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Research paper

Regularity of vectorial minimizers for non-uniformly elliptic anisotropic integrals

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ABSTRACT

We establish the local boundedness of the local minimizers $u : \Omega \to \mathbb{R}^m$ of non-uniformly elliptic integrals of the form $\int_\Omega f(x, Dv)\,dx$, where Ω is a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$ and the integrand satisfies anisotropic growth conditions of the type

$$\sum_{i=1}^{n} \lambda_i(x)|\xi_i|^{p_i} \leq f(x,\xi) \leq \mu(x)\left\{1+|\xi|^q\right\}$$

for some exponents $q \geq p_i > 1$ and with non-negative functions λ_i, μ fulfilling suitable summability assumptions. The main novelties here are the degenerate and anisotropic behavior of the integrand and the fact that we also address the case of vectorial minimizers $(m > 1)$. Our proof is based on the celebrated Moser iteration technique and employs an embedding result for anisotropic Sobolev spaces.

1. Introduction

In this paper we are interested in the regularity of local minimizers $u : \Omega \to \mathbb{R}^m$, $u \in W^{1,1}(\Omega;\mathbb{R}^m)$ with $m \geq 1$, of non-uniformly elliptic functionals of the form

$$\mathcal{F}(v) = \int_\Omega f(x, Dv)\,dx, \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. We assume that the energy density $f = f(x,\xi)$, $x \in \Omega$, $\xi \in \mathbb{R}^{m\times n}$, is a Carathéodory function, convex and of class C^1 with respect to ξ and satisfying the following *degenerate* and *anisotropic* behavior: for some exponents p_i, $i \in \{1,\dots,n\}$, and q with $1 < p_i \leq q$ and for some measurable functions $\lambda_i, \mu : \Omega \to [0,\infty)$, $i \in \{1,\dots,n\}$,

$$\sum_{i=1}^{n} \lambda_i(x)|\xi_i|^{p_i} \leq f(x,\xi) \leq \mu(x)\left\{1+|\xi|^q\right\} \qquad \text{for a.e. } x \in \Omega \text{ and for every } \xi \in \mathbb{R}^{m\times n}, \tag{1.2}$$

where

$$\lambda_i^{-1} \in L^{r_i}_{loc}(\Omega), \;\; i \in \{1,\dots,n\}, \qquad \mu \in L^s_{loc}(\Omega) \tag{1.3}$$

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for some $r_i \in [1,\infty]$ and $s \in (1,\infty]$. Throughout this paper ξ_i, $i \in \{1,\dots,n\}$, denotes the i-th column of the $m \times n$-matrix $\xi = (\xi_i^\alpha)$, $i \in \{1,\dots,n\}$, $\alpha \in \{1,\dots,m\}$, i.e.

$$\xi = (\xi_1, \xi_2, \dots, \xi_n) = \begin{pmatrix} \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 \\ \xi_1^2 & \xi_2^2 & \cdots & \xi_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^m & \xi_2^m & \cdots & \xi_n^m \end{pmatrix}.$$

In particular, if $\xi = Dv$, then $\xi_i = (v^1_{x_i}, \dots, v^m_{x_i})^T$. In addition, we assume that there exists a function $g : \Omega \times [0,\infty)^n \to [0,\infty)$ such that

$$f(x,\xi) = g(x, |\xi_1|, \dots, |\xi_i|, \dots, |\xi_n|) \tag{1.4}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{m\times n}$.

In the case $p := \min\{p_i\} < q$, with $\lambda_i \geq 1$ for every i and $\mu \in L^\infty(\Omega)$, the functional (1.1) belongs to the class of variational problems with p,q-growth conditions, introduced by Marcellini [1–4] and since then widely investigated. As for the case of anisotropic p_i,q-growth, we recall e.g. Boccardo–Marcellini–Sbordone [5], Stroffolini [6], Fusco–Sbordone [7] and Marcellini [8–10].

Anisotropic elliptic equations have been considered under many different aspects, for instance with respect to the maximum principle and the multiplicity of solutions: see e.g. Pucci, Rădulescu *et al.* [11–13].

Actually, the research on problems satisfying p,q-growth conditions is so intense that it is impossible to give an exhaustive and comprehensive list of references; for an overview on the subject and a detailed bibliography, see Mingione [14], Marcellini [15] and Mingione–Rădulescu [16].

In the vector-valued case, as suggested by well-known counterexamples by De Giorgi [17], Giusti–Miranda [18], Nečas [19] and Šverák–Yan [20], the structural assumption $f(x,Dv) = F(x,|Dv|)$ on the integrand is generally required for everywhere regularity. We point out that the condition $f(x,Dv) = F(x,|Dv|)$ is more specific than (1.4).

For the L^∞ regularity in the vectorial framework, see for example [21–24], where the authors established the local boundedness of solutions for some classes of quasilinear systems, which – in the variational context – may correspond to integrals as in (1.1). In particular, in [22] local L^∞-estimates were obtained for the local minimizers of (1.1), by assuming (1.2) and (1.4) with positive constants in place of the functions $\lambda_i(x)$ and $\mu(x)$, i.e. without imposing growth conditions on the integrand that are degenerate in x.

In the context of non-uniform ellipticity, starting from the celebrated paper by Trudinger [25], in [26] the authors proved the local boundedness for local minimizers of integrals of the form (1.1) under p,q-growth conditions of the type

$$\lambda(x)\,|\xi|^p \leq f(x,\xi) \leq \mu(x)\left\{1 + |\xi|^q\right\},$$

for some exponents $q \geq p > 1$ and with non-negative functions λ,μ satisfying suitable summability conditions. We also refer to [27], where the local boundedness is established for scalar-valued quasi-minimizers of non-uniformly elliptic integrals of the form

$$\mathcal{G}(v) = \int_\Omega \tilde{f}(x, v, Dv)\,dx,$$

where $\tilde{f} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory map satisfying a non-uniform growth condition of the type

$$\lambda(x)\,|\xi|^p \leq \tilde{f}(x,u,\xi) \leq \mu(x)\left\{|\xi|^p + |u|^q\right\} + a(x),$$

for $q \geq p > 1$ and non-negative functions a, λ, μ fulfilling appropriate summability assumptions.

In this paper we carry on the research conducted in the previous articles, extending the (local) L^∞-regularity results to the case of vectorial local minimizers ($m > 1$) of non-uniformly elliptic integrals under the anisotropic growth conditions (1.2), which may possibly be degenerate with respect to the x-variable. Precise assumptions and statements are given in Section 2, where we also impose some restrictions to the growth exponents $\{p_i\}$ and q in (1.2) and to the integrability exponents $\{r_i\}$ and s in (1.3). These restrictions are the natural generalizations of the various bounds that must be satisfied to ensure the regularity of local minimizers; for a knowledge of some of these bounds, we limit ourselves to pointing out the works [5,22,26–32] and the references therein, since there is a vast literature on this topic. In this regard, see also Remark 2.4 below, where we compare the main result of this paper (Theorem 2.2) with those obtained in [22,26,28,29].

The main novelty of our Theorem 2.2 is the degenerate and anisotropic behavior of the integrand (1.2) in the *vector case* $m > 1$. In this respect, we wish to mention the very recent work [33]: there, Feng, Gao and Zhang establish the local boundedness of vectorial local minimizers for a class of integral functionals with rank-one convex integrands and specific structural conditions. In particular, their result is applicable to integrals of the type

$$\int_\Omega \left\{ \sum_{\alpha=1}^m \lambda(x)\,|Dv^\alpha|^p + \mu(x)\,|Dv|^r \right\} dx, \tag{1.5}$$

with suitable $\lambda(x), \mu(x) > 0$ and $p, r > 1$. However, the result in [33] is not comparable to ours, since the functionals of the form (1.5) do not satisfy conditions (1.2) and (1.4), not even if $n = m \geq 2$, $\lambda_1 = \cdots = \lambda_n = \lambda$ and $p_1 = \cdots = p_n = p$.

Before describing the structure of this paper, we also want to point out the recent article [32], where Feo, Passarelli di Napoli and Posteraro prove the local boundedness of *scalar* minimizers of non-uniformly elliptic integrals of the form (1.1), assuming that the left inequality in (1.2) holds with monomial weights of the type

$$\lambda_i(x) = |x_i|^{\alpha_i p_i} \qquad \text{for some } \alpha_i \in [0,1) \qquad \text{for every } i \in \{1,\dots,n\}.$$

The proof methods used in [32,33] adapt the renowned De Giorgi iteration technique. Moreover, the approach employed in [32] relies essentially on an anisotropic Sobolev inequality with respect to the weights $|x_i|^{\alpha_i p_i}$.

It is worth noting that, to deal with the anisotropic behavior of the integrand, we base our estimates on an embedding result for anisotropic Sobolev spaces due to Troisi [34]. Other key ingredients in the proof of our result are the derivation of the Euler's equation for the functional (1.1) and a suitable Moser iteration procedure.

The paper is organized as follows. In the next section we give the complete statement of the main regularity result. Section 3 is devoted to the preliminaries: after a list of some classic notations and some essential lemmas, we recall the definition and properties of the anisotropic Sobolev spaces that will be needed to prove our result. In Section 4 we prove that an Euler's equation holds true. This is a main step in the proof of Theorem 2.2, which is given in Section 5. Finally, in Section 6 we present interesting examples of applicability of our main result.

2. Assumptions and statement of the main result

Let us define the integral functional

$$\mathcal{F}(v) := \int_\Omega f(x, Dv(x))\,dx, \tag{2.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, and $v \in W^{1,1}(\Omega;\mathbb{R}^m)$, $m \in \mathbb{N}$.

In the sequel, we denote by $\mathbb{R}_+$ the set $[0,\infty)$. Moreover, we assume that $f : \Omega\times\mathbb{R}^{m\times n} \to \mathbb{R}_+$ is a Carathéodory function satisfying the following conditions:

(A1) $f = f(x,\xi)$, $x \in \Omega$, $\xi \in \mathbb{R}^{m\times n}$, is convex and of class C^1 with respect to ξ;

(A2) there exists a function $g : \Omega \times (\mathbb{R}_+)^n \to \mathbb{R}_+$ such that

$$f(x,\xi) = g(x, |\xi_1|,\dots,|\xi_i|,\dots,|\xi_n|) \quad \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^{m\times n}; \tag{2.2}$$

(A3) there exists $\tau \geq 1$ such that

$$f(x,t\xi) \leq t^\tau f(x,\xi)$$

for every $t > 1$, for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{m\times n}$;

(A4) there exist some exponents p_i, $i \in \{1,\dots,n\}$, and q with $1 < p_i \leq q$ and some measurable functions $\lambda_i, \mu : \Omega \to \mathbb{R}_+$, $i \in \{1,\dots,n\}$, such that

$$\sum_{i=1}^n \lambda_i(x)|\xi_i|^{p_i} \leq f(x,\xi) \leq \mu(x)\left\{1 + |\xi|^q\right\} \quad \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^{m\times n}, \tag{2.3}$$

where

$$\lambda_i^{-1} \in L^{r_i}_{loc}(\Omega),\ \ i \in \{1,\dots,n\}, \qquad \mu \in L^s_{loc}(\Omega)$$

for some $r_i \in [1,\infty]$ and $s \in (1,\infty]$.

For every $i \in \{1,\dots,n\}$, we denote

$$\sigma_i = \begin{cases} \frac{p_i r_i}{r_i+1} & \text{if } r_i \in [1,\infty), \\ p_i & \text{if } r_i = \infty, \end{cases} \tag{2.4}$$

and additionally require that $\sigma_i \geq 1$, i.e. we require that $r_i \geq \frac{1}{p_i-1}$ if $1 < p_i < 2$.

We set

$$W^{1,\mathcal{F}}(\Omega;\mathbb{R}^m) := \{v \in W^{1,1}(\Omega;\mathbb{R}^m) \ : \ \mathcal{F}(v) < +\infty\}$$

and define a local minimizer of (2.1) as follows.

Definition 2.1. *A function u is a local minimizer of (2.1) if $u \in W^{1,\mathcal{F}}(\Omega;\mathbb{R}^m)$ and*

$$\mathcal{F}(u) \leq \mathcal{F}(u+\varphi)$$

for all $\varphi \in W^{1,\mathcal{F}}(\Omega;\mathbb{R}^m)$ with $\operatorname{supp}\varphi \Subset \Omega$.

Given a real number $\ell \geq 1$, ℓ^* is its Sobolev exponent, i.e.

$$\ell^* := \begin{cases} \frac{n\ell}{n-\ell} & \text{if } \ell < n, \\ \text{any value in } (\ell, \infty) & \text{if } \ell \geq n, \end{cases} \tag{2.5}$$

and ℓ' is the conjugate exponent of ℓ, i.e. $\ell' = \frac{\ell}{\ell-1}$ if $\ell > 1$, while $\ell' = \infty$ if $\ell = 1$. We set

$$\mathbf{p} := (p_1, \dots, p_n), \qquad \mathbf{pr} := (p_1 r_1, \dots, p_n r_n), \qquad \sigma := (\sigma_1, \dots, \sigma_n)$$

and let $\overline{\mathbf{p}}$ be the harmonic average of $\{p_i\}$, i.e.

$$\frac{1}{\overline{\mathbf{p}}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}. \tag{2.6}$$

Similarly, we will denote by $\overline{\mathbf{pr}}$ and $\overline{\sigma}$ the harmonic average of $\{p_i r_i\}$ and $\{\sigma_i\}$, respectively. As usual, $\frac{1}{\infty}$ has to be read as 0. Therefore, in the particular case $r_1 = \cdots = r_n = \infty$, we have $\frac{1}{\overline{\mathbf{pr}}} = 0$. Likewise, $\frac{1}{qs} = 0$ if $s = \infty$.

As we anticipated earlier, to prove the local boundedness of the local minimizers of (2.1), we need some restrictions on the exponents $\{p_i\}, q, \{r_i\}$ and s. More precisely, our main result reads as follows.

