varepsilon|^2 \leq CM_0$, so that $||f_\varepsilon(\cdot,s)||_{L^2(\mathbb{R}^2)} \leq \varepsilon^{-1/2}$ for any $s \in [s_1, s_2]\setminus A$, where $\text{meas}(A) \leq C(M_0)\varepsilon^{1/2}$. It follows from Proposition A.2 that

$$\int_{\mathbb{R}^2\times\{s\}} V_\varepsilon(u_\varepsilon) \leq C(M_0) \qquad \forall\, s \in [s_1, s_2] \setminus A,$$

and therefore integrating, we are led to

$$\int_{\mathbb{R}^2\times([s_1,s_2]\setminus A)} V_\varepsilon(u_\varepsilon) \le C(M_0)|s_2 - s_1|.$$

On the other hand,

$$\int_{\mathbb{R}^2\times A} V_\varepsilon(u_\varepsilon) \le C(M_0)\varepsilon^{1/2}|\log\varepsilon|,$$

and the conclusion follows. □

In view of Lemma 2.2, we may assume, passing possibly to a further subsequence $\varepsilon_n \to 0$ that

$$|\partial_t u_{\varepsilon_n}(x, s|\log\varepsilon|)|^2 dxds \rightharpoonup \omega_* \tag{2.8}$$

and

$$V_{\varepsilon_n}(u_{\varepsilon_n}(x, s|\log\varepsilon|))dxds \rightharpoonup W_* \tag{2.9}$$

in the sense of measures. It follows also from Lemma 2.1, that ω_* and W_* are supported in $\Sigma_{\mathfrak{v}}$.

In another direction, we obtained in [3] the following regularity property[6] for $\Sigma_{\mathfrak{v}}$.

Lemma 2.3 ([3] Theorem 2). *Let $s_0 > 0$ and $(a_i(s_0), s_0) \in \Sigma_{\mathfrak{v}}$, for $i = 1, ..., \ell(s_0)$. There exists a neighborhood $\mathcal{O}$ of $(a_i(s_0), s_0)$ in $\mathbb{R}^2 \times \mathbb{R}^+$ such that*

$$\Sigma_{\mathfrak{v}} \cap \mathcal{O} \subseteq \Sigma_{\mathfrak{v}} \cap \mathcal{C},$$

where $\mathcal{C}$ is the parabolic cone defined by

$$\mathcal{C} = \{(a, s) \in \mathbb{R}^2 \times \mathbb{R}^+ \text{ such that } |s - s_0| \ge \alpha|a - a_i(s_0)|^2\},$$

and where $\alpha > 0$ is a constant depending only on M_0.

In our proofs, we will require in some places a more quantitative version of Lemma 2.3.

Lemma 2.4. *Let $s_0 > 0$ and $r > 0$ be given. There exist constants σ_0, γ_0 depending only on M_0, such that if*

$$\Sigma_{\mathfrak{v}}^{s_0} \subset \cup_{i=1}^{n(s_0)} B(x_i, r),$$

[6] Which is clearly superseded by Theorem 2

where the points $x_1, .., x_{n(s_0)} \in \mathbb{R}^2$ *verify*

$$|x_i - x_j| \geq \sigma_0 r \qquad \forall i \neq j = 1, ..., n(s_0),$$

then

$$\Sigma_{\mathfrak{v}}^s \subset \cup_{i=1}^{n(s_0)} B(x_i, \frac{\sigma_0 r}{8})$$

for every $s_0 \leq s \leq s_0 + \gamma_0 r^2$.

Proof. Let $R > 0$ such that $\Sigma_{\mathfrak{v}}^{s_0} \subset B(0, R)$. Set

$$\Omega^\varepsilon(t) = \left\{ x \in \mathbb{R}^2 \, , \, \int_{B(x, r_\varepsilon)} e_\varepsilon(u_\varepsilon(\cdot, t)) \geq \frac{\eta_0}{2} |\log \varepsilon| \right\},$$

where $r_\varepsilon = |\log \varepsilon|^{-1/6}$ and η_0 is some constant provided in [3] Theorem 2 (actually a lower bound on the densities θ_i which appear in (5)). Since by assumption $\Sigma_{\mathfrak{v}}^{s_0} \subset \cup_{i=1}^{n(s_0)} B(x_i, r)$ it follows that, for ε sufficiently small,

$$\Omega^\varepsilon(s_0 |\log \varepsilon|) \cap B(0, R) \subset \cup_{i=1}^{n(s_0)} B(x_i, r + r_\varepsilon).$$

Applying Proposition 4.4 of [3] (also called the cylinders lemma) we obtain

$$\Omega^\varepsilon(s |\log \varepsilon|) \cap B(0, R) \subset \cup_{i=1}^{n(s_0)} B(x_i, \frac{\sigma_0}{8}(r + r_\varepsilon))$$

for every $s_0 \leq s \leq s_0 + \gamma r^2$. On the other hand, if $x \in \Sigma_{\mathfrak{v}}^s$ then for each neighborhood $\mathcal{U}_x$ of x one has

$$\int_{\mathcal{U}_x} e_\varepsilon(\mathfrak{u}_\varepsilon(\cdot, s)) \geq \frac{\eta_0}{2} |\log \varepsilon|$$

for sufficiently small ε, so that $\mathcal{U}_x \cap \Omega^\varepsilon(s |\log \varepsilon|) \neq \emptyset$, and the conclusion follows. $\square$

A rather direct consequence of Lemma 2.4 is

Lemma 2.5. *Set*

$$r(s_0) = \frac{1}{4} \inf\{|a_i(s_0) - a_j(s_0)|, \; 1 \leq i < j \leq \ell(s_0)\}. \tag{2.10}$$

We have, for every $0 \leq r \leq r(s_0)$

$$\Sigma_{\mathfrak{v}} \cap \mathbb{R}^2 \times [s_0, s_0 + \alpha r^2] \subseteq \cup_{j=1}^{\ell(s_0)} B(a_j(s_0), r) \times [s_0, s_0 + \alpha r^2], \tag{2.11}$$

where $\alpha = \frac{\gamma_0}{\sigma_0^2}$, *and* σ_0, γ_0 *are as in Lemma 2.4.*

Remark 2.1. Consider, for $0 \le r \le r(s_0)$ and $s_0 \le s \le s_0 + \alpha r^2$, the total degree

$$d(a_i(s_0), s, r) \equiv \deg(\mathfrak{u}_*(\cdot, s), \partial B(a_i(s_0), r)).$$

It follows from the continuity properties of the degree and Lemma 2.5 that

$$d(a_i(s_0), s, r) = d_i(s_0).$$

In particular, if $d_i(s) \neq 0$, then $d(a_i(s_0), s, r) \neq 0$ so that

$$\Sigma_{\mathfrak{v}}^s \cap B(a_i(s_0), r) \neq \emptyset \tag{2.12}$$

for any $0 \le r \le r(s_0)$ and $s_0 \le s \le s_0 + \alpha r^2$.

3 Properties of the limiting measures

3.1 Leading order quantization

This section contains a first step towards the quantization result stated in Theorem 5 and Lemma 1: we establish that the limiting energy density and potential are quantized[7]. The argument is mainly of elliptic nature and is developed in the Appendix. On the other hand, the precise relation of the densities with the degrees of the vortices will be a consequence of the dynamical law discussed in Section 3.2 (see Proposition 4.1). We have first

Proposition 3.1. *For all but finitely many $s > 0$ we have*

$$\theta_i(s) \in \pi\mathbb{N}. \tag{3.1}$$

In particular, there exists a finite set $\{0 = \tau_0 < \tau_1 < ... < \tau_q < \tau_{q+1} = +\infty\}$ such that

$$\mathfrak{v}_*^s(\mathbb{R}^2) = \pi n_k \qquad \text{for every } s \in (\tau_k, \tau_{k+1})\,,\ k = 0, ..., q, \tag{3.2}$$

where $n_k \in \mathbb{N}$. Moreover, $n_k > n_{k+1}$ for all $k = 0, \cdots, q$.

Proof. We write again $(\mathrm{PGL})_\varepsilon$ in the elliptic form (2.7) with perturbation term $f_\varepsilon = -\partial_t u_\varepsilon$. Our aim is to apply Theorem A.2. In order to do so, we fix first some arbitrary time $S > 0$ and then choose $R > 0$, sufficiently large so that

$$\Sigma_{\mathfrak{v}}^s \subset B_R \qquad \text{for every } s \in [0, S]. \tag{3.3}$$

[7] The precise value of the quantization will be established later in Section 4.

We next check the validity of the bounds (A.5) and (A.6) for some suitable constant M_0 independent of ε. First notice that (A.5) are standard consequences of $(\mathrm{PGL})_\varepsilon$ and assumption (H_0) on the initial datum, and are actually valid for any time $s > 0$, on the whole of $\mathbb{R}^2$.

On the other hand, by (3.3) and Lemma 2.1,

$$|e_\varepsilon(u_\varepsilon)| \leq C(M_0, S) \qquad \forall z \in B_{2R} \setminus B_R,$$

so that (A.6) holds (with B_1 replaced by B_R). Finally, we claim that, for any given $\sigma > 0$ there exists a subset $F_\varepsilon \subset [0, S]$ such that

$$\mathrm{meas}([0, S] \setminus F_\varepsilon) \leq \sigma,$$

and

$$\int_{B_R} |\partial_t u_\varepsilon|^2(x, s) + V_\varepsilon(u_\varepsilon(x, s)) \leq C(\sigma, S), \qquad \forall\, s \in F_\varepsilon.$$

This is an immediate consequence of Lemma 2.2. Consider the set

$$F = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} F_{\varepsilon_k},$$

where $(\varepsilon_k)_k \in \mathbb{N}$ is the sequence in Theorem 1. It follows that $\mathrm{meas}(F) = \lim_{n\to+\infty} \mathrm{meas} \cup_{k\geq n} F_{\varepsilon_k} \geq S - \sigma$, so that

$$\mathrm{meas}([0, S] \setminus F) \leq \sigma.$$

Next let $s_0 \in F$. It follows from the definition of F that s_0 belongs to F_{ε_k} for infinitely many $k \in \mathbb{N}$, so that for some subsequence (depending possibly on s_0) denoted ε_n, we have, for all $n \in \mathbb{N}$,

$$\int_{B_R} |\partial_t u_{\varepsilon_n}(x, s_0)|^2 + V_{\varepsilon_n}(u_{\varepsilon_n}(x, s_0))dx \leq C(\sigma, S).$$

Let $i \in \{1, ..., \ell(s_0)\}$, $r(s_0)$ be given by (2.10), and $\chi_i \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ such that $0 \leq \chi_i \leq 1$, $\chi_i \equiv 1$ on $B(a_i(s_0), r(s_0))$, and $\chi_i \equiv 0$ outside $B(a_i(s_0), 2r(s_0))$. We have, by (5)

$$\mathfrak{v}_{\varepsilon_n}^{s_0}(\chi_i) \to \theta_i(s_0).$$

On the other hand, it follows from Theorem A.2 in the Appendix, applied to $\mathfrak{u}_{\varepsilon_n}(\cdot - a_i(s_0), s_0)$ with $R = r(s_0)$, that there exists $n(s_0) \in \mathbb{N}$ such that

$$\mathfrak{v}_{\varepsilon_n}^{s}(B(a_i(s_0), r(s_0))) = \pi n(s_0) + o(1) \qquad \text{as } n \to +\infty.$$

Moreover, since $(B(a_i(s_0), 2r(s_0)) \setminus B(a_i(s_0), r(s_0))) \cap \Sigma_{\mathfrak{v}}^{s_0} = \emptyset$, we have

$$\mathfrak{v}_{\varepsilon_n}^{s_0} \left(B(a_i(s_0), 2r(s_0)) \setminus B(a_i(s_0), r(s_0))\right) = o(1) \qquad \text{as } n \to +\infty.$$

Hence

$$\mathfrak{v}_{\varepsilon_n}^{s_0}(\chi_i) \to \pi n(s_0) \qquad \text{as } n \to +\infty,$$

and therefore $\theta_i(s_0) = \pi n(s_0)$. Since S and σ were arbitrarily chosen, we infer that $\theta_i(s) \in \pi\mathbb{N}$ for almost all $s > 0$ and for all $i = 1, ..., \ell(s)$.

Recall that, by [3], Theorem 5 ii), the function $s \to ||\mathfrak{v}_*^s||$ is non-increasing. On the other hand, we just proved that for a.e. $s > 0$, $||\mathfrak{v}_*^s|| \in \pi\mathbb{N}$. Therefore, there exists a finite number of times $\tau_1 < .. < \tau_q \in \mathbb{R}^+$ such that $||\mathfrak{v}_*^s||$ is constant on any interval not containing any of the τ_i, $i = 1, .., q$. This proves the global identity (3.2).

We show next that $\theta_i(s) \in \pi\mathbb{N}$ for any $s \notin \{\tau_1, ..., \tau_q\}$, and $i \in \{1, ..., \ell(s)\}$ (so far we proved that fact only for a.e. s), thus proving the local equality (3.1). Set $r(s)$ as in (2.10) and define $\chi_i \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ such that $0 \leq \chi_i \leq 1$, $\chi_i \equiv 1$ on $B(a_i(s), r(s))$, and $\chi_i \equiv 0$ outside $B(a_i(s_0), 2r(s_0))$. We have

$$\mathfrak{v}_*^s(\chi_i) = \theta_i(s).$$

On the other hand, by [3], Theorem 5 ii), both of the functions $\tau \mapsto \mathfrak{v}_*^\tau(\chi_i)$ and $\tau \mapsto \mathfrak{v}_*^\tau(1-\chi_i)$ are non-increasing in a right neighborhood of s, whereas their sum is constant. They are therefore constant on this neighborhood. Since we already know that $\mathfrak{v}_*^\tau(\chi_i) \in \pi\mathbb{N}$ for almost every τ, it follows that $\theta_i(s) \in \pi\mathbb{N}$ and the proof is complete. □

As a byproduct of the previous discussion, we deduce the following

Corollary 3.1. *We have*

$$\omega_* = \sum_{k=1}^{q} \sum_{i=1}^{\ell(\tau_k)} \beta_i(\tau_k)\delta_{(a_i(\tau_k),\tau_k)},$$

where $\beta_i(\tau_k) \in \pi\mathbb{N}$ *for* $k = 1, ..., q$ *and* $i = 1, ..., \ell(\tau_k)$. *In particular, for every* $k = 0, ..., q$,

$$\omega_* \left(\mathbb{R}^2 \times (\tau_k, \tau_{k+1})\right) = 0.$$

The measure W_* can be directly deduced from the energy density in view of the following

Proposition 3.2. *The following identity holds.*

$$W_* = \frac{1}{2}\mathfrak{v}_*^s \otimes ds = \left(\sum_{i=1}^{\ell(s)} \frac{\theta_i(s)}{2}\delta_{a_i(s)}\right) \otimes ds.$$

Proof. We first notice that, as an immediate consequence of Lemma 2.1, *supp* $(W_*) \subset \Sigma_{\mathfrak{v}}$. We next divide the proof into two steps.

Step 1. *Let* $s_0 > 0$ *and* $0 < r \leq r(s_0)$*, where* $r(s_0)$ *is given by* (2.10)*. Then, for* $0 < \delta \leq \alpha r^2/4$*, where the constant* α *is defined in Lemma 2.5, and for* $i = 1, ..., \ell(s_0)$*,*

$$W_*(B(a_i(s_0), r) \times [s_0, s_0 + \delta]) = \delta \cdot \frac{\theta_i(s_0)}{2}. \tag{3.4}$$

By Lemma 2.5, $\Sigma_{\mathfrak{v}}^s \cap B(a_i(s_0), 3r) \subset B(a_i(s_0), r/2)$ for every $s \in [s_0, s_0 + \delta]$. It follows from Lemma 2.1 that the energy density $e_\varepsilon(u_\varepsilon)$ is uniformly bounded on $(B(a_i(s_0), 2r) \setminus B(a_i(s_0), r)) \times [s_0, s_0 + \delta]$. Our aim is to apply[8] Theorem A.1 and Theorem A.2 of the Appendix to u_ε restricted to $B(a_i(s_0), 2r) \times \{s\}$, for $s \in [s_0, s_0 + \delta]$, in order to prove that

$$\int_{B(a_i(s_0), r) \times [s_0, s_0+\delta]} V_\varepsilon(u_\varepsilon) = \delta \cdot \frac{\theta_i(s_0)}{2} + o(1). \tag{3.5}$$

For that purpose, we split the time interval of integration $[s_0, s_0 + \delta]$ into three disjoint parts. Let $M_2 > 0$ be given, set

$$F^\varepsilon = \{s \in [s_0, s_0 + \delta],\ ||\partial_t u_\varepsilon(\cdot, s)||_{L^2(\mathbb{R}^2)} \leq M_2\},$$

$$A_1^\varepsilon = \{s \in [s_0, s_0 + \delta],\ M_2 \leq ||\partial_t u_\varepsilon(\cdot, s)||_{L^2(\mathbb{R}^2)} \leq \varepsilon^{-1/2}\},$$
$$A_2^\varepsilon = \{s \in [s_0, s_0 + \delta],\ ||\partial_t u_\varepsilon(\cdot, s)||_{L^2(\mathbb{R}^2)} \geq \varepsilon^{-1/2}\}.$$

Since $\int_{\mathbb{R}^2 \times \mathbb{R}^+} |\partial_t u_\varepsilon|^2 \leq M_0$, we deduce

$$\text{meas}(A_1^\varepsilon) \leq C(M_0) M_2^{-2}, \qquad \text{meas}(A_2^\varepsilon) \leq M_0 \varepsilon.$$

By Theorem A.1,

$$\int_{B(a_i(s_0), 2r) \times A_1^\varepsilon} V_\varepsilon(u_\varepsilon) \leq C(M_0)\text{meas}(A_1^\varepsilon) \leq C(M_0) M_2^{-2}, \tag{3.6}$$

[8] after a suitable scaling

whereas, by (H_0),

$$\int_{B(a_i(s_0),2r)\times A_2^\varepsilon} V_\varepsilon(u_\varepsilon) \leq C(M_0)\varepsilon|\log\varepsilon|\,. \tag{3.7}$$

We turn finally to F_ε. It follows from Theorem A.2 that for $s \in F_\varepsilon$ we have

$$\left|\int_{B(a_i(s_0),r)\times\{s\}} V_\varepsilon(u_\varepsilon) - \frac{e_\varepsilon(u_\varepsilon)}{2|\log\varepsilon|}\right| \leq \frac{C(M_0,M_2)}{\sqrt{|\log\varepsilon|}}.$$

Collecting the previous estimates we obtain, setting $\Lambda_0 = B(a_i(s_0),r)\times [s_0, s_0+\delta]$,

$$\begin{aligned}
&\left|\int_{\Lambda_0} V_\varepsilon(u_\varepsilon) \; - \delta\cdot\frac{\theta_i(s_0)}{2}\right| \\
&\leq \left|\int_{\Lambda_0}\frac{e_\varepsilon(u_\varepsilon)}{2|\log\varepsilon|} - \delta\cdot\frac{\theta_i(s_0)}{2}\right| + \left|\int_{B(a_i(s_0),r)\times(F_\varepsilon\cup A_1^\varepsilon\cup A_2^\varepsilon)} V_\varepsilon(u_\varepsilon) - \frac{e_\varepsilon(u_\varepsilon)}{2|\log\varepsilon|}\right| \\
&\leq \left|\int_{\Lambda_0}\frac{e_\varepsilon(u_\varepsilon)}{2|\log\varepsilon|} - \delta\cdot\frac{\theta_i(s_0)}{2}\right| + \frac{C(M_0,M_2)}{\sqrt{|\log\varepsilon|}} + C(M_0)M_2^{-2} + C(M_0)\varepsilon|\log\varepsilon|.
\end{aligned}$$

We conclude by letting first $\varepsilon \to 0$, and then $M_2 \to +\infty$.

