GRADIENT REGULARITY FOR A CLASS OF NON-AUTONOMOUS FUNCTIONALS WITH DINI OR NON-DINI CONTINUOUS COEFFICIENTS

PAOLO BARONI AND ALESSANDRA COSCIA

Abstract. We prove C^1 regularity for local vectorial minimizers of the non-autonomous functional

$$w \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N) \longmapsto \int_\Omega b(x)\big[|Dw|^p + a(x)|Dw|^p\log(e+|Dw|)\big]\,dx\,,$$

with Ω open subset of $\mathbb{R}^n$, $n \geq 2$, $p > 1$, $0 \leq a(\cdot) \leq \|a\|_{L^\infty(\Omega)} < \infty$ and $0 < \nu \leq b(\cdot) \leq L$. The result is obtained provided that the function $a(\cdot)$ is log-Dini continuous and that the coefficient $b(\cdot)$ is Dini continuous or it is weakly differentiable and its gradient locally belongs to the Lorentz space $L^{n,1}(\Omega;\mathbb{R}^n)$.

Keywords: non-autonomous functionals, gradient continuity, Dini continuous coefficients.

1. Introduction and results

A recent and important object of investigation in the Calculus of Variations is the study of the regularity of minimizers of non-autonomous functionals of the type

$$u \in W^{1,1}_{\mathrm{loc}}(\Omega) \longmapsto \int_\Omega f(x,u,Du)\,dx\,, \tag{1.1}$$

being Ω a bounded domain in $\mathbb{R}^n$ and $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ a Carathéodory function satisfying unbalanced growth conditions of (p,q)-type:

$$\frac{1}{c}\,|\xi|^p \leq f(x,u,\xi) \leq c\big(1+|\xi|^q\big) \qquad 1 < p < q\,, \tag{1.2}$$

for almost every $x \in \Omega$ and all $(u,\xi) \in \mathbb{R}\times\mathbb{R}^n$, with $c \geq 1$. General energies of this type (i.e., when no structural assumptions are made) were initially introduced and studied by Marcellini [41, 42, 43] in the late eighties but the theory has seen deep and substantial contributions in the recent years, see for instance the Schauder estimates in [29, 30], the series of regularity results under sharp bounds on p and q by Bella and Schäffner [8, 9, 35, 45], the interesting approach developed in [7] also covering more general cases and the new boundary regularity results in [11, 31].

Starting from [18, 19, 20], a substantial amount of work has been put in the study of a special structure having (p,q)-growth, the so-called double phase structure, where the peculiar form of the energy has allowed a much more precise, clean and careful analysis of several aspects of regularity. We refer with double phase structure the fact that the energy density has the form

$$f(x,u,\xi) \approx |\xi|^p + a(x)|\xi|^q \tag{1.3}$$

with p and q as in (1.2) and $a(\cdot) \geq 0$ continuous and sufficiently regular to compensate the non-uniform ellipticity of the functional [28]; see [6] for a regularity theory for local minimizers in a general framework, [25, 26] for interesting borderline cases, [27] for extension to manifold-valued minimizers, [44] for ω-minimizers, [22, 24] for the fully nonlinear counterpart of the theory and [33, 34] for far-reaching extensions of such structures.

The object of study of this paper is the borderline version of the double phase functional (1.1)-(1.3) that was introduced in [4]:

$$
\begin{aligned}
(1.4)\quad u \in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N) \longmapsto \mathcal{P}(u,\Omega) &:= \int_\Omega b(x)H(x,Du)\,dx \\
&= \int_\Omega b(x)\Big[|Du|^p + a(x)|Du|^p\log(e+|Du|)\Big]\,dx
\end{aligned}
$$

for $p > 1, N \geq 1$, and the aspect of regularity we are interested in is the plain continuity of the gradient of local minimizers. The basic assumptions we impose on $a(\cdot), b(\cdot)$ are as follows: $a : \Omega \to \mathbb{R}$ continuous and $b : \Omega \to \mathbb{R}$ measurable with

$$
(1.5)\qquad 0 \leq a(x) \leq \|a\|_{L^\infty} < +\infty, \quad 0 < \nu \leq b(x) \leq L < +\infty \qquad \text{for a.e. } x \in \Omega.
$$

Due to the soft non-uniform ellipticity of the energy in (1.4), the regularity of minimizers is strictly entangled with the behavior of the quantity

$$
(1.6)\qquad \omega_{\log}(R) = \omega_a(R)\log\left(\frac{1}{R}\right)
$$

where $\omega_a : [0,1] \to [0, 2\|a\|_{L^\infty}]$ is a modulus of continuity for $a(\cdot)$: a concave (thus continuous in $(0,1)$) and increasing function, continuous in zero and with $\omega_a(0) = 0$, such that

$$
(1.7)\qquad |a(x) - a(y)| \leq \omega_a(|x-y|) \qquad \text{for all } x, y \in \Omega \text{ with } |x-y| \leq 1\,.
$$

We define in a totally analogous way $\omega_b : [0,1] \to [0,2]$ as the modulus of continuity of $b(\cdot)$, if b is (uniformly) continuous (note that all the results we are mentioning and proving are local; it is therefore not restrictive to assume that if b is continuous, then it has a modulus of continuity):

$$
(1.8)\qquad |b(x) - b(y)| \leq L\,\omega_b(|x-y|) \qquad \text{for all } x, y \in \Omega \text{ with } |x-y| \leq 1\,.
$$

More in detail, schematically summarizing the results in [4, 5, 21], one has that scalar local minimizer to (1.4) (i.e., $N = 1$) in the case (1.5) holds are such that

- if $\limsup_{\rho\searrow 0}\omega_{\log}(R) < \infty$, then
$$
u \in C^{0,\alpha}_{\rm loc}(\Omega), \qquad Du \in L^{p(1+\delta_0)}_{\rm loc}(\Omega;\mathbb{R}^n)
$$
for some constants $\alpha, \delta_0 \in (0,1)$ depending on the data and the Harnack's inequality holds for positive solutions;
- if $\limsup\limits_{R\searrow 0}\omega_{\log}(R) = 0$ and $b(\cdot)$ is continuous (i.e., it has a modulus of continuity), then
$$
u \in C^{0,\alpha}_{\rm loc}(\Omega) \qquad \text{for every } \alpha \in (0,1);
$$
- if $\omega_{\log}(R) + \omega_b(R) \leq c\,R^\gamma$ for some $\gamma \in (0,1)$ and $c \geq 1$, then
$$
u \in C^{1,\beta}_{\rm loc}(\Omega)
$$
for some exponent $\beta \in (0,1)$ depending on the data.