Theorem 2.2. *Let us assume that* (**A1**), (**A2**), (**A3**) *and* (**A4**) *hold under the summability conditions*

$$\lambda_i^{-1} \in L^{r_i}_{loc}(\Omega), \;\; i \in \{1, \dots, n\}, \qquad \mu \in L^s_{loc}(\Omega),$$

for some $r_i \in [1, \infty]$ and $s \in (1, \infty]$ such that $\max\{\sigma_i\} < \overline{\sigma}^*$ and

$$\frac{1}{\overline{\mathbf{pr}}} + \frac{1}{qs} + \frac{1}{\overline{\mathbf{p}}} - \frac{1}{q} < \frac{1}{n}. \tag{2.7}$$

For every $i \in \{1, \dots, n\}$, if $1 < p_i < 2$ we also require $r_i \geq \frac{1}{p_i - 1}$. Then, every local minimizer u of (2.1) is locally bounded. Moreover, for every ball $B_{R_0}(x_0) \Subset \Omega$ with $R_0 \in (0, 1]$,

(1) *there exists a constant $c_1 > 1$, depending on m, n, $\mathbf{p}$, q, $\mathbf{r}$, s, τ if $\overline{\sigma} < n$, and also on R_0 if $\overline{\sigma} \geq n$, such that for every $R \in (0, R_0)$ we have*

$$\|u\|_{L^\infty(B_{R/2}(x_0))} \leq \frac{c_1}{R^{\vartheta_3}} \left[1 + \|\mu\|^{\frac{1}{\overline{\mathbf{p}}}}_{L^s(B_R)} \prod_{i=1}^{n} \|\lambda_i^{-1}\|^{\frac{1}{np_i}}_{L^{r_i}(B_R)} \right]^{\vartheta_1} \left\| |u| + 1 \right\|^{\vartheta_2}_{L^{qs'}(B_R)}, \tag{2.8}$$

(2) *there exists a constant $c_2 > 0$, depending on m, n, $\mathbf{p}$, q, $\mathbf{r}$, s, τ and R_0, such that for every $R \in (0, R_0)$ we have*

$$\begin{aligned} \|u - u_R\|_{L^\infty(B_{R/(2\sqrt{n})}(x_0))} \leq \; & \frac{c_2}{R^{\vartheta_3}} \left[1 + \|\mu\|^{\frac{1}{\overline{\mathbf{p}}}}_{L^s(B_{R/\sqrt{n}})} \prod_{i=1}^{n} \|\lambda_i^{-1}\|^{\frac{1}{np_i}}_{L^{r_i}(B_{R/\sqrt{n}})} \right]^{\vartheta_1} \\ & \cdot \left[1 + \left(1 + \int_{B_R(x_0)} f(x, Du)\, dx \right)^{\frac{1}{p}} \sum_{i=1}^{n} \|\lambda_i^{-1}\|^{\frac{1}{p_i}}_{L^{r_i}(B_R)} \right]^{\vartheta_2}, \end{aligned} \tag{2.9}$$

where $u_R := \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u\, dx$, $p := \min_{1 \leq i \leq n} \{p_i\}$ and

$$\vartheta_1 := \frac{\overline{\sigma}^*}{\overline{\sigma}^* - qs'}, \qquad \vartheta_2 := \frac{q(\overline{\sigma}^* - \overline{\mathbf{p}} s')}{\overline{\mathbf{p}}(\overline{\sigma}^* - qs')}, \qquad \vartheta_3 := \frac{\overline{\sigma}^* [q^2 s' + n(q - \overline{\mathbf{p}})]}{\overline{\mathbf{p}} qs' (\overline{\sigma}^* - qs')}.$$

Remark 2.3. If $n > \overline{\sigma}$, then condition (2.7) is equivalent to

$$qs' < \overline{\sigma}^*. \tag{2.10}$$

In the case $n \leq \overline{\sigma}$, we can still suppose that $qs' < \overline{\sigma}^*$, provided we choose a sufficiently large value for $\overline{\sigma}^*$. For our purposes, in the sequel we will always assume that $qs' < \overline{\sigma}^*$.

Also note that if $p = q$ (i.e. $p_1 = \cdots = p_n = q$), then $\vartheta_2 = 1$ and $\vartheta_3 = \vartheta_1$.

Remark 2.4 (*Comparison with Other Results*). In the previous paper [26], the authors assume that there exist measurable functions $\lambda, \mu : \Omega \to \mathbb{R}_+$ such that

$$\begin{cases} \lambda(x)\, |\xi|^p \leq f(x, \xi) \leq \mu(x) \left\{ 1 + |\xi|^q \right\} \\ \lambda^{-1} \in L^r_{loc}(\Omega), \quad \mu \in L^s_{loc}(\Omega) \end{cases} \tag{2.11}$$

for $1 < p \leq q$, for some exponents $r \in [1, \infty]$, $s \in (1, \infty]$ and for every $\xi \in \mathbb{R}^{m \times n}$. Therefore, the left-hand side of (2.3) is an anisotropic version of the left inequality in (2.11). Furthermore, condition (2.7) in Theorem 2.2 is the evident counterpart of condition (2.6) in [26, Theorem 2.1], which reads as follows:

$$\frac{1}{pr} + \frac{1}{qs} + \frac{1}{p} - \frac{1}{q} < \frac{1}{n}.$$

In this regard, also observe that (2.10) is a generalization of the restriction $q < \overline{\mathbf{p}}^*$ adopted, for example, in [22,28,29], where the authors assume that μ is a positive constant and $\lambda_i \equiv 1$ for any $i \in \{1, \dots, n\}$. Moreover, the requirement $\max\{\sigma_i\} < \overline{\sigma}^*$ in Theorem 2.2 is the natural generalization of the condition $\max\{p_i\} < \overline{\mathbf{p}}^*$ in [29, Theorem 2.3]. In fact, if $r_1 = \cdots = r_n = \infty$, then $\sigma_i = p_i$ for every $i \in \{1, \dots, n\}$, and therefore $\overline{\sigma} = \overline{\mathbf{p}}$.

3. Notations and preliminaries

In this paper we shall denote by C or c a general positive constant that may vary on different occasions. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. The norm we use on $\mathbb{R}^k$, $k \in \mathbb{N}$, will be the standard Euclidean one and it will be denoted by $|\cdot|$. In particular, for the vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^k$, we write $\langle \mathbf{v}, \mathbf{w} \rangle$ for the usual inner product and $|\mathbf{v}| := \langle \mathbf{v}, \mathbf{v} \rangle^{\frac{1}{2}}$ for the corresponding Euclidean norm.

In what follows, $B_r(x_0) = \left\{x \in \mathbb{R}^n : \left|x - x_0\right| < r\right\}$ will denote the n-dimensional open ball with radius $r > 0$ and center $x_0 \in \mathbb{R}^n$. We shall sometimes omit the dependence on the center when all balls occurring in a proof are concentric. Unless otherwise stated, different balls in the same context will have the same center.

If $E \subseteq \mathbb{R}^k$ is a Lebesgue-measurable set, then we will denote by $|E|$ its k-dimensional Lebesgue measure. When $0 < |E| < \infty$, the mean value of a function $v \in L^1(E)$ is defined by

$$\fint_E v(x)\,dx \;:=\; \frac{1}{|E|} \int_E v(x)\,dx\,.$$

Now we gather some results that will be needed later on. Let us start with a lemma on the properties of the functions f and g considered in Section 2.

Lemma 3.1. *Assume* (**A1**) *and* (**A2**). *Then, for a.e. $x \in \Omega$ and every $i \in \{1, \dots, n\}$,*
(i) *$g(x, t_1, \dots, t_n)$ is of class C^1 with respect to $(t_1, \dots, t_n) \in (\mathbb{R}_+)^n$;*
(ii) *$g_{t_i}(x, z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_n) = 0$ for any $(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n) \in (\mathbb{R}_+)^{n-1}$;*
(iii) *$D_\xi f(x, 0) = 0$;*
(iv) *$g(x, t_1, \dots, t_n)$ is convex with respect to $t_i \in \mathbb{R}_+$;*
(v) *$g(x, t_1, \dots, t_n)$ is non-decreasing with respect to $t_i \in \mathbb{R}_+$;*
(vi) *$\rho \mapsto f(x, \rho\,\xi)$ is non-decreasing in $\mathbb{R}_+$ for every $\xi \in \mathbb{R}^{m \times n}$.*

Proof. We first prove (i), (ii) and (iii). Let $\{\mathrm{e}_\alpha\}_{1 \le \alpha \le m}$ be the standard basis of $\mathbb{R}^m$. Fix $\alpha \in \{1, \dots, m\}$ and let $\tilde{g} : \Omega \times \mathbb{R}^n \to \mathbb{R}_+$ be the function defined by

$$\tilde{g}(x, t_1, t_2, \dots, t_n) := f\left(x, \left(t_1\,\mathrm{e}_\alpha^T, t_2\,\mathrm{e}_\alpha^T, \dots, t_n\,\mathrm{e}_\alpha^T\right)\right), \qquad x \in \Omega,\ (t_1, \dots, t_n) \in \mathbb{R}^n. \tag{3.1}$$

Then, for a.e. $x \in \Omega$, the function $\tilde{g}(x, \cdot)$ is of class C^1 on $\mathbb{R}^n$, since it is a composition of C^1 functions. Moreover, for every $i \in \{1, \dots, n\}$ and every $(z_1, \dots, z_n) \in \mathbb{R}^n$, we get

$$\tilde{g}_{t_i}(x, z_1, \dots, z_i, \dots, z_n) \;=\; \frac{\partial f}{\partial \xi_i^\alpha}\left(x, \left(z_1\,\mathrm{e}_\alpha^T, \dots, z_i\,\mathrm{e}_\alpha^T, \dots, z_n\,\mathrm{e}_\alpha^T\right)\right). \tag{3.2}$$

Now observe that, for a.e. $x \in \Omega$, the partial map

$$(t_1, \dots, t_n) \in (\mathbb{R}_+)^n \longmapsto g(x, t_1, \dots, t_n)$$

is the restriction of $\tilde{g}(x, \cdot)$ to $(\mathbb{R}_+)^n$. Indeed, by (**A2**) and (3.1), for a.e. $x \in \Omega$ and every $(t_1, \dots, t_n) \in (\mathbb{R}_+)^n$ we have

$$g(x, t_1, \dots, t_n) \;=\; f\left(x, \left(t_1\,\mathrm{e}_\alpha^T, \dots, t_n\,\mathrm{e}_\alpha^T\right)\right) \;=\; \tilde{g}(x, t_1, \dots, t_n).$$

Therefore, for a.e. $x \in \Omega$, the map $g(x, \cdot)$ is of class C^1 on $(\mathbb{R}_+)^n$.

Now also fix $i \in \{1, \dots, n\}$, $(\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_n) \in \mathbb{R}^{m \times (n-1)}$ and $x \in \Omega$, and consider the function $F : \mathbb{R} \to \mathbb{R}_+$ defined by

$$F(t) := f(x, (\xi_1, \dots, \xi_{i-1}, t\,\mathrm{e}_\alpha^T, \xi_{i+1}, \dots, \xi_n)), \qquad t \in \mathbb{R},$$

with the usual modifications in the limit cases $i \in \{1, n\}$. Then F is of class C^1 on $\mathbb{R}$, because it is a composition of C^1 functions. Furthermore, F is an even function, since

$$\begin{aligned} F(t) &= g(x, |\xi_1|, \dots, |\xi_{i-1}|, |t|, |\xi_{i+1}|, \dots, |\xi_n|) \\ &= g(x, |\xi_1|, \dots, |\xi_{i-1}|, |-t|, |\xi_{i+1}|, \dots, |\xi_n|) = F(-t) \qquad \text{for every } t \in \mathbb{R}. \end{aligned}$$

Hence,

$$F'(0) \;=\; \frac{\partial f}{\partial \xi_i^\alpha}(x, (\xi_1, \dots, \xi_{i-1}, \underbrace{0^T}_{\in \mathbb{R}^m}, \xi_{i+1}, \dots, \xi_n)) = 0\,. \tag{3.3}$$

Since $(\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_n) \in \mathbb{R}^{m \times (n-1)}$ is arbitrary, from (3.3) and (3.2) with $z_i = 0$ we deduce

$$g_{t_i}(x, z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_n) \;=\; \frac{\partial f}{\partial \xi_i^\alpha}\left(x, \left(z_1\,\mathrm{e}_\alpha^T, \dots, z_{i-1}\,\mathrm{e}_\alpha^T, 0^T, z_{i+1}\,\mathrm{e}_\alpha^T, \dots, z_n\,\mathrm{e}_\alpha^T\right)\right) = 0$$

for every $(z_1,\dots,z_{i-1},z_{i+1},\dots,z_n) \in (\mathbb{R}_+)^{n-1}$, where we have used the fact that $g(x,\cdot)$ is the restriction of $\tilde{g}(x,\cdot)$ to $(\mathbb{R}_+)^n$. The conclusions (ii) and (iii) then follow from the arbitrariness of $x \in \Omega$, $i \in \{1,\dots,n\}$, $\alpha \in \{1,\dots,m\}$ and $(z_1,\dots,z_{i-1},z_{i+1},\dots,z_n) \in (\mathbb{R}_+)^{n-1}$.

We now prove (iv) and (v). Let $w \in \mathbb{R}^m$ with $|w| = 1$ and fix $i \in \{1,\dots,n\}$. Moreover, let

$$(t_1,\dots,t_{i-1},a,b,t_{i+1},\dots,t_n) \in (\mathbb{R}_+)^{n+1},$$

$$\zeta = (t_1\,w^T,\dots,t_{i-1}\,w^T,a\,w^T,t_{i+1}\,w^T,\dots,t_n\,w^T),$$

$$\eta = (t_1\,w^T,\dots,t_{i-1}\,w^T,b\,w^T,t_{i+1}\,w^T,\dots,t_n\,w^T).$$

Then, for a.e. $x \in \Omega$ and for all $\theta \in [0,1]$ we have

$$\begin{aligned} & g(x,t_1,\dots,t_{i-1},\theta a + (1-\theta)b,t_{i+1},\dots,t_n) \\ &\quad = f(x,\theta\,\zeta + (1-\theta)\,\eta) \\ &\quad \le \theta\, f(x,\zeta) + (1-\theta)\, f(x,\eta) \\ &\quad = \theta\, g(x,t_1,\dots,t_{i-1},a,t_{i+1},\dots,t_n) + (1-\theta)\, g(x,t_1,\dots,t_{i-1},b,t_{i+1},\dots,t_n). \end{aligned}$$

This implies that $g(x,t_1,\dots,t_n)$ is convex with respect to each variable $t_i \in \mathbb{R}_+$. Therefore, the partial maps

$$z \in \mathbb{R}_+ \longmapsto g_{t_i}(x,t_1,\dots,t_{i-1},z,t_{i+1},\dots,t_n), \qquad i \in \{1,\dots,n\},$$

are non-decreasing and this monotonicity property, together with (i) and (ii), entails that $g(x,t_1,\dots,t_n)$ is non-decreasing with respect to each variable $t_i \in \mathbb{R}_+$.

