ADAPTIVE FINITE ELEMENT APPROXIMATION FOR QUASI-STATIC CRACK GROWTH

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ABSTRACT. We provide an adaptive finite element approximation for a model of quasi-static crack growth in dimension two. The discrete setting consists of integral functionals that are defined on continuous, piecewise affine functions, where the triangulation is a part of the unknown of the problem and adaptive in each minimization step. The limit passage is conducted simultaneously in the vanishing mesh size and discretized time step, and results in an evolution for the continuum Griffith model of brittle fracture with isotropic surface energy [33] which is characterized by an irreversibility condition, a global stability, and an energy balance. Our result corresponds to an evolutionary counterpart of the static Γ-convergence result in [3] for which, as a byproduct, we provide an alternative proof.

1. INTRODUCTION

The fracture behavior of brittle materials has been a central focus of research in mechanical engineering, beginning with GRIFFITH's pioneering work in the 1920s [41]. Griffith's theory revolutionized the understanding of crack formation and propagation by framing it as a competition between the elastic bulk energy of a material and the energy needed to increase the area of the cracked surface. Building on this foundation, FRANCFORT and MARIGO [27] introduced a variational approach to fracture, where displacement fields and crack paths are determined by minimizing so-called *Griffith energies*. They proposed an evolutionary model in the framework of rate independent systems, called an *irreversible quasi-static crack evolution*, which is governed by three fundamental principles: irreversibility of the crack, static equilibrium at every time, and an energy balance that ensures that ensures that the process is non-dissipative. Unlike traditional fracture theories, this framework does not rely on prescribed crack paths and provides a more effective description of crack initiation. The present paper is devoted to an approximation result of such fracture evolutions based on adaptive finite elements. We start by giving a nonexhaustive account on the relevant literature.

Existence of quasi-static crack evolutions: The mathematical well-posedness of the model from [27] was initiated in [24] for a $2d$ antiplane model with restrictive assumptions on the crack topology. The topological restrictions have then been removed in the breakthrough result [26] by passing to the so-called *weak formulation*, i.e., expressing the problem in the functional setting of SBV-functions [2] and replacing the crack by the jump set of the displacement u. This study was subsequently generalized to nonlinear elasticity [21, 22], including the setting of non-interpenetration [23]. We also refer to [25, 43] for some results based on local minimization. All such existence results are derived from solving certain time-incremental problems where one fundamental challenge consists in proving that the static equilibrium property at all times is conserved in the passage to time-continuous solutions. The extension of this strategy to the Griffith energy in linearized elasticity gives rise to several additional difficulties inherent to

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the presence of symmetrized gradients. Indeed, due to the lack of Korn's inequality, there is only control on the symmetric part of the gradient $e(u) = \frac{1}{2}(\nabla u^T + \nabla u)$ but not on the full gradient ∇u, leading to an analytically more intricate formulation in the larger space of special functions of bounded deformation. Only recently, departing from DAL MASO's seminal paper [20] on the generalized space $GSBD$, there have been significant developments for the analysis of linear Griffith models. We refer the reader to static existence results, both in the weak [1, 13, 30] and the strong [10, 12, 29] setting. In [33], the existence of quasi-static crack evolutions has been established in dimension two, generalizing the seminal work [26] to the vectorial, geometrically linear setting.

Approximation of the Griffith functional: Due to the presence of unknown surfaces, minimization problems for the Griffith functional are notoriously difficult to be solved numerically in an efficient and robust way. Consequently, the regularization of free-discontinuity problems by computationally more viable models is of fundamental importance. With the aim of rigorously proving convergence of such approximation schemes, one usually resorts to the variational notion of Γ-convergence [19] that ensures convergence of minimizers. Over the last years, a variety of different ways to approximate the Griffith energy has been proposed, including so-called phase-field approximations where the sharp discontinuity is smoothened into a diffuse crack in terms of an auxiliary phase-field variable. This well-known approach was not only extensively studied from a mathematical point of view, see [8, 9, 11, 42] for recent results in the linearly elastic setting, but constitutes also one of the most popular computational methods for simulating brittle fracture. Practically, this approach is combined with an additional spatial discretization of both variables in terms of finite-difference or finite-elements (for rigorous results by means of Γ-convergence see [4, 5, 18]). One major drawback lies in the fact that the two-parameter approximation in terms of the diffuse crack and the mesh-size parameter gives rise to a multiscale numerical problem requiring the use of a very thin mesh. Similar issues may appear for approximations by non-local functionals, where the energy density depends on some 'averaged behavior' of u in a neighborhood of vanishing size, see [44, 46, 49] for results in the setting of linear fracture models. To avoid dealing with a further approximation parameter, one therefore searches for discrete approximations of single-scale type, see [28, 47, 48].

In this paper, we want to focus on one prominent example of those, namely a discrete finite-element approximation that makes use of adaptive mesh-refinements and was first introduced for the Mumford-Shah functional by DAL MASO AND CHAMBOLLE [14]. The main feature of this approach is that it involves an implicit mesh-optimization and hence gives enough flexibility to approximate isotropic crack energies. As described in [6], this is a real advantage as this method does not require to use very fine meshes. In [45], this approximation scheme was then generalized to the Griffith functional in linearized elasticity. Yet, in this result, the approximating sequence of functionals still depends on the full gradient ∇u in an unnatural way. Only recently, BABADJIAN AND BONHOMME [3] succeeded in extending the results from [14] to the linear case which is exclusively written in terms of the symmetric gradient $e(u)$ of the displacement u. We refer to [3, Introduction] for more details and for a review of related discrete-to-continuous approximations.

Evolutionary approximation results: All the aforementioned results are purely static and do not take into account the irreversible nature of crack growth. In fact, the literature on approximation of evolutionary fracture models is comparably scarce. In [36], a phase-field approximation of quasi-static fracture evolution was provided for the antiplane case, and in the setting of finite elasticity a result on discontinuous-finite-element approximation is available [38, 39]. Additionally, crack evolutions have been identified as the effective variational limit of sequences of problems in various settings of applicative relevance such as atomistic models [32], homogenization [40], linearization [35], or a cohesive-to-brittle passage in the limit of infinite specimen size [37]. However, rigorous approximation schemes for quasi-static crack growth in the framework of linearized elasticity are still pending. In the present paper, we move a first step in this direction by providing a discrete-to-continuous passage for the adaptive finite element model analyzed in [3, 14]. Besides providing a first approximation result for an evolutionary Griffith model, we believe that the techniques developed in this paper may also contribute to tackle other relevant approximation schemes in the future, such as (spatially discretized) phase-field models.

The present paper: Following the setting in [3, 14], we call a triangulation of the reference domain $\Omega \subset \mathbb{R}^2$ admissible if the angles of all contained triangles exceed a given value θ_0 and the size of their edges lies between ε and a given function $\omega(\varepsilon) > \varepsilon$, with $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. We consider functionals defined on the family $\mathcal{A}_\varepsilon(\Omega)$ of continuous functions $u\colon \Omega \to \mathbb{R}^2$ which are piecewise affine on admissible triangulations of Ω, and take the form

$$E_\varepsilon(u) := \frac{1}{\varepsilon}\int_\Omega f(\varepsilon\,\mathbb{C}e(u)\colon e(u))\,\mathrm{d}x\,.$$

Here, f is a nondecreasing function behaving as the identity near 0 and as a constant near ∞, see (2.2), and by $\mathbb{C}$ we denote a fourth order elasticity tensor, bounded from above and below, see (2.3). For simplicity, we will mainly focus on the special case $f(t) = t \wedge \kappa$ and $\mathbb{C} = \mathrm{Id}_{2\times2\times2\times2}$. For any function $u \in \mathcal{A}_\varepsilon(\Omega)$, we regard a triangle T as 'cracked' if the absolute value of the constant symmetric gradient $|e(u)_T|$ on the triangle exceeds the threshold $\sqrt{\kappa/\varepsilon}$. This criterion enables us to set up a time-discrete evolution by incrementally minimizing the energy at discrete time sets $(t^k_\delta)_k \subset [0,T]$ depending on a time-discretization parameter δ. In each step t^k_δ, the energy should depend on the displacements $(u^j_{\varepsilon,\delta})_{j<k}$ at all previous time steps. To this end, we consider the union of all 'cracked triangles' at previous time steps as the discrete crack set $\Omega^{\rm crack}_{\varepsilon,\delta,k}$, and define the energy

$$E_\varepsilon(u,(u^j_{\varepsilon,\delta})_{j<k}) = \int_{\Omega\setminus\Omega^{\rm crack}_{\varepsilon,\delta,k}} |e(u)|^2 + \kappa\frac{|\Omega^{\rm crack}_{\varepsilon,\delta,k}|}{\varepsilon}\,. \tag{1.1}$$

This corresponds to the energy E_ε, accounting additionally for 'cracked triangles' at all the previous time steps. In order to implement an irreversibility condition, we however have to require additional properties for the admissible triangulations, depending on the previous time steps. More precisely, for each t^k_δ, we assume that all triangles that have been 'cracked' at previous time steps $t^j_\delta < t^k_\delta$ are contained in the triangulation. This requirement restricts the flexibility in the choice of the mesh and gives rise to a notion of an increasing crack set on the time-discrete level. Secondly, we need an additional technical condition related to a background mesh, which is inspired by the construction of recovery sequences in [14].

Starting from an incremental minimization scheme of the history-dependent energy in (1.1), we obtain a time-discrete evolution that is piecewise constant in time and piecewise affine in space. In our main result (Theorem 4.2), we show that in the simultaneous limit $\delta \to 0, \varepsilon \to 0$ this evolution converges to an irreversible quasi-static crack evolution in the sense of [27]. This means that we find a pair $t \mapsto (u(t), K(t))$ for $t \in [0,T]$, where $u(t)$ lies in the space $GSBD^2(\Omega)$, see [20], and $K(t)$ is a rectifiable set containing the jump set of $u(t)$, such that $t \mapsto (u(t), K(t))$ satisfies: (a) an irreversibility condition, (b) a global stability at all times (sometimes referred to as unilateral minimality), and (c) an energy balance law, see Definition 4.1 for details. As a byproduct, we get that the energies converge at all times. Moreover, we show the convergence of crack sets in the sense of σ-convergence introduced in [40] (see Definition 5.1) and we obtain strong convergence of the linear strains. Besides providing an approximation result for fracture evolution, our main theorem also provides an alternative proof of the existence result in [33].

Proof strategy: In the proof, we essentially face three challenges: (1) we need a suitable compactness argument to identify the continuum crack $K(t)$. (2) This notion needs to be compatible for deriving a $\liminf$-inequality for the history-dependent energy E_ε in (1.1). (3) Eventually, we need to show the stability of the unilateral minimality property, which corresponds to constructing a mutual recovery sequence for displacements and crack sets. We now briefly sketch our strategies to tackle these issues.

(1) A suitable notion for convergence of crack sets, the so-called σ*-convergence*, is already available and has been employed successfully for proving the stability of unilateral minimality [40]. We adapt this notion to the $GSBD$-setting by resorting to tools developed in [31] which allow to separate effects of bulk and surface energies for linear Griffith functionals, see Section 5.2.

(2) The proof of the lower bound in the Γ-convergence result [3] relies on a careful blow-up analysis. As it appears to be incompatible to combine this strategy with σ-convergence, we approach the problem from a different perspective: we consider the energy as split into a bulk and a surface part, called *displacement-void representation*, where the surface part is essentially of the form $c\mathcal{H}^1(\partial\Omega^{\rm crack}_{\varepsilon,\delta,k})$ for a suitable constant $c > 0$, cf. (1.1). Since then the energy can be represented as a pair of function and set, we can resort to recent results on such functionals [17], obtained in connection with models for material voids in elastically stressed solids. Note that a similar proof strategy was adopted already in [6] for the Mumford-Shah functional. Roughly speaking, the proof therein relies on the fact that the 'averaged volume' of the set of 'cracked triangles' can be bounded from below by the boundary of a slightly smaller set of triangles, where the deformation gradient can still be controlled on the removed triangles. We are able to derive an analogous statement for the vector-valued case and the symmetric gradient $e(u)$ which provides a sharp lower bound for the boundary of void sets after a suitable modification, see Theorem 2.1. The original argument in [6] strongly relies on a scalar-valued displacement field and consists in locally controlling the gradient in 'cracked triangles', see [14, Remark 3.5] or [3, Introduction]. Our proof instead is considerably more involved as it is inherently nonlocal and uses some techniques from planar graph theory. This theorem on 'void modifications' lies at the core of our analysis and, in our opinion, represents the crucial novelty of our work. As a byproduct, this result also allows us to give an alternative short proof of the Γ-convergence result in [3], see Theorem 2.2.

(3) The third major issue in the proof of the main result lies in showing the stability of unilateral minimizers. Here, relying on some arguments from [32], we adapt the *jump-transfer construction* from [26] to our discrete setting. One challenge is to find an admissible triangulation for the construction of a piecewise affine interpolation of the function with 'transferred jump'. As this triangulation has to contain the 'cracked triangles' of previous time steps, we set up a triangulation by combining the triangulation of the former time step with the construction of [14]. Since our crack set consists of boundaries of voids, we also have to make sure to choose the correct interpolation of u on these triangles.

Organization of the paper: The paper is organized as follows. In Section 2 we introduce the finite-element approximation for the Griffith functional and we state the result on modification of voids, which is fundamental for our analysis. With this at hand, we then give a short proof of the (static) Γ-liminf inequality from [3]. Section 3 is devoted to the proof of the void modification by combining results from planar graph theory with extension arguments relying on suitable Korn-type inequalities. Then, in Section 4, we introduce the quasi-static adaptive finite-element model. After setting up a time-discrete evolution by inductively minimizing the history-dependent energy, we state our main result on the approximation of quasi-static crack growth. Subsequently, in Section 5, we establish some preparatory compactness and semicontinuity results in our functional setting and recall the notion of σ-convergence, together with proofs of the corresponding irreversibility and compactness properties. Section 6 is devoted to the proof of our main result, where we first derive properties of the time-discrete evolutions and then pass to the continuum limit. To confirm that the latter is indeed a quasi-static crack evolution, we depend critically on the aforementioned stability of unilateral minimality, whose proof is deferred to Section 7.

Notation: We close the introduction by introducing some notation: The 2-dimensional Lebesgue and the 1-dimensional Hausdorff measures in $\mathbb{R}^2$ are denoted by $\mathcal{L}^2$ or $|\cdot|$ and $\mathcal{H}^1$ respectively. Depending on the context $\tilde{\subset}$ stands for inclusions up to sets that are negligible with respect to either $\mathcal{H}^1$ or $\mathcal{L}^2$. The interior and closure of a set $A \subset \mathbb{R}^2$ are denoted by $\mathrm{int}(A)$ and $\overline{A}$, by $\partial^* A$ we denote its reduced boundary, and by χ_A the corresponding characteristic function. By $A \triangle B$ we indicate the symmetric difference of two sets $A, B \subset \mathbb{R}^2$. The set of 2×2 matrices is denoted by $\mathbb{R}^{2\times 2}$ and the subset of symmetric and skew-symmetric matrices by $\mathbb{R}^{2\times 2}_{\rm sym}$, respectively $\mathbb{R}^{2\times 2}_{\rm skew}$. For an open and bounded set $U \subset \mathbb{R}^2$, we denote by $L^0(U;\mathbb{R}^2)$ the $\mathcal{L}^2$-measurable functions from U to $\mathbb{R}^2$. By $a \wedge b$ we denote the minimum of $a, b \in \mathbb{R}$. Finally, in the following $C > 0$ denotes a universal constant that may change from line to line.

2. Finite element approximation of Griffith: The displacement-void approach

In this section we revisit the recent Γ-convergence result [3] on the approximation of the Griffith functional by adaptive finite elements. We follow a different approach, based on a *displacement-void representation* of the energy, which will form the basis for our study for quasi-static fracture evolution starting in Section 4.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. We call a *triangulation* of Ω a collection of closed triangles that only intersect on common edges or vertices and whose union contains Ω. The vertices of the triangles are called *nodes* of the triangulation. For a given angle $\theta_0 > 0$ and a function $\omega\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega(\varepsilon) \geq 6\varepsilon$ for all $\varepsilon > 0$ and $\lim_{\varepsilon\to 0}\omega(\varepsilon) = 0$, we denote by $\mathcal{T}_\varepsilon(\Omega) := \mathcal{T}_\varepsilon(\Omega, \theta_0, \omega)$ all triangulations of Ω whose edges have length between ε and $\omega(\varepsilon)$ and whose angles are greater than or equal to θ_0. We say that u is *piecewise affine on a triangulation* $\mathbf{T} \in \mathcal{T}_\varepsilon(\Omega)$ if u is affine on each triangle $T \in \mathbf{T}$. The corresponding constant symmetrized gradient of u on each T is denoted by $e(u)_T$. We then define

$$\mathcal{A}_\varepsilon(\Omega) := \{u\colon \Omega \to \mathbb{R}^2 \text{ continuous: there exists a } \mathbf{T} \in \mathcal{T}_\varepsilon(\Omega) \text{ such that } u \text{ is piecewise affine on } \mathbf{T}\}\,. \tag{2.1}$$

We will associate to each function $u \in \mathcal{A}_\varepsilon(\Omega)$ a possibly non-unique triangulation $\mathbf{T}(u) \in \mathcal{T}_\varepsilon(\Omega)$ as the ambiguity in the choice of $\mathbf{T}(u)$ does not pose any issues in the following.

Let $f\colon [0,\infty) \to [0,\infty)$ be a nondecreasing continuous function, which is differentiable at 0 and satisfies

$$f(0) = 0\,, \quad \lim_{t\to 0^+}\frac{f(t)}{t} = 1\,, \quad \lim_{t\to\infty} f(t) = \kappa \tag{2.2}$$

for some $\kappa > 0$. Moreover, let $\mathbb{C} \in \mathbb{R}^{2\times 2\times 2\times 2}$ be a symmetric fourth order tensor such that

$$c_1|\xi|^2 \leq \mathbb{C}\xi : \xi \leq c_2|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^{2\times 2}_{\rm sym} \tag{2.3}$$

for some constants $0 < c_1 \leq c_2$. We define an energy $E_\varepsilon\colon L^0(\Omega;\mathbb{R}^2) \to \mathbb{R}$ by

$$E_\varepsilon(u) := \begin{cases} \frac{1}{\varepsilon}\int_\Omega f(\varepsilon\,\mathbb{C}e(u) : e(u))\,\mathrm{d}x & \text{if } u \in \mathcal{A}_\varepsilon(\Omega)\,, \\ +\infty & \text{if } u \in L^0(\Omega;\mathbb{R}^2) \setminus \mathcal{A}_\varepsilon(\Omega)\,. \end{cases} \tag{2.4}$$

In [3], the authors showed that E_ε Γ-converges to the *Griffith functional* $E\colon L^0(\Omega;\mathbb{R}^2) \to \mathbb{R}$ defined by

$$E(u) := \begin{cases} \int_\Omega \mathbb{C}e(u) : e(u)\,\mathrm{d}x + \kappa\sin(\theta_0)\,\mathcal{H}^1(J_u) & \text{if } u \in GSBD^2(\Omega)\,, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.5}$$

in terms of the topology induced by measure convergence. For basic notation and properties of $GSBD$ functions, we refer the reader to [20]. Our first goal is to rederive this result based on a different technique.

2.1. Displacement-void representation and modification of voids.

To explain the idea, we consider the special case

$$f(t) = t \wedge \kappa \quad \text{for } t \geq 0 \quad \text{and} \quad \mathbb{C} = \mathrm{Id}_{2\times 2\times 2\times 2}. \tag{2.6}$$

For a function $u \in \mathcal{A}_\varepsilon(\Omega)$ and a corresponding triangulation $\mathbf{T}(u) \in \mathcal{T}_\varepsilon(u)$, we define $\mathbf{T}_\varepsilon^{\rm big}(u) := \{T \in \mathbf{T}(u) \colon \varepsilon |e(u)_T|^2 \geq \kappa\}$, corresponding to the set of triangles where $|e(u)_T|$ is big. We define their union in Ω as

$$\Omega_\varepsilon^{\rm big}(u) := \mathrm{int}\Big(\bigcup_{T \in \mathbf{T}_\varepsilon^{\rm big}(u)} T\Big) \cap \Omega. \tag{2.7}$$

Note that κ corresponds to the point where f is not differentiable. With this choice, the energy can be split into

$$E_\varepsilon(u) = \sum_{T \in \mathbf{T}(u) \setminus \mathbf{T}_\varepsilon^{\rm big}(u)} |T \cap \Omega| |e(u)_T|^2 \, + \, \kappa \frac{|\Omega_\varepsilon^{\rm big}(u)|}{\varepsilon} \, . \tag{2.8}$$

Our idea relies on rewriting this as an energy featuring bulk and surface terms of the form

$$E_\varepsilon(u) \sim \int_{\Omega \setminus \Omega_\varepsilon^{\rm big}(u)} |e(u)|^2 \, \mathrm{d}x + \frac{\kappa \sin\theta_0}{2} \mathcal{H}^1(\partial \Omega_\varepsilon^{\rm big}(u)). \tag{2.9}$$

This will allow us to directly apply Γ-convergence results for a class of energies defined on pairs of function-set [17]. Since in [17] the main motivation was a model for material voids inside elastically stressed solids, we call this a *displacement-void representation* of the energy.

However, in general (2.9) is not an identity as one can only guarantee $\frac{|\Omega_\varepsilon^{\rm big}|}{\varepsilon} \geq \frac{\sin\theta_0}{2} c \mathcal{H}^1(\partial\Omega_\varepsilon^{\rm big})$ for some $0 < c < 1$. Our first main result states that a sharp lower bound up to an arbitrarily small error can be achieved by a suitable modification of $\Omega_\varepsilon^{\rm big}$. In the following, we say that a set E is induced by (a subset of triangles) $\mathbf{T}_E \subset \mathbf{T}$ if it is given by the interior of the union of $\mathbf{T}_E$ intersected with Ω, i.e.,

$$E = \mathrm{int}\Big(\bigcup_{T \in \mathbf{T}_E} T\Big) \cap \Omega. \tag{2.10}$$

Theorem 2.1 (Void modification)**.** *Let $u \in H^1(\Omega;\mathbb{R}^2)$ be a function which is piecewise affine on a triangulation $\mathbf{T} \in \mathcal{T}_\varepsilon(\Omega)$, and suppose that, for a given subset $\mathbf{T}_A \subset \mathbf{T}$ and A induced by $\mathbf{T}_A$, it holds that*

$$\int_{\Omega \setminus A} |e(u)|^2 \, \mathrm{d}x + \frac{2}{\varepsilon \sin\theta_0} |A| \leq C_0. \tag{2.11}$$

Then, given $\eta > 0$ there exist $A_{\rm mod}$ induced by some subset of triangles in $\mathbf{T}$ and $u_{\rm mod} \in H^1(\Omega_{\varepsilon,\eta};\mathbb{R}^2)$ such that

$$|A_{\rm mod}| \leq C_\eta \, \varepsilon, \qquad |\{u \neq u_{\rm mod}\} \cap \Omega_{\varepsilon,\eta}| \leq C_\eta \, \varepsilon \tag{2.12}$$

and

$$\|e(u_{\rm mod})\|_{L^2(\Omega_{\varepsilon,\eta} \setminus A_{\rm mod})} \leq C_\eta, \qquad \mathcal{H}^1(\partial A_{\rm mod}) \leq \frac{2}{\varepsilon \sin\theta_0} |A| + C\eta, \tag{2.13}$$

where $C > 0$ depends only on C_0 in (2.11), C_η is a constant times a negative power of η, and $\Omega_{\varepsilon,\eta} := \{x \in \Omega \colon \mathrm{dist}(x, \partial\Omega) > 2\omega(\varepsilon) + \frac{\varepsilon}{\eta^3}\}$. Furthermore, if $\mathcal{C}(A_{\rm mod})$ denotes the set of all connected components of $A_{\rm mod}$, we have

$$\#\mathcal{C}(A_{\rm mod}) \leq C \frac{\eta}{\varepsilon}. \tag{2.14}$$

The result will be proven in Section 3. We now show that this allows to obtain a short proof of the Γ-liminf inequality in [3].

2.2. **Γ-convergence.** The upper bound follows from a density result (see [11, 16]) and an explicit construction, relying on the construction of the recovery sequence in [14]. The lower bound in [3] is based on a careful blow-up analysis. We provide an alternative proof based on Theorem 2.1.

Theorem 2.2. *Let $(u_\varepsilon)_\varepsilon \subset L^0(\Omega;\mathbb{R}^2)$ with $u_\varepsilon \to u$ in measure on Ω. Then it holds*

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq E(u)\,. \tag{2.15}$$

Proof. It is not restrictive to assume that $E_\varepsilon(u_\varepsilon) \leq C_0$ for some $C_0 > 0$. We start by expressing the energy similarly as in (2.8) for general densities f. To this end, given $R > 0$, we define $\mathbf{T}^{\rm big}_\varepsilon(u) \subset \mathbf{T}(u)$ by

$$\mathbf{T}^{\rm big}_\varepsilon(u) := \{T \in \mathbf{T}(u) \colon \varepsilon|e(u)_T|^2_{\mathbb{C}} \geq R\}\,,$$

where for shorthand we set $|e(u)_T|_{\mathbb{C}} := (\mathbb{C}e(u)_T \colon e(u)_T)^{1/2}$. Correspondingly, we define the set $\Omega^{\rm big}_\varepsilon(u)$ as in (2.7). Since f is non-decreasing, we obtain the estimate

$$E_\varepsilon(u) \geq \sum_{T\in\mathbf{T}(u)\setminus\mathbf{T}^{\rm big}_\varepsilon(u)} \frac{|T\cap\Omega|}{\varepsilon} f(\varepsilon|e(u)_T|^2_{\mathbb{C}}) \,+\, f(R)\,\frac{|\Omega^{\rm big}_\varepsilon(u)|}{\varepsilon}\,. \tag{2.16}$$

Note that in the general case we will need to consider the limit $R \to \infty$ whereas in the special case (2.6) one can fix $R = \kappa$.

We first deal with the second term in (2.16). We let $\eta > 0$, $\varepsilon_0 > 0$, and fix $\Omega_* \subset\subset \Omega$ such that $\Omega_{\varepsilon,\eta} \supset \Omega_*$ for all $0 < \varepsilon \leq \varepsilon_0$, with $\Omega_{\varepsilon,\eta} = \{x \in \Omega \colon \mathrm{dist}(x,\partial\Omega) > 2\omega(\varepsilon) + \frac{\varepsilon}{\eta^3}\}$. We apply Theorem 2.1 for u_ε and $A = \Omega^{\rm big}_\varepsilon(u_\varepsilon)$ to obtain $(u_\varepsilon)_{\rm mod}$ and $\Omega^{\rm mod}_\varepsilon(u_\varepsilon) := A_{\rm mod}$. Then, we define $v_\varepsilon \colon \Omega_* \to \mathbb{R}^2$ by

$$v_\varepsilon(x) := \begin{cases} (u_\varepsilon)_{\rm mod}(x) & \text{if } x \in \Omega_* \setminus \Omega^{\rm mod}_\varepsilon(u_\varepsilon)\,,\\ 0 & \text{if } x \in \Omega^{\rm mod}_\varepsilon(u_\varepsilon) \cap \Omega_*\,. \end{cases}$$

Then, in view of the energy bound and (2.13), we have

$$\sup_{0<\varepsilon\leq\varepsilon_0} \Big(\|e(v_\varepsilon)\|_{L^2(\Omega_*)} + \mathcal{H}^1(J_{v_\varepsilon})\Big) < \infty.$$

Since $u_\varepsilon \to u$ in measure, and $\{v_\varepsilon \neq u_\varepsilon\} \leq |\Omega^{\rm mod}_\varepsilon(u_\varepsilon)| + |\{u_\varepsilon \neq (u_\varepsilon)_{\rm mod}\}| \leq C_\eta\varepsilon$ by (2.12), we obtain $v_\varepsilon \to u$ in measure on Ω_* and then by compactness (see e.g. [17, Theorem 3.5] or [13]) that $u \in GSBD^2(\Omega_*)$. Moreover, from $|\Omega^{\rm mod}_\varepsilon(u_\varepsilon)| \leq C_\eta\varepsilon$ we have $\chi_{\Omega^{\rm mod}_\varepsilon(u_\varepsilon)} \to 0$ in $L^1(\Omega)$. Therefore, we can apply the lower semicontinuity result for surface measures of voids stated in [17, Theorem 5.1]. This together with the second estimate in (2.13) gives

$$\liminf_{\varepsilon\to 0} \frac{|\Omega^{\rm big}_\varepsilon(u_\varepsilon)|}{\varepsilon} + C\eta \geq \liminf_{\varepsilon\to 0} \frac{\sin(\theta_0)}{2}\mathcal{H}^1(\partial\Omega^{\rm mod}_\varepsilon(u_\varepsilon)\cap\Omega_*) \geq \sin(\theta_0)\mathcal{H}^1(J_u\cap\Omega_*)\,. \tag{2.17}$$

Now, we prove the lower semicontinuity of the elastic part of the energy. By (2.2) and a Taylor expansion we obtain, for $s \geq 0$,

$$f(s) = f(0) + f'(0)s + \gamma(s) = s + \gamma(s)\,,$$

where $\gamma \colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\frac{\gamma(s)}{s} \to 0$ for $s \to 0$. For any subset $\mathbf{T}' \subset \mathbf{T}(u_\varepsilon)$ this leads to

$$\begin{aligned} &\sum_{T\in\mathbf{T}'\setminus\mathbf{T}^{\rm big}_\varepsilon(u_\varepsilon)} \frac{|T\cap\Omega|}{\varepsilon} f(\varepsilon|e(u_\varepsilon)_T|^2_{\mathbb{C}}) \\ &\quad= \sum_{T\in\mathbf{T}'\setminus\mathbf{T}^{\rm big}_\varepsilon(u_\varepsilon)} |T\cap\Omega||e(u_\varepsilon)_T|^2_{\mathbb{C}} + \sum_{T\in\mathbf{T}'\setminus\mathbf{T}^{\rm big}_\varepsilon(u_\varepsilon)} \frac{|T\cap\Omega|}{\varepsilon}\gamma(\varepsilon|e(u_\varepsilon)_T|^2_{\mathbb{C}}). \end{aligned} \tag{2.18}$$

We define the function $\chi_\varepsilon := \chi_{[0,\varepsilon^{-1/4})}(|e(u_\varepsilon)|_{\mathbb{C}})$ and observe that, for ε small enough, we have $\varepsilon^{-1/4} \le \sqrt{R}\varepsilon^{-1/2}$, i.e., in particular $\chi_\varepsilon \le \chi_{\Omega\setminus\Omega_\varepsilon^{\rm big}(u_\varepsilon)}$. Altogether, from (2.18) we thus obtain

$$\sum_{T\in\mathbf{T}(u_\varepsilon)\setminus\mathbf{T}_\varepsilon^{\rm big}(u_\varepsilon)} \frac{|T\cap\Omega|}{\varepsilon} f(\varepsilon|e(u_\varepsilon)_T|^2_{\mathbb{C}}) \ge \int_{\Omega_*} \chi_\varepsilon |e(u_\varepsilon)|^2_{\mathbb{C}}\,\mathrm{d}x \; + \frac{1}{\varepsilon}\int_{\Omega_*} \chi_\varepsilon \gamma(\varepsilon|e(u_\varepsilon)|^2_{\mathbb{C}})\,\mathrm{d}x\,. \tag{2.19}$$

We first deal with the second term in (2.19). Note that we can write

$$\frac{1}{\varepsilon}\int_{\Omega_*} \chi_\varepsilon \gamma(\varepsilon|e(u_\varepsilon)|^2_{\mathbb{C}})\,\mathrm{d}x = \int_{\Omega_*} \chi_\varepsilon |e(u_\varepsilon)|^2_{\mathbb{C}} \frac{\gamma(\varepsilon|e(u_\varepsilon)|^2_{\mathbb{C}})}{\varepsilon|e(u_\varepsilon)|^2_{\mathbb{C}}}\,\mathrm{d}x\,.$$

By the definition of χ_ε and γ we get that $\chi_\varepsilon \frac{\gamma(\varepsilon|e(u_\varepsilon)|^2_{\mathbb{C}})}{\varepsilon|e(u_\varepsilon)|^2_{\mathbb{C}}}$ converges uniformly to zero as $\varepsilon\to 0$. Since by the energy bound $\chi_\varepsilon|e(u_\varepsilon)|_{\mathbb{C}}$ is bounded in $L^2(\Omega)$, we conclude

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Omega_*} \chi_\varepsilon \gamma(\varepsilon|e(u_\varepsilon)|^2_{\mathbb{C}})\,\mathrm{d}x = 0\,. \tag{2.20}$$

By the energy bound on u_ε we know that $|\Omega_\varepsilon^{\rm big}(u_\varepsilon)| \le C\varepsilon$ and that $\chi_{\Omega\setminus\Omega_\varepsilon^{\rm big}(u_\varepsilon)} e(u_\varepsilon)$ is bounded in $L^2(\Omega;\mathbb{R}^{2\times2}_{\rm sym})$. We hence conclude that $\chi_\varepsilon\to 1$ in measure and $\chi_\varepsilon e(u_\varepsilon) \rightharpoonup e(u)$ weakly in $L^2(\Omega;\mathbb{R}^{2\times2}_{\rm sym})$ by [17, Theorem 3.5, (3.7)(ii)]. In view of (2.19) and (2.20), we obtain

$$\liminf_{\varepsilon\to0} \sum_{T\in\mathbf{T}(u_\varepsilon)\setminus\mathbf{T}_\varepsilon^{\rm big}(u_\varepsilon)} \frac{|T\cap\Omega|}{\varepsilon} f(\varepsilon|e(u_\varepsilon)_T|^2_{\mathbb{C}}) \ge \liminf_{\varepsilon\to0}\int_{\Omega_*} \chi_\varepsilon |e(u_\varepsilon)|^2_{\mathbb{C}}\,\mathrm{d}x \ge \int_{\Omega_*} |e(u)|^2_{\mathbb{C}}\,\mathrm{d}x\,. \tag{2.21}$$

Finally, combining (2.16), (2.17), and (2.21), we see that, for any $R>0$, it holds

$$\liminf_{\varepsilon\to0} E_\varepsilon(u_\varepsilon) \ge \int_{\Omega_*} |e(u)|^2_{\mathbb{C}}\,\mathrm{d}x + \sin(\theta_0) f(R)\,\mathcal{H}^1(J_u\cap\Omega_*) - C\eta\,.$$

Sending $R\to\infty$ and $\eta\to0$, by using (2.2) and the arbitrariness of $\Omega_*\subset\subset\Omega$ we conclude $u\in GSBD^2(\Omega)$ and that (2.15) holds. ☐

We mention that with this technique we could prove also Γ-convergence under Dirichlet boundary conditions and the convergence of minimizers, as done in [3, Section 4]. We omit this here but refer to Proposition 6.6 and Corollary 6.8 later where this is performed in the evolutionary framework.

3. Void modification: Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. In this section, we will also show the following corollary on the inclusion of modified void sets.

Corollary 3.1. *Let A^1, A^2 be induced by $\mathbf{T}_{A^1}$ and $\mathbf{T}_{A^2}$. Suppose that $\mathbf{T}_{A^1}\subset\mathbf{T}_{A^2}$. Then, the sets $A^1_{\rm mod}$, $A^2_{\rm mod}$ in Theorem 2.1 can be chosen such that $A^1_{\rm mod}\subset A^2_{\rm mod}$ and $\mathbf{T}^{\rm mod}_{A^1}\subset\mathbf{T}^{\rm mod}_{A^2}$, where $\mathbf{T}^{\rm mod}_{A^j} := \{T\in\mathbf{T}_{A^j}: T\subset A^j_{\rm mod}\}$ for $j=1,2$.*

We start by presenting the main idea of the proof of Theorem 2.1. For $T\in\mathbf{T}$, we denote by $\mathcal{N}(T)$ the three nodes (corners) of T and by $\partial^j T$, $j=1,2,3$, the three edges of T. Using that ε is the minimal length on an edge and θ_0 is the minimal interior angle, an elementary computation shows

$$|T| \ge \frac{1}{2}\varepsilon\sin\theta_0 \max_{j=1,2,3}\mathcal{H}^1(\partial^j T). \tag{3.1}$$

Fixing ideas, for the moment we assume for simplicity that the sidelength of all triangles is $\sim \varepsilon$, and that $A \subset\subset \Omega$. Then, by using (3.1) one can estimate

$$\frac{2}{\varepsilon \sin\theta_0}|A| \geq \mathcal{H}^1(\partial A) - C\varepsilon\#\mathbf{T}_A^{\rm ex}, \tag{3.2}$$

where $\mathbf{T}_A^{\rm ex} \subset \mathbf{T}_A$ denotes the triangles for which more than one side is 'exposed' to $\mathbb{R}^2 \setminus A$, see the triangles highlighted in Figure 1. Now, to validate the second inequality in (2.13) it would be enough to show that $\#\mathbf{T}_A^{\rm ex} \leq C\eta/\varepsilon$, which however in general does not hold. Another option is to 'heal' the triangles $\mathbf{T}_A^{\rm ex}$, i.e., we define $A_{\rm mod} = A \setminus \bigcup_{T\in\mathbf{T}_A^{\rm ex}} T$ and observe that indeed it holds

$$\frac{2}{\varepsilon \sin\theta_0}|A| \geq \mathcal{H}^1(\partial A_{\rm mod}).$$

Yet, in order to do so, we need to ensure that

$$\|e(u)\|_{L^2(\Omega\setminus A_{\rm mod})} \leq C\|e(u)\|_{L^2(\Omega\setminus A)}.$$

Such a strategy has been implemented in [6] for a scalar-valued problem. In the present vectorial setting, however, this procedure only works partially. For 'good' exposed triangles $T \in \mathbf{T}_A^{\rm ex,good}$ (highlighted dark blue in Figure 1), one can show that $\|e(u)\|_{L^2(T)} \leq C\|e(u)\|_{L^2(N_T)}$ for a suitable neighborhood $N_T \subset \mathbb{R}^2 \setminus A$, see Lemma 3.3 below. For 'bad' exposed triangles $T \in \mathbf{T}_A^{\rm ex,bad}$ (highlighted in light blue in Figure 1), in contrast to [6], the fact that only the symmetric gradient $e(u)$ is controlled prohibits to obtain a similar estimate.