Note that some of the previous results have also a vectorial counterpart, but some do not. Other noticeable results for minimizers of the functionals in (1.4) and measure data problems associated to its Euler equation can be found in [12, 13, 14] by Byun, Youn and collaborators; see also [1] for a significant counterexample. A natural borderline question would be regarding the assumptions to be imposed on $b(\cdot), a(\cdot)$ in order to ensure gradient continuity for local minimizers and a first answer is given in the next theorem:

Theorem 1.1. *Let $u \in W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^N)$, $N \geq 1$, a local minimizer of* (1.4) *under the assumption* (1.5)*; suppose moreover that $b(\cdot)$ is Dini continuous and that $a(\cdot)$ is* log*-Dini, in the sense, respectively, that both $\omega_b(\rho)$ and $\omega_{\log}(\rho)$ are integrable in zero with respect to the measure $d\rho/\rho$:*

$$\int_0^1 \omega_b(\rho)\,\frac{d\rho}{\rho} < \infty, \qquad \int_0^1 \omega_{\log}(\rho)\,\frac{d\rho}{\rho} = \int_0^1 \omega_a(\rho)\log\left(\frac{1}{\rho}\right)\frac{d\rho}{\rho} < \infty\,. \tag{1.9}$$

Du is continuous in Ω and the following local boundedness estimate holds: for every $K \Subset \Omega$, there exists a radius R_0 depending on $n, N, p, L/\nu, \omega_b(\cdot), \omega_a(\cdot), \operatorname{dist}(K,\partial\Omega)$ and $\|H(\cdot,Du)\|_{L^1(K)}$ such that for almost every $x_0 \in K$ and each ball $B_R(x_0) \subset K$ with radius $R \in (0, R_0]$ it holds

$$H(x_0, Du(x_0)) \leq c(n,N,p,L/\nu) \fint_{B_R(x_0)} H(x, Du(x))\,dx\,. \tag{1.10}$$

Note that the scalar case of the previous result is a consequence of [13, Theorem 1.2] and the fact that an energy, weak solution to the Euler equation of $\mathcal{P}$ is also a SOLA; the extension of the results in [13] to the vectorial case is completely non-trivial and therefore we decided to present and prove Theorem 1.1, which follows without effort from the approach we developed to prove the forthcoming Theorem 1.2. Dini continuity of the coefficient is a natural assumption ensuring gradient continuity and pointwise potential estimates whose best possible consequence is gradient continuity, see [13, 38, 39] and the counterexample in [36]; the log-Dini continuity of $a(\cdot)$ is also quite natural in view of the fact that, at least heuristically, it is simply needed a logarithmic correction on the assumption on $b(\cdot)$ for the regularity of $a(\cdot)$ [13]; see also [3] for a declination of this principle from the point of view of Sobolev-like assumptions.

Maybe more interesting and less expected of the previous one is the forthcoming Theorem 1.2 where we replace the assumption of Dini continuity of $b(\cdot)$ with an assumption of integral-Sobolev type. In particular we shall assume that $b(\cdot)$ is weakly differentiable and that its gradient belongs to the Lorentz space $L^{n,1}$ locally in Ω, which means that for every $K \Subset \Omega$ there holds

$$\int_0^\infty \big|\{x \in K : |Db(x)| > \lambda\}\big|^{\frac{1}{n}}\,d\lambda < \infty\,.$$

We emphasize that a by-now classic Sobolev-type sharp embedding in Lorentz spaces ensures that in this case $b(\cdot)$ is continuous but not Dini continuous; see [17, Remark 3.6]. Therefore, Theorem 1.1 does not follow by embedding from

Theorem 1.2. *Let $u \in W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^N)$ a minimizer of* (1.4) *under the assumption* (1.5)*. Suppose that $a(\cdot)$ is* log*-Dini continuous and that $b \in W^{1,1}_{\rm loc}(\Omega)$ with $Db \in L^{n,1}_{\rm loc}(\Omega;\mathbb{R}^n)$; then Du is continuous in Ω and a local boundedness estimate as* (1.10) *holds with the sole difference that R_0 here depends on $|Db(\cdot)|/\nu$ instead of $\omega_b(\cdot)$ (the other dependencies remain unchanged).*

We stress that it is also possible to impose Sobolev-Lorentz-Zygmund type assumptions on the switching coefficient $a(\cdot)$ and this was done in [3] by the first author. In particular in [3] the case without coefficient is treated (i.e., $b \equiv 1$) but gradient continuity of minimizers of (1.4) is proved not supposing $a(\cdot)$ log-Dini continuous, but weakly differentiable with

$$\int_0^\infty \big|\{x \in \Omega : |Da(x)| > \lambda\}\big|^{\frac{1}{n}} \log^+\lambda\,d\lambda < \infty, \qquad (\log^+\lambda = \max\{\log\lambda, 0\})\,,$$

that is, by a nice result by Bennett and Rudnick [10], $Da \in L^{n,1}\log L(\Omega;\mathbb{R}^n)$ (or at least locally in Ω). We believe that it is possible to consider on both $a(\cdot), b(\cdot)$

assumptions of integral type, but this goes beyond the scopes of this work and will be investigated in the future.

The concept allowing to prove gradient continuity under the assumptions of Theorem 1.1 finds its roots in the basic perturbation idea going back to Campanato: we fix a ball $B_R \equiv B_R(x_0) \Subset \Omega$ and we quantify, in integral terms, how much our minimizer is distant from a regular minimizer of a reference variational problem (see (3.3)-(3.4)). This is measured in terms of the quantities $\omega_b(R)$ and $\omega_{\log}(R)$ (see (5.1)). Both the gradient boundedness and its continuity come from a telescopic summation argument and this ultimately boils down to testing the summability of $\omega_b + \omega_{\log}$ along a sequence of radii $R_j = \delta^j R$ for some starting radius $R > 0$ and a very small constant $\delta \ll 1$. A simple computation shows that

$$\sum_{j=0}^{\infty} \omega_b(R_j) \approx \int_0^{2R} \omega_b(\rho)\, \frac{d\rho}{\rho}, \qquad \sum_{j=0}^{\infty} \omega_{\log}(R_j) \approx \int_0^{2R} \omega_{\log}(\rho)\, \frac{d\rho}{\rho}$$

(see (4.2) and (5.2)); it is immediate now to realize why the assumptions of Dini and log-Dini continuity come into play.

Theorem 1.2 is based on a much more refined, and quite new, perturbation argument. Thanks to the use of higher integrability, the aforementioned distance from the regular minimizer is measured, for what concern the coefficient $b(\cdot)$ (the switching coefficient is indeed treated exactly as in Theorem 1.1), in terms of the L^s-excess, for $s \gg 1$ a very large constant:

$$\left(\fint_{B_R} \left| b - (b)_{B_R} \right|^s dx \right)^{1/s}, \qquad (b)_{B_R} = \fint_{B_R} b(y)\, dy\,. \tag{1.11}$$

One has here to realize that in the case $Db \in L^{n,1}$, the dyadic summation of this quantity is bounded, in view of Sobolev-Poincaré's inequality and Lemma 4.1 (which is a simple consequence of the characterization of $L^{n,1}$ in terms of rearrangements): for $q = s_* = ns/(n+s) < n$ as in Lemma 3.2 and $B_j \equiv B_{R_j}(x_0)$

$$\sum_{j=0}^{\infty} \left(\fint_{B_j} \left| b - (b)_{B_j} \right|^s dx \right)^{1/s} \lesssim \sum_{j=0}^{\infty} R_j \left(\fint_{B_j} |Db|^q\, dx \right)^{1/q} \lesssim \|Db\|_{L^{n,1}(B_{2R})}\,.$$