Finally, let us prove (vi). Fix $\rho_1,\rho_2 \in \mathbb{R}_+$ with $\rho_1 < \rho_2$. Then, for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{m\times n}$ we have

$$\begin{aligned} f(x,\rho_1\,\xi) &= g(x,\rho_1|\xi_1|,\rho_1|\xi_2|,\dots,\rho_1|\xi_{n-1}|,\rho_1|\xi_n|) \\ &\le g(x,\rho_2|\xi_1|,\rho_1|\xi_2|,\dots,\rho_1|\xi_{n-1}|,\rho_1|\xi_n|) \\ &\le \dots \le g(x,\rho_2|\xi_1|,\rho_2|\xi_2|,\dots,\rho_2|\xi_{n-1}|,\rho_1|\xi_n|) \\ &\le g(x,\rho_2|\xi_1|,\rho_2|\xi_2|,\dots,\rho_2|\xi_{n-1}|,\rho_2|\xi_n|) = f(x,\rho_2\,\xi), \end{aligned}$$

where, in the last three lines, we have repeatedly used the property (v). This concludes the proof. □

If f is as in Section 2, then $W^{1,\mathcal{F}}(\Omega;\mathbb{R}^m)$ is a vector space; this is a consequence of the following lemma.

Lemma 3.2. *Assume* (**A1**), (**A2**) *and* (**A3**). *Then, for a.e.* $x \in \Omega$,

(i) $f(x,\gamma\xi) \le \max\{1,\gamma^\tau\} f(x,\xi)$ *for every* $\gamma > 0$ *and every* $\xi \in \mathbb{R}^{m\times n}$;

(ii) $f(x,\xi+\eta) \le 2^{\tau-1}\,[f(x,\xi) + f(x,\eta)]$ *for every* $\xi,\eta \in \mathbb{R}^{m\times n}$,
where $\tau \ge 1$ *is the constant in* (**A3**).

Proof. Let us prove (i). If $\xi = 0$, then $f(x,\gamma\xi) = f(x,\xi)$ for every $\gamma > 0$ and the conclusion immediately follows.

Assume $|\xi| > 0$. We consider the cases $\gamma > 1$ and $0 < \gamma \le 1$ separately.

Let $\gamma > 1$. Then (**A3**) implies $f(x,\gamma\xi) \le \gamma^\tau f(x,\xi)$.

If instead $0 < \gamma \le 1$, by Lemma 3.1 (vi) we get $f(x,\gamma\xi) \le f(x,\xi)$ and the conclusion follows.

Let us now prove (ii). If $\xi,\eta \in \mathbb{R}^{m\times n}$, by (**A1**) and (**A3**) we obtain

$$f(x,\xi+\eta) \le \frac{1}{2}\,[f(x,2\xi) + f(x,2\eta)] \le 2^{\tau-1}\,[f(x,\xi) + f(x,\eta)].$$

This completes the proof. □

We now recall the following elementary result, whose proof can be found, for example, in [28, Lemma 3.1].

Lemma 3.3. *Consider* $h : \mathbb{R}_+ \to \mathbb{R}_+$ *of class* C^1. *Assume that there exists* $\tau \ge 1$ *such that*

$$h(\gamma t) \le \gamma^\tau h(t) \quad \text{for all } \gamma > 1 \text{ and } t \ge 0.$$

Then

$$h'(t)\,t \le \tau\, h(t) \quad \text{for all } t \ge 0.$$

3.1. Anisotropic Sobolev spaces

Let $q_i \ge 1$ for all $i \in \{1,\dots,n\}$. For any open subset Ω of $\mathbb{R}^n$, we consider the anisotropic Sobolev space

$$W^{1,(q_1,\dots,q_n)}(\Omega;\mathbb{R}^m) := \left\{ v \in W^{1,1}(\Omega;\mathbb{R}^m) : v_{x_i} \in L^{q_i}(\Omega;\mathbb{R}^m), \text{ for all } i = 1,\dots,n \right\}$$

endowed with the natural norm

$$\|v\|_{W^{1,(q_1,\dots,q_n)}(\Omega;\mathbb{R}^m)} := \|v\|_{L^1(\Omega;\mathbb{R}^m)} + \sum_{i=1}^{n} \|v_{x_i}\|_{L^{q_i}(\Omega;\mathbb{R}^m)}\,.$$

Sometimes, when no confusion may arise, we will omit the target space $\mathbb{R}^m$. Let us denote $\mathbf{q} = (q_1, \dots, q_n)$ and

$$W_0^{1,(q_1,\dots,q_n)}(\Omega;\mathbb{R}^m) = W_0^{1,1}(\Omega;\mathbb{R}^m) \cap W^{1,(q_1,\dots,q_n)}(\Omega;\mathbb{R}^m)\,.$$

These spaces are studied in [34] (see also [35]). We now report an embedding theorem for this class of spaces, whose proof can be obtained by a straightforward adaptation of that of [34, Theorem 1.2].

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $v \in W_0^{1,(q_1,\dots,q_n)}(\Omega;\mathbb{R}^m)$, $q_i \geq 1$ for all $i \in \{1,\dots,n\}$. Then,*

(1) *if $\overline{\mathbf{q}} < n$, we have*

$$\|v\|_{L^{\overline{\mathbf{q}}^*}(\Omega;\mathbb{R}^m)} \leq c \prod_{i=1}^{n} \|v_{x_i}\|_{L^{q_i}(\Omega;\mathbb{R}^m)}^{\frac{1}{n}}$$

for a constant $c = c(m,n,\mathbf{q}) > 0$, where $\overline{\mathbf{q}}^*$ is defined by (2.5)–(2.6) with $\mathbf{p} = \mathbf{q}$ and $\ell = \overline{\mathbf{q}}$;

(2) *if $\overline{\mathbf{q}} \geq n$, for every $1 \leq \chi < \infty$ we have*

$$\|v\|_{L^{\chi}(\Omega;\mathbb{R}^m)} \leq c\, |\Omega|^{\frac{1}{\chi}+\frac{1}{n}-\frac{1}{\overline{\mathbf{q}}}} \prod_{i=1}^{n} \|v_{x_i}\|_{L^{q_i}(\Omega;\mathbb{R}^m)}^{\frac{1}{n}}$$

for a constant $c = c(m,n,\mathbf{q},\chi) > 0$.

The following embedding result is proved in [35].

Theorem 3.5. *Let $Q \subset \mathbb{R}^n$ be a cube with edges parallel to the coordinate axes and consider a function $v \in W^{1,(q_1,\dots,q_n)}(Q;\mathbb{R}^m)$, $q_i \geq 1$ for all $i \in \{1,\dots,n\}$. Let $\max\{q_i\} < \overline{\mathbf{q}}^*$. Then $v \in L^{\overline{\mathbf{q}}^*}(Q;\mathbb{R}^m)$. Moreover, there exists a positive constant c depending on n and $\mathbf{q}$ if $\overline{\mathbf{q}} < n$, and additionally on Q if $\overline{\mathbf{q}} \geq n$, such that*

$$\|v\|_{L^{\overline{\mathbf{q}}^*}(Q)} \leq c \left\{ \|v\|_{L^1(Q)} + \sum_{i=1}^{n} \|v_{x_i}\|_{L^{q_i}(Q)} \right\}.$$

The next proposition is a consequence of the above theorem.

Proposition 3.6. *For every $i \in \{1,\dots,n\}$, let $p_i \in (1,\infty)$, $r_i \in [1,\infty]$, with $r_i \geq \frac{1}{p_i-1}$ if $1 < p_i < 2$, and let σ_i be defined according to (2.4). Moreover, assume that $\max\{\sigma_i\} < \overline{\sigma}^*$ and that the left inequality in (2.3) holds under the summability conditions*

$$\lambda_i^{-1} \in L^{r_i}_{loc}(\Omega), \quad i \in \{1,\dots,n\}.$$

If $u \in W^{1,\mathcal{F}}(\Omega;\mathbb{R}^m)$, then we get $|u| \in L^{\overline{\sigma}^*}_{loc}(\Omega)$.

Proof. Let $Q \Subset \Omega$ be a cube with edges parallel to the coordinate axes and fix $j \in \{1,\dots,n\}$.

Let us first assume that $r_j < \infty$. Recalling that $\sigma_j = \frac{p_j r_j}{r_j+1}$, by Hölder's inequality we obtain

$$\|u_{x_j}\|_{L^{\sigma_j}(Q)}^{p_j} = \left(\int_Q \lambda_j^{\frac{r_j}{r_j+1}} |u_{x_j}|^{\frac{p_j r_j}{r_j+1}} \lambda_j^{-\frac{r_j}{r_j+1}} \, dx \right)^{\frac{r_j+1}{r_j}} \leq \|\lambda_j^{-1}\|_{L^{r_j}(Q)} \int_Q \lambda_j\, |u_{x_j}|^{p_j}\, dx\,. \tag{3.4}$$

Combining the previous estimate with the left inequality in (2.3), we have

$$\|u_{x_j}\|_{L^{\sigma_j}(Q)}^{p_j} \leq \|\lambda_j^{-1}\|_{L^{r_j}(Q)} \sum_{i=1}^{n} \int_\Omega \lambda_i\, |u_{x_i}|^{p_i}\, dx \leq \|\lambda_j^{-1}\|_{L^{r_j}(Q)} \int_\Omega f(x,Du)\, dx\,,$$

and the last term is finite, because $\lambda_j^{-1} \in L^{r_j}_{loc}(\Omega)$ and $u \in W^{1,\mathcal{F}}(\Omega;\mathbb{R}^m)$.

If $r_j = \infty$, then $\sigma_j = p_j$ and we find

$$\|u_{x_j}\|_{L^{p_j}(Q)}^{p_j} = \int_Q \lambda_j\, |u_{x_j}|^{p_j}\, \lambda_j^{-1}\, dx \leq \|\lambda_j^{-1}\|_{L^\infty(Q)} \int_Q \lambda_j\, |u_{x_j}|^{p_j}\, dx\,, \tag{3.5}$$

which combined with the left inequality in (2.3) gives

$$\|u_{x_j}\|_{L^{p_j}(Q)}^{p_j} \leq \|\lambda_j^{-1}\|_{L^\infty(Q)} \int_\Omega f(x,Du)\, dx \; < +\infty\,.$$

We have thus proved that $u \in W^{1,(\sigma_1,\dots,\sigma_n)}(Q;\mathbb{R}^m)$. Since $\sigma_i \geq 1$ for all $i \in \{1,\dots,n\}$ and $\max\{\sigma_i\} < \overline{\sigma}^*$, by Theorem 3.5 we get $u \in L^{\overline{\sigma}^*}(Q;\mathbb{R}^m)$. The conclusion then follows from the arbitrariness of Q. □

We conclude this section with the following proposition about a Poincaré–Sobolev type inequality.

Proposition 3.7. *For every $i \in \{1,\dots,n\}$, let $p_i \in (1,\infty)$, $r_i \in [1,\infty]$, with $r_i \geq \frac{1}{p_i-1}$ if $1 < p_i < 2$, and let σ_i be defined according to (2.4). Consider a bounded open set $\Omega \subset \mathbb{R}^n$ and let $v \in W_0^{1,(\sigma_1,\dots,\sigma_n)}(\Omega;\mathbb{R}^m)$. Moreover, for every $i \in \{1,\dots,n\}$, let $\lambda_i : \Omega \to [0,\infty)$ be a measurable function such that $\lambda_i^{-1} \in L^{r_i}(\Omega)$. Then, there exists a positive constant c, depending on m, n and $\sigma = (\sigma_1,\dots,\sigma_n)$, such that*

$$\left(\int_\Omega |v|^{\overline{\sigma}^*}\,dx\right)^{\frac{1}{\overline{\sigma}^*}} \leq c\,\mathfrak{M} \prod_{i=1}^n \|\lambda_i^{-1}\|_{L^{r_i}(\Omega)}^{\frac{1}{np_i}} \left(\int_\Omega \lambda_i\,|v_{x_i}|^{p_i}\,dx\right)^{\frac{1}{np_i}},$$

where

$$\mathfrak{M} := \begin{cases} 1 & if\ \overline{\sigma} < n, \\ |\Omega|^{\frac{1}{\overline{\sigma}^*}+\frac{1}{n}-\frac{1}{\overline{\sigma}}} & if\ \overline{\sigma} \geq n. \end{cases}$$

Proof. Notice that the assumptions imply $\sigma_i \geq 1$ for all $i \in \{1,\dots,n\}$. Therefore, by Theorem 3.4 we get

$$\|v\|_{L^{\overline{\sigma}^*}(\Omega)} \leq c\,\mathfrak{M} \prod_{i=1}^n \|v_{x_i}\|_{L^{\sigma_i}(\Omega)}^{\frac{1}{n}}$$

for a positive constant c depending on m, n and σ. Arguing as in (3.4)–(3.5), we obtain

$$\|v\|_{L^{\overline{\sigma}^*}(\Omega)} \leq c\,\mathfrak{M} \prod_{i=1}^n \|v_{x_i}\|_{L^{\sigma_i}(\Omega)}^{\frac{1}{n}} \leq c\,\mathfrak{M} \prod_{i=1}^n \|\lambda_i^{-1}\|_{L^{r_i}(\Omega)}^{\frac{1}{np_i}} \left(\int_\Omega \lambda_i\,|v_{x_i}|^{p_i}\,dx\right)^{\frac{1}{np_i}}.$$

This concludes the proof. □

4. The Euler's equation

In this section, we prove that an Euler's equation holds true. This equation will be our starting point in the proof of Theorem 2.2.