Therefore, roughly speaking, our argument will feature both: (1) healing of triangles in $\mathbf{T}_A^{\rm ex,good}$ and (2) estimating the number $\#\mathbf{T}_A^{\rm ex,bad}$ in terms of $C\eta/\varepsilon$ which allows to obtain a small error in (3.2). The latter counting argument will borrow some arguments from planar graph theory and will be based on healing small components, see Lemma 3.4.

3.1. **Preparations.** We start by introducing some notions. First, we define the *saturation* of a connected set $Z \subset \mathbb{R}^2$, denoted by $\mathrm{sat}(Z)$, as $\mathrm{int}(\overline{Z} \cup h_Z)$, where h_Z denotes the union of the bounded connected components of $\mathbb{R}^2 \setminus \overline{Z}$. Note that $\mathrm{sat}(Z)$ arises from Z by 'filling its holes'. We call a connected set Z *saturated* if it holds $\mathrm{sat}(Z) = Z$. In the following, we consider generic sets H of the form (2.10). We extend the notation for specific triangles introduced for A to a generic set H of the form (2.10). Whenever H, K are of the form (2.10), with $H \subset K$, we set $\mathbf{T}_H^\bullet := \mathbf{T}_K^\bullet \cap \mathbf{T}_H$, with $\bullet$ a standpoint for ex, ex, bad, ex, good.

Graph related to a set H: We denote the (open) connected components of H by $\mathcal{C}(H) = \{H_1, \ldots, H_n\}$. (Here, note that the connected components of H are in general different from the ones of $\overline{H}$, see e.g. Figure 1.) We introduce a graph related to H. We define the *vertices* $\mathcal{V}(H)$ and the *edges* $\mathcal{E}(H)$ of the graph as

$$\mathcal{V}(H) := \{v \in \mathcal{N}(T) \colon T \in \mathbf{T}_H,\ v \in \bigcup\nolimits_{j=1}^n \partial H_j\}, \quad \mathcal{E}(H) := \big\{\partial^i T \colon T \in \mathbf{T}_H,\, \partial^i T \subset \bigcup\nolimits_{j=1}^n \partial H_j\big\}.$$

Note that any saturated connected component H_j with $\partial H_j \cap \partial H_k = \emptyset$ for all $k \neq j$ is represented by a closed cycle where each vertex has exactly two edges. Whenever for two components H_j and H_k the boundaries ∂H_j and ∂H_k have a nonempty intersection, this is related to a vertex with four edges, see Figure 1. More generally, if l different connected components meet at a vertex, this vertex has $2l$ edges. Let us denote

$$\mathcal{V}_{2l}(H) := \{v \in \mathcal{V}(H) \colon n(v) = 2l\}, \quad \text{for } n(v) := \#\{S \in \mathcal{E}(H) \colon v \in S\}. \tag{3.3}$$

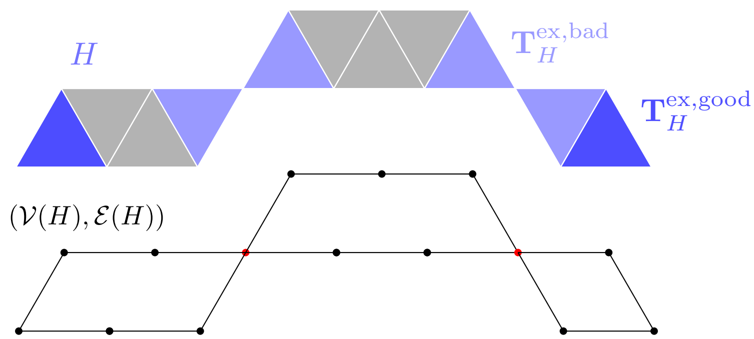

FIGURE 1. An example for H consisting of three components and the corresponding graph $(\mathcal{V}(H), \mathcal{E}(H))$. Note that $\overline{H}$, instead, is connected. The triangles highlighted in blue are part of $\mathbf{T}_H^{\mathrm{ex}} = \mathbf{T}_H^{\mathrm{ex,good}} \cup \mathbf{T}_H^{\mathrm{ex,bad}}$. The vertices depicted in red are part of $\mathcal{V}_4(H)$ whereas the vertices depicted in black are all part of $\mathcal{V}_2(H)$. $\mathcal{E}(H)$ is the set of edges of triangles contained in ∂H. Here and in the following figures, we always use subsets of a regular triangular lattice for illustration purposes.

We can understand $(\mathcal{V}(H), \mathcal{E}(H))$ as a planar graph. Denoting by $\mathcal{F}(H)$ the bounded faces of this graph (i.e., the bounded planar regions delimited by edges in $\mathcal{E}(H)$), we observe that each component in $\mathcal{C}(H)$ corresponds to such a face, and additional faces come from the bounded connected components of $\mathbb{R}^2 \setminus \overline{H}$, see Figure 2.

We recall the *Euler formula* for planar graphs:

$$\#\mathcal{V}(H) - \#\mathcal{E}(H) + \#\mathcal{F}(H) = \nu(\mathcal{V}(H), \mathcal{E}(H)), \tag{3.4}$$

where $\nu(\mathcal{V}(H), \mathcal{E}(H))$ denotes the number of connected components of the planar graph $(\mathcal{V}(H), \mathcal{E}(H))$.

We cover ∂H by cycles in the graph. To this end, fix $H_i \in \mathcal{C}(H)$. For two vertices $v_1, v_2 \in \mathcal{V}(H) \cap \partial H_i$, we denote by $\overline{v_1\, v_2}$ the segment with endpoints v_1 and v_2. We now consider vertices $v_i^j \in \mathcal{V}(H) \cap \partial H_i$ that fulfill $\{\overline{v_i^j\, v_i^{j+1}}\}_{j=1}^{J-1} \subset \mathcal{E}(H)$ and $\overline{v_i^J\, v_i^1} \subset \mathcal{E}(H)$. If ∂H_i is connected, we can choose a tuple in such a way that the points in the tuple $(v_i^1, \ldots, v_i^J)$ coincide with $\mathcal{V}(H) \cap \partial H_i$. Note that $(v_i^1, \ldots, v_i^J)$ can contain the same vertex multiple times, which corresponds to additional cycles in the graph, see the first example in Figure 2. If ∂H_i consists of several connected components (see e.g. the second example in Figure 2), we repeat the argument for each component, and for simplicity collect all tuples in a single tuple, still denoted by $(v_i^1, \ldots, v_i^J)$. For $l \geq 0$, we define

$$\mathcal{D}_l(H) := \Big\{ H_i \in \mathcal{C}(H) \colon \#\{j = 1, \ldots, J \colon n(v_i^j) \geq 4\} = l \Big\}. \tag{3.5}$$

Roughly speaking, $\mathcal{D}_l(H)$ collects the set of components which touch other components at l different vertices. (Also self-intersections of one component are possible and taken into account depending on how often the vertex appears in $(v_i^1, \ldots, v_i^J)$, see Figure 2.) By an elementary computation we have that

$$\sum_{l \geq 1} l \#\mathcal{D}_l(H) = \sum_{k \geq 2} k \#\mathcal{V}_{2k}(H). \tag{3.6}$$

Indeed, the left-hand side can be written as $\sum_{i=1}^n \#\{j = 1 \ldots J \colon n(v_i^j) \geq 4\}$, and each $v \in \mathcal{V}_{2k}(H)$, $k \geq 2$, appears k-times in these cycles.

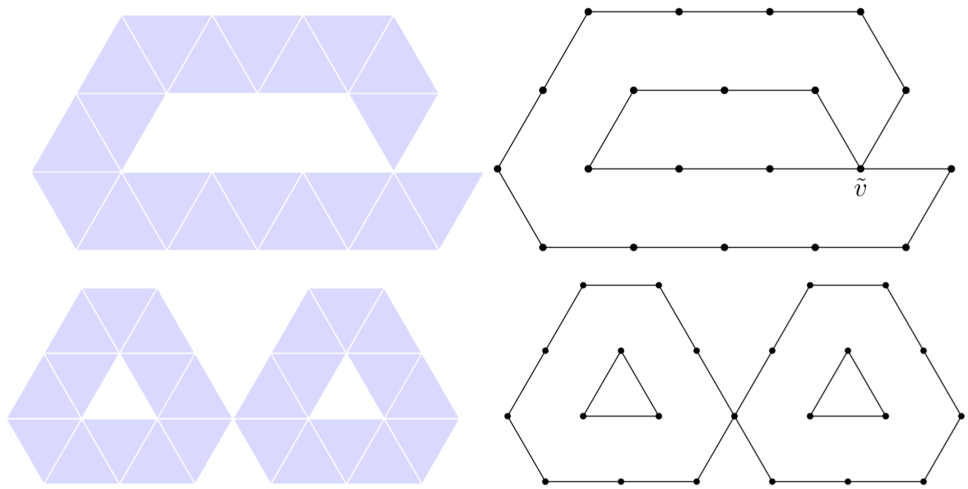

FIGURE 2. The first graphic depicts an example for a self-intersecting component $H_i \in \mathcal{D}_2(H)$ and the corresponding graph. Note that the vertex $\tilde{v}$ is contained twice in the corresponding tuple $(v_i^1, ..., v_i^J)$ and that $\mathbb{R}^2 \setminus \overline{H_i}$ consists of two components. In the second example, the two components $H_1, H_2 \in \mathcal{D}_1(H)$ have boundaries $\partial H_1, \partial H_2$ that consist of two connected components each.

In the proof of Theorem 2.1, our strategy will be to bound $\#\mathcal{V}_{2k}(H)$ for $k \geq 2$ since these vertices are related to triangles in $\mathbf{T}_H^{\mathrm{ex,bad}}$ (see Figure 1). In view of (3.6), this can be achieved by finding a suitable bound on $\#\mathcal{D}_l(H)$, $l \geq 1$. The following lemma shows that, under suitable assumptions on the components of $\mathbb{R}^2 \setminus \overline{H}$, it actually suffices to control $\mathcal{D}_1(H)$ and $\mathcal{D}_2(H)$.

Lemma 3.2. *Let H satisfy $\mathcal{H}^1(\partial H) \leq M$ and, denoting by $(F_i)_i$ the connected components of $\mathbb{R}^2 \setminus \overline{H}$, assume that $|F_i| > \varepsilon^2/\eta^2$ for every i. Then,*

$$\sum_{l \geq 3} l \# \mathcal{D}_l(H) \leq C \# \mathcal{D}_1(H) + C \frac{\eta}{\varepsilon}, \tag{3.7}$$

where $C > 0$ only depends on M.

Proof. By the assumption on F_i, using that $(F_i)_i$ are pairwise disjoint and the isoperimetric inequality, we get $\frac{\varepsilon}{\eta} \#(F_i)_i \leq \sum_i |F_i|^{1/2} \leq C \sum_i \mathcal{H}^1(\partial F_i) \leq C\mathcal{H}^1(\partial H)$. Noticing that $\mathcal{F}(H) = \mathcal{C}(H) \cup (F_i)_i$, we thus have

$$\#\mathcal{F}(H) \leq \#\mathcal{C}(H) + C\frac{\eta}{\varepsilon}\mathcal{H}^1(\partial H) \leq \#\mathcal{C}(H) + C\frac{\eta}{\varepsilon}. \tag{3.8}$$

Further, we note that

$$\#\mathcal{E}(H) = \sum_{k \geq 1} k \# \mathcal{V}_{2k}(H) \tag{3.9}$$

as each edge is associated to exactly two vertices. Then, from (3.4) we obtain

$$\#\mathcal{V}(H) + \big(\#\mathcal{F}(H) - \#\mathcal{C}(H)\big) + \#\mathcal{C}(H) = \nu(\mathcal{V}(H), \mathcal{E}(H)) + \#\mathcal{E}(H).$$

Using the definition of $(\mathcal{V}_{2k}(H))_k$ and $(\mathcal{D}_l(H))_l$ in this formula, as well as (3.9), we get

$$\sum_{k\geq 1}\#\mathcal{V}_{2k}(H) + \big(\#\mathcal{F}(H) - \#\mathcal{C}(H)\big) + \sum_{l\geq 0}\#\mathcal{D}_l(H) = \nu(\mathcal{V}(H),\mathcal{E}(H)) + \sum_{k\geq 1} k\#\mathcal{V}_{2k}(H).$$

As $k-1 \geq k/2$ for $k\geq 2$, this yields

$$\big(\#\mathcal{F}(H) - \#\mathcal{C}(H)\big) + \sum_{l\geq 0}\#\mathcal{D}_l(H) \geq \nu(\mathcal{V}(H),\mathcal{E}(H)) + \sum_{k\geq 2}\frac{k}{2}\#\mathcal{V}_{2k}(H).$$

This along with (3.6) shows

$$\big(\#\mathcal{F}(H) - \#\mathcal{C}(H)\big) + \sum_{l\geq 0}\#\mathcal{D}_l(H) \geq \nu(\mathcal{V}(H),\mathcal{E}(H)) + \sum_{l\geq 1}\frac{l}{2}\#\mathcal{D}_l(H),$$

and thus, as $l/2 - 1 \geq l/6$ for $l \geq 3$, we deduce

$$\big(\#\mathcal{F}(H) - \#\mathcal{C}(H)\big) + \#\mathcal{D}_0(H) + \frac{1}{2}\#\mathcal{D}_1(H) \geq \nu(\mathcal{V}(H),\mathcal{E}(H)) + \sum_{l\geq 3}\frac{l}{6}\#\mathcal{D}_l(H).$$

Eventually, we observe that clearly $\nu(\mathcal{V}(H),\mathcal{E}(H)) \geq \#\mathcal{D}_0(H)$ as each component in $\mathcal{D}_0(H)$ induces a connected component of the planar graph. This along with (3.8) shows (3.7). □

Healing of suitable sets. Let $\mathbf{N}(T;H)$ denote the triangles in $\overline{H\setminus T}$ sharing an edge with $T\in\mathbf{T}$. We let

$$\mathbf{M}_j(H) := \{T\in\mathbf{T}_H\colon \#\mathbf{N}(T;H) = j\} \quad \text{for } j=0,1,2,3. \tag{3.10}$$

In other words, $\mathbf{M}_j(H)$ consists of triangles in $\overline{H}$ that expose $3-j$ edges to $\Omega\setminus H$ (see Figure 3). We let

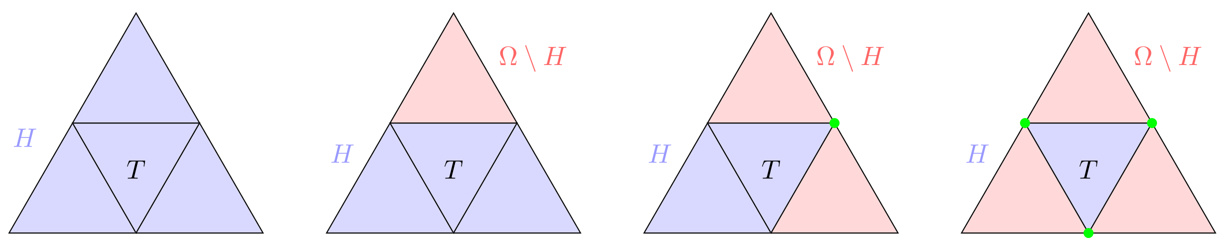

FIGURE 3. Triangles in the sets $\mathbf{M}_j(H)$. If we suppose that in all four pictures the other triangles which are not depicted are contained in $\Omega\setminus H$, the green vertices correspond to the vertices given in (3.11).

$$\mathbf{M}_j^{\rm heal}(H) := \{T\in\mathbf{M}_j(H)\colon \text{there exists } v\in\mathcal{N}(T) \text{ s.t. } v\notin\mathcal{N}(T') \text{ for all } T'\in\mathbf{T}_H\setminus\{T\}\} \quad \text{for } j=0,1. \tag{3.11}$$

We notice that for $T\in\mathbf{M}_1^{\rm heal}(H)$ the vertex v as in (3.11) is necessarily the one which is not contained in the edge shared with any other (different) triangle in $\mathbf{T}_H$, while for $\mathbf{M}_0^{\rm heal}(H)$ there could be more vertices satisfying the condition $v\notin\mathcal{N}(T')$ for all $T'\in\mathbf{T}_H\setminus\{T\}$, see Figure 3. In Figure 1, there is a set H with six triangles in $\mathbf{M}_1(H)$, of which two (the external ones in dark blue) are in $\mathbf{M}_1^{\rm heal}(H)$.

We also define the triangles which 'cannot be healed' (in the sense of Lemma 3.3 below) by

$$\mathbf{M}_j^{\rm nh}(H) := \mathbf{M}_j(H)\setminus\mathbf{M}_j^{\rm heal}(H) \quad \text{for } j=0,1, \tag{3.12}$$

and

$$H_{\rm heal} := H \setminus \bigcup_{j=0,1} \bigcup_{T\in \mathbf{M}_j^{\rm heal}(H)} T. \tag{3.13}$$

Each $T \in \mathbf{M}_0^{\rm heal}(H)$ satisfies $\mathcal{H}^1(\overline{H_{\rm heal}} \cap T) = 0$ which yields

$$\begin{aligned}\mathcal{H}^1(\partial H) &= \sum_{T\in\mathbf{T}_H} \mathcal{H}^1(\partial H\cap T)\\ &\geq \mathcal{H}^1(\partial H_{\rm heal}) + \sum_{T\in\mathbf{M}_1^{\rm heal}(H)} \big(\mathcal{H}^1(\partial H\cap T) - \mathcal{H}^1(\partial H_{\rm heal}\cap T)\big) + \sum_{T\in\mathbf{M}_0^{\rm heal}(H)} \mathcal{H}^1(\partial H\cap T).\end{aligned}$$

For $T \in \mathbf{M}_1^{\rm heal}(H)$ we have that $T \cap \overline{H_{\rm heal}}$ coincides with one edge of T or is empty, and thus by (3.1) we get $\mathcal{H}^1(\partial H_{\rm heal} \cap T) \leq \frac{2}{\varepsilon \sin\theta_0}|T|$, so that

$$\mathcal{H}^1(\partial H) \geq \mathcal{H}^1(\partial H_{\rm heal}) - \sum_{j=0,1}\sum_{T\in\mathbf{M}_j^{\rm heal}(H)} \Big(\frac{2}{\varepsilon\sin\theta_0}|T| - \mathcal{H}^1(\partial H\cap T)\Big). \tag{3.14}$$

Moreover, we observe that

$$\#\mathbf{M}_0^{\rm nh}(H) + \#\mathbf{M}_1^{\rm nh}(H) \leq \sum_{k\geq 2} k\#\mathcal{V}_{2k}(H). \tag{3.15}$$

In fact, setting $\mathcal{V}^{\rm nh}(H) := \bigcup_{l\geq 2}\mathcal{V}_{2l}(H)$, for each $T\in\mathbf{M}_1^{\rm nh}(H)$, the set $\mathcal{V}^{\rm nh}(H)\cap\mathcal{N}(T)$ contains the unique vertex v of T which is contained in both edges exposed to $\mathbb{R}^2\setminus H$, otherwise T would lie in $\mathbf{M}_1^{\rm heal}(H)$ by (3.11). Similarly, any $T\in\mathbf{M}_0^{\rm nh}(H)$ necessarily contains at least one vertex in $\mathcal{V}^{\rm nh}(H)$, by definition.

Recall that in Theorem 2.1 one considers a function $u\in H^1(\Omega;\mathbb{R}^2)$ that is piecewise affine on $\mathbf{T}$ with $\|e(u)\|_{L^2(\Omega\setminus H)} \leq C$. The symmetric gradient of such a function can still be controlled on $\Omega\setminus H_{\rm heal}$, as the following lemma shows.

Lemma 3.3 (Healing of triangles)**.** *Let $u\in H^1(\Omega;\mathbb{R}^2)$ be piecewise affine on $\mathbf{T}$ and let H be induced by $\mathbf{T}_H$, for a given subset $\mathbf{T}_H\subset\mathbf{T}$. Suppose that* $\mathrm{dist}(H,\partial\Omega)\geq\omega(\varepsilon)$*. Then, there exists a uniform constant $C>0$ such that*

$$\|e(u)\|_{L^2(\Omega\setminus H_{\rm heal})} \leq C\|e(u)\|_{L^2(\Omega\setminus H)}.$$

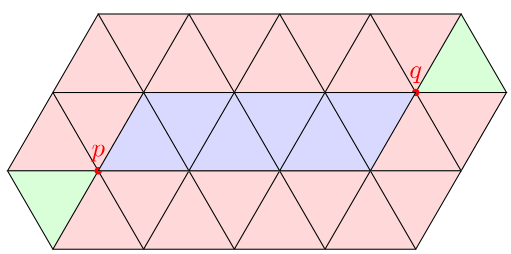

FIGURE 4. The sets Z (violet), Y (green), $N_Z\setminus Y$ (red), and $\{p,q\} = \overline{Y}\cap\overline{Z}$.

This result will be crucial in the proof of Theorem 2.1 as it enables us to 'heal away' the triangles in $H\setminus H_{\rm heal}$ without loosing control on the symmetric gradient. The condition on $\mathrm{dist}(H,\partial\Omega)$ ensures that a suitable neighborhood of H lies in Ω. We will also employ the following result, allowing to even heal suitable unions of triangles.

Lemma 3.4 (Healing of entire components). *Let $\mathbf{T}_Z \subset \mathbf{T}$ be such that the set Z induced by $\mathbf{T}_Z$ is connected with $|Z| \le \varepsilon^2/\eta^2$ and $Z = \mathrm{sat}(Z)$, i.e., Z has no holes. Suppose that $\mathrm{dist}(Z, \partial\Omega) \ge \omega(\varepsilon)$. Let $\mathbf{T}_{N_Z} \subset \mathbf{T}$ denote the set of all $T \in \mathbf{T} \setminus \mathbf{T}_Z$ with $T \cap \overline{Z} \neq \emptyset$, and let N_Z be the set induced by $\mathbf{T}_{N_Z}$. Moreover, let $\mathbf{T}_Y \subset \mathbf{T}_{N_Z}$ be such that $\overline{Y} \cap \overline{Z}$ consists of at most two points, where Y is induced by $\mathbf{T}_Y$, see Figure 4. Then, given $u \in H^1(N_Z \setminus \overline{Y}; \mathbb{R}^2)$ being piecewise affine on $\mathbf{T}_{N_Z} \setminus \mathbf{T}_Y$, there exists $u_{\mathrm{heal}} \in H^1(\mathrm{int}(\overline{Z \cup (N_Z \setminus Y)}); \mathbb{R}^2)$ with $u_{\mathrm{heal}} = u$ on $N_Z \setminus \overline{Y}$ such that*

$$\|e(u_{\mathrm{heal}})\|_{L^2(Z \cup (N_Z \setminus Y))} \le \frac{C}{\eta^\alpha} \|e(u)\|_{L^2(N_Z \setminus Y)}$$

for a universal constant $C > 0$ and some $\alpha \in \mathbb{N}$.

Note that the case $Y = \emptyset$ corresponds to the situation that $e(u)$ is controlled in an entire neighborhood N_Z of Z which allows to extend u inside Z according to classical extension theorems. The main point of this lemma is that such an extension of u from N_Z to $N_Z \cup Z$ is still possible in the presence of Y, as long as $\overline{Y} \cap \overline{Z}$ consists of *at most two points* (see Figure 4). Already for $\#(\overline{Y} \cap \overline{Z}) \ge 3$, the situation is less rigid and a statement as in Lemma 3.4 cannot be expected. Note that we will apply this result for sets Z which are connected components $H_j \in \mathcal{C}(H)$ for suitable H. In particular, we observe that the lemma can be applied for all components in $H_j \in \mathcal{D}_l(H)$, $l = 0, 1, 2$, which satisfy $|H_j| \le \varepsilon^2/\eta^2$ and have no holes. For convenience, we postpone the proofs of Lemma 3.3 and Lemma 3.4 to Section 3.3 below.

3.2. **Proof of Theorem 2.1 and Corollary 3.1.** We proceed with the proof of Theorem 2.1.

Proof of Theorem 2.1. For convenience, we first prove the result under the additional assumption that $\mathrm{dist}(A, \partial\Omega) \ge \omega(\varepsilon)$. We indicate the necessary adaptations for the general case at the end of the proof. Moreover, it is not restrictive to prove the result only for $\eta \le \eta_0$ for some universal η_0 chosen in (3.25) below.

First relation between area and boundary of A: From (3.1) and (3.10) we find

$$\begin{aligned} \frac{2}{\varepsilon \sin\theta_0}|A| &= \sum_{T \in \mathbf{T}_A} \frac{2}{\varepsilon \sin\theta_0}|T| \ge \sum_{T \in \mathbf{M}_2(A)} \mathcal{H}^1(\partial A \cap T) + \sum_{j=0,1} \sum_{T \in \mathbf{M}_j(A)} \frac{2}{\varepsilon \sin\theta_0}|T| \\ &= \mathcal{H}^1(\partial A) + \sum_{j=0,1} \sum_{T \in \mathbf{M}_j(A)} \Big(\frac{2}{\varepsilon \sin\theta_0}|T| - \mathcal{H}^1(\partial A \cap T) \Big). \end{aligned} \tag{3.16}$$

At this point, the energy bound (2.11) and (3.1) already give us the (unsharp) bounds

$$\#\mathbf{T}_A \le C/\varepsilon, \qquad \mathcal{H}^1(\partial A) \le C \tag{3.17}$$

for some $C > 0$ depending on C_0, where we used that $|T| \ge c\varepsilon^2$ for some $c > 0$ depending on θ_0.

Modification 1: Connected components and filling holes: We recall that A is an open set by definition. In the sequel, we will consider connected components of A and its complement. We will also consider connected components of $\overline{A}$ and its complement which may lead to different objects: while in connected components of A adjacent triangles share an edge, in connected components of $\overline{A}$ triangles may be linked solely by a vertex, see Figure 1.

For technical reasons related to Lemma 3.2 and Lemma 3.4, we need to avoid that $\overline{A}$ contains small holes. Therefore, we fill small holes of $\overline{A}$ as follows. We denote by $\mathcal{A}_{\mathrm{small}}$ the collection of connected components A_c of $\mathbb{R}^2 \setminus \overline{A}$ satisfying $|A_c| \le \varepsilon^2/\eta^2$. We define

$$B := \mathrm{int}\Big(\overline{A} \cup \bigcup_{A_c \in \mathcal{A}_{\mathrm{small}}} A_c \Big), \tag{3.18}$$

and denote by $\mathbf{T}_B$ the triangles contained in B. (For the illustration of a hole, we refer to Figure 2.) Then, we clearly have that B is induced by $\mathbf{T}_B$, i.e., $B = \mathrm{int}(\bigcup_{T\in\mathbf{T}_B} T)$. Moreover, each connected component of A is contained in a connected component of B. Note that by the definition of B we have $\mathbf{M}_j(B) \subset \bigcup_{i=0}^{j} \mathbf{M}_i(A)$ for $j = 0, 1, 2$. From (3.1), $\partial B \subset \partial A$, and arguing as in the first line of (3.16) we then obtain

$$\frac{2}{\varepsilon\sin\theta_0}|A| \geq \sum_{T\in\mathbf{M}_2(B)} \mathcal{H}^1(\partial B\cap T) + \sum_{j=0,1}\sum_{T\in\mathbf{M}_j(B)} \frac{2}{\varepsilon\sin\theta_0}|T|$$
$$= \mathcal{H}^1(\partial B) + \sum_{j=0,1}\sum_{T\in\mathbf{M}_j(B)} \Big(\frac{2}{\varepsilon\sin\theta_0}|T| - \mathcal{H}^1(\partial B\cap T)\Big). \tag{3.19}$$

The isoperimetric inequality along with (3.17) and $\sqrt{|A_c|} \leq \varepsilon/\eta$ for all $A_c \in \mathcal{A}_{\mathrm{small}}$ yield

$$\sum_{A_c\in\mathcal{A}_{\mathrm{small}}} |A_c| \leq \frac{\varepsilon}{\eta} \sum_{A_c\in\mathcal{A}_{\mathrm{small}}} \sqrt{|A_c|} \leq C\frac{\varepsilon}{\eta} \sum_{A_c\in\mathcal{A}_{\mathrm{small}}} \mathcal{H}^1(\partial A_c) \leq C\frac{\varepsilon}{\eta}.$$

Again using (3.17) and the fact that $|T| \geq c\varepsilon^2$, this shows

$$\#\mathbf{T}_B \leq \#\mathbf{T}_A + C\frac{1}{\varepsilon\,\eta} \leq C\frac{1}{\varepsilon\,\eta}, \qquad |B| \leq C\frac{\varepsilon}{\eta}, \qquad \mathcal{H}^1(\partial B) \leq C \tag{3.20}$$

for some $C > 0$ depending on C_0 and θ_0. We also note that, by the assumption on A, it clearly holds $B \subset \Omega$ with $\mathrm{dist}(B, \partial\Omega) \geq \omega(\varepsilon)$.

Motivation of next steps: For motivating the next steps of the proof, let us also introduce B_{heal} related to B as defined in (3.13). Taking $H = B$ in (3.14), along with (3.12) and (3.19) we deduce

$$\frac{2}{\varepsilon\sin\theta_0}|A| \geq \mathcal{H}^1(\partial B_{\mathrm{heal}}) + \sum_{j=0,1}\sum_{T\in\mathbf{M}_j^{\mathrm{nh}}(B)} \Big(\frac{2}{\varepsilon\sin\theta_0}|T| - \mathcal{H}^1(\partial T\cap B)\Big).$$

We observe that $\frac{2}{\varepsilon\sin\theta_0}|T| - \mathcal{H}^1(\partial T) \geq -C\varepsilon$ for each $T \in \mathbf{M}_0^{\mathrm{nh}}(B)\cup\mathbf{M}_1^{\mathrm{nh}}(B)$. In fact, it is elementary to show that $|T| \geq c(\mathcal{H}^1(\partial T))^2$ for c only depending on θ_0. Then, it suffices to observe that the minimum of the function $x \mapsto \frac{2}{\varepsilon\sin\theta_0}cx^2 - x$ is larger than $-C\varepsilon$. Therefore, we obtain

$$\frac{2}{\varepsilon\sin\theta_0}|A| \geq \mathcal{H}^1(\partial B_{\mathrm{heal}}) - C\varepsilon\big(\#\mathbf{M}_0^{\mathrm{nh}}(B) + \#\mathbf{M}_1^{\mathrm{nh}}(B)\big)\,, \tag{3.21}$$

and Lemma 3.3 gives the corresponding control of the function on $\Omega\setminus B_{\mathrm{heal}}$. By (3.6) and (3.15), in order to control $\#\mathbf{M}_0^{\mathrm{nh}}(B) + \#\mathbf{M}_1^{\mathrm{nh}}(B)$, it would be enough to control $l\#\mathcal{D}_l$ for every $l \in \mathbb{N}$. At this stage, one could use Lemma 3.2 (which applies to B in view of Modification 1) to control $l\#\mathcal{D}_l$ for $l \geq 3$ in terms of $\#\mathcal{D}_1$. Concerning the components in $\mathcal{D}_1$ and $\mathcal{D}_2$, we will treat them differently depending on whether they are *small* or *large*, according to the following definition: for a generic set H and for $l = 0, 1, 2$ we let

$$\mathcal{D}_l^{\mathrm{small}}(H) := \Big\{H_j \in \mathcal{D}_l(H)\colon |\mathrm{sat}(H_j)| \leq \frac{\varepsilon^2}{\eta^2}\Big\}, \quad \mathcal{D}_l^{\mathrm{large}}(H) := \mathcal{D}_l(H)\setminus\mathcal{D}_l^{\mathrm{small}}(H), \tag{3.22}$$

where $\mathrm{sat}(\cdot)$ is defined at the beginning of Section 3.1. Now the idea is that, to bound $\#\mathcal{D}_1$ and $\#\mathcal{D}_2$, it suffices to apply Lemma 3.4 to treat the small components in $\mathcal{D}_1^{\mathrm{small}}$, $\mathcal{D}_2^{\mathrm{small}}$, since the cardinality of the remaining ones is less than $C\eta/\varepsilon$. In fact, for H with $\mathcal{H}^1(\partial H) \leq C$ (as it holds for B, see (3.18) and (3.20)) it follows that

$$\#\mathcal{D}_l^{\mathrm{large}}(H) \leq C\frac{\eta}{\varepsilon}, \qquad l = 0, 1, 2. \tag{3.23}$$

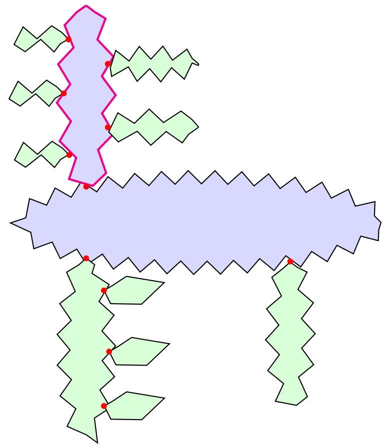

FIGURE 5. Schematic illustration of different components of $\overline{B}$. Note that $\overline{B}$ is connected and $\#\mathcal{C}(B) = 12$. The red dots depict the separating vertices. In this example the green parts correspond to the 7 elements in $\mathcal{G}_{\mathrm{sep}}^{\mathrm{small}}$. The component with magenta border depicts a possible component in $\mathcal{D}_1^{\mathrm{small}}(B_{\mathrm{sep}})$.

To see this, for each $H_j \in \mathcal{D}_l^{\mathrm{large}}(H)$, we apply the isoperimetric inequality to obtain $\mathcal{H}^1(\partial H_j) \geq \mathcal{H}^1(\partial\,\mathrm{sat}(H_j)) \geq c|\mathrm{sat}(H_j)|^{1/2} \geq c\varepsilon/\eta$ for some universal $c > 0$. Then, $\#\mathcal{D}_l^{\mathrm{large}}(H) \leq C\mathcal{H}^1(\partial H)\,\eta/\varepsilon \leq C\eta/\varepsilon$.

This program, however, cannot be pursued for B because $\#\mathcal{D}_1(B)$ explicitly appears on the right-hand side of (3.7), so simply healing the small components would not be enough. Unfortunately, in general, there is no direct bound available for $\#\mathcal{D}_1(B)$ and we need to perform another preliminary modification to control *a priori* $\#\mathcal{D}_1$ on a suitable subset of B. Roughly speaking, we need to get rid of $\mathcal{D}_1^{\mathrm{small}}$ *before* applying Lemma 3.2. Therefore, we will never actually use B_{heal} in the proof but only the object (3.33) below which is obtained after the following preliminary modification.

Modification 2: Dealing with unions of components in $\mathcal{D}_1(B)$ in terms of separating vertices. A vertex v in $\bigcup_{k\geq 2}\mathcal{V}_{2k}(B)$ is called a *separating vertex* if removing v and the associated edges from $(\mathcal{V}(B),\mathcal{E}(B))$ would increase the number of connected components of the graph $(\mathcal{V}(B),\mathcal{E}(B))$. Equivalently, in the topology of subsets of $\mathbb{R}^2$, this corresponds to considering the connected components of $\overline{B}\setminus\{v\}$ whose number would increase compared to the number of connected components of $\overline{B}$. We denote the set of *separating vertices* in B by $\mathcal{V}_{\mathrm{sep}}(B) \subset \mathcal{V}(B)$, see Figure 5.

For $v \in \mathcal{V}_{\mathrm{sep}}(B)$ let $\mathcal{G}_{\mathrm{sep}}(v)$ be the connected components of $\overline{B}\setminus\{v\}$. We define

$$\mathcal{G}_{\mathrm{sep}} := \bigcup_{v\in\mathcal{V}_{\mathrm{sep}}(B)} \mathcal{G}_{\mathrm{sep}}(v) \cup \mathcal{C}(\overline{B})\,.$$

Here, we explicitly add also $\mathcal{C}(\overline{B})$, i.e., all connected components of $\overline{B}$. (Note that this is redundant whenever there are at least two vertices in $\mathcal{V}_{\mathrm{sep}}(B)$ in different components of $\mathcal{C}(\overline{B})$.)