The idea of considering coefficient with controlled excess (in the sense just described) is not totally new: equations and variational problems with VMO coefficients - that is, for which their excess as defined in (1.11) is vanishing as $R \searrow 0^+$, uniformly with respect to the center of the balls - have been largely studied since the seminal works [15, 16]. Much more recent is the observation that one can also consider summability properties of the excess along dyadic sequences of radii beyond its qualitative smallness. In [38] it is proved that systems of p-Laplacian type have C^1_{loc} solutions provided the right-hand side belongs to $L^{n,1}$:

$$\operatorname{div}\Big(|Du|^{p-2} Du \Big) = f, \qquad p > 1\,.$$

This is a consequence, for equations, of the celebrated linear potential estimates in [37] and can also be proved as corollary of the more general vectorial potential estimates of [40]. In any case, this is the affirmative response to a long-standing conjecture by Uraltseva, claiming that the condition on f ensuring Lipschitz regularity for solution should be independent of p. We choose to mention [38] since the telescopic summation technique therein introduced was the main inspiration for the papers [2, 3] of the first author and also for the present one. A further step is the extension of this idea to the coefficients of uniformly elliptic problems, and this can be done (see [2] and the current paper) realizing that, at least by formal differentiation, coefficients can be treated as right-hand sides. In [28, Theorem 1.8] it is shown that solutions to uniformly elliptic vectorial problems of the type

$$\operatorname{div}\Big(b(x)\frac{\varphi'(|Du|)}{|Du|}Du\Big) = f$$

are Lipschitz (and therefore C^1, after computations of standard flavor) regular if both $|f|$ and $|Db|$ belong to $L^{n,1}$; the approach therein is based on Moser's iteration. See [2] again for the two dimensional case and a perturbative approach to the question.

2. Preliminaries

2.1. **Local and global minimizers.** A *local minimizer* to $\mathcal{P}$ is a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, $N \geq 1$ such that $H(\cdot, Du) \in L^1_{\mathrm{loc}}(\Omega)$ and the minimality condition

$$\mathcal{P}(u, \operatorname{supp}(u-v)) \leq \mathcal{P}(v, \operatorname{supp}(u-v))$$

is satisfied whenever $v \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ is such that $\operatorname{supp}(u-v) \Subset \Omega$.

Due to the local nature of our results, we may assume that Ω is a bounded domain and that local minimizers u belong to $W^{1,p}(\Omega;\mathbb{R}^N)$ with $H(\cdot, Du) \in L^1(\Omega)$ too. As a consequence, we will write $\|Du\|_{L^p}$ for $\|Du\|_{L^p(\Omega;\mathbb{R}^N)}$ and $\|H(\cdot, Du)\|_{L^1}$ for $\|H(\cdot, Du)\|_{L^1(\Omega)}$. The minor changes that lead from this situation to the case described in Theorems 1.1-1.2 are left to the reader.

2.2. **Notation and elementary properties.** In what follows we denote by c a general positive constant possibly varying from line to line; special occurrences will be denoted by $c_0, \bar{c}$, etc. All such constants will always be *larger or equal than one*; relevant dependencies on parameters will be highlighted using parentheses, i.e., $c \equiv c(n,p,\delta)$ will mean that c depends on n, p and δ. We denote by

$$B_r(x_0) := \{x \in \mathbb{R}^n \,:\, |x - x_0| < r\}$$

the open ball with center x_0 and radius $r > 0$; when not important, or clear from the context, we shall omit denoting the center writing $B_r \equiv B_r(x_0)$. Unless otherwise stated, different balls in the same context will have the same center. With $\mathcal{B} \subset \mathbb{R}^n$ being a measurable set with positive, finite measure $|\mathcal{B}| > 0$, and with $g\colon \mathcal{B} \to \mathbb{R}^\ell$, $\ell \geq 1$, being a locally integrable map, we shall denote by

$$(g)_{\mathcal{B}} := \fint_{\mathcal{B}} g(x)\,dx = \frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} g(x)\,dx$$

its integral average. A well-known property is the following: for any $g \in L^p(\mathcal{B};\mathbb{R}^\ell)$, $p \geq 1, \ell \geq 1$, the estimate

$$\fint_{\mathcal{B}} |g(x) - (g)_{\mathcal{B}}|^p\,dx \leq 2^p \fint_{\mathcal{B}} |g(x) - \zeta|^p\,dx \tag{2.1}$$

holds for each $\zeta \in \mathbb{R}^\ell$. We use the agreement that $\mathbb{N}$ is the set $\{1,2,3,\dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The Sobolev conjugate exponent p^* is $np/(n-p)$ when $p < n$.

We recall some useful properties of the logarithm function of later frequent use:

$$\log(e + Ax) \leq A\log(e+x) \qquad \text{for every } x \geq 0, A \geq 1\,, \tag{2.2}$$

$$\log(e+xy) \leq \log(e+x) + \log(e+y) \qquad \text{for every } x, y \geq 0\,, \tag{2.3}$$

$$\log(e + x^\sigma) \leq 1 + \max\{1,\sigma\}\log(e+x) \qquad \text{for every } x \geq 0, \sigma > 0\,. \tag{2.4}$$

The proofs are very simple, we only highlight for (2.4) that distinguishing the cases $\sigma < 1$, where $\log(e+x^\sigma) \leq \log(2(e+x)) \leq 1 + \log(e+x)$, and $\sigma \geq 1$, where $\log(e + x^\sigma) \leq \sigma\log(e+x)$, leads to the result.

The following lemma will be very useful in order to treat the logarithmic part of the energy.

Lemma 2.1. *Let $s > 1$, $\sigma, \beta, \theta \geq 0$ and let f be a positive function in $L^s(B_r)$ for some ball $B_r(x_0)$ with radius $r \leq e^{-1}$. Then there exists a constant c depending on $n, \beta, \sigma, \theta, s$ such that*

$$\fint_{B_r} f \log^\beta\left(e + f^\sigma\right) dx \leq c\big(1 + r^\theta \, \|f\|_{L^1(B_r)}\big)^\beta \log^\beta\left(\frac{1}{r}\right)\Big(\fint_{B_r} f^s \, dx\Big)^{1/s}.$$

Proof. Recalling that $r \leq e^{-1}$, by (2.4), (2.3), the forthcoming (2.38), basic properties of the logarithm function (for instance (2.2)) we estimate