Proposition 4.1. *Assume* (**A1**)–(**A3**) *and let u be a local minimizer of (2.1). Then*

$$\int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^m \frac{\partial f}{\partial \xi_i^\alpha}(x,Du)\,\varphi_{x_i}^\alpha\,dx = 0 \tag{4.1}$$

for all $\varphi \in W^{1,\mathcal{F}}(\Omega;\mathbb{R}^m)$ with $\operatorname{supp}\varphi \Subset \Omega$.

Proof. Let $\varphi \in W^{1,\mathcal{F}}(\Omega;\mathbb{R}^m)$ with $\operatorname{supp}\varphi \Subset \Omega$. By (2.2) also $-\varphi$ is in $W^{1,\mathcal{F}}(\Omega;\mathbb{R}^m)$. By Lemma 3.2 we get $u + t\varphi \in W^{1,\mathcal{F}}(\Omega;\mathbb{R}^m)$ for every $t \in \mathbb{R}$. By the local minimality of u,

$$\mathcal{F}(u) \leq \mathcal{F}(u+t\varphi) \qquad \forall\, t \in \mathbb{R}.$$

To prove (4.1) it suffices to prove that

$$\frac{d}{dt}\mathcal{F}(u+t\varphi)\Big|_{t=0} = \int_\Omega \frac{d}{dt} f(x, Du(x) + t\,D\varphi(x))\Big|_{t=0}\,dx.$$

Therefore, we need to prove that

$$\Big|\sum_{i=1}^n \sum_{\alpha=1}^m \frac{\partial f}{\partial \xi_i^\alpha}(x, Du + t\,D\varphi)\varphi_{x_i}^\alpha\Big| \leq H(x) \qquad \forall\, t \in (-1,1)$$

with $H \in L^1(\Omega)$. By the convexity of $f(x,\cdot)$, we obtain

$$f(x,\xi_0) - f(x, 2\xi_0 - \xi) \leq \sum_{i=1}^n \sum_{\alpha=1}^m \frac{\partial f}{\partial \xi_i^\alpha}(x,\xi_0)(\xi_i^\alpha - (\xi_0)_i^\alpha) \leq f(x,\xi) - f(x,\xi_0).$$

If $\xi_0 = Du(x) + t\,D\varphi(x)$ and $\xi = Du(x) + (1+t)\,D\varphi(x)$, we have

$$2\xi_0 - \xi = Du(x) + (t-1)\,D\varphi(x)$$

and

$$\begin{aligned} f(x, Du + t\,D\varphi) - f(x, Du + (t-1)\,D\varphi) &\leq \sum_{i=1}^n \sum_{\alpha=1}^m \frac{\partial f}{\partial \xi_i^\alpha}(x, Du + t\,D\varphi)\varphi_{x_i}^\alpha \\ &\leq f(x, Du + (1+t)\,D\varphi) - f(x, Du + t\,D\varphi). \end{aligned}$$

Therefore, since f is non-negative,

$$\Big|\sum_{i=1}^n \sum_{\alpha=1}^m \frac{\partial f}{\partial \xi_i^\alpha}(x, Du + t\,D\varphi)\varphi_{x_i}^\alpha\Big| \leq f(x, Du + (1+t)\,D\varphi) + f(x, Du + (t-1)\,D\varphi).$$

Moreover, since $t \in (-1,1)$, we can use Lemma 3.2 to estimate the first term on the right-hand side of the previous inequality, thus obtaining

$$\begin{aligned} f(x, Du + (1+t)\,D\varphi) &\le 2^{\tau-1}\,[f(x,Du) + f(x,(1+t)\,D\varphi)] \\ &\le 2^{\tau-1}\left[f(x,Du) + 2^{\tau} f(x,D\varphi)\right]. \end{aligned}$$

To estimate $f(x, Du + (t-1)\,D\varphi)$, we now consider the cases $t \in [0,1)$ and $t \in (-1,0)$ separately. Let us first assume $t \in [0,1)$. Then, using again the convexity of $f(x,\cdot)$ and Lemma 3.2, we get

$$\begin{aligned} f(x, Du + (t-1)\,D\varphi) &\le t\, f(x,Du) + (1-t) f(x, Du - D\varphi) \\ &\le f(x,Du) + f(x, Du - D\varphi) \\ &\le (1+2^{\tau-1}) f(x,Du) + 2^{\tau-1} f(x,-D\varphi). \end{aligned}$$

Now assume $t \in (-1,0)$. Since $1 < 1-t < 2$, by Lemma 3.2 we have

$$\begin{aligned} f(x, Du + (t-1)\,D\varphi) &\le 2^{\tau-1}\,[f(x,Du) + f(x,(t-1)\,D\varphi)] \\ &\le 2^{\tau-1}\left[f(x,Du) + 2^{\tau} f(x,-D\varphi)\right] \\ &= 2^{\tau-1} f(x,Du) + 2^{2\tau-1} f(x,-D\varphi). \end{aligned}$$

We have so proved that

$$\Big|\sum_{i=1}^{n}\sum_{\alpha=1}^{m} \frac{\partial f}{\partial \xi_i^{\alpha}}(x, Du + t\,D\varphi)\varphi^{\alpha}_{x_i}\Big| \le (1+2^{\tau}) f(x,Du) + 2^{2\tau-1}[f(x,D\varphi) + f(x,-D\varphi)] =: H(x).$$

Since $u, \varphi, -\varphi \in W^{1,\mathcal{F}}(\Omega;\mathbb{R}^m)$, we conclude. □

5. Proof of the main result

We first state a lemma useful for the proof of Theorem 2.2. In the statement, the functions λ_i, μ and the exponents $\{p_i\}, q, \{r_i\}, s$ are the same as in the statement of Theorem 2.2. Moreover, $x_0 \in \Omega$ and $R_0 \in (0,1]$ are such that $B_{R_0} := B_{R_0}(x_0) \Subset \Omega$. Fixed $0 < \rho < R \le R_0$, the function $\eta \in C_c^{\infty}(B_R(x_0))$ denotes a cut-off function satisfying

$$0 \le \eta \le 1, \qquad \eta \equiv 1 \ \text{ on } B_\rho(x_0), \qquad |D\eta| \le \frac{2}{R-\rho}. \tag{5.1}$$

Lemma 5.1. *Let us assume that* (**A1**)–(**A4**) *hold under the summability conditions*

$$\lambda_i^{-1} \in L^{r_i}_{loc}(\Omega), \ \ i \in \{1,\dots,n\}, \qquad \mu \in L^{s}_{loc}(\Omega),$$

for some $r_i \in [1,\infty]$ and $s \in (1,\infty]$ such that

$$\max\{\sigma_i\} < \overline{\sigma}^* \qquad \textit{and} \qquad \frac{1}{\overline{\mathbf{pr}}} + \frac{1}{qs} + \frac{1}{\overline{\mathbf{p}}} - \frac{1}{q} < \frac{1}{n}.$$

For every $i \in \{1,\dots,n\}$, if $1 < p_i < 2$ we also require $r_i \ge \frac{1}{p_i-1}$. Moreover, let u be a local minimizer of (2.1). Then, for every $\gamma \ge 0$ and every $i \in \{1,\dots,n\}$ we have

$$\int_{B_R} \lambda_i(x)\,|u|^{p_i\gamma}\,|u_{x_i}|^{p_i}\,\eta^q\,dx \le \frac{c}{(R-\rho)^q}\int_{B_R} \mu(x)\,(\max\{|u|,1\})^{q+p_i\gamma}\,dx \tag{5.2}$$

for a positive constant c depending on n, q and τ, but independent of γ, u, R and ρ.

Proof. We begin by defining a class of suitable test functions for the Euler's equation (4.1). Let us approximate the identity function $\mathrm{id} : \mathbb{R}_+ \to \mathbb{R}_+$ with an increasing sequence of non-decreasing C^1 functions $h_k : \mathbb{R}_+ \to \mathbb{R}_+$ having the following properties:

$$h_k(t) = 0 \quad \forall\, t \in \left[0, \frac{1}{k}\right], \qquad h_k(t) = k \quad \forall\, t \in [k+1,\infty), \qquad 0 \le h_k'(t) \le 2 \ \text{ in } \mathbb{R}_+.$$

Fixed $k, i \in \mathbb{N}$, $i \le n$, and $\gamma > 0$, let $\Phi_k^{(i,\gamma)} : \mathbb{R}_+ \to \mathbb{R}_+$ be the increasing function defined as follows

$$\Phi_k^{(i,\gamma)}(t) := h_k(t^{p_i\gamma}).$$

Define $\varphi_k^{(i,\gamma)} : B_R(x_0) \to \mathbb{R}^m$ by

$$\varphi_k^{(i,\gamma)}(x) := \Phi_k^{(i,\gamma)}(|u(x)|)\,u(x)\,[\eta(x)]^q.$$

From now on, we omit the dependence of Φ_k and φ_k on i and γ, i.e. $\Phi_k = \Phi_k^{(i,\gamma)}$ and $\varphi_k = \varphi_k^{(i,\gamma)}$. We have that Φ_k is in $C^1(\mathbb{R}_+)$, bounded and with bounded derivative. Precisely, define a_k and b_k positive real numbers, such that $a_k^{p_i\gamma} = \frac{1}{k}$ and $b_k^{p_i\gamma} = k+1$. In particular,

$$\Phi_k'(s) = \begin{cases} 0 & \text{if } \ s \in \mathbb{R}_+ \setminus [a_k, b_k], \\ p_i\gamma\, h_k'(s^{p_i\gamma}) s^{p_i\gamma-1} & \text{if } \ s \in [a_k, b_k], \end{cases}$$

and

$$\|\Phi_k'\|_{L^\infty(\mathbb{R}_+)} \le 2p_i\gamma \max\{a_k^{p_i\gamma-1}, b_k^{p_i\gamma-1}\} < +\infty\,.$$

As a consequence, taking into account that $u \in W^{1,1}(\Omega;\mathbb{R}^m)$, we have that $\Phi_k(|u|)\,u$ is in $W^{1,1}(\Omega;\mathbb{R}^m)$. This implies that $\varphi_k \in W^{1,1}(\Omega;\mathbb{R}^m)$ too. Moreover, $\operatorname{supp}\varphi_k \Subset B_R(x_0)$. Therefore, by Lemma 3.2 (ii), φ_k is a test function for the Euler's equation (4.1) if we prove that

$$\begin{aligned}
I_1 &:= \int_{B_R\cap\{a_k<|u|<b_k\}} f\left(x, \Phi_k'(|u|)\,\frac{u(x)}{|u(x)|}\,\langle u, u_{x_1}\rangle\,\eta^q, \dots, \Phi_k'(|u|)\,\frac{u(x)}{|u(x)|}\,\langle u, u_{x_n}\rangle\,\eta^q\right) dx < +\infty\,,\\
I_2 &:= \int_{B_R\cap\{|u|>a_k\}} f(x, \eta^q\,\Phi_k(|u|)\,Du)\,dx < +\infty\,,\\
I_3 &:= \int_{B_R\cap\{|u|>a_k\}} f(x, \Phi_k(|u|)\,u\,q\eta^{q-1}\,\eta_{x_1}, \dots, \Phi_k(|u|)\,u\,q\eta^{q-1}\,\eta_{x_n})\,dx < +\infty\,.
\end{aligned}$$

Of course, by the definitions of Φ_k, a_k, b_k and by (2.3), we have

$$\begin{aligned}
&\int_{B_R\cap\{|u|\notin(a_k,b_k)\}} f\left(x, \Phi_k'(|u|)\,\frac{u(x)}{|u(x)|}\,\langle u, u_{x_1}\rangle\,\eta^q, \dots, \Phi_k'(|u|)\,\frac{u(x)}{|u(x)|}\,\langle u, u_{x_n}\rangle\,\eta^q\right) dx\\
&\quad = \int_{B_R} f(x,0)\,dx \le \int_{B_R} \mu(x)\,dx < +\infty\,,\\
&\int_{B_R\cap\{|u|\le a_k\}} f(x, \eta^q\,\Phi_k(|u|)\,Du)\,dx = \int_{B_R} f(x,0)\,dx < +\infty\,,\\
&\int_{B_R\cap\{|u|\le a_k\}} f(x, \Phi_k(|u|)\,u\,q\eta^{q-1}\,\eta_{x_1}, \dots, \Phi_k(|u|)\,u\,q\eta^{q-1}\,\eta_{x_n})\,dx = \int_{B_R} f(x,0)\,dx < +\infty\,.
\end{aligned}$$

We now estimate I_1, I_2 and I_3 separately. Let us first consider I_1. Using Lemma 3.2 (i), (**A2**), the Cauchy–Schwarz inequality, Lemma 3.1 (v) and (**A3**), we find

$$\begin{aligned}
I_1 &\le \max\{1, \|\Phi_k'\|^\tau_{L^\infty(\mathbb{R}_+)}\} \int_{B_R\cap\{a_k<|u|<b_k\}} f\left(x, \frac{u}{|u|}\langle u, u_{x_1}\rangle, \dots, \frac{u}{|u|}\langle u, u_{x_n}\rangle\right) dx\\
&\le \max\{1, \|\Phi_k'\|^\tau_{L^\infty(\mathbb{R}_+)}\} \int_{B_R\cap\{a_k<|u|<b_k\}} g(x, |u|\,|u_{x_1}|, \dots, |u|\,|u_{x_n}|)\,dx\\
&\le \max\{1, \|\Phi_k'\|^\tau_{L^\infty(\mathbb{R}_+)}\} \int_{B_R\cap\{a_k<|u|<b_k\}} g(x, b_k\,|u_{x_1}|, \dots, b_k\,|u_{x_n}|)\,dx\\
&= \max\{1, \|\Phi_k'\|^\tau_{L^\infty(\mathbb{R}_+)}\} \int_{B_R\cap\{a_k<|u|<b_k\}} f(x, b_k\,Du)\,dx\\
&\le \max\{1, \|\Phi_k'\|^\tau_{L^\infty(\mathbb{R}_+)}\}\, b_k^\tau \int_{B_R} f(x, Du)\,dx
\end{aligned}$$

and the last integral is finite, since $u \in W^{1,\mathcal{F}}(\Omega;\mathbb{R}^m)$. To prove that I_2 is bounded, we apply Lemma 3.2 (i) and the definition of Φ_k, thus obtaining

$$I_2 = \int_{B_R\cap\{|u|>a_k\}} f(x, \eta^q\,\Phi_k(|u|)\,Du)\,dx \le k^\tau \int_{B_R} f(x, Du)\,dx < +\infty\,.$$