Note that each element in $\mathcal{G}_{\mathrm{sep}}$ consists of unions of components in $\mathcal{C}(B) = (B_j)_j$ (up to a set of negligible measure). We observe that two elements $G_1, G_2 \in \mathcal{G}_{\mathrm{sep}}$ satisfy $|G_1 \cap G_2| = 0$ or, up to relabeling, $G_1 \subset G_2$.

We define

$$\mathcal{G}_{\rm sep}^{\rm small} := \big\{G \in \mathcal{G}_{\rm sep} \colon |{\rm sat}(G)| \le \varepsilon^2/\eta^2, \text{ there is no } \tilde{G} \in \mathcal{G}_{\rm sep} \text{ with } |{\rm sat}(\tilde{G})| \le \varepsilon^2/\eta^2 \text{ and } G \subset \tilde{G}\big\}. \tag{3.24}$$

For each $G \in \mathcal{G}_{\rm sep}^{\rm small}$, it holds $\mathcal{H}^1(\partial G) \le \varepsilon/\eta^3$. In fact, $G = {\rm sat}(G)$ by the definition of B, see (3.18). As $c\varepsilon^2 \le |T|$ for some $c > 0$, we obtain $\#\mathbf{T}_G \le \frac{1}{c\eta^2}$. Thus, by the discrete Hölder inequality

$$\mathcal{H}^1(\partial G) \le \sum_{T \in \mathbf{T}_G} \mathcal{H}^1(\partial T) \le C \sum_{T \in \mathbf{T}_G} \sqrt{|T|} \le C\big(\#\mathbf{T}_G\big)^{1/2}\Big(\sum_{T \in \mathbf{T}_G} |T|\Big)^{1/2} \le C\frac{\varepsilon}{\eta^2} \le \frac{\varepsilon}{\eta^3}\,, \tag{3.25}$$

where we used that $\mathcal{H}^1(\partial T)^2 \le C|T|$ for some $C > 0$ only depending on θ_0, and the last step holds for $\eta \le \eta_0$ with η_0 small enough. Note that on each element in $\mathcal{G}_{\rm sep}^{\rm small}$ the assumptions of Lemma 3.4 are satisfied: if $G \in \mathcal{G}_{\rm sep}^{\rm small} \cap \mathcal{G}_{\rm sep}(v)$, in the notation of Lemma 3.4, $G = {\rm sat}(G)$ corresponds to $\overline{Z}$ and Y is the union of the triangles containing v and included in $(\overline{B} \setminus G) \cup \{v\}$ (therefore $\overline{Y} \cap \overline{G} = \{v\}$) (see Figure 5). The condition ${\rm dist}(B, \partial\Omega) \ge \omega(\varepsilon)$ ensures that $N_G \setminus Y \subset \Omega \setminus B$. If $G \in \mathcal{G}_{\rm sep}^{\rm small} \cap \mathcal{C}(\overline{B})$, the assumptions of Lemma 3.4 are fulfilled with $Y = \emptyset$. We define

$$B_{\rm sep} := B \setminus \bigcup_{G \in \mathcal{G}_{\rm sep}^{\rm small}} G, \tag{3.26}$$

and modify u on each $G \in \mathcal{G}_{\rm sep}^{\rm small}$ as in Lemma 3.4. This leads to a function $u_{\rm sep} \in H^1(\Omega;\mathbb{R}^2)$ with

$$\|e(u_{\rm sep})\|_{L^2(\Omega\setminus B_{\rm sep})} \le \frac{C}{\eta^\alpha}\|e(u)\|_{L^2(\Omega\setminus B)}. \tag{3.27}$$

Above, we used that the neighborhoods N_G in Lemma 3.4 related to different components $G \in \mathcal{G}_{\rm sep}^{\rm small}$ overlap only a bounded number of times depending on θ_0: in fact, a triangle belongs to N_G if it has nonempty intersection with some triangle in G, and for every $G \ne G' \in \mathcal{G}_{\rm sep}^{\rm small}$ it holds $G \cap G' = \emptyset$. Thus, since any $T \in \mathbf{T}$ has nonempty intersection with at most c (c depending only on θ_0) different triangles in $\mathbf{T}$, then any $T \in \mathbf{T}$ belongs to at most c different neighborhoods N_G.

Moreover, we have $\partial B_{\rm sep} \subset \partial B$ and by using (3.19) we get

$$\frac{2}{\varepsilon \sin\theta_0}|A| \ge \mathcal{H}^1(\partial B_{\rm sep}) + \sum_{j=0,1}\sum_{T \in \mathbf{M}_j(B_{\rm sep})}\Big(\frac{2}{\varepsilon\sin\theta_0}|T| - \mathcal{H}^1(\partial B_{\rm sep} \cap T)\Big). \tag{3.28}$$

This simply follows from the fact that we remove entire components from B: indeed, it holds that $\mathbf{M}_j(B_{\rm sep}) = \mathbf{M}_j(B) \cap \mathbf{T}_{\rm sep}$ for $\mathbf{T}_{\rm sep} := \{T \in \mathbf{T} \colon T \subset \overline{B_{\rm sep}}\}$. Moreover, for $j = 0, 1, 2$, we have that $\mathcal{H}^1(\partial B \cap T) = \mathcal{H}^1(\partial B_{\rm sep} \cap T)$ for every $T \in \mathbf{M}_j(B_{\rm sep})$ and $\mathcal{H}^1(\partial B \cap T) = \mathcal{H}^1((\partial B \setminus \partial B_{\rm sep}) \cap T)$ for every $T \in \mathbf{M}_j(B) \setminus \mathbf{M}_j(B_{\rm sep})$. Therefore, $\mathcal{H}^1(\partial B) - \mathcal{H}^1(\partial B_{\rm sep}) = \mathcal{H}^1(\partial B \setminus \partial B_{\rm sep})$ and hence

$$\begin{aligned}
&\mathcal{H}^1(\partial B) + \sum_{j=0,1}\sum_{T \in \mathbf{M}_j(B)}\Big(\frac{2}{\varepsilon\sin\theta_0}|T| - \mathcal{H}^1(\partial B \cap T)\Big) \\
&\quad - \Big(\mathcal{H}^1(\partial B_{\rm sep}) + \sum_{j=0,1}\sum_{T \in \mathbf{M}_j(B_{\rm sep})}\Big(\frac{2}{\varepsilon\sin\theta_0}|T| - \mathcal{H}^1(\partial B_{\rm sep} \cap T)\Big)\Big) \\
&= \mathcal{H}^1(\partial B \setminus \partial B_{\rm sep}) + \sum_{j=0,1}\sum_{T \in \mathbf{M}_j(B)\setminus\mathbf{M}_j(B_{\rm sep})}\Big(\frac{2}{\varepsilon\sin\theta_0}|T| - \mathcal{H}^1((\partial B \setminus \partial B_{\rm sep}) \cap T)\Big) \\
&= \sum_{T \in \mathbf{M}_2(B)\setminus\mathbf{M}_2(B_{\rm sep})} \mathcal{H}^1((\partial B \setminus \partial B_{\rm sep}) \cap T) + \sum_{j=0,1}\sum_{T \in \mathbf{M}_j(B)\setminus\mathbf{M}_j(B_{\rm sep})}\frac{2}{\varepsilon\sin\theta_0}|T| \ge 0.
\end{aligned}$$

This along with (3.19) shows (3.28). By Lemma 3.2 applied to $H = B_{\rm sep}$ we get

$$\sum_{l\geq 3} l\#\mathcal{D}_l(B_{\rm sep}) \leq C\#\mathcal{D}_1(B_{\rm sep}) + C\frac{\eta}{\varepsilon}. \tag{3.29}$$

Here, by using (3.20), the constant C only depends on C_0. The crucial point is that, differently from $\mathcal{D}_1(B)$, now $\mathcal{D}_1(B_{\rm sep})$ satisfies the additional fundamental property

$$\#\mathcal{D}_1(B_{\rm sep}) \leq C\eta/\varepsilon. \tag{3.30}$$

Indeed, recalling definition (3.22), by (3.20) and (3.23) we have

$$\#\mathcal{D}_l^{\rm large}(B_{\rm sep}) \leq C\eta/\varepsilon \quad \text{for } l = 0, 1, 2. \tag{3.31}$$

Now, consider a component $B^i_{\rm sep} \in \mathcal{D}_1^{\rm small}(B_{\rm sep})$ and the corresponding $v \in \mathcal{V}(B^i_{\rm sep})$ with $v \in \partial B^i_{\rm sep}$ and $n(v) \geq 4$, see (3.5). Note that v is a separating vertex of the set B considered above, i.e., $v \in \mathcal{V}_{\rm sep}(B)$. Then, the only reason why this component has not been removed from B in the construction of $B_{\rm sep}$ is the fact that there is some $G_i \in \mathcal{G}_{\rm sep}(v)$ with $G_i \supset B^i_{\rm sep}$ and $|{\rm sat}(G_i)| > \varepsilon^2/\eta^2$. For an example of such $B^i_{\rm sep}$, see the magenta-bordered component in Figure 5. As the sets $(G_i)_i$ are pairwise disjoint for different $B^i_{\rm sep} \in \mathcal{D}_1^{\rm small}(B_{\rm sep})$, by (3.20) and repeating the argument below (3.23) we can compute

$$\#\mathcal{D}_1^{\rm small}(B_{\rm sep})\frac{\varepsilon}{\eta} \leq \sum_{B^i_{\rm sep}\in\mathcal{D}_1^{\rm small}(B_{\rm sep})} |{\rm sat}(G_i)|^{1/2} \leq C \sum_{B^i_{\rm sep}\in\mathcal{D}_1^{\rm small}(B_{\rm sep})} \mathcal{H}^1(\partial G_i) \leq C\mathcal{H}^1(\partial B) \leq C\,,$$

and thus, together with (3.31) we get (3.30). Summarizing, in view of (3.29)–(3.31), we obtain

$$\#\mathcal{D}_0^{\rm large}(B_{\rm sep}) + \#\mathcal{D}_1(B_{\rm sep}) + \#\mathcal{D}_2^{\rm large}(B_{\rm sep}) + \sum_{l\geq 3} l\#\mathcal{D}_l(B_{\rm sep}) \leq C\frac{\eta}{\varepsilon}. \tag{3.32}$$

Modification 3: Healing. Eventually, we define

$$\widehat{B}_{\rm sep} := B_{\rm sep} \setminus \bigcup_{B^j_{\rm sep}\in\mathcal{D}_2^{\rm small}(B_{\rm sep})} B^j_{\rm sep},$$

i.e., we remove the small components $\mathcal{D}_2^{\rm small}(B_{\rm sep})$, and, recalling (3.13),

$$A_{\rm mod} := \Big(\widehat{B}_{\rm sep}\Big)_{\rm heal}. \tag{3.33}$$

Note that by construction $A_{\rm mod}$ cannot have 'holes' smaller than ε^2/η^2. Hence, we can use first Lemma 3.4 for components in $\mathcal{D}_2^{\rm small}(B_{\rm sep})$ and then Lemma 3.3 for triangles in $\mathbf{M}_j^{\rm heal}(\widehat{B}_{\rm sep})$, $j = 0, 1$, to find a function $u_{\rm mod} \in H^1(\Omega;\mathbb{R}^2)$ with

$$\|e(u_{\rm mod})\|_{L^2(\Omega\setminus A_{\rm mod})} \leq \frac{C}{\eta^\alpha}\|e(u_{\rm sep})\|_{L^2(\Omega\setminus B_{\rm sep})}.$$

(Here, for Lemma 3.4, we again use that neighborhoods only overlap a finite number of times, see the argument below (3.27).) Using that $A \subset B$ and then $\|e(u)\|_{L^2(\Omega\setminus B)} \leq \|e(u)\|_{L^2(\Omega\setminus A)}$, with (3.27) we get

$$\|e(u_{\rm mod})\|_{L^2(\Omega\setminus A_{\rm mod})} \leq \frac{C}{\eta^{2\alpha}}\|e(u)\|_{L^2(\Omega\setminus A)}. \tag{3.34}$$

Since we assumed (2.11), this gives the first part of (2.13).

Let us now confirm the second part: in view of (3.32), the main property of $\widehat{B}_{\rm sep}$ is that

$$\sum_{l\geq 1} l\#\mathcal{D}_l(\widehat{B}_{\rm sep}) \leq C\frac{\eta}{\varepsilon}. \tag{3.35}$$

Therefore, taking $H = \widehat{B}_{\rm sep}$ in (3.14), along with (3.28) and (3.12), by arguing as in (3.21), we deduce

$$\frac{2}{\varepsilon \sin\theta_0}|A| \geq \mathcal{H}^1(\partial A_{\rm mod}) - C\varepsilon\big(\#\mathbf{M}_0^{\rm nh}(\widehat{B}_{\rm sep}) + \#\mathbf{M}_1^{\rm nh}(\widehat{B}_{\rm sep})\big). \tag{3.36}$$

Taking $\widehat{B}_{\rm sep}$ as H in (3.6) and (3.15), we then get

$$\frac{2}{\varepsilon \sin\theta_0}|A| \geq \mathcal{H}^1(\partial A_{\rm mod}) - C\varepsilon \sum_{l\geq 1} l\#\mathcal{D}_l(\widehat{B}_{\rm sep}). \tag{3.37}$$

This along with (3.35) shows the second part of (2.13). Next, (2.12) follows from the fact that $\{u \neq u_{\rm mod}\} \subset (B\setminus A)\cup A = B$, $A_{\rm mod} \subset B$, and (3.20). Finally, to validate (2.14), in view of (3.35), we are left to estimate the number of components $\mathcal{D}_0(\widehat{B}_{\rm sep})$. First, $\#\mathcal{D}_0^{\rm large}(\widehat{B}_{\rm sep})$ is already controlled by (3.32). Each $A \in \mathcal{D}_0^{\rm small}(\widehat{B}_{\rm sep})$ is either some element of $\mathcal{G}_{\rm sep}^{\rm small}$, see the definition in (3.24), or a component in $\bigcup_{l\geq 1}\mathcal{D}_l(B_{\rm sep}) \setminus \mathcal{D}_2^{\rm small}(B_{\rm sep})$. In the first case, such small isolated components were already removed in Modification 2 (see (3.26)), i.e., it holds $\mathcal{D}_0^{\rm small}(\widehat{B}_{\rm sep}) \subset \bigcup_{l\geq 1}\mathcal{D}_l(B_{\rm sep}) \setminus \mathcal{D}_2^{\rm small}(B_{\rm sep})$. Then, the desired control follows from (3.32).

General case: Components close to $\partial\Omega$. Recall that, so far, we assumed that $\mathrm{dist}(A,\partial\Omega) \geq \omega(\varepsilon)$ as this allowed us to apply Lemmas 3.3–3.4 throughout the proof. In particular, we have healed the components $\mathcal{G}_{\rm sep}^{\rm small}$ in Modification 2 and the components $\mathcal{D}_2^{\rm small}(B_{\rm sep})$ in Modification 3. If such components have distance from $\partial\Omega$ smaller than $\omega(\varepsilon)$, the extension in Lemma 3.4 cannot be performed. Yet, we observe that all such components have diameter smaller than ε/η^3, see (3.25) for $\mathcal{G}_{\rm sep}^{\rm small}$ (the computation for $\mathcal{D}_2^{\rm small}(B_{\rm sep})$ is exactly the same). Thus, such components are contained in $\{x \in \Omega\colon \mathrm{dist}(x,\partial\Omega) \leq \omega(\varepsilon) + \varepsilon/\eta^3\}$, i.e., have empty intersection with $\Omega_{\varepsilon,\eta}$. In a similar fashion, triangles in Lemma 3.3 cannot be healed if their distance from $\partial\Omega$ is smaller than $\omega(\varepsilon)$ which means they are contained in $\{x\in\Omega\colon \mathrm{dist}(x,\partial\Omega)\leq 2\omega(\varepsilon)\} \subset \Omega\setminus\Omega_{\varepsilon,\eta}$. Summarizing, for sets $A\subset\Omega$ which do not satisfy $\mathrm{dist}(A,\partial\Omega)\geq\omega(\varepsilon)$ we get that (3.34) holds with $\Omega_{\varepsilon,\eta}\setminus A_{\rm mod}$ in place of $\Omega\setminus A_{\rm mod}$ which shows (2.13). □

Remark 3.5. The construction implies that $\partial T \cap \partial A_{\rm mod} \,\tilde{=}\, \emptyset$ for all $T \subset A_{\rm mod}$ with $T \notin \mathbf{T_A}$. In fact, for the set B defined in Modification 1 (see (3.18)) it clearly holds $\partial T \cap \partial B \,\tilde{=}\, \emptyset$ for all $T \in \mathbf{T}$ with $T \subset B\setminus A$. Then, we recall that in Modifications 2 and 3 we only remove entire saturated components or heal away triangles.

We now proceed with the proof of Corollary 3.1.

Proof of Corollary 3.1. Let $\mathbf{T}_{A^1} \subset \mathbf{T}_{A^2}$. First, we apply Theorem 2.1 to A^1 and A^2 and obtain $A^1_{\rm mod}, A^2_{\rm mod}$ as well as $u^1_{\rm mod}$ and $u^2_{\rm mod}$. We note that in general the monotonicity is not preserved in all the different modification steps carried out in the proof of Theorem 2.1. In fact, as will will explain below, this is the case for Modifications 1–2 of the construction above, but in Modification 3 one might remove components $D \in \mathcal{D}_2^{\rm small}(B^2_{\rm sep})$ with $A^1_{\rm mod}\cap D \neq \emptyset$. However, as we will point out, such components could have been 'healed away' already before in the construction of $A^1_{\rm mod}$.

Before we start, we observe that it suffices to show $A^1_{\rm mod} \subset A^2_{\rm mod}$. Indeed, due to $\mathbf{T}_{A^1} \subset \mathbf{T}_{A^2}$ and $A^1_{\rm mod}\subset A^2_{\rm mod}$, $T \in \mathbf{T}^{\rm mod}_{A^1}$ implies $T \in \mathbf{T}^{\rm mod}_{A^2}$, and thus $\mathbf{T}^{\rm mod}_{A^1} \subset \mathbf{T}^{\rm mod}_{A^2}$ directly follows.

Modification 1: For each connected component $A_c \in \mathcal{A}^1_{\rm small}$ of $\mathbb{R}^2\setminus\overline{A^1}$ with $|A_c| \leq \varepsilon^2/\eta^2$, we have that either $A_c \subset A^2$ or $A_c \setminus \overline{A^2}$ is a connected component of $\mathbb{R}^2\setminus\overline{A^2}$ with $|A_c\setminus\overline{A^2}| \leq \varepsilon^2/\eta^2$. Thus, $A_c\setminus\overline{A^2} \in \mathcal{A}^2_{\rm small}$. By the definition of B^1 and B^2 in Modification 1, see (3.18), we hence obtain $B^1 \subset B^2$.

Modification 2: Next, we need to show that $B^1_{\rm sep} \subset B^2_{\rm sep}$. Recalling the definition in (3.26) it is enough to show that

$$\bigcup_{G \in \mathcal{G}^{\rm small,2}_{\rm sep}} G \cap \overline{B^1} \subset \bigcup_{G \in \mathcal{G}^{\rm small,1}_{\rm sep}} G \tag{3.38}$$

For notational convenience, we formally denote the connected components $\mathcal{C}(\overline{B^i})$ of $\overline{B^i} = \overline{B^i} \setminus \emptyset$ with $\mathcal{G}^i_{\rm sep}(\emptyset)$. Then, by a slight abuse of notation, we add $\{\emptyset\}$ as a placeholder for a vertex to the set of separating vertices, i.e., $\tilde{\mathcal{V}}_{\rm sep}(B^i) = \mathcal{V}_{\rm sep}(B^i) \cup \{\emptyset\}$ and write $G \in \mathcal{G}^i_{\rm sep} = \bigcup_{v \in \tilde{\mathcal{V}}_{\rm sep}(B^i)} \mathcal{G}^i_{\rm sep}(v)$, and accordingly $\mathcal{G}^{\rm small,i}_{\rm sep}$ as in (3.24).

Let $x \in G \cap \overline{B^1}$ for some $G \in \mathcal{G}^{\rm small,2}_{\rm sep}$. By definition we know that $G = {\rm sat}(G) \in \mathcal{G}^2_{\rm sep}(v)$ for some $v \in \tilde{\mathcal{V}}_{\rm sep}(B^2)$, i.e., G is a connected component of $\overline{B^2} \setminus \{v\}$ with $|G| \le \varepsilon^2/\eta^2$. Note that, as $B^1 \subset B^2$, the set $G \cap \overline{B^1}$ possibly consists of different connected components of $\overline{B^1} \setminus \{v\}$, i.e., $G \cap \overline{B^1} = \bigcup_j G_j$ for suitable components $(G_j)_j$.

If $v \in \tilde{\mathcal{V}}_{\rm sep}(B^1)$, i.e., v is also a separating vertex of B^1, we can conclude by definition that $G_j \in \mathcal{G}^1_{\rm sep}$ for all j. As $|{\rm sat}(G_j)| \le |{\rm sat}(G \cap B^1)| \le |G| \le \varepsilon^2/\eta^2$, we get $G_j \in \mathcal{G}^{\rm small,1}_{\rm sep}$ for all j. Since $G \cap \overline{B^1} = \bigcup_j G_j$, we have $x \in G_j$ for some j.

If instead $v \notin \tilde{\mathcal{V}}_{\rm sep}(B^1)$, the connected components of $\overline{B^1}$ intersected with $\mathbb{R}^2 \setminus \{v\}$ are exactly the components of $\overline{B^1} \setminus \{v\}$, i.e., G_j as above are components of $\overline{B^1}$ intersected with $\mathbb{R}^2 \setminus \{v\}$. Hence, by setting accordingly $\tilde{G}_j = G_j$ or $\tilde{G}_j = G_j \cup \{v\}$, for all j we find a set $\tilde{G}_j \in \mathcal{G}^1_{\rm sep}(\emptyset) \subset \mathcal{G}^1_{\rm sep}$. Since $|G \cup \{v\}| = |G|$, we have in particular that $|{\rm sat}(\tilde{G}_j)| \le |G| \le \varepsilon^2/\eta^2$ and thus $\tilde{G}_j \in \mathcal{G}^{\rm small,1}_{\rm sep}$ for all j. As $G \cap \overline{B^1} = \bigcup_j \tilde{G}_j \setminus \{v\}$, we have $x \in G_j$ for some j, which concludes the proof of (3.38).

Modification 3: Note that $\widehat{B}^1_{\rm sep} \subset \widehat{B}^2_{\rm sep}$ is in principle not true as for a component $D \in \mathcal{D}^{\rm small}_2(B^2_{\rm sep})$ we might have $A^1_{\rm mod} \cap D \neq \emptyset$. However, because $D \in \mathcal{D}^{\rm small}_2(B^2_{\rm sep})$ we know that $\overline{D \cap B^1_{\rm sep}}$ touches $\overline{B^1_{\rm sep}} \setminus (D \cap B^1_{\rm sep})$ at most at two points. Since $D = {\rm sat}(D)$, we further get $|{\rm sat}(D \cap B^1_{\rm sep})| \le |D| \le \frac{\varepsilon^2}{\eta^2}$. In particular, ${\rm sat}(D \cap B^1_{\rm sep}) = D \cap B^1_{\rm sep}$ due to the construction of B^1, and the set $D \cap B^1_{\rm sep}$ (or, respectively, all its connected components) fulfill the assumptions of Lemma 3.4. This means we could have healed $D \cap B^1_{\rm sep}$ already in the construction of $\widehat{B}^1_{\rm sep}$, which then ensures $\widehat{B}^1_{\rm sep} \subset \widehat{B}^2_{\rm sep}$. Since the healing of triangles as in (3.33) also preserves the monotonicity, we have $A^1_{\rm mod} \subset A^2_{\rm mod}$. □

3.3. **Proof of healing lemmas.** It now remains to prove Lemma 3.3 and Lemma 3.4.

Proof of Lemma 3.3. For each $T \in \mathbf{M}^{\rm heal}_0(H) \cup \mathbf{M}^{\rm heal}_1(H)$, let N^*_T be the union of triangles in $\Omega \setminus H$ having nonempty intersection with T. Then, let N_T be the connected component of N^*_T containing the vertex in the definition (3.11). Moreover, let T^1 and T^2 be two adjacent triangles to T (i.e., sharing an edge with T) that are contained in N_T. (Notice that the choice is unique for $\mathbf{M}^{\rm heal}_1(H)$, and that there are up to three different choices of pairs $\{T^1, T^2\}$ for $\mathbf{M}^{\rm heal}_0(H)$.) For every u affine on any triangle T, we denote by $e(u)_T$ and $(\nabla u)_T$ the constant matrices $e(u)$ and ∇u on T, respectively.

By Korn's inequality we get a function $z(x) = u(x) - Ax$, where $A \in \mathbb{R}^{2\times 2}_{\rm skew}$, such that

$$\|\nabla z\|^2_{L^2(N_T)} \le K_{N_T} \|e(u)\|^2_{L^2(N_T)} \,. \tag{3.39}$$

We notice that the Korn constant K_{N_T} corresponding to N_T depends only on the parameter θ_0 associated to the family of admissible triangulations. Defining

$$A^j := (\nabla u)_{T^j} - e(u)_{T^j}$$

for $j = 1, 2$, we deduce

$$|A - A^j|^2\, |T^j| = \|A - A^j\|^2_{L^2(T^j)} \le 2\|\nabla z\|^2_{L^2(T^j)} + 2\|e(u)\|^2_{L^2(T^j)} \le 2(K_{N_T} + 1)\|e(u)\|^2_{L^2(N_T)}, \tag{3.40}$$

by the identity $\|e(u)\|^2_{L^2(T^j)} = |T^j||e(u)_{T^j}|^2$, the triangle inequality, and (3.39). Denoting by l_j the unit vectors parallel to the edge in common between T and T^j for $j = 1, 2$, we get

$$\begin{aligned}\|e(u)\|^2_{L^2(T)} &= |T||e(u)_T|^2 = |T||e(z)_T|^2 \le |T||(\nabla z)_T|^2 \le \bar{C}|T|\Big(|(\nabla z)_T \cdot l_1|^2 + |(\nabla z)_T \cdot l_2|^2\Big)\\ &= \bar{C}|T|\Big(|(\nabla z)_{T^1} \cdot l_1|^2 + |(\nabla z)_{T^2} \cdot l_2|^2\Big)\end{aligned}$$

for $\bar{C} > 0$ depending only on θ_0, where we used that by the continuity of z it holds $(\nabla z)_T \cdot l_j = (\nabla z)_{T^j} \cdot l_j$ for $j = 1, 2$. Since $(\nabla z)_{T^j} = (\nabla u)_{T^j} - A = e(u)_{T^j} + A^j - A$, we hence obtain by (3.40)

$$\begin{aligned}\|e(u)\|^2_{L^2(T)} &\le \bar{C}|T|\Big(|(\nabla z)_{T^1} \cdot l_1|^2 + |(\nabla z)_{T^2} \cdot l_2|^2\Big) \le \bar{C}|T| \sum_{j=1,2} \Big(|e(u)_{T^j}|^2 + |A - A^j|^2\Big)\\ &\le \bar{C} \sum_{j=1,2} \frac{|T|}{|T^j|}\Big(\|e(u)\|^2_{L^2(T_j)} + 2(K_{N_T} + 1)\|e(u)\|^2_{L^2(N_T)}\Big) \le \bar{C}'\|e(u)\|^2_{L^2(N_T)},\end{aligned}$$

where $\bar{C}' > 0$ just depends on θ_0, recalling that both K_{N_T} and the volume ratio between adjacent triangles depend only on θ_0. We conclude by summing over $T \in \mathbf{M}_0^{\mathrm{heal}}(H) \cup \mathbf{M}_1^{\mathrm{heal}}(H)$, observing that each triangle in $\Omega \setminus H$ could belong at most to a bounded number (depending on θ_0) of different N_T. □

Proof of Lemma 3.4. Along the proof we denote by N, $\mathbf{T}_N$ the sets N_Z, $\mathbf{T}_{N_Z}$ and by C a universal positive constant, possibly varying from line to line, depending only on the parameter of the triangulation θ_0. Moreover, C_η denotes a generic constant of the form $C\eta^{-\alpha}$ for some $\alpha \in \mathbb{N}$.

We first show that

$$\#\mathbf{T}_N \le \frac{C}{\eta^2}, \qquad |T| \le C\varepsilon^2/\eta^2 \text{ for all } T \in \mathbf{T}_N\,. \tag{3.41}$$

Indeed, recall that each edge in $\mathbf{T}$ has at least length ε, so that we obtain $\varepsilon\#\mathcal{V}(Z) \le \mathcal{H}^1(\partial Z)$. Using $|Z| \le \varepsilon^2/\eta^2$, by repeating the calculation in (3.25), we get $\mathcal{H}^1(\partial Z) \le C\varepsilon/\eta^2$, and thus $\#\mathcal{V}(Z) \le \varepsilon^{-1}\mathcal{H}^1(\partial Z) \le C/\eta^2$. Since each vertex in $\mathcal{V}(Z)$ is contained in only a bounded number of triangles in $\mathbf{T}_N$ (depending on θ_0), we obtain the estimate $\#\mathbf{T}_N \le C/\eta^2$. Each $T \in \mathbf{T}$, $T \subset Z$, satisfies $|T| \le \varepsilon^2/\eta^2$. Thus, as the area of adjacent triangles is comparable by a constant depending on θ_0, we conclude the second part of (3.41).

Since Z has no holes, we can suppose (up to enlarging Y) that $N \setminus Y$ consists of two connected components N_1 and N_2 whose closures contain the two touching points p, $q \in \overline{Y} \cap \overline{Z}$ (see Figure 4). In fact, if $N \setminus Y$ had further components, their closure would only intersect with Z at one of the touching points and not share an edge with a triangle in $\overline{Z}$. We now claim that there are $A_1, A_2 \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ such that

$$\|\nabla u - A_j\|_{L^2(T)} \le C_\eta \|e(u)\|_{L^2(N_j)} \qquad \text{for each triangle } T \subset \overline{N}_j,\ \ j = 1, 2. \tag{3.42}$$

To see this, for every $T \subset \overline{N}_j$, let

$$(\nabla u)_T = e(u)_T + A_T, \tag{3.43}$$

for $(\nabla u)_T$, $e(u)_T$, and A_T suitable matrices representing the constant values of ∇u, $e(u)$, and $\nabla u - e(u)$ on T. Given two adjacent triangles $T_1, T_2 \in \mathbf{T}_N$ in N_j, i.e., sharing an edge, let us consider the circle $C_{1,2}$ with the maximal radius among those circles centered on a point of the common edge and included in $T_1 \cup T_2$. The radius of any $C_{1,2}$ is larger than ε/C, where C depends only on θ_0. Applying Korn's inequality on $C_{1,2}$, we find $A_{1,2} \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ and $K > 0$ (the Korn constant for a circle) such that

$$\|\nabla u - A_{1,2}\|_{L^2(C_{1,2})} \leq K\|e(u)\|_{L^2(C_{1,2})},$$

and then

$$\|A_{T_j} - A_{1,2}\|_{L^2(C_{1,2}\cap T_j)} \leq (K+1)\|e(u)\|_{L^2(C_{1,2})}.$$

Since A_{T_j} are constant matrices, noticing that $\|A_{T_j} - A_{1,2}\|^2_{L^2(C_{1,2}\cap T_j)} = \frac{|C_{1,2}|}{2}|A_{T_j} - A_{1,2}|^2$, we deduce by the triangle inequality that

$$|A_{T_1} - A_{T_2}|^2 \leq \frac{8}{|C_{1,2}|}(K+1)^2\|e(u)\|^2_{L^2(C_{1,2})}. \tag{3.44}$$

Being N_j connected, given any $\tilde{T}_1$, $\tilde{T}_2 \subset \overline{N}_j$, there are triangles $\tilde{\mathbf{T}} := (\hat{T}_j)_{j=1}^n \subset \mathbf{T}$ included in $\overline{N_j}$ with $\hat{T}_1 = \tilde{T}_1$, $\hat{T}_n = \tilde{T}_2$, such that $\hat{T}_j$, $\hat{T}_{j+1}$ are adjacent for all $j = 1, \dots, n-1$. We apply the estimate (3.44) for pairs of adjacent triangles a finite number of times (less than $\#\mathbf{T}_N$). Then, using the triangle inequality and also noting that the sets $C_{1,2}$ are such that $|C_{1,2}| \geq \varepsilon^2/C$ and that circles corresponding to different pairs overlap at most twice, we find that

$$|A_{\tilde{T}_1} - A_{\tilde{T}_2}| \leq \frac{C}{\varepsilon}\sqrt{\#\mathbf{T}_N}\|e(u)\|_{L^2(N_j)} \leq \frac{C_\eta}{\varepsilon}\|e(u)\|_{L^2(N_j)}, \tag{3.45}$$

where in the last step we used (3.41). We now confirm (3.42) by fixing A_j as one of the A_T, $T \subset \overline{N_j}$. Notice that for every $T \subset \overline{N}_j$, recalling the notation (3.43), we indeed have by (3.41) and (3.45)

$$\|\nabla u - A_j\|_{L^2(T)} \leq \|\nabla u - A_T\|_{L^2(T)} + \|A_j - A_T\|_{L^2(T)} = \|e(u)\|_{L^2(T)} + \|A_j - A_T\|_{L^2(T)} \leq C_\eta\|e(u)\|_{L^2(N_j)}.$$

This concludes the proof of (3.42).

By the fact that u is affine on each $T \in \mathbf{T}_N$, $|T| \geq \varepsilon^2/C$, and by (3.42) it also follows that

$$|(\nabla u)_T - A_j| = \|\nabla u - A_j\|_{L^\infty(T)} \leq \frac{C_\eta}{\varepsilon}\|e(u)\|_{L^2(N_j)} \qquad \text{for } j = 1,2. \tag{3.46}$$

The fundamental point in the proof is now that

$$|A_2 - A_1| \leq \frac{C_\eta}{\varepsilon}\|e(u)\|_{L^2(N\setminus Y)}. \tag{3.47}$$

In fact, recall that p and q are connected by a path consisting of less than $\#\mathbf{T}_N$ edges of triangles in $\mathbf{T}_N$. Then, by (3.46) and the Fundamental Theorem of Calculus applied on any edge of the path connecting p and q to the scalar-valued functions $(u - A_j\cdot)_i$, $j = 1,2$, $i = 1,2$ (here, $(u - A_j\cdot)_i$ are the two components of $x \mapsto u(x) - A_j x$ for $j = 1,2$), we get

$$\big|\big(u(q) - u(p) - A_j(q-p)\big)_i\big| \leq C_\eta\|e(u)\|_{L^2(N_j)}, \tag{3.48}$$

where we used that the path of edges has a length of order $\sim \frac{\varepsilon}{\eta^2}$, because $\mathcal{H}^1(\partial Z) \leq C\varepsilon/\eta^2$, see below (3.41). Subtracting the two terms in (3.48) for $j = 1,2$ and for fixed i, we get

$$|\big((A_2 - A_1)(q-p)\big)_i| \leq C_\eta\|e(u)\|_{L^2(N\setminus Y)},$$

which confirms (3.47) since A_j are skew symmetric and $|q-p| \in [\varepsilon, C\eta^{-2}\varepsilon]$.

Combining (3.46) and (3.47) we deduce that

$$\|\nabla u - A_1\|_{L^\infty(N\setminus Y)} \leq \frac{C_\eta}{\varepsilon}\|e(u)\|_{L^2(N\setminus Y)}. \tag{3.49}$$

By McShane's theorem we find a Lipschitz extension $\widetilde{w}$ of $u - A_1\cdot$ from $N \setminus \overline{Y}$ to $\mathrm{int}(\overline{Z \cup (N \setminus Y)})$ whose components have the same Lipschitz constant as $u - A_1\cdot$. In particular,

$$\|\nabla \widetilde{w}\|_{L^\infty((N\setminus Y)\cup Z)} \le \frac{C_\eta}{\varepsilon}\|e(u)\|_{L^2(N\setminus Y)}, \qquad \widetilde{w} = u - A_1\cdot \text{ on } N \setminus \overline{Y}. \tag{3.50}$$

We set

$$u_{\mathrm{heal}} := \widetilde{w} + A_1\cdot\,.$$

The second condition in (3.50) immediately gives that $u_{\mathrm{heal}} = u$ on $N \setminus \overline{Y}$. Moreover,

$$\begin{aligned}\|e(u_{\mathrm{heal}})\|_{L^2((N\setminus Y)\cup Z)} &= \|e(\widetilde{w})\|_{L^2((N\setminus Y)\cup Z)} \le \|\nabla \widetilde{w}\|_{L^2((N\setminus Y)\cup Z)}\\ &\le |(N \setminus Y)\cup Z|^{1/2}\|\nabla\widetilde{w}\|_{L^\infty((N\setminus Y)\cup Z)} \le C_\eta\|e(u)\|_{L^2(N\setminus Y)},\end{aligned}$$

where in the last inequality we used (3.50) and the fact that $|Z| \le \varepsilon^2/\eta^2$ by assumption as well as that $|N|$ is controlled by $C\varepsilon^2/\eta^4$ due to (3.41), respectively. This concludes the proof. □

4. Approximation of quasi-static crack growth

This section is devoted to the formulation of our main result. We present a convergence result for an evolutionary problem with respect to the adaptive finite-element approximation introduced in Section 2. More precisely, we set up a time-incremental minimization scheme and prove the convergence to a continuum quasi-static crack growth in the spirit of Francfort and Larsen [26]. In particular, we recover the existence result of a fracture evolution in linearized elasticity [33]. The main issue compared to the Γ-convergence result in Section 2.2 consists in dealing with the irreversibility of the fracture process.