$$\begin{aligned}
&\fint_{B_r} f \log^\beta(e + f^\sigma) \, dx \leq c(\sigma,\beta) \fint_{B_r} f \Big[1 + \log\left(e + f\right)\Big]^\beta dx \\
&\leq c(\sigma,\beta) \fint_{B_r} f \Big[1 + \log\Big(e + \frac{f}{(f)_{B_r}}\Big) + \log\left(e + (f)_{B_r}\right)\Big]^\beta dx \\
&\leq c(\sigma,\beta)\Big[\fint_{B_r} f \, dx + \fint_{B_r} f \, \log^\beta\Big(e + \frac{f}{(f)_{B_r}}\Big) dx + \log^\beta\Big(e + \fint_{B_r} f \, dx\Big) \fint_{B_r} f \, dx\Big] \\
&\leq c(n,\sigma,\beta,s)\Big(\fint_{B_r} f^s \, dx\Big)^{1/s}\Big[1 + \log^\beta\Big(e + \fint_{B_r} f \, dx\Big)\Big] \\
&\leq c(n,\sigma,\beta,s)\Big(\fint_{B_r} f^s \, dx\Big)^{1/s}\Big[1 + \log^\beta\Big(e + \frac{1}{r^n}\int_{B_r} f \, dx\Big)\Big] \\
&\leq c(n,\sigma,\beta,s)\Big(\fint_{B_r} f^s \, dx\Big)^{1/s}\Big[1 + \log^\beta\Big(e + (1 + r^\theta \, r^{-\theta} \, \|f\|_{L^1(B_r)})\frac{1}{r^n}\Big)\Big] \\
&\leq c(n,\sigma,\beta,s)(1 + r^\theta \, \|f\|_{L^1(B_r)})^\beta\Big(\fint_{B_r} f^s \, dx\Big)^{1/s}\Big[1 + \log^\beta\Big(e + \frac{1}{r^{n+\theta}}\Big)\Big] \\
&\leq c(n,\sigma,\beta,s)(1 + r^\theta \, \|f\|_{L^1(B_r)})^\beta\Big(\fint_{B_r} f^s \, dx\Big)^{1/s}\Big[\log^\beta\Big(\frac{1}{r}\Big) + \Big[2(n+\theta) \, \log\Big(\frac{1}{r}\Big)\Big]^\beta\Big] \\
&\leq c(n,\sigma,\beta,\theta,s)(1 + r^\theta \, \|f\|_{L^1(B_r)})^\beta \log^\beta\Big(\frac{1}{r}\Big)\Big(\fint_{B_r} f^s \, dx\Big)^{1/s}
\end{aligned}$$

and the proof is complete. □

2.3. **N-functions setting.** In the following we are going to introduce a general class of tools, related to the so-called general class of N-functions. For the results we mention here see for instance [23, 32].

We consider a convex function $\varphi : [0,\infty) \to [0,\infty)$, such that

$$\begin{aligned}
&\varphi \in C^1([0,\infty)) \cap C^2((0,\infty)), \qquad \varphi(0) = \varphi'(0) = 0, \\
&\varphi'(t) \text{ is monotone increasing and } \lim_{t\to\infty} \varphi'(t) = \infty\,.
\end{aligned} \tag{2.5}$$

In addition we assume that there exists a constant $c_\varphi \geq 1$ such that

$$\frac{1}{c_\varphi} \leq \frac{\varphi''(t)t}{\varphi'(t)} \leq c_\varphi, \qquad \text{for all } t > 0\,. \tag{2.6}$$

If the function φ verifies (2.5) and (2.6), then we call φ as an N-function. Notice that every non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ satisfies the following property

$$\varphi(t + s) \leq \varphi(2t) + \varphi(2s)\,. \tag{2.7}$$

We define the auxiliary vector field $V_\varphi \colon \mathbb{R}^\ell \to \mathbb{R}^\ell, \ell \in \mathbb{N}$ by

$$V_\varphi(z) := \left(\frac{\varphi'(|z|)}{|z|}\right)^{1/2} z\,,$$

where V_φ is continuously extended to zero when $z = 0$; V_φ turns out to be a bijection of $\mathbb{R}^\ell$ by (2.5) and under the assumption (2.6) V_φ describes the monotonicity properties of the map $[\varphi'(|z|)/|z|]z$: for $z_1, z_2 \in \mathbb{R}^\ell$, $z_1, z_2 \neq 0$ we have

(2.8)
$$\frac{1}{c}\left|V_\varphi(z_1) - V_\varphi(z_2)\right|^2 \leq \langle \frac{\varphi'(|z_1|)}{|z_1|} z_1 - \frac{\varphi'(|z_2|)}{|z_2|} z_2, z_1 - z_2\rangle \leq c\left|V_\varphi(z_1) - V_\varphi(z_2)\right|^2,$$

for a constant $c \geq 1$ depending on c_φ. For another constant $c \equiv c(c_\varphi)$ the following relations (see [32, Lemma 2.4]) hold for every $z, z_1, z_2 \in \mathbb{R}^\ell$ with z_1 or z_2 different from zero (which means $|z_1| + |z_2| > 0$):

$$\frac{1}{c}\,\varphi(|z|) \leq \left|V_\varphi(z)\right|^2 \leq c\,\varphi(|z|)\,, \tag{2.9}$$

$$\frac{1}{c}\,\varphi''(|z_1|+|z_2|)|z_1 - z_2|^2 \leq \left|V_\varphi(z_1) - V_\varphi(z_2)\right|^2 \leq c\,\varphi''(|z_1|+|z_2|)|z_1 - z_2|^2\,. \tag{2.10}$$

For a constant $\bar a \geq 0$, we are interested in

$$\varphi_p(t) = \frac{t^p}{p}\,, \quad \varphi_{\log}(t) = \frac{t^p}{p}\log(e+t)\,, \quad \text{and} \quad \varphi_{\bar a}(t) = \varphi_p(t) + \bar a\,\varphi_{\log}(t)\,, \tag{2.11}$$

which verify all the assumptions (2.5). In addition for $t > 0$

$$\frac{t\,\varphi_p''(t)}{\varphi_p'(t)} = p - 1\,, \qquad p - 1 \leq \frac{t\,\varphi_{\log}''(t)}{\varphi_{\log}'(t)} \leq 2p\,,$$
$$(p-1)\,\varphi_{\bar a}'(t) \leq t\,\varphi_{\bar a}''(t) \leq 2p\,\varphi_{\bar a}'(t)\,,$$

so (2.6) is satisfied with a constant $c_\varphi = c_\varphi(p)$ depending only on p and independent of $\bar a$. It is also easy to estimate

$$t^{p-1}\log(e+t) \leq \varphi_{\log}'(t) \leq 2\,t^{p-1}\log(e+t)\,, \tag{2.12}$$

$$(p-1)\,t^{p-2}\log(e+t) \leq \varphi_{\log}''(t) \leq (p+1)\,t^{p-2}\log(e+t)\,. \tag{2.13}$$

Adopting the notation

$$h_{\bar a}(t) = t^{p-1} + \bar a\,t^{p-1}\log(e+t) \qquad \text{for every } t \geq 0\,, \tag{2.14}$$

from (2.12) and (2.13) we deduce the following estimates:

$$h_{\bar a}(t) \leq \varphi_{\bar a}'(t) \leq 2\,h_{\bar a}(t)\,, \tag{2.15}$$

$$\frac{1}{c(p)}\,h_{\bar a}(t) \leq t\,\varphi_{\bar a}''(t) \leq c(p)\,h_{\bar a}(t)\,. \tag{2.16}$$

which will be useful in the proof of Lemma 2.4.