Step 2. Proof of Proposition 3.2 completed. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^2\times\mathbb{R}^+)$. We wish to prove that

$$\int \chi dW_* = \int_{R^+}\sum_{i=1}^{\ell(s)}\frac{\theta_i(s)}{2}\chi(a_i(s))ds. \tag{3.8}$$

By Step 2, we know that (3.8) holds when χ is the characteristic function of a cylinder of the form

$$B(a_i(s_0),r)\times[s_0,s_0+\delta], \tag{3.9}$$

for $s_0 > 0$, $r < r(s_0)$, $i = 1,...,\ell(s_0)$ and $\delta < \alpha r^2$. The proof of (3.8) in the general case follows by approximating χ with piecewise constant functions, (constant on cylinders of the type of (3.9))[9], the absolute continuity in time of W_* (given by Lemma 2.2) and Step 1. □

[9] A covering of $\Sigma_{\mathfrak{v}}$ by a disjoint union of such cylinders is given in [3], Lemma 5.3 .

3.2 Motion law for the limiting energy density

Our purpose is to describe the evolution of the energy integral $\int \chi d\mathfrak{v}_*^s$, where χ is a smooth test function with compact support. We impose moreover some special property, which was already introduced in [3], namely that there exists some $r > 0$ such that

$$H_r(s) \qquad\qquad \frac{\partial^2 \chi}{\partial \bar{z}^2} = 0 \qquad \text{on } \cup_{i=1}^{\ell(s)} B(a_i(s), \frac{r}{8}).$$

Proposition 3.3. *Let $0 < s_0 < s_1$ and assume that χ satisfies $H_r(s)$ for any $s \in [s_0, s_1]$. Then, for $s \in (s_0, s_1)$,*

$$\frac{d}{ds}\left(\int_{\mathbb{R}^2} \chi d\mathfrak{v}_*^s\right) = \mathcal{F}^s_{\rm core}(\chi) + \mathcal{F}^s_{\rm inter}(\chi) + \mathcal{F}^s_{\rm drift}(\chi) + \mathcal{F}^s_{\rm dissip}(\chi), \tag{3.10}$$

where

$$\mathcal{F}^s_{\rm core}(\chi) = \sum_{i=1}^{\ell(s)} [\pi d_i^2(s) - \theta_i(s)] \frac{\Delta \chi}{2}(a_i(s)) \tag{3.11}$$

$$\mathcal{F}^s_{\rm inter}(\chi) = 2 \sum_{i \neq j=1}^{\ell(s)} \pi d_i(s) d_j(s) \nabla_{a_i} (\log |a_i(s) - a_j(s)|) \cdot \nabla \chi(a_i(s)) \tag{3.12}$$

$$\mathcal{F}^s_{\rm drift}(\chi) = \sum_{i=1}^{\ell(s)} \pi d_i(s)\, \vec{c}(s)^\perp \cdot \nabla \chi(a_i(s)) \tag{3.13}$$

and

$$\mathcal{F}^s_{\rm dissip}(\chi) = \int_{\mathbb{R}^2 \times \{s\}} \chi(x) d\omega_*(x, s). \tag{3.14}$$

The first term $\mathcal{F}^s_{\rm core}(\chi)$ on the r.h.s. of (3.10) stems from the fine core structure of the vortices at the ε-level, in the same spirit as the analysis performed in the Appendix (taking $f_\varepsilon = -\partial_t v_\varepsilon$). We see for instance that it vanishes for $|d_i(s)| = 1$, whereas its contribution is more delicate to analyze in the multiple degree case. The second term $\mathcal{F}^s_{\rm inter}(\chi)$ represents the mutual interaction of the vortices, and was already derived in earlier works on the subject (see [12], [11], [15], [13], [8]). Finally, the third term $\mathcal{F}^s_{\rm drift}(\chi)$ stems from the interaction of the vortices with the field $\vec{c}$, the residual, low frequency part of the phase.

Proof of Proposition 3.3. Using Pohozaev's identity, formula (2.1) may be rewritten as (see e.g. Lemma 2.3 of [3])

$$\frac{d}{ds} \int_{\mathbb{R}^2} \chi(x)\, d\mathfrak{v}_\varepsilon^s = - \int_{\mathbb{R}^2 \times \{s|\log \varepsilon|\}} \chi(x) |\partial_t u_\varepsilon|^2\, dx + \mathcal{F}_S(s, \chi, \mathfrak{u}_\varepsilon),$$

where

$$\mathcal{F}_S(s,\chi,\mathfrak{u}_\varepsilon)=\int_{\mathbb{R}^2\times\{s\}}\Big(D^2\chi\nabla\mathfrak{u}_\varepsilon\cdot\nabla\mathfrak{u}_\varepsilon-\Delta\chi e_\varepsilon(\mathfrak{u}_\varepsilon)\Big)\,dx.$$

A little algebra (see [3] Section 2.1) shows that

$$\mathcal{F}_S(s,\chi,\mathfrak{u}_\varepsilon)=-\int_{\mathbb{R}^2\times\{s\}}\Delta\chi\,dW_\varepsilon^s+2\mathrm{Re}\int_{\mathbb{R}^2\times\{s\}}\omega(\mathfrak{u}_\varepsilon)\,\frac{\partial^2\chi}{\partial\bar{z}^2},$$

where the Hopf differential $\omega(v)$ is defined by

$$\omega(v)=|v_{x_1}|^2-|v_{x_2}|^2-2iv_{x_1}\cdot v_{x_2}.$$

Since convergence in (1) holds in $\mathcal{C}^1$ (for space variables) out of the support of $\frac{\partial^2\chi}{\partial\bar{z}^2}$, by assumption $(H_r(s))$ and Theorem 1, we obtain

$$\mathcal{F}_S(s,\chi,\mathfrak{u}_\varepsilon)\,ds\to\Big(-\int_{\mathbb{R}^2\times\{s\}}\Delta\chi\,dW_*^s+2\mathrm{Re}\int_{\mathbb{R}^2\times\{s\}}\omega(\mathfrak{u}_*)\,\frac{\partial^2\chi}{\partial\bar{z}^2}\Big)\,ds$$

as $\varepsilon\to 0$. An explicit computation shows that

$$\omega(\mathfrak{u}_*)(z,s)=-\Big(\sum_{i=1}^{\ell(s)}\frac{d_i(s)}{z-a_i(s)}\Big)^2-2\sum_{i=1}^{\ell(s)}i\bar{c}(s)\frac{d_i(s)}{z-a_i(s)}+|c(s)|^2.$$

Further explicit computations (see e.g. Proposition 7.1 of [3]) then lead to

$$\begin{aligned}-2\mathrm{Re}\int_{\mathbb{R}^2\times\{s\}}\Big(\sum_{i=1}^{\ell(s)}\frac{d_i(s)}{z-a_i(s)}\Big)^2\frac{\partial^2\chi}{\partial\bar{z}^2}&=\sum_{i=1}^{\ell(s)}\pi d_i^2(s)\frac{\Delta\chi}{2}(a_i(s))\\&+2\sum_{i\neq j=1}^{\ell(s)}\pi d_i(s)d_j(s)\nabla_{a_i}(\log|a_i(s)-a_j(s)|)\cdot\nabla\chi(a_i(s)).\end{aligned}$$

Similarly,

$$-4\mathrm{Re}\int_{\mathbb{R}^2\times\{s\}}\sum_{i=1}^{\ell(s)}i\bar{c}(s)\frac{d_i(s)}{z-a_i(s)}\,\frac{\partial^2\chi}{\partial\bar{z}^2}=\sum_{i=1}^{\ell(s)}\pi d_i(s)\,\vec{c}(s)^\perp\cdot\nabla\chi(a_i(s)),$$

and

$$2\mathrm{Re}\int_{\mathbb{R}^2\times\{s\}}|c(s)|^2\,\frac{\partial^2\chi}{\partial\bar{z}^2}=0.$$

On the other hand, in view of Proposition 3.2 we have

$$W_*^s ds = \sum_{i=1}^{\ell(s)} \frac{\theta_i(s)}{2} \Delta\chi(a_i(s)) \otimes ds,$$

whereas by (2.8), for any interval $I \subset \mathbb{R}^+$

$$\int_I \int_{\mathbb{R}^2} \chi(x) |\partial_t u_\varepsilon|^2 (x, s|\log \varepsilon|)\, dxds \to \int_{I\times\mathbb{R}^2} \chi(x) d\omega_*.$$

The conclusion follows by summation. □

Formula (3.10) as well as the energy quantization are the starting points in order to derive the motion law for the vortices. Before we turn directly to this issue, we emphasize first in the next section some further regularity properties of $\Sigma_{\mathfrak{v}}$.

3.3 Backward continuity for $\Sigma_{\mathfrak{v}}$

In view of Proposition 3.1 and Corollary 3.1 the energy is conserved and dissipation vanishes asymptotically on the strips $\mathbb{R}^2 \times (\tau_k, \tau_{k+1})$. This fact yields some nice reversibility properties for the limiting equations: in particular we will derive in this section some backward continuity for $\Sigma_{\mathfrak{v}}$.

Lemma 3.1. *Let $s_0 \in (\tau_k, \tau_{k+1})$ and $r(s_0)$ be given by (2.10). There exists a constant α_0 depending only on M_0 such that for every $0 < r \le r(s_0)$*

$$\Sigma_{\mathfrak{v}} \cap \mathbb{R}^2 \times (s', s_0] \subseteq \cup_{j=1}^{\ell(s_0)} B(a_j(s_0), r) \times (s', s_0], \tag{3.15}$$

where $s' = \max\{s_0 - \alpha_0 r^2, \tau_k\}$.

The proof is based on the forward continuity property expressed in the Cylinders Lemma 2.4 and Lemma 2.5, as well as the local conservation of energy, a consequence of Proposition 3.1.

Proof. Let $0 < r \le r(s_0)$ be given and define $\tilde{r} = (2\sigma_0)^{-C(M_0)-1} r$, and $\alpha_0 = \gamma_0 (2\sigma_0)^{-2C(M_0)-2}$ where $C(M_0)$ is defined in (2) and σ_0, γ_0 in Lemma 2.5. Set $s' = \max\{s_0 - \alpha_0 r^2, \tau_k\}$ and let $s' < s'' < s_0$. Applying Lemma 5.2 of [3] to $\Sigma_{\mathfrak{v}}^{s''} = \{a_i(s''),\ i = 1, ..., \ell(s'') \le \ell_0\}$ with $\sigma = \sigma_0$ and $r_0 = r$, we deduce that

$$\Sigma_{\mathfrak{v}}^{s''} \subset \cup_{j\in J} B(a_j(s''), \tilde{r}'),$$

for some $\tilde{r}'$ such that $\tilde{r} \le \tilde{r}' \le \tilde{r}(2\sigma_0)^{C(M_0)}$, and moreover

$$|a_j(s'') - a_k(s'')| \ge \sigma_0 \tilde{r}' \qquad \text{for any } j \ne k \in J.$$

Hence Lemma 2.4 applies to $\Sigma_{\mathfrak{v}}^{s''}$, so that

$$\Sigma_{\mathfrak{v}}^{s} \subset \cup_{j\in J} B(a_j(s''), \frac{\sigma_0 \tilde{r}'}{8}) \qquad \text{for any } s'' \le s \le s'' + \gamma_0(\tilde{r}')^2, \tag{3.16}$$

and in particular, since by definition of α_0

$$\gamma_0(\tilde{r}')^2 \ge \gamma_0(\tilde{r})^2 \ge \alpha_0 r^2,$$

we obtain

$$\Sigma_{\mathfrak{v}}^{s_0} \subset \cup_{j\in J} B(a_j(s''), \frac{\sigma_0 \tilde{r}'}{8}).$$

Set

$$J' = \{j \in J,\ B(a_j(s''), \frac{\sigma_0 \tilde{r}'}{8}) \cap \Sigma_{\mathfrak{v}}^{s_0} = \emptyset\}.$$

Assume that $J' \neq \emptyset$, and let $j \in J'$. Since $s'' > \tau_k$ by assumption, we obtain by (3.16)

$$\mathfrak{v}_*^{s''}(B(a_j(s''), \frac{\sigma_0 \tilde{r}'}{8})) = \mathfrak{v}_*^{s_0}(B(a_j(s''), \frac{\sigma_0 \tilde{r}'}{8})) = 0,$$

a contradiction. Therefore, $J' = \emptyset$ so that

$$\cup_{j\in J} B(a_j(s''), \frac{\sigma_0 \tilde{r}'}{8}) \subseteq \cup_{j=1}^{\ell(s_0)} B(a_j(s_0), \frac{\sigma_0 \tilde{r}'}{4}) \subseteq \cup_{j=1}^{\ell(s_0)} B(a_j(s_0), r), \tag{3.17}$$

where we have used the inequality $\tilde{r}' \le (2\sigma_0)^{C(M_0)} \tilde{r}$ and the definition of $\tilde{r}$. Combining (3.16) and (3.17) the conclusion (3.15) follows. $\square$

4 Quantization formula and the cone property

In this section we prove two results which are central in the arguments of the paper, in particular they will allow us to complete the proofs of Theorem 5 and Theorem 6. The first one is an explicit formulation of the quantization, whereas the second deals with the regularity properties of $\Sigma_{\mathfrak{v}}$. More precisely, we have shown so far that near each point $a_i(s_0)$, for $s_0 \notin \{\tau_1, ..., \tau_q\}$, the set of the trajectories $\Sigma_{\mathfrak{v}}$ is included in a two-sided parabolic cone. Here, we improve that property and prove that we may actually replace the parabolic cone by a straight cone.

Let $s_0 > 0$ be given such that $s_0 \notin \{\tau_1, ..., \tau_q\}$, and $r(s_0)$ given by (2.10). In view of the continuity properties of $\Sigma_{\mathfrak{v}}$ stated in Lemma 2.5 and Lemma

3.1, there exists $\Delta s_0 > 0$[10], such that for every $s \in [s_0 - \Delta s_0, s_0 + \Delta s_0]$ and $i \in 1, ..., \ell(s_0)$,

$$\Sigma_{\mathfrak{v}}^s \cap B(a_i(s_0), r(s_0)) \setminus B(a_i(s_0), r(s_0)/2) = \emptyset. \tag{4.1}$$

For $i = 1, ..., \ell(s_0)$, we set $B_i \equiv B(a_i(s_0), r(s_0))$.

Proposition 4.1. *i) For every $s \notin \{\tau_1, ..., \tau_q\}$,*

$$\theta_i(s) = \pi d_i^2(s). \tag{4.2}$$

In particular, for $s \notin \{\tau_1, ..., \tau_q\}$, $\mathcal{F}^s_{\rm core}(\chi) \equiv 0 \; \forall \chi \in \mathcal{C}^\infty_c(\mathbb{R}^2)$.

ii) There exists $K(M_0, s_0) > 0$ such that for every $s \in [s_0 - \Delta s_0, s_0 + \Delta s_0]$ and every $a_k(s) \in B_i$,

$$|a_k(s) - a_i(s_0)| \leq K(M_0, s_0)|s - s_0|.$$

Moreover,

$$K(M_0, s_0) = C(M_0)[\frac{1}{r(s_0)} + ||\vec{c}||_{L^\infty}].$$

Remark. Notice however that the parameters of the cone (as well as the O.D.E.) obey the parabolic scaling. In particular, this is a major obstacle on the way to prove the finiteness of the $\mathcal{H}^1$ measure of Σ_v.

The proof of Proposition 4.1 relies on Lemmas 4.1 and 4.2 below, which are both consequences of the evolution equation for the limiting energy density. More precisely, we apply formula (3.10) with test functions with compact support near a given vortex $a_i(s_0)$. Let us emphasize however that the test functions allowed are strongly constrained by condition $H_r(s_0)$. Therefore, we have essentially to restrict ourselves to two kinds of functions. First, we consider functions χ that are affine near $a_i(s_0)$, that is of the form $\chi(x) = \langle \vec{v}, x \rangle + b$. This will give us information on the local center of mass of the measure and its drift. Secondly, we consider functions that near a_i coincide with the squared distance to $a_i(s_0)$, i.e. $\chi(x) = |x - a_i(s_0)|^2$. This gives specific information concerning the core of the vortex (i.e., possible splitting and recombinations). In particular it allows to relate $\theta_i(s_0)$ to $d_i^2(s_0)$.

