Regularity and convergence of critical points of an Ambrosio-Tortorelli functional with linear growth and of its Γ-limit

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March 16, 2025

Abstract

In the one-dimensional setting we consider an Ambrosio-Tortorelli functional $F_\varepsilon(u, v)$ which has linear growth with respect to u'. We prove that under suitable conditions on the fidelity term, minimizers and critical points of F_ε are Sobolev regular, and that the same is true for the Γ-limit F of F_ε. As a corollary, we obtain that the functional $A_w(u)$ computing the length of the generalized graph of a function of bounded variation u, under the same conditions on the fidelity term, admits a unique minimizer of class C^1. This solves a conjecture by De Giorgi [15] in the one-dimensional case.

Key words: Relaxation, phase-field approximation, area functional, regularity of minimizers, critical points, Γ-convergence.

AMS (MOS) 2020 Subject Classification: 49J45, 49Q20, 28A75.

1 Introduction

The Mumford-Shah functional was introduced as a main tool for image segmentation [23, 24], and after employed for other applications, such as in fracture mechanics [20]; for numerical implementation it is of utmost importance the work of Ambrosio-Tortorelli [4, 5], where the authors introduced a phase-field functional that, by mean of Γ-convergence, approximates the Mumford-Shah one. In its best known version, the Ambrosio-Tortorelli functional has the form

$$AT_\varepsilon(u, v) = \int_\Omega \Big(v^2|\nabla u|^2 + \frac{(v-1)^2}{4\varepsilon} + \varepsilon|\nabla v|^2\Big)dx, \tag{1.1}$$

valid for $(u, v) \in H^1(\Omega) \times H^1(\Omega)$, with Ω an open bounded subset of $\mathbb{R}^n$. As $\varepsilon \to 0$, the functional $AT_\varepsilon(u, v)$ tends to the Mumford-Shah functional $MS(u)$ in terms of Γ-convergence, where

$$MS(u) = \int_\Omega |\nabla u|^2 dx + \mathcal{H}^{n-1}(S_u),$$

where now u is a special function of bounded variation, ∇u is its approximate gradient, and $\mathcal{H}^{n-1}(S_u)$ is the $(n-1)$-dimensional Hausdorff measure of its jump set S_u. As usual in the

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Γ-convergence of Ambrosio-Tortorelli type energies, the Γ-limit is $+\infty$ when the phase-field variable v is not constantly 1, and for this reason we omit the dependence on v.
In the years some attention has been payed to several variants of (1.1), also with different powers of the gradient of u; for instance, in [2] the growth of the first term in (1.1) with respect to ∇u is linear. Besides the applications in mechanics, the Ambrosio-Tortorelli approximation and related ones have been used successfully also in other fields, as the analysis of liquid crystals [6], Steiner type problems [12], optimal transportation problem [11, 25, 18].
Both from the implementation point of view and for studying the corresponding time-dependent evolution of critical points, or minima, of the Γ-limit, it is important that such critical points can be obtained as limits of critical points of the Ambrosio-Tortorelli approximating energies. For this type of results, see [19, 21] and more recently [9, 26, 10].
In this paper we consider the problem of convergence of critical points of the energy F_ε to critical points of the corresponding Γ-limit. Here F_ε is an Ambrosio-Tortorelli type energy with linear growth in u', namely

$$F_\varepsilon(u,v) = \int_a^b v^2 f(|u'|)\,dx + \frac{1}{4\varepsilon}\int_a^b (v-1)^2\,dx + \varepsilon\int_a^b (v')^2 dx + \int_a^b |u-w|^2\,dx$$

for $(u,v) \in W^{1,1}((a,b));\mathbb{R}^k) \times H^1((a,b))$. We assume that $f : [0,+\infty) \to \mathbb{R}$ is a non-negative and increasing convex function of class C^1 satisfying $f'(0) = 0$ and $\lim_{t\to\infty} f(t)/t = 1$. According to [2], F_ε Γ-converges to

$$F(u) = \int_a^b f(|u'|)\,dx + \sum_{x\in S_u} \frac{|u^+(x)-u^-(x)|}{1+|u^+(x)-u^-(x)|} + |D^c u|((a,b)) + \int_a^b |u-w|^2\,dx.$$

Notice that, due to the linear growth condition of F_ε, existence of minimizers of F_ε in $H^1((a,b);\mathbb{R}^k) \times H^1((a,b))$ is not guaranteed, as well as the domain of F will be the space of function of bounded variation. The energy of the form F can be seen, for instance, as a prototype energy for a 1-dimensional mechanical model for cohesive fracture in the framework of nonlinear elasticity. The presence of the fidelity term is crucial for our purpose, which is twofold. On the one hand we show regularity of minimizers and critical points of F_ε under the assumption that w is sufficiently small in $L^2((a,b))$, and on the other hand we show the convergence of these critical points to critical points of F. More precisely, setting $\Omega := (a,b)$, we show the following first main result:

Theorem 1.1. *There are constants $\bar\varepsilon > 0$ and $\beta > 0$ depending only on Ω such that the following holds: for all $\varepsilon \in (0,\bar\varepsilon)$ and $w \in L^2(\Omega;\mathbb{R}^k)$ with $\|w\|_{L^2} \le \beta$, there exist minimizers $(u_\varepsilon, v_\varepsilon)$ of F_ε in $W^{1,\infty}(\Omega;\mathbb{R}^k) \times H^2(\Omega)$ such that, as $\varepsilon \to 0$, converge to a couple $(u,1)$, where u minimizes F, and still belongs to $W^{1,\infty}(\Omega;\mathbb{R}^k)$. Moreover, if $\widetilde{w} := \frac{1}{b-a}\int_a^b w\,dx$, and $(u_\varepsilon, v_\varepsilon)$ are critical points of F_ε such that $F_\varepsilon(u_\varepsilon, v_\varepsilon) \le F_\varepsilon(\widetilde{w},1)$ for $\varepsilon < \varepsilon_0$, then $u_\varepsilon \in W^{1,\infty}(\Omega;\mathbb{R}^k)$ and it converges weakly star in $W^{1,\infty}(\Omega;\mathbb{R}^k)$ to u minimizer of F.*

In the second part of the paper we notice that, in the special case $f(|y|) = \sqrt{1+y^2}$, F is related to the relaxed area functional A_w, and as a byproduct of our main result we obtain regularity of minimizers of A_w. In general, given a map $u \in C^1(\Omega;\mathbb{R}^k)$, the area functional measures the area of the graph of u; the relaxation of this functional has been attracted attention in the last years, especially for its application to the Cartesian Plateau problem [15, 1, 13]. Restricting our attention to the one-dimensional case, the area functional reduces to the length functional $A_w(u)$, measuring the length of the generalized graph of a given curve u in $\mathbb{R}^k$. Precisely, for all $u \in BV((a,b);\mathbb{R}^k)$, the relaxed length functional is

$$A_w(u) = \int_a^b \sqrt{1+|u'|^2}\,dx + \sum_{x\in S_u} |u^+(x)-u^-(x)| + |D^c u|((a,b)) + \int_\Omega |u-w|^2\,dx, \qquad (1.2)$$

where $w \in L^2((a,b);\mathbb{R}^k)$ is a given map acting as fidelity term, and ensuring that a minimizer of A_w exists in $BV((a,b);\mathbb{R}^k)$. Due to the strict convexity of A_w such minimizer is unique; according to De Giorgi [15], its conjecture asserts that this minimizer is of class C^1 if the L^∞-norm of w is sufficiently small.
Comparing F with the length functional A_w, we obtain the following:

Theorem 1.2. *There is a constant $\beta > 0$ depending only on Ω such that the following holds: for all $w \in L^2(\Omega;\mathbb{R}^k)$ with $\|w\|_{L^2} \leq \beta$ the unique minimizer u of A_w is of class $C^1(\Omega;\mathbb{R}^k)$.*

The proof of the first part of Theorem 1.1, concerning the regularity of the minimizers of F, relies on a regularization approach and a Γ-convergence argument.
As we have already mentioned, due to the linear growth condition of F_ε, the existence of minimizers of F_ε in $H^1((a,b);\mathbb{R}^k) \times H^1((a,b))$ is not guaranteed. To address this issue, we introduce $F_{\varepsilon,\delta} := F_\varepsilon(u,v) + \frac{\delta}{2}\int_\Omega |u'|^2\, dx$, a regularitation of F_ε, for which, by employing the direct method of the calculus of variations, we are able to establish the existence of minimizers in the desired domain. This is proven in Theorem 4.1. The added term depending on δ allows us to obtain the crucial bound $\|u'\|^2_{L^2} \leq C$, a bound that would not otherwise be achievable. Indeed, while F_ε already contains a term involving u', namely $\int_\Omega v^2 f(|u'|)\, dx$, in cases where $v = 0$, we are unable to deduce any information about u'. Even when $v \neq 0$, the best we can conclude is that $u \in W^{1,1} \subset BV$, which is insufficient for our purposes.
Lemma 4.3 allows us to deduce that, given $(u_{\varepsilon,\delta}, v_{\varepsilon,\delta})$ as minimizers of $F_{\varepsilon,\delta}$, under suitable conditions on the L^2-norm of the fidelity term w, we have $v_{\varepsilon,\delta} \geq \frac{1}{4}$. The fact that $v_{\varepsilon,\delta}$ stays away from zero will be a crucial ingredient in the proof of what follows. Moreover, the same result holds even if $\delta = 0$.
The Γ-convergence Theorem 4.6 ensures that, under the same conditions on the L^2-norm of w, minimizers $(u_{\varepsilon,\delta}, v_{\varepsilon,\delta})$ of the functional $F_{\varepsilon,\delta}$ converge in $L^1(\Omega;\mathbb{R}^k) \times L^1(\Omega)$ as $\delta \to 0^+$ to minimizers of the functional $\hat{F}_\varepsilon$ defined in (4.1).
However, in Theorem 4.7, we prove that, up to subsequences, $u_{\varepsilon,\delta} \rightharpoonup u_\varepsilon$ weakly* in $W^{1,\infty}$, $v_{\varepsilon,\delta} \rightharpoonup v_\varepsilon$ weakly in H^1 as $\delta \to 0$. Thus, $(u_\varepsilon, v_\varepsilon)$ are minimizers in $W^{1,\infty}(\Omega;\mathbb{R}^k) \times H^1(\Omega)$ of $\hat{F}_\varepsilon = F_\varepsilon$. Moreover, we show the existence of a constant C independent of ε and δ, such that $\|u_\varepsilon\|_{W^{1,\infty}} + \|v_\varepsilon\|_{H^1} \leq C$.
Finally, by using the Γ-convergence result from Theorem 2.2, which guarantees that the minimizers $(u_\varepsilon, v_\varepsilon)$ of F_ε converge to minimizers (u,v) of F as $\varepsilon \to 0$, and by applying the above estimate and the semicontinuity of the norm, we conclude that (u,v) are minimizers of F in $W^{1,\infty}(\Omega;\mathbb{R}^k) \times H^1(\Omega)$.