In the sequel for $p > 1$, for every $x \in \Omega$ and $z \in \mathbb{R}^\ell$, we adopt the notation

$$H(x,z) := |z|^p + a(x)|z|^p\log(e+|z|) \;=\; p\big[\varphi_p(|z|) \;+\; a(x)\,\varphi_{\log}(|z|)\big]\,, \tag{2.17}$$

$$H_{\bar a}(z) := |z|^p + \bar a\,|z|^p\log(e+|z|) = p\,\varphi_{\bar a}(|z|)\,; \tag{2.18}$$

let us remark that the relations

$$H_{\bar a}(z) = h_{\bar a}(|z|)\,|z|\,, \qquad \partial_z H_{\bar a}(z) = p\,\varphi_{\bar a}'(|z|)\,\frac{z}{|z|} \tag{2.19}$$

hold, where the function $h_{\bar a}(\cdot)$ is defined in (2.14). Let us denote by $V_p(\cdot)$ and $V_{\log}(\cdot)$ the vector fields generated by φ_p and $\varphi_{\log}$, and by

$$V_{\bar a}(z) := \sqrt{\frac{\varphi_{\bar a}'(|z|)}{|z|}}\; z\,, \tag{2.20}$$

the one generated by $\varphi_{\bar{a}}(t)$, all continuously extended to zero when $z = 0$. It is easy to verify that

$$|V_p(z)|^2 = |z|^p = p\,\varphi_p(|z|)\,,$$
$$p\,\varphi_{\log}(|z|) = |z|^p \log(e+|z|) \le |V_{\log}(z)|^2 \le 2\,|z|^p \log(e+|z|) = 2p\,\varphi_{\log}(|z|)\,,$$
$$p\,\varphi_{\bar{a}}(|z|) \le |V_{\bar{a}}(z)|^2 \le 2p\,\varphi_{\bar{a}}(|z|)\,, \tag{2.21}$$

thus in all cases (2.9) holds for a constant c depending only on p. In particular, if $a(\cdot)$ is a function satisfying (1.5), from (2.21) it follows that for every $x_0 \in \Omega$

$$H(x_0, z) \le \left|V_{a(x_0)}(z)\right|^2 \le 2\,H(x_0, z)\,. \tag{2.22}$$

In addition, since $\varphi_p''(t) = (p-1)t^{p-2}$ the estimate in (2.10), which holds whenever $z_1, z_2 \in \mathbb{R}^\ell$, $|z_1| + |z_2| > 0$, becomes:

$$\frac{1}{c}|z_1 - z_2|^2\,(|z_1| + |z_2|)^{p-2} \le |V_p(z_1) - V_p(z_2)|^2 \le c|z_1 - z_2|^2\,(|z_1| + |z_2|)^{p-2} \tag{2.23}$$

where c depends only on p. In particular, when $p \ge 2$,

$$|z_1 - z_2|^p \le c|V_p(z_1) - V_p(z_2)|^2$$

holds, while for $1 < p \le 2$ (see [38, Lemma 2]) we will use that

$$|z_1 - z_2| \le c\left|V_p(z_1) - V_p(z_2)\right|^{2/p} + c|z_1|^{(2-p)/2}\left|V_p(z_1) - V_p(z_2)\right| \tag{2.24}$$

both for a suitable constant $c \equiv c(p)$. Finally, let us remark that combining (2.10) for $V_p(\cdot)$ and $V_{\bar{a}}(\cdot)$ it is easy to verify that for every $z_1, z_2 \in \mathbb{R}^\ell$

$$\left|V_p(z_1) - V_p(z_2)\right|^2 \le c(p)\left|V_{\bar{a}}(z_1) - V_{\bar{a}}(z_2)\right|^2. \tag{2.25}$$

2.4. **Preliminary results.** First of all, let us prove that log-Dini continuity (1.9) of $a(\cdot)$ guarantees that $a(\cdot)$ is log-Hölder vanishing (that is, (2.41) holds); later this will allow the application of Theorem 2.7 to the minimizers of $\mathcal{P}$ in (1.4).

Lemma 2.2. *Let $a(\cdot)$ be a function satisfying (1.5) and let $\omega_a(\cdot)$ be a modulus of continuity of $a(\cdot)$ as in (1.7). If*

$$\int_0^1 \omega_a(\rho)\log\big(\frac{1}{\rho}\big)\,\frac{d\rho}{\rho} < \infty\,, \tag{2.26}$$

then $\lim_{R\searrow 0^+} \omega_a(R)\log(1/R) = 0$.

Proof. It suffices to prove that $\limsup_{R\searrow 0^+} \omega_a(R)\log\big(1/R\big) = 0$. Reasoning by contradiction, let us assume that there exists a decreasing sequence $\{R_k\} \subset (0, 1/2]$ such that $R \searrow 0^+$ and

$$\lim_{k\to+\infty} \omega_a(R_k)\log\big(\frac{1}{R_k}\big) = l \in (0, +\infty]\,; \tag{2.27}$$

we assume for now that l is finite. Up to a subsequence, we may assume in addition that for every k

$$\omega_a(R_k)\log\big(\frac{1}{R_k}\big) \ge \frac{l}{2} \qquad \text{and} \qquad \lim_{k\to+\infty}\frac{R_{k+1}}{R_k} = 0\,. \tag{2.28}$$

We want to prove that the integral (2.26) is divergent at 0. We can estimate the improper integral (2.26) with a series of integrals on $[R_{k+1}, R_k]$:

$$\int_0^1 \omega_a(\rho)\log\big(\frac{1}{\rho}\big)\,\frac{d\rho}{\rho} \ge \sum_{k=0}^{\infty}\int_{R_{k+1}}^{R_k} \omega_a(\rho)\log\big(\frac{1}{\rho}\big)\,\frac{d\rho}{\rho}\,. \tag{2.29}$$

Using the monotonicity of both $\omega_a(R)/R$ and $\log(1/R)$ and (2.28), we estimate

$$\sum_{k=0}^{\infty}\int_{R_{k+1}}^{R_k}\frac{\omega_a(\rho)}{\rho}\log\big(\frac{1}{\rho}\big)\,d\rho \geq \sum_{k=0}^{\infty}\frac{\omega_a(R_k)}{R_k}\log\big(\frac{1}{R_k}\big)\int_{R_{k+1}}^{R_k}d\rho \geq \frac{l}{2}\sum_{k=0}^{\infty}\big(1-\frac{R_{k+1}}{R_k}\big)\,.$$

By (2.28) the series with non-negative terms $\displaystyle\sum_{k=0}^{\infty}\big(1-\frac{R_{k+1}}{R_k}\big)$ is divergent, thus by (2.29) the integral (2.26) is divergent at 0, against the assumption of the Lemma. Finally, if the limit in (2.27) is $l=+\infty$, fixed $T>0$ it suffices to replace $l/2$ in (2.28) with T to gain the same conclusion with the same calculations. □

First, we need the following estimates on the function $H_{\bar a}(\cdot)$, defined in (2.18) and on its gradient. For the proof see [21, (2.44),(2.45) and Lemma 2.2], where all estimates are proved for the function $H(\cdot,\cdot)$ defined in (2.17), of which the function $H_{\bar a}(\cdot)$ is a particular case.

Lemma 2.3. *Let $H_{\bar a}\colon\mathbb{R}^\ell\to\mathbb{R}$ be the function defined in* (2.18)*. Then the following estimates hold for every $A\geq 1$ and every $z,\lambda\in\mathbb{R}^\ell$:*

$$H_{\bar a}(Az)\leq A^{p+1}H_{\bar a}(z)\,, \tag{2.30}$$

$$H_{\bar a}(z_1\pm z_2)\leq 2^{p+1}(\,H_{\bar a}(z_1)+H_{\bar a}(z_2))\,, \tag{2.31}$$

$$\big|\langle\partial_z H_{\bar a}(z),\lambda\rangle\big|\leq (p+1)\,\big(H_{\bar a}(z)\,+\,H_{\bar a}(\lambda)\big)\,. \tag{2.32}$$

We collect in next lemma some monotonicity and growth results that link the vector field $\partial_z H_{\bar a}$ and the related nonlinear expression $V_{\bar a}$; these will be particularly useful in order to prove the comparison Lemma 3.2.