Lemma 4.1. *With $s_0 > 0$, $a_i(s_0)$ and Δs_0 as above, for every $s \in [s_0 - \Delta s_0, s_0 + \Delta s_0]$ we have for the energy*

$$\sum_{a_k(s) \in B_i} \theta_k(s) = \theta_i(s_0), \tag{4.3}$$

[10]More precisely, $\Delta s_0 = \inf\{\frac{1}{4}\alpha_0 r^2(s_0), s_0 - \tau_k, \tau_{k+1} - s_0\}$

and for the center of mass $\bar{a}_i(s) \equiv \frac{1}{\theta_i(s_0)} \sum_{a_k(s)\in B_i} a_k(s)\,\theta_k(s)$

$$\frac{d}{ds}\bar{a}_i(s) = \frac{\pi d_i(s_0)}{\theta_i(s_0)}\big[\vec{c}(s)^{\perp} + \sum_{\substack{a_j(s)\notin B_i \\ a_k(s)\in B_i}} 2d_j(s)\nabla_{a_k}(\log|a_k(s)-a_j(s)|)\big]. \quad (4.4)$$

Proof. Let $\chi \in \mathcal{C}_c^{\infty}(\mathbb{R}^2)$ be such that $\operatorname{supp}\chi \subset B_i$ and such that χ is affine near $a_i(s_0)$, i.e.

$$\chi(x) = \langle \vec{v}, x\rangle + b \qquad \text{on } B(a_i(s_0), r(s_0)/2),$$

for some $\vec{v} \in \mathbb{R}^2$ and $b \in \mathbb{R}$. Since $\Delta\chi \equiv 0$ on $B(a_i(s_0), r(s_0)/2)$, from (3.11) we deduce that $\mathcal{F}^s_{\text{core}}(\chi) = 0$. By conservation of the degree on B_i,

$$\sum_{a_k(s)\in B_1} d_k(s) = d_i(s_0). \quad (4.5)$$

Moreover, by antisymmetry,

$$\sum_{\substack{a_k(s),a_j(s)\in B_i \\ a_k(s)\neq a_j(s)}} d_k(s)d_j(s)\langle\nabla_{a_k}(\log|a_k(s)-a_j(s)|),\vec{v}\rangle = 0, \quad (4.6)$$

so that the interaction term reads

$$\mathcal{F}_{\text{inter}}(\chi) = 2\pi d_i(s_0) \sum_{\substack{a_k(s)\in B_i \\ a_j(s)\notin B_i}} \langle d_j(s)\nabla_{a_k}(\log|a_k(s)-a_j(s)|),\vec{v}\rangle. \quad (4.7)$$

By choosing $b = 0$ we obtain formula (4.4), since the vector $\vec{v}$ is arbitrary, while choosing $b = 1$ and $\vec{v} = 0$ yields (4.3). □

Next we consider the second type of test functions. This yields

Lemma 4.2. *With* $s_0 > 0$, $a_i(s_0)$ *and* $\Delta s_0 > 0$ *as above, for every* $s \in [s_0 - \Delta s_0, s_0 + \Delta s_0]$ *and for every* $b \in \mathbb{R}^2$ *we have*

$$\begin{aligned}\frac{d}{ds}\int_{B_i} |x-b|^2 d\mathfrak{v}^s_* = {}& 2(\pi d_i^2(s_0) - \theta_i(s_0)) \\ &+ 4\pi \sum_{\substack{a_k(s)\in B_i \\ a_j(s)\notin B_i}} d_k(s)d_j(s)\langle a_k(s)-b, \nabla_{a_k}\log|a_k(s)-a_j(s)|\rangle \\ &+ 2\pi \sum_{a_k(s)\in B_i} d_k(s)\langle a_k(s)-b, \vec{c}(s)^{\perp}\rangle.\end{aligned} \quad (4.8)$$

Proof. We may assume, changing possibly the origin, that $b = 0$. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ be such that $\operatorname{supp}\chi \subset B_i$ and $\chi(x) = |x|^2$ on $B(a_i(s_0), r(s_0)/2)$. We have $\Delta\chi(x) = 4$ and $\nabla\chi(x) = 2x$ on $B(a_i(s_0), r(s_0)/2)$. Inserting these relations in (3.11), (3.12), (3.13) we have, for any $s \in [s_0 - \Delta s_0, s_0 + \Delta s_0]$,

$$\mathcal{F}^s_{\text{core}}(\chi) = 2 \sum_{a_k(s)\in B_i} \pi d_k^2(s) - \theta_k(s), \tag{4.9}$$

$$\mathcal{F}^s_{\text{inter}}(\chi) = 4\pi \sum_{a_k(s)\in B_i} \sum_{j=1}^{\ell(s)} d_k(s)d_j(s)\langle \nabla_{a_k} \log|a_k(s) - a_j(s)|, a_k(s)\rangle, \tag{4.10}$$

and

$$\mathcal{F}^s_{\text{drift}}(\chi) = 2\pi \sum_{a_k(s)\in B_i} d_k(s)\langle a_k(s), \vec{c}(s)^\perp\rangle. \tag{4.11}$$

Since

$$\nabla_{a_k} \log|a_k(s) - a_j(s)| = \frac{a_k(s) - a_j(s)}{|a_k(s) - a_j(s)|^2},$$

we observe that

$$\begin{aligned}\sum_{\substack{a_k(s),a_j(s)\in B_i\\ a_k(s)\neq a_j(s)}} & d_k(s)d_j(s)\langle \nabla_{a_k} \log|a_k(s) - a_j(s)|, a_k(s)\rangle \\ &= \sum_{k<j} d_k(s)d_j(s)\langle \nabla_{a_k} \log|a_k(s) - a_j(s)|, a_k(s) - a_j(s)\rangle = \sum_{k<j} d_k(s)d_j(s)\end{aligned}$$

so that

$$\begin{aligned}\mathcal{F}^s_{\text{inter}}(\chi) = 2\pi & \sum_{\substack{a_k(s),a_j(s)\in B_i\\ a_k(s)\neq a_j(s)}} d_k(s)d_j(s) \\ &+ 4\pi \sum_{\substack{a_k(s)\in B_i\\ a_j(s)\notin B_i}} d_k(s)d_j(s)\langle a_k(s), \nabla_{a_k} \log|a_k(s) - a_j(s)|\rangle.\end{aligned}$$

Since

$$\sum_{a_k(s)\in B_i} d_k^2(s) + \sum_{\substack{a_k(s),a_j(s)\in B_i\\ a_k(s)\neq a_j(s)}} d_j(s)d_k(s) = (\sum_{a_k(s)\in B_i} d_k(s))^2 = d_i^2(s_0),$$

and $\sum_{a_k(s)\in B_i} \theta_k(s) = \theta_i(s_0)$, the conclusion follows. □

Proof of Proposition 4.1. We begin with i). Choose $b = a_i(s_0)$ and set

$$f(s) = \int_{B_i} |x - a_i(s_0)|^2 d\mathfrak{v}_*^s.$$

By (4.8) we have, for $s \in [s_0 - \Delta s_0, s_0 + \Delta s_0)]$,

$$\begin{aligned} f'(s) &= 2(\pi d_i^2(s_0) - \theta_i(s_0)) \\ &\quad + 4\pi \sum_{\substack{a_k(s)\in B_i \\ a_j(s)\notin B_i}} d_k(s) d_j(s) \langle a_k(s) - a_i(s_0), \nabla_{a_k} \log |a_k(s) - a_j(s)| \rangle \\ &\quad + 2\pi \sum_{a_k(s)\in B_i} d_k(s) \langle a_k(s) - a_i(s_0), \vec{c}(s)^\perp \rangle \, . \end{aligned} \tag{4.12}$$

In view of the continuity properties of $\Sigma_{\mathfrak{v}}$, as stated in Lemma 2.3, the r.h.s. of (4.12) is a continuous function of s. Therefore $f \in \mathcal{C}^1([s_0 - \Delta s_0, s_0 + \Delta s_0])$ and going back to (4.12) we obtain

$$f'(s_0) = 2(\pi d_i^2(s_0) - \theta_i(s_0)).$$

Since $f(s_0) = 0$ and $f(s) \geq 0 \ \forall\, s$, $f(s_0)$ is a local minimum, and therefore $f'(s_0) = 0$, which yields i).

ii) Assume now that $\theta_i(s_0) = \pi d_i^2(s_0)$. Set

$$g(s) = \max_{a_k(s)\in B_i} |a_k(s) - a_i(s_0)|^2.$$

Since the measure $\mathfrak{v}_*^s$ is a sum of quantized Dirac masses centered at the points $a_i(s)$, we have

$$C(M_0) g(s) \leq f(s)$$

for some constant $C(M_0)$ depending only on M_0. On the other hand, by (4.12), we have, for $s \in [s_0 - \Delta s_0, s_0 + \Delta s_0]$,

$$\begin{aligned} |f'(s)| &\leq C(M_0) \Big(\frac{1}{\inf\limits_{a_k(s)\in B_i, a_j(s)\notin B_i} |a_k(s) - a_j(s)|} + ||\vec{c}(s)|| \Big) g(s) \\ &\leq C(M_0) \Big(\frac{1}{\inf_{k\neq j} |a_k(s_0) - a_j(s_0)|} + ||\vec{c}(s)|| \Big) g(s) \\ &\leq K(M_0, s_0) g(s), \end{aligned} \tag{4.13}$$

where we have set $K(M_0, s_0) = C(M_0)(\frac{1}{r(s_0)} + ||\vec{c}||_{L^\infty})$. Integrating (4.13) we obtain

$$C(M_0) g(s) \leq f(s) \leq K(M_0, s_0) |s - s_0| \sqrt{\max_{t\in[s_0,s]} g(t)},$$

so that

$$\sqrt{\max_{t\in[s_0,s]} g(t)} \leq C(M_0)(\frac{1}{r(s_0)} + ||\vec{c}||_{L^\infty})|s-s_0|,$$

and the conclusion follows. □

Proof of Theorem 5. It follows combining Proposition 4.1 i) with Proposition 3.2. □

At this stage we are also able to carry out the proof of Theorem 6, which is somewhat similar to the proof of Lemma 4.2.

Proof of Theorem 6. If κ_1 is chosen sufficiently small and $0 < \kappa \leq \kappa_1$, then there exists a constant $\gamma > 0$ such that

$$\Sigma_{\mathfrak{v}}^s \cap B(a,r) \subset B(a, C\sqrt{\kappa}r) \qquad \forall\ s \in [s_0 - \gamma\kappa r^2, s_0 + \gamma\kappa r^2] \cap (\tau_k, \tau_{k+1}), \tag{4.14}$$

where C depends only on M_0. Indeed, for $s \geq s_0$, this follows from Lemma 2.4, whereas for $s \leq s_0$ we argue as in the proof of Lemma 3.1, using forward continuity and conservation of the degree.

We set $I = [s_0 - \gamma\kappa r^2, s_0 + \gamma\kappa r^2] \cap (\tau_k, \tau_{k+1})$. As a consequence of (4.14), as well as conservation of the degree and of energy, we obtain

$$\Gamma(s) \equiv \sum_{a_i(s)\in B(a,r)} d_i^2(s) - \left(\sum_{a_i(s)\in B(a,r)} d_i(s) \right)^2 = \Gamma, \qquad \forall\ s \in I. \tag{4.15}$$

A computation very similar to the one leading to (4.8) yields, for $s \in I$,

$$\begin{aligned} \frac{d}{ds}\int_{B(a,r)} |x-a|^2 d\mathfrak{v}_*^s + 2\pi\Gamma \\ = 4\pi \sum_{\substack{a_k(s)\in B(a,r)\\ a_j(s)\notin B(a,r)}} d_k(s)d_j(s)\langle a_k(s) - a, \nabla_{a_k} \log|a_k(s) - a_j(s)|\rangle \\ + 2\pi \sum_{a_k(s)\in B(a,r)} d_k(s)\langle a_k(s) - a, \vec{c}(s)^\perp\rangle, \end{aligned} \tag{4.16}$$

so that using (4.14) we obtain

$$\left| \frac{d}{ds}\int_{B(a,r)} |x-a|^2 d\mathfrak{v}_*^s + 2\pi\Gamma \right| \leq C\sqrt{\kappa}(1+r) \qquad \forall\ s \in I. \tag{4.17}$$

Assume that $\Gamma \neq 0$. We show that, for some suitable constant $\gamma_1 > 0$, we have $\tau_{k+1} - s_0 < \gamma_1\kappa^2 r^2$ if Γ is positive (resp. $s_0 - \tau_k < \gamma_1\kappa^2 r^2$ if Γ negative). We consider only the case Γ positive, the other case being handled similarly.

First, we choose κ_1 sufficiently small so that $C\sqrt{\kappa_1}(1+r) \leq \pi$. Inequality (4.17) yields

$$\frac{d}{ds}\int_{B(a,r)} |x-a|^2 d\mathfrak{v}_*^s \leq -\pi \tag{4.18}$$

for all $s \in I$. Let $\gamma_1 > 0$ to be determined later, decreasing κ_1 if necessary we may assume that $\kappa_1 \leq \gamma/(\gamma_1^2)$ so that $\gamma\kappa \geq \gamma_1\kappa^2$ for $0 < \kappa \leq \kappa_1$.

Assume by contradiction that $\tau_{k+1} > s_0 + \gamma_1\kappa^2 r^2$. Integrating (4.18) from s_0 to s_1, where $s_1 = s_0 + \gamma_1\kappa^2 r^2$, we obtain

$$0 \leq \int_{B(a,r)} |x-a|^2 d\mathfrak{v}_*^{s_1} \leq \int_{B(a,r)} |x-a|^2 d\mathfrak{v}_*^{s_0} - \pi\gamma_1\kappa^2 r^2 \leq (C - \pi\gamma_1)\kappa^2 r^2. \tag{4.19}$$

The contradiction follows choosing γ_1 large enough so that $C - \pi\gamma_1 < 0$. □

5 Motion law in the classical sense

As stated in Theorem 4, instead of the weak formulation (4), we are able to recover the classical motion in a number of cases.

We consider next a time $s_0 \notin \{\tau_1, ..., \tau_q\}$ and recall that Δs_0 is defined in Section 4.

Lemma 5.1. *Let $s_0 \notin \{\tau_1, ..., \tau_q\}$. Then for every $s \in [s_0 - \Delta s_0, s_0 + \Delta s_0]$,*

$$\ell(s) \geq \ell(s_0).$$

Proof. This is a consequence of the fact that $d_i(s_0) \neq 0$ and of the conservation of the degree, so that for $i = 1, ...\ell(s_0)$ and $s \in [s_0 - \Delta s_0, s_0 + \Delta s_0]$ we have $\Sigma_{\mathfrak{v}}^s \cap B(a_i(s_0), r(s_0)) \neq \emptyset$, and therefore card $[\Sigma_{\mathfrak{v}}^s \cap B(a_i(s_0), r(s_0))\,] \geq 1$. □

We introduce, for $i = 1, ..., \ell(s_0)$, the local number of vortices, defined by

$$\bar{\ell}_i(s) = \text{card}\,[\Sigma_{\mathfrak{v}}^s \cap B(a_i(s_0), r(s_0))\,].$$

In view of the argument above,

$$\bar{\ell}_i(s) \geq \bar{\ell}_i(s_0) \equiv 1 \qquad \text{for any } s \in [s - \Delta s_0, s + \Delta s_0].$$

If we assume moreover $\bar{\ell}_i(s) \equiv 1$, then we recover the classical motion law.

Lemma 5.2. *Assume that* $\bar{\ell}_i(s) = 1$ *for all* $s \in [s_0, s_1]$ *for some* $s_0 < s_1 < s_0 + \Delta s_0$. *Then for* $s \in [s_0, s_1]$ *the intersection* $B(a_i(s_0), r(s_0)) \cap \Sigma^s_{\mathfrak{v}}$ *is reduced to a single point, which we may therefore label* $\{a_i(s)\}$, *and*

$$\frac{d}{ds} a_i(s) = \frac{1}{d_i(s_0)} \left(\vec{c}(s)^{\perp} + 2 \sum_{i \neq j = 1}^{\ell(s)} d_j(s) \nabla_{a_i} (\log |a_i(s) - a_j(s)|) \right) \qquad (5.1)$$

for $s \in (s_0, s_1)$.

Proof. The fact that the intersection is reduced to a single point is an assumption. Combining equation (4.4) in Lemma 4.1 with (4.2), we are led to

$$\frac{d}{ds} \bar{a}_i(s) = \frac{1}{d_i(s_0)} \big[\vec{c}(s)^{\perp} + \sum_{\substack{a_j(s) \notin B_i \\ a_k(s) \in B_i}} 2 d_j(s) \nabla_{a_k} (\log |a_k(s) - a_j(s)|) \big].$$

Since in our situation we have to deal with a single point, the center of mass $\bar{a}_i(s)$ coincides with $a_i(s)$; equation (5.1) follows. □

In view of possible splittings and recombinations it is of course difficult to decide when $\bar{\ell}_i(s)$ is locally constant. At this stage we are only able to handle the case $d_i = \pm 1$. More precisely, we have

Lemma 5.3. *Assume that for some* i *we have*

$$|d_i(s_0)| = 1 \,.$$

Then for every $s \in [s_0 - \Delta s_0, s_0 + \Delta s_0]$,

$$\bar{\ell}_i(s) = 1$$

and therefore (5.1) *holds.*

Proof. We have, as a consequence of the quantization result (6),

$$\pi = \theta_i(s_0) = \sum_{a_k(s) \in B_i} \theta_k(s) \geq \pi \bar{\ell}_i(s),$$

so that the conclusion follows. □

We are now in position to present the proof of Proposition 4.

Proof of Proposition 4. Let $s_0 \notin \{\tau_1, ..., \tau_q\}$. We claim that for each $0 < \delta < \Delta s_0$ there exists $s_0 \leq s_1 \leq s_0 + \delta$ such that $\ell(s) = \ell(s_1)$ for

s in a suitably small right hand-side neighborhood of s_1. Indeed, if not, iterating Lemma 5.1 we would be able to construct a sequence $s_0 < s_1 < s_2 < \cdots < s_n < \cdots < s_0 + \delta$ verifying $\ell(s_{n+1}) > \ell(s_n)$ for all $n \in \mathbb{N}$, so that $\ell(s_n) \to +\infty$, a contradiction with the bound (2). Proposition 4 i) then follows from the claim and Lemma 5.2.

In order to prove ii), it suffices to show that if $\tau_k < s_0 < \tau_{k+1}$ and if s^* denotes the maximal time of existence of the ordinary differential equation (4) starting at s_0, then $s^* \leq \tau_{k+1}$. If not, then by closedness of $\Sigma_{\mathfrak{v}}$, the set $\Sigma_{\mathfrak{v}}^{s^*}$ is included in the union of the $\ell(s_0)$ distinct points $\tilde{a}_i(\tau_{q+1}) = \lim_{s \nearrow \tau_{q+1}} a_i(s)$. Moreover, since $d_i(s) \neq 0$ for all $i = 1, ..., \ell(s_0)$ and the points $\bar{a}_i(\tau_{k+1})$ are separated, we infer, by continuity of the degree, that $\ell(s) \geq \ell(s_0)$ in a right hand-side neighborhood of τ_{k+1}, so that $\limsup_{s \searrow \tau_{k+1}} \|\mathfrak{v}_*^s\| \geq \pi\ell(s_0)$. On the other hand, $\|\mathfrak{v}_*^s\| = \pi\ell(s_0)$ for $s_0 \leq s < \tau_{k+1}$ and the function $s \mapsto \|\mathfrak{v}_*^s\|$ is globally non increasing. The definition of τ_{k+1} leads to a contradiction.

Finally, iii) is an easy consequence of ii) and the annihilation result proved in [3] Theorem 3. □

6 Properties of the trajectories in the general case

In this section we prove Theorem 2 and Theorem 3. The main ingredient in the argument is the cone property stated in Proposition 4.1 ii). This kind of property has already been used in order to prove rectifiability by slicing in various contexts (see e.g. [19],[10]). The cone property is of course only effective if some bound on its width is available. In our case it depends strongly on the configuration at time s, more precisely the number $r(s)$ defined in Lemma 2.5. Therefore, for $\delta > 0$, we introduce the "truncated" sets $\Sigma(\delta)$ defined, for $\delta > 0$ by

$$\Sigma(\delta) = \cup_{s>0} \Sigma^s(\delta),$$

where $\Sigma^s(\delta) = \Sigma_{\mathfrak{v}}^s$ if $r(s) \geq 2\delta$, and $\Sigma^s(\delta) = \emptyset$ otherwise. Moreover, let $k \in \{1, ..., q\}$, $s \in (\tau_k, \tau_{k+1})$ and set

$$\Lambda^s(\delta) = \Sigma(\delta) \cap \mathbb{R}^2 \times [s, s + \sigma],$$

where $\sigma(\delta) = \min\{\alpha_0\delta^2, (s - \tau_k)/2, (\tau_{k+1} - s)/2\}$, and where $\alpha_0 > 0$ has been defined in Lemma 2.3. The main step in the proof of Theorem 2 is

Proposition 6.1. *For every $\delta > 0$ and every $\bar{s} \notin \{\tau_1, ..., \tau_q\}$, the set $\Lambda^{\bar{s}}(\delta)$ is contained in a finite union of graphs of Lipschitz maps $f_i : \mathbb{R} \to \mathbb{R}^2$, $i = 1, ..., \ell$. Moreover, there exists a constant $K = K(\delta)$ depending only on δ such that $||f_i||_{Lip} \leq K(\delta)$ for any $i = 1, ..., \ell$.*

Proof of Theorem 2. Observe that if $\delta' > \delta$ then $\Sigma(\delta') \subset \Sigma(\delta)$, and that $\Sigma_{\mathfrak{v}} = \cup_{\delta>0}\Sigma(\delta)$. Hence we may write

$$\Sigma_{\mathfrak{v}} = \bigcup_{k=0}^{q}\left(\Sigma_{\mathfrak{v}}^{\tau_k} \cup \bigcup_{s\in\mathbb{Q}\cap(\tau_k,\tau_{k+1})}\bigcup_{n\in\mathbb{N}}\Lambda^s(1/n)\right).$$

In view of Proposition 6.1 $\Lambda^s(1/n)$ is rectifiable for every $s > 0$, $n \in \mathbb{N}$, whereas $\Sigma_{\mathfrak{v}}^{\tau_k}$ is a finite set. Therefore $\Sigma_{\mathfrak{v}}$ is a countable union of rectifiable sets, hence it is rectifiable. □

In the proof of Proposition 6.1 we will use the following properties of the sets $\Sigma(\delta)$.