4.1. **Quasi-static adaptive finite element model.** We define an arbitrary sequence $(\varepsilon_n)_n \subset (0,\infty)$ with $\varepsilon_n \to 0$ as $n \to \infty$. Instead of considering general densities f with properties (2.2), we consider for simplicity only the special case $f(t) = t \wedge \kappa$ and $\mathbb{C} = \mathrm{Id}_{2\times2\times2\times2}$. The case of general $\mathbb{C}$ can be treated in the same way, adjusting the notation accordingly. Moreover, we assume that the function $\omega\colon \mathbb{R}^+ \to \mathbb{R}^+$ is given by

$$\omega(\varepsilon_n) = 10^6\varepsilon_n. \tag{4.1}$$

The constant 10^6 is chosen for definiteness only. This assumption could be removed at the expense of additional estimates which we omit for simplicity.

In order to introduce boundary conditions on a part $\partial_D\Omega \subset \partial\Omega$ of the boundary, we impose boundary conditions in a *neighborhood* of the boundary. More precisely, we suppose that there exists another Lipschitz set $\Omega' \supset \Omega$ with $\partial_D\Omega = \partial\Omega \cap \Omega'$ such that also $\Omega' \setminus \overline{\Omega}$ is Lipschitz. For a given boundary datum $g \in W^{2,\infty}(\Omega';\mathbb{R}^2)$ and a triangulation $\mathbf{T}_n \in \mathcal{T}_{\varepsilon_n}(\Omega')$, we define

$$g_{\mathbf{T}_n} \text{ as the piecewise affine interpolation of } g \text{ on } \mathbf{T}_n. \tag{4.2}$$

Recalling (2.4), we then consider the energy

$$E_n(u) := \int_\Omega |e(u)|^2 \wedge \frac{\kappa}{\varepsilon_n}\,\mathrm{d}x \tag{4.3}$$

if $u \in \mathcal{A}_{\varepsilon_n}(\Omega')$ and if for the (possibly non uniquely chosen) triangulation $\mathbf{T}_n(u) \in \mathcal{T}_{\varepsilon_n}(\Omega')$ (see (2.1)) it holds $u = g_{\mathbf{T}_n(u)}$ on each triangle $T \in \mathbf{T}_n(u)$ such that $T \cap \overline{\Omega} = \emptyset$. Otherwise, we set $E_n(u) = +\infty$. We emphasize that the energy is still defined as an integral over Ω although the functions u are defined on the larger set Ω'.

Now we introduce a time discrete evolution which is driven by time-dependent boundary conditions $g \in W^{1,1}(0,T;W^{2,\infty}(\Omega';\mathbb{R}^2))$. Given a sequence $(\delta_n)_n \subset (0,\infty)$ with $\delta_n \to 0$ and for each δ_n, we consider the subdivision $0 = t_n^0 < \cdots < t_n^{T/\delta_n} = T$ of the interval $[0,T]$ with step size δ_n. (Without restriction, we assume that $T/\delta_n \in \mathbb{N}$.) Correspondingly, let $(g(t_n^k))_k$ be the sequence of boundary data at different time steps $k \in \{0,\ldots,T/\delta_n\}$.

We assume for the moment that a displacement history $(u_n^j)_{j<k}$ at time steps $(t_n^j)_{j<k}$ is given, and introduce admissible competitors and the energy for the next time step, taking into account the irreversibility of the process.

Consider $u \in \mathcal{A}_{\varepsilon_n}(\Omega')$ and the corresponding triangulation $\mathbf{T}_n(u)$. Recall the definition of $\mathbf{T}_{\varepsilon_n}^{\rm big} = \{T \in \mathbf{T}_n(u) \colon \varepsilon_n |e(u)_T|^2 \geq \kappa\}$ and the definition of $\Omega_{\varepsilon_n}^{\rm big}(u)$ in (2.7). In place of $\Omega_{\varepsilon_n}^{\rm big}(u)$, we now define a possibly larger 'crack set' $\Omega_n^{\rm crack}(u)$ by considering also triangles that are very far away from a regular background mesh. More precisely, let $\mathbf{Z}_n$ be the triangulation that is based on the square grid of size $\varepsilon_n' := 2\varepsilon_n \cos(\theta_0)$ with nodes contained in $\varepsilon_n'\mathbb{Z}^2 \cap \Omega'$, where each square is then cut into two triangles along the diagonal (see [14, Figure 5.11]). For a function $u \in \mathcal{A}_{\varepsilon_n}(\Omega')$ with triangulation $\mathbf{T}_n(u)$, we define $\mathbf{Z}_n(u) := \mathbf{T}_n(u) \cap \mathbf{Z}_n$ as the part of the triangulation that belongs to this regular background mesh and let ${\rm dist}(T, \mathbf{Z}_n(u)) := \min\{{\rm dist}(T,\tilde{T}) \colon \tilde{T} \in \mathbf{Z}_n(u)\}$. We then define

$$\mathbf{T}_n^{\rm crack}(u) := \big\{T \in \mathbf{T}_n(u) \colon \varepsilon_n|e(u)_T|^2 \geq \kappa \quad \text{or} \quad {\rm dist}(T, \mathbf{Z}_n(u)) \geq 10^6\varepsilon_n\big\}. \tag{4.4}$$

The associated crack set is then defined as the union of all such triangles in Ω', this means

$$\Omega_n^{\rm crack}(u) := {\rm int}\Big(\bigcup_{T \in \mathbf{T}_n^{\rm crack}(u)} T\Big) \cap \Omega'\,. \tag{4.5}$$

Note that, additionally to the condition on the gradient, we also regard triangles as 'cracked' if they are far away from a fixed background mesh. This means in particular that, if $\mathbf{Z}_n(u) = \emptyset$, we would have $\Omega_n^{\rm crack}(u) = \Omega'$. This condition is inspired by the construction of recovery sequences in [14, Appendix] where all triangles are close to a background mesh, i.e., in that situation the additional condition is not active. In our evolutionary setting, we expect the same, and thus the condition is merely of technical nature. Let us also emphasize that the constant 10^6 is chosen for definiteness only and could be chosen arbitrarily large, but fixed. Both requirements in (4.4) will turn out to be crucial for our proof of the stability of the static equilibrium property, see Theorem 6.7 below.

Given a displacement history $(u_n^j)_{j<k}$, we define

$$\mathbf{T}_{n,k-1}^{\rm crack} := \bigcup_{j<k} \mathbf{T}_n^{\rm crack}(u_n^j) \qquad \Omega_{n,k-1}^{\rm crack} := \bigcup_{j<k} \Omega_n^{\rm crack}(u_n^j)\,. \tag{4.6}$$

For a given displacement $u \in \mathcal{A}_{\varepsilon_n}(\Omega')$ we also set

$$\mathbf{T}_{n,k-1}^{\rm crack}(u) := \mathbf{T}_{n,k-1}^{\rm crack} \cup \mathbf{T}_n^{\rm crack}(u) \qquad \Omega_{n,k-1}^{\rm crack}(u) := \Omega_{n,k-1}^{\rm crack} \cup \Omega_n^{\rm crack}(u)\,.$$

Similar to the splitting in (2.8), we define the corresponding history-dependent energy by

$$\mathcal{E}_n(u,(u_n^j)_{j<k}) := \int_{\Omega\setminus\Omega_{n,k-1}^{\rm crack}(u)} |e(u)|^2 \,+\, \kappa\,\frac{|\Omega_{n,k-1}^{\rm crack}(u)|}{\varepsilon_n} =: \mathcal{E}_n^{\rm elast}(u,(u_n^j)_{j<k}) + \mathcal{E}_n^{\rm crack}(u,(u_n^j)_{j<k}). \tag{4.7}$$

Note that the set $\Omega_{n,k-1}^{\rm crack}$ and thus the energy take the 'cracked triangles' of all previous time steps into account. In general, the triangulations at each time could be different and without additional requirements it is not guaranteed that the union $\Omega_{n,k-1}^{\rm crack}$ is consistent with an admissible triangulation. In particular, $\mathbf{T}_{n,k-1}^{\rm crack}$ is not necessarily a triangulation partitioning $\Omega_{n,k-1}^{\rm crack}$. For this reason, we introduce a further restriction as we set up the time-incremental minimization scheme. More precisely, recalling (2.1), we set $\hat{\mathcal{A}}_n^0(\Omega') := \mathcal{A}_{\varepsilon_n}(\Omega')$ and for $k \geq 1$ we introduce the set $\hat{\mathcal{A}}_n^k(\Omega') \subset \mathcal{A}_{\varepsilon_n}(\Omega')$ that depends on the displacement history $(u_n^j)_{j<k}$ and consists of all functions $u\colon \Omega' \to \mathbb{R}^2$ such that there exists a triangulation $\mathbf{T}_n \in \mathcal{T}_{\varepsilon_n}(\Omega')$ with u being piecewise affine on $\mathbf{T}_n$ and such that $\mathbf{T}_n$ fulfills

$$\mathbf{T}^{\mathrm{crack}}_{n,k-1} \subset \mathbf{T}_n. \tag{4.8}$$

We then define

$$\mathcal{A}_n^k := \big\{ v \in \hat{\mathcal{A}}_n^k(\Omega') \text{ and } v = g(t_n^k)_{\mathbf{T}_n(v)} \text{ on all } T \in \mathbf{T}_n(v) \text{ with } T \cap \overline{\Omega} = \emptyset \big\}. \tag{4.9}$$

Inductively, provided that $u_n^j \in \mathcal{A}_n^j$ for $0 \le j \le k-1$, we see that $\hat{\mathcal{A}}_n^k(\Omega') \neq \emptyset$ and then also $\mathcal{A}_n^k \neq \emptyset$ for all $k \ge 1$ since the triangulation $\mathbf{T}_n(u_n^{k-1})$ satisfies $\mathbf{T}^{\mathrm{crack}}_{n,k-1} \subset \mathbf{T}_n(u_n^{k-1})$.

We suppose that the *initial value* $u_n^0 \in \mathcal{A}_n^0$ is a minimum configuration in the sense that

$$u_n^0 \in \operatorname{argmin}\big\{ E_n(v) \colon v \in \mathcal{A}_n^0 \big\}\,, \tag{4.10}$$

with E_n given as in (4.3). We inductively define an evolution as follows: given $(u_n^j)_{0 \le j \le k-1}$, we let

$$u_n^k \in \operatorname{argmin}\big\{ \mathcal{E}_n(v, (u_n^j)_{j<k}) : v \in \mathcal{A}_n^k \big\}, \tag{4.11}$$

i.e., the minimization problem involves the previous time steps, according to the definition in (4.7). The existence of minimizers in (4.11) immediately follows from the direct method since $\mathcal{A}_n^k \neq \emptyset$, the problem is finite dimensional for a fixed triangulation, $\mathcal{E}_n(\cdot, (u_n^j)_{j<k})$ is continuous, and the set of admissible interpolations $\mathcal{A}_n^k$ is compact.

4.2. **Quasi-static fracture evolution.** We consider the *Griffith energy*

$$\mathcal{E}(u, K) := \int_\Omega |e(u)|^2 \,\mathrm{d}x + \kappa \sin(\theta_0)\, \mathcal{H}^1(K)\,, \tag{4.12}$$

for each $u \in GSBD^2(\Omega')$ and each rectifiable set $K \subset \Omega \cup \partial_D \Omega$ with $\mathcal{H}^1(K) < +\infty$, where $e(u)$ denotes the approximate symmetric gradient and J_u is the jump set of u, which is subject to the constraint $J_u \,\tilde{\subset}\, K$. (Here and in the following, $\tilde{\subset}$ stands for inclusions up to $\mathcal{H}^1$-negligible sets.) We highlight that, although the elastic energy is defined on Ω, the functions are defined on the larger set Ω' and the crack sets K may intersect the Dirichlet boundary $\partial_D\Omega$.

By $AD(g, H)$ we denote all functions $v \in GSBD^2(\Omega')$ such that

$$v = g \text{ on } \Omega' \setminus \overline{\Omega}, \quad J_v \,\tilde{\subset}\, H. \tag{4.13}$$

Definition 4.1. We define an *irreversible quasi-static crack evolution* with respect to the boundary condition $g \in W^{1,1}(0, T; W^{2,\infty}(\Omega'; \mathbb{R}^2))$ as any mapping $t \to (u(t), \Gamma(t))$ with $u(t) \in AD(g(t), \Gamma(t))$ for all $t \in [0, T]$ such that the following four conditions hold:

(a) *Initial condition*: $u(0)$ minimizes $\mathcal{E}(u, J_u)$ given in (4.12) among all $v \in GSBD^2(\Omega')$ with $v = g(0)$ on $\Omega' \setminus \overline{\Omega}$.
(b) *Irreversibility*: $\Gamma(t_1) \,\tilde{\subset}\, \Gamma(t_2)$ for all $0 \le t_1 \le t_2 \le T$.
(c) *Global stability*: For every $t \in (0, T]$, for every H with $\Gamma(t) \,\tilde{\subset}\, H$, and for every $v \in AD(g(t), H)$ it holds that

$$\mathcal{E}(u(t), \Gamma(t)) \le \mathcal{E}(v, H)\,. \tag{4.14}$$

(d) *Energy balance*: The function $t \mapsto \mathcal{E}(u(t), \Gamma(t))$ is absolutely continuous and it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(u(t), \Gamma(t)) = \int_\Omega e(u(t)) : \nabla \partial_t g(t) \,\mathrm{d}x \quad \text{for a.e. } t \in [0, T]\,, \tag{4.15}$$

where by ∂_t we denote the time derivative of g.

In [33] (see [26] for the scalar case), the existence of an irreversible quasi-static crack evolution with respect to the boundary displacement g has been shown. In this present work, our goal consists in approximating such an evolution with the time-discretized evolution defined in Section 4.1.

4.3. **Main result: Approximation of quasi-static crack growth.** To formulate our main result, we need to introduce some further notation. First, we associate a 'crack surface' to the finite element evolution $(u_n^k)_k$. To this end, we choose $\eta_n \to 0$ depending on ε_n such that

$$(C_{\eta_n})^7 \varepsilon_n \to 0 \quad \text{as } n \to \infty, \tag{4.16}$$

where $C_{\eta_n} \geq 1$ denotes the constant in Theorem 2.1, and we apply Theorem 2.1 to η_n, u_n^k, and $A = \Omega_{n,k}^{\rm crack}$. (Here, we consider Ω' as the ambient space in place of Ω, in particular we replace Ω with Ω' in (2.10), see also (4.5).) We obtain a set $\Omega_{n,k}^{\rm mod}$ and a function $u_{n,k}^{\rm mod}$ satisfying

$$\begin{aligned} |\Omega_{n,k}^{\rm mod}| \leq C_{\eta_n}\varepsilon_n, \qquad |\{u_n^k \neq u_{n,k}^{\rm mod}\} \cap \Omega'_{\varepsilon_n,\eta_n}| \leq C_{\eta_n}\varepsilon_n, \\ \|e(u_{n,k}^{\rm mod})\|_{L^2(\Omega'_{\varepsilon_n,\eta_n}\setminus \Omega_{n,k}^{\rm mod})} \leq C_{\eta_n}, \quad \mathcal{H}^1(\partial \Omega_{n,k}^{\rm mod}) \leq \frac{2}{\varepsilon_n \sin\theta_0}|\Omega_{n,k}^{\rm crack}| + C\eta_n, \end{aligned} \tag{4.17}$$

where the constants C and C_{η_n} also depend on $\max_{0\leq k \leq T/\delta_n} \mathcal{E}_n(u_n^k, (u_n^j)_{j<k})$.

We define the evolution $u_n\colon [0,T] \times \Omega' \to \mathbb{R}$, piecewise affine in space and piecewise constant in time, by

$$u_n(t) := u_n^k \chi_{\Omega' \setminus \Omega_{n,k}^{\rm crack}} \text{ for } t \in [t_n^k, t_n^{k+1})\,. \tag{4.18}$$

The crack set K_n is defined by

$$K_n(t) := \partial \Omega_{n,k}^{\rm mod} \quad \text{for } t \in [t_n^k, t_n^{k+1})\,. \tag{4.19}$$

For the crack sets, we use the notion of σ-*convergence* recalled in Definition 5.1 below, which is a suitable notion of convergence for crack sets. In particular, below we will obtain the existence of $K(t) \tilde{\subset} \overline{\Omega} \cap \Omega'$ for $t \in [0,T]$ such that $K_n(t)$ σ-converges to $K(t)$ for $t \in [0,T]$.

As a final preparation, we identify sets on which convergence of displacement fields can be guaranteed. For a crack set $\Gamma(t) \subset \overline{\Omega} \cap \Omega'$ with $\mathcal{H}^1(\Gamma(t)) < \infty$, by $B(t) \subset \Omega$ we denote the largest set of finite perimeter (with respect to set inclusion) which satisfies $\partial^* B(t) \cap \Omega' \tilde{\subset} \Gamma(t)$. This set represents the 'broken off pieces', and by $G(t) := \Omega' \setminus B(t)$ instead we denote the 'good set', which in particular satisfies $\Omega' \setminus \overline{\Omega} \subset G(t)$. Note that convergence of the displacements can only be expected on $G(t)$, see [35, Subsection 2.4] for a thorough discussion. Moreover, we note that for an evolution $t \mapsto (u(t), \Gamma(t))$ in the sense of Definition 4.1 it holds that

$$e(u(t)) = 0 \quad \text{ on } B(t) \quad \text{ for all } t \in [0,T]. \tag{4.20}$$

In fact, this follows by applying (4.14) with test set $H = \Gamma(t)$ and test function $v = u(t)\chi_{G(t)}$.

The main result of this paper is the following convergence theorem.

Theorem 4.2 (Approximation of quasi-static crack growth)**.** *There exists a quasi-static crack evolution $t \to (u(t), K(t))$ with respect to the boundary condition g such that, up to a subsequence, we have that*

$$K_n(t) \ \sigma\text{-converges to } K(t) \text{ for all } t \in [0,T] \tag{4.21}$$

as $n \to \infty$, $K(0) = J_{u(0)}$, and that, for each $t \in [0,T]$,

$$u_n(t) \to u(t) \text{ as } k \to \infty \text{ in measure on } G(t), \quad e(u_n(t)) \to e(u(t)) \text{ in } L^2(\Omega; \mathbb{R}^{2\times 2}_{\rm sym}), \tag{4.22}$$

where $G(t)$ is the set corresponding to $K(t)$ defined before (4.20)*. Moreover, for all $t \in [0,T]$ we have that*

$$\mathcal{E}_n(u_n(t), (u_n^j)_{j<k(t)}) \to \mathcal{E}(u(t), K(t)) \quad \text{as } n \to \infty, \tag{4.23}$$

where for each n the (n-dependent) index $k(t) \in \mathbb{N}$ is chosen such that $t \in [t_n^{k(t)}, t_n^{k(t)+1})$.

Remark 4.3. We proceed with two comments on the result:

(i) The energy convergence (4.23) can be improved to separate energy convergence in the sense that, for all $t \in [0,T]$,
$$\mathcal{E}_n^{\rm elast}\big(u_n(t), (u_n^j)_{j<k(t)}\big) \to \int_\Omega |e(u(t))|^2 \,\mathrm{d}x$$
(recall (4.7)) and
$$\lim_{n\to\infty} \mathcal{E}_n^{\rm crack}\big(u_n(t), (u_n^j)_{j<k(t)}\big) = \frac{\kappa \sin(\theta_0)}{2} \lim_{n\to\infty} \mathcal{H}^1(K_n(t)) = \kappa \sin(\theta_0)\,\mathcal{H}^1(K(t))\,. \tag{4.24}$$

(ii) The identity (4.24) is the reason why crack sets along the sequence are defined in terms of $\partial\Omega_{n,k}^{\rm mod}$ and not in terms of $\partial\Omega_{n,k}^{\rm crack}$. In fact, for the latter the identity (4.24) does not hold in general.

5. Preparations

In this section, we collect some tools and remarks that we will need to prove Theorem 4.2 in the next section. Before providing the necessary compactness and semicontinuity statements, we recall a suitable notion of convergence for crack sets and prove, respectively recall, the necessary compactness and irreversiblity results.

5.1. **Convergence of sets.** Let $\Omega \subset \mathbb{R}^d$ for $d \geq 2$ and denote by e_1 the first unit vector. We let
$$\mathrm{PC}(\Omega) := \{v \in L^1(\Omega;\mathbb{R}^d)\colon v = e_1\chi_T \text{ with } T \subset \Omega \text{ of finite perimeter}\} \tag{5.1}$$
be the collection of piecewise constant functions taking values in $\{0, e_1\}$. In this subsection, by $\mathcal{A}(\Omega)$ we denote the family of open subsets of Ω. We recall the notion of σ-convergence from [40].

Definition 5.1. [40, Definition 5.1] A sequence of rectifiable sets $(K_n)_n$ in Ω σ-converges in Ω to K if the functionals $\mathcal{H}_n^-\colon \mathrm{PC}(\Omega)\times\mathcal{A}(\Omega) \to [0,+\infty)$ defined by
$$\mathcal{H}_n^-(u,A) := \mathcal{H}^{d-1}\big((J_u \setminus K_n)\cap A\big) \tag{5.2}$$
are such that, for every $A \in \mathcal{A}(\Omega)$, $\mathcal{H}_n^-(\cdot, A)$ Γ-converges with respect to the strong topology of $L^1(\Omega)$ to $\mathcal{H}^-(\cdot, A)$, where $\mathcal{H}^-\colon \mathrm{PC}(\Omega)\times\mathcal{A}(\Omega) \to [0,+\infty)$ is given by
$$\mathcal{H}^-(u,A) := \int_{J_u\cap A} h^-(x,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}(x), \tag{5.3}$$
and K is the maximal (with respect to $\tilde{\subset}$) rectifiable set in Ω such that
$$h^-(x,\nu_K(x)) = 0 \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in K. \tag{5.4}$$

Remark 5.2. (i) More precisely, if $\mathcal{H}^{d-1}(K_n) \leq C$ for all $n \in \mathbb{N}$, by [40, Proposition 3.3] the density h^- in (5.3) is characterized by
$$h^-(x,\nu) = \limsup_{\varrho\to0^+} \liminf_{n\to+\infty} \frac{\mathbf{m}^{\rm PC}_{\mathcal{H}_n^-}(\overline{u}_{x,\nu}, Q_\varrho^\nu(x))}{\varrho^{d-1}}, \tag{5.5}$$
for $x \in \Omega$ and $\nu \in \mathbb{S}^{d-1} = \{y \in \mathbb{R}^d\colon |y| = 1\}$, where $Q_\varrho^\nu(x)$ denotes a suitable cube with sidelength ϱ and two sides orthogonal to ν, $\overline{u}_{x,\nu} := e_1\chi_{Q_\varrho^{\nu,-}(x)}$ (with $Q_\varrho^{\nu,-}(x) = \{y \in Q_\varrho^\nu(x)\colon (y-x)\cdot\nu < 0\}$), and
$$\mathbf{m}^{\rm PC}_{\mathcal{H}}(\overline{v}, A) := \inf_{v\in \mathrm{PC}(A)} \{\mathcal{H}(v,A)\colon v = \overline{v} \text{ in a neighborhood of } \partial A\} \quad \text{for } \overline{v} \in \mathrm{PC}(A),\ A \in \mathcal{A}(\Omega). \tag{5.6}$$

In particular, using $\overline{u}_{x,\nu}$ as competitor, we get $h^- \leq 1$.

(ii) Let $(K_n)_n$ be a sequence of rectifiable sets in Ω such that K_n σ-converges to K in Ω and let $(\tilde{K}_n)_n$ be another sequence of rectifiable sets such that $\tilde{K}_n$ σ-converges to $\tilde{K}$ in Ω, and $\tilde{K}_n \tilde{\subset} K_n$ for all $n \in \mathbb{N}$. Then, we have $\tilde{K} \tilde{\subset} K$.

(iii) If F is a closed set such that $K_n \tilde{\subset} F$, then the σ-limit K of $(K_n)_n$, satisfies $K \tilde{\subset} F$: in fact, $\mathcal{H}^-_n(u, \Omega \setminus F) = \mathcal{H}^{d-1}(J_u \cap (\Omega \setminus F))$ for every $n \in \mathbb{N}$, and thus $\mathcal{H}^-(u, \Omega \setminus F) = \mathcal{H}^{d-1}(J_u \cap (\Omega \setminus F))$, i.e., h^- cannot be 0 on a subset of $\Omega \setminus F$ of positive $\mathcal{H}^{d-1}$-measure.

In the following, we need the compactness and lower semicontinuity properties of σ-convergence, see [40, Propositions 5.3, proof of Theorem 8.1].

Proposition 5.3 (Compactness and lower semicontinuity)**.** *Let $(K_n)_n$ be a sequence of rectifiable sets in Ω with $\mathcal{H}^{d-1}(K_n) \leq C$. Then there exists a subsequence $(n_k)_k$ and a rectifiable set K in Ω such that K_{n_k} σ-converges in Ω to K. Moreover,*

$$\mathcal{H}^{d-1}(K) \leq \liminf_{n\to\infty} \mathcal{H}^{d-1}(K_n)\,.$$

Theorem 5.4 (Variant of Helly's theorem for σ-convergence)**.** *Let $t \mapsto K_n(t)$ be a sequence of increasing set functions defined on an interval $I \subset \mathbb{R}$ with values contained in Ω, i.e., $K(s) \subset K(t) \subset \Omega$ for every $s, t \in I$ with $s < t$. Assume that $\mathcal{H}^{d-1}(K_n(t))$ is bounded uniformly with respect to n and t. Then, there exist a subsequence $(K_{n_k})_k$ and an increasing set function $t \mapsto K(t)$ on I such that for every $t \in I$ we have*

(a) *$K_{n_k}(t) \to K(t)$ in the sense of σ-convergence,*
(b) *$\mathcal{H}^{d-1}(K(t)) \leq \liminf_{k\to\infty} \mathcal{H}^{d-1}(K_{n_k}(t))$.*

For definition and properties of $SBV^p(\Omega)$, $p \in (1, \infty)$, we refer the reader to [2]. In the following, we say $v_n \rightharpoonup v$ weakly in $SBV^p(\Omega)$ if $v_n \to v$ in $L^1(\Omega)$ and $\sup_n(\|\nabla v_n\|_{L^p(\Omega)} + \mathcal{H}^{d-1}(J_{v_n})) < +\infty$. We will also make use of the following property of σ-convergence (see [40, Proposition 5.8]).

Proposition 5.5. *Let $(K_n)_n$ be a sequence of rectifiable sets in Ω such that K_n σ-converges to K in Ω. Let $(v_n)_n$ be a sequence $SBV^p(\Omega)$ with $v_n \rightharpoonup v$ weakly in $SBV^p(\Omega)$ and $\mathcal{H}^{d-1}(J_{v_n} \setminus K_n) \to 0$. Then $J_v \tilde{\subset} K$.*

We close this section with a lower semicontinuity result for the boundaries of void sets.

Lemma 5.6 (Lower semicontinuity for void sets)**.** *Let K_n be a sequence of rectifiable sets of the form $K_n = \partial V_n$ for closed sets $V_n \subset \mathbb{R}^d$, $n \in \mathbb{N}$, with finite perimeter and $|V_n| \to 0$. Suppose that K_n σ-converges to K. Then*

$$\liminf_{n\to\infty} \mathcal{H}^{d-1}(K_n) \geq 2\mathcal{H}^{d-1}(K).$$

Proof. Given $\mu > 0$, by a covering argument there exist an open and smooth set $U \subset \Omega$ and a function $v \in \mathrm{PC}(\Omega)$ such that

$$\mathcal{H}^{d-1}(K \setminus U) \leq \mu, \qquad \mathcal{H}^{d-1}((K \triangle J_v) \cap U) \leq \mu\,. \tag{5.7}$$

Let $(v_n)_n \subset \mathrm{PC}(U)$ be a recovery sequence in $L^1(U;\mathbb{R}^d)$ for the restriction of v to U with respect to the functionals (5.2) and (5.3). Since $h^- \leq 1$, see Remark 5.2(i), we have

$$\limsup_{n\to+\infty} \mathcal{H}^{d-1}((J_{v_n} \setminus K_n) \cap U) \leq \int_{(J_v \setminus K) \cap U} h^-(x, \nu_v)\,\mathrm{d}\mathcal{H}^{d-1}(x) \leq \mathcal{H}^{d-1}((J_v \setminus K) \cap U) \leq \mu\,. \tag{5.8}$$

We notice that the function

$$\tilde{v}_n \coloneqq \chi_{U \setminus V_n} v_n$$

is such that

$$\tilde{v}_n \to v \quad \text{in } L^1(U;\mathbb{R}^d), \tag{5.9}$$

since $|V_n| \to 0$. Notice that the function $\tilde{v}_n$ might also jump outside of ∂V_n because $J_{v_n} \cap (U \setminus V_n) \neq \emptyset$ is in general possible. Therefore, we apply [7, Proposition 9] to the open set $\tilde{\Omega}_n := U \setminus V_n$ (in place of Ω therein) to obtain a sequence of approximating pairs $((z_h, F_h))_h$, with $z_h \in SBV^2(\tilde{\Omega}_n;\mathbb{R}^d)$ and F_h of class C^∞, such that

$$J_{z_h} \tilde{\subset} \partial F_h, \quad z_h = 0 \text{ in } F_h, \quad z_h \to \tilde{v}_n = v_n \ \text{ in } L^1(\tilde{\Omega}_n;\mathbb{R}^d), \quad F_h \to \emptyset,$$

and

$$\limsup_{h\to\infty} \mathcal{H}^{d-1}(\partial F_h) = 2\mathcal{H}^{d-1}(J_{\tilde{v}_n} \cap \tilde{\Omega}_n) \le 2\mathcal{H}^{d-1}((J_{v_n} \setminus K_n) \cap U). \tag{5.10}$$

For $h_n \in \mathbb{N}$ such that

$$\|z_{h_n} - v_n\|_{L^1(\tilde{\Omega}_n)} + |F_{h_n}| + \mathcal{H}^{d-1}(\partial F_{h_n}) - 2\mathcal{H}^{d-1}((J_{v_n} \setminus K_n) \cap U) \le \frac{1}{n}, \tag{5.11}$$

setting

$$w_n := z_{h_n}\chi_{\tilde{\Omega}_n} \in SBV^2(U;\mathbb{R}^d), \qquad G_n := F_{h_n} \cup (V_n \cap U),$$

it holds that

$$w_n = 0 \text{ in } G_n, \quad J_{w_n} \tilde{\subset} \partial G_n, \quad w_n \to v \quad \text{in } L^1(U;\mathbb{R}^d).$$

Therefore, [7, Proposition 3] gives

$$2\mathcal{H}^{d-1}(J_v \cap U) \le \liminf_{n\to+\infty} \mathcal{H}^{d-1}(\partial G_n). \tag{5.12}$$

By (5.8) and (5.11) it holds that

$$\limsup_{n\to+\infty} \mathcal{H}^{d-1}(\partial F_{h_n}) \le \limsup_{n\to+\infty} 2\mathcal{H}^{d-1}((J_{v_n} \setminus K_n) \cap U) \le 2\mu.$$

Since $\mathcal{H}^{d-1}(\partial G_n) \le \mathcal{H}^{d-1}(K_n \cap U) + \mathcal{H}^{d-1}(\partial F_{h_n})$, we get

$$\liminf_{n\to+\infty} \mathcal{H}^{d-1}(K_n \cap U) \ge \liminf_{n\to+\infty} \mathcal{H}^{d-1}(\partial G_n) - 2\mu. \tag{5.13}$$

By (5.7), (5.12), and (5.13) we deduce

$$\liminf_{n\to+\infty} \mathcal{H}^{d-1}(K_n) \ge \liminf_{n\to+\infty} \mathcal{H}^{d-1}(K_n \cap U) \ge 2\mathcal{H}^{d-1}(J_v \cap U) - 2\mu \ge 2\mathcal{H}^{d-1}(K) - 6\mu,$$

and we conclude by the arbitrariness of $\mu > 0$. □

5.2. **Compactness.** From now on we restrict ourselves to the case $d = 2$. For technical reasons, we need to consider the space of $GSBD^p$ functions (with $p \in (1,\infty)$) which may also attain a limit value ∞. Following [17, Subsection 4.4], we define $\bar{\mathbb{R}}^2 := \mathbb{R}^2 \cup \{\infty\}$, and for $U \subset \mathbb{R}^2$ open, we let

$$GSBD^p_\infty(U) := \Big\{ u \in L^0(U;\bar{\mathbb{R}}^2) \colon \ A^\infty_u := \{u = \infty\} \text{ satisfies } \mathcal{H}^1(\partial^* A^\infty_u) < +\infty,$$
$$\tilde{u}_t := u\chi_{U\setminus A^\infty_u} + t\chi_{A^\infty_u} \in GSBD^p(U) \ \text{ for all } t \in \mathbb{R}^2 \Big\}. \tag{5.14}$$

Symbolically, we also write

$$u = u\chi_{U\setminus A^\infty_u} + \infty\chi_{A^\infty_u},$$

and for any $u \in GSBD^p_\infty(U)$, we set

$$e(u) = 0 \text{ in } A^\infty_u, \qquad J_u = J_{u\chi_{\Omega\setminus A^\infty_u}} \cup (\partial^* A^\infty_u \cap U). \tag{5.15}$$

We also define the subspaces $PR(U) \subset GSBD^p(U)$ and $PR_\infty(U) \subset GSBD^p_\infty(U)$ as the functions in $u \in GSBD^p(U)$ and $u \in GSBD^p_\infty(U)$, respectively, with $e(u) \equiv 0$. Functions $a \in PR_\infty(U)$ are *piecewise rigid* in the sense that they can be represented as

$$a = \sum_{j\in\mathbb{N}} a^j \chi_{P^j} + \infty\chi_{A^\infty_a},$$

where a^j are affine mappings with $e(a^j) = 0$ and $(P^j)_{j\in\mathbb{N}}$ is a Caccioppoli partition of $U \setminus A^\infty_a$. If $a \in PR(U)$, then $A^\infty_a = \emptyset$.

There exists a metric $\bar{d}$ on $GSBD^p_\infty(U)$, see [17, Equation (3.13)], which induces the following convergence: $\bar{d}(u_n, u) \to 0$ if and only if

$$u_n \to u \quad \text{in measure on } U \setminus A^\infty_u, \qquad |u_n| \to \infty \quad \text{on } A^\infty_u.$$

In the following, we say that a sequence $(u_n)_n \subset GSBD^p_\infty(U)$ *converges weakly* to $u \in GSBD^p_\infty(U)$ if

$$\sup\nolimits_{n\in\mathbb{N}} \big(\|e(u_n)\|_{L^p(U)} + \mathcal{H}^1(J_{u_n})\big) < +\infty \tag{5.16}$$

and

$$\bar{d}(u_n, u) \to 0, \qquad e(u_n) \rightharpoonup e(u) \text{ weakly in } L^p(U \setminus A^\infty_u; \mathbb{R}^{2\times 2}_{\mathrm{sym}}).$$

Proposition 5.7 (Compactness)**.** *Let $(u_n)_n \subset GSBD^p(U)$ satisfy (5.16). Then there exists $u \in GSBD^p_\infty(U)$ such that, up to a subsequence, u_n converges weakly to u in $GSBD^p_\infty(U)$. If additionally $u_n \in PR_\infty(U)$ for all $n \in \mathbb{N}$, we get $u \in PR_\infty(U)$.*

Proof. The result follows from [17, Theorem 3.5 and Lemma 3.6] which itself is a consequence of [13, Theorem 1.1]. The closedness of PR_∞ follows by repeating the argument in [34, Lemma 3.3]. □

For the next result, we again consider Lipschitz sets $\Omega \subset \Omega' \subset \mathbb{R}^2$ such that $\Omega' \setminus \overline{\Omega}$ is Lipschitz, as considered in Section 4.

Proposition 5.8. *Let $(K_n)_n$ be a sequence of rectifiable sets in $\Omega' \cap \overline{\Omega}$, with $\mathcal{H}^1(K_n) \le C$ which σ-converges to $K \subset \Omega'$. Let $(v_n)_n$ be a sequence converging weakly in $GSBD^p_\infty(\Omega')$ to $v \in GSBD^p_\infty(\Omega')$ such that $\mathcal{H}^1(J_{v_n} \setminus K_n) \to 0$ as $n \to \infty$ and $v_n|_{\Omega'\setminus\overline{\Omega}}$ is a bounded sequence in $W^{1,p}(\Omega' \setminus \overline{\Omega}; \mathbb{R}^2)$. Then the following hold:*

- (i) $J_v \mathrel{\tilde{\subset}} K \mathrel{\tilde{\subset}} \Omega' \cap \overline{\Omega}$.
- (ii) *It holds that $|A^\infty_v \cap G_K| = 0$, where G_K denotes the smallest set of finite perimeter with $|\Omega' \setminus (\overline{\Omega} \cup G_K)| = 0$ and $\partial^* G_K \cap \Omega' \mathrel{\tilde{\subset}} K$.*

Later we will apply this proposition to sequences satisfying Dirichlet boundary conditions which guarantees the boundedness assumption on $\Omega' \setminus \overline{\Omega}$. The set $G(t)$ introduced before (4.20) will play the role of G_K. The result ensures the compatibility of the notion of σ-convergence with the weak notion of $GSBD^p_\infty$-convergence for the displacement fields, and can be seen as the $GSBD$-analog of [40, Proposition 5.8]. More precisely, it enables us to show that limiting evolutions $t \mapsto u(t)$ fulfill $J_{u(t)} \mathrel{\tilde{\subset}} K(t)$ and are thus admissible with respect to $t \mapsto K(t)$. Before we come to the proof, we show the following preliminary result.