Lemma 2.4. *Let $\ell\in\mathbb{N}$ and let $V_{\bar a}\colon\mathbb{R}^\ell\to\mathbb{R}^\ell$, $H_{\bar a}\colon\mathbb{R}^\ell\to\mathbb{R}$, and $h_{\bar a}\colon[0,+\infty)\to\mathbb{R}$ be the functions defined in* (2.20)*,* (2.18)*, and* (2.14) *respectively. Then the following estimates hold for every $z,z_1,z_2\in\mathbb{R}^\ell$:*

$$\frac{1}{c(p)}\,\big|V_{\bar a}(z_1)-V_{\bar a}(z_2)\big|^2\leq\langle\partial_z H_{\bar a}(z_1)-\partial_z H_{\bar a}(z_2)\,,z_1-z_2\rangle\,, \tag{2.33}$$

$$|\partial_z H_{\bar a}(z)|\leq c(p)h_{\bar a}(|z|)\,, \tag{2.34}$$

$$\big|V_{\bar a}(z_1)-V_{\bar a}(z_2)\big|^2\geq\frac{1}{c(p)}\,\frac{h_{\bar a}(|z_1|+|z_2|)}{|z_1|+|z_2|}|z_1-z_2|^2\,, \tag{2.35}$$

$$(|z_1|+|z_2|)\,h_{\bar a}(|z_1|+|z_2|)\leq 2^{p+1}\big(H_{\bar a}(z_1)+\,H_{\bar a}(z_2)\big)\,, \tag{2.36}$$

$$|z_1-z_2|\,h_{\bar a}(|z_1|)\leq c(p)\,\big|V_{\bar a}(z_1)-V_{\bar a}(z_2)\big|\Big(\sqrt{H_{\bar a}(z_1)}\,+\,\sqrt{H_{\bar a}(z_2)}\Big)\,, \tag{2.37}$$

under the further assumption $|z_1|+|z_2|>0$ in (2.35) *and* (2.37)*.*

Proof. Inequality (2.33) follows immediately by (2.19), rewriting (2.8) for the vector field $V_{\bar a}(\cdot)$. Again by (2.19), we get $|\partial_z H_{\bar a}(z)|=p\,\varphi'_{\bar a}(|z|)$ and inequality (2.34) follows from (2.15). Inequality (2.35) can be obtained rewriting (2.10) for $V_{\bar a}(\cdot)$ together with (2.16). In order to prove (2.36), recalling (2.14), (2.11), (2.7) and (2.30), we calculate

$$\begin{aligned}(|z_1|+|z_2|)h_{\bar a}(|z_1|+|z_2|)&=p\,\varphi_{\bar a}(\,|z_1|+|z_2|\,)\leq p\big[\varphi_{\bar a}(2|z_1|)+\varphi_{\bar a}(2|z_2|)\big]\\&=H_{\bar a}(2z_1)\,+\,H_{\bar a}(2z_2)\leq 2^{p+1}\Big[H_{\bar a}(z_1)+H_{\bar a}(z_2)\Big]\,.\end{aligned}$$

Finally, let us prove inequality (2.37). Let us fix $z_1,z_2\in\mathbb{R}^\ell$ with $|z_1|+|z_2|>0$, assuming without loss of generality that $z_1\neq 0$; by (2.35), the monotonicity of $h_{\bar a}(t)$, the monotonicity and the subadditivity of the function $\sqrt{t}$ on $[0,+\infty[$, we estimate

$$
\begin{aligned}
|z_1 - z_2| &= |z_1 - z_2| \sqrt{\frac{h_{\bar{a}}(\,|z_1| + |z_2|\,)}{|z_1| + |z_2|}} \sqrt{\frac{|z_1| + |z_2|}{h_{\bar{a}}(\,|z_1| + |z_2|\,)}} \\
&\le c(p)\,\big|V_{\bar{a}}(z_1) - V_{\bar{a}}(z_2)\big| \sqrt{\frac{|z_1| + |z_2|}{h_{\bar{a}}(\,|z_1| + |z_2|\,)}} \\
&\le c(p)\,\big|V_{\bar{a}}(z_1) - V_{\bar{a}}(z_2)\big| \sqrt{\frac{|z_2 - z_1| + 2|z_1|}{h_{\bar{a}}(|z_1|)}} \\
&\le c(p)\,\big|V_{\bar{a}}(z_1) - V_{\bar{a}}(z_2)\big| \left[\sqrt{\frac{|z_1 - z_2|}{h_{\bar{a}}(|z_1|)}} + \sqrt{\frac{|z_1|}{h_{\bar{a}}(|z_1|)}}\,\right].
\end{aligned}
$$

Therefore, by (2.19) we obtain that

$$
\begin{aligned}
|z_1 - z_2|\, h_{\bar{a}}(|z_1|) &\le c(p)\big|V_{\bar{a}}(z_1) - V_{\bar{a}}(z_2)\big| \left[\sqrt{h_{\bar{a}}(|z_1|)\,|z_1 - z_2|} + \sqrt{H_{\bar{a}}(z_1)}\right] \\
&\le c(p)\left[\big|V_{\bar{a}}(z_1) - V_{\bar{a}}(z_2)\big|\sqrt{H_{\bar{a}}(z_1)} + \sqrt{h_{\bar{a}}(|z_1|)\,|z_2|}\right].
\end{aligned}
$$

Using again monotonicity, (2.36), and subadditivity we can estimate

$$
\sqrt{h_{\bar{a}}(|z_1|)\,|z_2|} \le \sqrt{\big(|z_1| + |z_2|\big)\, h_{\bar{a}}(|z_1| + |z_2|)} \le c(p)\Big(\sqrt{H_{\bar{a}}(z_1)} + \sqrt{H_{\bar{a}}(z_2)}\Big)
$$

and the proof of (2.37) is complete. □

The following lemma will be useful to prove an excess-like decay estimate for our minimizer u, see Proposition 3.6.

Lemma 2.5. *For every $a_1, a_2 \ge 0$ and every $z \in \mathbb{R}^\ell, \ell \in \mathbb{N}$ the following estimate holds:*

$$
\big|V_{a_1}(z) - V_{a_2}(z)\big| \le |a_1 - a_2||z|^{p/2} \log(e + |z|)\,.
$$

Proof. By the definition of the vector fields V_{a_1} and V_{a_2}, using the Lipschitz regularity (with Lipschitz constant 1/2) of the function $t \in [0, +\infty) \to \sqrt{1+t}$, we estimate

$$
\begin{aligned}
|V_{a_1}(z) - V_{a_2}(z)| &= \Bigg|\sqrt{\frac{\varphi'_{a_1}(|z|)}{|z|}}\; z - \sqrt{\frac{\varphi'_{a_2}(|z|)}{|z|}}\; z\Bigg| \\
&= |z|^{(p-1)/2}|z|^{1/2}\,\Bigg|\sqrt{1 + a_1\, \log(e + |z|) + \frac{a_1}{p}\frac{|z|}{e + |z|}} \\
&\qquad\qquad - \sqrt{1 + a_2\, \log(e + |z|) + \frac{a_2}{p}\frac{|z|}{e + |z|}}\;\Bigg| \\
&\le \frac{1}{2}\,|z|^{p/2}\,|a_1 - a_2|\Big[\,\log(e + |z|) + \frac{1}{p}\,\frac{|z|}{(e + |z|)}\,\Big] \\
&\le |z|^{p/2}\,|a_1 - a_2|\log(e + |z|)\,.
\end{aligned}
$$

□

We conclude this paragraph by recalling the following approximation lemma (see [21, Lemma 3.2 and Remark 3.4]) which will be useful to specify the class of admissible test functions in the weak formulation of the Euler-Lagrange equation (Remark 3.3).