The second part of the theorem, concerning the convergence of the critical points of F_ε, is based on the following observations, which are stated in Corollary 4.5 and Corollary 4.8. The fact that $v_\varepsilon \geq 1/4$ and the existence of a constant $C > 0$, independent of ε, such that $\|u_\varepsilon\|_{W^{1,\infty}} + \|v_\varepsilon\|_{H^1} \leq C$ also hold when $(u_\varepsilon, v_\varepsilon)$ are critical points of F_ε, provided that we assume $F_\varepsilon(u_\varepsilon, v_\varepsilon)$ remains below a certain threshold. Indeed, this assumption, together with the use of the Euler-Lagrange equations, is the only essential ingredient. The other key element is the application of a standard result on maximal monotone operators, which allows us to conclude that the Euler-Lagrange equation of F_ε converges to that of F.

Theorem 1.2 follows from the observation that if $u \in W^{1,\infty}(\Omega;\mathbb{R}^k)$ is a minimizer of F, then it is also a minimizer of A_w. Thus, up to this point, we have shown that the minimizers of A_w are Sobolev regular. Actually, it is possible to gain regularity $C^1(\Omega;\mathbb{R}^k)$. The idea behind the proof is as follows. We know that the minimizer u of A_w satisfies the Euler-Lagrange equation with the boundary conditions as expressed in Theorem 3.1. By integrating this equation we express the derivative of u in terms of a new function Φ, which turns out to be continuous.

2 Notation and preliminaries

We consider a one-dimensional setting, where $\Omega := (a, b)$ represents the domain of interest. We use standard notation for Sobolev and Lebesgue spaces. $\mathcal{L}^1$ will denote the Lebsegue measure in $\mathbb{R}$. When μ is a measure on Ω and B is a Borel subset of Ω, we denote by $\mu \llcorner B$ *the restriction of* μ *to the set* B, i.e., the measure given by $(\mu \llcorner B)(E) := \mu(E \cap B)$, for every Borel set $E \subset \Omega$. With $|\mu|(\Omega)$ we denote the total variation on Ω of the measure.

2.1 Functions of bounded variation

Let $u \in L^1(\Omega)$. We say that u is a *function of bounded variation* in Ω if its distributional derivative is representable by a finite Radon measure in Ω; i.e., if

$$\int_\Omega u\varphi' dx = -\int_\Omega \varphi \, dDu \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega),$$

for some Radon measure Du. The space of all functions of bounded variation in Ω will be denoted by $BV(\Omega)$.

Given $u \in BV(\Omega)$, we define the *jump set* of u, denoted by S_u, as the complement of the set of Lebesgue points of u. In dimension 1 there is always a precise representative of u, which is still denoted by u, and it will be continuous expect in its jump set. If $x \in S_u$ the *traces* $u^\pm$ of u at x are defined as

$$u^\pm(x) = \lim_{y \to x^\pm} u(y).$$

If $u \in BV(\Omega)$, we define the three measures $D^a u$, $D^J u$, and $D^c u$ as follows. By the Radon-Nikodym Theorem we set $Du = D^a u + D^s u$, where $D^a u \ll \mathcal{L}^1$ and $D^s u$ is the singular part of Du with respect to $\mathcal{L}^1$. Here $D^a u$ is the *absolutely continuous part* of Du with respect to the Lebesgue measure, while $D^s u = D^J u + D^c u$ where $D^J u = Du \llcorner S_u$ is the *jump part of* Du, and $D^c u = D^s u \llcorner (\Omega \setminus S_u)$ is the *Cantor part of* Du. We can write then

$$Du = D^a u + D^J u + D^c u.$$

For a detailed study of the properties of BV-functions we refer to [7, 16, 17]. For an introduction to the study of free-discontinuity problems in the BV setting we refer to [7].

2.2 Relaxation and Γ-convergence

Let (X, d) be a metric space. We now recall the concept of a *relaxed functional.* Given a function $F : X \to \mathbb{R} \cup \{+\infty\}$, the relaxed functional $\overline{F}$ of F, or *the relaxation of* F, is the greatest d-lower semicontinuous functional less than or equal to F.

We say that a sequence $F_j : X \to [-\infty, +\infty]$ Γ-*converges* to $F : X \to [-\infty, +\infty]$ (as $j \to +\infty$) if for all $u \in X$ we have

(i) (*lower limit inequality*) for every sequence (u_j) converging to u,

$$F(u) \le \liminf_j F_j(u_j); \tag{2.1}$$

(ii) (*existence of a recovery sequence*) there exists a sequence (u_j) converging to u such that

$$F(u) \ge \limsup_j F_j(u_j), \tag{2.2}$$

or, equivalently by (2.1),

$$F(u) = \lim_j F_j(u_j).$$

The function F is called the Γ-*limit of* (F_j) (with respect to d), and we write $F = \Gamma\text{-}\lim_j F_j$. If (F_ε) is a family of functionals indexed by $\varepsilon > 0$, then we say that F_ε Γ-converges to F as $\varepsilon \to 0^+$ if $F = \Gamma\text{-}\lim_{j\to\infty} F_{\varepsilon_j}$ for all ε_j converging to 0 as $j \to \infty$.

The importance of introducing this notion is highlighted by the following fundamental result.

Theorem 2.1. *Let $F = \Gamma - \lim_j F_j$, and let a compact set $K \subset X$ exist such that $\inf_X F_j = \inf_K F_j$ for all j. Then*

$$\exists \min_X F = \lim_j \inf_X F_j.$$

Moreover, if (u_j) is a converging sequence such that $\lim_j F_j(u_j) = \lim_j \inf_X F_j$ then its limit is a minimum point for F.

For an introduction to $\Gamma-$convergence we refer to [14].

2.3 A result of Γ-convergence

We recall the following result, proven by Alicandro and Focardi (see [3, Theorem 3.2, Remark 3.4]). As in our case, their framework is vectorial, and generalizes the result of [2, Theorem 5.1], in which the authors consider the particular case $k = 1$.

Theorem 2.2. *Let $\Omega = (a, b)$, $u \in W^{1,1}(\Omega;\mathbb{R}^k)$ and $v \in W^{1,2}(\Omega)$; suppose that $f : [0, +\infty) \to [0, +\infty)$ is a C^1 function of $|u'|$, convex, increasing with $\lim_{t\to+\infty} \frac{f(t)}{t} = 1$; let $W : [0, 1] \to [0, +\infty)$ be a continuous function such that $W(1) = 0$ and $W(t) > 0$ if $t \in [0, 1)$; let $\psi : [0, 1] \to [0, 1]$ be a lower semicontinuous increasing function with $\psi(0) = 0$, $\psi(1) = 1$, and $\psi(t) > 0$ if $t > 0$. Suppose that $F_\varepsilon : L^1(\Omega;\mathbb{R}^k) \times L^1(\Omega) \to [0, +\infty]$ be defined by*

$$F_\varepsilon(u, v) = \begin{cases} \displaystyle\int_\Omega \left(\psi(v)f(|u'|) + \frac{1}{\varepsilon}W(v) + \varepsilon|v'|^2 + |u - w|^2\right) dx & \text{if } u \in W^{1,1}(\Omega;\mathbb{R}^k), v \in W^{1,2}(\Omega) \\ & \quad 0 \le v \le 1 \text{ a.e.}, \\ +\infty & \text{otherwise.} \end{cases}$$

Then there exists the $\Gamma\text{-}\lim_{\varepsilon\to0^+} F_\varepsilon(u, v) = F(u, v)$ with respect to the $L^1(\Omega;\mathbb{R}^k) \times L^1(\Omega) \to [0, +\infty]$, where

$$F(u, v) = \begin{cases} \displaystyle\int_\Omega f(|u'|)\, dx + \int_{S_u} g(|u^+ - u^-|)\, d\mathcal{H}^0 + |D^c u|(\Omega) + \int_\Omega |u - w|^2\, dx \\ \qquad\qquad \text{if } u \in BV(\Omega;\mathbb{R}^k) \text{ and } v = 1 \text{ a.e}, \\ +\infty \qquad\qquad \text{otherwise,} \end{cases}$$

and

$$g(z) := \min\{\psi(t)z + 2c_W(t) : 0 \le t \le 1\},$$

with

$$c_W(t) := 2\int_t^1 \sqrt{W(s)}\, ds.$$

Notice that under the assumptions of Theorem 2.2 on f it holds that there is a constant $C > 0$ such that

$$f(|p|) \le C(1 + |p|), \qquad \forall p \in \mathbb{R}^k. \tag{2.3}$$

2.4 Other preliminary results

Theorem 2.3. *[22, Theorem 1.8.1] Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ be a non-negative C^1 function that is convex in the third variable. Then the functional*

$$\int_\Omega F(x, v(x), p(x))\, dx$$

is sequentially lower semicontinuous in $L^1(\Omega)\times L^1(\Omega;\mathbb{R}^k)_w$. *More precisely, if* $v_j\to v$ *in* $L^1(\Omega)$ *and* $p_j \rightharpoonup p$ *weakly in* $L^1(\Omega;\mathbb{R}^k)$*, then*

$$\int_\Omega F(x,v(x),p(x))\,dx \le \liminf_{j\to\infty}\int_\Omega F(x,v_j(x),p_j(x))\,dx.$$

Theorem 2.4 (see [13])**.** *If* $u\in BV(\Omega;\mathbb{R}^k)$ *and* $v\in H^1(\Omega)$*, define*

$$I(u,v)=\int_\Omega v^2 f(|u'|)\,dx+\int_\Omega v^2\,d|D^c u|+\sum_{x\in S_u} v^2(x)|u^+(x)-u^-(x)|$$

with f *as in Theorem 2.2. Then, for all* $v\in H^1(\Omega)$*, it holds that*

$$I(u,v)\le \liminf_{k\to\infty} I(u_j,v)$$

whenever $u_j\to u$ *weakly* in* $BV(\Omega;\mathbb{R}^k)$*.*

In the theorem above, the variable $v\in H^1(\Omega)$ is held fixed, and the weak* convergence $u_j\to u$ in $BV(\Omega;\mathbb{R}^k)$ is the sole variable under consideration. The result establishes that the functional $I(u,v)$ is lower semicontinuous with respect to the weak* convergence in $BV(\Omega;\mathbb{R}^k)$, for a fixed v.