We now turn our attention to I_3. Using Lemma 3.2 (i), the definition of Φ_k, (**A2**), (5.1), Lemma 3.1 (v) and (2.3), we get

$$\begin{aligned}
I_3 &\le (qk)^\tau \int_{B_R} f(x, u\,\eta_{x_1}, \dots, u\,\eta_{x_n})\,dx\\
&= (qk)^\tau \int_{B_R} g(x, |u|\,|\eta_{x_1}|, \dots, |u|\,|\eta_{x_n}|)\,dx\\
&\le (qk)^\tau \int_{B_R} g\left(x, \frac{2|u|}{R-\rho}, \dots, \frac{2|u|}{R-\rho}\right) dx\\
&\le \left(\frac{2qk}{R-\rho}\right)^\tau \int_{B_R} \mu(x)\left\{1 + n^{\frac{q}{2}}\,|u|^q\right\} dx\,,
\end{aligned}$$

where we have also used $\frac{2}{R-\rho} > 2$. Since $|u| \in L^{\overline{\sigma}^*}_{loc}(\Omega)$ by Proposition 3.6 and $qs' < \overline{\sigma}^*$ by assumption (see Remark 2.3), the last integral in the preceding estimate is finite. Indeed, by Hölder's inequality,

$$\int_{B_R} \mu(x)|u|^q\,dx \le \|\mu\|_{L^s(B_{R_0})}\,\|u\|^q_{L^{qs'}(B_{R_0})} < +\infty\,.$$

Let us now consider the Euler's equation (4.1) with test function φ_k. We obtain

$$I_4 + I_5 := \sum_{j=1}^n \sum_{\alpha=1}^m \int_{B_R} \frac{\partial f}{\partial \xi_j^\alpha}(x, Du)\,u^\alpha_{x_j}\,\Phi_k(|u|)\,\eta^q\,dx$$

$$+\sum_{j=1}^{n}\sum_{\alpha,\beta=1}^{m}\int_{B_R}\frac{\partial f}{\partial\xi_j^\alpha}(x,Du)\,u^\alpha\,\frac{u^\beta}{|u|}\,u_{x_j}^\beta\,\Phi_k'(|u|)\,\eta^q\,dx$$

$$\le q\left|\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\int_{B_R}\frac{\partial f}{\partial\xi_j^\alpha}(x,Du)\,\Phi_k(|u|)\,u^\alpha\,\eta^{q-1}\eta_{x_j}\,dx\right| =: I_6\,. \tag{5.3}$$

At this stage, we estimate I_4, I_5 and I_6 separately.

Estimate of I_4

As far as I_4 is concerned, we use the convexity of $f(x,\cdot)$, (2.3) and (5.1). Thus,

$$I_4 \ge \int_{B_R}[f(x,Du)-f(x,0)]\,\Phi_k(|u|)\,\eta^q\,dx$$
$$\ge \int_{B_R} f(x,Du)\,\Phi_k(|u|)\,\eta^q\,dx - \int_{B_R}\mu(x)\,\Phi_k(|u|)\,dx\,. \tag{5.4}$$

Estimate of I_5

We claim that $I_5 \ge 0$. Indeed, by (2.2) and the properties (i) and (v) in Lemma 3.1, we get

$$\sum_{j=1}^{n}\sum_{\alpha,\beta=1}^{m}\frac{\partial f}{\partial\xi_j^\alpha}(x,Du)\,u^\alpha\,u^\beta\,u_{x_j}^\beta = \sum_{j=1}^{n}\frac{\partial g}{\partial t_j}(x,|u_{x_1}|,\dots,|u_{x_n}|)\,\frac{\left(\sum_{\alpha=1}^m u^\alpha\,u_{x_j}^\alpha\right)^2}{|u_{x_j}|}\ge 0\,.$$

Thus, by the monotonicity of Φ_k we have

$$I_5 = \int_{B_R}\sum_{j=1}^{n}\frac{\partial g}{\partial t_j}(x,|u_{x_1}|,\dots,|u_{x_n}|)\,\frac{\left(\sum_{\alpha=1}^m u^\alpha\,u_{x_j}^\alpha\right)^2}{|u_{x_j}|\,|u|}\,\Phi_k'(|u|)\,\eta^q\,dx\ \ge 0\,. \tag{5.5}$$

Estimate of I_6

Using again (2.2), the properties (i) and (v) in Lemma 3.1 and (5.1), we have

$$I_6 = q\left|\sum_{j=1}^{n}\int_{B_R}\frac{\partial g}{\partial t_j}(x,|u_{x_1}|,\dots,|u_{x_n}|)\,\frac{\langle u,u_{x_j}\rangle}{|u_{x_j}|}\,\Phi_k(|u|)\,\eta^{q-1}\,\eta_{x_j}\,dx\right|$$
$$\le q\sum_{j=1}^{n}\int_{B_R}\frac{\partial g}{\partial t_j}(x,|u_{x_1}|,\dots,|u_{x_n}|)\,|u|\,\Phi_k(|u|)\,\eta^{q-1}\,|D\eta|\,dx$$
$$\le \frac{2q}{R-\rho}\sum_{j=1}^{n}\int_{A_{R,j}^-\cup A_{R,j}^+}\frac{\partial g}{\partial t_j}(x,|u_{x_1}|,\dots,|u_{x_n}|)\,|u|\,\Phi_k(|u|)\,\eta^{q-1}\,dx\,, \tag{5.6}$$

where

$$A_{R,j}^- := B_R\cap\left\{\eta\ne 0,\ |u_{x_j}|\le\frac{2qL|u|}{\eta(R-\rho)}\right\}$$

and

$$A_{R,j}^+ := B_R\cap\left\{\eta\ne 0,\ |u_{x_j}|>\frac{2qL|u|}{\eta(R-\rho)}\right\}$$

with $L>0$ to be chosen later.

For a.e. $x\in A_{R,j}^-$ define $H_j(x,\cdot):\mathbb{R}_+\to\mathbb{R}_+$,

$$H_j(x,\varrho) := g(x,|u_{x_1}(x)|,\dots,|u_{x_{j-1}}(x)|,\varrho,|u_{x_{j+1}}(x)|,\dots,|u_{x_n}(x)|)\,,$$

of class C^1 w.r.t. ϱ due to Lemma 3.1 (i). By the properties (i) and (iv) in Lemma 3.1 and by the assumption $x\in A_{R,j}^-$, the following inequality holds:

$$\frac{\partial g}{\partial t_j}(x,|u_{x_1}|,\dots,|u_{x_n}|)\,\frac{2q|u|}{\eta(R-\rho)}\ \le\ \frac{1}{L}\,\frac{\partial H_j}{\partial\varrho}\left(x,\frac{2qL|u|}{\eta(R-\rho)}\right)\frac{2qL|u|}{\eta(R-\rho)}\,. \tag{5.7}$$

Now, let $\omega>1$ and denote by e_1 the vector $(1,0,\dots,0)$ in $\mathbb{R}^m$. By the definition of H_j, (2.2), (**A3**) and Lemma 3.1 (v), for a.e. $x\in A_{R,j}^-$ and for every $\varrho\in\mathbb{R}_+$ we have

$$H_j(x,\omega\varrho) = f\left(x,\omega\frac{u_{x_1}(x)}{\omega},\dots,\omega\frac{u_{x_{j-1}}(x)}{\omega},\omega\varrho\,\mathsf{e}_1^T,\omega\frac{u_{x_{j+1}}(x)}{\omega},\dots,\omega\frac{u_{x_n}(x)}{\omega}\right)$$
$$\le \omega^\tau f\left(x,\frac{u_{x_1}(x)}{\omega},\dots,\frac{u_{x_{j-1}}(x)}{\omega},\varrho\,\mathsf{e}_1^T,\frac{u_{x_{j+1}}(x)}{\omega},\dots,\frac{u_{x_n}(x)}{\omega}\right)$$
$$= \omega^\tau g\left(x,\frac{|u_{x_1}(x)|}{\omega},\dots,\frac{|u_{x_{j-1}}(x)|}{\omega},\varrho,\frac{|u_{x_{j+1}}(x)|}{\omega},\dots,\frac{|u_{x_n}(x)|}{\omega}\right)$$

$$\le \omega^\tau g(x, |u_{x_1}(x)|, \dots, |u_{x_{j-1}}(x)|, \varrho, |u_{x_{j+1}}(x)|, \dots, |u_{x_n}(x)|) = \omega^\tau H_j(x, \varrho).$$

Therefore, we can apply Lemma 3.3 with $h = H_j(x, \cdot)$, thus obtaining

$$\frac{1}{L}\frac{\partial H_j}{\partial \varrho}\left(x, \frac{2qL|u|}{\eta(R-\rho)}\right)\frac{2qL|u|}{\eta(R-\rho)} \le \frac{\tau}{L} H_j\left(x, \frac{2qL|u|}{\eta(R-\rho)}\right). \tag{5.8}$$

Using again the definition of H_j, Lemma 3.1 (v) and (2.2), we get

$$\begin{aligned} H_j\left(x, \frac{2qL|u|}{\eta(R-\rho)}\right) &\le g\left(x, |u_{x_1}|, \dots, |u_{x_{j-1}}|, |u_{x_j}| + \frac{2qL|u|}{\eta(R-\rho)}, |u_{x_{j+1}}|, \dots, |u_{x_n}|\right) \\ &= f\left(x, u_{x_1}, \dots, u_{x_{j-1}}, \left(|u_{x_j}| + \frac{2qL|u|}{\eta(R-\rho)}\right)\mathrm{e}_1^T, u_{x_{j+1}}, \dots, u_{x_n}\right). \end{aligned} \tag{5.9}$$

Since

$$\begin{aligned} &\left(u_{x_1}, \dots, u_{x_{j-1}}, \left(|u_{x_j}| + \frac{2qL|u|}{\eta(R-\rho)}\right)\mathrm{e}_1^T, u_{x_{j+1}}, \dots, u_{x_n}\right) \\ &= \frac{1}{2}\left(2u_{x_1}, \dots, 2u_{x_{j-1}}, 2|u_{x_j}|\,\mathrm{e}_1^T, 2u_{x_{j+1}}, \dots, 2u_{x_n}\right) \\ &\quad + \frac{1}{2}\left(\underbrace{0^T}_{\in\mathbb{R}^m}, \dots, \underbrace{0^T}_{\in\mathbb{R}^m}, \frac{4qL|u|}{\eta(R-\rho)}\mathrm{e}_1^T, \underbrace{0^T}_{\in\mathbb{R}^m}, \dots, \underbrace{0^T}_{\in\mathbb{R}^m}\right) =: \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}, \end{aligned}$$

by the convexity of $f(x, \cdot)$ we obtain

$$f\left(x, u_{x_1}, \dots, u_{x_{j-1}}, \left(|u_{x_j}| + \frac{2qL|u|}{\eta(R-\rho)}\right)\mathrm{e}_1^T, u_{x_{j+1}}, \dots, u_{x_n}\right) \le \frac{1}{2} f(x, \mathbf{v}) + \frac{1}{2} f(x, \mathbf{w}). \tag{5.10}$$

Of course, using (2.2) and (**A3**) we have

$$f(x, \mathbf{v}) = g(x, 2|u_{x_1}|, \dots, 2|u_{x_j}|, \dots, 2|u_{x_n}|) = f(x, 2\,Du(x)) \le 2^\tau f(x, Du(x)). \tag{5.11}$$

Let us deal with $f(x, \mathbf{w})$. By (2.2), Lemma 3.1 (v) and (**A4**), we deduce

$$\begin{aligned} f(x, \mathbf{w}) &= g\left(x, \underbrace{0, \dots, 0}_{j-1}, \frac{4qL|u|}{\eta(R-\rho)}, \underbrace{0, \dots, 0}_{n-j}\right) \le g\left(x, \frac{4qL|u|}{\eta(R-\rho)}, \dots, \frac{4qL|u|}{\eta(R-\rho)}\right) \\ &\le \mu(x)\left\{1 + n^{\frac{q}{2}}\left[\frac{4qL|u|}{\eta(R-\rho)}\right]^q\right\} \le (4q\sqrt{n})^q\,\mu(x)\left\{1 + \left[\frac{L|u|}{\eta(R-\rho)}\right]^q\right\}. \end{aligned} \tag{5.12}$$

Without loss of generality, we can assume $L \ge 1$ so that $\frac{L}{R-\rho} > 1$. Therefore, collecting (5.7)–(5.12) and using $0 \le \eta \le 1$, we get

$$\begin{aligned} &\frac{2q}{R-\rho}\sum_{j=1}^n \int_{A_{R,j}^-} \frac{\partial g}{\partial t_j}(x, |u_{x_1}|, \dots, |u_{x_n}|)\,|u|\,\Phi_k(|u|)\,\eta^{q-1}\,dx \\ &\le \frac{2^{\tau-1}\tau n}{L}\int_{B_R} f(x, Du)\,\Phi_k(|u|)\,\eta^q\,dx + \frac{(4q\sqrt{n})^q\,\tau n L^{q-1}}{2\,(R-\rho)^q}\int_{B_R}\mu(x)\left\{1 + |u|^q\right\}\Phi_k(|u|)\,dx. \end{aligned} \tag{5.13}$$