Lemma 5.9. *Let $(K_n)_n$ be a sequence of rectifiable sets in $\Omega' \cap \overline{\Omega}$, with $\mathcal{H}^1(K_n) \le C$ which σ-converges to $K \subset \Omega'$. Consider the functionals*

$$\mathcal{E}'_n(u) := \int_{\Omega'} |e(u)|^p \,\mathrm{d}x + \mathcal{H}^1(J_u \setminus K_n) \tag{5.17}$$

if $u \in GSBD^p(\Omega')$ and $+\infty$ otherwise in $L^0(\Omega';\mathbb{R}^2)$. Then $\mathcal{E}'_n(u)$ Γ-converges, with respect to the convergence in measure in Ω', to the functional

$$\mathcal{E}'(u) := \int_{\Omega'} |e(u)|^p \,\mathrm{d}x + \int_{J_u} h^-(x,\nu_u)\,\mathrm{d}\mathcal{H}^1(x) \tag{5.18}$$

if $u \in GSBD^p(\Omega')$ and $+\infty$ otherwise in $L^0(\Omega';\mathbb{R}^2)$, where h^- is the density given in Remark 5.2(i) for $d = 2$.

Proof of Lemma 5.9. Our strategy is to apply the abstract Γ-convergence result in [31] to the sequence $\mathcal{E}'_n$. As a pointwise lower bound on the density of the surface integral is needed, cf. [31, Assumption (2.12)], we consider a suitable perturbation. Precisely, given $\varepsilon > 0$, let us define the functionals $\mathcal{E}'_{n,\varepsilon}(u) := \mathcal{E}'_n(u) + \varepsilon\mathcal{H}^1(J_u \cap K_n)$, i.e.,

$$\mathcal{E}'_{n,\varepsilon}(u) = \int_{\Omega'} |e(u)|^p \,\mathrm{d}x + \mathcal{H}^1(J_u \setminus K_n) + \varepsilon\mathcal{H}^1(J_u \cap K_n)$$

if $u \in GSBD^p(\Omega')$ and $+\infty$ otherwise in $L^0(\Omega';\mathbb{R}^2)$. The characterization of the Γ-limit of $\mathcal{E}'_{n,\varepsilon}$ with respect to the convergence in measure follows from [31, Theorem 2.4, Remark 3.15]: the Γ-limit of $\mathcal{E}'_{n,\varepsilon}$ is

$$\mathcal{E}'_\varepsilon(u) := \int_{\Omega'} |e(u)|^p \,\mathrm{d}x + \int_{J_u} h^-_\varepsilon(x,\nu_u)\,\mathrm{d}\mathcal{H}^1(x), \tag{5.19}$$

if $u \in GSBD^p(\Omega')$ and $+\infty$ otherwise in $L^0(\Omega';\mathbb{R}^2)$, with

$$h^-_\varepsilon(x,\nu) = \limsup_{\varrho\to 0^+} \liminf_{n\to+\infty} \frac{\mathbf{m}^{\mathrm{PC}}_{\mathcal{H}^-_{n,\varepsilon}}(\overline{u}_{x,\nu}, Q^\nu_\varrho(x))}{\varrho} \tag{5.20}$$

for $x \in \Omega'$ and $\nu \in \mathbb{S}^1$, where $\mathcal{H}^-_{n,\varepsilon}(u,A) := \mathcal{H}^-_n(u,A) + \varepsilon\mathcal{H}^1(J_u \cap K_n \cap A)$. Here, we recall the definition of $\mathcal{H}^-_n$ and $\mathbf{m}^{\mathrm{PC}}_{\mathcal{H}^-_{n,\varepsilon}}$ in (5.2) and (5.6), respectively.

We notice that [31, Theorem 2.4] proves an integral representation result for the Γ-limit of $\mathcal{E}'_{n,\varepsilon}$, and [31, Remark 3.15] shows that the surface density is exactly h^-_ε. Strictly speaking, the result has been explicitly detailed for $p = 2$, but it holds with minor changes in the proof also for any $p > 1$, cf. [31, Remark 5.3].

As $\mathcal{H}^-_n \le \mathcal{H}^-_{n,\varepsilon}$ and $\mathcal{H}^-_{n,\varepsilon}$ is increasing in ε, it is immediate that

$$h^- \le \lim_{\varepsilon\to 0} h^-_\varepsilon,$$

for h^- defined in (5.5). We observe that $H(x) := \limsup_{\varrho\to 0^+} \liminf_{n\to+\infty} \varrho^{-1}\mathcal{H}^1(K_n \cap Q^\nu_\varrho(x))$ is finite up to a set of negligible $\mathcal{H}^1$-measure. Indeed, if $H(x) = +\infty$ on B with $\mathcal{H}^1(B) > 0$, then the weak* limit of $\mu_n := \mathcal{H}^1 \llcorner_{K_n}$, denoted by μ, would satisfy $\mu(B) = +\infty$ by [2, Theorem 2.56], which contradicts the weak* lower semicontinuity of the total variation $\mu(\Omega') \le \liminf_{n\to\infty} \mu_n(\Omega') < +\infty$. Recalling (5.5) and (5.20) this yields

$$h^-_\varepsilon(x,\nu) \le h^-(x,\nu) + \varepsilon H(x) \quad \text{for all } x \in \Omega' \text{ and } \nu \in \mathbb{S}^1,$$

and then

$$h^- \ge \lim_{\varepsilon\to 0} h^-_\varepsilon.$$

Therefore, for any $A \in \mathcal{A}(\Omega')$, the functionals $\mathcal{E}'_\varepsilon(\cdot,A)$ pointwise (and monotonically) converge to $\mathcal{E}'(\cdot,A)$, $\mathcal{E}'_{n,\varepsilon} - \mathcal{E}'_n \in (0, M\varepsilon)$ for $M := \sup_n \mathcal{H}^1(K_n)$, and then the representation result holds true also for the functional $\mathcal{E}'$ given in (5.18). □

Proof of Proposition 5.8. (i) We first remark that $K \tilde{\subset} \Omega' \cap \overline{\Omega}$ by Remark 5.2(iii). By assumption we get $v|_{\Omega'\setminus\overline{\Omega}} \in W^{1,p}(\Omega'\setminus\overline{\Omega};\mathbb{R}^2)$ and $v_n \to v \in L^p(\Omega'\setminus\overline{\Omega};\mathbb{R}^2)$, up to a subsequence. Let us apply the compactness result [31, Theorem 3.8] to v_n (which, similarly to the integral representation result employed in the proof of Lemma 5.9, has been proven only for $p = 2$ but holds for every $p > 1$, with minor changes in the proof). Recalling the precise form of the modifications, see [33, Theorem 6.1] and also [35, Theorem 3.1], there are functions y_n with $y_n = v_n$ on $\Omega' \setminus \overline{\Omega}$, r_n with $|\{r_n \neq 0\}| \leq \frac{1}{n}$, $a_n = \sum_j a_n^j \chi_{P_n^j} \in \mathrm{PR}(\Omega')$, and $u \in GSBD^p(\Omega')$ such that

$$y_n = v_n + r_n + a_n, \tag{5.21}$$

and

$$\mathcal{H}^1\big((J_{a_n} \cup J_{r_n}) \setminus J_{v_n}\big) \leq \frac{1}{n}, \qquad \mathcal{E}'_n(y_n) \leq \mathcal{E}'_n(v_n) + \frac{1}{n}, \qquad y_n \to u \text{ in measure on } \Omega', \tag{5.22}$$

where $\mathcal{E}'_n$ is defined as in (5.17). Since $a_n \in \mathrm{PR}(\Omega')$ and (5.22) holds, there exists $a = \sum_j a_j \chi_{P_j} \in \mathrm{PR}_\infty(\Omega')$ such that $\bar{\mathrm{d}}(a_n, a) \to 0$ by Proposition 5.7. We notice that $a = 0$ on $\Omega' \setminus \overline{\Omega}$ by (5.21) and since $y_n = v_n$ on $\Omega' \setminus \overline{\Omega}$ as well as $|\{r_n \neq 0\}| \to 0$. Next, we observe that

$$A_v^\infty = A_a^\infty. \tag{5.23}$$

Indeed, otherwise $|y_n| \to +\infty$ a.e. on $A_v^\infty \triangle A_a^\infty$ since $A_v^\infty = \{|v_n| \to +\infty\}$, $A_a^\infty = \{|a_n| \to +\infty\}$, and $|\{r_n \neq 0\}| \to 0$. This contradicts the third property in (5.22). Since a_n, a are piecewise rigid functions, denoting by $B_n := \bigcup\{P_n^j \colon P_n^j \cap A_v^\infty \neq \emptyset\}$, it holds that

$$|B_n \triangle A_v^\infty| \to 0, \qquad \partial^* B_n \cap \Omega' \tilde{\subset} J_{a_n}, \tag{5.24}$$

where for the second property we assumed without restriction that the affine mappings $(a_n^j)_j$ are pairwise distinct, cf. [34, (3.1)]. Then, for every $b \in \mathbb{R}^2$,

$$y_n^b := y_n + b\chi_{B_n} - a_n \chi_{\Omega' \setminus B_n}$$

is such that, by (5.21)–(5.24) and the fact that v_n converges weakly in $GSBD^p_\infty(\Omega')$ to v,

$$y_n^b \to u^b := v\chi_{\Omega' \setminus A_v^\infty} + (u + b)\chi_{A_v^\infty} \quad \text{a.e. in } \Omega', \tag{5.25}$$

and

$$\partial^* A_v^\infty \cap \Omega' \tilde{\subset} J_{u^b} \text{ for a.e. } b \in \mathbb{R}^2. \tag{5.26}$$

Let us fix some b satisfying (5.26). Recalling (5.15) and combining (5.25)–(5.26) we get

$$J_v \tilde{\subset} J_{u^b}. \tag{5.27}$$

We now apply the result [31, Lemma 7.1] to (the surface parts of) $\mathcal{E}'_n$ and $\mathcal{E}$ as in Lemma 5.9, and to the converging sequence $y_n^b \to u^b$, see (5.25): this gives that

$$\int_{J_{u^b}} h^-(x, \nu_{u^b})\, \mathrm{d}\mathcal{H}^1(x) \leq \liminf_{n\to\infty} \mathcal{H}^1(J_{y_n^b} \setminus K_n).$$

Using the definition of y_n^b, by the first property in (5.22), (5.24), and the assumption that $\mathcal{H}^1(J_{v_n} \setminus K_n) \to 0$, we get

$$\int_{J_{u^b}} h^-(x, \nu_{u^b})\, \mathrm{d}\mathcal{H}^1(x) = 0.$$

By Definition 5.1 we find $J_{u^b} \tilde{\subset} K$ and by (5.27) we conclude the proof of (i).

(ii) Since $v_n \to v$ in $L^p(\Omega' \setminus \overline{\Omega};\mathbb{R}^2)$, up to a subsequence we have $v_n \to v$ a.e. in $\Omega' \setminus \overline{\Omega}$ and $|v_n| \to +\infty$ a.e. in A_v^∞, by definition of weak convergence in $GSBD^p_\infty(\Omega')$. This shows $|(\Omega' \setminus \overline{\Omega}) \cap A_v^\infty| = 0$. Moreover, the fact that $J_{u^b} \tilde{\subset} K$ and (5.26) give $\partial^* A_v^\infty \cap \Omega' \tilde{\subset} K$.

Let G_K be the smallest set with (a) $|\Omega' \setminus (\overline{\Omega} \cup G_k)| = 0$ and (b) $\partial^* G_k \cap \Omega' \tilde{\subset} K$. As $|(\Omega' \setminus \overline{\Omega}) \cap A_v^\infty| = 0$, also $G_K \setminus A_v^\infty$ is a set satisfying (a) and (b). Therefore, the minimality of G_K implies $|A_v^\infty \cap G_K| = 0$. $\square$

6. Proof of the main result

This section is devoted to the proof of Theorem 4.2. We start by establishing a uniform energy bound on $u_n(t)$, defined in (4.18), which will enable us to pass to the limit by the compactness and semicontinuity results that were established in the previous section. In order to conclude that (4.14) holds, we need a result on the stability of unilateral minimizers (Theorem 6.7) whose proof is deferred to Section 7. With this result at hand, we are finally able to conclude the proof of Theorem 4.2.

6.1. **Energy bound.** In this section, our goal is to show that the energy of the evolution is bounded uniformly in time. For this, we need to prove an energy estimate on the time-discrete level, which will be also crucial to establish the energy balance.

We recall the notation for the time discretization $\{0 = t_n^0 < t_n^1 < \cdots < t_n^{T/\delta_n} = T\}$ of the interval $[0, T]$ with step size δ_n, and introduce the following shorthand notation. Given an arbitrary $v \in \mathcal{A}_n^k$, we write

$$\mathcal{E}_n^0(v) = E_n(v) \quad \text{and} \quad \mathcal{E}_n^k(v) := \mathcal{E}_n(v, (u_n^j)_{j<k}) \quad \text{for } k \geq 1, \tag{6.1}$$

where E_n and $\mathcal{E}_n$ are defined in (4.3) and (4.7), respectively, and $(u_n^j)_{j<k}$ denote the displacements that have been found at previous time steps $(t_n^j)_{j<k}$. We start by proving a bound on the elastic part of the energy.

Proposition 6.1 (Elastic energy)**.** *Let $t \mapsto u_n(t)$ be the discrete evolution defined in* (4.18)*. There exists a constant $C > 0$ depending on g such that $\mathcal{E}_n^0(u_n^0) \leq C$ and*

$$\int_{\Omega'} |e(u_n(t))|^2 \,\mathrm{d}x \leq C \qquad \textit{for all } t \in [0, T] \textit{ and for all } n \in \mathbb{N}.$$

Proof. For the time step $t_n^0 = 0$, we consider the background mesh $\mathbf{Z}_n \in \mathcal{T}_n(\Omega')$ as introduced before (4.4) and the test function $g_n^0 := g(0)_{\mathbf{Z}_n}$, see (4.2). We then have $g_n^0 \in \mathcal{A}_n^0$, see (4.9). Since $g \in W^{1,1}(0, T; W^{2,\infty}(\Omega; \mathbb{R}^2))$, we can deduce that $e(g_n^0)_T$ is uniformly bounded on each triangle T, hence we obtain $\mathcal{E}_n^0(g_n^0) \leq C$ for some $C > 0$ independently of n. By (4.10) this implies that $\mathcal{E}_n^0(u_n^0) \leq C$ uniformly in n.

In a similar fashion, we can test with a function $g_n^k \in \mathcal{A}_n^k$, namely $g_n^k = g(t_n^k)_{\mathbf{T}_n}$, where $\mathbf{T}_n$ is given by $\mathbf{T}_n(u_n^{k-1})$. As before, the regularity of g implies that $e(g_n^k)_T$ is uniformly bounded, and thus $\varepsilon_n |e(g_n^k)_T|^2 < \kappa$ for all $T \in \mathbf{T}_n$ for n large enough. In particular, recalling (4.4)–(4.5) this means $\Omega_{n,k-1}^{\mathrm{crack}}(g_n^k) \subset \Omega_{n,k-1}^{\mathrm{crack}}(u_n^k)$. As $\mathcal{E}_n^k(u_n^k) \leq \mathcal{E}_n^k(g_n^k)$ by (4.11), using (4.7) we can deduce for all $t \in [t_n^k, t_n^{k+1})$

$$\begin{aligned}\int_\Omega |e(u_n(t))|^2 \,\mathrm{d}x &= \int_{\Omega \setminus \Omega_{n,k-1}^{\mathrm{crack}}(u_n^k)} |e(u_n^k)|^2 \,\mathrm{d}x \leq \int_{\Omega \setminus \Omega_{n,k-1}^{\mathrm{crack}}(g_n^k)} |e(g_n^k)|^2 \,\mathrm{d}x + \kappa \frac{|\Omega_{n,k-1}^{\mathrm{crack}}(g_n^k)|}{\varepsilon_n} - \kappa \frac{|\Omega_{n,k-1}^{\mathrm{crack}}(u_n^k)|}{\varepsilon_n} \\ &\leq \int_{\Omega \setminus \Omega_{n,k-1}^{\mathrm{crack}}(g_n^k)} |e(g_n^k)|^2 \,\mathrm{d}x \leq C\,.\end{aligned}$$

Eventually, we have $\|e(u_n(t))\|_{L^2(\Omega' \setminus \overline{\Omega})} \leq C$ by the definition of $\mathcal{A}_n^k$ in (4.9) and the regularity of g. In fact, if $T \cap \overline{\Omega} = \emptyset$, then $u_n(t) = g(t_n^k)_{\mathbf{T}_n(u_n(t))}$ in T by the definition of $\mathcal{A}_n^k$. Instead, for $\mathbf{T}_n^{\mathrm{bdy}}(u_n(t)) :=$

$\{T \in \mathbf{T}_n(u_n(t))\colon T \cap (\Omega' \setminus \overline{\Omega}) \neq \emptyset,\, T \cap \Omega \neq \emptyset\}$, it holds $\#\mathbf{T}_n^{\mathrm{bdy}}(u_n(t)) \leq \frac{\tilde{C}}{\varepsilon_n}$ with $\tilde{C}$ depending on the Lipschitz constant of $\partial\Omega$. Therefore, in view of (4.4), $\|e(u_n(t))\|^2_{L^2(\Omega_n^{\mathrm{bdy}}(u_n(t)))} \leq \tilde{C}$, where

$$\Omega_n^{\mathrm{bdy}}(u_n(t)) \coloneqq \mathrm{int}\Big(\bigcup_{T \in \mathbf{T}_n^{\mathrm{bdy}}(u_n(t))} T\Big).$$

This concludes the proof. □

As an immediate consequence, we obtain the following corollary.

Corollary 6.2. *Let $t \mapsto u_n(t)$ be the discrete evolution defined in* (4.18)*. Then, there exists a constant $C > 0$ depending on g, but independent of k and n, such that*

$$\int_0^{t_n^k} \int_\Omega e(u_n(\tau)) : e(\partial_t g(\tau))\,\mathrm{d}x\,\mathrm{d}\tau \leq C \quad \textit{for all } k = 0, \dots, T/\delta_n. \tag{6.2}$$

Proof. By Hölder's inequality we find

$$\int_0^{t_n^k} \int_\Omega e(u_n(\tau)) : e(\partial_t g(\tau))\,\mathrm{d}x\,\mathrm{d}\tau \leq \ \|e(\partial_t g)\|_{L^1(0,T;L^2(\Omega))}\ \|e(u_n)\|_{L^\infty([0,t_n^k);L^2(\Omega))}\,.$$

Using Proposition 6.1 and $g \in W^{1,1}(0,T;W^{2,\infty}(\Omega;\mathbb{R}^2))$ we deduce (6.2). □

We continue with the following discrete energy estimate that will be fundamental to establish a uniform bound on the energy and to prove the energy-balance law.

Lemma 6.3 (Discrete energy estimate)**.** *Let $t \mapsto u_n(t)$ be the discrete evolution defined in* (4.18)*. Let $k = 0, \dots, T/\delta_n$. Then, there exists $(\beta_n)_n$ independent of k with $\beta_n \to 0$ as $n \to \infty$ such that*

$$\mathcal{E}_n^k(u_n^k) - \mathcal{E}_n^0(u_n^0) \leq 2\int_0^{t_n^k} \int_\Omega e(u_n(\tau)) : e(\partial_t g(\tau))\,\mathrm{d}x\,\mathrm{d}\tau + \beta_n. \tag{6.3}$$

Proof. The argumentation follows a well-known strategy, see e.g. [26, Section 3.2], [21, Lemma 6.1], or [32, Lemma 4.3] for an application in a discrete setting. We notice that we have a quadratic bulk energy as in [26], which simplifies the computations. We start by introducing a notation: for the boundary function $g \in W^{1,1}(0,T;W^{2,\infty}(\Omega;\mathbb{R}^2))$ and a time step $0 \leq l \leq T/\delta_n$, we define $\tilde{g}_n^l(t) := g(t)_{\mathbf{T}_n(u_n^l)}$ as the piecewise affine interpolation of $g(t)$ with respect to $\mathbf{T}_n(u_n^l)$.

Given the time t_n^l, we define the test function $\xi_n^l := u_n^{l-1} + \tilde{g}_n^{l-1}(t_n^l) - \tilde{g}_n^{l-1}(t_n^{l-1})$. Note that ξ_n^l is piecewise affine with respect to $\mathbf{T}_n(u_n^{l-1})$ and by definition we have $\xi_n^l \in \mathcal{A}_n^l$. In view of (4.11), we obtain $\mathcal{E}_n^l(u_n^l) \leq \mathcal{E}_n^l(\xi_n^l)$. Our goal is to prove that there exists a bounded sequence $(\vartheta_n)_n$ in $L^\infty([0,T] \times \Omega)$ with $\|\vartheta_n(\tau)\|_{L^2(\Omega)} \to 0$ uniformly in τ such that, for each l, we have

$$\mathcal{E}_n^l(\xi_n^l) - \mathcal{E}_n^l(u_n^{l-1}) \leq 2\int_{t_n^{l-1}}^{t_n^l} \int_\Omega (e(u_n(\tau)) + \vartheta_n(\tau)) : \partial_t e(g(\tau))\,\mathrm{d}x\,\mathrm{d}\tau + \,C\sigma_n^l \varepsilon_n\,, \tag{6.4}$$

where $\sigma_n^l := \int_{t_n^{l-1}}^{t_n^l} \|\partial_t g\|_{W^{2,\infty}(\Omega)}\,\mathrm{d}s$. Once this is shown, we can deduce (6.3) as follows. Note first that $\mathcal{E}_n^{l-1}(u_n^{l-1}) = \mathcal{E}_n^l(u_n^{l-1})$ for each step l. Since $\mathcal{E}_n^l(u_n^l) \leq \mathcal{E}_n^l(\xi_n^l)$, we can sum up over all time steps $1 \leq l \leq k$ to obtain a telescopic sum on the left-hand side of (6.4) which leads to

$$\mathcal{E}_n^k(u_n^k) - \mathcal{E}_n^0(u_n^0) \leq 2\int_0^{t_n^k} \int_\Omega (e(u_n(\tau)) + \vartheta_n(\tau)) : \partial_t e(g(\tau))\,\mathrm{d}x\,\mathrm{d}\tau + C\varepsilon_n, \tag{6.5}$$

where we used $\sum_l \sigma_n^l \le C$. Since $\|\vartheta_n(\tau)\|_{L^2(\Omega)} \to 0$ uniformly in $\tau \in [0,T]$, we get by Hölder's inequality

$$\int_0^{t_n^k} \int_\Omega |\vartheta_n(\tau) : \partial_t e(g(\tau))| \,\mathrm{d}x \,\mathrm{d}\tau \le \int_0^{t_n^k} \|\vartheta_n(\tau)\|_{L^2(\Omega)} \|\partial_t e(g(\tau))\|_{L^2(\Omega)} \,\mathrm{d}\tau \to 0\,.$$

In view of (6.5), setting

$$\beta_n := C\varepsilon_n + \int_0^T \int_\Omega |\vartheta_n(\tau) : \partial_t e(g(\tau))| \,\mathrm{d}x \,\mathrm{d}\tau,$$

we obtain (6.3).

It now remains to prove (6.4). By definition we have $\Omega^{\mathrm{crack}}_{n,l-1}(u_n^{l-1}) = \Omega^{\mathrm{crack}}_{n,l-1} \subset \Omega^{\mathrm{crack}}_{n,l-1}(\xi_n^l)$ and hence (recall (4.7))

$$\mathcal{E}_n^l(\xi_n^l) - \mathcal{E}_n^l(u_n^{l-1}) = \int_{\Omega\setminus\Omega^{\mathrm{crack}}_{n,l-1}(\xi_n^l)} |e(\xi_n^l)|^2 \,\mathrm{d}x - \int_{\Omega\setminus\Omega^{\mathrm{crack}}_{n,l-1}} |e(u_n^{l-1})|^2 \,\mathrm{d}x + \frac{\kappa\, |\Omega^{\mathrm{crack}}_{n,l-1}(\xi_n^l) \setminus \Omega^{\mathrm{crack}}_{n,l-1}|}{\varepsilon_n}\,. \tag{6.6}$$

We can split the first term into

$$\int_{\Omega\setminus\Omega^{\mathrm{crack}}_{n,l-1}(\xi_n^l)} |e(\xi_n^l)|^2 \,\mathrm{d}x = \int_{\Omega\setminus\Omega^{\mathrm{crack}}_{n,l-1}} |e(\xi_n^l)|^2 \,\mathrm{d}x - \int_{\Omega^{\mathrm{crack}}_{n,l-1}(\xi_n^l)\setminus\Omega^{\mathrm{crack}}_{n,l-1}} |e(\xi_n^l)|^2 \,\mathrm{d}x\,. \tag{6.7}$$

Note that $\Omega^{\mathrm{crack}}_{n,l-1}(\xi_n^l)\setminus\Omega^{\mathrm{crack}}_{n,l-1}$ consists of all triangles $T \in \mathbf{T}^{\mathrm{crack}}_{n,l-1}(\xi_n^l)\setminus\mathbf{T}^{\mathrm{crack}}_{n,l-1}$, i.e., it holds that $\varepsilon_n |e(u_n^j)_T|^2 < \kappa$ and $\mathrm{dist}(T, \mathbf{Z}_n(u_n^j)) < 10^6 \varepsilon_n'$ for all $0 \le j \le l-1$ such that $T \in \mathbf{T}_n(u_n^j)$. Since $\mathbf{T}_n(\xi_n^l) = \mathbf{T}_n(u_n^{l-1})$, we hence have $\mathrm{dist}(T, \mathbf{Z}_n(\xi_n^l)) = \mathrm{dist}(T, \mathbf{Z}_n(u_n^{l-1})) < 10^6\varepsilon_n'$ for all $T \in \mathbf{T}^{\mathrm{crack}}_{n,l-1}(\xi_n^l) \setminus \mathbf{T}^{\mathrm{crack}}_{n,l-1}$. Therefore we conclude by (4.4) that $\varepsilon_n |e(\xi_n^l)_T|^2 \ge \kappa$ for all $T \in \mathbf{T}^{\mathrm{crack}}_{n,l-1}(\xi_n^l) \setminus \mathbf{T}^{\mathrm{crack}}_{n,l-1}$. This leads to

$$\int_{\Omega^{\mathrm{crack}}_{n,l-1}(\xi_n^l)\setminus\Omega^{\mathrm{crack}}_{n,l-1}} |e(\xi_n^l)|^2 \,\mathrm{d}x \ge \kappa \frac{|\Omega^{\mathrm{crack}}_{n,l-1}(\xi_n^l) \setminus \Omega^{\mathrm{crack}}_{n,l-1}|}{\varepsilon_n}\,.$$

Combining this with (6.6) and (6.7) we obtain

$$\mathcal{E}_n^l(\xi_n^l) - \mathcal{E}_n^l(u_n^{l-1}) \le \int_{\Omega\setminus\Omega^{\mathrm{crack}}_{n,l-1}} (|e(\xi_n^l)|^2 - |e(u_n^{l-1})|^2) \,\mathrm{d}x\,. \tag{6.8}$$

Next, recalling the definition $\xi_n^l := u_n^{l-1} + \tilde{g}_n^{l-1}(t_n^l) - \tilde{g}_n^{l-1}(t_n^{l-1})$, we apply the mean value theorem to the function $h\colon [0,1] \to \mathbb{R};\ s \mapsto |e(u_n^{l-1}) + s\,(e(\tilde{g}_n^{l-1}(t_n^l) - \tilde{g}_n^{l-1}(t_n^{l-1})))|^2$ to obtain some $\rho_n^{l-1} \in [0,1]$ with

$$\begin{aligned} &\int_{\Omega\setminus\Omega^{\mathrm{crack}}_{n,l-1}} (|e(\xi_n^l)|^2 - |e(u_n^{l-1})|^2) \,\mathrm{d}x \\ &= 2\int_{\Omega\setminus\Omega^{\mathrm{crack}}_{n,l-1}} \big(e(u_n^{l-1}) + \rho_n^{l-1}\, e(\tilde{g}_n^{l-1}(t_n^l) - \tilde{g}_n^{l-1}(t_n^{l-1}))\big) : \big(e(\tilde{g}_n^{l-1}(t_n^l) - \tilde{g}_n^{l-1}(t_n^{l-1}))\big) \,\mathrm{d}x\,. \end{aligned} \tag{6.9}$$

We can now define the function $\vartheta_n\colon [0,T] \to L^2(\Omega;\mathbb{R}^{2\times2}_{\mathrm{sym}})$ by setting, for $s \in [t_n^{l-1}, t_n^l)$,

$$\vartheta_n(s) := \rho_n^{l-1}\big(e(\tilde{g}_n^{l-1}(t_n^l)) - e(\tilde{g}_n^{l-1}(t_n^{l-1}))\big) = \rho_n^{l-1} \int_{t_n^{l-1}}^{t_n^l} \partial_t e(\tilde{g}_n^{l-1}(\tau)) \,\mathrm{d}\tau \quad \text{in } \Omega \setminus \Omega^{\mathrm{crack}}_{n,l-1}\,, \tag{6.10}$$

and 0 outside of it. Since $g \in W^{1,1}(0,T;W^{2,\infty}(\Omega;\mathbb{R}^2))$, the function $\tau \mapsto \partial_t e(\tilde{g}_n^{l-1}(\tau))$ belongs to $L^1(0,T;L^\infty(\Omega;\mathbb{R}^{2\times2}_{\mathrm{sym}}))$ and we can use the absolute continuity of the integral and $|t_n^l - t_n^{l-1}| = \delta_n \to 0$ to conclude that $\|\vartheta_n(s)\|_{L^2(\Omega)} \to 0$ uniformly in s. Moreover, we write

$$e(\tilde{g}_n^{l-1}(t_n^l)) - e(\tilde{g}_n^{l-1}(t_n^{l-1})) = \int_{t_n^{l-1}}^{t_n^l} \partial_t e(g(\tau))\,\mathrm{d}\tau + \zeta_n^{l-1}\,, \tag{6.11}$$

with $\zeta_n^{l-1} := \int_{t_n^{l-1}}^{t_n^l} \partial_t e(\tilde{g}_n^{l-1}(\tau) - g(\tau))\,\mathrm{d}\tau$. By the regularity of g and the definition of $\tilde{g}_n^l(t)$ we obtain

$$\|\zeta_n^{l-1}\|_{L^\infty(\Omega)} \le C\sigma_n^l\,\omega(\varepsilon_n) \le C\sigma_n^l\,\varepsilon_n, \tag{6.12}$$

where σ_n^l is given in (6.4), and the last step follows from (4.1). Noting that ϑ_n is piecewise constant in time, we deduce from (6.9)–(6.11) and by Fubini's theorem

$$\begin{aligned}\int_{\Omega\setminus\Omega^{\mathrm{crack}}_{n,l-1}} (|e(\xi_n^l)|^2 - |e(u_n^{l-1})|^2)\,\mathrm{d}x = 2\int_{t_n^{l-1}}^{t_n^l}\int_{\Omega\setminus\Omega^{\mathrm{crack}}_{n,l-1}} (e(u_n^{l-1}) + \vartheta_n(\tau)) : e(\partial_t g(\tau))\,\mathrm{d}x\,\mathrm{d}\tau \\ +2\int_{\Omega\setminus\Omega^{\mathrm{crack}}_{n,l-1}} (e(u_n^{l-1}) + \vartheta_n(\tau)) : \zeta_n^{l-1}\,\mathrm{d}x\,.\end{aligned} \tag{6.13}$$

By Proposition 6.1, the definition in (4.18), and the fact that $\|\vartheta_n(s)\|_{L^2(\Omega)} \to 0$, we can estimate the last term by (6.12) and obtain

$$\int_{\Omega\setminus\Omega^{\mathrm{crack}}_{n,l-1}} (e(u_n^{l-1}) + \vartheta_n(\tau)) : \zeta_n^{l-1}\,\mathrm{d}x \le C\sigma_n^l\,\varepsilon_n\,. \tag{6.14}$$

By (4.18), (6.8), (6.13), and (6.14) we obtain

$$\mathcal{E}_n^l(\xi_n^l) - \mathcal{E}_n^l(u_n^{l-1}) \le 2\int_{t_n^{l-1}}^{t_n^l}\int_\Omega (e(u_n(\tau)) + \vartheta_n(\tau)) : e(\partial_t g(\tau))\,\mathrm{d}x\,\mathrm{d}\tau + C\sigma_n^l\varepsilon_n\,.$$

This yields (6.4) and concludes the proof. ☐

As a direct consequence, we obtain the following bound on the energy.

Corollary 6.4 (Energy bound)**.** *Let $t \mapsto u_n(t)$ be the discrete evolution defined in* (4.18)*. Then, there exists a constant $C > 0$ only depending on g such that*

$$\mathcal{E}_n^k(u_n^k) \le C \quad \text{for all } k = 0, \dots, T/\delta_n \text{ and } n \in \mathbb{N}.$$

Proof. The proof follows by combining $\mathcal{E}_n^0(u_n^0) \le C$ (see Proposition 6.1), (6.2), and Lemma 6.3. ☐

6.2. **Compactness and lower semicontinuity.** Based on the energy bound in Corollary 6.4, we can pass to the limit in the crack sets and displacements by compactness arguments. We start with the crack sets.

Proposition 6.5 (Convergence of crack sets)**.** *Let $t \mapsto (u_n(t), K_n(t))$ be the evolution defined in* (4.18) *and* (4.19)*. There exists an increasing set function $t \mapsto K(t)$ in $\Omega' \cap \overline{\Omega}$ and a subsequence (not relabeled) such that, for every $t \in [0, T]$, the set $K_n(t)$ σ-converges to $K(t)$.*

Proof. By (4.17), (4.19), and the energy bound in Corollary 6.4 we have that $\mathcal{H}^1(K_n(t)) \le C$ for all $t \in [0, T]$. We therefore can apply Proposition 5.3 for each $t \in [0, T]$ to obtain the limiting set function $K(t) \tilde{\subset} \Omega'$ such that $K_n(t)$ σ-converges to $K(t)$ for a t-dependent subsequence. Since $K_n(t) \tilde{\subset} \Omega' \cap \overline{\Omega}$, the definition of σ-convergence directly implies $K(t) \tilde{\subset} \Omega' \cap \overline{\Omega}$, see Remark 5.2(iii). Our goal is to apply Helly's theorem in the version of Theorem 5.4 to show that the subsequence can be chosen independently of t

and that $t \mapsto K(t)$ is increasing. This is however impeded by the fact that $K_n(t)$ is not an increasing set function. As a remedy, we define

$$\tilde{K}_n(t) := \bigcup_{T \in \mathbf{T}^{\rm mod}_{n,k}} \partial T \quad \text{for } t \in [t^k_n, t^{k+1}_n)\,,$$

where $\mathbf{T}^{\rm mod}_{n,k} := \{T \in \mathbf{T}^{\rm crack}_{n,k} \colon T \subset \Omega^{\rm mod}_{n,k}\}$. As $\Omega^{\rm crack}_{n,k-1} \subset \Omega^{\rm crack}_{n,k}$, see (4.6), we get $\mathbf{T}^{\rm mod}_{n,k-1} \subset \mathbf{T}^{\rm mod}_{n,k}$ by Corollary 3.1, and thus the set function $t \mapsto \tilde{K}_n(t)$ in fact fulfills $\tilde{K}_n(s) \tilde{\subset} \tilde{K}_n(t)$ for $s \le t$. Moreover, $\mathcal{H}^1(\tilde{K}_n(t)) \le C$ for all $t \in [0,T]$, again by the energy bound in Corollary 6.4. (Use $|T| \ge c\varepsilon\mathcal{H}^1(\partial T)$ for all triangles T for c depending on θ_0 to get $\#\mathbf{T}^{\rm mod}_{n,k} \le C/\varepsilon_n$, as well as (4.1).) Hence, we can apply Theorem 5.4 to obtain an increasing set function $t \mapsto \tilde{K}(t)$ such that $\tilde{K}_n(t)$ σ-converges to $\tilde{K}(t)$ for every $t \in [0,T]$, up to extracting a subsequence (not relabeled). We now want to show that $\tilde{K}(t) \tilde{=} K(t)$ for each $t \in [0,T]$. Note that each triangle $T \in \{T \notin \mathbf{T}^{\rm crack}_{n,k} \colon T \subset \Omega^{\rm mod}_{n,k}\}$ fulfills $\partial T \cap K_n \tilde{=} \emptyset$, by Remark 3.5. Therefore, we have $K_n(t) \tilde{\subset} \tilde{K}_n(t)$ because

$$K_n(t) = \partial\Omega^{\rm mod}_{n,k} \setminus \bigcup_{\substack{T \subset \Omega^{\rm mod}_{n,k} \\ T \notin \mathbf{T}^{\rm crack}_{n,k}}} \partial T \subset \bigcup_{\substack{T \subset \Omega^{\rm mod}_{n,k} \\ T \in \mathbf{T}^{\rm crack}_{n,k}}} \partial T = \tilde{K}_n(t)\,.$$

By Remark 5.2(ii), this implies $K(t) \tilde{\subset} \tilde{K}(t)$. To show the reverse inclusion, we fix a time $t \in [0,T]$. We denote by $\mathcal{H}^-_{\tilde{K}(t)}$ the Γ-limit of $\mathcal{H}^-_{\tilde{K}_n(t)}$ given by $\mathcal{H}^-_{\tilde{K}_n(t)}(u,A) = \mathcal{H}^1((J_u \setminus \tilde{K}_n(t)) \cap A)$, and let $h^-_{\tilde{K}(t)}$ be the corresponding density function from (5.3). In the analogous way, we denote $\mathcal{H}^-_{K(t)}$ and $h^-_{K(t)}$.