Lemma 6.1. *For any $\delta > 0$, the set $\Sigma(\delta)$ is closed in $\mathbb{R}^2 \times \mathbb{R}^+$, and so are the sets $\Lambda^s(\delta)$ for any $s > 0$.*

Proof. Fix $\delta > 0$ and let $(s_n)_{n\in\mathbb{N}}$ be a sequence such that $s_n \to s_0$ and $\Sigma^{s_n}(\delta) \neq \emptyset$ for any $n \in \mathbb{N}$. We claim that

$$\Sigma^{s_n}(\delta) \to \Sigma_{\mathfrak{v}}^{s_0} \tag{6.1}$$

in the Hausdorff distance. This is a consequence of Lemma 2.5 and Lemma 3.1. Indeed, by (2.11) , (3.15) it follows that if $0 < r \leq r(s_0)$, then for n sufficiently large (depending on r)

$$\Sigma_{\mathfrak{v}}^{s_n} \subset \cup_{i=1}^{\ell(s_0)} B(a_i(s_0), r)$$

and

$$B(a_i(s_0), r) \cap \Sigma_{\mathfrak{v}}^{s_n} \neq \emptyset \qquad \text{for any } i = 1, ..., \ell(s_0),$$

so that

$$\Sigma_{\mathfrak{v}}^{s_0} \subset \cup_{i=1}^{\ell(s_n)} B(a_i(s_n), 2r),$$

this proves the claim. In particular, $r(s_0) \geq \liminf r(s_n) \geq 2\delta$ so that $\Sigma_{\mathfrak{v}}^{s_0} = \Sigma^{s_0}(\delta)$ and the conclusion follows. □

Lemma 6.2. *i) For $s \in (\tau_k, \tau_{k+1})$, we have*

$$\Lambda^s(\delta) \subset \cup_{i=1}^{\ell(s_0)} B(a_i(s_0), \delta) \times [s, s+\sigma], \tag{6.2}$$

where $s_0 = \min\{s' \in [s, s+\sigma]\ ,\ \ \Sigma^{s'}(\delta) \cap \Lambda^{s'}(\delta) \neq \emptyset\}$.

ii) The balls $B(a_i(s_0), \delta)$ are disjoint, and for each $s' \in [s, s+\sigma]$, $B(a_i(s_0), \delta) \cap \Sigma^{s'}(\delta)$ contains at most one element.

Proof. Let $s \in (\tau_k, \tau_{k+1})$. In view of Lemma 6.1 the set $\Lambda^s(\delta)$ is compact and hence its projection on the time axis

$$S = \{s' \in [s, s+\sigma]\,,\ \Sigma^{s'}(\delta) \cap \Lambda^{s'}(\delta) \neq \emptyset\}$$

is a compact subset of $[s, s+\sigma]$. Therefore its infimum s_0 is achieved. Since $\sigma \leq \alpha\delta^2 \leq \alpha r^2(s_0)$ and $\Lambda^s(\delta) \subset \Sigma_{\mathfrak{v}} \times [s_0, s+\sigma]$, this implies that $\Lambda^s(\delta) \subset \cup_{i=1}^{\ell(s_0)} B(a_i(s_0), \delta) \times [s_0, s+\sigma]$, and since $\Lambda^s(\delta) \cap \mathbb{R}^2 \times [s, s_0] = \emptyset$, (6.2) follows.
Statement ii) is an immediate consequence of the definition of $\Sigma^{s'}(\delta)$, $s' \in [s, s+\sigma]$. □

Proof of Proposition 6.1. Fix $s \in (\tau_k, \tau_{k+1})$, and let s_0 be as defined in Lemma 6.2. For any $i = 1, ...\ell(s_0)$, by Lemma 6.1, the sets $\Lambda_i^s \equiv \Lambda^s(\delta) \cap B(a_i(s_0), \delta) \times [s, s+\sigma]$ are closed, as well as their projections S_i on the time axis. Moreover, by Lemma 6.2 ii), for any $s' \in S_i$, the set $\Lambda_i^s \cap \Sigma^{s'}(\delta) \times \{s'\}$ consists exactly of one element $(a_i(s'), s')$.

Let $f_i : S_i \to B(a_i(s_0), \delta) \subset \mathbb{R}^2$ the function uniquely determined by $f_i(s) = a_i(s)$. For $s_1 < s_2 \in S_i$, we have, by definition of S_i,

$$s_2 - s_1 \leq \sigma \leq \alpha\delta^2 \leq \alpha r(s_1)^2,$$

whereas, by the continuity properties of $\Sigma_{\mathfrak{v}}$,

$$|a_i(s_2) - a_i(s_1)| \leq 2\delta \leq r(s_1).$$

We are now in position to apply Proposition 4.1, ii), obtaining that f_i is a Lipschitz function for any $i = 1, ..., \ell(s_0)$, and $||f_i||_{Lip} \leq C(M_0)(\delta^{-1} + ||\vec{c}||_{L^\infty})$. MacShane extension Theorem allows to extend f_i to a lipschitz map $f_i : \mathbb{R}^+ \to \mathbb{R}^2$. □

Proof of Theorem 3. As for the proof of Theorem 2, the argument relies heavily on Proposition 6.1 and the constructions related to the sets $\Sigma(\delta)$. We first define the set $\mathcal{Z}$ of times $s > 0$ where the statement of Theorem 3 may fail. To that aim, we write

$$\Sigma_{\mathfrak{v}} = \cup_{n \in \mathbb{N}^*} \Sigma(\frac{1}{n}),$$

and take advantage of the fact that, by Proposition 6.1, $\Sigma(1/n)$ is contained locally in a finite union of lipschitz graphs. Let π be the projection from $\mathbb{R}^2 \times \mathbb{R}^+$ onto the time axis $\mathbb{R}^+$, and consider $S_n = \pi(\Sigma(1/n))$.

Firstly, notice that, by Rademacher Theorem, there exists a set $\mathcal{Z}_n^0$ of measure zero such that if $s \in S_n \setminus \mathcal{Z}_n^0$, then s is a point of differentiability for all the lipschitz functions f_i given in Proposition 6.1, whose graphs contain locally $\Sigma(1/n)$. On the other hand, there exists a set $\mathcal{Z}_n^1$ of measure zero such that every $s \in S_n \setminus \mathcal{Z}_n^1$ is of density one for S_n, i.e.

$$\lim_{\Delta s \to 0} \frac{|(s - \Delta s, s + \Delta s) \cap S_n|}{2\Delta s} = 1.$$

This is indeed a consequence of the fact that a.e. s in $\mathbb{R}^+$ is a Lebesgue point of 1_{S_n}. We finally set $\mathcal{Z}_n = \mathcal{Z}_n^0 \cup \mathcal{Z}_n^1$. and

$$\mathcal{Z} = \cup_{n \in \mathbb{N}^*} \mathcal{Z}_n \cup \{\tau_1, .., \tau_k\},$$

so that in particular $|\mathcal{Z}| = 0$.

Next consider $s_0 \notin \mathcal{Z}$, $s_0 \in (\tau_q, \tau_{q+1})$.

Step 1. There exists $n_0 \in \mathbb{N}^*$, $\tau > 0$ and a set E_{s_0} such that

$$[s_0 - \tau, s_0 + \tau] = S_{n_0} \cup E_{s_0},$$

where s_0 is a point of zero density for the set E_{s_0}.

Proof. Since $s_0 \notin \mathcal{Z}$, there exists $n_0 \in \mathbb{N}^*$ such that s_0 is of density 1 for S_{n_0}. Choose $\tau < \min\{s_0 - \tau_q, \tau_{q+1} - s_0\}$, and set $E_{s_0} = [s_0 - \tau, s_0 + \tau] \setminus S_{n_0}$. □

Next let $i \in \{1, ..., \ell(s_0)\}$. In view of Lemma 6.2 and the proof of Proposition 6.1, there exists a lipschitz function $f_i : [s_0 - \tau, s_0 + \tau] \to \mathbb{R}^2$ such that

$$\Sigma^s(1/n_0) \cap (B(a_i(s_0), r(s_0)) \times [s_0 - \tau, s_0 + \tau]) = \mathrm{graph} f_i. \tag{6.3}$$

Since $s_0 \notin \mathcal{Z}_n$, f_i is differentiable at s_0. We may thus construct a function $f : \mathbb{R}^+ \to \mathbb{R}^2$ verifying the assumptions of Definition 1. Indeed, choose $f(s) = f_i(s)$ if $s \in [s_0 - \tau, s_0 + \tau]$, and arbitrarily in $\Sigma^s_{\mathfrak{v}}$ elsewhere. Hence $f_i'(s_0)$ is an approximate speed for $a_i(s_0)$. We show next that this is the only possible approximate speed.

Step 2. Let $f : [s_0 - \tau, s_0 + \tau] \to \mathbb{R}^2$ such that $f(s) \in \Sigma^s_{\mathfrak{v}}$ and $f(s) \to a_i(s_0)$ as $s \to s_0$. Then f is approximately differentiable at s_0 and its approximate derivative at s_0 coincides with $f_i'(s_0)$.

Proof. We have to prove that

$$\mathrm{ap} \lim_{\Delta s \to 0} Q(\Delta s) = f_i'(s_0), \qquad \text{where} \qquad Q(\Delta s) = \frac{f(s_0 + \Delta s) - f(s_0)}{\Delta s},$$

or, by definition[11], that for any neighborhood $W \subset \mathbb{R}^2$ of $f_i'(s_0)$, the set

[11] See [9] 2.9.12.

$\mathbb{R}^+ \setminus Q^{-1}(W)$ has density zero at s_0. Remark first that, since $f(s) \to a_i(s_0)$ as $s \to s_0$, in view of (6.3) and Step 1, we deduce that $f(s) = f_i(s)$ for $s \in S_{n_0} \cap [s_0 - \tau, s_0 + \tau]$, so that in particular

$$Q(\Delta s) = Q_i(\Delta s) \equiv \frac{f_i(s_0 + \Delta s) - f_i(s_0)}{\Delta s} \tag{6.4}$$

for any $s_0 + \Delta s \in [s_0 - \tau, s_0 + \tau] \setminus E_{s_0}$. Equation (6.4) implies that $\mathbb{R}^+ \setminus Q^{-1}(W) \subseteq \Big(\mathbb{R}^+ \setminus Q_i^{-1}(W)\Big) \cup E_{s_0} \cup (\mathbb{R}^+ \setminus [s_0 - \tau, s_0 + \tau])$, and this last set has density zero at the point s_0. The proof is complete. □

7 Quantization revisited

In Section 4, we proved that the energy is asymptotically quantized in the $|\log \varepsilon|$ scale. In the present section we estimate the next term in the expansion of the energy as $\varepsilon \to 0$, at least locally in space. In contrast to the results in Section 4, based essentially on elliptic estimates (the perturbed equation studied in the Appendix), the results in this section use more the very parabolic nature of the equation, and especially Theorem 6.

We begin with the following ε-version of Theorem 6, which may also be compared with Proposition A.9 in the Appendix.

Proposition 7.1. *Let u_ε be a solution of $(PGL)_\varepsilon$ verifying (H_0), and κ_1, γ_1 the constants provided in Theorem 6. Let $0 < \kappa \leq \kappa_1$ be given. Assume that there exist $\varepsilon^{1/2} < R_1 < R_2$, $b \in \mathbb{R}^2$, and l points $\{b_1, ..., b_l\} \subset \mathbb{R}^2$ such that, for some $s_0 > R_2^2$,*

1. $\{|\mathfrak{u}_\varepsilon(., s_0)| \leq \frac{1}{2}\} \cap B(b, \kappa^{-1} R_2) \subset \cup_{i=1}^{l} B(b_i, R_1) \subset B(b, R_2)$,
2. $\operatorname{dist}(b_i, b_j) \geq \kappa^{-1} R_1$ *for* $i \neq j$,
3. $\displaystyle\int_{B(b,\kappa^{-1}R_2)\times[s_0-\gamma_1 R_2^2, s_0+\gamma_1 R_2^2]} |\partial_t u_\varepsilon(x, s|\log\varepsilon|)|^2 \, dx ds \leq \frac{\pi}{2}.$

Then, there exists $\varepsilon_0 > 0$, depending only on M_0, κ and R_2/R_1 such that if $\varepsilon/R_2 \leq \varepsilon_0$ then

$$\sum_{i=1}^{l} d_i^2 = \left(\sum_{i=1}^{l} d_i\right)^2, \tag{7.1}$$

where $d_i = \deg(\mathfrak{u}_\varepsilon(., s_0), \partial B(b_i, R_1))$.

Proof. By scaling [12] and translation invariance, we may assume that $R_2 = 1$ and $b = 0$. The proof goes then by contradiction : we fix $0 < R_1 < 1$ and assume that for a sequence $\varepsilon_n \to 0$ there exist solutions u_{ε_n} and points $\{b_i^n\}_{1\leq i\leq l_n}$ which satisfy conditions 1, 2 and 3 but violate equality (7.1). The points b_i^n and their number l_n may depend on n, but passing to a subsequence we may assume that $l_n \equiv l$ is constant and that the points b_i^n converge: we therefore argue as if they were equally constant. We apply Theorem 1 to the sequence $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}}$. This defines in particular the limiting vortices $a_i(s_0)$ and their degrees $d_i(s_0)$, whereas Lemma 1 defines the collision times τ_k. Passing to the limit $n \to +\infty$ in condition 1, we obtain

$$\cup_{i=1}^{l(s_0)}\{a_i(s_0)\} \cap B(a, \kappa^{-1}R_2) \subset \cup_{i=1}^{l} B(b_i, R_1) \subset B(b, R_2), \tag{7.2}$$

and similarly passing to the limit in condition 3 we infer from the quantization of dissipation (see Corollary 3.1) that

$$\mathrm{dist}(s_0, \{\tau_1, ..., \tau_q\}) \geq \gamma_1 R_2^2.$$

We apply Theorem 6 first with $a = b_i$ and $r = R_1$ and then with $a = b$ and $r = R_2$. This yields first

$$\sum_{a_i(s_0)\in B(b_i,R_1)} d_i^2(s_0) = \left(\sum_{a_i(s_0)\in B(b_i,R_1)} d_i(s_0)\right)^2 = d_i^2, \tag{7.3}$$

and then

$$\sum_{a_i(s_0)\in B(b,R_2)} d_i^2(s_0) = \left(\sum_{a_i(s_0)\in B(b,R_2)} d_i(s_0)\right)^2 = \left(\sum_{i=1}^{l} d_i\right)^2. \tag{7.4}$$

Since by (7.2)

$$\sum_{i=1}^{l}\left(\sum_{a_i(s_0)\in B(b_i,R_1)} d_i^2(s_0)\right) = \sum_{a_i(s_0)\in B(b,R_2)} d_i^2(s_0),$$

we obtain from (7.3) and (7.4) that equality (7.1) is satisfied, a contradiction since we assumed it to be violated. □

[12] Under the change of scale $x \mapsto R_2 x,\ t \mapsto R_2^2 t$, the new function $\tilde{u}_{\tilde{\varepsilon}}(x,t) = u_\varepsilon(R_2 x, R_2^2 t)$ satisfies $(PGL)_{\tilde{\varepsilon}}$ where $\tilde{\varepsilon} = \varepsilon/R_2$. Notice in particular that $\varepsilon \leq \tilde{\varepsilon} \leq \varepsilon^{1/2}$ so that $E_{\tilde{\varepsilon}}(\tilde{u}_{\tilde{\varepsilon}}(.,0)) = E_\varepsilon(u_\varepsilon(.,0) \leq M_0|\log\varepsilon| \leq 2M_0|\log\tilde{\varepsilon}|$ and (H_0) remains valid after doubling M_0.

Proof of Theorem 7 Without loss of generality, we may assume that $\Omega = B(1)$ and that $B(2) \setminus B(1/4) \cap \Sigma_{\mathfrak{v}}^{s_0} = \emptyset$, the general case following by scalings and suitable coverings.

It follows from Lemma 2.1 that $e_\varepsilon(u_\varepsilon)$ is uniformly bounded on $(B(1) \setminus B(1/2)) \times \{s_0|\log\varepsilon|\}$ so that we are in position to apply Theorem A.2 of the Appendix. Writing $f_\varepsilon = \frac{\partial}{\partial t} u_\varepsilon(\cdot\,, s_0|\log\varepsilon|)$, this yields the estimate

$$\left| \int_{B(1)\times\{s_0\}} e_\varepsilon(\mathfrak{u}_\varepsilon)dx - \pi n|\log\varepsilon| \right| \leq C(M_0)(1 + \|f_\varepsilon\|^2_{L^2} + \log(2 + ||f_\varepsilon||_{L^2}\sqrt{|\log\varepsilon|})), \tag{7.5}$$

for some $n \in \mathbb{N}$. We first notice that if $\|f_\varepsilon\|^2_{L^2} \geq C|\log\varepsilon|$, there is nothing to prove since we may choose the constant $C(M_0)$ in (10) sufficiently large. On the other hand, if $\|f_\varepsilon\|^2_{L^2} = o(|\log\varepsilon|)$, then it follows from our first quantization results (Theorem 5 and Lemma 1) that necessarily $n = \sum_{a_i(s_0)\in B(1)} d_i^2(s_0) = n_\Omega$. Finally, we observe that if $\|f_\varepsilon\|^2_{L^2} \geq \log|\log\varepsilon|$ then (7.5) implies (10) so that it remains to study the case $\|f_\varepsilon\|^2_{L^2} \leq \log|\log\varepsilon|$ which we assume throughout the rest of the proof.

In order to improve estimate (7.5) to (10), we have to recast the machinery of the proof of Theorem A.2 of the Appendix in the parabolic situation considered here: in particular we will show that Proposition 7.1 allows to improve the estimates of Proposition A.10. For that purpose, recall first that by Proposition A.4

$$\left| \int_{B(1)\times\{s_0\}} e_\varepsilon(\mathfrak{u}_\varepsilon)dx - \frac{1}{2}\int_{\Omega_\varepsilon(1/2)} |\nabla\Psi_\varepsilon|^2 \right| \leq C(M_0)(1 + \|f_\varepsilon\|^2_{L^2(\Omega)}). \tag{7.6}$$

We show next that

$$\left| \frac{1}{2}\int_{\Omega_\varepsilon(1/2)} |\nabla\Psi_\varepsilon|^2 - n_\Omega|\log\varepsilon| \right| \leq C(M_0)(1 + \|f_\varepsilon\|^2_{L^2(\Omega)}). \tag{7.7}$$

The starting point is again the expansion (A.80) of $\int |\nabla\Psi_\varepsilon|^2$, using once more the clustering process of Lemma A.9. The improvement here comes from the fact that whereas we had to stop the iterations in Proposition A.10 at the scale $R_{\max}$ given by (A.92), we may now continue thanks to Proposition 7.1.