Let us now deal with $A_{R,j}^+$. By the properties (v), (i) and (iv) in Lemma 3.1 we have

$$\begin{aligned} &g(x, 2|u_{x_1}|, \dots, 2|u_{x_j}|, \dots, 2|u_{x_n}|) \\ &\quad\ge g(x, |u_{x_1}|, \dots, 2|u_{x_j}|, \dots, |u_{x_n}|) \\ &\quad\ge g(x, |u_{x_1}|, \dots, |u_{x_j}|, \dots, |u_{x_n}|) + \frac{\partial g}{\partial t_j}(x, |u_{x_1}|, \dots, |u_{x_j}|, \dots, |u_{x_n}|)\,|u_{x_j}| \\ &\quad\ge \frac{\partial g}{\partial t_j}(x, |u_{x_1}|, \dots, |u_{x_j}|, \dots, |u_{x_n}|)\,|u_{x_j}|. \end{aligned}$$

Using the above inequality together with (2.2) and (**A3**), for a.e. $x \in A_{R,j}^+$ we obtain

$$\begin{aligned} \frac{\partial g}{\partial t_j}(x, |u_{x_1}|, \dots, |u_{x_n}|)\,\frac{2q|u|}{\eta(R-\rho)} &\le \frac{1}{L}\frac{\partial g}{\partial t_j}(x, |u_{x_1}|, \dots, |u_{x_n}|)\,|u_{x_j}| \\ &\le \frac{1}{L} g(x, 2|u_{x_1}|, \dots, 2|u_{x_j}|, \dots, 2|u_{x_n}|) \\ &= \frac{1}{L} f(x, 2\,Du(x)) \le \frac{2^\tau}{L} f(x, Du(x)). \end{aligned}$$

Thus

$$\frac{2q}{R-\rho}\sum_{j=1}^n \int_{A_{R,j}^+} \frac{\partial g}{\partial t_j}(x, |u_{x_1}|, \dots, |u_{x_n}|)\,|u|\,\Phi_k(|u|)\,\eta^{q-1}\,dx \le \frac{2^\tau n}{L}\int_{B_R} f(x, Du)\,\Phi_k(|u|)\,\eta^q\,dx, \tag{5.14}$$

and joining estimates (5.6), (5.13) and (5.14), we get

$$I_6 \le \frac{2^\tau n(\tau+1)}{L} \int_{B_R} f(x,Du)\,\Phi_k(|u|)\,\eta^q\,dx + \frac{(4q\sqrt{n})^q\,\tau n L^{q-1}}{2\,(R-\rho)^q} \int_{B_R} \mu(x)\left\{1+|u|^q\right\}\Phi_k(|u|)\,dx\,. \tag{5.15}$$

Collecting (5.3), (5.4), (5.5) and (5.15), and using $\frac{1}{R-\rho} > 1$, we find

$$\int_{B_R} f(x,Du)\,\Phi_k(|u|)\,\eta^q\,dx \le \frac{2^\tau n(\tau+1)}{L} \int_{B_R} f(x,Du)\,\Phi_k(|u|)\,\eta^q\,dx + \frac{(4q\sqrt{n})^q\,\tau n L^{q-1}+1}{(R-\rho)^q} \int_{B_R} \mu(x)\left\{1+|u|^q\right\}\Phi_k(|u|)\,dx\,. \tag{5.16}$$

Now we choose $L = 2^{\tau+1}\,n(\tau+1) > 1$ and reabsorb the first integral in the right-hand side of (5.16) by the left-hand side, thus obtaining

$$\int_{B_R} f(x,Du)\,\Phi_k(|u|)\,\eta^q\,dx \le \frac{c_0}{(R-\rho)^q} \int_{B_R} \mu(x)\left\{1+|u|^q\right\}\Phi_k(|u|)\,dx\,, \tag{5.17}$$

where c_0 is a positive constant depending only on n, q and τ. Inequalities (2.3) and (5.17) imply

$$\int_{B_R} \lambda_i(x)\,|u_{x_i}|^{p_i}\,\Phi_k(|u|)\,\eta^q\,dx \le \frac{c_0}{(R-\rho)^q} \int_{B_R} \mu(x)\left\{1+|u|^q\right\}\Phi_k(|u|)\,dx\,.$$

At this point, we recall that $\Phi_k = \Phi_k^{(i,\gamma)}$. Using the monotone convergence theorem, we let $k \to +\infty$ and, by definition of Φ_k, we get

$$\begin{aligned}\int_{B_R} \lambda_i(x)\,|u|^{p_i\gamma}\,|u_{x_i}|^{p_i}\,\eta^q\,dx &\le \frac{c_0}{(R-\rho)^q} \int_{B_R} \mu(x)\left\{1+|u|^q\right\}|u|^{p_i\gamma}\,dx \\ &\le \frac{2c_0}{(R-\rho)^q} \int_{B_R} \mu(x)\,(\max\{|u|,1\})^{q+p_i\gamma}\,dx\,,\end{aligned} \tag{5.18}$$

where, in the last line, we have used

$$|u|^{q+p_i\gamma} + |u|^{p_i\gamma} \le 2\,(\max\{|u|,1\})^{q+p_i\gamma}\,.$$

In a similar way to what has been done up to now, it is easy to prove that $\varphi := u\eta^q$ is a test function for the Euler's equation (4.1). Moreover, using this test function and arguing exactly as above, we obtain

$$\int_{B_R} \lambda_i(x)\,|u_{x_i}|^{p_i}\,\eta^q\,dx \le \frac{2c_0}{(R-\rho)^q} \int_{B_R} \mu(x)\,(\max\{|u|,1\})^q\,dx\,,$$

so that (5.18) also holds for $\gamma = 0$.

Therefore, for every $\gamma \ge 0$ and every $i \in \{1,\dots,n\}$ we have

$$\int_{B_R} \lambda_i(x)\,|u|^{p_i\gamma}\,|u_{x_i}|^{p_i}\,\eta^q\,dx \le \frac{c}{(R-\rho)^q} \int_{B_R} \mu(x)\,(\max\{|u|,1\})^{q+p_i\gamma}\,dx\,,$$

where $c = c(n,q,\tau) > 0$. This concludes the proof. □

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. Let $u \in W^{1,\mathcal{F}}(\Omega;\mathbb{R}^m)$ be a local minimizer of (2.1) and consider $x_0 \in \Omega$ and $R_0 \in (0,1]$ such that $B_{R_0} := B_{R_0}(x_0) \Subset \Omega$. Also fix $0 < \rho < R < R_0$ and consider a cut-off function $\eta \in C_c^\infty(B_R(x_0))$ satisfying (5.1). To shorten our notation, we now set

$$G(x) := \max\{|u(x)|,1\}\,. \tag{5.19}$$

We split the proof into three steps.

Step 1. First we prove that, if $\delta \ge 1$ and $|u|^\delta \in W^{1,(\sigma_1,\dots,\sigma_n)}(B_{R_0})$, then

$$\|G^\delta\|_{L^{\overline{\sigma}^*}(B_\rho)} \le \frac{C\,\delta}{(R-\rho)^{\frac{q}{\overline{\mathbf{p}}}}}\,\|\mu\|_{L^s(B_R)}^{\frac{1}{\overline{\mathbf{p}}}}\,\|G\|_{L^{qs'}(B_R)}^{\frac{q}{\overline{\mathbf{p}}}-1}\,\|G^\delta\|_{L^{qs'}(B_R)} \prod_{i=1}^n \|\lambda_i^{-1}\|_{L^{r_i}(B_R)}^{\frac{1}{np_i}} \tag{5.20}$$

for a constant $C > 1$ depending on $m, n, \mathbf{p}, q, \mathbf{r}, \tau$ if $\overline{\sigma} < n$, and also on R_0 if $\overline{\sigma} \ge n$.

To prove the above inequality, we notice that for every $\gamma := \delta - 1 \ge 0$ and every $i \in \{1,\dots,n\}$ we have

$$\int_{B_R} \left|\left[(|u|^{\gamma+1}+1)\,\eta^q\right]_{x_i}\right|^{p_i} \lambda_i(x)\,dx \le 2^{p_i-1}\int_{B_R} (q\eta^{q-1}|D\eta|)^{p_i}\,(|u|^{\gamma+1}+1)^{p_i}\,\lambda_i(x)\,dx + 2^{p_i-1}\,(\gamma+1)^{p_i}\int_{B_R} \lambda_i(x)\,\eta^{qp_i}\,|u|^{\gamma p_i}\,|u_{x_i}|^{p_i}\,dx$$

$$=: J_{1,i} + J_{2,i}\,. \tag{5.21}$$

To estimate $J_{1,i}$, we observe that

$$J_{1,i} \le \frac{2^{2q-1}q^q}{(R-\rho)^{p_i}} \int_{B_R} (|u|^{\gamma+1}+1)^{p_i}\,\lambda_i(x)\,dx \le \frac{2^{3q-1}q^q}{(R-\rho)^q} \int_{B_R} G^{q+\gamma p_i}\,\lambda_i(x)\,dx\,,$$

where we have used $p_i \le q$, $\frac{1}{R-\rho} > 1$ and

$$(|u|^{\gamma+1}+1)^{p_i} \le (2\,G^{\gamma+1})^{p_i} \le 2^q\,G^{q+\gamma p_i}\,.$$

From (2.3) it easily follows that

$$\lambda_i \le 2\,n^{\frac{q}{2}}\,\mu \qquad \text{a.e. in } \Omega$$

for every $i \in \{1,\dots,n\}$. Therefore,

$$J_{1,i} \le \frac{2^{3q}q^q n^{\frac{q}{2}}}{(R-\rho)^q} \int_{B_R} \mu(x)\,G^{q+\gamma p_i}\,dx\,. \tag{5.22}$$

Since $\eta^{qp_i} \le \eta^q$, we can estimate $J_{2,i}$ using inequality (5.2). Thus,

$$\begin{aligned} J_{2,i} &\le 2^{q-1}\,(\gamma+1)^{p_i} \int_{B_R} \lambda_i(x)\,\eta^q\,|u|^{\gamma p_i}\,|u_{x_i}|^{p_i}\,dx \\ &\le C_0\,\frac{(\gamma+1)^{p_i}}{(R-\rho)^q} \int_{B_R} \mu(x)\,G^{q+\gamma p_i}\,dx\,, \end{aligned} \tag{5.23}$$

where $C_0 = C_0(n,q,\tau) > 0$. By Proposition 3.7 applied to $v = (|u|^{\gamma+1}+1)\,\eta^q$, we obtain

$$\left(\int_{B_R} [(|u|^{\gamma+1}+1)\,\eta^q]^{\overline{\sigma}^*}\,dx\right)^{\frac{n}{\overline{\sigma}^*}} \le C_1 \prod_{i=1}^n \|\lambda_i^{-1}\|_{L^{r_i}(B_R)}^{\frac{1}{p_i}} \left(\int_{B_R} \left|\left[(|u|^{\gamma+1}+1)\,\eta^q\right]_{x_i}\right|^{p_i} \lambda_i(x)\,dx\right)^{\frac{1}{p_i}}$$

for a positive constant C_1 depending on $m, n, \mathbf{p}, \mathbf{r}$ if $\overline{\sigma} < n$, and also on R_0 if $\overline{\sigma} \ge n$. Collecting this inequality, (5.21), (5.22) and (5.23), we get

$$\left(\int_{B_R} [(|u|^{\gamma+1}+1)\,\eta^q]^{\overline{\sigma}^*}\,dx\right)^{\frac{n}{\overline{\sigma}^*}} \le C_2 \prod_{i=1}^n \|\lambda_i^{-1}\|_{L^{r_i}(B_R)}^{\frac{1}{p_i}} \left[\frac{1+(\gamma+1)^{p_i}}{(R-\rho)^q} \int_{B_R} \mu(x)\,G^{q+\gamma p_i}\,dx\right]^{\frac{1}{p_i}},$$

where C_2 is a positive constant depending on $m, n, \mathbf{p}, q, \mathbf{r}, \tau$ if $\overline{\sigma} < n$, and additionally on R_0 if $\overline{\sigma} \ge n$. Using (5.1) and (2.6), we then find

$$\begin{aligned} &\left(\int_{B_\rho} (|u|^{\gamma+1}+1)^{\overline{\sigma}^*}\,dx\right)^{\frac{n}{\overline{\sigma}^*}} \\ &\le C_2 \prod_{i=1}^n \frac{\gamma+1}{(R-\rho)^{\frac{q}{p_i}}}\,\|\lambda_i^{-1}\|_{L^{r_i}(B_R)}^{\frac{1}{p_i}} \left(\int_{B_R} \mu(x)\,G^{q+\gamma p_i}\,dx\right)^{\frac{1}{p_i}} \\ &\le \frac{C_2\,(\gamma+1)^n}{(R-\rho)^{\frac{nq}{\overline{\mathbf{p}}}}} \prod_{i=1}^n \|\lambda_i^{-1}\|_{L^{r_i}(B_R)}^{\frac{1}{p_i}} \left(\int_{B_R} \mu(x)\,G^{q+\gamma p_i}\,dx\right)^{\frac{1}{p_i}}, \end{aligned} \tag{5.24}$$

for a different constant $C_2 > 0$. By Hölder's inequality and $\mu \in L^s_{loc}(\Omega)$, we obtain

$$\left(\int_{B_R} \mu(x)\,G^{q+\gamma p_i}\,dx\right)^{\frac{1}{p_i}} \le \|\mu\|_{L^s(B_R)}^{\frac{1}{p_i}} \left(\int_{B_R} G^{(q+\gamma p_i)s'}\,dx\right)^{\frac{1}{p_i s'}}. \tag{5.25}$$

If $p_i = q$ for some $i \in \{1,\dots,n\}$, the above estimate gives

$$\left(\int_{B_R} \mu(x)\,G^{(\gamma+1)q}\,dx\right)^{\frac{1}{q}} \le \|\mu\|_{L^s(B_R)}^{\frac{1}{q}}\,\|G^{\gamma+1}\|_{L^{qs'}(B_R)}\,.$$

If $p_i < q$ for some $i \in \{1,\dots,n\}$, we apply Hölder's inequality again to the last integral in (5.25). We thus obtain

$$\left(\int_{B_R} G^{(q+\gamma p_i)s'}\,dx\right)^{\frac{1}{p_i s'}} = \left(\int_{B_R} G^{(q-p_i)s'}\,G^{(\gamma+1)p_i s'}\,dx\right)^{\frac{1}{p_i s'}} \le \|G\|_{L^{qs'}(B_R)}^{\frac{q}{p_i}-1}\,\|G^{\gamma+1}\|_{L^{qs'}(B_R)}\,. \tag{5.26}$$

Then, up to redefine the constant $C_2 > 0$, estimates (5.24), (5.25) and (5.26) give

$$\left\||u|^{\gamma+1}+1\right\|_{L^{\overline{\sigma}^*}(B_\rho)} \le \frac{C_2\,(\gamma+1)}{(R-\rho)^{\frac{q}{\overline{\mathbf{p}}}}}\,\|\mu\|_{L^s(B_R)}^{\frac{1}{\overline{\mathbf{p}}}}\,\|G\|_{L^{qs'}(B_R)}^{\frac{q}{\overline{\mathbf{p}}}-1}\,\|G^{\gamma+1}\|_{L^{qs'}(B_R)} \prod_{i=1}^n \|\lambda_i^{-1}\|_{L^{r_i}(B_R)}^{\frac{1}{np_i}},$$

where we have used (2.6) again. Since $\delta = \gamma+1$ and $G^\delta \le |u|^\delta + 1$, inequality (5.20) holds true.