We suppose by contradiction that $\mathcal{H}^1(\tilde{K}(t) \setminus K(t)) > 0$. As $K(t)$ is the maximal set on which $h^-_{K(t)}$ is $\mathcal{H}^1$-a.e. equal to 0 (cf. Definition 5.1), we find $\mu > 0$ such that

$$\int_{\tilde{K}(t)} h^-_{K(t)}(x, \nu_{\tilde{K}(t)}(x))\,\mathrm{d}\mathcal{H}^1(x) > 3\mu. \tag{6.15}$$

We use a covering argument to find an open set $U \subset \Omega'$ and a function $v \in \mathrm{PC}(U)$ such that

$$\mathcal{H}^1(\tilde{K}(t) \setminus U) \le \mu \qquad \mathcal{H}^1\big((\tilde{K}(t) \triangle J_v) \cap U\big) \le \mu\,. \tag{6.16}$$

Let $(v_n)_n$ be a recovery sequence for v with respect to the functional $\mathcal{H}^-_{\tilde{K}(t)}(\cdot, U)$, i.e., particularly

$$\limsup_{n\to\infty} \mathcal{H}^1((J_{v_n} \setminus \tilde{K}_n(t)) \cap U) \le \int_{J_v \cap U} h^-_{\tilde{K}(t)}(x, \nu_v(x))\,\mathrm{d}\mathcal{H}^1(x) \le \mu$$

by (6.16) and the fact that $h^-_{\tilde{K}(t)} \le 1$, see Remark 5.2(i). We define a modified sequence by

$$\tilde{v}_n(x) := \begin{cases} v_n(x) & x \in \Omega' \setminus \Omega^{\rm mod}_{n,k}, \\ 0 & x \in \Omega^{\rm mod}_{n,k}\,. \end{cases} \tag{6.17}$$

Then, recalling (4.19) we have $J_{\tilde{v}_n} \setminus K_n(t) \tilde{\subset} J_{v_n} \setminus \tilde{K}_n(t)$ and thus $\limsup_{n\to\infty} \mathcal{H}^1((J_{\tilde{v}_n} \setminus K_n(t)) \cap U) \le \mu$. In view of (4.16)–(4.17), we obtain $|\{\tilde{v}_n \neq v_n\}| \le |\Omega^{\rm mod}_{n,k}| \le C_{\eta_n}\varepsilon_n \to 0$. As $v_n \to v$ in $L^1(U;\mathbb{R}^2)$, we thus get $\tilde{v}_n \to v$ in $L^1(U;\mathbb{R}^2)$. Using the Γ-liminf inequality for the σ-convergence of $K_n(t)$ to $K(t)$ we get

$$\mathcal{H}^-_{K(t)}(v,U) = \int_{J_v \cap U} h^-_{K(t)}(x, \nu_v(x))\,\mathrm{d}\mathcal{H}^1(x) \le \limsup_{n\to\infty} \mathcal{H}^1((J_{\tilde{v}_n} \setminus K_n(t)) \cap U) \le \mu. \tag{6.18}$$

Now, (6.16) and the fact that $h^-_{K(t)} \le 1$ show

$$\int_{\tilde K(t)} h^-_{K(t)}(x, \nu_{\tilde K(t)}(x))\, \mathrm{d}\mathcal{H}^1(x) \le \mathcal{H}^-_{K(t)}(v, U) + 2\mu \le 3\mu,$$

which contradicts (6.15) and concludes the proof. □

In the sequel, it will be convenient to express some of the quantities considered so far in terms of the time t in place of the iteration step. As before, let $\{0 = t_n^0 < t_n^1 < \cdots < t_n^{T/\delta_n} = T\}$ be the time discretization of the interval $[0,T]$, and let $(u_n^j)_{j<k}$ be a corresponding displacement history. We define the set of functions $\mathcal{A}_n(t) := \mathcal{A}_n^k$ for $t \in [t_n^k, t_n^{k+1})$, see (4.9). Recalling (4.7) and (6.1), we define

$$\mathcal{E}_n(v_n; t) := \mathcal{E}_n^k(v_n) = \mathcal{E}_n(v_n; (u_n^j)_{j<k}) \quad \text{for } t \in [t_n^k, t_n^{k+1}) \tag{6.19}$$

for each $v_n \in \mathcal{A}_n(t)$. We use similar notation for the parts of the energy introduced in (4.7). Recall also the Griffith energy defined in (4.12).

Proposition 6.6. *Let $t \mapsto (u_n(t), K_n(t))$ be the evolution defined in* (4.18) *and* (4.19)*. Let $K(t)$ be the σ-limit of $K_n(t)$ given by Proposition 6.5, and let $G(t)$ be the set corresponding to $K(t)$ defined before* (4.20)*. Then, there exists a function $u(t) \in AD(g(t), K(t))$ with $u(t) = 0$ on $\Omega' \setminus G(t)$, and a subsequence $(n_l)_l$ depending on t, such that $u_{n_l}(t) \to u(t)$ in measure on $G(t)$ and $e(u_{n_l}(t)) \rightharpoonup e(u(t))$ weakly in $L^2(G(t); \mathbb{R}^{2\times 2}_{\mathrm{sym}})$. Moreover,*

$$\liminf_{n\to\infty} \mathcal{E}_n^{\mathrm{crack}}(u_n(t); t) \ge \kappa \sin\theta_0 \mathcal{H}^1(K(t))\,. \tag{6.20}$$

Proof. We fix $t \in [0,T]$ and for each $n \in \mathbb{N}$ we choose k_n such that $t \in [t_n^{k_n}, t_n^{k_n+1})$. Recall the definition of $\Omega^{\mathrm{crack}}_{n,k_n}$ and $\Omega^{\mathrm{mod}}_{n,k_n}$ in (4.6) and (4.17), respectively. By u_n^{mod} we denote the function given by Theorem 2.1 applied on $u_n(t)$, which by (2.13) and the energy bounds in Proposition 6.1, and Corollary 6.4 satisfies

$$\|e(u_n^{\mathrm{mod}})\|_{L^2(\Omega'_{\varepsilon_n,\eta_n} \setminus \Omega^{\mathrm{mod}}_{n,k_n})} \le C_{\eta_n}. \tag{6.21}$$

Choose $\Omega_* \subset\subset \Omega'$ and suppose that n is large enough such that $\Omega_* \subset \Omega'_{\varepsilon_n,\eta_n}$. We define $v_n \in GSBD^2(\Omega_*)$ by $v_n := (1 - \chi_{\Omega^{\mathrm{mod}}_{n,k_n}})u_n^{\mathrm{mod}}$. Then, by (4.16), (4.17), and Corollary 6.4 we get

$$|D_n| \le C_{\eta_n}\varepsilon_n + C\varepsilon_n \le C_{\eta_n}\varepsilon_n \to 0, \qquad \text{where } D_n := \big(\{u_n(t) \ne v_n\} \cap \Omega_*\big) \cup \Omega^{\mathrm{crack}}_{n,k_n}. \tag{6.22}$$

For $p := 3/2$ we have by Hölder's inequality, (6.21)–(6.22), and the energy bound in Proposition 6.1 that

$$\begin{aligned}\|e(v_n)\|_{L^p(\Omega_*)} &= \|e(v_n)\|_{L^p(D_n)} + \|e(u_n^{k_n})\|_{L^p(\Omega' \setminus \Omega^{\mathrm{crack}}_{n,k_n})} \\ &\le C|D_n|^{1/6}\|e(u_n^{\mathrm{mod}})\|_{L^2(D_n \setminus \Omega^{\mathrm{mod}}_{n,k_n})} + C\|e(u_n^{k_n})\|_{L^2(\Omega' \setminus \Omega^{\mathrm{crack}}_{n,k_n})} \le C_{\eta_n}^{7/6}\varepsilon_n^{1/6} + C \le C,\end{aligned}$$

where in the last step we used (4.16). By (4.17) and the energy bound we get $\mathcal{H}^1(J_{v_n}) \le \mathcal{H}^1(K_n(t)) \le C$. Thus, by Proposition 5.7 we find $u(t) \in GSBD^p_\infty(\Omega_*)$ such that (up to a subsequence, not relabeled)

$$v_n \to u(t) \text{ in measure on } \Omega_* \setminus A^\infty_{u(t)}. \tag{6.23}$$

By Proposition 5.8(i) and the σ-convergence of $K_n(t)$ to $K(t)$, we get $J_{u(t)} \cap \Omega_* \tilde\subset K(t)$. Then, by a diagonal argument (letting $\Omega_* \nearrow \Omega'$), we find $u(t) \in GSBD^p_\infty(\Omega')$ such that $v_n \to u(t)$ in measure on $\Omega' \setminus A^\infty_{u(t)}$ with $J_{u(t)} \tilde\subset K(t)$.

Note that $g(t_n^{k_n})_{\mathbf{T}(u_n^{k_n})} \to g(t)$ in $W^{1,2}(\Omega'; \mathbb{R}^2)$ by the regularity of g (recall (4.2)). As $u_n(t) \in \mathcal{A}_n(t)$ (see (4.9)), it is elementary to check that $u(t) = g(t)$ on $\Omega' \setminus \overline{\Omega}$. In particular, we find $A^\infty_{u(t)} \subset \Omega$. Then, we can apply Proposition 5.8(ii) which along with the definition of $G(t)$ before (4.20) implies $|A^\infty_{u(t)} \cap G(t)| = \emptyset$.

Thus, in view of (6.23), we conclude $v_n \to u(t)$ in measure on $G(t)$. Next, by (6.22) we get $u_n(t) \to u(t)$ in measure on $G(t)$. Moreover, by Proposition 6.1, $|\Omega^{\rm crack}_{n,k_n}| \to 0$, and by weak compactness we get $e(u_n(t)) \rightharpoonup e(u(t))$ weakly in $L^2(G(t); \mathbb{R}^{2\times 2}_{\rm sym})$. Up to replacing $u(t)$ by 0 in $\Omega' \setminus G(t)$ (not relabeled), we thus obtain $u(t) \in GSBD^2(\Omega')$, and $J_{u(t)} \tilde{\subset} K(t)$ as well as $u(t) = g(t)$ on $\Omega' \setminus \overline{\Omega}$ still hold. Summarizing, we have shown that $u(t) \in AD(g(t), K(t))$, see (4.13).

It remains to show (6.20). We recall from (4.17) and (4.19) that

$$\mathcal{H}^1(K_n(t)) \leq \frac{2}{\varepsilon_n \sin\theta_0} |\Omega^{\rm crack}_{n,k}| + C\eta_n \leq \frac{2}{\kappa \sin\theta_0} \mathcal{E}^{\rm crack}_n(u_n(t); t) + C\eta_n.$$

For $n \to \infty$ we have $\eta_n \to 0$ and $|\Omega^{\rm mod}_{n,k}| \to 0$ (see (4.16)–(4.17)). Then, in view of $K_n(t) = \partial\Omega^{\rm mod}_{n,k}$, by means of Lemma 5.6 we derive

$$2\mathcal{H}^1(K(t)) \leq \frac{2}{\kappa \sin\theta_0} \liminf_{n\to\infty} \mathcal{E}^{\rm crack}_n(u_n(t); t)\,.$$

This concludes the proof of (6.20). ☐

6.3. **Proof of Theorem 4.2.** In this section we give the proof of the main result. We need a final preparation. Recall the Griffith energy $\mathcal{E}$ defined in (4.12).

Theorem 6.7 (Stability)**.** *Let $t \mapsto (u_n(t), K_n(t))$ be the evolution defined by* (4.18)–(4.19) *and let $K(t)$ be the σ-limit of $K_n(t)$. For any $\psi \in GSBD^2(\Omega')$ with $\psi = g(t)$ on $\Omega' \setminus \overline{\Omega}$ there exists a sequence $(\psi_n)_n$ of piecewise affine displacements with $\psi_n \in \mathcal{A}_n(t)$ converging to ψ in measure on Ω' such that*

$$\limsup_{n\to\infty} \Big(\mathcal{E}^{\rm crack}_n(\psi_n; t) - \mathcal{E}^{\rm crack}_n(u_n(t); t)\Big) \leq \kappa \sin(\theta_0)\, \mathcal{H}^1(J_\psi \setminus K(t)), \tag{6.24}$$

and

$$\lim_{n\to\infty} \mathcal{E}^{\rm elast}_n(\psi_n; t) \leq \int_\Omega |e(\psi)|^2 \,\mathrm{d}x. \tag{6.25}$$

Corollary 6.8 (Recovery sequence)**.** *For each $\psi \in GSBD^2(\Omega')$ with $\psi = g(0)$ on $\Omega' \setminus \overline{\Omega}$ there exists a sequence $(\psi_n)_n$ with $\psi_n \in \mathcal{A}^0_n$ such that ψ_n converges to ψ in measure on Ω' and*

$$\limsup_{n\to\infty} \mathcal{E}^0_n(\psi_n) \leq \mathcal{E}(\psi, J_\psi).$$

This result is essential to pass from the minimality condition (4.11) in the finite element model to the global stability in (4.14). For this reason, estimates of this kind are often referred to as *stability of unilateral minimizers*, see e.g. [40]. The proof of Theorem 6.7 will be deferred to the next section; it could be directly adapted (indeed, simplified: it is enough to drop the dependence on t and formally consider $K_n = \emptyset$) to prove Corollary 6.8, which in turn confirms the upper bound in the Γ-convergence result [3, Theorem 3.1]. The strategy follows the one of the *jump transfer lemma* introduced by Francfort and Larsen in [26], see the works [21, 23, 32, 33, 37] for several variants. Our situation is slightly different and more delicate because we have to construct an admissible mesh associated to the recovery sequence. For this, we will exploit the ideas by Dal Maso and Chambolle [14] for the explicit construction of a minimizing adaptive mesh.

We are now in the position to prove the main result of this paper.

Proof of Theorem 4.2. We split the proof into five parts. First, we prove the existence of a limiting evolution and validate the irreversibility of the crack sets. Subsequently, in a second step, we use Theorem 6.7 to prove the stability property (4.14). Afterwards, we show the convergence of displacement fields without the necessity of passing to t-dependent subsequences, see Step 3. Next, we want to confirm that the mapping $t \to (u(t), K(t))$ actually fulfills the energy balance (4.15) which is content of Step 4. In the last step, we then prove the convergence of energies and thus the strong convergence of the linear strains.

Step 1: Limiting evolution. Let $K(t)$ be the σ-limit of $K_n(t)$ for $t \in [0,T]$ given by Proposition 6.5. Note that $t \mapsto K(t)$ is an increasing set function. For each fixed time $t \in [0,T]$, by virtue of Proposition 6.6, there exists $u(t) \in AD(g(t), K(t))$ with $u(t) = 0$ on $\Omega' \setminus G(t)$ and a subsequence n_l depending on t such that $u_{n_l}(t)$ converges to $u(t)$ in measure on $G(t)$ and $e(u_{n_l}(t)) \rightharpoonup e(u(t))$ weakly in $L^2(G(t); \mathbb{R}^{2\times 2}_{\mathrm{sym}})$.

Step 2: Stability (4.14). Fix $t \in [0,T]$. Let H with $K(t) \tilde{\subset} H$ and $\psi \in AD(g(t), H)$. We employ Theorem 6.7 to obtain a sequence of piecewise affine displacements $\psi_n \in \mathcal{A}_n(t)$ approximating ψ and satisfying (6.24)–(6.25). By the minimality property of the solution $u_n(t)$, see (4.10)–(4.11), (4.18), and the shorthand notation in (6.19) we have

$$\mathcal{E}_n(u_n(t); t) \leq \mathcal{E}_n(\psi_n; t)\,.$$

We now split the energy on both sides like in (4.7): subtracting the crack energy on the left-hand side gives us

$$\mathcal{E}_n^{\mathrm{elast}}(u_n(t); t) \leq \mathcal{E}_n^{\mathrm{elast}}(\psi_n; t) + \mathcal{E}_n^{\mathrm{crack}}(\psi_n; t) - \mathcal{E}_n^{\mathrm{crack}}(u_n(t); t). \tag{6.26}$$

Passing to the limit in (6.26), by employing (6.24) and (6.25), we find

$$\limsup_{n\to\infty} \mathcal{E}_n^{\mathrm{elast}}(u_n(t); t) \leq \int_\Omega |e(\psi)|^2 \,\mathrm{d}x + \kappa \sin(\theta_0)\, \mathcal{H}^1(J_\psi \setminus K(t)). \tag{6.27}$$

By Proposition 6.1, Proposition 6.6, and a compactness argument, we get $e(u_{n_l}(t)) \rightharpoonup W(t)$ in $L^2(\Omega'; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ for some $W(t) \in L^2(\Omega'; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ with $W(t) = e(u(t))$ on $G(t)$. As $u(t) = 0$ on $\Omega' \setminus G(t)$, we derive

$$\int_\Omega |e(u(t))|^2 \,\mathrm{d}x + \int_{\Omega \setminus G(t)} |W(t)|^2 \,\mathrm{d}x \leq \int_\Omega |e(\psi)|^2 \,\mathrm{d}x + \kappa \sin(\theta_0)\, \mathcal{H}^1(J_\psi \setminus K(t)). \tag{6.28}$$

Since $\psi \in AD(\psi, H)$, we have $J_\psi \tilde{\subset} H$, and thus

$$\int_\Omega |e(u(t))|^2 \,\mathrm{d}x \leq \int_\Omega |e(\psi)|^2 \,\mathrm{d}x + \kappa \sin(\theta_0)\, \mathcal{H}^1(H \setminus K(t)). \tag{6.29}$$

Because $K(t) \tilde{\subset} H$ we conclude that (4.14) holds. By particularly choosing $\psi = u(t)$ in (6.28) and recalling $J_{u(t)} \tilde{\subset} K(t)$, we get $W(t) \equiv 0$ on $\Omega \setminus G(t)$, and thus $e(u_{n_l}(t)) \rightharpoonup e(u(t))$ weakly in $L^2(\Omega'; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$.

Step 3: Uniqueness of limiting displacements. We argue that the obtained limit $u(t)$ is uniquely determined on $G(t)$ and satisfies $e(u(t)) = 0$ on $\Omega' \setminus G(t)$ for all $t \in [0,T]$. This along with Uyrsohn's principle shows that the subsequence $(n_l)_l$ in Step 1 can be chosen independently of t. This shows (4.22) except for strong convergence, which we defer to Step 5.

Let us suppose that $\hat{u}(t) \in AD(g(t), K(t))$ denotes another limit of the sequence $u_n(t)$ found in Step 1. As in Step 2, one can show that (6.29) holds for $\hat{u}(t)$ in place of $u(t)$. This shows that both $u(t)$ and $\hat{u}(t)$ are minimizers of the strictly convex minimization problem $v \mapsto \int_\Omega |e(v)|^2 \,\mathrm{d}x$ for $v \in AD(g(t), K(t))$. Consequently, $e(u(t)) = e(\hat{u}(t))$ on Ω'. Moreover, $e(u(t)) = e(\hat{u}(t)) = 0$ on $\Omega' \setminus G(t)$, see (4.20). By a piecewise rigidity argument (see [15]), taking the definition of $G(t)$ and the fact $u(t) = \hat{u}(t) = g(t)$ on $\Omega' \setminus \overline{\Omega} \subset G(t)$ into account and using the fact that adding infinitesimal rigid motions is only admissible on the connected components of $\Omega' \setminus K(t)$ inside $B(t)$, we then see that $u(t) = \hat{u}(t)$ on $G(t)$. We refer to [35, Proof of Theorem 2.2] for details.

Moreover, arguing as in Step 2 and above in Step 3 (cf. also [26, Lemma 3.8] and [33, Theorem 7.5]), we can ensure that, for each $t \in [0,T]$ outside of an at most countable subset and every $t_n \nearrow t$, we have that $u(t_n) \to u(t)$ in measure on $G(t)$ and $e(u(t_n)) \rightharpoonup e(u(t))$ in $L^2(\Omega'; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$.

Step 4: Energy balance: From Step 1 and Step 3 we recall that $e(u_n(\tau)) \rightharpoonup e(u(\tau))$ in $L^2(\Omega';\mathbb{R}^{2\times 2}_{\rm sym})$ for all $\tau \in [0,T]$. As $e(\partial_t g(\tau,\cdot)) \in L^2(\Omega',\mathbb{R}^{2\times 2}_{\rm sym})$ is uniformly bounded in time, using Proposition 6.1 we can apply the reverse Fatou's lemma to deduce that

$$\limsup_{n\to\infty} \int_0^t \int_\Omega e(u_n(\tau)) : (\partial_t g(\tau))\,\mathrm{d}x\,\mathrm{d}\tau \le \int_0^t \int_\Omega e(u(\tau)) : (\partial_t g(\tau))\,\mathrm{d}x\,\mathrm{d}\tau\,. \tag{6.30}$$

By Lemma 6.3 we find $(\beta_n)_n$ with $\beta_n \to 0$ such that for any $t \in [t_n^k, t_n^{k+1})$ we have

$$\mathcal{E}_n(u_n(t);t) - \mathcal{E}^0_n(u^0_n) \le 2\int_0^{t_n^k} \int_\Omega e(u_n(\tau)) : e(\partial_t g(\tau))\,\mathrm{d}x\,\mathrm{d}\tau + \beta_n.$$

By passing to another sequence $(\beta_n)_n$, still satisfying $\beta_n \to 0$, for every $t \in [0,T]$ we have

$$\mathcal{E}_n(u_n(t);t) \le 2\int_0^{t} \int_\Omega e(u_n(\tau)) : e(\partial_t g(\tau))\,\mathrm{d}x\,\mathrm{d}\tau + C\beta_n + \mathcal{E}^0_n(u^0_n)\,, \tag{6.31}$$

where we have used $e(u_n) \in L^\infty(0,T;L^2(\Omega,\mathbb{R}^{2\times 2}_{\rm sym}))$ (see Proposition 6.1) and $g \in W^{1,1}(0,T;W^{2,\infty}(\Omega;\mathbb{R}^2))$ to estimate the integral from t_n^k to t in terms of β_n. Recalling (4.10), (6.1), and Corollary 6.8 we have

$$\mathcal{E}(u(0),K(0)) \le \limsup_{n\to\infty} \mathcal{E}^0_n(u^0_n) = \limsup_{n\to\infty} \min_{\mathcal{A}^0_n} \mathcal{E}^0_n \le \mathcal{E}(\psi, J_\psi)$$

for all $\psi \in GSBD^2(\Omega')$ with $\psi = g(0)$ on $\Omega' \setminus \overline{\Omega}$, where the first inequality follows from Proposition 6.6 and weak lower semicontinuity of norms. As $J_{u(0)} \,\tilde{\subset}\, K(0)$, this shows that condition (a) in Definition 4.1 holds. By choosing $\psi = u(0)$ and using again $J_{u(0)} \,\tilde{\subset}\, K(0)$, we get $J_{u(0)} = K(0)$ and $\limsup_{n\to\infty} \mathcal{E}^0_n(u^0_n) \le \mathcal{E}(u(0),K(0))$. Combining this with Proposition 6.6 and (6.30)–(6.31) we get

$$\begin{aligned}\mathcal{E}(u(t),K(t)) &\le \liminf_{n\to\infty} \mathcal{E}_n(u_n(t);t) \le \limsup_{n\to\infty} \mathcal{E}_n(u_n(t);t) \le \limsup_{n\to\infty} \mathcal{E}^0_n(u^0_n) \\ &+ \limsup_{n\to\infty} 2\int_0^t \int_\Omega e(u_n(\tau)) : e(\partial_t g(\tau))\,\mathrm{d}x\,\mathrm{d}\tau \le \mathcal{E}(u(0),K(0)) + 2\int_0^t \int_\Omega e(u(\tau)) : e(\partial_t g(\tau))\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned} \tag{6.32}$$

Thus, it remains to prove the reverse inequality

$$\mathcal{E}(u(t),K(t)) \ge \mathcal{E}(u(0),K(0)) + 2\int_0^t \int_\Omega e(u(\tau)) : e(\partial_t g(\tau))\,\mathrm{d}x\,\mathrm{d}\tau. \tag{6.33}$$

Fixed $t \in [0,T]$, let $(s_k^i)_i$ be a partition of $[0,t]$ with

$$\lim_{k\to+\infty} \max_{1\le i\le k} (s_k^i - s_k^{i-1}) = 0.$$

For any i and k, we take $H = K(s_k^i)$ and $v = (u(s_k^i) - g(s_k^i) + g(s_k^{i-1})) \in AD(g(s_k^{i-1}), K(s_k^i))$ as an admissible competitor at time s_k^{i-1}. This yields

$$\mathcal{E}(u(s_k^{i-1}), K(s_k^{i-1})) \le \mathcal{E}\big(u(s_k^i) + \big(g(s_k^{i-1}) - g(s_k^i)\big), K(s_k^i)\big),$$

that is

$$\mathcal{E}(u(s_k^i), K(s_k^i)) \ge \mathcal{E}(u(s_k^{i-1}), K(s_k^{i-1})) + 2\int_\Omega e(u(s_k^i)) \colon e(g(s_k^i) - g(s_k^{i-1}))\,\mathrm{d}x - \int_\Omega |e(g(s_k^i) - g(s_k^{i-1}))|^2\,\mathrm{d}x.$$

Summing over i, and observing that $g(s_k^i) - g(s_k^{i-1}) = \int_{s_k^{i-1}}^{s_k^i} \partial_t g(\tau)\,\mathrm{d}\tau$ are functions on $L^2(\Omega';\mathbb{R}^2)$ (where the integral is in the Bochner sense), we get

$$\mathcal{E}(u(t),K(t)) \ge \mathcal{E}(u(0),K(0)) + 2\int_0^t \int_\Omega e(\overline{u}(\tau)) \colon e(\partial_t g(\tau))\,\mathrm{d}x\,\mathrm{d}\tau + o_{k\to+\infty}(1),$$

with $\overline{u}(\tau) := u(s_k^i)$ for $\tau \in (s_k^{i-1}, s_k^i]$. Now, we use $g \in W^{1,1}(0,T; W^{1,2}(\Omega'; \mathbb{R}^2))$ together with the uniform bound from Proposition 6.1 and the weak convergence in $L^2(\Omega; \mathbb{R}^{2\times 2}_{\rm sym})$, as $k \to \infty$, of $e(\overline{u}(\tau))$ to $e(u(\tau))$ for all $\tau \in [0,t]$, except for at most countable many (see end of Step 3). This shows (6.33) and concludes the proof of the energy balance (4.15).

Step 5: Energy convergence and strong convergence of displacements. Gathering (6.32) with (6.33), and recalling (6.19) we deduce (4.23). Moreover, since the two parts of the energy are separately lower semi-continuous, we obtain the statement in Remark 4.3(i). In particular, $\|e(u_n(t))\|_{L^2(\Omega)} \to \|e(u(t))\|_{L^2(\Omega)}$, which shows $e(u_n(t)) \to e(u(t))$ strongly in $L^2(\Omega'; \mathbb{R}^{2\times 2}_{\rm sym})$ and concludes the proof of (4.22). □

7. Unilateral stability: Proof of Theorem 6.7

This section is entirely devoted to the proof of the stability result in Theorem 6.7.

7.1. **Preparations.** We start with some preparations for the proof.

Density argument. We first observe that it suffices to prove the statement for functions ψ with more regularity, employing a suitable density argument. Let $\mathcal{W}(\Omega') \subset GSBD^2(\Omega')$ be the collection of functions v such that J_v is closed and included in a finite union of closed and connected pieces of C^1-curves and v lies in $W^{2,\infty}(\Omega' \setminus J_v; \mathbb{R}^2)$. By the density result [35, Theorem 3.2] we can choose a sequence $(v_n)_n \subset \mathcal{W}(\Omega')$ with $v_n = g(t)$ on $\Omega' \setminus \overline{\Omega}$ such that

$$\begin{cases} v_n \to v \text{ in measure on } \Omega' \,, \\ \|e(v_n) - e(v)\|_{L^2(\Omega')} \to 0 \,, \\ \mathcal{H}^1(J_{v_n} \triangle J_v) \to 0. \end{cases} \tag{7.1}$$

Bearing in mind this density result, by a diagonal argument it suffices to construct a sequence as in Theorem 6.7 for a function $\psi \in \mathcal{W}(\Omega')$ with $\psi = g(t)$ on $\Omega' \setminus \overline{\Omega}$. Furthermore, for simplicity we only treat the case that $J_\psi \cap \partial_D\Omega = \emptyset$ (no jump along the boundary), for the general case follows by minor adaptations of the construction at the boundary (see [26]) which would merely overburden notation in the sequel.

Choose $k(t) \in \mathbb{N}$ such that $t \in [t_n^{k(t)}, t_n^{k(t)+1})$ and define $\mathcal{A}_n^{k(t)}(g(t))$ as in (4.9) with $g(t)$ in place of $g(t_n^k)$. We fix $\theta > 0$ and observe that it suffices to construct a sequence $(\psi_n)_n \in \mathcal{A}_n^{k(t)}(g(t))$ of displacements such that

$$\limsup_{n\to\infty} |\{|\psi_n - \psi| \geq \delta\}| \leq C\theta \quad \text{for all } \delta > 0 \,, \tag{7.2}$$

$$\limsup_{n\to\infty} \mathcal{E}_n^{\rm crack}(\psi_n; t) - \mathcal{E}_n^{\rm crack}(u_n(t); t) \leq \kappa \sin(\theta_0)\, \mathcal{H}^1(J_\psi \setminus K(t)) + C\theta \,, \tag{7.3}$$

$$\limsup_{n\to\infty} \mathcal{E}_n^{\rm elast}(\psi_n; t) \leq \int_\Omega |e(\psi)|^2 \, {\rm d}x + C\theta, \tag{7.4}$$

where $C > 0$ is a universal constant. Strictly speaking, we need to construct a sequence ψ_n that attains the boundary values $g(t_n^{k(t)})$, i.e., lies in $\mathcal{A}_n(t) = \mathcal{A}_n^{k(t)} = \mathcal{A}_n^{k(t)}(g(t_n^{k(t)}))$. Therefore, we eventually need to replace the sequence $(\psi_n)_n$ by $\psi_n - g(t)_{\mathbf{T}(\psi_n)} + g(t_n^{k(t)})_{\mathbf{T}(\psi_n)}$. Due to the regularity of g, this still leads to (7.2)–(7.4). Then, the statement follows again by a diagonal argument, sending $\theta \to 0$.

Besicovitch covering. We follow the procedure from [32], which in turn stems from [26], i.e., we introduce a fine cover of J_ψ with closed squares satisfying certain additional properties. By ν_ψ we denote a measure-theoretic unit normal at J_ψ. We split our considerations into (1) $K(t) \cap J_\psi$ and (2) $J_\psi \setminus K(t)$.

(1) By a covering argument, for given $\theta > 0$ there exists an open set $U \subset \Omega'$ and a function $v \in \mathrm{PC}(U)$ such that

$$\mathcal{H}^1(K(t) \setminus U) \leq \theta, \qquad \mathcal{H}^1((K(t) \triangle J_v) \cap U) \leq \theta\,. \tag{7.5}$$

By the σ-convergence of $K_n(t)$ to $K(t)$, there exists a sequence $(v_n)_n \in PC(U)$ with $v_n \to v$ in $L^1(U)$ and

$$\limsup_{n\to\infty} \mathcal{H}^1(J_{v_n} \setminus K_n(t)) \leq \mathcal{H}^1(J_v \setminus K(t)) \leq \theta\,, \tag{7.6}$$

see (5.8) for a similar argument. We can choose a suitable subset $G_j \subset J_v$ as done preceding [26, (2.2)] with $\mathcal{H}^1(J_v \setminus G_j) \leq \theta$ such that

$$\mathcal{H}^1(K(t) \setminus G_j) \leq \mathcal{H}^1((K(t) \setminus G_j) \cap U) + \theta \leq \mathcal{H}^1(J_v \setminus G_j) + 2\theta \leq 3\theta\,. \tag{7.7}$$

For each $x \in G_j \cap J_\psi$ we consider closed squares $Q_r(x)$ with sidelength $2r$ and two sides orthogonal to $\nu_\psi(x)$ which are contained in U and satisfy [26, (2.3), (2.5)].

(2) For closed squares $Q_r(x) \subset \Omega'$ with a center $x \in J_\psi \setminus K(t)$, still oriented in direction $\nu_\psi(x)$, we can assume that for $\mathcal{H}^1$-a.e. $x \in J_\psi \setminus K(t)$ and for r sufficiently small it holds

$$\mathcal{H}^1\big(K(t) \cap Q_r(x)\big) \leq \theta r\,. \tag{7.8}$$

Here, we use the fact that $K(t)$ has $\mathcal{H}^1$-density 0 in a subset of $J_\psi \setminus K(t)$ of full $\mathcal{H}^1$ measure.

As J_ψ is contained in a finite union of closed C^1-curves, for a.e. $x \in J_\psi$, possibly passing to smaller r, the above squares in cases (1) and (2) can be chosen such that they also satisfy

$$2r \leq \mathcal{H}^1\big(J_\psi \cap Q_r(x)\big) \leq 4r, \tag{7.9}$$

$$J_\psi \cap Q_r(x) \subset \{y \colon |(y-x)\cdot \nu_\psi(x)| \leq \theta r\}. \tag{7.10}$$

With this, we obtain a fine cover of $\Gamma := (G_j \cap J_\psi) \cup (J_\psi \setminus K(t))$ to which we can apply the Besicovitch covering theorem with respect to the Radon measure $\mathcal{L}^2 + \mathcal{H}^1|_\Gamma$. For $\theta > 0$ fixed as above, we hereby find a finite and disjoint subcollection $\mathcal{B} := (Q_{r_i}(x_i))_i$, or shortly denoted by $(Q_i)_i$, such that $(Q_i)_i$ satisfy the properties (7.8)–(7.10) ((7.8) only for $x_i \notin K(t)$) as well as

$$\mathcal{L}^2\big(\bigcup\nolimits_{\mathcal{B}} Q_i\big) \leq \theta, \qquad \mathcal{H}^1\big(\Gamma \setminus \bigcup\nolimits_{\mathcal{B}} Q_i\big) \leq \theta\,. \tag{7.11}$$

Here and in the following, we use $\bigcup_{\mathcal{B}} Q_i$ as a shorthand for $\bigcup_{Q_{r_i}(x_i)\in\mathcal{B}} Q_{r_i}(x_i)$. Using (7.7), (7.11), and the definition of Γ we get

$$\mathcal{H}^1\big(J_\psi \setminus \bigcup\nolimits_{\mathcal{B}} Q_i\big) \leq 4\theta. \tag{7.12}$$

Without further notice, we will frequently use that the squares are pairwise disjoint.

By $\mathcal{B}_{\mathrm{good}} \subset \mathcal{B}$ we denote the collection of closed squares $Q_i = Q_{r_i}(x_i)$ with $x_i \in J_\psi \setminus K(t)$, and similarly we let $\mathcal{B}_{\mathrm{bad}} \subset \mathcal{B}$ be the collection of all squares $Q_i = Q_{r_i}(x_i)$ with $x_i \in J_\psi \cap G_j$. Clearly, we have $\mathcal{B} = \mathcal{B}_{\mathrm{good}} \cup \mathcal{B}_{\mathrm{bad}}$. We also define the sets

$$B_{\mathrm{good}} = \bigcup\nolimits_{\mathcal{B}_{\mathrm{good}}} Q_i, \qquad B_{\mathrm{bad}} = \bigcup\nolimits_{\mathcal{B}_{\mathrm{bad}}} Q_i.$$

Note that by construction we have $B_{\mathrm{bad}} \subset U$. In order to construct a sequence $\psi_n \colon \Omega' \to \mathbb{R}^2$ of piecewise affine functions satisfying (7.2)–(7.4), we need to specify the triangulation $\mathbf{T}_n(\psi_n)$ that is associated to ψ_n. For this, we split Ω' into three different subsets B_{good}, B_{bad} and $B_{\mathrm{rest}} = \Omega' \setminus (B_{\mathrm{good}} \cup B_{\mathrm{bad}})$. Inside of B_{good} and B_{bad} we will need two kinds of a *transfer of jump sets* which we discuss next.

Jump transfer. We again consider the good and bad squares separately.