Lemma 2.6. *Let us consider a ball $B = B_R(x_0) \Subset \Omega$ and $\phi \in W^{1,p}_0(B_R;\mathbb{R}^N)$ a function such that $H(x, D\phi) \in L^1(B_R)$, with H defined in* (2.17) *and the modulus of continuity $\omega_a(\cdot)$ of the function $a(\cdot)$ satisfying $\lim_{R\searrow 0^+} \omega_a(R)\log(1/R) < +\infty$.*

Then there exists a sequence $\{\phi_k\} \subset C^\infty_0(B_R;\mathbb{R}^N)$ such that $D\phi_k \to D\phi$ a.e., $\phi_k \to \phi$ strongly in $W^{1,p}(B_R;\mathbb{R}^N)$ and

$$H(x, D\phi_k) \to H(x, D\phi) \qquad \textit{strongly in } L^1(B_R)\,.$$

2.5. **$L^p \log^\gamma L$ spaces.** Given an open bounded set $\Omega \subset \mathbb{R}^n$ and a Young function $\varphi : [0,\infty) \to [0,\infty)$ (φ is convex, strictly monotone increasing, $\lim_{t\to 0} \frac{\varphi(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$) the Orlicz space $L^\varphi(\Omega;\mathbb{R}^\ell), \ell \in \mathbb{N}$ is the set of measurable maps $f : \Omega \to \mathbb{R}^\ell$ such that $\int_\Omega \varphi(\lambda|f(x)|)\,dx < \infty$ for some $\lambda > 0$. When $\varphi(t) = t^p/p$, the previous quantity defines an averaged L^p norm; when $\varphi(t) = (t^p/p)\log^\gamma(e+t)$, for $p > 1$, $\gamma \in \mathbb{R}$ or $p = 1$ and $\gamma \ge 0$, the Orlicz space $L^\varphi(\Omega;\mathbb{R}^\ell)$ is denoted by $L^p \log^\gamma L(\Omega;\mathbb{R}^\ell)$ and it consists of the measurable functions such that

$$\int_\Omega |f|^p \log^\gamma(e + |f|)\,dx < \infty\,.$$

In particular, we want to stress here that a classic inequality shows that for every $q > 1$ and $f \in L^q(\Omega;\mathbb{R}^\ell)$ we have

$$\fint_\Omega |f| \log^\gamma \Big(e + \frac{|f|}{(|f|)_\Omega}\Big)\,dx \le c(n,\ell,\gamma,q)\Big(\fint_\Omega |f|^q\,dx\Big)^{1/q} \tag{2.38}$$

(see [2, 3, 13]); being the quantity on the left-hand side equivalent to the Luxemburg norm in $L\log^\gamma L$, as a consequence of (2.38) we deduce that

$$f \in L^q(\Omega;\mathbb{R}^\ell)\,,\, q > p \quad \Longrightarrow \quad f \in L^p \log L(\Omega;\mathbb{R}^\ell)\,. \tag{2.39}$$

2.6. **Basic estimates for minimizers of $\mathcal{P}$.** In this paragraph we describe a higher integrability result available for local minimizers of $\mathcal{P}$ that holds under the weak assumption

$$\limsup_{R\searrow 0^+} \omega_a(R)\log\Big(\frac{1}{R}\Big) = \limsup_{R\searrow 0^+} \omega_{\log}(R) < \infty \ : \tag{2.40}$$

by Lemma 2.2 our assumption of log-Dini continuity on $a(\cdot)$ ensures the stronger

$$\lim_{R\searrow 0^+} \omega_a(R)\log\Big(\frac{1}{R}\Big) = 0\,, \tag{2.41}$$

so we can assume that there exists a threshold $\bar{R} > 0$ such that

$$\sup_{R\in(0,\bar{R}]} \omega_a(R)\log\Big(\frac{1}{R}\Big) \le 1\,. \tag{2.42}$$

As a consequence of (1.5) and (2.40) one has that local minimizers of $\mathcal{P}$ are higher integrable, that is $H(\cdot, Du)$ belongs to Lebesgue's space smaller than L^1 . The fact in (2.41) allows us to state such result in a slightly simpler form; in particular allows to get rid of the dependence of the constants on the energy and this will simplify later our approach.

Theorem 2.7 (Gradient's higher integrability)**.** *Let $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{P}$ defined in* (1.4) *and suppose that* (1.5) *and* (2.41) *hold true. Then there exists a positive integrability exponent $\bar{\delta}_g > 0$, depending only on n, N, p and L/ν such that $H(\cdot, Du) \in L^{1+\bar{\delta}_g}_{\mathrm{loc}}(\Omega)$.*

Moreover if $M \ge 1$ is such that

$$M \ge \|Du\|_{L^p} \tag{2.43}$$

then there exists a threshold

$$R_0 = \min\left\{\frac{1}{e+M}, \bar{R}\right\} \tag{2.44}$$

such that if $r \le R_0$ *and* $B_r(x_0) \subset \Omega$ *the reverse Hölder's inequality*

$$\left(\fint_{B_{\vartheta r}(x_0)} [H(x,Du)]^{1+\delta}\,dx\right)^{\frac{1}{1+\delta}} \le c\fint_{B_r(x_0)} H(x,Du)\,dx \tag{2.45}$$

holds true for every $\vartheta \in [1/2, 3/4]$ *and every* $\delta \in [0, \bar{\delta}_g]$, *for constant* c *depending only on* n, N, p *and* L/ν.