The following corollary generalizes this result by allowing both variables u and v to vary simultaneously. Specifically

Corollary 2.5. *If* $v_j\to v$ *weakly in* $H^1(\Omega)$ *and* $u_k\to u$ *weakly* in* $BV(\Omega;\mathbb{R}^k)$*, then*

$$I(u,v)\le \liminf_{j\to\infty} I(u_j,v_j).$$

Proof. Since $H^1(\Omega)$ is compactly embedded in $C^0(\Omega)$, it follows that

$$v_j\to v \quad \text{uniformly.} \tag{2.4}$$

We write

$$I(u_j,v_j)=I(u_j,v)+\big(I(u_j,v_j)-I(u_j,v)\big),$$

and, since the liminf of the sum is greater than or equal to the sum of the liminf values, we deduce

$$\begin{aligned}\liminf_{j\to+\infty} I(u_j,v_j) &\ge \liminf_{j\to+\infty} I(u_j,v)+\liminf_{j\to+\infty}\big(I(u_j,v_j)-I(u_j,v)\big)\\ &:= \mathrm{I}+\mathrm{II}.\end{aligned}$$

We begin by proving that $\mathrm{II}\to 0$. Indeed,

$$\begin{aligned}|I(u_j,v_j)-I(u_j,v)| &= \Big|\int_\Omega (v_j^2-v^2)f(|u_j'|)\,dx+\int_\Omega (v_j^2-v^2)\,d|D^c u_j|\\ &\quad+\sum_{x\in S_{u_j}}(v_j^2(x)-v^2(x))|u_j^+(x)-u_j^-(x)|\Big|\\ &\le \int_\Omega |v_j^2-v^2|f(|u_j'|)\,dx+\int_\Omega |v_j^2-v^2|\,d|D^c u_j|\\ &\quad+\sum_{x\in S_{u_j}}|v_j^2(x)-v^2(x)||u_j^+(x)-u_j^-(x)|\\ &\le \|v_j^2-v^2\|_{L^\infty}\Big(\int_\Omega f(|u_j'|)\,dx+|D^c u_j|(\Omega)+|D^J u_j|(\Omega)\Big),\end{aligned} \tag{2.5}$$

where in the last inequality we used the fact that $|v_j^2 - v^2| \le \|v_j^2 - v^2\|_{L^\infty}$.
Now, recalling (2.3) we obtain

$$|I(u_j, v_j) - I(u_j, v)| \le \|v_j^2 - v^2\|_{L^\infty} \Big(C(b-a) + C \int_\Omega |u_j'| \, dx + |D^c u_j|(\Omega) + |D^J u_j|(\Omega) \Big) = \|v_j^2 - v^2\|_{L^\infty} C \Big((b-a) + \|u_j\|_{BV} \Big),$$

which tends to zero as $j \to \infty$ due to (2.4).
By invoking Theorem 2.4 for term I, we obtain the desired result. □

3 Main results

Recall that, in our setting and throughout the following, we consider $\Omega = (a,b)$, $u \in W^{1,1}(\Omega; \mathbb{R}^k)$, and $v \in H^1(\Omega)$. Moreover, let us refer to Theorem 2.2, taking $w \in L^2(\Omega; \mathbb{R}^k)$, $W(v) = \frac{(1-v)^2}{4}$, so that $c_W(t) = \frac{(1-t)^2}{2}$ and $\psi(v) = v^2$. With these choices, F_ε can be rewritten as

$$F_\varepsilon(u,v) = \int_\Omega v^2 f(|u'|) \, dx + \frac{1}{4\varepsilon} \int_\Omega (1-v)^2 \, dx + \varepsilon \int_\Omega (v')^2 dx + \int_\Omega |u - w|^2 \, dx,$$

while the Γ-limit F becomes

$$F(u,v) = \begin{cases} \displaystyle\int_\Omega f(|u'|) \, dx + \int_{S_u} g(|u^+ - u^-|) \, d\mathcal{H}^0 + |D^c u|(\Omega) + \int_\Omega |u - w|^2 \, dx & \text{if } u \in BV(\Omega; \mathbb{R}^k), \\ & \text{and } v \equiv 1; \\ +\infty & \text{otherwise.} \end{cases}$$

The function $g(z)$ can be explicitly computed and is equal to

$$g(z) = \frac{|z|}{1 + |z|}. \tag{3.1}$$

See also [2, Example 4.6].
Here and below we make the following assumptions on f:

- $f : [0, \infty) \to [0, \infty)$ is of class C^1, it is increasing and strictly convex;
- $f'(0) = \lim_{t \to 0^+} \frac{f(t) - f(0)}{t} = 0$, and $\lim_{t \to +\infty} \frac{f(t)}{t} = 1$.

Under these hypothesis (2.3) holds. Our first main result reads as follows.

Theorem 3.1. *Let $\Omega = (a,b) \subset \mathbb{R}$ and $\beta = \min\left\{ \sqrt{\frac{1}{128}}, \frac{1}{68(b-a)^{1/2}} \right\}$. Given $w \in L^2(\Omega; \mathbb{R}^k)$ such that $\|w\|_{L^2} \le \beta$, there exists a minimizer u of F in $W^{1,\infty}(\Omega; \mathbb{R}^k)$ with $f'(|u'|) \frac{u'}{|u'|} \in H^1(\Omega; \mathbb{R}^k)$, and satisfying the equation*

$$-\frac{d}{dx} \Big(f'(|u'|) \frac{u'}{|u'|} \Big) + 2(u - w) = 0,$$

with the boundary conditions

$$f'(|u'(x)|) \frac{u'(x)}{|u'(x)|} = 0 \quad \textit{for } x \in \{a, b\}.$$

We also consider critical points of the functional F_ε; namely, we say that $(u,v) \in W^{1,1}(\Omega;\mathbb{R}^k) \times H^2(\Omega)$ is a critical point for F_ε if the following equations are satisfied

$$\begin{aligned} &2vf(|u'|) + \frac{1}{2\varepsilon}(v-1) - 2\varepsilon v'' = 0, \\ &-\frac{d}{dx}\left(v^2 f'(|u'|)\frac{u'}{|u'|}\right) + 2(u-w) = 0, \end{aligned} \tag{3.2}$$

with boundary conditions

$$\begin{cases} v'(x) = 0 & \text{for } x \in \{a,b\}, \\ v^2(x) f'(|u'(x)|)\frac{u'(x)}{|u'(x)|} = 0 & \text{for } x \in \{a,b\}. \end{cases}$$

Theorem 3.2. *Let $\Omega = (a,b) \subset \mathbb{R}$, $\beta := \min\left\{\sqrt{\frac{1}{128}}, \frac{1}{68(b-a)^{1/2}}\right\}$ and let $w \in L^2(\Omega;\mathbb{R}^k)$ be such that $\|w\|_{L^2} \leq \beta$. Let $\widetilde{w} = \frac{1}{b-a}\int_a^b w\,dx$. Then there is a constant $\overline{\varepsilon} > 0$ such that the following holds: if $(u_\varepsilon, v_\varepsilon)$ are critical points for F_ε satisfying*

$$F_\varepsilon(u_\varepsilon, v_\varepsilon) \leq F_\varepsilon(\widetilde{w}, 1), \tag{3.3}$$

for $\varepsilon \leq \overline{\varepsilon}$, then $(u_\varepsilon, v_\varepsilon) \in W^{1,\infty}(\Omega;\mathbb{R}^k) \times H^2(\Omega)$, and there is $u \in W^{1,\infty}(\Omega;\mathbb{R}^k)$ minimizer of F such that

$$u_\varepsilon \rightharpoonup u \qquad \text{weakly* in } W^{1,\infty}(\Omega;\mathbb{R}^k), \tag{3.4}$$

$$v_\varepsilon \rightharpoonup 1 \qquad \text{weakly in } H^1(\Omega). \tag{3.5}$$

Theorem 1.1 follows by combining the two theorems stated above.

4 Proof of Theorem 3.1 and Theorem 3.2

The proof of theorems above will be a consequence of the propositions in the rest of the section. Before entering into the details of the proof, we define a regularization of F_ε as follows: for $\delta > 0$, let

$$F_{\varepsilon,\delta}(u,v) = F_\varepsilon(u,v) + \frac{\delta}{2}\int_\Omega |u'|^2\,dx.$$

We also define

$$\hat{F}_{\varepsilon,\delta}(u,v) = \begin{cases} F_{\varepsilon,\delta}(u,v) & \text{on } H^1(\Omega;\mathbb{R}^k) \times H^1(\Omega) \text{ and } v \geq \frac{1}{4} \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\hat{F}_\varepsilon(u,v) = \begin{cases} F_\varepsilon(u,v) + \sum_{x \in S_u} v(x)^2|u^+ - u^-| + \int_\Omega v(x)^2\,d|D^c u| & \text{if } u \in BV(\Omega;\mathbb{R}^k), \\ & v \in H^1(\Omega),\ v \geq \frac{1}{4}; \\ +\infty & \text{otherwise.} \end{cases} \tag{4.1}$$