Bad squares. The sequence $(v_n)_n \subset PC(U)$ defined above, satisfying $\limsup_{n\to\infty} \mathcal{H}^1(J_{v_n} \setminus K_n(t)) \leq \theta$ and $v_n \to v$ in $L^1(U;\mathbb{R}^2)$, can be used to transfer the jump set of ψ inside $B_{\rm bad}$. First, we recall the main points from [26], for any $Q_i = Q_{r_i}(x_i) \in \mathcal{B}_{\rm bad}$: there are two lines $H_i^{n,+}$ and $H_i^{n,-}$ with normal $\nu_\phi(x_i)$ which lie above and below the middle line H_i containing the point x_i, also with normal $\nu_\phi(x_i)$, such that

$$R_i^n \supset Q_i \cap \{y\colon |(y-x_i)\cdot \nu_\phi(x_i)| \leq 2\theta r_i\}, \tag{7.13}$$

$$\mathcal{H}^1\big(\bigcup\nolimits_{\mathcal{B}_{\rm bad}} L_i^n\big) \leq C\theta \tag{7.14}$$

for a universal constant $C > 0$, where R_i^n denotes the rectangular subset that lies between $H_i^{n,+}$ and $H_i^{n,-}$, and $L_i^n := \partial R_i^n \setminus (H_i^{n,+} \cup H_i^{n,-})$ denotes its lateral boundaries, cf. [26, (2.10)–(2.11)] and Figure 6. Moreover, the construction provides a set of finite perimeter $P_i^n \subset Q_i = Q_{r_i}(x_i) \in \mathcal{B}_{\rm bad}$ such that

$$\Gamma_i^n := \partial^* P_i^n \cap Q_i \subset R_i^n \tag{7.15}$$

satisfies

$$\mathcal{H}^1\big((\Gamma_i^n \cup L_i^n) \setminus J_{v_n}\big) \leq C\theta r_i. \tag{7.16}$$

Note that, without restriction, P_i^n can be chosen such that both P_i^n and $Q_i \setminus P_i^n$ are connected sets and hence Γ_i^n is a curve, see [32, below (5.17)] for the precise argument. We refer to [26, Section 2] or [32, Section 5] for more details and also refer to Figure 6 for an illustration of the construction. We mention that the construction is simplified compared to [32, 26] as the sequence $(v_n)_n$ lies in $PC(U)$ and thus $\nabla v_n \equiv 0$.

In [26], the approximating functions ϕ_n are defined by $\phi_n = \psi$ on $\Omega' \setminus B_{\rm bad}$ and, in $B_{\rm bad}$, as $\phi_n = \psi$ outside of $\bigcup_{\mathcal{B}_{\rm bad}} R_i^n$ and inside $\bigcup_{\mathcal{B}_{\rm bad}} R_i^n$ by reflection ensuring that

$$J_{\phi_n} \cap Q_i \mathrel{\tilde{\subset}} \Gamma_i^n \cup L_i^n \qquad \text{for all } Q_i \in \mathcal{B}_{\rm bad}. \tag{7.17}$$

Here, we will use a variant of this construction, see [32, Section 5], which is based on a cut-off construction and further ensures that, for a constant $C > 0$,

$$\begin{aligned} &\|\nabla \phi_n\|_{L^\infty(\Omega')} \leq C \|\nabla \psi\|_{L^\infty(\Omega')}, \\ &\nabla \phi_n \in SBV(\Omega';\mathbb{R}^{2\times 2}) \ \text{ with } \|\nabla(\nabla\phi_n)\|_{L^\infty(\Omega')} \leq C\theta^{-1}(\min\nolimits_i r_i)^{-1}\|\nabla\psi\|_{L^\infty(\Omega')} + C\|\nabla(\nabla\psi)\|_{L^\infty(\Omega')}, \end{aligned} \tag{7.18}$$

and that, for a sufficiently small constant $c_\theta > 0$ depending on θ,

$$\text{(i)} \ \bigcup_{\mathcal{B}_{\rm bad}} \Gamma_i^n \cup L_i^n \mathrel{\tilde{\subset}} J_{\phi_n}\,, \qquad \text{(ii)} \ \mathcal{H}^1\big(\big\{x \in J_{\phi_n} \cap B_{\rm bad}\colon |[\phi_n](x)| \leq c_\theta\big\}\big) \leq C\theta. \tag{7.19}$$

We refer the reader to [32, (5.16), (5.21), (5.22)].

For later purposes, we stress that we can assume without restriction that each connected component of $(\Gamma_i^n \setminus K_n(t)) \cap Q_i$ connects two different connected components of $\Omega^{\rm mod}_{n,k}$. (Here and in the sequel, we write k in place of $k(t)$.) In fact, if the start and endpoint of a component lie in the same connected component of $\Omega^{\rm mod}_{n,k}$, in the construction of ϕ_n the curve Γ_i^n could be replaced by a curve that lies inside $K_n(t)$, i.e., on the boundary of $\Omega^{\rm mod}_{n,k}$, and such that (7.16) would still hold. For the same reason, it is not restrictive to assume that each connected components of $\Omega^{\rm mod}_{n,k}$ meets at most two connected components of $(\Gamma_i^n \setminus K_n(t)) \cap Q_i$ as otherwise the curve could be replaced accordingly without affecting (7.16). Furthermore, we can assume that for each component H of $\Omega^{\rm mod}_{n,k}$, the set $\overline{H}$ is not self-intersecting (cf., e.g., Figure 2), since otherwise we could modify $K_n(t)$ (and thus J_{v_n}) such that no self-intersections appear but (7.16) still holds.

For future notational purposes, we denote the two connected sets P_i^n and $Q_i \setminus P_i^n$ and the restriction of the function on these sets by

$$P_i^{n,+} := P_i^n, \quad P_i^{n,-} := Q_i \setminus P_i^n, \qquad \phi_i^{n,\pm} := \phi_n|_{P_i^{n,\pm}}. \tag{7.20}$$

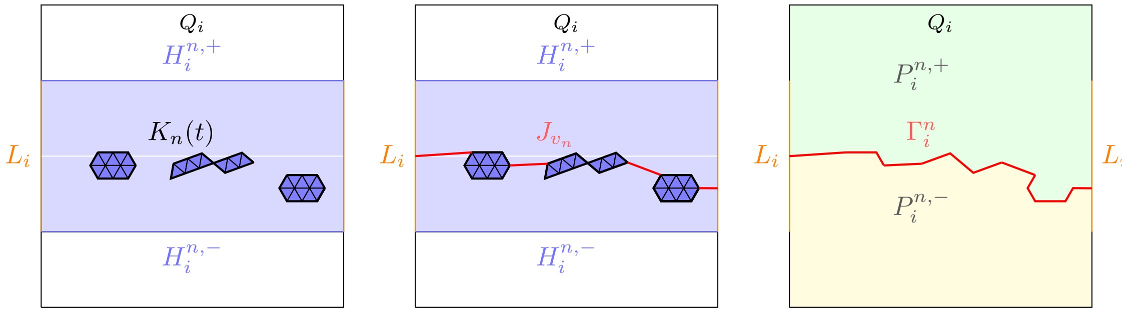

FIGURE 6. Construction of the continuous curve $\Gamma_i^n \subset R_i^n$ from J_{v_n}, for $Q_i \in \mathcal{B}_{\rm bad}$. In general, the curve Γ_i^n is a subset of J_{v_n} that is almost contained in K_n, which in turn is the boundary of the 'modified crack set' $\Omega^{\rm mod}_{n,k}$. Note that the figure is only a schematic illustration: in fact, the actual thickness of L_i^n is much smaller.

Good squares. In the good squares we employ a similar construction to guarantee that the jump sets of the approximating functions are flat in suitable squares of $\mathcal{B}_{\rm good}$ and that also $\nabla\phi_n$ lies in $W^{1,\infty}(\Omega'\setminus J_{\phi_n};\mathbb{R}^2)$. Such additional properties will be crucial in our following constructions.

First, we define the collection of squares

$$\mathcal{B}^{\rm large}_{\rm good} := \{Q_i \in \mathcal{B}_{\rm good} \colon |\Omega^{\rm crack}_{n,k}\cap Q_i| > \theta^{-1} r_i\varepsilon_n\}, \quad \mathcal{B}^{\rm small}_{\rm good} := \mathcal{B}_{\rm good}\setminus \mathcal{B}^{\rm large}_{\rm good}, \tag{7.21}$$

and the sets

$$B^{\rm large}_{\rm good} := \bigcup_{\mathcal{B}^{\rm large}_{\rm good}} Q_i, \qquad B^{\rm small}_{\rm good} := \bigcup_{\mathcal{B}^{\rm small}_{\rm good}} Q_i\,.$$

(We notice that the sets defined above depend on n and k, but we neglect this dependence in the notation.) We state a technical lemma, whose proof is deferred to Section 7.5.

Lemma 7.1. *Let $\theta>0$, let $E\subset\Omega'$ be induced by a union of triangles $\mathbf{T}_E\in\mathcal{T}_{\varepsilon_n}(\Omega')$ (cf. (2.10)), and $Q_i\in\mathcal{B}_{\rm good}$ with $|E\cap Q_i|\le\theta^{-1}r_i\varepsilon_n$. Then, there exists $\xi\in(-\theta r_i,\theta r_i)$ and a constant $\hat C>0$ such that the set*

$$A_i^n(\xi;E) := \big\{y\in Q_i\colon (y-x_i)\cdot\nu_\psi(x_i)\in(\xi-10^8\varepsilon_n,\xi+10^8\varepsilon_n)\big\}$$

fulfills

$$\#\big\{T\in\mathbf{T}_E\colon T\cap A_i^n(\xi;E)\neq\emptyset\big\}\le\frac{\hat C}{\theta^2}\,. \tag{7.22}$$

For $Q_i\in\mathcal{B}^{\rm small}_{\rm good}$, we define

$$R_i^n = \{y\in Q_i\colon |(y-x)\cdot\nu_\psi(x)|\le\theta r_i\}, \qquad H_i^{n,\pm} = \{y\in Q_i\colon (y-x)\cdot\nu_\psi(x) = \pm\theta r_i\}.$$

(The sets are actually independent of n. Yet, n is added to have the same notation as for bad squares.) For any $Q_i = Q_{r_i}(x_i)\in\mathcal{B}^{\rm small}_{\rm good}$, let $A_i^n := A_i^n(\xi_i^n;\Omega^{\rm crack}_{n,k})$ from Lemma 7.1, and for

$$\Gamma_i^n := \{y\in Q_i\colon (y-x_i)\cdot\nu_\psi(x_i) = \xi_i^n\}, \quad P_i^{n,\pm} := \{y\in Q_i\colon \pm(y-x_i)\cdot\nu_\psi(x_i) > \xi_i^n\}\,, \tag{7.23}$$

we apply the reflection lemma [35, Lemma 3.4] to the (restriction of the) function $\psi \in W^{2,\infty}(\Omega' \setminus J_\psi;\mathbb{R}^2)$ to $Q_i \setminus R_i^n$, which has Sobolev regularity by (7.10). The line $\{x_2 = 0\}$ in the lemma corresponds to each of the two lines $H_i^{n,+}$ and $H_i^{n,-}$. By this, we get ϕ_i^{n-}, $\phi_i^{n,+}$ of class $W^{2,\infty}$ and set

$$\phi_n := \phi_i^{n,-}\chi_{P_i^{n,-}} + \phi_i^{n,+}\chi_{P_i^{n,+}}, \tag{7.24}$$

see also Figure 8 for an illustration. By the above construction we get that

$$J_{\phi_n} \cap Q_i \subset \Gamma_i^n \cup L_i^n, \quad \|\nabla\phi_n\|_{L^\infty(\Omega')} \le C\|\nabla\psi\|_{L^\infty(\Omega')}, \quad \|\phi_n\|_{W^{2,\infty}(Q_i\setminus\Gamma_i^n)} \le C_{\psi,\theta}\|\psi\|_{W^{2,\infty}(Q_i\setminus J_\psi)}\,, \tag{7.25}$$

where L_i^n denotes the lateral boundary also for $Q_i \in \mathcal{B}_{\rm good}^{\rm small}$, i.e., $L_i^n := \partial R_i^n \setminus (H_i^{n,+} \cup H_i^{n,-})$, and where for shortness we denote by $C_{\psi,\theta}$ the constant on the right-hand side of (7.18). Following again the strategy of [32, Section 5] as done for squares in $\mathcal{B}_{\rm bad}$, we get that for a small constant c_θ only depending on θ it holds

$$\text{(i)} \bigcup_{\mathcal{B}_{\rm good}^{\rm small}} \Gamma_i^n \cup L_i^n \,\tilde{\subset}\, J_{\phi_n}\,, \qquad \text{(ii)}\ \mathcal{H}^1\big(\big\{x \in J_{\phi_n} \cap B_{\rm good}^{\rm small} \colon |[\phi_n](x)| \le c_\theta\big\}\big) \le C\theta. \tag{7.26}$$

Definition of ϕ_n. Eventually, we define the function ϕ_n on $B_{\rm bad}$ and $B_{\rm good}^{\rm small}$ as discussed above, and let

$$\phi_n = \psi \text{ on } \Omega' \setminus (B_{\rm bad} \cup B_{\rm good}^{\rm small}). \tag{7.27}$$

Let us collect some of the main properties that we will use in the following. First, by the above construction it also holds for all $Q_i \in \mathcal{B}_{\rm bad} \cup \mathcal{B}_{\rm good}^{\rm small}$ that

$$\phi_n = \psi \quad \text{in } Q_i \setminus R_i^n\,. \tag{7.28}$$

In view of (7.18) and (7.25), there is $C_\psi > 0$ depending on ψ and $C_{\psi,\theta}$ additionally depending on θ such that

$$\|\nabla\phi_n\|_{L^\infty(\Omega'\setminus J_{\phi_n})} \le C_\psi, \qquad \|\nabla^2\phi_n\|_{L^\infty(\Omega'\setminus J_{\phi_n})} \le C_{\psi,\theta}. \tag{7.29}$$

By (7.19) and (7.26), for a sufficiently small constant $c_\theta > 0$ depending on θ, it holds that

$$\text{(i)} \bigcup_{\mathcal{B}_{\rm bad}\cup\mathcal{B}_{\rm good}^{\rm small}} \Gamma_i^n \cup L_i^n \,\tilde{\subset}\, J_{\phi_n}\,, \quad \text{(ii)}\ \mathcal{H}^1\big(\big\{x \in J_{\phi_n} \cap (B_{\rm bad} \cup B_{\rm good}^{\rm small}) \colon |[\phi_n](x)| \le c_\theta\big\}\big) \le C\theta. \tag{7.30}$$

Moreover, by construction and recalling (7.17), (7.25) we find

$$J_{\phi_n} \cap \mathrm{int}(Q_i) \,\tilde{\subset}\, \Gamma_i^n, \quad J_{\phi_n} \cap \partial Q_i \subset L_i^n \quad \text{for any } Q_i \in \mathcal{B}_{\rm bad} \cup \mathcal{B}_{\rm good}^{\rm small} \tag{7.31}$$

such that by (7.9) it holds

$$\mathcal{H}^1(\Gamma_i^n \cup L_i^n) \le (1+2\theta)\mathcal{H}^1(J_\psi \cap Q_i) \quad \text{for all } Q_i \in \mathcal{B}_{\rm good}^{\rm small}\,. \tag{7.32}$$

7.2. **Definition of ψ_n and $\mathbf{T}_n(\psi_n)$, and proof of (7.2).** We now proceed with the definition of the sequence of displacements $(\psi_n)_n$ and the corresponding triangulations $\mathbf{T}_n(\psi_n)$. For $k = k(t)$ such that $t \in [t_n^k, t_n^{k+1})$, we consider $u_n(t) = u_n^k \in \mathcal{A}_n^k$ and the corresponding triangulation $\mathbf{T}_n(u_n^k)$. In the following, we will use the identity $\mathbf{T}_{n,k-1}^{\rm crack}(u_n^k) = \mathbf{T}_{n,k}^{\rm crack}$. Recalling the partition of Ω' into $B_{\rm good}$, $B_{\rm bad}$, and $B_{\rm rest}$, we let

$$\mathbf{T}_{n,k}^{\rm bad} := \big\{T \in \mathbf{T}_n(u_n^k) \colon T \cap B_{\rm bad} \neq \emptyset\big\}, \qquad \mathbf{T}_{n,k}^{\rm rest} := \big\{T \in \mathbf{T}_n(u_n^k) \colon T \cap (B_{\rm good} \cup B_{\rm bad}) = \emptyset\big\},$$

and define the set $B_{\rm good}^n := \Omega' \setminus \bigcup_{T \in \mathbf{T}_{n,k}^{\rm bad} \cup \mathbf{T}_{n,k}^{\rm rest}} T$. Note that $B_{\rm good}^n \cap B_{\rm bad} = \emptyset$ and for n small enough we have $B_{\rm good} \subset B_{\rm good}^n$. We will define the triangulation $\mathbf{T}_n(\psi_n)$ as

$$\mathbf{T}_n(\psi_n) = \mathbf{T}_{n,k}^{\rm bad} \cup \mathbf{T}_{n,k}^{\rm rest} \cup \mathbf{T}_{n,k}^{\rm good}, \tag{7.33}$$

where $\mathbf{T}_{n,k}^{\rm good}$ denotes a suitable triangulation of $B^n_{\rm good}$, which will be specified below. On every $T \in \mathbf{T}_{n,k}^{\rm rest}$ we choose ψ_n as the affine interpolation of ψ.

The main steps consist now in (a) defining ψ_n on the triangles of $\mathbf{T}_{n,k}^{\rm bad}$ and in (b) defining $\mathbf{T}_{n,k}^{\rm good}$ and the interpolation ψ_n on $\mathbf{T}_{n,k}^{\rm good}$.

(a) Bad squares. We define the sequence of displacements $(\psi_n)_n$ on triangles in $\mathbf{T}_{n,k}^{\rm bad}$. We fix $Q_i \in \mathcal{B}_{\rm bad}$, and define the set of vertices

$$\mathcal{V}_{i,n} \coloneqq \left\{x \in \mathcal{V}(T)\colon\ T \in \mathbf{T}_{n,k}^{\rm bad},\ T \cap Q_i \neq \emptyset\right\}, \tag{7.34}$$

where we denote the three vertices of the triangle T by $\mathcal{V}(T)$. We specify the value $\psi_n(x)$ on each $x \in \mathcal{V}_{i,n}$ in order to compute the piecewise affine interpolation on each triangle formed by the vertices $\mathcal{V}_{i,n}$. Recall that inside the closed set Q_i the jump J_{ϕ_n} is included in the curve Γ_i^n and the two lines L_i^n, see (7.17). For any vertex $x \in \mathcal{V}_{i,n}$ with $x \notin \Gamma_i^n \cup L_i^n$, we simply set $\psi_n(x) \coloneqq \phi_n(x)$. Instead, if $x \in \Gamma_i^n \cup L_i^n$, we proceed as follows. Since $\phi_n \in W^{1,\infty}(\Omega \setminus J_{\phi_n};\mathbb{R}^2)$, we know that $\phi_i^{n,+}(x)$ and $\phi_i^{n,-}(x)$ are well defined as boundary traces for each $x \in \Gamma_i^n$, and as boundary traces also on $P_i^{n,\pm} \cap L_i^n$, respectively, cf. (7.20). For $x \in L_i^n \cap P_i^{n,\pm}$, we set $\psi_n(x) \coloneqq \phi_i^{n,\pm}(x)$. For $x \in \Gamma_i^n \setminus K_n(t)$, we choose one of the values, e.g., $\psi_n(x) \coloneqq \phi_i^{n,+}(x)$. For $x \in \Gamma_i^n \cap K_n(t)$, there exists at least one triangle T with $x \in T$ and $|T \cap \Omega_{n,k}^{\rm mod}| = 0$ because $K_n(t) = \partial\Omega_{n,k}^{\rm mod}$. We collect all such triangles in the set

$$\mathbf{T}_{\rm out}^x \coloneqq \left\{T \in \mathbf{T}^x : |T \cap \Omega_{n,k}^{\rm mod}| = 0\right\}, \quad \text{where} \quad \mathbf{T}^x \coloneqq \left\{T \in \mathbf{T}_{n,k}^{\rm bad}\colon x \in T\right\}. \tag{7.35}$$

Moreover, we let $N_{\rm out}^x \coloneqq \bigcup_{T \in \mathbf{T}_{\rm out}^x} T$ be the neighborhood of x consisting of these surrounding triangles. If $N_{\rm out}^x \tilde{\subset} P_i^{n,+}$, we set $\psi_n(x) \coloneqq \phi_i^{n,+}(x)$ (see the point x_1 in Figure 7). If $N_{\rm out}^x \tilde{\subset} P_i^{n,-}$, we set $\psi_n(x) \coloneqq \phi_i^{n-}(x)$ (see the point x_4 in Figure 7). Otherwise, we again choose an arbitrary value, e.g., $\psi_n(x) \coloneqq \phi_i^{n,+}(x)$ (see e.g. the point x_3 in Figure 7). For later purposes, we also define

$$N_{\rm out}(T) = \bigcup_{x \in \mathcal{V}(T)} N_{\rm out}^x. \tag{7.36}$$

Now, we can define ψ_n on the union of the triangles in $\mathbf{T}_{n,k}^{\rm bad}$ (and thus in particular on $B_{\rm bad}$) by taking the piecewise affine interpolation on each triangle $T \in \mathbf{T}_{n,k}^{\rm bad}$.

(b) Good squares. We come to the definition of $\mathbf{T}_{n,k}^{\rm good}$ and the corresponding interpolation ψ_n on $B^n_{\rm good}$. As the argument is local, without restriction we suppose that $\mathcal{B}_{\rm good}$ consists of a single square Q_i. In the remaining part of this section, we will therefore drop the index i for simplicity, and write Q, r, as well as A^n (see before (7.23)).

For $Q \in \mathcal{B}_{\rm good}^{\rm large}$, we set $\mathbf{T}_{n,k}^{\rm good} = \{T \in \mathbf{T}_n(u_n^k)\colon T \cap Q \neq \emptyset\}$, and we define ψ_n as the piecewise affine interpolation of ψ on each $T \in \mathbf{T}_{n,k}^{\rm good}$.

Let us now come to the construction in the second case $Q \in \mathcal{B}_{\rm good}^{\rm small}$, i.e., $|\Omega_{n,k}^{\rm crack} \cap Q| \leq \theta^{-1} r \varepsilon_n$. As the underlying triangulation in $Q \setminus A^n$ we will use $\mathbf{T}_n(u_n^k)$, whereas inside A^n we use the explicit construction of an optimal triangulation from [14]. However, in order to ensure that the triangulation $\mathbf{T}_n(\psi_n)$ is admissible, it has to fulfill the condition $\mathbf{T}_{n,k-1}^{\rm crack} \subset \mathbf{T}_n(\psi_n)$, see (4.8). Therefore, we need to treat the parts of A^n that are contained in the 'pre-crack' $\Omega_{n,k-1}^{\rm crack}$ differently, which leads to a more involved construction, see Figure 8.

We start by considering neighborhoods of $\Omega_{n,k}^{\rm crack}$ in Q, by adding a buffer zone of order $10^7\varepsilon_n'$ with $\varepsilon_n' = 2\varepsilon_n \cos(\theta_0)$ around each 'broken triangle'. More precisely, we define the 'enlarged crack sets' $N_{n,k}^{\rm crack}$

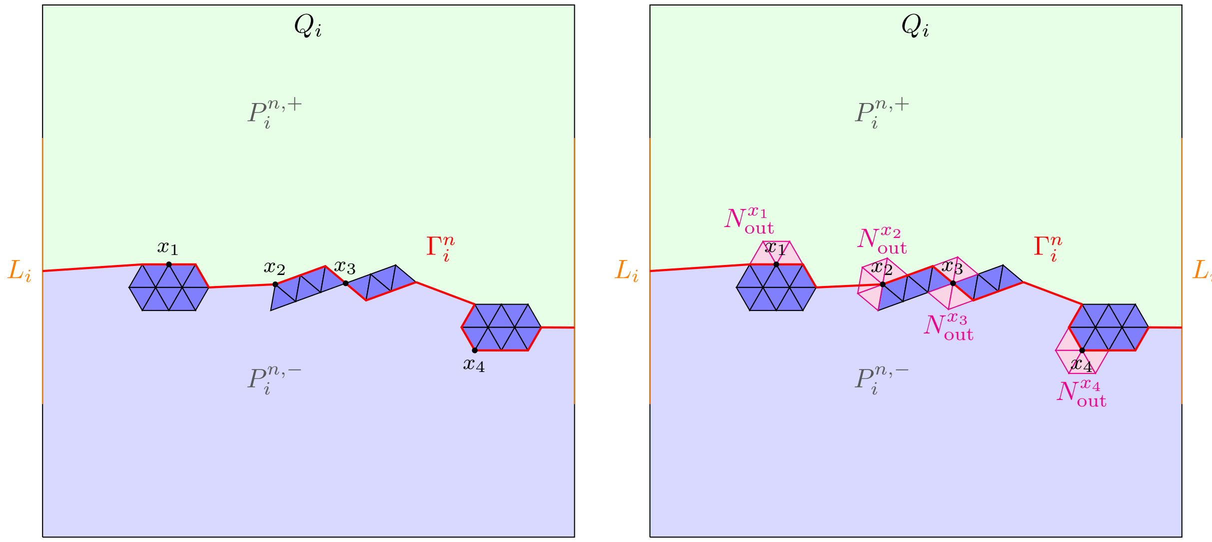

FIGURE 7. Construction of ψ_n depending on the position of N_{out}^x relative to Γ_i^n, for $Q_i \in \mathcal{B}_{\mathrm{bad}}$. In this case, we have e.g. $\psi_n(x_1) = \phi_i^{n,+}(x_1)$ and $\psi_n(x_4) = \phi_i^{n,-}(x_4)$. For x_2 and x_3 we choose one of the two values.

and the corresponding collection of triangles as

$$\mathbf{T}_{n,k}^{\mathrm{neigh}} := \big\{T \in \mathbf{T}_n(u_n^k)\colon \operatorname{dist}\big(T, (\Omega_{n,k}^{\mathrm{crack}} \cap Q)\big) \le 10^7 \varepsilon_n'\big\} \quad \text{and} \quad N_{n,k}^{\mathrm{crack}} := \bigcup_{T \in \mathbf{T}_{n,k}^{\mathrm{neigh}}} T\,. \tag{7.37}$$

By the definition of $\Omega_{n,k}^{\mathrm{crack}}$, we know that outside of $N_{n,k}^{\mathrm{crack}}$ the triangles in $\mathbf{T}_n(u_n^k)$ are part of a regular 'background triangulation' $\mathbf{Z}_n$ with size ε_n'. By the construction of Γ_i^n in $B_{\mathrm{good}}^{\mathrm{small}}$ and due to Lemma 7.1, the set $(J_{\phi_n} \setminus N_{n,k}^{\mathrm{crack}}) \cap A^n$ consists of at most $M := \hat{C}\theta^{-2} + 1$ straight segments $(S_k^n)_{k=1}^M$. According to [14, Proof of Proposition 4.1] (see in particular [14, (4.9) and Figure 4.10]) there exists a triangulation $\tilde{\mathbf{T}}_n \in \mathcal{T}_{\varepsilon_n}(\mathbb{R}^2)$ such that, setting $\tilde{\mathbf{T}}_n' := \{T \in \tilde{\mathbf{T}}_n \colon T \cap \bigcup_{k=1}^M S_k^n \neq \emptyset\}$, we have

(1) $J_{\phi_n} \cap \mathcal{V}(T) = \emptyset$ for all $T \in \tilde{\mathbf{T}}_n$.
(2) All triangles $T \in \tilde{\mathbf{T}}_n$ with $\operatorname{dist}(T, \bigcup_{k=1}^M S_k^n) \ge 5\varepsilon_n'$ are part of the 'background' triangulation $\mathbf{Z}_n$.
(3) It holds

$$\limsup_{n\to\infty} \sum_{T \in \tilde{\mathbf{T}}_n'} \frac{|T|}{\varepsilon_n} \le \limsup_{n\to\infty} \sin(\theta_0)\mathcal{H}^1\Big(\bigcup_{k=1}^M S_k^n\Big) \le (1+2\theta)\sin(\theta_0)\mathcal{H}^1(J_\psi \cap Q)\,, \tag{7.38}$$

where the second inequality in (7.38) follows from the construction of the segments $(S_k^n)_k$ and (7.32).

We now define $\mathbf{T}_{n,k}^{\mathrm{good}}$ as the set of triangles of the following type, see also Figure 8 for an illustration:

(i) $T \in \mathbf{T}_n(u_n^k)$ with $T \cap Q \neq \emptyset$ and $T \cap A^n = \emptyset$;
(ii) $T \in \mathbf{T}_{n,k}^{\mathrm{neigh}}$ with $T \cap A^n \neq \emptyset$;
(iii) $T \in \tilde{\mathbf{T}}_n$ with $T \cap A^n \neq \emptyset$ and $T \notin \mathbf{T}_{n,k}^{\mathrm{neigh}}$.

Recalling (7.37), we notice that this leads to a triangulation of $B^n_{\rm good}$. In fact, by construction all triangles $T \in \tilde{\mathbf{T}}_n$ with $\mathrm{dist}(T, \bigcup_{k=1}^N S^n_k) \geq 5\varepsilon'_n$ are part of $\mathbf{Z}_n$ (see point (2) above) and by the definition in (4.4) all triangles of $\mathbf{T}_n(u^k_n)$ inside $\{x \in N^{\rm crack}_{n,k} \colon \mathrm{dist}(x, \Omega^{\rm crack}_{n,k}) \geq 10^6\varepsilon'_n\}$ are part of $\mathbf{Z}_n$.

Eventually, on the triangles in $\mathbf{T}^{\rm good}_{n,k}$, we can define ψ_n as the interpolation of ϕ_n on $\mathbf{T}^{\rm good}_{n,k}$. Note that also for $T \in \mathbf{T}^{\rm good}_{n,k} \cap \tilde{\mathbf{T}}_n$ the interpolation is well defined because by construction none of the vertices of T belongs to $\bigcup_k S^n_k$ (see point (1) above).

(c) Conclusion. We have now concluded the definition of the function ψ_n and of the triangulation $\mathbf{T}_n(\psi_n)$. As by construction $\mathbf{T}_n(\psi_n)$ fulfills $\mathbf{T}^{\rm crack}_{n,k-1} \subset \mathbf{T}_n(\psi_n)$ (actually, it even holds $\mathbf{T}^{\rm crack}_{n,k} \subset \mathbf{T}_n(\psi_n)$), we have $\psi_n \in \hat{\mathcal{A}}^k_n(\Omega')$, see before (4.8). For later purposes, we note that

$$\varepsilon_n|e(\psi_n)_T|^2 \geq \kappa \qquad \text{for all } T \in \mathbf{T}^{\rm crack}_{n,k-1}(\psi_n) \setminus \mathbf{T}^{\rm crack}_{n,k}. \tag{7.39}$$

Indeed, since $T \notin \mathbf{T}^{\rm crack}_{n,k}$, by (4.4) we know that $\varepsilon_n|e(u^j_n)_T|^2 < \kappa$ and $\mathrm{dist}(T, \mathbf{Z}_n(u^j_n)) < 10^6\varepsilon_n$ for all $j \leq k$ such that $T \in \mathbf{T}_n(u^j_n)$. As $T \in \mathbf{T}^{\rm crack}_{n,k-1}(\psi_n)$, we can conclude that either $\varepsilon_n|e(\psi_n)_T|^2 \geq \kappa$ or $\mathrm{dist}(T, \mathbf{Z}_n(\psi_n)) \geq 10^6\varepsilon_n$. If $T \notin \mathbf{T}_n(u^k_n)$ we have, by the construction of $\mathbf{T}_n(\psi_n)$, that $T \in \tilde{\mathbf{T}}_n$ and thus by (2) it holds $\mathrm{dist}(T, \mathbf{Z}_n(\psi_n)) < 10^6\varepsilon_n$. However, if $T \in \mathbf{T}_n(u^k_n)$, we have by the above argument that $\mathrm{dist}(T, \mathbf{Z}_n(u^k_n)) < 10^6\varepsilon_n$, which leads to $\mathrm{dist}(T, \mathbf{Z}_n(\psi_n)) < 10^6\varepsilon_n$. In both cases, this yields $\varepsilon_n|e(\psi_n)_T|^2 \geq \kappa$, i.e., (7.39) holds.

Since we assumed that $\partial_D\Omega \cap J_\psi = \emptyset$, we can also suppose that for n large enough we have $(B_{\rm bad} \cup B^n_{\rm good}) \cap \partial_D\Omega = \emptyset$. Therefore, we obtain $\psi_n = g(t)_{\mathbf{T}_n(\psi_n)}$ on all $T \in \mathbf{T}_n(\psi_n)$ with $T \cap \overline{\Omega} = \emptyset$, which yields $\psi_n \in \mathcal{A}^k_n(g(t))$, see the definition before (7.2). Moreover, by the regularity of $\psi \in \mathcal{W}(\Omega')$ we have that $\psi_n \to \psi$ in measure on $\Omega' \setminus (B_{\rm good} \cup B_{\rm bad})$. Therefore, by the first estimate in (7.11) it follows that, for all $\delta > 0$,

$$\limsup_{n\to\infty} |\{|\psi_n - \psi| > \delta\}| \leq |(B_{\rm good} \cup B_{\rm bad})| \leq \theta.$$

We thus have validated that (7.2) holds for ψ_n.

7.3. **Proof of** (7.3)**: Stability estimate of the crack part.** We now proceed with the proof of (7.3). Again let $t \in [0, T]$ be given and, for each $n \in \mathbb{N}$, choose $k = k(t)$ such that $t \in [t^k_n, t^{k+1}_n)$. Let $(\psi_n)_n$ be the sequence defined above with the associated triangulation $\mathbf{T}_n(\psi_n)$. Note that by definition we have

$$\mathcal{E}^{\rm crack}_n(\psi_n; t) - \mathcal{E}^{\rm crack}_n(u_n(t); t) \leq \sum_{T \in \mathbf{T}^{\rm crack}_{n,k-1}(\psi_n) \setminus \mathbf{T}^{\rm crack}_{n,k}} \kappa \frac{|T|}{\varepsilon_n}\,. \tag{7.40}$$

We split $\mathbf{T}^{\rm crack}_{n,k-1}(\psi_n)$ into three different parts, namely $\mathbf{T}^{\rm cr,g}_{n,k}(\psi_n)$, $\mathbf{T}^{\rm cr,b}_{n,k}(\psi_n)$, and $\mathbf{T}^{\rm cr,r}_{n,k}(\psi_n)$, which corresponds to the intersection of $\mathbf{T}^{\rm crack}_{n,k-1}(\psi_n)$ with $\mathbf{T}^{\rm good}_{n,k}$, $\mathbf{T}^{\rm bad}_{n,k}$, and $\mathbf{T}^{\rm rest}_{n,k}$, respectively, see (7.33).

Bad squares. We start with the bad squares and aim at estimating

$$\limsup_{n\to\infty} \sum_{T \in \mathbf{T}^{\rm cr,b}_{n,k}(\psi_n) \setminus \mathbf{T}^{\rm crack}_{n,k}} \frac{|T|}{\varepsilon_n} \leq C\theta \tag{7.41}$$

for a universal constant $C > 0$. This will rely on the following two lemmas whose proofs will be deferred to Section 7.5 below. For their formulation, we recall the definition of the neighborhood $N_{\rm out}(T)$ in (7.36)

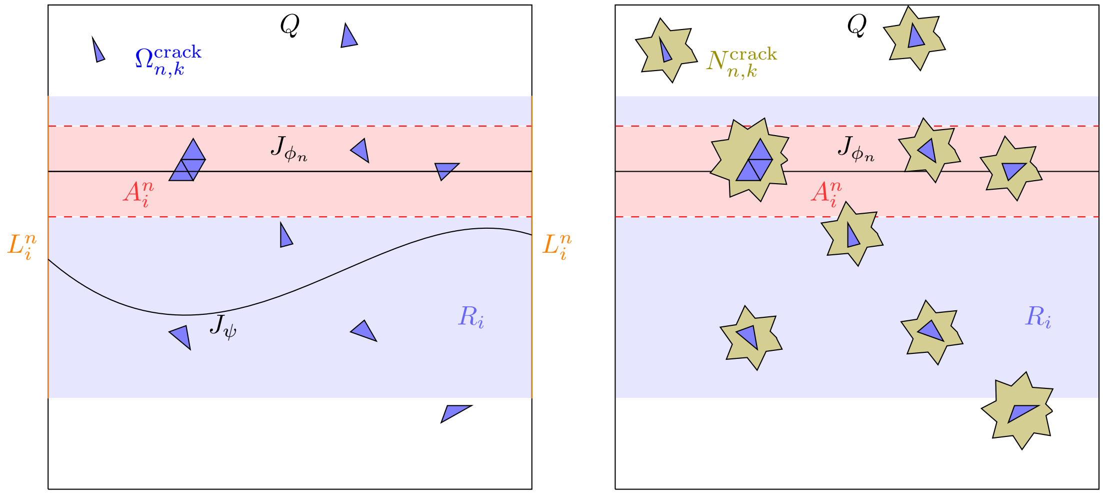

FIGURE 8. Construction of $\mathbf{T}_n(\psi_n)$ in $Q \in \mathcal{B}^{\rm small}_{\rm good}$. First, we transfer the jump from J_ψ to the flat set J_{ϕ_n}, whose vertical coordinate is determined by Lemma 7.1. In the golden neighborhoods $N^{\rm crack}_{n,k}$ of connected components of $\Omega^{\rm crack}_{n,k}$ (which are actually of size much larger than the triangles), we keep the triangulation $\mathbf{T}_n(u^k_n)$, and in the red region we use the triangulation $\tilde{\mathbf{T}}_n$ given by [14].

and of the sets $P_i^{n,\pm}$ in (7.20), see also Figure 7. We then introduce the set of triangles, where $N_{\rm out}(T)$ is contained in one of the sets $P_i^{n,+}$ or $P_i^{n,-}$, namely

$$\mathbf{T}^{\rm side}_n = \Big\{ T \in \mathbf{T}^{\rm bad}_{n,k} \colon \ \ N_{\rm out}(T) \subset \overline{P_i^{n,+}} \text{ or } N_{\rm out}(T) \subset \overline{P_i^{n,-}} \text{ for some } Q_i \in \mathcal{B}_{\rm bad} \Big\}. \tag{7.42}$$

We also recall the definition of L_i^n and Γ_i^n in (7.14) and (7.15), respectively.