More precisely, from the scale ε to the scale $R_{\max}$ we proceed with the clustering process of Lemma A.9, with the constant $\kappa = \kappa_0$ given by (A.84). This leads to an error term of order $|\log\kappa_0|$ (see inequality (A.93)). In contrast, from scale $R_{\max}$ to scale $O(1)$, we use Lemma A.9 with $\kappa = \kappa_1$. We notice firstly that $R_{\max} \geq \varepsilon^{1/2}$, since $\|f_\varepsilon\|^2_{L^2} \leq \log|\log\varepsilon|$. Secondly, conditions 1 and 2 in Proposition 7.1 are automatically satisfied for the clusters obtained. Finally, s_0 is not a collision time, so that setting δ to be half the distance between s_0 and the set of collision times we have

$$\int_{B(2)\times[s_0-\delta,s_0+\delta]} |\partial_t u_\varepsilon|^2 \leq o(1)$$

and condition 3 of Proposition 7.1 is satisfied for ε small enough. We may therefore apply this proposition to conclude that

$$\sum_{i\neq j\in J_n} d_i d_j = 0,$$

so that from scale $R_{\max}$ to scale $O(1)$ we obtain an error term of order $O(1)$. Combining the two steps, we are led to

$$\left|\frac{1}{2}\int_{\Omega_\varepsilon(1/2)} |\nabla\Psi_\varepsilon|^2 - n_\Omega|\log\varepsilon|\right| \leq C(M_0)(1+|\log\kappa_0|),$$

and we obtain (7.7) from the definition of κ_0. □

Appendix A : Core analysis for perturbed elliptic Ginzburg-Landau equations

In this Appendix[13], we consider solutions v_ε of the perturbed elliptic Ginzburg-Landau equation

$$-\Delta v_\varepsilon = \frac{1}{\varepsilon^2} v_\varepsilon(1-|v_\varepsilon|^2) + f_\varepsilon \qquad \text{on } \Omega. \tag{A.1}$$

We assume throughout that the following additional condition is satisfied, for some constants M_0 :

$$E_\varepsilon(v_\varepsilon) \equiv E_\varepsilon(v_\varepsilon,\Omega) = \int_\Omega e_\varepsilon(v_\varepsilon) \leq M_0|\log\varepsilon|. \tag{A.2}$$

Our first result is

Theorem A.1. *Assume v_ε satisfies* (A.1), (A.2), $|v_\varepsilon| \leq M_0$ *on* $\Omega \equiv B_1$ *and* f_ε *the bound*

$$||f_\varepsilon||_{L^2(B_1)} \leq \varepsilon^{-\beta}, \tag{A.3}$$

for some constant $0<\beta<1$. *Then, for every* $0<R<1$,

$$\int_{B_R} |\nabla|v_\varepsilon||^2 + \frac{(1-|v_\varepsilon|^2)^2}{4\varepsilon^2} \leq C(M_0,\beta,R). \tag{A.4}$$

[13] A related analysis of (A.1) is carried out in [17].

Whereas our first result shows in particular that, under mild assumption on f_ε, the potential remains bounded, our second result asserts a quantization for the energy and the potential when (A.3) is replaced by a stronger assumption.

Theorem A.2. *i) Assume that v_ε verifies* (A.2) *on $\Omega \equiv B_1$ and*

$$|v_\varepsilon| \le 1 + M_0\varepsilon, \qquad |\nabla v_\varepsilon| \le \frac{M_0}{\varepsilon} \qquad \text{on } \Omega, \tag{A.5}$$

$$e_\varepsilon(v_\varepsilon) \le M_0 \qquad \text{on } B_1 \setminus B_{\frac{1}{2}}. \tag{A.6}$$

Then there exists an integer $n \in \mathbb{N}$ such that

$$|E_\varepsilon(v_\varepsilon) - \pi n|\log\varepsilon|| \le C(M_0)(1 + \|f_\varepsilon\|^2_{L^2(\Omega)} + \log(2 + ||f_\varepsilon||_{L^2(\Omega)}\sqrt{|\log\varepsilon|})) \tag{A.7}$$

$$\left|\int_\Omega \frac{(1-|v_\varepsilon|^2)^2}{4\varepsilon^2} - \frac{\pi}{2}n\right| \le \frac{C(M_0)}{|\log\varepsilon|}(1 + \varepsilon^\alpha\|f_\varepsilon\|^2_{L^2(\Omega)}) \tag{A.8}$$

where the constants $C(M_0)$ and α depend only on M_0.

ii) Moreover, if $n = 1$, then we have

$$|E_\varepsilon(v_\varepsilon) - \pi|\log\varepsilon|\,| \le C(M_0)(1 + \|f_\varepsilon\|^2_{L^2(\Omega)}). \tag{A.9}$$

This result is a variant of the main result in [7], where the ball B_1 is replaced by a bounded simply connected domain Ω, $f_\varepsilon \equiv 0$, and moreover some smooth boundary value g of modulus 1 was imposed on $\partial\Omega$. In this case they obtained the remarkable identity

$$n = d^2 = \deg{(g, \partial\Omega)}^2. \tag{A.10}$$

This result however does not remain valid if the equation is perturbed, (i.e. $f_\varepsilon \neq 0$) even by a term f_ε of order $O(\frac{1}{\sqrt{|\log\varepsilon|}})$ in L^2. Indeed, take the parabolic equation with initial data having two vortices of degree $+1$ and -1 at distance of order 1, they will not collide before a time s of order 1, so that one may find some intermediate time such that $||\partial_t u||_{L^2} \le \frac{C}{\sqrt{|\log\varepsilon|}}$.

A.1 Proof of Theorem A.1

We start with the now classical observation (see e.g. [5], Proposition 1 and Lemma B.2).

Proposition A.1. *Assume* v_ε *satisfies* (A.2) *and* (A.3) *on* $\Omega \equiv B_1$. *Then, for every* $0 < R < 1$ *and* $0 < \sigma_0 < 1/2$, *there exist constants* $\ell_0 > 0$ *and* $\lambda > 0$ *depending only on* M_0, β, σ_0 *and* R *and at most* ℓ_0 *points* $a_1^\varepsilon, ..., a_\ell^\varepsilon$, $(\ell \leq \ell_0)$ *such that*

$$|1 - |v_\varepsilon|| \leq \sigma_0 \qquad \text{on } B_R \setminus \cup_{i=1}^{\ell} B(a_i^\varepsilon, \lambda\varepsilon).$$

Moreover, for every $z \in B_R$ *and every* $0 < \beta < \alpha < 1$,

$$\int_{B(z,\varepsilon^\alpha)} \frac{(1 - |v_\varepsilon|^2)^2}{4\varepsilon^2} \leq C(M_0, \alpha, R). \tag{A.11}$$

Notice that (A.11) holds for every $z \in B_R$, in particular on the vorticity set. The next result yields an improved estimate when z is far from the vorticity set, the proof is adapted from [4].

Proposition A.2. *Assume* v_ε *verifies* (A.1) *on* $\Omega \equiv B_1$. *There exists a positive constant* $\sigma_0 \leq 1/2$ *such that if*

$$|1 - |v_\varepsilon|| \leq \sigma_0 \qquad \text{on } B_1, \tag{A.12}$$

then, for any $0 < R < 1$,

$$\int_{B_R} |\nabla |v_\varepsilon||^2 + \frac{(1 - |v_\varepsilon|^2)^2}{4\varepsilon^2} \leq C(R)\varepsilon E_\varepsilon^{\frac{1}{2}}(v_\varepsilon)(E_\varepsilon(v_\varepsilon) + ||f_\varepsilon||^2_{L^2(B_1)}).$$

Proof of Proposition A.2. By assumption (A.12), we may write $v_\varepsilon \equiv \rho_\varepsilon \exp(i\varphi_\varepsilon)$ on B_1, and changing φ_ε possibly by a constant, we may impose the additional condition

$$\frac{1}{|B_1|} \int_{B_1} \varphi_\varepsilon = 0. \tag{A.13}$$

Vector multiplication of (A.1) by v_ε leads to the elliptic equation

$$-\text{div}(\rho_\varepsilon^2 \nabla \varphi_\varepsilon) = f_\varepsilon \times v_\varepsilon \qquad \text{in } B_1. \tag{A.14}$$

We will handle (A.14) as a linear equation for the function φ_ε, ρ_ε being considered as a coefficient. In the sequel, we write $\varphi = \varphi_\varepsilon$ and $\rho = \rho_\varepsilon$ when this is not misleading. In order to avoid boundary conditions, we consider the truncated function $\tilde{\varphi}$ defined by $\tilde{\varphi} = \varphi\,\chi$, where $\chi \in \mathcal{C}_c^\infty(B_1)$ is a smooth cut-off function such that $\chi \equiv 1$ on $B_{\frac{1+R}{2}}$. The function $\tilde{\varphi}$ then verifies the equation

$$-\text{div}(\rho^2 \nabla \tilde{\varphi}) = \text{div}(\rho^2 \varphi \nabla \chi) + \rho^2 \nabla \chi \cdot \nabla \varphi + (f_\varepsilon \times v_\varepsilon)\chi \qquad \text{in } B_1. \tag{A.15}$$

Since by assumption ρ is close to 1, it is natural to treat the l.h.s. of (A.15) as a perturbation of the Laplace operator, and to rewrite (A.15) as follows

$$-\Delta\tilde{\varphi} = \mathrm{div}((\rho^2-1)\nabla\tilde{\varphi}) + \mathrm{div}(\rho^2\varphi\nabla\chi) + \rho^2\nabla\chi\cdot\nabla\varphi + f_\varepsilon \times v_\varepsilon\chi \qquad \text{in } B_1.$$

We introduce the function φ_0 defined on B_1 as the solution of

$$\begin{cases} -\Delta\varphi_0 &= \mathrm{div}(\rho^2\varphi\nabla\chi) + \rho^2\nabla\chi\cdot\nabla\varphi + f_\varepsilon\times v_\varepsilon\chi & \text{in } B_1, \\ \varphi_0 &= 0 & \text{on } \partial B_1. \end{cases} \tag{A.16}$$

We set $\varphi_1 = \tilde{\varphi} - \varphi_0$, i.e.

$$\tilde{\varphi} = \varphi_0 + \varphi_1.$$

We will show that φ_1 is essentially a perturbation term. At this stage, we divide the estimates into several steps. We start with linear estimates for φ_0.

Step 1 : Estimates for φ_0. We claim that

$$\|\nabla\varphi_0\|^2_{L^q(B_1)} \le C_q\left[||e_\varepsilon(v_\varepsilon)||_{L^1(B_1)} + ||f_\varepsilon||^2_{L^2(B_1)}\right] \qquad \forall\, 2\le q<+\infty. \tag{A.17}$$

Proof. The estimate follows from the linear theory for the Laplace operator and (A.13) □

Step 2 : The equation for φ_1. The function φ_1 verifies the elliptic problem

$$\begin{cases} -\Delta\varphi_1 &= \mathrm{div}((\rho^2-1)\nabla\tilde{\varphi}) & \text{in } B_1, \\ \varphi_1 &= 0 & \text{on } \partial B_1. \end{cases} \tag{A.18}$$

It is convenient to rewrite equation (A.18) as

$$-\Delta\varphi_1 = \mathrm{div}((\rho^2-1)\nabla\varphi_1) + \mathrm{div}\,(g_0), \tag{A.19}$$

where we have set $g_0 = (\rho^2-1)\nabla\varphi_0$. Since $|\rho_\varepsilon| \le C$ on B_1, we obtain, for any $q \ge 2$ the estimate for g_0

$$\|g_0\|_{L^q(B_1)} \le C_q||\nabla\varphi_0||_{L^q(B_1)}. \tag{A.20}$$

We now estimate φ_1 from (A.19) through a fixed point argument.

Step 3 : The fixed point argument. Equation (A.19) may be rewritten as

$$\varphi_1 = \mathcal{T}(\mathrm{div}((\rho^2-1)\nabla\varphi_1)) + \mathcal{T}(\mathrm{div}\, g_0),$$

which is of the form

$$(\mathrm{Id} - A)\varphi_1 = b$$

where $\mathcal{T} = \Delta^{-1}$, A is the linear operator $v \mapsto \mathcal{T}(\mathrm{div}((\rho^2 - 1)\nabla v))$ and $b = \mathcal{T}(\mathrm{div}\, g_0)$. Consider the Banach space $X_q = W_0^{1,q}(B_1)$. It follows from the linear theory for $\mathcal{T}$ that $A : X_q \to X_q$ is linear continuous and that

$$\|A\|_{\mathcal{L}(X_q)} \leq C(q)\|1 - \rho\|_{L^\infty(B_1)}.$$

In particular, we may choose the constant $\sigma_0 > 0$ such that

$$C_q\|1 - \rho\|_{L^\infty(B(1))} \leq C_q\sigma_0 < \frac{1}{2}.$$

With this choice of σ_0, we deduce that $I - A$ is invertible on X_q and

$$\|\varphi_1\|_{X_q} \leq C_q\|b\|_{X_q}. \tag{A.21}$$

Finally, by (A.20) we obtain

$$\|b\|_{X_q} = \|\mathcal{T}(\mathrm{div}\, g_0)\|_{X_q} \leq C_q\|g_0\|_{L^q(B_1)} \leq C_q||\nabla\varphi_0||_{L^q(B_1)}.$$

Going back to (A.21) we deduce

$$\|\nabla\varphi_1\|_{L^q(B_1)} \leq C_q||\nabla\varphi_0||_{L^q(B_1)}. \tag{A.22}$$

We now combine the estimates for φ_0 and φ_1.

Step 4 : Improved integrability of $\nabla\tilde{\varphi}$. Combining (A.17) and (A.22), we obtain

$$\|\nabla\tilde{\varphi}\|_{L^q(B_1)} \leq C_q(\|e_\varepsilon(v_\varepsilon)\|^{\frac{1}{2}}_{L^1(B_1)} + ||f_\varepsilon||_{L^2(B_1)}), \qquad \forall\ q \geq 2. \tag{A.23}$$

Step 5 : Estimates for the modulus and potential terms.

Recall that the function ρ satisfies the equation

$$-\Delta\rho + \rho|\nabla\varphi|^2 = \rho\frac{(1 - \rho^2)}{\varepsilon^2} + f_\varepsilon\, v_\varepsilon. \tag{A.24}$$

Let $\xi \in \mathcal{C}_c^\infty(B_{R'})$, $R' = (1 + R)/2$, be such that $\xi \equiv 1$ on B_R, so that $\varphi = \tilde{\varphi}$ on $\mathrm{supp}\xi$. Multiplying (A.24) by $(1-\rho^2)\xi$ and integrating by parts we obtain

$$\int_{B_1} 2\rho|\nabla\rho|^2\xi + \rho\frac{(1 - \rho^2)^2}{\varepsilon^2}\xi = \int_{\mathrm{supp}\xi} \nabla\rho\cdot\nabla\xi(1-\rho^2) + \rho(1-\rho^2)|\nabla\tilde{\varphi}|^2\xi - f_\varepsilon\, v_\varepsilon(1-\rho^2)\xi,$$

and since $|1-\rho| \leq \frac{1}{2}$ on B_1 we obtain

$$\int_{B_R} |\nabla|v_\varepsilon||^2 + V_\varepsilon(v_\varepsilon) \leq C\varepsilon \left(\int_{B_1} V_\varepsilon(v_\varepsilon)\right)^{\frac{1}{2}} \left(\int_{B_1} |\nabla|v_\varepsilon||^2 + |\nabla\tilde{\varphi}|^4 + |f_\varepsilon|^2\right)^{\frac{1}{2}}.$$

The conclusion follows using (A.23).

Proof of Theorem A.1 completed.

Step1: Global estimates for the potential. Let $\sigma_0 \leq 1/2$ be given by Proposition A.2 and consider the points $a_1^\varepsilon, .., a_\ell^\varepsilon$ provided by Proposition A.1 for this choice of σ_0. Consider a point $x_0 \in B_R$ such that

$$\mathrm{dist}(x_0, \{a_1^\varepsilon, ..., a_\ell^\varepsilon\}) \geq 10\varepsilon^\alpha,$$

where $\alpha = (1+\beta)/2$. Consider the rescaled function $\tilde{v}_\varepsilon(x) = v_\varepsilon(\varepsilon^\alpha x + x_0)$ which verifies

$$-\Delta\tilde{v}_\varepsilon = \frac{1}{\tilde{\varepsilon}^2}\tilde{v}_\varepsilon(1-|\tilde{v}_\varepsilon|^2) + \tilde{f}_{\tilde{\varepsilon}} \qquad \text{on } B_1,$$

where $\tilde{f}_{\tilde{\varepsilon}}(x) = \varepsilon^{2\alpha} f_\varepsilon(\varepsilon^\alpha x + x_0)$, and $\tilde{\varepsilon} = \varepsilon^{1-\alpha}$. In particular,

$$||\tilde{f}_\varepsilon||^2_{L^2(B_1)} = \varepsilon^{2\alpha}||f_\varepsilon||^2_{L^2(B(x_0,\varepsilon^\alpha))}.$$

Applying Proposition A.2 we are led to

$$\int_{B_R} |\nabla|\tilde{v}_\varepsilon||^2 + V_{\tilde{\varepsilon}}(\tilde{v}_\varepsilon) \leq C(R)\tilde{\varepsilon}\|e_{\tilde{\varepsilon}}(\tilde{v}_\varepsilon)\|^{\frac{1}{2}}_{L^1(B(1))}(\|e_{\tilde{\varepsilon}}(\tilde{v}_\varepsilon)\|_{L^1(B(1))} + ||\tilde{f}_\varepsilon||^2_{L^2}),$$

so that going back to v_ε we obtain,

$$\int_{B(x_0,R\varepsilon^\alpha)} |\nabla|v_\varepsilon||^2 + V_\varepsilon(v_\varepsilon) \leq C(R)\varepsilon^{\frac{1-\alpha}{2}}(\|e_\varepsilon(v_\varepsilon)\|_{L^1(B(x_0,\varepsilon^\alpha))} + \varepsilon^{2\alpha}||f_\varepsilon||^2_{L^2(B(x_0,\varepsilon^\alpha))}.$$

By a standard covering argument we obtain

$$\int_{B_R\setminus\cup_{i=1}^\ell B(a_i^\varepsilon,10\varepsilon^\alpha)} |\nabla|v_\varepsilon||^2 + V_\varepsilon(v_\varepsilon) \leq C(R)\varepsilon^{(1-\alpha)}(\|e_\varepsilon(v_\varepsilon)\|_{L^1(B_1)} + \varepsilon^{2\alpha}||f_\varepsilon||^2_{L^2(B(1))}). \tag{A.25}$$

Adding (A.25) and (A.11) for a suitable ε^α-covering of $\cup_{i=1}^\ell B(a_i, 10\varepsilon^\alpha)$, we are led to

$$\int_{B_R} V_\varepsilon(v_\varepsilon) \leq C(M_0, \beta, R).$$

Step 2: Global estimates for $\nabla|v_\varepsilon|$. The starting point is the elliptic equation for the function $\theta \equiv 1-\rho_\varepsilon^2 \equiv 1-|v_\varepsilon|^2$, namely

$$-\Delta\theta + 2\frac{\rho_\varepsilon^2}{\varepsilon^2}\theta = 2|\nabla v_\varepsilon|^2 + 2f_\varepsilon \cdot v_\varepsilon \qquad \text{on } B_1. \tag{A.26}$$

We proceed as in Step 5 of the proof of Proposition A.2 and multiply (A.26) by $\theta\,\xi$. By the same computation we obtain

$$\int_{B_R} |\nabla|v_\varepsilon|\,|^2 \leq C\int_{B_{R'}} |\theta|(|\nabla v_\varepsilon|^2 + |f_\varepsilon|) + \theta^2. \tag{A.27}$$

In view of Step 1, we deduce from (A.27) that

$$\int_{B_R} |\nabla|v_\varepsilon|\,|^2 \leq C\varepsilon\left(\varepsilon + ||\nabla v_\varepsilon||^2_{L^4(B_{R'})} + ||f_\varepsilon||_{L^2(B_{R'})}\right).$$

The conclusion follows from the next lemma. □

Lemma A.1. *Under the assumptions in Theorem A.1 we have*

$$\int_{B_{R'}} |\nabla v_\varepsilon|^4 \leq C\left(1 + ||f_\varepsilon||^2_{L^2(B_{R''})} + \frac{1}{\varepsilon^2}\int_{B_{R''}} V_\varepsilon(v_\varepsilon)\right), \tag{A.28}$$

where $C>0$ is a constant and $R'' = (1+R')/2$.