Theorem 4.1. *The functional $F_{\varepsilon,\delta}$ admits minimizers in $H^1(\Omega;\mathbb{R}^k) \times H^2(\Omega)$ for every $\varepsilon, \delta > 0$ and $w \in L^2(\Omega;\mathbb{R}^k)$. Moreover, if $(u_{\varepsilon,\delta}, v_{\varepsilon,\delta}) \in H^1(\Omega;\mathbb{R}^k) \times H^2(\Omega)$ are minimizers of $F_{\varepsilon,\delta}$, then $v_{\varepsilon,\delta}^2 f'(|u'_{\varepsilon,\delta}|)\frac{u'_{\varepsilon,\delta}}{|u'_{\varepsilon,\delta}|} + \delta u'_{\varepsilon,\delta} \in H^1(\Omega)$ and they satisfy the Euler-Lagrange equations*

$$\begin{aligned} &2v_{\varepsilon,\delta} f(|u'_{\varepsilon,\delta}|) + \frac{1}{2\varepsilon}(v_{\varepsilon,\delta} - 1) - 2\varepsilon v''_{\varepsilon,\delta} = 0, \\ &-\frac{d}{dx}\left(v_{\varepsilon,\delta}^2 f'(|u'_{\varepsilon,\delta}|)\frac{u'_{\varepsilon,\delta}}{|u'_{\varepsilon,\delta}|}\right) + 2(u_{\varepsilon,\delta} - w) - \delta u''_{\varepsilon,\delta} = 0, \end{aligned} \tag{4.2}$$

with boundary conditions

$$\begin{cases} 2\varepsilon v'_{\varepsilon,\delta}(x) = 0 & \textit{for } x \in \{a,b\}, \\ v^2_{\varepsilon,\delta}(x) f'(|u'_{\varepsilon,\delta}(x)|) \frac{u'_{\varepsilon,\delta}(x)}{|u'_{\varepsilon,\delta}(x)|} + \delta u'_{\varepsilon,\delta}(x) = 0 & \textit{for } x \in \{a,b\}. \end{cases}$$

Proof. To prove the first part of the theorem, we employ the direct method in the calculus of variations. Specifically, we prove the coercivity and lower semicontinuity of $F_{\varepsilon,\delta}$.

Coercivity and Compactness. Let $m > 0$ be such that

$$F_{\varepsilon,\delta}(u,v) \leq m.$$

In particular, we have $\dfrac{\delta}{2}\displaystyle\int_\Omega |u'|^2\,dx \leq m$ and $\|u-w\|^2_{L^2} \leq m$, which implies respectively

$$\|u'\|^2_{L^2} \leq C \tag{4.3}$$

and

$$\|u\|_{L^2} \leq \|u-w\|_{L^2} + \|w\|_{L^2} \leq \sqrt{m} + \|w\|_{L^2} \leq C, \tag{4.4}$$

where in the last inequality we have used the hypothesis $w \in L^2(\Omega;\mathbb{R}^k)$. Here and below C denotes a positive constant, independent of ε and δ, which may change from line to line. Combining (4.3) and (4.4), we deduce

$$\|u\|_{H^1} \leq C.$$

Similarly, for v, we have $\varepsilon\displaystyle\int_\Omega |v'|^2\,dx \leq m$ and $\dfrac{1}{4\varepsilon}\displaystyle\int_\Omega (1-v)^2\,dx \leq m$, which implies respectively

$$\|v'\|^2_{L^2} \leq C \tag{4.5}$$

and

$$\|v\|_{L^2} \leq \|1-v\|_{L^2} + \|1\|_{L^2} \leq \sqrt{m} + (b-a)^{1/2} \leq C. \tag{4.6}$$

Combining (4.5) and (4.6), we deduce

$$\|v\|_{H^1} \leq C.$$

Thus, the desired precompatcness in H^1.

Lower Semicontinuity. Let $(u_j, v_j) \in H^1(\Omega;\mathbb{R}^k) \times H^1(\Omega)$. We can assume without loss of generality that for all j

$$F_{\varepsilon,\delta}(u_j, v_j) \leq C,$$

and thus

$$\|u_j\|_{H^1} \leq C, \quad \|v_j\|_{H^1} \leq C \quad \forall j.$$

Therefore, up to a subsequence, $u_j \rightharpoonup u$ and $v_j \rightharpoonup v$ weakly in H^1. Weak convergence in H^1 ensures

$$\liminf_{j\to+\infty} \|u'_j\|_{L^2} \geq \|u'\|_{L^2}, \quad \liminf_{j\to+\infty} \|v'_j\|_{L^2} \geq \|v'\|_{L^2}.$$

Moreover, by the Rellich Theorem, weak convergence in H^1 implies strong convergence in L^2, i.e.,

$$u_j \to u \quad \text{strongly in } L^2(\Omega;\mathbb{R}^k), \quad v_j \to v \quad \text{strongly in } L^2(\Omega),$$

which in turn implies

$$\frac{1}{4\varepsilon}\int_\Omega (1-v)^2 dx = \frac{1}{4\varepsilon}\|1-v\|_{L^2}^2 = \lim_{j\to+\infty}\|1-v_j\|_{L^2}^2; \tag{4.7}$$

$$\int_\Omega |u-w|^2 dx = \|u-w\|_{L^2}^2 = \lim_{j\to+\infty}\|u_j-w\|_{L^2}^2. \tag{4.8}$$

Furthermore

$$\varepsilon\int_\Omega |v'|^2 dx = \varepsilon\|v'\|_{L^2}^2 \le \liminf_{j\to+\infty}\varepsilon\|v_j'\|_{L^2}^2; \tag{4.9}$$

$$\frac{\delta}{2}\int_\Omega |u'|^2 dx = \frac{\delta}{2}\|u'\|_{L^2}^2 \le \liminf_{j\to+\infty}\frac{\delta}{2}\|u_j'\|_{L^2}^2. \tag{4.10}$$

Finally, by Theorem 2.3, for $F(x,v,p) = v^2 f(|p|)$ we obtain

$$\int_\Omega v^2 f(|u'|)dx \le \liminf_{j\to+\infty}\int_\Omega v_j^2 f(|u_j'|)dx,$$

where we remarked that the weak convergences in H^1 of v_j, u_j imply $v_j \to v$ in $L^1(\Omega)$ and $u_j' \rightharpoonup u'$ in $L^1(\Omega;\mathbb{R}^k)$.
In conclusion,

$$F_{\varepsilon,\delta}(u,v) \le \liminf_{j\to+\infty} F_{\varepsilon,\delta}(u_j,v_j).$$

We now proceed to prove the second part of the statement. Let $(u_{\varepsilon,\delta}, v_{\varepsilon,\delta})$ a couple of minimizers of the functional. From the computation of the Euler-Lagrange equations, we obtain the conditions (4.11) and (4.12) stated below (which are the weak forms of (4.2)). By comparison, the condition

$$\int_\Omega 2v_{\varepsilon,\delta}\varphi f(|u_{\varepsilon,\delta}'|)\ dx + \frac{1}{2\varepsilon}\int_\Omega (v_{\varepsilon,\delta}-1)\varphi\ dx + 2\varepsilon\int_\Omega v_{\varepsilon,\delta}'\varphi'\ dx = 0 \tag{4.11}$$

valid for all $\varphi \in H^1(\Omega)$, shows that $v_{\varepsilon,\delta} \in H^2(\Omega)$ and it yields

$$\begin{cases} 2v_{\varepsilon,\delta} f(|u_{\varepsilon,\delta}'|) + \frac{1}{2\varepsilon}(v_{\varepsilon,\delta}-1) - 2\varepsilon v_{\varepsilon,\delta}'' = 0, & \\ 2\varepsilon v_{\varepsilon,\delta}'(x) = 0 & \text{for } x \in \{a,b\}. \end{cases}$$

Similarly, the condition

$$\int_\Omega \left(v_{\varepsilon,\delta}^2 f'(|u_{\varepsilon,\delta}'|)\frac{u_{\varepsilon,\delta}'}{|u_{\varepsilon,\delta}'|} + \delta u_{\varepsilon,\delta}'\right)\cdot\psi'\ dx + \int_\Omega 2(u_{\varepsilon,\delta}-w)\cdot\psi\ dx = 0 \tag{4.12}$$

valid for all $\psi \in H^1(\Omega;\mathbb{R}^k)$ implies $v_{\varepsilon,\delta}^2 f'(|u_{\varepsilon,\delta}'|)\frac{u_{\varepsilon,\delta}'}{|u_{\varepsilon,\delta}'|} + \delta u_{\varepsilon,\delta}' \in H^1(\Omega)$ and it yields

$$\begin{cases} -\frac{d}{dx}\left(v_{\varepsilon,\delta}^2 f'(|u_{\varepsilon,\delta}'|)\frac{u_{\varepsilon,\delta}'}{|u_{\varepsilon,\delta}'|}\right) + 2(u_{\varepsilon,\delta}-w) - \delta u_{\varepsilon,\delta}'' = 0, & \\ v_{\varepsilon,\delta}^2(x) f'(|u_{\varepsilon,\delta}'(x)|)\frac{u_{\varepsilon,\delta}'(x)}{|u_{\varepsilon,\delta}'(x)|} + \delta u_{\varepsilon,\delta}'(x) = 0 & \text{for } x \in \{a,b\}. \end{cases}$$

□

Adapting the results from [19] and [21] to our context, we obtain the following lemma:

Lemma 4.2. *Let $(u_{\varepsilon,\delta}, v_{\varepsilon,\delta}) \in H^1(\Omega;\mathbb{R}^k)\times H^2(\Omega)$ be minimizers of $F_{\varepsilon,\delta}$. Then the following properties hold:*

$$0 \le v_{\varepsilon,\delta}(x) \le 1 \quad \textit{for all } x \in \Omega, \tag{4.13}$$

$$\|u_{\varepsilon,\delta}\|_{L^2} \le \|w\|_{L^2}. \tag{4.14}$$

Moreover, if $(u_\varepsilon, v_\varepsilon) \in H^1(\Omega;\mathbb{R}^k)\times H^2(\Omega)$ are critical points of F_ε, then (4.14) *holds for u_ε.*