Step 2. Now we prove the local boundedness of G, and then of u, using the Moser iteration technique. To simplify our notation even more, we introduce the function

$$A(r) := \|\mu\|_{L^s(B_r)}^{\frac{1}{\mathbf{p}}} \|G\|_{L^{qs'}(B_r)}^{\frac{q}{\mathbf{p}}-1} \prod_{i=1}^n \|\lambda_i^{-1}\|_{L^{r_i}(B_r)}^{\frac{1}{np_i}}, \qquad r \in (0, R_0]. \tag{5.27}$$

Observe that $A(r)$ is non-decreasing, since

$$1 < p_i \le q \quad \forall\, i \in \{1, \dots, n\} \qquad \text{implies} \qquad \frac{q}{\mathbf{p}} - 1 \ge 0.$$

Using (5.27), inequality (5.20) reads as follows:

$$\|G\|_{L^{\delta\overline{\sigma}^*}(B_\rho)} \le \left\{ \frac{C\,\delta\,A(R)}{(R-\rho)^{\frac{q}{\mathbf{p}}}} \right\}^{\frac{1}{\delta}} \|G\|_{L^{\delta qs'}(B_R)}, \tag{5.28}$$

where $C > 1$. For all $h \in \mathbb{N}$, we define

$$\delta_h := \left(\frac{\overline{\sigma}^*}{qs'} \right)^{h-1}, \qquad R_h := \frac{R}{2}\left(1 + \frac{1}{2^{h-1}}\right), \qquad \rho_h := R_{h+1}. \tag{5.29}$$

Notice that δ_h has been chosen in such a way that $\delta_1 = 1$ and $\delta_h\, \overline{\sigma}^* = \delta_{h+1}\, qs'$. Moreover, δ_h diverges to $+\infty$ as $h \to \infty$, since $qs' < \overline{\sigma}^*$ by assumption. By (5.28), replacing δ, R and ρ with δ_h, R_h and ρ_h, respectively, we have that

$$G \in L^{\delta_h qs'}(B_{R_h}) \quad \text{implies} \quad G \in L^{\delta_{h+1} qs'}(B_{R_{h+1}}).$$

More precisely, the inequality

$$\|G\|_{L^{\delta_{h+1} qs'}(B_{R_{h+1}})} \le \left\{ \frac{C\,\delta_h\,A(R_h)}{(R_h-\rho_h)^{\frac{q}{\mathbf{p}}}} \right\}^{\frac{1}{\delta_h}} \|G\|_{L^{\delta_h qs'}(B_{R_h})}$$

holds true for every $h \in \mathbb{N}$. Since $A(R_h) \le A(R)$ for all $h \in \mathbb{N}$, we obtain

$$\|G\|_{L^{\delta_{h+1} qs'}(B_{R_{h+1}})} \le \left\{ \frac{C\,\delta_h\,A(R)}{(R_h-\rho_h)^{\frac{q}{\mathbf{p}}}} \right\}^{\frac{1}{\delta_h}} \|G\|_{L^{\delta_h qs'}(B_{R_h})} \tag{5.30}$$

for every $h \in \mathbb{N}$. In particular, if $h = 1$ we get

$$\|G\|_{L^{\overline{\sigma}^*}(B_{3R/4})} \le \frac{4^{\frac{q}{\mathbf{p}}}\,C}{R^{\frac{q}{\mathbf{p}}}} \|\mu\|_{L^s(B_R)}^{\frac{1}{\mathbf{p}}} \|G\|_{L^{qs'}(B_R)}^{\frac{q}{\mathbf{p}}} \prod_{i=1}^n \|\lambda_i^{-1}\|_{L^{r_i}(B_R)}^{\frac{1}{np_i}}. \tag{5.31}$$

Notice that the right-hand side of (5.31) is finite, because $|u| \in L^{\overline{\sigma}^*}(B_{R_0})$ by Proposition 3.6 and $qs' < \overline{\sigma}^*$ by assumption. Now, we define

$$M_h := \left(\int_{B_{R_h}} G^{\delta_h qs'}\,dx \right)^{\frac{1}{\delta_h qs'}}, \qquad h \in \mathbb{N}, \tag{5.32}$$

so that inequality (5.30) turns into

$$M_{h+1} \le \left\{ \frac{C\,\delta_h\,A(R)\,(2^{\frac{q}{\mathbf{p}}})^{h+1}}{R^{\frac{q}{\mathbf{p}}}} \right\}^{\frac{1}{\delta_h}} M_h. \tag{5.33}$$

Iterating the above estimate, we get

$$M_{h+1} \le M_1 \prod_{k=1}^h \left\{ \frac{C\,\delta_k\,A(R)\,(2^{\frac{q}{\mathbf{p}}})^{k+1}}{R^{\frac{q}{\mathbf{p}}}} \right\}^{\frac{1}{\delta_k}} \tag{5.34}$$

for any $h \in \mathbb{N}$. Now we observe that

$$\sum_{k=1}^\infty \frac{1}{\delta_k} = \sum_{k=0}^\infty \left(\frac{qs'}{\overline{\sigma}^*} \right)^k = \frac{\overline{\sigma}^*}{\overline{\sigma}^* - qs'},$$

so that

$$\prod_{k=1}^h \left(\frac{C\,A(R)}{R^{\frac{q}{\mathbf{p}}}} \right)^{\frac{1}{\delta_k}} = \left(\frac{C\,A(R)}{R^{\frac{q}{\mathbf{p}}}} \right)^{\sum_{k=1}^h \frac{1}{\delta_k}} < \left(\frac{C\,(1+A(R))}{R^{\frac{q}{\mathbf{p}}}} \right)^{\sum_{k=1}^h \frac{1}{\delta_k}} < \left(\frac{C\,(1+A(R))}{R^{\frac{q}{\mathbf{p}}}} \right)^{\frac{\overline{\sigma}^*}{\overline{\sigma}^* - qs'}}, \tag{5.35}$$

where we have also used $C > 1$ and $R \le 1$. Moreover, by the definition of δ_h in (5.29), we have

$$\prod_{k=1}^{h} \delta_k^{\frac{1}{\delta_k}} = \left(\frac{\overline{\sigma}^*}{qs'}\right)^{\sum_{k=1}^{h} \frac{k-1}{\delta_k}} < \left(\frac{\overline{\sigma}^*}{qs'}\right)^{\sum_{k=1}^{\infty} k\left(\frac{qs'}{\overline{\sigma}^*}\right)^k} = \left(\frac{\overline{\sigma}^*}{qs'}\right)^{\frac{\overline{\sigma}^* qs'}{(\overline{\sigma}^*-qs')^2}} \tag{5.36}$$

and

$$\prod_{k=1}^{h} (2^{\frac{q}{\mathbf{p}}})^{\frac{k+1}{\delta_k}} = 2^{\frac{q}{\mathbf{p}}\left(\frac{\overline{\sigma}^*}{qs'}\right)^2 \sum_{k=2}^{h+1} k\left(\frac{qs'}{\overline{\sigma}^*}\right)^k} < 2^{\frac{q}{\mathbf{p}}\left(\frac{\overline{\sigma}^*}{qs'}\right)^2 \sum_{k=1}^{\infty} k\left(\frac{qs'}{\overline{\sigma}^*}\right)^k} = 2^{\frac{(\overline{\sigma}^*)^3}{\mathbf{p}s'(\overline{\sigma}^*-qs')^2}}. \tag{5.37}$$

Combining estimates (5.34)–(5.37), recalling the definitions (5.27) and (5.32), and letting $h \to +\infty$, we obtain

$$\|G\|_{L^\infty(B_{R/2})} \le c_1 \left[\frac{1}{R^{\frac{q}{\mathbf{p}}}}\left(1 + \|\mu\|_{L^s(B_R)}^{\frac{1}{\mathbf{p}}} \|G\|_{L^{qs'}(B_R)}^{\frac{q}{\mathbf{p}}-1} \prod_{i=1}^{n} \|\lambda_i^{-1}\|_{L^{r_i}(B_R)}^{\frac{1}{np_i}}\right)\right]^{\frac{\overline{\sigma}^*}{\overline{\sigma}^*-qs'}} \left[\fint_{B_R} G^{qs'}\,dx\right]^{\frac{1}{qs'}},$$

where $c_1 > 1$ is a constant depending on $m, n, \mathbf{p}, q, \mathbf{r}, s, \tau$ if $\overline{\sigma} < n$, and also on R_0 if $\overline{\sigma} \ge n$. Now, using the fact that $G \ge 1$, $\frac{q}{\mathbf{p}} - 1 \ge 0$ and $R \le 1$, we deduce

$$1 \le \left[\fint_{B_R} G^{qs'}\,dx\right]^{\frac{1}{qs'}\left(\frac{q}{\mathbf{p}}-1\right)} = |B_R|^{\frac{1}{qs'}\left(1-\frac{q}{\mathbf{p}}\right)} \|G\|_{L^{qs'}(B_R)}^{\frac{q}{\mathbf{p}}-1} = \frac{\omega_n^{\frac{1}{qs'}\left(1-\frac{q}{\mathbf{p}}\right)}}{R^{\frac{n}{qs'}\left(\frac{q}{\mathbf{p}}-1\right)}} \|G\|_{L^{qs'}(B_R)}^{\frac{q}{\mathbf{p}}-1}$$

and

$$\|G\|_{L^{qs'}(B_R)}^{\frac{q}{\mathbf{p}}-1} \le \frac{1}{R^{\frac{n}{qs'}\left(\frac{q}{\mathbf{p}}-1\right)}} \|G\|_{L^{qs'}(B_R)}^{\frac{q}{\mathbf{p}}-1},$$

where ω_n denotes the n-dimensional Lebesgue measure of the unit ball $B_1(0)$. Joining the three previous estimates and recalling the definition of G in (5.19), we get

$$\|u\|_{L^\infty(B_{R/2})} \le \frac{c_1}{R^{\vartheta_3}} \left[1 + \|\mu\|_{L^s(B_R)}^{\frac{1}{\mathbf{p}}} \prod_{i=1}^{n} \|\lambda_i^{-1}\|_{L^{r_i}(B_R)}^{\frac{1}{np_i}}\right]^{\vartheta_1} \left\||u| + 1\right\|_{L^{qs'}(B_R)}^{\vartheta_2},$$

for a different constant $c_1 > 1$ and

$$\vartheta_1 := \frac{\overline{\sigma}^*}{\overline{\sigma}^* - qs'},$$

$$\vartheta_2 := \vartheta_1\left(\frac{q}{\mathbf{p}} - 1\right) + 1 = \frac{q\,(\overline{\sigma}^* - \mathbf{p}s')}{\mathbf{p}\,(\overline{\sigma}^* - qs')},$$

$$\vartheta_3 := \vartheta_1\left[\frac{q}{\mathbf{p}} + \frac{n}{qs'}\left(\frac{q}{\mathbf{p}} - 1\right)\right] = \frac{\overline{\sigma}^*\,[q^2 s' + n\,(q - \mathbf{p})]}{\mathbf{p}qs'\,(\overline{\sigma}^* - qs')}.$$

We have thus proved that $u \in L^\infty(B_{R/2}(x_0); \mathbb{R}^m)$ and estimate (2.8).

Step 3. Here we prove estimate (2.9). Fix $B_R(x_0) \subset B_{R_0}(x_0) \Subset \Omega$ with $R_0 \in (0, 1]$. Notice that if $Q_\ell(x_0)$ denotes the cube with edges parallel to the coordinate axes, centered at x_0 and with side length 2ℓ, then $B_{R/\sqrt{n}}(x_0) \subseteq Q_{R/\sqrt{n}}(x_0) \subseteq B_R(x_0)$.