Lemma 7.2. *Consider $T \in \mathbf{T}^{\rm bad}_{n,k}$ satisfying one of the conditions*

(a) $T \cap (L_i^n \cup \Gamma^i_n) = \emptyset$,
(b) $T \cap L_i^n = \emptyset$ *and* $T \in \mathbf{T}^{\rm side}_n \setminus \mathbf{T}^{\rm crack}_{n,k}$.

Then, it holds $|(\nabla\psi_n)_T| \le C$ for some C only depending on ψ, and in particular, for n large enough, we have by (7.39) *that $T \notin \mathbf{T}^{\rm crack}_{n,k-1}(\psi_n) \setminus \mathbf{T}^{\rm crack}_{n,k}$.*

Lemma 7.3. *For n large enough, the curves Γ_i^n satisfy the properties*

$$\sum_{Q_i \in \mathcal{B}_{\rm bad}} \#\big\{T \in \mathbf{T}_n(\psi_n)\colon \ \emptyset \neq T \cap \Gamma_i^n \subset K_n(t), \ T \notin \mathbf{T}^{\rm side}_n \cup \mathbf{T}^{\rm crack}_{n,k}\big\} \le \frac{C\,\theta}{\varepsilon_n}\,. \tag{7.43}$$

$$\sum_{Q_i \in \mathcal{B}_{\rm bad}} \#\big\{T \in \mathbf{T}_n(\psi_n)\colon \ (\Gamma_i^n \setminus K_n(t)) \cap T \neq \emptyset \textit{ or } L_i^n \cap T \neq \emptyset\big\} \le \frac{C\,\theta}{\varepsilon_n}\,. \tag{7.44}$$

With these auxiliary results we can prove (7.41). In view of (7.43), (7.44), and the fact that $|T| \le C\varepsilon_n^2$ for all $T \in \mathbf{T}^{\rm cr,b}_{n,k}(\psi_n)$ (recall (4.1)), it is enough to confirm that each $T \in \mathbf{T}^{\rm cr,b}_{n,k}(\psi_n) \setminus \mathbf{T}^{\rm crack}_{n,k}$ lies in the collection of triangles estimated in (7.43) or (7.44). Indeed, all triangles not lying in these collections either satisfy (a) or (b) of Lemma 7.2 or fulfill $T \in \mathbf{T}^{\rm crack}_{n,k}$. In both cases, it holds $T \notin \mathbf{T}^{\rm cr,b}_{n,k}(\psi_n) \setminus \mathbf{T}^{\rm crack}_{n,k}$ for n large enough, and thus we obtain a contradiction.

Good squares. For the good squares, our goal is to prove

$$\limsup_{n\to\infty} \sum_{T\in \mathbf{T}^{\rm cr,g}_{n,k}(\psi_n)\setminus \mathbf{T}^{\rm crack}_{n,k}} \frac{|T|}{\varepsilon_n} \le \sin(\theta_0)\mathcal{H}^1\big((J_\psi \setminus K(t)) \cap B_{\rm good}\big) + C\theta\,. \tag{7.45}$$

We define the 'cracked triangles' in $Q_i \in \mathcal{B}_{\rm good}$ according to ψ_n by

$$\mathbf{T}'_n(\psi_n, Q_i) = \big\{T \in \mathbf{T}_n(\psi_n) \colon T \cap (J_{\phi_n} \cap Q_i) \neq \emptyset\big\}\,. \tag{7.46}$$

We state an important property whose proof is again deferred to Section 7.5.

Lemma 7.4. *It holds that*

$$\mathbf{T}^{\rm cr,g}_{n,k}(\psi_n) \setminus \mathbf{T}^{\rm crack}_{n,k} \subset \bigcup_{Q_i\in\mathcal{B}_{\rm good}} \mathbf{T}'_n(\psi_n, Q_i)\,. \tag{7.47}$$

Recalling the distinction in (7.21), we first address squares $Q_i \in \mathcal{B}^{\rm large}_{\rm good}$. In fact, we want to prove that

$$\sum_{Q_i\in\mathcal{B}^{\rm large}_{\rm good}} \sum_{T\in\mathbf{T}'_n(\psi_n,Q_i)} \frac{|T|}{\varepsilon_n} \le C\theta. \tag{7.48}$$

By definition we find that for each $Q_i \in \mathcal{B}^{\rm large}_{\rm good}$ we have $r_i < \theta\,\varepsilon_n^{-1}|\Omega^{\rm crack}_{n,k} \cap Q_i|$ such that the energy bound in Corollary 6.4 implies $\sum_{Q_i\in\mathcal{B}^{\rm large}_{\rm good}} r_i \le C\theta$. By (7.9) we find

$$\mathcal{H}^1(J_\psi \cap Q_i) \le Cr_i \quad \text{and} \quad \mathcal{H}^1(J_\psi \cap B^{\rm large}_{\rm good}) \le C\theta. \tag{7.49}$$

Then, by construction and the regularity of J_ψ we have $\#\mathbf{T}'_n(\psi_n, Q_i) \le Cr_i/\varepsilon_n$ for all $Q_i \in \mathcal{B}^{\rm large}_{\rm good}$ (see (7.27)), i.e., by (4.1) we get

$$\sum_{Q_i\in\mathcal{B}^{\rm large}_{\rm good}} \sum_{T\in\mathbf{T}'_n(\psi_n,Q_i)} \frac{|T|}{\varepsilon_n} \le \sum_{Q_i\in\mathcal{B}^{\rm large}_{\rm good}} C\varepsilon_n \#\mathbf{T}'_n(\psi_n,Q_i) \le \sum_{Q_i\in\mathcal{B}^{\rm large}_{\rm good}} Cr_i \le C\theta\,,$$

and hence (7.48). We now proceed with $\mathcal{B}^{\rm small}_{\rm good}$. Recall that $J_{\phi_n} \cap Q_i \subset \Gamma^n_i \cup L^n_i$, see (7.23)–(7.25). Recalling also the construction of $\mathbf{T}_n(\psi_n)$ in Q_i, in particular the definition of $\tilde{\mathbf{T}}'_{n,i}$ for each Q_i, we have

$$\mathbf{T}'_n(\psi_n, Q_i) \subset \tilde{\mathbf{T}}'_{n,i} \cup \big\{T \in \mathbf{T}^{\rm neigh}_{n,k,i} \colon T \cap \Gamma^n_i \cap A^n_i \neq \emptyset\big\} \cup \{T \in \mathbf{T}_n(\psi_n) \colon T \cap L^n_i \neq \emptyset\}\,. \tag{7.50}$$

Here, $\mathbf{T}^{\rm neigh}_{n,k,i}$ and $\tilde{\mathbf{T}}'_{n,i}$ are defined as in (7.37) and above (7.38), where we also include the subscript i in the notation since the objects are related to Q_i. We estimate the energetic contribution for each of the three families of triangles in (7.50): for the first family, we can make use of (7.38) and we obtain, for any $Q_i \in \mathcal{B}^{\rm small}_{\rm good}$,

$$\limsup_{n\to\infty} \sum_{T\in\tilde{\mathbf{T}}'_{n,i}} \frac{|T|}{\varepsilon_n} \le (1+2\theta)\sin(\theta_0)\mathcal{H}^1(J_\psi \cap Q_i)\,. \tag{7.51}$$

For the second family, by definition of $\mathbf{T}^{\rm neigh}_{n,k,i}$ in (7.37) and Lemma 7.1, we get

$$\#\big\{T \in \mathbf{T}^{\rm neigh}_{n,k,i} \colon T \cap \Gamma^n_i \cap A^n_i \neq \emptyset\big\} \le C\#\big\{T \in \mathbf{T}^{\rm crack}_{n,k} \colon T \cap A^n_i \neq \emptyset\big\} \le C\frac{\hat{C}}{\theta^2}, \tag{7.52}$$

where the C depends on the constant 10^7. For the third family, since $\mathcal{H}^1(L_i^n) \le Cr_i\theta$, we deduce

$$\#\{T \in \mathbf{T}_n(\psi_n) \colon T \cap L_i^n \neq \emptyset\} \le C\frac{\theta r_i}{\varepsilon_n}\,. \tag{7.53}$$

Therefore, in view of the fact that $|T| \le C\varepsilon_n^2$ by (4.1), collecting (7.51)–(7.53) and using (7.9) we get that for each $Q_i \in \mathcal{B}_{\rm good}^{\rm small}$ and for n large enough it holds

$$\sum_{T\in \mathbf{T}'_n(\psi_n,Q_i)} \frac{|T|}{\varepsilon_n} \le \sin(\theta_0)\mathcal{H}^1(J_\psi \cap Q_i) + C\frac{\hat{C}}{\theta^2}\varepsilon_n + C\theta r_i.$$

From Lemma 7.4 and the fact that $\sum_i r_i \le C$ we can conclude that

$$\limsup_{n\to\infty} \sum_{T\in \mathbf{T}^{\rm cr,g}_{n,k}(\psi_n)\setminus \mathbf{T}^{\rm crack}_{n,k}} \frac{|T|}{\varepsilon_n} \le \limsup_{n\to\infty} \sum_{Q_i\in\mathcal{B}_{\rm good}} \sum_{T\in\mathbf{T}'_n(\psi_n,Q_i)} \frac{|T|}{\varepsilon_n} \le \sum_{Q_i\in\mathcal{B}_{\rm good}} \sin(\theta_0)\mathcal{H}^1(J_\psi\cap Q_i) + C\theta. \tag{7.54}$$

In view of (7.8) and using again that $\sum_i r_i \le C$, this implies (7.45).

The remaining part. Finally, we want to show that

$$\limsup_{n\to\infty} \sum_{T\in \mathbf{T}^{\rm cr,r}_{n,k}(\psi_n)\setminus \mathbf{T}^{\rm crack}_{n,k}} \frac{|T|}{\varepsilon_n} \le C\theta\,. \tag{7.55}$$

Recall that for all $T \in \mathbf{T}^{\rm rest}_{n,k}$ we have $\psi_n = \psi$ on the vertices of T, see below (7.33). In particular, for all triangles $T \in \mathbf{T}^{\rm rest}_{n,k}$ with $T \cap J_\psi = \emptyset$ we have $|e(\psi_n)_T| = |e(\psi)_T| \le |\nabla\psi_T| \le C$ and hence, $T \notin \mathbf{T}^{\rm cr,r}_{n,k}(\psi_n) \setminus \mathbf{T}^{\rm crack}_{n,k}$ for n large enough by (7.39). Therefore, it suffices to consider all $T \in \mathbf{T}^{\rm rest}_{n,k}$ with $T \cap J_\psi \neq \emptyset$. Due to (7.12) and the regularity of J_ψ, we obtain

$$\#\{T \in \mathbf{T}^{\rm rest}_{n,k} \colon T\cap J_\psi \neq \emptyset\} \le \frac{C}{\varepsilon_n}\mathcal{H}^1(J_\psi \cap B_{\rm rest}) \le C\frac{\theta}{\varepsilon_n}\,. \tag{7.56}$$

Altogether, we obtain $\#(\mathbf{T}^{\rm cr,r}_{n,k}(\psi_n) \setminus \mathbf{T}^{\rm crack}_{n,k}) \le C\frac{\theta}{\varepsilon_n}$. This along with (4.1) yields (7.55).

Conclusion. By putting together (7.41), (7.45), and (7.55) we conclude

$$\limsup_{n\to\infty} \sum_{T\in \mathbf{T}^{\rm crack}_{n,k-1}(\psi_n)\setminus \mathbf{T}^{\rm crack}_{n,k}} \frac{|T|}{\varepsilon_n} \le \sin(\theta_0)\mathcal{H}^1(J_\psi \setminus K(t)) + C\theta\,. \tag{7.57}$$

By (7.40) this finally validates (7.3).

7.4. **Proof of** (7.4)**: Stability estimate for elastic part.** Finally, we come to the proof of (7.4). Recall the definition of approximating functions ϕ_n from the paragraph ‘Jump transfer’ in Section 7.1. We define the collection of triangles which do not intersect the jump by

$$\mathbf{D}_n(\phi_n) := \{T \in \mathbf{T}_n(\psi_n) \colon T \cap J_{\phi_n} = \emptyset\}, \qquad D_n(\phi_n) := \bigcup_{T\in\mathbf{D}_n(\phi_n)} T\,.$$

Next, we introduce the set of triangles T outside $\mathbf{T}^{\rm crack}_{n,k-1}(\psi_n)$ where $|e(\psi_n)_T|^2$ is potentially not uniformly controlled. To this end, we let

$$G_{n,k}(\psi_n) := \bigcup_{T\in\mathcal{G}_{n,k}(\psi_n)} T,$$

where

$$\mathcal{G}_{n,k}(\psi_n) := \big\{T \in \mathbf{T}_n(\psi_n) \setminus \mathbf{T}^{\rm crack}_{n,k-1}(\psi_n) \colon T \cap (\Gamma^n_i \cup L^n_i) \neq \emptyset \text{ for } Q_i \in \mathcal{B}_{\rm bad} \cup \mathcal{B}^{\rm small}_{\rm good} \text{ or } T \notin \mathbf{D}_n(\phi_n)\big\}.$$

We split the elastic energy $\mathcal{E}^{\rm elast}_n(\psi_n, t)$ into the two parts

$$\int_{\Omega \setminus \Omega^{\rm crack}_{n,k-1}(\psi_n)} |e(\psi_n)|^2 \,\mathrm{d}x = \int_{\Omega \setminus (G_{n,k}(\psi_n) \cup \Omega^{\rm crack}_{n,k-1}(\psi_n))} |e(\psi_n)|^2 \,\mathrm{d}x + \int_{G_{n,k}(\psi_n)} |e(\psi_n)|^2 \,\mathrm{d}x \,. \tag{7.58}$$

In order to estimate the second term, we need the following lemma.

Lemma 7.5. *For ε_n small enough, it holds that*

$$\sum_{Q_i \in \mathcal{B}_{\rm bad} \cup \mathcal{B}^{\rm small}_{\rm good}} \#\big(\{T \in \mathbf{T}_n(\psi_n) \colon T \cap \Gamma^n_i \neq \emptyset\} \setminus \mathbf{T}^{\rm crack}_{n,k-1}(\psi_n)\big) \leq C\theta\varepsilon_n^{-1}. \tag{7.59}$$

We defer the proof to Section 7.5. In view of (7.44) and (7.53), we have

$$\sum_{Q_i \in \mathcal{B}_{\rm bad} \cup \mathcal{B}^{\rm small}_{\rm good}} \#\{T \in \mathbf{T}_n(\psi_n) \colon T \cap L^n_i \neq \emptyset\} \leq C\theta\varepsilon_n^{-1}. \tag{7.60}$$

Let $\mathbf{T}^{\rm good,large}_{n,k} = \{T \in \mathbf{T}_n(u^k_n) \colon T \cap B^{\rm large}_{\rm good} \neq \emptyset\}$. From (7.27), (7.49), and the argument in (7.56) we deduce

$$\#\big((\mathbf{T}^{\rm rest}_{n,k} \cup \mathbf{T}^{\rm good,large}_{n,k}) \setminus \mathbf{D}_n(\phi_n)\big) \leq C\theta\varepsilon_n^{-1} \,. \tag{7.61}$$

By Lemma 7.5, (7.60), (7.61), and (7.31), recalling $|T| \leq C\varepsilon_n^2$ for any $T \in \mathbf{T}_n(\psi_n)$ by (4.1) and using $|e(\psi_n)_T|^2 \leq \kappa/\varepsilon_n$ for all $T \notin \mathbf{T}^{\rm crack}_{n,k-1}(\psi_n)$, we obtain

$$|G_{n,k}(\psi_n)| \leq C\theta\varepsilon_n \quad \text{and} \quad \int_{G_{n,k}(\psi_n)} |e(\psi_n)|^2 \,\mathrm{d}x \leq C\theta \,. \tag{7.62}$$

Therefore, we are left to estimate the elastic energy contribution on $\Omega \setminus (G_{n,k}(\psi_n) \cup \Omega^{\rm crack}_{n,k-1}(\psi_n))$. We first show that for each $T \in \mathbf{T}_n(\psi_n) \setminus (\mathbf{T}^{\rm crack}_{n,k-1}(\psi_n) \cup \mathcal{G}_{n,k}(\psi_n))$, the function $\phi_n|_T$ has $W^{2,\infty}$-regularity. Indeed, if T intersects $Q_i \in \mathcal{B}_{\rm bad} \cup \mathcal{B}^{\rm small}_{\rm good}$ with $T \cap (\Gamma^n_i \cup L^n_i) = \emptyset$ and $T \subset Q_i$, it follows that either $T \subset P^{n,-}_i$ or $T \subset P^{n,+}_i$ (see (7.20) and (7.23)), and thus ϕ_n coincides with $\phi^{n,-}_i$ or $\phi^{n,+}_i$ in T, respectively (see (7.20) and (7.24)). If $T \cap (\Gamma^n_i \cup L^n_i) = \emptyset$ and $T \setminus Q_i \neq \emptyset$, then ϕ_n coincides with ψ in T by (7.28). If $T \in (\mathbf{T}^{\rm rest}_{n,k} \cup \mathbf{T}^{\rm good,large}_{n,k}) \cap \mathbf{D}_n(\phi_n)$, we have that $\phi_n = \psi$ on T, see (7.27).

In all cases, we get $\|\nabla\phi_n\|_{W^{1,\infty}(T)} \leq C_{\psi,\theta}$ by (7.29). Since on all such triangles ψ_n is the piecewise affine interpolation of ϕ_n, using (4.1) we derive

$$\|e(\psi_n) - e(\phi_n)\|_{L^\infty(T)} \leq \|\nabla\psi_n - \nabla\phi_n\|_{L^\infty(T)} \leq C_{\psi,\theta}\,\varepsilon_n \qquad \text{for all } T \in \mathbf{T}_n(\psi_n) \setminus (\mathbf{T}^{\rm crack}_{n,k-1}(\psi_n) \cup \mathcal{G}_{n,k}(\psi_n)).$$

By summing over all triangles this gives

$$\int_{\Omega \setminus (G_{n,k}(\psi_n) \cup \Omega^{\rm crack}_{n,k-1}(\psi_n))} |e(\psi_n) - e(\phi_n)|^2 \,\mathrm{d}x \leq |\Omega|\, C^2_{\psi,\theta}\, \varepsilon_n^2. \tag{7.63}$$

By the bound on $\nabla\phi_n$ in (7.29), the first property in (7.11), and (7.27) we further find

$$\int_\Omega |e(\phi_n)|^2 \,\mathrm{d}x \leq \int_\Omega |e(\psi)|^2 \,\mathrm{d}x + C\theta.$$

This and (7.63) control the limsup of the first integral on the right-hand side of (7.58). Therefore, in view of (7.62), the proof of (7.4) is concluded.

7.5. **Proof of lemmas.** Finally, we prove the lemmas used in the previous subsections.

Proof of Lemma 7.1. We consider $\xi_j := (-\theta r_i + 2j \cdot 10^8 \varepsilon_n)$ for $j = 0, \ldots, N := \lceil \frac{\theta r_i}{10^8 \varepsilon_n} \rceil$ and observe that for n large enough

$$\bigcup_{j=0}^{N} A_i^n(\xi_j) \supset R_i^n \quad \text{and} \quad |E \cap Q_i| \geq \sum_{j=1}^{N-2} |E \cap A_i^n(\xi_j)|,$$

denoting, here and below, $A_i^n(\xi_j; E)$ by $A_i^n(\xi_j)$. Therefore, we find $J \in \{2, \ldots, N-3\}$ such that

$$\sum_{j=J-1}^{J+1} |E \cap A_i^n(\xi_j)| \leq \tilde{C} N^{-1} |E \cap Q_i| \leq \tilde{C} \varepsilon_n^2 / \theta^2$$

for some universal $\tilde{C} > 0$, where we used the assumption that $|E \cap Q_i| \leq \theta^{-1} r_i \varepsilon_n$. As $|T| \geq c\varepsilon_n^2$ for c only depending on θ_0, we obtain

$$\#\Big\{ T \in \mathbf{T}_E \colon T \subset \bigcup_{j=J-1}^{J+1} A_i^n(\xi_j) \Big\} \leq \tilde{C}/\theta^2.$$

In view of (4.1), this shows the statement for $\xi = \xi_J$. □

Proof of Lemma 7.2. We first consider the case that $T \cap (L_i^n \cup \Gamma_i^n) = \emptyset$. As it holds $\|\nabla \phi_n\|_{L^\infty(\Omega' \setminus J_{\phi_n})} \leq C_\psi$ by (7.29), we find by (7.31) that

$$|\phi_n(x) - \phi_n(x')| \leq \|\nabla \phi_n\|_{L^\infty(T)} |x - x'| \leq C_\psi |x - x'|$$

for vertices $x, x' \in \mathcal{V}(T)$. In particular, as $\psi_n(x) = \phi_n(x)$ and $\psi_n(x') = \phi_n(x')$, this implies

$$|(\nabla \psi_n)_T (x - x')| = |\psi_n(x) - \psi_n(x')| \leq C_\psi |x - x'| \,. \tag{7.64}$$

Since this holds true along all three edges of T, we deduce that also $|(\nabla \psi_n)_T| \leq C$ for some $C > 0$, as desired.

Now, we assume that $T \cap L_i^n = \emptyset$ but $T \cap \Gamma_i^n \neq \emptyset$ and $T \in \mathbf{T}_n^{\mathrm{side}} \setminus \mathbf{T}_{n,k}^{\mathrm{crack}}$. By definition of $\mathbf{T}_n^{\mathrm{side}}$ in (7.42), we find that either $N_{\mathrm{out}}(T) \tilde{\subset} P_i^{n,+}$ or $N_{\mathrm{out}}(T) \tilde{\subset} P_i^{n,-}$ and $N_{\mathrm{out}}(T) \tilde{\subset} P_i^{n,\pm}$ if and only if $T \tilde{\subset} P_i^{n,\pm}$, i.e., ψ_n is the piecewise affine interpolation of $\phi_i^{n,\pm}$. Arguing as above, we deduce that $|(\nabla \psi_n)_T| \leq C$. □

Proof of Lemma 7.3. We start with the proof of (7.43). Consider $T \in \mathbf{T}_n^{\mathrm{bad}}$ such that $\emptyset \neq T \cap \Gamma_i^n \subset K_n(t)$ and $T \notin \mathbf{T}_n^{\mathrm{side}} \cup \mathbf{T}_{n,k}^{\mathrm{crack}}$. As $\partial T \cap \partial \Omega_{n,k}^{\mathrm{mod}} \neq \emptyset$ and $T \notin \mathbf{T}_{n,k}^{\mathrm{crack}}$, we first notice that, in view of Remark 3.5, we have $|T \cap \Omega_{n,k}^{\mathrm{mod}}| = 0$. Then we may assume that $T \subset N_{\mathrm{out}}^x$ for all $x \in \mathcal{V}(T)$, hence $N_{\mathrm{out}}(T)$ is a connected set. As $\emptyset \neq T \cap \Gamma_i^n \subset K_n(t)$, we get $N_{\mathrm{out}}(T) \cap \partial \Omega_{n,k}^{\mathrm{mod}} \neq \emptyset$. However, since $T \notin \mathbf{T}_n^{\mathrm{side}}$, we also have $N_{\mathrm{out}}(T) \cap P_i^{n,+} \neq \emptyset$ and at the same time $N_{\mathrm{out}}(T) \cap P_i^{n,-} \neq \emptyset$. This implies that T is either close to the intersection of a component of $\Gamma_i^n \setminus K_n(t)$ and a component of $\Omega_{n,k}^{\mathrm{mod}}$ or it is close to the intersection of two different components in $\Omega_{n,k}^{\mathrm{mod}}$ (see, e.g., x_2 in Figure 7 or x_3 in this picture, respectively).

Recall that each connected components of $\Omega_{n,k}^{\mathrm{mod}}$ meets at most two connected components of $\Gamma_i^n \setminus K_n(t)$, see the discussion before (7.20). For the same reason, we can assume that for each connected component H of $\Omega_{n,k}^{\mathrm{mod}}$, the set $H \cap \Gamma_i^n$ intersects at most two other components in $\Omega_{n,k}^{\mathrm{mod}}$. Summarizing, the number of triangles T of the above form is controlled (up to a multiplicative constant) by the total number of connected components $\mathcal{C}(\Omega_{n,k}^{\mathrm{mod}})$. By (2.14) and Corollary 6.4 we thus obtain

$$\#\big\{ T \in \mathbf{T}_n^{\mathrm{bad}} \colon \; \emptyset \neq T \cap \Gamma_i^n \subset K_n(t), \; T \notin \mathbf{T}_n^{\mathrm{side}} \cup \mathbf{T}_{n,k}^{\mathrm{crack}} \big\} \leq C \frac{\eta_n}{\varepsilon_n}. \tag{7.65}$$

Since $\eta_n < \theta$ for n large enough, we conclude the proof of (7.43).

We now come to the proof of (7.44). The triangles T with $L_i^n \cap T \neq \emptyset$ can be controlled exactly as in (7.53). By $(\Gamma_{i,l}^n)_l$ we denote the connected components of $(\Gamma_i^n \setminus K_n(t)) \cap Q_i$, and recall that we assumed without restriction that $\Gamma_{i,l}^n$ connects two different connected components of $\Omega_{n,k}^{\rm mod}$, see the discussion before (7.20). Note that $K_n(t)$ is the boundary of a union of triangles in $\mathbf{T}_n(u_n^k)$, and thus different connected components of $K_n(t)$ have distance at least $c\varepsilon_n$ for some $c > 0$ only depending on θ_0. Therefore, each $\Gamma_{i,l}^n$ fulfills $\mathcal{H}^1(\Gamma_{i,l}^n) \geq c\varepsilon_n$. The key point consists in showing

$$\#\{T \in \mathbf{T}_n(\psi_n)\colon T \cap \Gamma_{i,l}^n \neq \emptyset\} \leq \frac{C}{\varepsilon_n}\mathcal{H}^1(\Gamma_{i,l}^n). \tag{7.66}$$

Once this is shown, by summing over all components $(\Gamma_{i,l}^n)_l$, we get

$$\sum_{Q_i \in \mathcal{B}_{\rm bad}} \#\{T \in \mathbf{T}_n(\psi_n)\colon\ (\Gamma_i^n \setminus K_n(t)) \cap T \neq \emptyset\} \leq \sum_{Q_i \in \mathcal{B}_{\rm bad}} \frac{C}{\varepsilon_n}\mathcal{H}^1\big((\Gamma_i^n \setminus K_n(t)) \cap Q_i\big)\,. \tag{7.67}$$

By using (7.6) and (7.16) we have $\limsup_{n\to\infty} \mathcal{H}^1((\Gamma_i^n \setminus K_n(t)) \cap B_{\rm bad}) \leq C\theta$, which concludes the proof of (7.44).

To prove (7.66), we consider the $\omega(\varepsilon_n)$-neighborhood of $\Gamma_{i,l}^n$, i.e., $N_{\varepsilon_n}(\Gamma_{i,l}^n) := \{x \in \mathbb{R}^2\colon \mathrm{dist}(x, \Gamma_{i,l}^n) \leq \omega(\varepsilon_n)\}$. As $\mathcal{H}^1(\Gamma_{i,l}^n) \geq c\varepsilon_n$, an elementary geometric argument yields the existence of a universal constant $C > 0$ such that

$$\big|N_{\varepsilon_n}(\Gamma_{i,l}^n)\big| \leq C\,\omega(\varepsilon_n)\mathcal{H}^1(\Gamma_{i,l}^n) \leq C\varepsilon_n\mathcal{H}^1(\Gamma_{i,l}^n)\,,$$

where we used (4.1). Since $|T| \geq c\varepsilon_n^2$, we hence obtain

$$\#\big\{T \in \mathbf{T}_n(\psi_n)\colon\ \Gamma_{i,l}^n \cap T \neq \emptyset\big\} \leq \#\big\{T \in \mathbf{T}_n(\psi_n)\colon\ T \subset N_{\varepsilon_n}(\Gamma_{i,l}^n)\big\} \leq C\frac{\big|N_{\varepsilon_n}(\Gamma_{i,l}^n)\big|}{\varepsilon_n^2} \leq \frac{C}{\varepsilon_n}\mathcal{H}^1(\Gamma_{i,l}^n)\,,$$

which validates (7.66). This concludes the proof. □

Proof of Lemma 7.4. We argue by contradiction and assume that there exists $T \in \mathbf{T}_{n,k}^{\rm cr,g}(\psi_n) \setminus \mathbf{T}_{n,k}^{\rm crack}$ with $T \notin \mathbf{T}'_n(\psi_n, Q_i)$ for all $Q_i \in \mathcal{B}_{\rm good}$. In view of (7.39), we have $|e(\psi_n)_T|^2 \geq \kappa/\varepsilon_n$ for $T \in \mathbf{T}_{n,k}^{\rm cr,g}(\psi_n) \setminus \mathbf{T}_{n,k}^{\rm crack}$. Since $T \notin \mathbf{T}'_n(\psi_n, Q_i)$, i.e., $T \cap (J_{\phi_n} \cap Q_i) = \emptyset$, we can deduce $\|\phi_n\|_{W^{1,\infty}(T)} \leq C_\psi$ by (7.29). Now, consider vertices $x, x' \in \mathcal{V}(T)$ and recall that by definition $\psi_n(x) = \phi_n(x)$ and $\psi_n(x') = \phi_n(x')$. We obtain

$$|(\nabla\psi_n)_T(x - x')| = |\psi_n(x) - \psi_n(x')| = |\phi_n(x) - \phi_n(x')| \leq \|\nabla\phi_n\|_{L^\infty(T)}|x - x'| \leq C_\psi\varepsilon_n\,. \tag{7.68}$$

Since this holds along all three edges of T, there exists a constant $C > 0$, such that $|e(\psi_n)_T| \leq |(\nabla\psi_n)_T| \leq C$, which for large $n \in \mathbb{N}$ contradicts the fact that $|e(\psi_n)_T|^2 \geq \kappa/\varepsilon_n$. Hence, (7.47) holds. □

Proof of Lemma 7.5. We follow the argumentation in [32, Lemma 5.2], which relies on the fact that the measure of jump points with small jump height is small, cf. (7.30). Fix $T \in \mathbf{T}_n(\psi_n) \setminus \mathbf{T}_{n,k-1}^{\rm crack}(\psi_n)$ such that $T \cap \Gamma_i^n \neq \emptyset$, for some $Q_i \in \mathcal{B}_{\rm bad} \cup \mathcal{B}_{\rm good}^{\rm small}$. By (7.30)(i) we then get that also $J_{\phi_n} \cap T \neq \emptyset$. Let $\tilde{x} \in T \cap J_{\phi_n}$ and choose $x \in T \cap \overline{P_i^{n,+}}$ and $x' \in T \cap \overline{P_i^{n,-}}$ (recall (7.20) and (7.24)) in such a way that $\tilde{x}$ lies on the segment between x and x', and $|x - x'| \geq c\varepsilon_n$. (If $T \subset \overline{P_i^{n,+}}$, we choose $x' = \tilde{x}$, and if $T \subset \overline{P_i^{n,-}}$, we choose $x = \tilde{x}$. In the following, we assume that $\tilde{x} \neq x, x'$. In the other case, the argument is similar using the one-sided traces of ϕ_n at $\tilde{x}$.) From $T \notin \mathbf{T}_{n,k-1}^{\rm crack}(\psi_n)$, we have that $|e(\psi_n)_T| \leq \sqrt{\kappa/\varepsilon_n}$. Since ϕ_n

is Lipschitz on $Q_i \setminus \Gamma_i^n$ (see (7.29) and (7.31)), we get $|\phi_n(x) - \psi_n(x)|, |\phi_n(x') - \psi_n(x')| \le C_\psi \varepsilon_n + C\sqrt{\varepsilon_n}$ by (4.1), and therefore

$$\Big\langle \phi_n(x) - \phi_n(x'), \frac{x-x'}{|x-x'|} \Big\rangle \le \Big\langle \psi_n(x) - \psi_n(x'), \frac{x-x'}{|x-x'|} \Big\rangle + C\sqrt{\varepsilon_n} \le \Big\langle e(\psi_n)_T \cdot (x-x'), \frac{x-x'}{|x-x'|} \Big\rangle + C\sqrt{\varepsilon_n}\,.$$

By this fact and $|x - x'| \le \omega(\varepsilon_n) \le C\varepsilon_n$ (see (4.1)), we hence can estimate

$$\Big\langle \phi_n(x) - \phi_n(x'), \frac{x-x'}{|x-x'|} \Big\rangle \le |e(\psi_n)_T|\,\varepsilon_n + C\sqrt{\varepsilon_n} \le C\sqrt{\varepsilon_n}. \tag{7.69}$$

On the other hand, by the Fundamental Theorem of Calculus we have that

$$\Big\langle \phi_n(x) - \phi_n(x'), \frac{x-x'}{|x-x'|} \Big\rangle = \int_0^1 \Big\langle \nabla\phi_n(x + s(x'-x)) \cdot (x'-x), \frac{x'-x}{|x-x'|} \Big\rangle \,\mathrm{d}s + \Big\langle [\phi_n(\tilde{x})], \frac{x'-x}{|x-x'|} \Big\rangle. \tag{7.70}$$

By (7.29) and $|x - x'| \le C\varepsilon_n$, the first term on the right-hand side is of order ε_n. Putting together (7.69) and (7.70) we thus obtain

$$\Big\langle [\phi_n(\tilde{x})], \frac{x'-x}{|x-x'|} \Big\rangle \le C\sqrt{\varepsilon_n} + C\varepsilon_n \le C\sqrt{\varepsilon_n}.$$

Letting $\nu_1 := \frac{x'-x}{|x'-x|}$ this means $\langle [\phi_n(\tilde{x})], \nu_1 \rangle \le C\sqrt{\varepsilon_n}$. Now we can repeat the procedure for different $\hat{x}$ and $\hat{x}'$, where the segment between $\hat{x}$ and $\hat{x}'$ intersects J_{ϕ_n} also exactly at $\tilde{x}$ and $\nu_2 := \frac{\hat{x}-\hat{x}'}{|\hat{x}-\hat{x}'|}$ satisfies $\langle \nu_1, \nu_2 \rangle \ge c$ for a universal $c > 0$. We then have $\langle [\phi_n(\tilde{x})], \nu_i \rangle \le C\sqrt{\varepsilon_n}$ for $i = 1, 2$, which leads to $|[\phi_n(\tilde{x})]| \le C\sqrt{\varepsilon_n}$.

We consider $N_{\varepsilon_n}(T) := \{x \in \mathbb{R}^2 \colon \mathrm{dist}(x, T) \le \varepsilon_n\}$, where we use $\|\nabla\phi_n\|_{L^\infty(N_{\varepsilon_n}(T)\setminus J_{\phi_n})} \le C_\psi$ to find

$$|[\phi_n(z)]| \le C\sqrt{\varepsilon_n} + CC_\psi\varepsilon_n \quad \text{for all } z \in J_{\phi_n} \cap N_{\varepsilon_n}(T)\,.$$

(In fact, $\tilde{x}$ and each z can be connected by two different curves on different components of $N_{\varepsilon_n}(T) \setminus \overline{J_{\phi_n}}$ of length $\sim \varepsilon_n$.) As $\mathcal{H}^1(J_{\phi_n} \cap N_{\varepsilon_n}(T))$ is at least ε_n, we get for ε_n small enough depending on θ

$$\mathcal{H}^1\big(\{z \in J_{\phi_n} \cap N_{\varepsilon_n}(T) \colon |[\phi_n](z)| \le c_\theta\}\big) = \mathcal{H}^1(J_{\phi_n} \cap N_{\varepsilon_n}(T)) \ge \varepsilon_n.$$

Summing over all T of the form described above, observing that each neighborhood $N_{\varepsilon_n}(T)$ intersects only a bounded number of neighborhoods $N_{\varepsilon_n}(T')$, $T' \ne T$, and using (7.30) we conclude that (7.59) holds. □

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