Proof. Formulas (4.13) follows straightforwardly by a truncation argument. Let us prove (4.14). Multiplying (4.2) by $u_{\varepsilon,\delta}$ and integrating between a and b, we obtain

$$\int_a^b \left[\delta u''_{\varepsilon,\delta}\cdot u_{\varepsilon,\delta} + u_{\varepsilon,\delta}\cdot\left(\frac{d}{dx}\left(v^2_{\varepsilon,\delta} f'(|u'_{\varepsilon,\delta}|)\frac{u'_{\varepsilon,\delta}}{|u'_{\varepsilon,\delta}|}\right)\right)\right]\,dx = 2\int_a^b (|u_{\varepsilon,\delta}|^2 - u_{\varepsilon,\delta}\cdot w)\,dx,$$

hence

$$-\int_a^b \left(\delta|u'_{\varepsilon,\delta}|^2 + v^2_{\varepsilon,\delta}|u'_{\varepsilon,\delta}|f'(|u'_{\varepsilon,\delta}|)\right)\,dx = 2\int_a^b |u_{\varepsilon,\delta}|^2\,dx - 2\int_a^b u_{\varepsilon,\delta}\cdot w\,dx,$$

from which we deduce

$$2\int_a^b |u_{\varepsilon,\delta}|^2\,dx \le 2\int_a^b u_{\varepsilon,\delta}\cdot w\,dx \le \|u_{\varepsilon,\delta}\|^2_{L^2} + \|w\|^2_{L^2},$$

implying the desired result. For critical points, the same argument holds even if $\delta = 0$. $\square$

Lemma 4.3. *Let $w \in L^2(\Omega;\mathbb{R}^k)$ with $\|w\|_{L^2} \le \beta$ where $\beta = \min\left\{\sqrt{\frac{1}{128}}, \frac{1}{68(b-a)^{1/2}}\right\}$. Then there exists $\overline{\varepsilon} > 0$ depending only on Ω and $f(0)$ such that, for all $\varepsilon \in (0,\overline{\varepsilon})$ and $\delta \ge 0$, if $(u_{\varepsilon,\delta}, v_{\varepsilon,\delta}) \in H^1(\Omega;\mathbb{R}^k)\times H^2(\Omega)$ are minimizers of $F_{\varepsilon,\delta}$, it holds that $v_{\varepsilon,\delta} \ge \frac{1}{4}$ on Ω. Furthermore, the following equalities hold:*

$$\begin{aligned}\inf\left\{F_{\varepsilon,\delta}(u,v) : u,v \in H^1(\Omega;\mathbb{R}^k)\times H^2(\Omega)\right\} &= \inf\left\{F_{\varepsilon,\delta}(u,v) : u,v \in H^1(\Omega;\mathbb{R}^k)\times H^2(\Omega), v \ge 1/4\right\},\\ \inf\left\{F_{\varepsilon}(u,v) : u,v \in H^1(\Omega;\mathbb{R}^k)\times H^2(\Omega)\right\} &= \inf\left\{F_{\varepsilon}(u,v) : u,v \in H^1(\Omega;\mathbb{R}^k)\times H^2(\Omega), v \ge 1/4\right\}.\end{aligned} \tag{4.15}$$

Proof. Let $\widetilde{w} = \dfrac{1}{b-a}\displaystyle\int_a^b w\,dx$ and $v \equiv 1$. Then we have

$$F_{\varepsilon,\delta}(\widetilde{w},1) = f(0)(b-a) + \int_a^b |\widetilde{w} - w|^2\,dx = f(0)(b-a) + \int_a^b (|\widetilde{w}|^2 - 2w\cdot\widetilde{w} + |w|^2)\,dx.$$

Now,

$$\int_a^b |\widetilde{w}|^2\,dx = \int_a^b \left[\frac{1}{(b-a)^2}\left(\Big|\int_a^b w\,dx\Big|\right)^2\right]\,dx \le \int_a^b \left[\frac{1}{(b-a)^2}(b-a)\int_a^b |w|^2\,dx\right]\,dx = \int_a^b |w|^2\,dx,$$

and then

$$F_{\varepsilon,\delta}(\widetilde{w},1) \le f(0)(b-a) + 4\int_a^b |w|^2\,dx \le f(0)(b-a) + 4\beta^2.$$

Let $(u_{\varepsilon,\delta}, v_{\varepsilon,\delta}) \in H^1(\Omega;\mathbb{R}^k)\times H^2(\Omega)$ be such that

$$F_{\varepsilon,\delta}(u_{\varepsilon,\delta}, v_{\varepsilon,\delta}) \le F_{\varepsilon,\delta}(\widetilde{w},1) \le f(0)(b-a) + 4\beta^2 =: C_\beta.$$

Notice that $(u_{\varepsilon,\delta}, v_{\varepsilon,\delta})$ always exists and it may happen that $\tilde{w}$ and 1 are already the global minimizers of $F_{\varepsilon,\delta}$. In that case $u_{\varepsilon,\delta} = \tilde{w}, v_{\varepsilon,\delta} = 1$ and $v \ge \frac{1}{4}$.
Since $v_{\varepsilon,\delta} \in H^1((a,b))$ is continuous and can be extended to a continuous function on $[a,b]$, let $\overline{x} \in \operatorname{argmax} v_{\varepsilon,\delta}$ and $\overline{y} \in \operatorname{argmin} v_{\varepsilon,\delta}$. We estimate:

$$\frac{1}{4\varepsilon}\int_a^b (1 - v_{\varepsilon,\delta})^2\,dx \le F_{\varepsilon,\delta}(u_{\varepsilon,\delta}, v_{\varepsilon,\delta}) \le C_\beta,$$

which implies

$$\int_a^b (1 - v_{\varepsilon,\delta})^2\,dx \le 4\varepsilon C_\beta. \tag{4.16}$$

In particular,

$$(b-a)(1-v_{\varepsilon,\delta}(\overline{x}))^2 = \int_a^b (1-v_{\varepsilon,\delta}(\overline{x}))^2\,dx \le \int_a^b (1-v_{\varepsilon,\delta})^2\,dx \le 4\varepsilon C_\beta.$$

Thus,

$$\begin{aligned} 1-v_{\varepsilon,\delta}(\overline{x}) &\le \sqrt{\frac{4\varepsilon C_\beta}{b-a}} = \sqrt{4\varepsilon\Big(f(0)+4\frac{\beta^2}{b-a}\Big)}, \\ v_{\varepsilon,\delta}(\overline{x}) &\ge 1-\sqrt{4\varepsilon\Big(f(0)+4\frac{\beta^2}{b-a}\Big)}. \end{aligned}$$

Now, there exists $\varepsilon_0>0$ such that $\forall\varepsilon\in(0,\varepsilon_0)$, $\sqrt{4\varepsilon\Big(f(0)+4\frac{\beta^2}{b-a}\Big)} \le \frac{1}{4}$; it follows that

$$v_{\varepsilon,\delta}(\overline{x}) \ge \frac{3}{4}.$$

Next, we will use this inequality to prove, repeating the argument for $\overline{y}\in \operatorname{argmin} v_\varepsilon$, that $v_{\varepsilon,\delta}(\overline{y}) \ge \frac{1}{4}$. We start from the following estimate:

$$\begin{aligned} C_\beta - \int_a^b v_{\varepsilon,\delta}^2 f(|u'_{\varepsilon,\delta}|)\,dx - \int_a^b |u_{\varepsilon,\delta}-w|^2\,dx - \frac{\delta}{2}\int_a^b |u'_{\varepsilon,\delta}|^2\,dx & \\ \ge \int_a^b \frac{1}{4\varepsilon}(1-v_{\varepsilon,\delta})^2 + \varepsilon|v'_{\varepsilon,\delta}|^2\,dx & \\ \ge \int_a^b |v'_{\varepsilon,\delta}(1-v_{\varepsilon,\delta})|\,dx & \\ = \int_a^b \left|\frac{d}{dx}\frac{1}{2}(1-v_{\varepsilon,\delta})^2\right|\,dx & \\ \ge \int_{\overline{x}}^{\overline{y}} \left|\frac{d}{dx}\frac{1}{2}(1-v_{\varepsilon,\delta})^2\right|\,dx & \\ \ge \left|\int_{\overline{x}}^{\overline{y}} \frac{d}{dx}\frac{1}{2}(1-v_{\varepsilon,\delta})^2\;dx\right| & \\ = \frac{1}{2}(1-v_{\varepsilon,\delta}(\overline{y}))^2 - \frac{1}{2}(1-v_{\varepsilon,\delta}(\overline{x}))^2 & \\ \ge \frac{1}{2}(1-v_{\varepsilon,\delta}(\overline{y}))^2 - \frac{1}{32}. & \end{aligned}$$

We have thus obtained

$$\frac{1}{2}(1-v_{\varepsilon,\delta}(\overline{y}))^2 \le C_\beta - \int_a^b v_{\varepsilon,\delta}^2 f(|u'_{\varepsilon,\delta}|)\;dx + \frac{1}{32}. \tag{4.17}$$

Now, let us estimate the term $\displaystyle\int_a^b v_{\varepsilon,\delta}^2 f(|u'_{\varepsilon,\delta}|)\;dx$:

$$\begin{aligned} \int_a^b v_{\varepsilon,\delta}^2 f(|u'_{\varepsilon,\delta}|)\;dx &\ge f(0)\int_a^b v_{\varepsilon,\delta}^2 - 1 + 1\;dx \\ &= f(0)\int_a^b (v_{\varepsilon,\delta}-1)(v_{\varepsilon,\delta}+1)\;dx + f(0)(b-a) \\ &\ge f(0)(b-a) - f(0)\Big(\int_a^b (v_{\varepsilon,\delta}-1)^2\;dx\Big)^{1/2}\Big(\int_a^b (v_{\varepsilon,\delta}+1)^2\;dx\Big)^{1/2} \\ &\ge f(0)(b-a) - f(0)(4\varepsilon C_\beta)^{1/2}\;2(b-a)^{1/2}, \end{aligned} \tag{4.18}$$

where the last inequality follows from the bound obtained in (4.16). Substituting (4.18) into (4.17), we find

$$\begin{aligned}\frac{1}{2}(1-v_{\varepsilon,\delta}(\overline{y}))^2 &\leq C_\beta - f(0)(b-a) + 2f(0)(4\varepsilon C_\beta)^{1/2}(b-a)^{1/2} + \frac{1}{32}\\ &= 4\beta^2 + 2f(0)\sqrt{4\varepsilon C_\beta(b-a)} + \frac{1}{32}\\ &= 4\beta^2 + 2f(0)\sqrt{4\varepsilon\Big(f(0)(b-a)+4\beta^2\Big)(b-a)} + \frac{1}{32} =: K.\end{aligned}$$

Thus,

$$v_{\varepsilon,\delta}(\overline{y}) \geq 1 - \sqrt{2K}.$$

Let us estimate K: using the fact that $\beta \leq \sqrt{\frac{1}{128}}$, we find

$$4\beta^2 \leq \frac{1}{32}.$$

Additionally, there exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$,

$$2f(0)\sqrt{4\varepsilon\Big(f(0)(b-a)+4\beta^2\Big)(b-a)} \leq \frac{7}{32}.$$

Combining these bounds, we obtain

$$K \leq \frac{9}{32}.$$

Finally, for $\overline{\varepsilon} = \min(\varepsilon_0, \varepsilon_1)$, it follows that $\forall \varepsilon \in (0, \overline{\varepsilon})$,

$$v_{\varepsilon,\delta}(\overline{y}) \geq 1 - \sqrt{\frac{9}{16}} = \frac{1}{4}.$$

The equalities in (4.15) follow from the arbitrariness of (u, v), completing the proof. □

Remark 4.4.