Define $u_R := \fint_{B_R(x_0)} u\,dx$. Since $u - u_R$ is a local minimizer too, by (2.8) and Hölder's inequality we have

$$\begin{aligned}&\|u - u_R\|_{L^\infty(B_{R/(2\sqrt{n})}(x_0))} \\ &\quad\le \frac{c}{R^{\vartheta_3}} \left[1 + \|\mu\|_{L^s(B_{R/\sqrt{n}})}^{\frac{1}{\mathbf{p}}} \prod_{i=1}^{n} \|\lambda_i^{-1}\|_{L^{r_i}(B_{R/\sqrt{n}})}^{\frac{1}{np_i}}\right]^{\vartheta_1} \left[1 + \|u - u_R\|_{L^{\overline{\sigma}^*}(B_{R/\sqrt{n}})}\right]^{\vartheta_2},\end{aligned} \tag{5.38}$$

where $c = c(m, n, \mathbf{p}, q, \mathbf{r}, s, \tau, R_0) > 0$. By Proposition 3.6 and Theorem 3.5, we get

$$\|u - u_R\|_{L^{\overline{\sigma}^*}(B_{R/\sqrt{n}})} \le \|u - u_R\|_{L^{\overline{\sigma}^*}(Q_{R/\sqrt{n}}(x_0))} \le c\left[\|u - u_R\|_{L^1(B_R)} + \sum_{i=1}^{n} \|u_{x_i}\|_{L^{\sigma_i}(B_R)}\right].$$

Now we apply the Poincaré inequality as in [22, page 185] (see also [28, page 84]). Thus we obtain

$$\|u - u_R\|_{L^1(B_R)} \le c\left[1 + \sum_{i=1}^{n} \|u_{x_i}\|_{L^1(B_R)}\right],$$

and combining the two previous estimates, we find

$$\|u - u_R\|_{L^{\overline{\sigma}^*}(B_{R/\sqrt{n}})} \le c\left[1 + \sum_{i=1}^{n} \|u_{x_i}\|_{L^{\sigma_i}(B_R)}\right]. \tag{5.39}$$

At this point, arguing as in the proof of Proposition 3.6, we deduce

$$\sum_{i=1}^{n} \|u_{x_i}\|_{L^{\sigma_i}(B_R)} \leq \sum_{i=1}^{n} \|\lambda_i^{-1}\|_{L^{r_i}(B_R)}^{\frac{1}{p_i}} \left[\int_{B_R} f(x, Du)\, dx\right]^{\frac{1}{p_i}}$$
$$\leq \left[1 + \int_{B_R} f(x, Du)\, dx\right]^{\frac{1}{p}} \sum_{i=1}^{n} \|\lambda_i^{-1}\|_{L^{r_i}(B_R)}^{\frac{1}{p_i}}, \tag{5.40}$$

where $p := \min_{1\leq i\leq n} \{p_i\}$. The final estimate (2.9) follows by collecting (5.38), (5.39) and (5.40). □

6. Examples of applicability

In this section, we provide two examples of integrals to which Theorem 2.2 is applicable. Before presenting these examples, we need to make some preliminary considerations.

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. For every $i \in \{1, \dots, n\}$, let $c_i \in (0, +\infty)$, $\kappa_i \in [0, +\infty)$, $p_i \in (1, +\infty)$, $\alpha_i \in [0, n/p_i)$ and

$$\lambda_i(x) := c_i\, |x|^{\alpha_i p_i}, \quad x \in \Omega. \tag{6.1}$$

For any fixed $i \in \{1, \dots, n\}$, $\lambda_i^{-1} \in L^{r_i}(\Omega)$ whenever

$$\begin{cases} r_i \in \left[1, \frac{n}{\alpha_i p_i}\right) & \text{if } \alpha_i > 0, \\ r_i \in [1, \infty] & \text{if } \alpha_i = 0. \end{cases} \tag{6.2}$$

Now let $q = \max\{p_i\}$, $C_0 = \max\{c_i\}$, $K_0 = \max\{\kappa_i\}$, $\beta \in [0, n)$ and assume that:

- $0 \in \overline{\Omega}$ if $K_0 = 0$ or $\beta = 0$;
- $0 \in \mathbb{R}^n \setminus \Omega$ (possibly $0 \in \partial\Omega$) if $K_0 > 0$ and $\beta > 0$.

We now consider the functions $\mu : \Omega \to [0, \infty)$ and $f : \Omega \times \mathbb{R}^{m\times n} \to [0, \infty)$, $m \in \mathbb{N}$, defined respectively by

$$\mu(x) := 2^{q-1}\, n \left\{ C_0 \max_{1\leq j\leq n} |x|^{\alpha_j p_j} + K_0\, |x|^{-\beta} \right\} \tag{6.3}$$

and

$$f(x, \xi) := \sum_{i=1}^{n} \lambda_i(x)\, |\xi_i|^{p_i} + |x|^{-\beta} \sum_{i=1}^{n} \kappa_i\, |\xi_i|^{p_i}. \tag{6.4}$$

The function f in (6.4) trivially satisfies the assumptions (**A2**) and (**A3**) with $\tau = \max\{p_i\}$. Moreover, if $p_i \geq 2$ for every $i \in \{1, \dots, n\}$, then f also fulfills (**A1**).

We are now ready to give our two examples, using (2.4), (6.1)–(6.4) and the considerations above.

Example 6.1. Let $K_0 = 0$ or $\beta = 0$. In this case, the function μ in (6.3) belongs to $L^s(\Omega)$ for all $s \in [1, \infty]$. Therefore, if (6.2) is in force for every $i \in \{1, \dots, n\}$ and if $s = \infty$, the map f in (6.4) satisfies (**A4**) with the weights λ_i, μ defined as in (6.1) and (6.3). In light of what has just been said, choosing

$$n = 2, \quad p_1 = 2, \quad p_2 = q = \tau = 4, \quad \alpha_1 = \frac{4}{5}, \quad \alpha_2 = \frac{161}{400}, \quad r_1 = \frac{6}{5}, \quad r_2 = \frac{11}{10}, \quad s = \infty, \tag{6.5}$$

it is easy to verify that the function

$$f(x, \xi) = \left(c_1\, |x|^{\frac{8}{5}} + \kappa_1\right) |\xi_1|^2 + \left(c_2\, |x|^{\frac{161}{100}} + \kappa_2\right) |\xi_2|^4, \qquad x \in \Omega,\ \xi \in \mathbb{R}^{m\times 2},$$

and the parameters in (6.5) fulfill all the assumptions of Theorem 2.2. Therefore, in the case $n = 2$, the main result of this paper is applicable to integrals of the form

$$\mathcal{F}_1(v) = \int_{\Omega} \left\{ \left(c_1\, |x|^{\frac{8}{5}} + \kappa_1\right) |v_{x_1}|^2 + \left(c_2\, |x|^{\frac{161}{100}} + \kappa_2\right) |v_{x_2}|^4 \right\} dx.$$

Example 6.2. Let us now consider the case $K_0 > 0$ and $\beta \in (0, n)$. In this case, the function μ in (6.3) belongs to $L^s(\Omega)$ for all $s \in (1, \frac{n}{\beta})$. Hence, if (6.2) holds true for every $i \in \{1, \dots, n\}$ and $1 < s < \frac{n}{\beta}$, the function f in (6.4) fulfills (**A4**) with the weights λ_i, μ defined by (6.1) and (6.3). Thanks to the previous considerations, choosing

$$n = 2, \quad p_1 = 2, \quad p_2 = q = \tau = 4, \quad \alpha_1 = \frac{4}{5}, \quad \alpha_2 = \frac{161}{400}, \quad r_1 = \frac{6}{5}, \quad r_2 = \frac{11}{10}, \quad \beta = \frac{1}{5}, \quad s = 5, \tag{6.6}$$

it is immediate to realize that the function

$$f(x, \xi) = \left(c_1 |x|^{\frac{8}{5}} + \kappa_1\, |x|^{-\frac{1}{5}}\right) |\xi_1|^2 + \left(c_2\, |x|^{\frac{161}{100}} + \kappa_2\, |x|^{-\frac{1}{5}}\right) |\xi_2|^4, \qquad x \in \Omega,\ \xi \in \mathbb{R}^{m\times 2},$$

and the parameters in (6.6) also satisfy all the assumptions of Theorem 2.2. Therefore, in the case $n = 2$, our main result can also be applied to integrals of the type

$$\mathcal{F}_2(v) = \int_\Omega \left\{ \left(c_1 |x|^{\frac{8}{5}} + \kappa_1 |x|^{-\frac{1}{5}} \right) |v_{x_1}|^2 + \left(c_2 |x|^{\frac{161}{100}} + \kappa_2 |x|^{-\frac{1}{5}} \right) |v_{x_2}|^4 \right\} dx .$$

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Data availability

No data was used for the research described in the article.

References

[1] P. Marcellini, Un exemple de solution discontinue d'un problème variationnel dans le cas scalaire, Istituto Matematico U. Dini, Università di Firenze, 1987, Preprint 11.
[2] P. Marcellini, Regularity and existence of solutions of elliptic equations with p,q-growth conditions, J. Differ. Eq. 90 (1991) 1–30.
[3] P. Marcellini, Regularity for elliptic equations with general growth conditions, J. Differ. Eq. 105 (1993) 296–333.
[4] P. Marcellini, Regularity for some scalar variational problems under general growth conditions, J. Optim. Theory Appl. 90 (1996) 161–181.
[5] L. Boccardo, P. Marcellini, C. Sbordone, L^∞-Regularity for variational problems with sharp nonstandard growth conditions, Boll. Un. Mat. Ital. A 4 (7) (1990) 219–225, 2. English, with Italian summary.
[6] B. Stroffolini, Global boundedness of solutions of anisotropic variational problems, Boll. Un. Mat. Ital. A 7 (5) (1991) 345–352, (3).
[7] N. Fusco, C. Sbordone, Some remarks on the regularity of minima of anisotropic integrals, Comm. Partial Differential Equations 18 (1–2) (1993) 153–167.
[8] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions, Arch. Ration. Mech. Anal. 105 (3) (1989) 267–284.
[9] P. Marcellini, Everywhere regularity for a class of elliptic systems without growth conditions, Ann. Sc. Norm. Sup. Pisa Cl. Sci. 23 (1) (1996) 1–25, (4).
[10] P. Marcellini, Anisotropic and p,q-nonlinear partial differential equations, Rend. Fis. Acc. Lincei 31 (2020) 295–301.
[11] M.-M. Boureanu, P. Pucci, V. Rădulescu, Multiplicity of solutions for a class of anisotropic elliptic equations with variable exponent, Complex Var. Elliptic Equ. 56 (7–9) (2011) 755–767.
[12] R. Fortini, D. Mugnai, P. Pucci, Maximum principles for anisotropic elliptic inequalities, Nonlinear Anal. 70 (8) (2009) 2917–2929.
[13] M. Mihăilescu, P. Pucci, V. Rădulescu, Nonhomogeneous boundary value problems in anisotropic Sobolev spaces, C. R. Math. Acad. Sci. Paris 345 (10) (2007) 561–566.
[14] G. Mingione, Regularity of minima: an invitation to the dark side of the calculus of variations, Appl. Math. 51 (4) (2006) 355–426.
[15] P. Marcellini, Growth conditions and regularity for weak solutions to nonlinear elliptic pdes, J. Math. Anal. Appl. 501 (1) (2021) 124408.
[16] G. Mingione, V. Rădulescu, Recent developments in problems with nonstandard growth and nonuniform ellipticity, J. Math. Anal. Appl. 501 (1) (2021) 125197.
[17] E. De Giorgi, Un esempio di estremali discontinue per un problema variazionale di tipo ellittico, Boll. Un. Mat. Ital. 1 (1968) 135–137, 4.
[18] E. Giusti, M. Miranda, Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni, Boll. Un. Mat. Ital. (4) 1 (1968) 219–226.
[19] J. Nečas, Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity, in: Theory of Nonlinear Operators (Proc. Fourth Internat. Summer School, Acad. Sci., Berlin, 1975), Akademie-Verlag, Berlin, 1977, pp. 197–206, Abh. Akad. Wiss. DDR Abt. Math.-Natur.-Tech., Jahrgang 1977, 1.
[20] V. Šverák, X. Yan, A singular minimizer of a smooth strongly convex functional in three dimensions, calc. Var, Partial. Differ. Equations 10 (3) (2000) 213–221.
[21] G. Cupini, P. Marcellini, E. Mascolo, Local boundedness of solutions to quasilinear elliptic systems, Manuscripta Math. 137 (3–4) (2012) 287–315.
[22] G. Cupini, P. Marcellini, E. Mascolo, Local boundedness of solutions to some anisotropic elliptic systems, in: Contemp. Math., vol. 595, Amer. Math. Soc., Providence, RI, 2013, http://dx.doi.org/10.1090/conm/595/11803.
[23] A. Dall'Aglio, E. Mascolo, L^∞ estimates for a class of nonlinear elliptic systems with nonstandard growth, Atti Sem. Mat. Fis. Univ. Modena 50 (1) (2002) 65–83.
[24] F. Leonetti, E. Mascolo, Local boundedness for vector valued minimizers of anisotropic functionals, Z. Anal. Anwend. 31 (3) (2012) 357–378.
[25] N.S. Trudinger, On the regularity of generalized solutions of linear, Non- Uniformly Elliptic Equations, Arch. Ration. Mech. Anal. 42 (1971) 50–62.
[26] G. Cupini, P. Marcellini, E. Mascolo, Nonuniformly elliptic energy integrals with p,q-growth, Nonlinear Anal. 177 (2018) 312–324.
[27] S. Biagi, G. Cupini, E. Mascolo, Regularity of quasi-minimizers for non-uniformly elliptic integrals, J. Math. Anal. Appl. 485 (2) (2020) 123838.
[28] G. Cupini, P. Marcellini, E. Mascolo, Regularity under sharp anisotropic general growth conditions, Discrete Contin. Dyn. Syst. Ser. B 11 (1) (2009) 66–86.
[29] G. Cupini, P. Marcellini, E. Mascolo, Regularity of minimizers under limit growth conditions, Nonlinear Anal. 153 (2017) 294–310.
[30] G. Cupini, P. Marcellini, E. Mascolo, Local boundedness of weak solutions to elliptic equations with p,q-growth, Math. Eng. 5 (3) (2023) 1–28.
[31] G. Cupini, P. Marcellini, E. Mascolo, A. Passarelli di Napoli, Lipschitz regularity for degenerate elliptic integrals with p,q-growth, Adv. Calc. Var. 16 (2) (2023) 443–465.
[32] F. Feo, A. Passarelli di Napoli, M.R. Posteraro, Local boundedness for minimizers of anisotropic functionals with monomial weights, J. Optim. Theory Appl. 201 (2024) 1313–1332.
[33] A. Zhang, Z. Feng, H. Gao, Local boundedness for vectorial minimizers of non-uniform variational integrals, J. Math. Anal. Appl. 544 (2) (2025) 129074.
[34] M. Troisi, Teoremi di inclusione per spazi di Sobolev non isotropi, Ric. Mat. 18 (1969) 3–24.
[35] E. Acerbi, N. Fusco, Partial regularity under anisotropic (p,q) growth conditions, J. Differ. Eq. 107 (1) (1994) 46–67.