Proof. On the ball $B_{R''}$ we decompose the function v_ε as $v_\varepsilon \equiv v_\varepsilon^1 + w_\varepsilon$, where w_ε is a harmonic function and verifies $w_\varepsilon = v_\varepsilon$ on the boundary $\partial B_{R''}$. In view of the bound on $|v_\varepsilon|$, w_ε is uniformly bounded on $\partial B_{R''}$, so that, since it is also harmonic,

$$|\nabla w_\varepsilon| \leq C \qquad \text{on } B_{R'}. \tag{A.29}$$

Turning to v_ε^1, we notice that by construction $v_\varepsilon^1 = 0$ on $\partial B_{R''}$, and that

$$|\Delta v_\varepsilon^1| = \left|\frac{2}{\varepsilon}V_\varepsilon(v_\varepsilon)^{1/2} + f_\varepsilon\right|.$$

Hence by standard elliptic theory

$$||v_\varepsilon^1||^2_{H^2(B_{R''})} \leq C(||f_\varepsilon||^2_{L^2(B_{R''})} + \varepsilon^{-2}\int_{B_{R''}} V_\varepsilon(v_\varepsilon)). \tag{A.30}$$

Moreover, since v_ε and w_ε are uniformly bounded on $B_{R''}$, the same holds for v_ε^1, i.e.

$$||v_\varepsilon^1||_{L^\infty(B_{R''})} \leq C. \tag{A.31}$$

We next invoke a classical Gagliardo-Nirenberg type inequality which asserts that

$$||\nabla v_\varepsilon^1||^2_{L^4(B_{R''})} \leq C||v_\varepsilon^1||_{H^2(B_{R''})}||v_\varepsilon^1||_{L^\infty(B_{R''})}. \tag{A.32}$$

Combining (A.30), (A.31) and (A.32), we obtain

$$||\nabla v_\varepsilon^1||^4_{L^4(B_{R''})} \leq C\left(||f_\varepsilon||^2_{L^2(B_{R''})} + \frac{1}{\varepsilon^2}\int_{B_{R''}} V_\varepsilon(u_\varepsilon)\right). \tag{A.33}$$

Invoking finally (A.29) together with the decomposition $v_\varepsilon = v_\varepsilon^1 + w_\varepsilon$ we complete the proof. □

As a straightforward consequence of the clearing-out property (Lemma B.4 in [3]) and proposition A.2, we may also consider the case $\Omega = \mathbb{R}^2$ and deduce, by a covering argument

Proposition A.3. *Assume v_ε satisfies* (A.1) *and* (A.2) *on $\Omega \equiv \mathbb{R}^2$ and f_ε satisfies the bound*

$$||f_\varepsilon||_{L^2(\mathbb{R}^2)} \leq \varepsilon^{-\beta}$$

for some $0 < \beta < 1$. Then

$$\int_{\mathbb{R}^2} |\nabla|u_\varepsilon||^2 + \frac{(1-|v_\varepsilon|^2)^2)}{4\varepsilon^2} \leq C(M_0, \beta).$$

The rest of this Appendix is devoted to the proof of Theorem A.2

A.2 Confinement of the vorticity set

Our first purpose is to cover the vorticity set by balls of radius r, proportional to ε, whose mutual distance is larger than $\kappa^{-1}r$, with κ arbitrarily small. We have

Lemma A.2. *Let X be a metric space, and consider ℓ distinct points $a_1, ..., a_\ell$ in X. Let $\delta_0 > 0$ and $0 < \kappa \leq \frac{1}{2}$ be given. Then there exists $\delta > 0$ such that*

$$\delta_0 \leq \delta \leq (\frac{\kappa}{2})^{-\ell}\delta_0 \tag{A.34}$$

and a subset $\{a_j\}_{j\in J}$ of $\{a_i\}_{1\leq i\leq \ell}$ such that

$$\cup_{i=1}^{\ell} B(a_i, \delta_0) \subset \cup_{j\in J} B(a_i, \delta) \tag{A.35}$$

and

$$\text{dist}(a_i, a_j) \geq \kappa^{-1}\delta \qquad \forall j \neq k \text{ in } J. \tag{A.36}$$

Proof. The proof is by iteration, in at most ℓ steps. First, consider the collection $\{a_i\}_{1\leq i\leq \ell}$. If (A.35), (A.36) is verified with $\delta=\delta_0$ there is nothing else to do. Otherwise, take two points, say a_1, a_2 such that $\mathrm{dist}(a_1,a_2)\leq \kappa^{-1}\delta_0$, consider the collection $a_2, a_3, ..., a_l$, and set $\delta = 2\kappa^{-1}\delta_0$. If (A.35) is verified, we stop. Otherwise we go on in the same way. If the process does not stop in $\ell-1$ steps, at the ℓ^{th} step we are left with one single ball of radius $\delta = (\frac{\kappa}{2})^{-\ell}\delta_0$, and (A.35) is void. □

Combining Lemma A.2 and Proposition A.1 we are led to

Lemma A.3. *Assume* v_ε *satisfies* (A.1), (A.2) *and* (A.3). *Then, for* $0<\kappa\leq\frac{1}{2}$ *there exists* $\lambda_\kappa>0$ *such that*

$$\lambda\leq\lambda_\kappa\leq(\frac{\kappa}{2})^{-\ell}\lambda$$

and points $\{a_j^\varepsilon\}_{j\in J_{\varepsilon,\kappa}}\subset\{a_i^\varepsilon\}_{1\leq i\leq\ell}$ *such that*

$$|1-|v_\varepsilon(x)||\leq\sigma_0 \qquad \text{on } B_1\setminus\cup_{j\in J_{\varepsilon,\kappa}}B(a_j^\varepsilon,\lambda_\kappa\varepsilon)$$

and

$$|a_j^\varepsilon-a_k^\varepsilon|\geq\kappa^{-1}\lambda_\kappa\varepsilon \qquad \text{for } j\neq k \text{ in } J_{\varepsilon,\kappa}. \tag{A.37}$$

Here, σ_0 *is given by Proposition A.2, and the constants* ℓ,λ *and the points* $a_1^\varepsilon,...,a_l^\varepsilon\in B_{\frac{1}{2}}$ *are obtained by Proposition A.1 with* $R=1/2$.

A.3 The harmonic potential Ψ_ε

In this section we choose $\kappa=\frac{1}{2}$, and, for $i\in J_{\varepsilon,1/2}$, we set $d_i^\varepsilon=\deg\left(\frac{v_\varepsilon}{|v_\varepsilon|},\partial B(a_i^\varepsilon,\lambda_{\frac{1}{2}}\varepsilon)\right)$. Let

$$\Psi_\varepsilon(x)=-\sum_{i\in J_{\varepsilon,1/2}}d_i^\varepsilon\log|x-a_i^\varepsilon|, \tag{A.38}$$

so that

$$-\Delta\Psi_\varepsilon=2\pi\sum_{i\in J_{\varepsilon,1/2}}d_i^\varepsilon\delta_{a_i^\varepsilon}, \qquad \text{on } \mathbb{R}^2 \tag{A.39}$$

and therefore

$$\frac{\partial}{\partial x}\left(\frac{v_\varepsilon}{|v_\varepsilon|^2}\times\frac{\partial v_\varepsilon}{\partial y}+\frac{\partial\Psi_\varepsilon}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{v_\varepsilon}{|v_\varepsilon|^2}\times\frac{\partial v_\varepsilon}{\partial x}-\frac{\partial\Psi_\varepsilon}{\partial y}\right)=0 \tag{A.40}$$

on $\Omega_\varepsilon(1/2)\equiv B_1\setminus\cup_{j\in J_{\varepsilon,1/2}}B(a_j^\varepsilon,\lambda_{\frac{1}{2}}\varepsilon)$. Since by (A.38) the circulation around each circle $\partial B(a_j^\varepsilon,\lambda_{\frac{1}{2}}\varepsilon)$ is zero, it follows from (A.40) that there exists some real-valued function Φ_ε, defined on $\Omega_\varepsilon(1/2)$ such that

$$\frac{v_\varepsilon}{|v_\varepsilon|^2}\times\nabla v_\varepsilon=\nabla^\perp\Psi_\varepsilon+\nabla\Phi_\varepsilon. \tag{A.41}$$

Since

$$\operatorname{div}(v_\varepsilon \times \nabla v_\varepsilon) = f_\varepsilon \times v_\varepsilon \qquad \text{on } B_1, \tag{A.42}$$

we are led to the elliptic equation for Φ_ε

$$\operatorname{div}(|v_\varepsilon|^2 \nabla \Phi_\varepsilon) = f_\varepsilon \times v_\varepsilon + \operatorname{div}\Big((1 - |v_\varepsilon|^2)\nabla^\perp \Psi_\varepsilon\Big) \quad \text{on } \Omega_\varepsilon(1/2). \tag{A.43}$$

Since Φ_ε is defined up to an additive constant, we may moreover impose

$$\int_{\Omega_\varepsilon(1/2)} \Phi_\varepsilon = 0. \tag{A.44}$$

We will show that $e_\varepsilon(v_\varepsilon)$ is close in some suitable sense to $|\nabla \Psi_\varepsilon|^2$, more precisely, we have

Proposition A.4. *Assume v_ε satisfies* (A.1), (A.2), (A.5) *and* (A.6). *Then there exists a constant $C(M_0)$, depending only on M_0, such that*

$$\left| E_\varepsilon(v_\varepsilon, B_1) - \frac{1}{2} \int_{\Omega_\varepsilon(1/2)} |\nabla \Psi_\varepsilon|^2 \right| \leq C(M_0)(1 + \|f_\varepsilon\|^2_{L^2(\Omega)}).$$

In order to establish Proposition A.4, we need to show that the contribution to the energy of $\nabla \Phi_\varepsilon$, $\nabla |v_\varepsilon|$ and the potential remain controlled. For that purpose, we consider the external domains

$$\Omega^\mu_\varepsilon(\kappa) = B_1 \setminus \cup_{i \in J_{\varepsilon,\kappa}} B(a_i^\varepsilon, \lambda_\kappa \kappa^{-\mu} \varepsilon),$$

for $0 \leq \mu < 1$. Clearly, $\Omega^\mu_\varepsilon(\kappa) \subset \Omega^{\mu'}_\varepsilon(\kappa)$ if $\mu \geq \mu'$. Moreover, it is possible to perform the above construction so that the function $\kappa \to \lambda_\kappa$ is non-increasing. We have

Proposition A.5. *The function $\nabla \Phi_\varepsilon$ decomposes as*

$$\nabla \Phi_\varepsilon = g^1_\varepsilon + g^2_\varepsilon \qquad \text{on } \Omega^{1/2}_\varepsilon(\kappa),$$

where

$$||g^1_\varepsilon||_{L^4(\Omega^{1/2}_\varepsilon(\kappa))} \leq C(M_0)(1 + \|f_\varepsilon\|_{L^2(\Omega)})$$

and

$$||g^2_\varepsilon||_{L^2(\Omega^{1/2}_\varepsilon(\kappa))} \leq C(M_0)(|\log \kappa|^{-1} + \varepsilon^{\frac{1}{2}})(1 + \|f_\varepsilon\|_{L^2(\Omega)}).$$

In the same spirit,

Proposition A.6. *We have*

$$\int_{\Omega^{3/4}_\varepsilon(\kappa)} |\nabla \rho|^2 + \frac{(1-\rho^2)^2}{\varepsilon^2} \leq C(M_0)(|\log \kappa|^{-2} + \varepsilon)(1 + \|f_\varepsilon\|^2_{L^2(\Omega)}).$$

A.4 Estimates far from the vorticity set

In this section we present the proofs of Propositions A.4, A.5 and A.6. We begin with

Lemma A.4. *There exists a constant $C(M_0)$ such that*

$$\int_{\Omega_\varepsilon(1/2)} |\nabla\Phi_\varepsilon|^2 \leq C(M_0)(1 + \|f_\varepsilon\|^2_{L^2(\Omega)}). \tag{A.45}$$

Proof. We multiply first equation (A.43) by Φ_ε and integrate by parts on $\Omega_\varepsilon(1/2)$. This yields

$$\begin{aligned}\int_{\Omega_\varepsilon(1/2)} \rho^2|\nabla\Phi_\varepsilon|^2 = &- \int_{\Omega_\varepsilon(1/2)} f_\varepsilon \times v_\varepsilon \Phi_\varepsilon + \int_{\Omega_\varepsilon(1/2)} (1-\rho^2)\nabla^\perp\Psi_\varepsilon \nabla\Phi_\varepsilon \\ &- \sum_{i\in J_{\varepsilon,1/2}\cup\{0\}} \left[\int_{\partial B^i} (1-\rho^2)\nabla^\perp\Psi_\varepsilon \cdot \vec{n}\Phi_\varepsilon - \int_{\partial B^i} \rho^2 \frac{\partial\Phi_\varepsilon}{\partial n}\Phi_\varepsilon\right],\end{aligned} \tag{A.46}$$

where $B^i = B(a_i^\varepsilon, \lambda_{\frac{1}{2}}\varepsilon)$ for $i \in J_{\varepsilon,1/2}$ and $B^0 = B_1$. On the other hand, integrating equation (A.42) on B^i gives, using (A.41)

$$\int_{\partial B^i} \rho^2 \frac{\partial\Phi_\varepsilon}{\partial n} - (1-\rho^2)\nabla^\perp\Psi_\varepsilon \cdot \vec{n} = -\int_{B^i} f_\varepsilon \times v_\varepsilon$$

so that

$$\int_{\Omega_\varepsilon} \rho^2|\nabla\Phi_\varepsilon|^2 = A_1 + A_2 + A_3 + A_4$$

where

$$\begin{aligned} A_1 &= \int_{\Omega_\varepsilon(1/2)} f_\varepsilon \times v_\varepsilon \Phi_\varepsilon \\ A_2 &= \int_{\Omega_\varepsilon(1/2)} (1-\rho^2)\nabla^\perp\Psi_\varepsilon \nabla\Phi_\varepsilon \\ A_3 &= \sum_{i\in J_{\varepsilon,1/2}\cup\{0\}} \int_{\partial B^i} \left[(1-\rho^2)\nabla^\perp\Psi_\varepsilon \cdot \vec{n} - \rho^2\frac{\partial\Phi_\varepsilon}{\partial n}\right](\Phi_\varepsilon - \Phi_i) \\ A_4 &= \sum_{i\in J_{\varepsilon,1/2}\cup\{0\}} \int_{B^i} f_\varepsilon \times v_\varepsilon \Phi_i \end{aligned}$$

and where $\Phi_i = \frac{1}{|\partial B^i|}\int_{\partial B_i} \Phi_\varepsilon$. We estimate each of the terms separately. Firstly, we have

$$|A_1| \leq ||f_\varepsilon||_{L^2(B_1)}||\Phi_\varepsilon||_{L^2(\Omega_\varepsilon(1/2))},$$

so that

$$|A_1| \leq 10\|f_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{10}\int_{\Omega_\varepsilon(1/2)} |\nabla\Phi_\varepsilon|^2. \tag{A.47}$$

For A_2, we write, since $|\nabla\Psi_\varepsilon| \leq \frac{C}{\varepsilon}$ on $\Omega_\varepsilon(1/2)$,

$$|A_2| \leq ||\nabla\Phi_\varepsilon||_{L^2(\Omega_\varepsilon(1/2))}||\frac{1-\rho^2}{\varepsilon}||_{L^2(\Omega_\varepsilon(1/2))} \leq C(M_0)||\nabla\Phi_\varepsilon||_{L^2(\Omega_\varepsilon(1/2))}, \tag{A.48}$$

where we used Proposition A.1 for the last inequality.

For A_3, since $|\nabla^\perp\Psi_\varepsilon| + |\nabla\Psi_\varepsilon| \leq C/\varepsilon$ on ∂B_i for $i = 1, ..., n$, we have $|\Phi_\varepsilon - \Phi_i| \leq C$ on the same set. On the other hand, on ∂B^0 all these quantities are bounded. Therefore

$$|A_3| \leq C(M_0). \tag{A.49}$$

For A_4, we write

$$|A_4| \leq C\sum_{i\in J_{\varepsilon,1/2}} ||f_\varepsilon||_{L^2(B^i)}\varepsilon|\Phi_i| + C(M_0)||f_\varepsilon||_{L^2(B_1)}, \tag{A.50}$$

and since $|\nabla\Phi_\varepsilon| \leq \frac{C}{\varepsilon}$, it follows that $|\Phi_i| \leq \frac{C}{\varepsilon}$ on ∂B^i, and therefore

$$|A_4| \leq C(M_0)||f_\varepsilon||_{L^2(\Omega)}. \tag{A.51}$$

Combining (A.47), (A.48), (A.49) and (A.51), we derive the conclusion. □

Our next purpose is to provide higher integrability of the gradient of the phase, in order to prove Proposition A.5. To that aim, we consider three elliptic problems on B_1, firstly

$$\begin{cases} -\Delta\tilde{\varphi}^1_\varepsilon = 0 & \text{on } B_1 \\ \tilde{\varphi}^1_\varepsilon = \Phi_\varepsilon & \text{on } \partial B_1, \end{cases}$$

secondly,

$$\begin{cases} -\Delta\tilde{\varphi}^2_\varepsilon = f_\varepsilon \times v_\varepsilon & \text{on } B_1 \\ \tilde{\varphi}^2_\varepsilon = 0 & \text{on } \partial B_1, \end{cases}$$

and finally

$$\begin{cases} -\operatorname{div}(\tilde{\rho}^2\nabla\tilde{\varphi}^3_\varepsilon) = \operatorname{div}\Big((1-\tilde{\rho}^2)[\nabla^\perp\Psi_\varepsilon + \nabla\tilde{\varphi}^1_\varepsilon + \nabla\tilde{\varphi}^2_\varepsilon]\Big) & \text{on } B_1 \\ \tilde{\varphi}^3_\varepsilon = 0 & \text{on } \partial B_1. \end{cases}$$

Here the function $\tilde{\rho}$ is defined as

$$\begin{cases} \tilde{\rho} = \rho & \text{on } \Omega^{1/4}_\varepsilon(\kappa) \\ \tilde{\rho} = 1 & \text{on } \cup_{i\in J_{\varepsilon,\kappa}} B(a^\varepsilon_i, \lambda_\kappa\kappa^{-1/4}\varepsilon). \end{cases}$$

We set

$$\tilde\Phi_\varepsilon = \tilde\varphi^1_\varepsilon + \tilde\varphi^2_\varepsilon + \tilde\varphi^3_\varepsilon \qquad \text{on } B_1,$$

so that

$$\operatorname{div}(\tilde\rho^2\nabla\tilde\Phi_\varepsilon) = f_\varepsilon \times v_\varepsilon + \operatorname{div}((1-\tilde\rho^2)\nabla^\perp\Psi_\varepsilon) \qquad \text{on } B_1 \tag{A.52}$$

and hence

$$\begin{cases} \operatorname{div}(\tilde\rho^2\nabla(\tilde\Phi_\varepsilon - \Phi_\varepsilon)) = 0 & \text{on } \Omega_\varepsilon^{1/4}(\kappa) \\ \tilde\Phi_\varepsilon - \Phi_\varepsilon = 0 & \text{on } \partial B_1. \end{cases} \tag{A.53}$$

We estimate each of the terms $\tilde\varphi^i_\varepsilon$ separately on B_1, and then the difference $\tilde\Phi_\varepsilon - \Phi_\varepsilon$ on $\Omega_\varepsilon^{1/2}(\kappa)$.