(i) Note that $F_{\varepsilon,\delta}(\widetilde{w}, 1)$ does not depend on δ and ε.

(ii) For the proof, it is not strictly necessary that $(u_{\varepsilon,\delta}, v_{\varepsilon,\delta})$ are minimizers of the functional, but only that they solve the Euler-Lagrange equations and that $F_{\varepsilon,\delta}(u_{\varepsilon,\delta}, v_{\varepsilon,\delta}) \leq F_{\varepsilon,\delta}(\widetilde{w}, 1)$ is satisfied. Therefore, the following corollary holds.

Corollary 4.5. *Let $w \in L^2(\Omega; \mathbb{R}^k)$ with the property $\|w\|_{L^2} \leq \beta$ where $\beta = \min\left\{\sqrt{\frac{1}{128}}, \frac{1}{68(b-a)^{1/2}}\right\}$. Let $\widetilde{w} := \frac{1}{b-a}\int_a^b w\, dx$, and assume that $(u_\varepsilon, v_\varepsilon)$ are critical points for F_ε satisfying*

$$F_\varepsilon(u_\varepsilon, v_\varepsilon) \leq F_\varepsilon(\widetilde{w}, 1). \tag{4.19}$$

Then there exists $\overline{\varepsilon} > 0$ depending only on (a, b) and $f(0)$ such that, for all $\varepsilon \in (0, \overline{\varepsilon})$ and $\delta \geq 0$, $v_\varepsilon \geq \frac{1}{4}$ on Ω.

Theorem 4.6. *Let $\varepsilon > 0$ be fixed. If there exists a constant $\bar{C} > 0$ such that*

$$\sup_{\delta\in(0,1)} \hat{F}_{\varepsilon,\delta}(u_{\varepsilon,\delta}, v_{\varepsilon,\delta}) < \bar{C}, \tag{4.20}$$

then there exists $C' > 0$ such that

$$\sup_{\delta\in(0,1)} \|u_{\varepsilon,\delta}\|_{BV} + \sup_{\delta\in(0,1)} \|v_{\varepsilon,\delta}\|_{H^1} < C'.$$

Furthermore, the functional $\hat{F}_{\varepsilon,\delta}$ Γ-converges with respect to $L^1(\Omega; \mathbb{R}^k) \times L^1(\Omega)$, as $\delta \to 0^+$, to the functional $\hat{F}_\varepsilon$.

Proof. Let us prove the first part of the statement. Given that $\lim_{t\to+\infty}\frac{f(t)}{t}=1$, there exists t_0 such that, for $t\geq t_0$, we have $f(|t|)\geq\frac{1}{2}|t|$. Let us define $\Omega_0=\{x\in\Omega:|u'_{\varepsilon,\delta}(x)|\geq t_0\}$. By (4.20) we have that $v_{\varepsilon,\delta}\geq\frac{1}{4}$ for every δ, and moreover,

$$\bar{C}\geq\int_\Omega v^2_{\varepsilon,\delta}f(|u'_{\varepsilon,\delta}|)\,dx\geq\frac{1}{16}\int_\Omega f(|u'_{\varepsilon,\delta}|)\,dx\geq\frac{1}{32}\int_{\Omega_0}|u'_{\varepsilon,\delta}|\,dx.$$

Hence

$$\|u'_{\varepsilon,\delta}\|_{L^1}\leq C. \tag{4.21}$$

Additionally, since

$$\begin{aligned}\int_\Omega|u_{\varepsilon,\delta}|\,dx-\int_\Omega|w|\,dx&\leq\int_\Omega|u_{\varepsilon,\delta}-w|\,dx\\&\leq\left(\int_\Omega|u_{\varepsilon,\delta}-w|^2\,dx\right)^{\frac{1}{2}}(b-a)^{\frac{1}{2}}\\&\leq\frac{1}{2}\left(\int_\Omega|u_{\varepsilon,\delta}-w|^2\,dx\right)+(b-a)\\&\leq\bar{C}+(b-a),\end{aligned}$$

it follows that

$$\int_\Omega|u_{\varepsilon,\delta}|\,dx\leq\bar{C}+\int_\Omega|w|\,dx+(b-a). \tag{4.22}$$

Combining (4.21) and (4.22), we obtain

$$\|u_{\varepsilon,\delta}\|_{BV}\leq C_1$$

for some constant $C_1>0$ indipendent of δ. Furthermore,

$$\int_\Omega\varepsilon|v'_{\varepsilon,\delta}|^2\,dx\leq\bar{C}, \tag{4.23}$$

and observe that

$$\begin{aligned}&\frac{1}{4\varepsilon}\int_\Omega(1-v_{\varepsilon,\delta})^2\,dx<\bar{C},\\&\frac{1}{4\varepsilon}\int_\Omega 1-2v_{\varepsilon,\delta}+v^2_{\varepsilon,\delta}\,dx<\bar{C},\\&\frac{1}{4\varepsilon}\int_\Omega v^2_{\varepsilon,\delta}\,dx<\bar{C}-\frac{b-a}{4\varepsilon}+\frac{1}{4\varepsilon}\int_\Omega 2v_{\varepsilon,\delta}\,dx.\end{aligned}$$

Applying Young's inequality $2v_{\varepsilon,\delta}\leq\frac{v^2_{\varepsilon,\delta}}{4}+4$, the inequality above becomes

$$\frac{1}{4\varepsilon}\int_\Omega v^2_{\varepsilon,\delta}\,dx<\bar{C}-\frac{b-a}{4\varepsilon}+\frac{1}{4\varepsilon}\frac{1}{4}\int_\Omega v^2_{\varepsilon,\delta}\,dx+\frac{b-a}{\varepsilon},$$

that is

$$\frac{3}{16\varepsilon}\int_\Omega v^2_{\varepsilon,\delta}\,dx<\bar{C}-\frac{b-a}{4\varepsilon}+\frac{b-a}{\varepsilon},$$

from which it follows that

$$\|v_{\varepsilon,\delta}\|^2_{L^2}\leq C_2. \tag{4.24}$$

Combining (4.23) and (4.24), we deduce

$$\|v_{\varepsilon,\delta}\|_{H^1}\leq C_3.$$

We have thus established the first part of the statement with $C' = C_1 + C_3$. It remains to address the Γ-convergence, specifically proving the $\Gamma - \liminf$ and the $\Gamma - \limsup$ inequality. We may assume, without loss of generality, that

$$\sup_j \hat{F}_{\varepsilon,\delta}(u_{\delta_j}, v_{\delta_j}) < C,$$

which from the first statement implies $u_{\delta_j} \to u$ weakly* in $BV(\Omega;\mathbb{R}^k)$ and $v_{\delta_j} \to v$ weakly in $H^1(\Omega)$ up to subsequences.

For every $\delta > 0$, we have $\hat{F}_\varepsilon(u_{\delta_j}, v_{\delta_j}) \le \hat{F}_{\varepsilon,\delta}(u_{\delta_j}, v_{\delta_j})$. Thus, the $\Gamma - \liminf$ inequality will follow from

$$\hat{F}_\varepsilon(u, v) \le \liminf_{j\to\infty} \hat{F}_\varepsilon(u_{\delta_j}, v_{\delta_j}),$$

which is the lower semicontinuity of $\hat{F}_\epsilon$. Using (4.9), (4.7), and (4.8), it remains to prove the semicontinuity of

$$\int_\Omega v^2 f(|u'|)\,dx + \int_\Omega v^2\,d|Du^c| + \sum_{x\in S(u)} v^2(x)|u^+(x) - u^-(x)|,$$

which is assured by Corollary 2.5.

Thus, we are left with the construction of a recovery sequence $(u_{\delta_j}, v_{\delta_j})$ satisfying the $\Gamma - \limsup$ inequality.

If $u \in H^1(\Omega;\mathbb{R}^k)$, we simply set $u_{\delta_j} = u$ and $v_{\delta_j} = v$. Conversely, if $u \in BV(\Omega;\mathbb{R}^k) \setminus H^1(\Omega;\mathbb{R}^k)$, fixing $v_{\delta_j} = v$, we have to prove that:

$$\lim_{j\to\infty} \int_\Omega v^2 f(|u'_{\delta_j}|)\,dx = I(u, v), \tag{4.25}$$

$$\lim_{j\to\infty} \int_\Omega |u_{\delta_j} - w|^2\,dx = \int_\Omega |u - w|^2\,dx, \tag{4.26}$$

$$\lim_{j\to\infty} \frac{\delta_j}{2} \int_\Omega |u'_{\delta_j}|^2\,dx = 0. \tag{4.27}$$

We begin by noting that, for $v \ge \frac{1}{4}$,

$$I(u, v) = \inf\left\{ \liminf_{n\to\infty} \int_\Omega v^2 f(|u'_n|)\,dx : u_n \to u \text{ in } L^1(\Omega;\mathbb{R}^k), u_n \in H^1(\Omega;\mathbb{R}^k) \right\},$$

i.e., for a fixed $v \in H^1(\Omega)$, $I(u, v)$ is the relaxation of $\displaystyle\int_\Omega v^2 f(|u'_n|)\,dx$.