Lemma A.5.

$$\int_{B_1}\left(|\nabla\tilde\varphi^1_\varepsilon|^4 + |\nabla\tilde\varphi^2_\varepsilon|^4\right) \le C(M_0)(1 + \|f_\varepsilon\|^4_{L^2(\Omega)}).$$

Proof. For $\tilde\varphi^1_\varepsilon$ by standard estimates for the Laplacian we have

$$||\nabla\tilde\varphi^1_\varepsilon||_{L^4(B_1)} \le C||\Phi_\varepsilon||_{\dot W^{3/4,4}(B_1)} \le C||\nabla\Phi_\varepsilon||_{L^2(\partial B_1)} \le C(M_0). \tag{A.54}$$

Similarly for $\tilde\varphi^2_\varepsilon$ we have

$$||\nabla\tilde\varphi^2_\varepsilon||_{L^4(B_1)} \le C||\tilde\varphi^2_\varepsilon||_{H^2(B_1)} \le C||f_\varepsilon||_{L^2(B_1)}.$$

□

Lemma A.6. *We have*

$$||\nabla\tilde\varphi^3_\varepsilon||_{L^2(B_1)} \le C(M_0)(\kappa^{1/8} + \varepsilon^{1/2})(1 + \|f_\varepsilon\|_{L^2(\Omega)}).$$

Proof. It suffices to estimate the quantity $(1-\tilde\rho^2)(\nabla^\perp\Psi_\varepsilon + \nabla\tilde\varphi^1_\varepsilon + \nabla\tilde\varphi^2_{\varepsilon,\kappa})$ in $L^2(B_1)$. Since $1-\tilde\rho^2 = 0$ on $\cup_{i\in J_{\varepsilon,\kappa}} B(a_i^\varepsilon, 2\lambda_\kappa\kappa^{-1/4}\varepsilon)$ we have

$$\begin{aligned} &||(1-\tilde\rho^2)(\nabla^\perp\Psi_\varepsilon + \nabla\tilde\varphi^1_\varepsilon + \nabla\tilde\varphi^2_\varepsilon)||_{L^2(B_1)} \\ \le& C||1-\tilde\rho^2||_{L^4(B_1)}\left[||\nabla\Psi_\varepsilon||_{L^4(\Omega_\varepsilon^{1/4}(1/4))} + ||\nabla\tilde\varphi^1_\varepsilon||_{L^4(B_1)} + ||\nabla\tilde\varphi^2_\varepsilon||_{L^4(B_1)}\right] \\ \le& C(M_0)\varepsilon^{1/2}\left[||\nabla\Psi_\varepsilon||_{L^4(\Omega_\varepsilon^{1/4}(\kappa))} + C(M_0)\right]. \end{aligned} \tag{A.55}$$

On the other hand,

$$\int_{\Omega_\varepsilon^{1/4}(\kappa)} |\nabla\Psi_\varepsilon|^4 \le C\int_{\lambda_\kappa\kappa^{1/4}\varepsilon}^1 \frac{1}{r^4} r\,dr \le C(\lambda_\kappa^{-2}\kappa^{1/2}\varepsilon^{-2} + 1). \tag{A.56}$$

The conclusion follows. □

Summarizing Lemma A.5 and A.6 we obtain

$$\nabla\tilde{\Phi}_\varepsilon = \tilde{g}^1_\varepsilon + \tilde{g}^2_\varepsilon \qquad \text{on } B_1, \tag{A.57}$$

where

$$||\tilde{g}^1_\varepsilon||_{L^4(B_1)} \le C(M_0)(1 + \|f_\varepsilon\|_{L^2(\Omega)}) \tag{A.58}$$

and

$$||\tilde{g}^2_\varepsilon||_{L^2(B_1)} \le C(M_0)(\kappa^{1/8} + \varepsilon^{1/2})(1 + \|f_\varepsilon\|_{L^2(\Omega)}). \tag{A.59}$$

Lemma A.7. *We have*

$$\int_{\Omega_\varepsilon^{1/2}(\kappa)} |\nabla(\tilde{\Phi}_\varepsilon - \Phi_\varepsilon)|^2 \le \frac{C(M_0)}{|\log\kappa|}(1 + \|f_\varepsilon\|^2_{L^2(\Omega)}).$$

Proof. At this stage, we already know, in view of Lemma A.4, (A.57), (A.58) and (A.59), that

$$\int_{\Omega_\varepsilon^{1/4}(\kappa)} |\nabla(\tilde{\Phi}_\varepsilon - \Phi_\varepsilon)|^2 \le C(M_0)(1 + \|f_\varepsilon\|^2_{L^2(\Omega)}), \tag{A.60}$$

and that $\tilde{\Phi}_\varepsilon - \Phi_\varepsilon$ is solution of the (outer) boundary value problem (A.53). In order to add an inner boundary condition, we first claim that for every $\lambda_\kappa\kappa^{-1/2}\varepsilon \ge r > \lambda_\kappa\kappa^{-1/4}\varepsilon$, we have

$$\int_{\partial B(a_i^\varepsilon, r)} \rho^2 \frac{\partial(\tilde{\Phi}_\varepsilon - \Phi_\varepsilon)}{\partial n} = 0 \qquad \text{for any } i \in J_{\varepsilon,\kappa}. \tag{A.61}$$

Indeed, (A.52) yields

$$\int_{\partial B(a_i^\varepsilon, r)} \rho^2 \frac{\partial\tilde{\Phi}_\varepsilon}{\partial n} = \int_{B(a_i^\varepsilon, r)} f_\varepsilon \times v_\varepsilon + \int_{\partial B(a_i^\varepsilon, r)} (1-\rho^2)\partial_\tau\Psi_\varepsilon,$$

whereas (A.43) yields the same estimate for Φ_ε. Next, by an averaging argument (in logarithmic scale) we may choose some r_κ in $(\lambda_\kappa\kappa^{-1/4}, \lambda_\kappa\kappa^{-1/2})$ such that

$$r_\kappa \int_{\partial B(a_i^\varepsilon, r_\kappa)} |\nabla(\tilde{\Phi}_\varepsilon - \Phi_\varepsilon)|^2 \le \frac{C(M_0)}{|\log\kappa|}(1 + \|f_\varepsilon\|^2_{L^2(\Omega)}). \tag{A.62}$$

Going back to (A.53) we deduce, multiplying (A.53) by $(\tilde{\Phi}_\varepsilon - \Phi_\varepsilon)$ and integrating by parts,

$$\int_{B_1 \setminus \cup_{i\in J_{\varepsilon,\kappa}} B(a_i^\varepsilon, r_\kappa)} \rho^2 |\nabla(\tilde{\Phi}_\varepsilon - \Phi_\varepsilon)|^2 = \sum_{i\in J_{\varepsilon,\kappa}} \int_{\partial B(a_i^\varepsilon, r_\kappa)} \rho^2 \frac{\partial(\tilde{\Phi}_\varepsilon - \Phi_\varepsilon)}{\partial n}(\tilde{\Phi}_\varepsilon - \Phi_\varepsilon).$$

In view of (A.61) and Poincaré inequality we deduce (here m_i denotes the mean-value of $\tilde{\Phi}_{\varepsilon,\kappa} - \Phi_{\varepsilon,\kappa}$ on $\partial B(a_i^\varepsilon, r_\kappa)$)

$$\begin{aligned}
&\left| \int_{\partial B(a_i^\varepsilon, r_\kappa)} \rho^2 \frac{\partial(\tilde{\Phi}_\varepsilon - \Phi_\varepsilon)}{\partial n} (\tilde{\Phi}_\varepsilon - \Phi_\varepsilon) \right| \\
\leq & ||\nabla(\tilde{\Phi}_\varepsilon - \Phi_\varepsilon)||_{L^2(\partial B(a_i^\varepsilon, r_\kappa))} ||\tilde{\Phi}_\varepsilon - \Phi_\varepsilon - m_i||_{L^2(\partial B(a_i^\varepsilon, r_\kappa))} \\
\leq & r_\kappa ||\nabla(\tilde{\Phi}_\varepsilon - \Phi_\varepsilon)||^2_{L^2(\partial B(a_i^\varepsilon, r_\kappa))}.
\end{aligned} \tag{A.63}$$

The conclusion follows from (A.62). □

Proof of Proposition A.5. It follows combining Lemma A.7, (A.57), (A.58) and (A.59). □

We now turn to the modulus $\rho = |v_\varepsilon|$. It satisfies the equation

$$-\Delta\rho + \rho^{-3}|v_\varepsilon \times \nabla v_\varepsilon|^2 = \frac{1}{\varepsilon^2}\rho(1-\rho^2) + f_\varepsilon \cdot \frac{v_\varepsilon}{|v_\varepsilon|},$$

so that in view of (A.41) we have

$$-\Delta\rho + \rho(\nabla^\perp\Psi_\varepsilon + \nabla\Phi_\varepsilon)^2 = \frac{1}{\varepsilon^2}\rho(1-\rho^2) + f_\varepsilon \cdot \frac{v_\varepsilon}{|v_\varepsilon|} \qquad \text{on } \Omega_\varepsilon^{1/2}(\kappa). \tag{A.64}$$

Proof of Proposition A.6. For $r \in (\lambda_\kappa \kappa^{-1/2}\varepsilon, \lambda_\kappa \kappa^{-3/4}\varepsilon)$ we have, multiplying (A.64) by $(1-\rho)$, integrating by parts, and using the fact that $|\rho| \geq 1/2$ on $\Omega_\varepsilon^{1/2}(\kappa)$,

$$\begin{aligned}
\int_{B_1 \setminus \cup_{i\in J_{\varepsilon,\kappa}} B(a_i^\varepsilon, r)} |\nabla\rho|^2 + \frac{(1-\rho^2)^2}{\varepsilon^2} \leq & C \int_{B_1 \setminus \cup_{i\in J_{\varepsilon,\kappa}} B(a_i^\varepsilon, r)} (1-\rho^2)(|\nabla\Phi_\varepsilon|^2 + |\nabla\Psi_\varepsilon|^2) \\
& + C \int_{B_1 \setminus \cup_{i\in J_{\varepsilon,\kappa}} B(a_i^\varepsilon, r)} |f_\varepsilon| \cdot |1-\rho| \\
& + \sum_{i\in J_{\varepsilon,\kappa}} \int_{\partial B(a_i^\varepsilon, r)} \left|\frac{\partial\rho}{\partial n}\right| \cdot |1-\rho|.
\end{aligned} \tag{A.65}$$

By an averaging argument in logarithmic scale, we choose some radius $r = r_\kappa \in (\lambda_\kappa \kappa^{-1/2}\varepsilon, \lambda_\kappa \kappa^{-3/4}\varepsilon)$ so that

$$\begin{aligned}
r \int_{\partial B(a_i^\varepsilon, r)} \left|\frac{\partial\rho}{\partial n}\right| \cdot |1-\rho| \leq & \frac{1}{|\log\kappa|} \int_{B_1} |\nabla\rho| \cdot |1-\rho| \\
\leq & \frac{1}{|\log\kappa|} \left(\int_{B_1} |\nabla\rho|^2\right)^{1/2} \left(\int_{B_1} |1-\rho|^2\right)^{1/2} \\
\leq & \frac{C(M_0)\varepsilon}{|\log\kappa|},
\end{aligned} \tag{A.66}$$

where we used Theorem A.1 for the last inequality. Since $r > \lambda_\kappa \kappa^{-1/2}\varepsilon$, it follows that

$$\int_{\partial B(a_i^\varepsilon, r)} \left|\frac{\partial \rho}{\partial n}\right| \cdot |1-\rho| \le \frac{C(M_0)}{\kappa^{1/2}|\log\kappa|}. \tag{A.67}$$

For the first term in (A.65), we write, decomposing $\nabla\Phi_\varepsilon = g_\varepsilon^1 + g_\varepsilon^2$ as in Proposition A.5,

$$\begin{aligned} &\int_{B_1\setminus\cup_{i\in J_{\varepsilon,\kappa}} B(a_i^\varepsilon,r)} (1-\rho^2)|\nabla\Phi_{\varepsilon,\kappa}|^2 \\ &\le C\int_{B_1\setminus\cup_{i\in J_{\varepsilon,\kappa}} B(a_i^\varepsilon,r)} (1-\rho^2)(|f_\varepsilon^1|^2 + |f_\varepsilon^2|^2) \\ &\le C\,||1-\rho^2||_{L^2(B_1)} \left(||f_\varepsilon^1||^2_{L^4(\Omega_\varepsilon^{1/2}(\kappa))} + ||f_\varepsilon^2||^2_{L^2(\Omega_\varepsilon^{1/2}(\kappa))}\right) \\ &\le C(M_0)(|\log\kappa|^{-2} + \varepsilon)(1 + \|f_\varepsilon\|^2_{L^2(\Omega)}). \end{aligned} \tag{A.68}$$

Finally, arguing as in (A.56),

$$\int_{B_1\setminus\cup_{i\in J_{\varepsilon,\kappa}}} (1-\rho^2)|\nabla\Psi_\varepsilon|^2 \le ||1-\rho^2||_{L^2(B_1)}||\nabla\Psi_\varepsilon||^2_{L^4(\Omega_\varepsilon^{1/2}(\kappa))} \le C(M_0)\lambda_\kappa^{-1}\kappa^{1/2},$$

and the conclusion follows. □

A.5 Estimates near the vorticity set

The purpose of this section is to show that the integral of the potential is (almost) quantized near the vortices.

Proposition A.7. *Let $a \in B_r$, $r > \lambda_{1/2}\varepsilon$, and $0 < \kappa < 1/2$ be given such that*

$$B(a, \kappa^{-1}r/2) \setminus B(a,r) \subset \Omega_\varepsilon(1/2). \tag{A.69}$$

Then, we have

$$\begin{aligned} &\left|\int_{B(a,\kappa^{-3/4}r)} V_\varepsilon(v_\varepsilon) - \frac{\pi}{2} d_\varepsilon^2(a,r)\right| \\ &\le C(M_0)((|\log\kappa|^{-1} + \varepsilon^{1/2})(1 + r^2\|f_\varepsilon\|^2_{L^2}) + \kappa^{-3/4} r||f_\varepsilon||_{L^2}|\log\varepsilon|^{1/2}), \end{aligned}$$

where

$$d_\varepsilon(a,r) = \deg(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(a,r)) = \sum_{i\in J_{\varepsilon,1/2},\ a_i^\varepsilon \in B(a,r)} d_i^\varepsilon.$$

The proof relies on Pohozaev's identity, which we recall next, and the decay estimates obtained in the previous section.

Lemma A.8. *We have, for any $r > 0$ and $a \in B_1$ such that $B(a, r) \subset B_1$,*

$$2\int_{B(a,r)} V_\varepsilon(v_\varepsilon) + \frac{r}{2}\int_{\partial B(a,r)} \left|\frac{\partial v_\varepsilon}{\partial r}\right|^2 d\tau = \frac{r}{2}\int_{\partial B(a,r)} \left|\frac{\partial v_\varepsilon}{\partial \tau}\right|^2 d\tau + \int_{B(a,r)} x\cdot\nabla v_\varepsilon\, f_\varepsilon + r\int_{\partial B(a,r)} 2V_\varepsilon(v_\varepsilon). \tag{A.70}$$

The proof is standard and follows by multiplication of the equation by $x \cdot \nabla v_\varepsilon$ and then integrating by parts.

Proof of Proposition A.7. We apply Lemma A.8 with some $r = r_0 \in (\kappa^{-3/4}r, 2\kappa^{-3/4}r)$ to be determined later. We handle next each of the terms on the r.h.s. separately: firstly,

$$\left|\int_{B(a,r_0)} x\cdot\nabla v_\varepsilon\, f_\varepsilon\right| \le r||\nabla v_\varepsilon||_{L^2(B_1)}||f_\varepsilon||_{L^2(B_1)} \le \kappa^{-3/4}|\log\varepsilon|^{1/2} r||f_\varepsilon||_{L^2(B_1)}. \tag{A.71}$$

For the boundary terms, denoting by $\partial \equiv \partial_e$ the directional derivative in an arbitrary direction e, $|e| = 1$ we first expand

$$|\partial v_\varepsilon|^2 = |v_\varepsilon|^2\left|\frac{v_\varepsilon}{|v_\varepsilon|^2}\times\partial v_\varepsilon\right|^2 + |\partial\rho|^2 = |v_\varepsilon|^2|\partial^\perp\Psi_\varepsilon + \partial\Phi_\varepsilon|^2 + |\partial\rho|^2 = |\partial^\perp\Psi_\varepsilon|^2 + R_e, \tag{A.72}$$

where the remainder term R_e is given by

$$R_e = (1 - |v_\varepsilon|^2)|\partial^\perp\Psi_\varepsilon|^2 + |v_\varepsilon|^2|\partial\Phi_\varepsilon|^2 + 2|v_\varepsilon|^2\partial^\perp\Psi_\varepsilon\cdot\partial\Phi_\varepsilon + |\partial\rho|^2,$$

so that $|R_e| \le C\cdot$

$$\left(|\nabla\rho|^2 + \frac{(1-|v_\varepsilon|^2)^2}{\varepsilon^2} + \varepsilon^2|\nabla\Psi_\varepsilon|^4 + |g_\varepsilon^1|^2 + |g_\varepsilon^2|^2 + |\nabla\Psi_\varepsilon|(|g_\varepsilon^1| + |g_\varepsilon^2|)|\right). \tag{A.73}$$

Invoking (A.56) (or similar computations), Proposition A.5 and A.6, we obtain, by various Hölder inequalities, for every e, $|e| = 1$,

$$\int_{B(a,2\kappa^{-3/4}r)\setminus B(a,\kappa^{-3/4}r)} |R_e| \le C(M_0)(|\log\kappa|^{-1} + \varepsilon^{1/2})(1 + r^2\|f_\varepsilon\|_{L^2}^2). \tag{A.74}$$

In view of (A.74), we are now in position to choose a value of radius $r_0 \in (\kappa^{-3/4}r, 2\kappa^{-3/4}r)$ such that for every e, $|e| = 1$,

$$r_0 \int_{\partial B(a,r_0)} |R_e| + \frac{(1-|v_\varepsilon|^2)^2}{\varepsilon^2} \leq C(M_0)(|\log\kappa|^{-1} + \varepsilon^{1/2})(1 + r^2\|f_\varepsilon\|^2_{L^2}). \quad \text{(A.75)}$$

In order to complete the estimates of the boundary terms in (A.70), it remains, in view of (A.72), to estimate $|\partial_e \Psi_\varepsilon|$ on $\partial B(a, r_0)$. Recall that Ψ_ε is explicitly given by formula (A.38). Moreover, in view of (A.69), for $x \in \partial B(a, r_0)$ and $i \in J_{\varepsilon,1/2}$,

$$\kappa^{-3/4}r \leq |x - a_i^\varepsilon| \leq 2\kappa^{-3/4}r \qquad \text{for } a_i^\varepsilon \in B(a,r)$$

whereas

$$|x - a_i^\varepsilon| \geq \frac{1}{2}\kappa^{-1}r \qquad \text{for } a_i^\varepsilon \notin B(a,r).$$

Therefore, explicit computations show that on $\partial B(a, r_0)$,

$$\begin{aligned} |\frac{\partial\Psi_\varepsilon}{\partial\tau}| + |\frac{\partial\Psi_\varepsilon}{\partial r} - \frac{d_\varepsilon(a,r)}{r_0}| &\leq \frac{C}{r_0} \sup_{a_i^\varepsilon \in B(a,r)} \frac{|a - a_i^\varepsilon|}{r_0} + \frac{C\kappa}{r} \\ &\leq \frac{C\kappa^{3/4}}{r_0} + \frac{C\kappa^{1/4}}{r_0} \leq \frac{C\kappa^{1/4}}{r_0}. \end{aligned}$$

Hence, on $\partial B(a, r)$ we are led to

$$\Big|\frac{\partial\Psi_\varepsilon}{\partial\tau}\Big|^2 + \Big||\frac{\partial\Psi_\varepsilon}{\partial r}|^2 - \frac{d_\varepsilon^2(a,r)}{r_0^2}| \leq C(M_0)\frac{\kappa^{1/4}}{r_0^2}. \quad \text{(A.76)}$$

Combining (A.70), (A.71) with $e = \tau$ and $e = e_r$, (A.72), (A.73), (A.74), (A.75) and (A.76), we deduce

$$\begin{aligned} &\left|\int_{B(a,r_0)} V_\varepsilon(v_\varepsilon) - \frac{\pi}{2} d_\varepsilon(a,r)^2\right| \\ &\leq C(M_0)((|\log\kappa|^{-1} + \varepsilon^{1/2})(1 + r^2\|f_\varepsilon\|^2_{L^2}) + \kappa^{-3/4}r|\log\varepsilon|^{1/2}||f_\varepsilon||_{L^2}). \end{aligned}$$

On the other hand

$$\int_{B(a,2\kappa^{-3/4}r)\setminus B(a,\kappa^{-3/4}r)} \frac{(1-|v_\varepsilon|^2)^2}{4\varepsilon^2} \leq C(M_0)(|\log\kappa|^{-1} + \varepsilon^{1/2})(1 + r^2\|f_\varepsilon\|^2_{L^2}),$$

so that the conclusion follows.