Thus, there exists a sequence $u_n \in H^1(\Omega;\mathbb{R}^k)$, $n > 0$, such that

$$\lim_{n\to\infty} \int_\Omega v^2 f(|u'_n|)\,dx = I(u, v). \tag{4.28}$$

For $n = 0$ set $u_0 \equiv 0$, and for $n > 0$ consider the sequence u_n such that (4.28) holds. For each k, consider the indices in

$$J_j := \left\{ n \in \mathbb{N} : \|u_n\|^2_{H^1} \le \frac{1}{\sqrt{\delta_j}} \right\}.$$

Note that $J_j \ne \emptyset$ because $u_0 \in J_j$. For each j, choose $n_j := \max\{n : n \in J_j\}$ and set $u_{\delta_j} := u_{n_j}$. We now prove (4.27): indeed

$$\delta_j \|u'_{\delta_j}\|^2_{L^2} \le \delta_j \|u_{\delta_j}\|^2_{H^1} = \delta_j \|u_{n_j}\|^2_{H^1} \le \sqrt{\delta_j}.$$

Therefore (4.26) follows from $u_{\delta_j} \to u$ weakly* in $BV(\Omega;\mathbb{R}^k)$, which implies $u_{\delta_j} \to u$ in $L^2(\Omega;\mathbb{R}^k)$.

To establish (4.25), we must show that u_{n_j} is still a recovery sequence for the relaxation, i.e.,

$$\lim_{j\to\infty}\int_\Omega v^2 f(|u'_{n_j}|)\,dx = I(u,v).$$

We note that:

$$n_j \to \infty \quad \text{as } j\to\infty. \tag{4.29}$$

Indeed, $J_j \subseteq J_{j+1}$ and $\bigcup_j J_j = \mathbb{N}$, since as $j\to\infty$, $\delta_j \to 0$ and $\frac{1}{\sqrt{\delta_j}} \to \infty$, meaning that every $n\in\mathbb{N}$ belongs to some J_j. Observe that n_j is a non-decreasing sequence of natural numbers, which cannot stabilize; if it did, there would exist a constant C such that $|J_j| < C$ for all j, contradicting the fact that $\bigcup_j J_j = \mathbb{N}$.

Thus, from (4.29), we conclude:

$$\lim_{j\to\infty}\int_\Omega v^2 f(|u'_{n_j}|)\,dx = \lim_{h\to\infty}\int_\Omega v^2 f(|u'_h|)\,dx = I(u,v).$$

□

From Lemma 4.3, the following theorem, ensuring that there are regular minimizers of F_ε, follows:

Theorem 4.7. *Let $\Omega = (a,b)$ and $\overline{\varepsilon} > 0$ be as in Lemma 4.3 and let $\varepsilon \in (0,\overline{\varepsilon})$. Let $w \in L^2(\Omega;\mathbb{R}^k)$ such that $\|w\|_{L^2} \le \beta$ where $\beta = \min\left\{\sqrt{\frac{1}{128}}, \frac{1}{68(b-a)^{1/2}}\right\}$. If $(u_{\varepsilon,\delta}, v_{\varepsilon,\delta}) \in H^1(\Omega;\mathbb{R}^k)\times H^2(\Omega)$ are minimizers of $F_{\varepsilon,\delta}$, then there exists a constant C, independent of ε and δ, such that*

$$\|u_{\varepsilon,\delta}\|_{W^{1,\infty}} + \|v_{\varepsilon,\delta}\|_{H^1} \le C, \tag{4.30}$$

and therefore, up to a subsequence,

$$u_{\varepsilon,\delta} \rightharpoonup u_\varepsilon \text{ weakly* in } W^{1,\infty}, \quad v_{\varepsilon,\delta} \rightharpoonup v_\varepsilon \quad \text{weakly in } H^1 \text{ as } \delta \to 0.$$

Moreover, u_ε and v_ε are minimizers of F_ε, and

$$\|u_\varepsilon\|_{W^{1,\infty}} + \|v_\varepsilon\|_{H^1} \le C, \tag{4.31}$$

for all $\varepsilon \in (0,\overline{\varepsilon})$.

Proof. Set $P_{\varepsilon,\delta} = 2\int_a^x (u_{\varepsilon,\delta} - w)\,dt$. Integrating (4.2) between a and x and multiplying by $u'_{\varepsilon,\delta}(x)$, we obtain

$$v^2_{\varepsilon,\delta} f'(|u'_{\varepsilon,\delta}|)\frac{u'_{\varepsilon,\delta}}{|u'_{\varepsilon,\delta}|}\cdot u'_{\varepsilon,\delta} \le v^2_{\varepsilon,\delta} f'(|u'_{\varepsilon,\delta}|)\frac{u'_{\varepsilon,\delta}}{|u'_{\varepsilon,\delta}|}\cdot u'_{\varepsilon,\delta} + \delta|u'_{\varepsilon,\delta}|^2 = \left[2\int_a^x (u_{\varepsilon,\delta} - w)\,dt\right]\cdot u'_{\varepsilon,\delta} \le |P_{\varepsilon,\delta}||u'_{\varepsilon,\delta}|.$$

Since we can estimate $P_{\varepsilon,\delta}$ as

$$|P_{\varepsilon,\delta}| \le 2\int_a^b |u_{\varepsilon,\delta} - w|\,dt \le 2\|u_{\varepsilon,\delta} - w\|_{L^2}(b-a)^{1/2} \le 4\|w\|_{L^2}(b-a)^{1/2} < \frac{1}{17}$$

and $v_{\varepsilon,\delta} \ge \frac{1}{4}$, we can write

$$\frac{1}{16}|u'_{\varepsilon,\delta}|\ f'(|u'_{\varepsilon,\delta}|) \le \frac{1}{17}|u'_{\varepsilon,\delta}|.$$

Hence

$$f'(|u'_{\varepsilon,\delta}|) \le \frac{16}{17} < 1$$

from which we deduce that

$$|u'_{\varepsilon,\delta}| \in \left\{t : f'(|t|) \leq \frac{16}{17}\right\} \subseteq [0, C],$$

and hence we have proved the existence of a constant $C > 0$ such that

$$\|u'_{\varepsilon,\delta}\|_{L^\infty} \leq C.$$

In particular

$$\|u'_{\varepsilon,\delta}\|_{L^2} \leq C.$$

Combining this estimate with (4.14), we obtain

$$\|u_{\varepsilon,\delta}\|_{H^1} \leq C$$

with C independent of ε and δ and due to Sobolev immersions

$$\|u_{\varepsilon,\delta}\|_{L^\infty} \leq C.$$

Thus

$$\|u_{\varepsilon,\delta}\|_{W^{1,\infty}} \leq C.$$

We now prove that $(u_\varepsilon, v_\varepsilon)$ are minimizers of F_ε. This is equivalent to prove

$$F_\varepsilon(u_\varepsilon, v_\varepsilon) \leq F_\varepsilon(u, v) \quad \forall (u, v) \in W^{1,\infty}(\Omega; \mathbb{R}^k) \times H^1(\Omega). \tag{4.32}$$

In virtue of (4.15), we can restrict ourselves to the case where $v \geq \frac{1}{4}$. Then

$$F_\varepsilon(u, v) = \hat{F}_\varepsilon(u, v) \tag{4.33}$$

and

$$F_\varepsilon(u_\varepsilon, v_\varepsilon) = \hat{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq \hat{F}_\varepsilon(u, v) \quad \forall u \in BV(\Omega; \mathbb{R}^k), v \in H^1(\Omega). \tag{4.34}$$

Thus,

$$F_\varepsilon(u_\varepsilon, v_\varepsilon) \overset{(4.34)}{\leq} \hat{F}_\varepsilon(u, v) \overset{(4.33)}{=} F_\varepsilon(u, v).$$

□

Corollary 4.8. *Let $(u_\varepsilon, v_\varepsilon)$ be critical points for F_ε. Under the same assumptions of Corollary 4.5, there exists a constant $C > 0$, independent of ε, such that*

$$\|u_\varepsilon\|_{W^{1,\infty}} + \|v_\varepsilon\|_{H^1} \leq C. \tag{4.35}$$

We are now ready to prove our main results.

Proof of Theorem 3.1. Theorem 2.2 tells us that the minimizers $(u_\varepsilon, v_\varepsilon)$ of F_ε converge to minimizers (u, v) of F. Now, from (4.31) and the semicontinuity of the norm, it follows that the minimizers of F are in $W^{1,\infty}(\Omega; \mathbb{R}^k) \times H^1(\Omega)$. Moreover, from (4.15), we have $v \geq \frac{1}{4}$.

□

Proof of Theorem 3.2. We begin by proving (3.5). From assumption (3.3), it follows that

$$\frac{1}{4\varepsilon} \int_a^b (v_\varepsilon - 1)^2 \leq F_\varepsilon(\widetilde{w}, 1),$$

which implies that $v_\varepsilon \to 1$ uniformly as $\varepsilon \to 0$.
Next, we proceed to (3.4). Since

$$\|u_\varepsilon\|_{W^{1,\infty}} \leq C$$

for some constant C independent of ε (see (4.35)), and given the assumptions on the function f (see Theorem 2.2), we obtain

$$\left| f'(|u'_\varepsilon|) \frac{u'_\varepsilon}{|u'_\varepsilon|} \right| \le f'(|u'_\varepsilon|) \le C.$$

Define the function $H : \mathbb{R}^k \to \mathbb{R}$ as

$$H(u'_\varepsilon) := f(|u'_\varepsilon|),$$

and observe that

$$|\nabla H(u'_\varepsilon)| = \left| f'(|u'_\varepsilon|) \frac{u'_\varepsilon}{|u'_\varepsilon|} \right| = f'(|u'_\varepsilon|) \le C. \tag{4.36}$$

The Euler-Lagrange equation for F_ε, after integration on (a, x), can thus be rewritten as

$$v_\varepsilon^2(x) \nabla H(u'_\varepsilon(x)) = \int_a^x 2(u_\varepsilon - w)\ ds, \qquad \forall x \in (a, b).$$

Since $v_\varepsilon \ge \frac{1}{4}$ (by Corollary 4.5), it follows that

$$\nabla H(u'_\varepsilon(x)) = \frac{1}{v_\varepsilon^2} \int_a^x 2(u_\varepsilon - w)\ ds.$$

Denoting by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(\Omega; \mathbb{R}^k)$, multiplying both sides by u'_ε, we deduce by (4.36) that there exists some $\eta \in L^\infty(\Omega; \mathbb{R}^k)$ such that, passing to the limit as $\varepsilon \to 0$,

$$\langle \nabla H(u'_\varepsilon), u'_\varepsilon \rangle = \langle \frac{1}{v_\varepsilon^2} \int_a^x 2(u_\varepsilon - w) ds, u'_\varepsilon \rangle \quad \to \quad \langle \eta, u' \rangle = \langle \int_a^x 2(u - w) ds, u' \rangle.$$

The fact that the right-hand side has this form can be seen by observing that $\frac{1}{v_\varepsilon^2} \int_a^x 2(u_\varepsilon - w) ds$ tends, strongly in $L^2(\Omega; \mathbb{R}^k)$, to $\int_a^x 2(u - w) ds$. If we now prove that

$$\eta = \nabla H(u') \tag{4.37}$$

then the proof is complete, as we have shown that the Euler-Lagrange equation of F_ε converges to that of F.
This follows from a standard result on maximal monotone operators ([8, Lemma 3.57]), whose assumptions are satisfied because:

(i) The gradient ∇H is a maximal monotone operator.