A.6 First global estimates

As a consequence of the results of the two previous sections we have

Proposition A.8. *There exists $n \in \mathbb{N}$ and $\alpha > 0$ such that*

$$\left| \int_{B_1} \frac{(1-|v_\varepsilon|^2)^2}{4\varepsilon^2} - \frac{\pi}{2} n \right| \leq C(M_0)|\log \varepsilon|^{-1}(1+\varepsilon^\alpha \|f_\varepsilon\|_{L^2}^2).$$

Proof. Let $0 < \kappa < \frac{1}{2}$ to be determined later, $a_i^\varepsilon \in J_{\varepsilon,\kappa}$ and set $r = \lambda_\kappa \varepsilon$. By (A.37) and the fact that λ_κ is non-increasing with κ, we have

$$B(a_i^\varepsilon, \kappa^{-1} r/2) \setminus B(a_i, \varepsilon, r) \subset \Omega_\varepsilon(1/2).$$

Therefore, we may apply Proposition A.7, which yields, for $d_{i,\kappa} = d(a_i, \lambda_\kappa \varepsilon)$,

$$\begin{aligned} &\left| \int_{B(a_i, \lambda_\kappa \kappa^{-3/4}\varepsilon)} V_\varepsilon(v_\varepsilon) - \frac{\pi}{2} d_{i,\kappa}^2 \right| \\ &\leq C(M_0)((|\log \kappa|^{-1} + \varepsilon^{1/2})(1 + r^2 \|f_\varepsilon\|_{L^2}^2) + \kappa^{-3/4}\lambda_\kappa \varepsilon |\log \varepsilon|) \\ &\leq C(M_0)((|\log \kappa|^{-1} + \varepsilon^{1/2})(1 + r^2 \|f_\varepsilon\|_{L^2}^2) + \varepsilon^{1/2}\kappa^{-c_0(M_0)}). \end{aligned} \tag{A.77}$$

Here we have used the inequality $\lambda_\kappa \leq C\kappa^{-\ell/2}$, so that $c_0(M_0) > 0$. Next choose $\kappa = \varepsilon^\tau$, with $\tau = \frac{1}{4c_0(M_0)}$, so that $\varepsilon^{1/2}\kappa^{-c_0(M_0)} \leq \varepsilon^{1/4}$. Therefore, setting $n = \sum_{i \in J_{\varepsilon,\kappa}} d_{i,\kappa}^2$ we are led to

$$\left| \sum_{i \in J_{\varepsilon,\kappa}} \int_{B(a_i, \lambda_\kappa \kappa^{-3/4}\varepsilon)} V_\varepsilon(v_\varepsilon) - \frac{\pi}{2} d_{i,\kappa}^2 \right| \leq C(M_0)|\log \varepsilon|^{-1}(1 + \varepsilon^\alpha \|f_\varepsilon\|_{L^2}^2). \tag{A.78}$$

The conclusion then follows using the external estimate given in Proposition A.6. □

Proof of Proposition A.4.
Expanding $|\nabla v_\varepsilon|^2$ as in (A.72) and (A.73), we may write on $\Omega_\varepsilon(1/2)$

$$\begin{aligned} &|e_\varepsilon(v_\varepsilon) - \frac{1}{2}|\nabla \Psi_\varepsilon|^2| \\ &\leq C\left(V_\varepsilon(v_\varepsilon) + |\nabla \rho|^2 + \varepsilon^2 |\nabla \Psi_\varepsilon|^4 + |g_\varepsilon^1|^2 + |g_\varepsilon^2|^2 + |\nabla \Psi_\varepsilon|(|g_\varepsilon^1| + |g_\varepsilon^2|)\right). \end{aligned} \tag{A.79}$$

We then argue as in (A.74) to conclude that

$$\int_{\Omega_\varepsilon(1/2)} |e_\varepsilon(v_\varepsilon) - \frac{1}{2}|\nabla \Psi_\varepsilon|^2| \leq C(M_0)(1 + \|f_\varepsilon\|_{L^2(\Omega)}^2).$$

On the other hand, since $e_\varepsilon(v_\varepsilon) \le C(M_0)\varepsilon^{-2}$ and $|B_1 \setminus \Omega_\varepsilon(1/2)| \le C(M_0)\varepsilon^2$, the conclusion follows. $\square$

The rest of this Appendix is devoted to the computation of $\int_{\Omega_\varepsilon} |\nabla \Psi_\varepsilon|^2$. In view of its definition, it follows from (A.38) that

$$\nabla \Psi_\varepsilon = \sum_{i \in J_{\varepsilon,1/2}} d_i^\varepsilon \frac{x - a_\varepsilon}{|x - a_\varepsilon|^2},$$

and therefore, for every $0 < \kappa \le 1/2$, there exists some constant $C(M_0, \kappa)$ such that

$$\int_{\Omega_\varepsilon} |\nabla \Psi_\varepsilon|^2 = 2\pi (\sum_{i \in J_{\varepsilon,\kappa}} d_{i,\kappa}^2) |\log \varepsilon| - 2\pi \sum_{i \neq j \in J_{\varepsilon,\kappa}} d_{i,\kappa} d_{j,\kappa} \log |a_i - a_j| + R_{\varepsilon,\kappa}, \tag{A.80}$$

where $|R_{\varepsilon,\kappa}| \le C(M_0, \kappa)$. Here $d_{i,\kappa} = \deg(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(a_i^\varepsilon, \lambda_\kappa \varepsilon))$, for $i \in J_{\varepsilon,\kappa}$. The constant κ will be fixed thanks to Proposition A.9. We will then show that the contribution of $2\pi \sum_{i \neq j \in J_{\varepsilon,\kappa}} d_{i,\kappa} d_{j,\kappa} \log |a_i - a_j| + R_{\varepsilon,\kappa}$ is small compared to $|\log \varepsilon|$.

A.7 Confinement and algebraic equilibrium relation

The central idea in this section is borrowed from [7] (see also related result in [14]). Our first result deals with a "two-scale" confinement.

Proposition A.9. *Let $a \in B_1$, $R_2 > R_1 \ge \lambda_{1/2}\varepsilon$, and consider ℓ points $a_1, ..., a_\ell$, and $0 < \kappa \le 1/2$ such that*

$$a_i \in B(a, R_2) \qquad \text{for } i = 1, ..., \ell \tag{A.81}$$

$$|a_i - a_j| \ge \kappa^{-1} R_1 \qquad \text{for } i \neq j \tag{A.82}$$

$$|1 - |v_\varepsilon|| \le \sigma_0 \qquad \text{on } B(a, \kappa^{-1} R_2) \setminus \cup_{i=1}^{\ell} B(a_i, R_1). \tag{A.83}$$

There exist constants $\gamma_0 > 0$ and $\varepsilon_1 > 0$ depending only on M_0 such that if $0 < \varepsilon \le \varepsilon_1$,

$$0 < \kappa \le \kappa_0 \equiv \exp(-\frac{1 + R_2^2 \|f_\varepsilon\|_{L^2}^2}{\gamma_0}) \tag{A.84}$$

and

$$\kappa^{-3/4} R_2 |\log \varepsilon|^{1/2} ||f_\varepsilon||_{L^2} \le \gamma_0, \tag{A.85}$$

then

$$(\sum_{i=1}^{\ell} d_i)^2 = \sum_{i=1}^{\ell} d_i^2, \tag{A.86}$$

where
$$d_i = \deg(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(a_i, R_1)).$$

Proof. We apply Proposition A.7 on the two different scales, first with $r = R_1$ and $a = a_i$, for some $i = 1, ..., \ell$, then with $r = R_2$ and a. This yields

$$\left| \int_{B(a_i, \kappa^{-3/4} R_1)} V_\varepsilon(v_\varepsilon) - \frac{\pi}{2} d_i^2 \right| \leq$$
$$C(M_0)((|\log \kappa|^{-1} + \varepsilon^{1/2})(1 + R_1^2 \|f_\varepsilon\|_{L^2}^2) + \kappa^{-3/4} R_1 |\log \varepsilon|^{1/2} ||f_\varepsilon||_{L^2})$$

and

$$\left| \int_{B(a, \kappa^{-3/4} R_2)} V_\varepsilon(v_\varepsilon) - \frac{\pi}{2} (\sum_{i=1}^{\ell} d_i)^2 \right| \leq$$
$$C(M_0)((|\log \kappa|^{-1} + \varepsilon^{1/2})(1 + R_2^2 \|f_\varepsilon\|_{L^2}^2) + \kappa^{-3/4} R_2 |\log \varepsilon|^{1/2} ||f_\varepsilon||_{L^2}),$$

since

$$\deg(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(a, \kappa^{-1} R_2)) = \sum_{i=1}^{\ell} \deg(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(a, \kappa^{-1} R_1)) = \sum_{i=1}^{\ell} d_i.$$

On the other hand, by Proposition A.6 and scaling,

$$\left| \int_{B(a, \kappa^{-3/4} R_2) \setminus \cup_{i=1}^{\ell} B(a_i, \kappa^{-3/4} R_1)} V_\varepsilon(v_\varepsilon) \right| \leq C(M_0)(|\log \kappa|^{-2} + \varepsilon)(1 + R_2^2 \|f_\varepsilon\|_{L^2}^2).$$

Therefore, combining the three inequalities we obtain

$$\left| \sum_{i=1}^{\ell} d_i^2 - (\sum_{i=1}^{\ell} d_i)^2 \right| \leq$$
$$C(M_0)((|\log \kappa|^{-1} + \varepsilon^{1/2})(1 + R_2^2 \|f_\varepsilon\|_{L^2}^2) + \kappa^{-3/4} R_2 |\log \varepsilon|^{1/2} ||f_\varepsilon||_{L^2}).$$

Since the left-hand side is an integer, it vanishes if in particular

$$C(M_0)((|\log \kappa|^{-1} + \varepsilon^{1/2})(1 + R_2^2 \|f_\varepsilon\|_{L^2}^2) + \kappa^{-3/4} R_2 |\log \varepsilon|^{1/2} ||f_\varepsilon||_{L^2}) \leq \frac{1}{2}.$$

The conclusion follows. □

Our next result is of purely combinatorial nature

Lemma A.9. *Let X be a normed space, let $0 < \kappa < 1/2$ and $0 < r < R_{max}$ be given. Consider ℓ distinct points $a_1, ..., a_\ell$ in X such that*

$$|a_i - a_j| \geq \kappa^{-1} r.$$

Then one of the following two situations holds

i) $\inf_{i \neq j} |a_i - a_j| \geq R_{max}$

ii) There exists a partition $\{1, ..., \ell\} = \cup_{i=1}^k J_k$ with $k < \ell$, and for each $i \in \{1, ..., k\}$ some $b_i \in \cup_{j \in J_i} \{a_j\}$ and $r < R \leq (\frac{\kappa}{2})^{-\ell/2} R_{max}$ such that

$$\cup_{i=1}^\ell B(a_i, r) \subset \cup_{i=1}^k B(b_i, R) \tag{A.87}$$

$$|b_i - b_j| \geq \kappa^{-1} R, \qquad \text{for any } i \neq j \in \{1, ..., k\}, \tag{A.88}$$

and, for every $d_1, ..., d_\ell \in \mathbb{R}^\ell$,

$$\begin{aligned} &\left| \sum_{i \neq j = 1}^{\ell} d_i d_j \log |a_i - a_j| - \sum_{i \neq j = 1}^{k} D_i D_j \log |b_i - b_j| \right| \\ &\leq C(\ell) (\sup_i |d_i|^2) |\log \kappa| + \sum_{n=1}^{k} \left(\sum_{i \neq j \in J_n} d_i d_j \right) \log R \end{aligned} \tag{A.89}$$

where $D_i = \sum_{n \in J_i} d_n$, and where the constant C depends only on ℓ.

Proof. If i) holds there is nothing left to prove. Otherwise set

$$\delta_0 = \inf_{i \neq j} |a_i - a_j|.$$

Applying Lemma A.2 we obtain a subset $\{b_1, ..., b_k\}$ of $\{a_1, ..., a_\ell\}$ and $\delta_0 \leq \delta \leq (\kappa/2)^{-\ell} \delta_0$ such that

$$\cup_{i=1}^\ell B(a_i, \delta_0) \subset \cup_{i=1}^k B(b_i, \delta). \tag{A.90}$$

and

$$|b_i - b_j| \geq \kappa^{-1} \delta \qquad \forall i \neq j. \tag{A.91}$$

We choose $R = \delta$. It follows from the definition of δ_0 and (A.91) that $k < \ell$, whereas (A.87) and (A.88) follow directly from (A.90) and (A.91) respectively. We set, for $i = 1, ..., k$,

$$J_k = \{i : \ a_i \in B(b_k, R)\}$$

and turn finally to (A.89). For $i \neq j$ in $\{1, ..., \ell\}$ we distinguish two cases:
- i, j belong to the same J_n, for some $n \in \{1, ..., k\}$: then

$$|\log|a_i - a_j| - \log R| \leq c(\ell)|\log\kappa|$$

which follows from the fact that $\delta_0 \leq |a_i - a_j| \leq 2R \leq 2(\kappa/2)^{-\ell}\delta_0$.
- $i \in J_n$ and $j \in J_m$, $n \neq m$. Then

$$|\log|a_i - a_j| - \log|b_n - b_m|| \leq C\kappa.$$

The proof of (A.89) follows by summation. □

We are now in position to state, going back to (A.80),

Proposition A.10. *We have, for some constant $C(M_0)$ depending only on M_0,*

$$\Big|\sum_{i\neq j\in J_{\varepsilon,\kappa_0}} d_{i,\kappa_0}d_{j,\kappa_0}\log|a_i-a_j|\Big| \leq C(M_0)\Big(1+\|f_\varepsilon\|^2_{L^2}+\log(2+||f_\varepsilon||_{L^2}\sqrt{|\log\varepsilon|})\Big).$$

Proof. We set

$$R_{max} = \min\{1, \frac{\gamma_0\kappa_0^{3/4}}{(2+||f_\varepsilon||_{L^2}\sqrt{|\log\varepsilon|})}\} \tag{A.92}$$

where the constants κ_0 and γ_0 are provided in Proposition A.9 with $R_2 = 1$. We apply Lemma A.9 with the points a_i^ε in J_{ε,κ_0}, and $r = \lambda_{\kappa_0}\varepsilon$. If case i) occurs then for each $i \neq j$

$$\log|a_i - a_j| \geq \log(R_{max})$$

and the conclusion follows. Otherwise, (A.89) holds. Expanding (A.86) in Proposition A.9, we have, for each $n \in \{1, ..., k\}$,

$$\sum_{i\neq j\in J_n} d_i d_j = 0.$$

Therefore

$$\begin{aligned}&\left|\sum_{i\neq j=1}^{\ell} d_i d_j \log|a_i - a_j| - \sum_{i\neq j=1}^{k} D_i D_j \log|b_i - b_j|\right| \\ &\leq C(M_0)|\log\kappa_0| \leq C(M_0)(1+\|f_\varepsilon\|^2_{L^2}).\end{aligned} \tag{A.93}$$

If $\inf_{i\neq j}|b_i - b_j| \geq R_{max}$ we have finished. Otherwise we use lemma A.9 with the b_i's instead of the a_i's and $r = R$, the number provided in Lemma A.9, ii). We iterate this procedure until i) is met. Since at each step the number of balls decreases, we are done in at most ℓ steps. □

A.8 Proof of Theorem A.2 completed

Proof of i). Combining (A.80) (for $\kappa = \kappa_0$) with Proposition A.10 we are led to

$$\left| \int_{\Omega_\varepsilon(1/2)} |\nabla\Psi_\varepsilon|^2 - 2\pi\Big(\sum_{i\in J_{\varepsilon,\kappa_0}} d_{i,\kappa_0}^2\Big)|\log\varepsilon| \right| \\ \leq C(M_0)\Big(1 + \|f_\varepsilon\|_{L^2}^2 + \log(2 + ||f_\varepsilon||_{L^2}\sqrt{|\log\varepsilon|})\Big).$$

On the other hand, by Proposition A.4,

$$\left| \int_{B_1} e_\varepsilon(v_\varepsilon) - \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla\Psi_\varepsilon|^2 \right| \leq C(M_0)(1 + \|f_\varepsilon\|_{L^2}^2 + \log(2 + ||f_\varepsilon||_{L^2}\sqrt{|\log\varepsilon|})) \tag{A.94}$$

and (A.7) follows with

$$n = \sum_{i\in J_{\varepsilon,\kappa_0}} d_{i,\kappa_0}^2.$$

Concerning (A.8), Proposition A.9 shows that

$$\left| \int_{B_1} \frac{(1 - |v_\varepsilon|^2)}{4\varepsilon^2} - \frac{\pi}{2}m \right| \leq \frac{C(M_0)}{|\log\varepsilon|}(1 + \varepsilon^\alpha\|f_\varepsilon\|_{L^2}^2)$$

for some $m \in \mathbb{N}$. The important point is that $m = n$. This can be checked by choosing $\kappa = \kappa_0$ in the proof of Proposition A.9.

Proof of ii). If $n = m = 1$, then one checks that there is actually only one ball at level $\kappa = \kappa_0$. In this case, $\frac{1}{2}\int |\nabla\Psi_\varepsilon|^2 = \pi|\log\varepsilon| + O(1)$ and the conclusion follows from (A.79). □

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