(ii) The weak convergences hold:

$$u'_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2, \quad \nabla H(u'_\varepsilon) \rightharpoonup \eta \quad \text{weakly in } L^2.$$

(iii) The upper limit satisfies

$$\limsup_{\varepsilon \to 0} \langle \nabla H(u'_\varepsilon), u'_\varepsilon \rangle = \langle \int_a^x 2(u - w) ds, u' \rangle = \langle \eta, u' \rangle.$$

Eventually we observe that the functional F restricted to $H^1(\Omega; \mathbb{R}^k)$ is strictly convex, and thus has a unique critical point which is the unique minimizer, hence u is the minimizer of F. $\square$

5 Application to the length functional

Let us consider the length functional

$$A_w(u) = \int_\Omega \sqrt{1 + |u'|^2}\, dx + \sum_{x \in S_u} |u^+ - u^-| + |D^c u|(\Omega) + \int_\Omega |u - w|^2\, dx,$$

defined for $u \in BV(\Omega; \mathbb{R}^k)$. We denote

$$\mathcal{L}(u) := \int_\Omega \sqrt{1 + |u'|^2}\, dx + \sum_{x \in S_u} |u^+ - u^-| + |D^c u|(\Omega),$$

so $A_w(u) = \mathcal{L}(u) + \int_\Omega |u - w|^2$.
Take $f(u') = \sqrt{1 + |u'|^2}$ and let $u \in W^{1,\infty}(\Omega; \mathbb{R}^k)$ be a minimizer of F. Recalling (3.1), we have $g(z) \leq |z|$. Hence it holds that

$$F(u) \leq F(\hat{u}) \leq A_w(\hat{u}) \quad \forall \hat{u} \in BV(\Omega; \mathbb{R}^k). \tag{5.1}$$

Moreover, $F(u) = A_w(u)$, and thus from (5.1) we deduce that

$$A_w(u) \leq A_w(\hat{u}) \quad \forall \hat{u} \in BV(\Omega; \mathbb{R}^k).$$

Therefore, u is a minimizer of A_w and belongs to $W^{1,\infty}(\Omega; \mathbb{R}^k)$.
We have thus proven the following theorem:

Theorem 5.1. *Let $\Omega = (a, b)$ and $\beta = \min\left\{\sqrt{\frac{1}{128}}, \frac{1}{68(b-a)^{1/2}}\right\}$. Given $w \in L^2(\Omega; \mathbb{R}^k)$ such that $\|w\|_{L^2} \leq \beta$, there exist minimizers u of A_w in $W^{1,\infty}(\Omega; \mathbb{R}^k)$.*

We now show that the minimizer as in Theorem 5.1 is indeed of class C^1. Precisely, the minimizer u of A_w solves, thanks to Theorem 3.1, the equation

$$-\frac{d}{dx} \frac{u'}{\sqrt{1 + (u')^2}} + 2(u - w) = 0.$$

with

$$\frac{u'(a)}{\sqrt{1 + (u'(a))^2}} = \frac{u'(b)}{\sqrt{1 + (u'(b))^2}} = 0.$$

Integrating on (a, x), for $x \in (a, b)$, we infer

$$\frac{u'(x)}{\sqrt{1 + (u'(x))^2}} = \Phi(x), \tag{5.2}$$

where $\Phi(x)$ is a primitive of $2(u - w)$. Notice that, being the primitive of an L^2 function, Φ belongs to H^1 and, consequently, is continuous. Since the left-hand-side of (5.2) is strictly less than 1, it follows that

$$\Phi(x) < 1.$$

Hence we can conclude $u'(x)^2 = \frac{\Phi(x)^2}{1 - \Phi(x)^2}$, and then

$$u'(x) = \frac{\Phi(x)}{|\Phi(x)|} \sqrt{\frac{\Phi(x)^2}{1 - \Phi(x)^2}},$$

that is a continuous function. We have then proved Theorem 1.2 with

$$\beta := \min\left\{\sqrt{\frac{1}{128}}, \frac{1}{68(b-a)^{1/2}}\right\}. \tag{5.3}$$

5.1 An example

We conclude this section with an example showing that in general, if $\|w\|_{L^2} > \beta$, the regularity of the minimizer of A_w is not true.
Following [15], the function $u : (-2,2) \to \mathbb{R}$ given by

$$u(x) = \begin{cases} -2 & \text{if } x < -1, \\ -1 - \sqrt{-2x - x^2} & \text{if } -1 \le x < 0, \\ 1 + \sqrt{2x - x^2} & \text{if } 0 \le x < 1, \\ 2 & \text{if } 1 \le x, \end{cases} \tag{5.4}$$

minimizes the functional $L(v,(-2,2)) + \int_{-2}^{2} |v - g| dx$, where

$$g(x) = \begin{cases} -2 & \text{if } x < 0, \\ 2 & \text{if } x \ge 0. \end{cases} \tag{5.5}$$

We want to modify this example for the functional

$$A_w(v) = \mathcal{L}(v,(-2,2)) + \int_{-2}^{2} |v - w|^2 dx, \tag{5.6}$$

where we choose w as

$$w(x) = \begin{cases} -3 & \text{if } x < 0, \\ 3 & \text{if } x \ge 0. \end{cases}$$

Theorem 5.2. *The unique minimizer u of (5.6) is convex on $(-2,0)$, concave on $(0,2)$, and has a jump at $x = 0$ of amplitude $u^+(0) - u^-(0) \ge 2$.*

Proof. The uniqueness of the minimizer is guaranteed by the strict convexity of the functional. We will now show some properties of u:
Step 1: the function u satisfies $-2 \le u(x) \le 2$ for a.e. x and u is non-decreasing. The first property follows from the fact that, if not, $(-2) \vee u \wedge 2$ would have smaller energy than u, contradicting the minimality. For the second one, assume that $0 \le a < b \le 2$ and $u(a) > u(b)$; then the function

$$\widetilde{u}(x) := \begin{cases} u(x) & \text{for } x \le a, \\ u(x) \vee u(a) & \text{for } x > a, \end{cases}$$

would have smaller energy than u, again a contradiction. Similarly we can show that u is non-decreasing in $(-2,0)$. Observing that the previous argument also works for $u(a) = \lim_{x\to 0^-} u(x)$, we deduce that u is non-decreasing in the whole domain.
Step 2: u is convex on $(-2,0)$ and concave on $(0,2)$. Let us show the first assertion, and assume by contradiction that for some Lebesgue points a, b, y with $0 \le a < y < b \le 2$, there holds

$$u(y) < u(a) + (y - a)\frac{u(b) - u(a)}{b - a}.$$

Then it is easily seen that

$$\widetilde{u}(x) := \begin{cases} u(x) & \text{for } x \notin (a,b), \\ u(x) \vee \left(u(a) + (x-a)\frac{u(b)-u(a)}{b-a}\right) & \text{for } x \in (a,b), \end{cases}$$

provides a minimizer better than u, absurd.
Step 3: it holds

$$\lim_{x\to 2^-} u(x) \ge 2, \qquad \lim_{x\to -2^+} u(x) \le -2.$$

Again, we prove the first inequality by contradiction (the second being similar) and assume that $\ell := \lim_{x\to 2^-} u(x) < 2$. Then we replace u by

$$\widetilde{u}(x) := \begin{cases} u(x) & \text{for } x < 0, \\ u(x) + (2-\ell) & \text{for } x \in [0,2), \end{cases}$$

and the energy of $\widetilde{u}$ will satisfy

$$\begin{aligned} A_w(\widetilde{u}) - A_w(u) &= (2-\ell) + \int_0^2 |u-3+2-\ell|^2 - |u-3|^2 dx \\ &= (2-\ell) + \int_0^2 (2-\ell)(2u-4-\ell)dx \\ &\leq (2-\ell) + \int_0^2 (2-\ell)(\ell-4)dx \\ &< (2-\ell) - 4(2-\ell) < 0, \end{aligned}$$

where in the first inequality we have used that $u \leq \ell$ and in the last but one that $\ell < 2$. This is a contradiction with the minimality of u.
Step 4: it holds

$$\lim_{x\to 0^+} u(x) \geq 1, \qquad \lim_{x\to 0^-} u(x) \leq -1.$$

Let $\ell := \lim_{x\to 2^-} u(x)$ assume that $\lim_{x\to 0^+} u(x) < 1$. This means that there exists $y > 0$ such that $u(y) < 1$; then we define

$$\widetilde{u}(x) := \begin{cases} u(x) & \text{for } x < 0, \\ u(x+y) & \text{for } x \in [0, 2-y), \\ \ell & \text{for } x \in [2-y, 2). \end{cases}$$

We estimate

$$\begin{aligned} A_w(\widetilde{u}) - A_w(u) &\leq y + \int_{2-y}^2 |\ell-3|^2 dx - \int_0^y |u-3|^2 dx \\ &\leq y + y|\ell-3|^2 - 4y \leq 2y - 4y < 0, \end{aligned}$$

where in the last but one inequality we have used that $\ell \geq 2$. This leads to a contradiction and the thesis follows. □

Acknowledgements: VL and RS are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). RS joins the project 2025 CUP E5324001950001. We also acknowledge the partial financial support of PRIN 2022PJ9EFL "Geometric Measure Theory: Structure of Singular Measures, Regularity Theory and Applications in the Calculus of Variations". The latter has been funded by the European Union under NextGenerationEU. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or The European Research Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

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