Proof. We outline the changes the proof in [4] requires, keeping the notation here employed. The starting point for showing (2.45) is the proof of a reverse Hölder inequality

$$\fint_{B_{r/2}(\bar{x})} H(x,Du)\,dx \le \tilde{c}\Big(\fint_{B_r(\bar{x})} [H(x,Du)]^d\,dx\Big)^{1/d} \tag{2.46}$$

for every ball $B_r(\bar{x}) \subset \Omega$ with $r \le e^{-1}$ and with the exponent $d \in (0,1)$ depending on $n, N, p, L/\nu$ and the constant $\tilde{c}$ depending on $n, N, p, L/\nu, \tilde{L}$ and $\|Du\|_{L^p}$, see (4.11) in [4]. However, the dependence of $\tilde{c}$ on $\tilde{L}$ can be avoided taking radii smaller than $\bar{R}$ (see (2.42) and compare with [4, Equation (4.5)]) and the dependence on $\|Du\|_{L^p}$ in [4] only comes from the estimate (4.13), in particular when estimating

$$\log\left(e + \frac{\|Du\|_{L^p}^p}{r^n}\right) \le \big(1 + \|Du\|_{L^p}^p\big)\log\left(e + \frac{1}{r^n}\right) \le 2n\big(1 + \|Du\|_{L^p}^p\big)\log\left(\frac{1}{r}\right)$$

using (2.2), also compare with [4, Remark 4.3]. If, on the other hand, we assume (2.44) we can estimate $\|Du\|_{L^p} \le r^{-1}$ if $r \le R_0$ and therefore

$$\log\left(e + \frac{\|Du\|_{L^p}^p}{r^n}\right) \le \log\left(e + \frac{1}{r^{n+p}}\right) \le 2(n+p)\log\left(\frac{1}{r}\right);$$

the proof now continues as in [4] but with $\tilde{c}$ *not anymore depending* on $\tilde{L}$ and $\|Du\|_{L^p}$. In view of (2.46) a standard application of Gehring's Lemma yields the result. Notice indeed that a simple scaling argument ensures that both the exponent and the constant in (2.45) do not depend on R_0: we fix $B_r(x_0) \subset \Omega$ and we write (2.46) setting $f(x) = H(x_0 + rx, Du(x_0 + rx))$ as

$$\fint_{B_{\rho/2}(\tilde{x})} |f|\,dx \le \tilde{c}\Big(\fint_{B_\rho(\tilde{x})} |f|^d\,dx\Big)^{1/d}$$

for every $\tilde{x} \in B_1(0)$ and $\rho \le 1$ such that $B_\rho(\tilde{x}) \subset B_1(0)$; notice that now $\tilde{c}$ does not depend on $\|Du\|_{L^p}$. Using Gehring's lemma (see for instance [4, Lemma 3.5]) gives

$$\Big(\fint_{B_{3/4}(0)} |f|^{1+\bar{\delta}_g}\,dx\Big)^{\frac{1}{1+\bar{\delta}_g}} \le c\fint_{B_1(0)} |f|\,dx$$

with $\bar{\delta}_g, c$ as in the statement, in particular not depending on R_0; scaling back to $H(\cdot, Du)$ gives (2.45) when $\vartheta = 3/4$. If $\vartheta < 3/4$ the results follows simply enlarging the integral on the left-hand side:

$$\fint_{B_{\vartheta r}(x_0)} [H(x,Du)]^{1+\delta}\,dx \le c(n)\fint_{B_{3r/4}(x_0)} [H(x,Du)]^{1+\delta}\,dx.$$

□

2.7. **Estimates for reference functionals.** This paragraph concerns the frozen functional obtained by freezing the switching coefficients $a(\cdot), b(\cdot)$ in $\mathcal{P}$, defined in (1.4). In particular, for $A \subset \mathbb{R}^n$ bounded domain (that in our case will always be a ball inside Ω), we consider minimizers of functionals of the type

$$\mathcal{P}_{\bar{a}}(w, A) := \int_A \left[|Dw|^p + \bar{a}|Dw|^p \log(e + |Dw|)\right] dx = \int_A H_{\bar{a}}(Dw)\, dx\,, \tag{2.47}$$

where $\bar{a} \geq 0$ is a constant.

The result we want to recall is the following excess decay estimate, which encodes the local $C^{1,\alpha}$ regularity of minimizers; it can be found in [32, Theorem 6.4]. Notice that the Hölder regularity **Assumption 2.2** necessary in [32] is satisfied in our case, see [4, Section 6].

Theorem 2.8. *Let $v \in W^{1,p}(A; \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{P}_{\bar{a}}$ defined in* (2.47) *such that $H_{\bar{a}}(Dv) \in L^1(A)$, and let $B_r \equiv B_r(x_0) \subset A$. The excess-decay estimate*

$$\fint_{B_\varrho} \left|V_{\bar{a}}(Dv) - \left(V_{\bar{a}}(Dv)\right)_{B_\rho}\right|^2 dx \leq c \left(\frac{\rho}{r}\right)^{2\alpha} \fint_{B_r} \left|V_{\bar{a}}(Dv) - \left(V_{\bar{a}}(Dv)\right)_{B_r}\right|^2 dx$$

holds for every couple of concentric balls $B_\rho \subset B_r$, for a constant $c \geq 1$ and an exponent $\alpha \in (0, 1)$ both depending only on n, N and p.

3. Comparison estimates and excess decay

In this section $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ will always be a local minimizer of the functional $\mathcal{P}$ defined in (1.4). We are going to define two more regular comparison maps and first deduce an integral comparison estimate; then we shall show how this does imply an excess-decay estimate with a correction term. We choose in (2.43)-(2.44)

$$M = \left(\frac{L}{\nu}\, \|H(\cdot, Du)\|_{L^1}\right)^{1/p} \quad \text{and} \quad R_0 = \min\left\{\frac{1}{e + M}, \bar{R}\right\}, \tag{3.1}$$

$\bar{R}$ being defined in (2.42), and we will work on a ball $B_R \equiv B_R(x_0)$ such that $B_{2R}(x_0) \subset \Omega$, with radius $0 < R \leq R_0/2$.

3.1. **Comparison lemma.** In this paragraph we prove a comparison lemma (see Lemma 3.2), where we estimate the distance between a minimizer of $\mathcal{P}$ and a minimizer of a frozen functional obtained by considering the case in which the coefficients $a(\cdot)$ and $b(\cdot)$ are constant.

Let us denote

$$\bar{a} := \inf_{B_{2R}} a(\cdot) \quad \text{and} \quad b_{\mathrm{av}} := (b)_{B_{R/2}(x_0)} = \fint_{B_{R/2}(x_0)} b(x)\, dx\,; \tag{3.2}$$

then, recalling (2.18), let $\bar{v} \in W^{1,p}(B_R; \mathbb{R}^N)$ and $v \in W^{1,p}(B_{R/2}; \mathbb{R}^N)$ be the solutions of the following Dirichlet problems:

$$\begin{cases} \bar{v} \longmapsto & \displaystyle\min_w \int_{B_R} b(x) H_{\bar{a}}(Dw)\, dx \\ & w \in u + W^{1,p}_0(B_R) \end{cases}, \tag{3.3}$$

$$\begin{cases} v \longmapsto & \displaystyle\min_w \int_{B_{R/2}} b_{\mathrm{av}}\, H_{\bar{a}}(Dw)\, dx \\ & w \in \bar{v} + W^{1,p}_0(B_{R/2}) \end{cases}. \tag{3.4}$$

Remark 3.1. Problems (3.3) and (3.4) are well-posed. Thanks to the definition (3.2) of $\bar{a}$ and to the bounds (1.5) on $b(\cdot)$ we get

$$\int_{B_R} b(x)\, H_{\bar{a}}(Du)\, dx \le L \int_{B_R} H(x, Du)\, dx < \infty\,; \tag{3.5}$$

further, by the minimality of $\bar{v}$ and (3.5) we obtain

$$\begin{aligned}\int_{B_{R/2}} H_{\bar{a}}(D\bar{v})\, dx \le \int_{B_R} H_{\bar{a}}(D\bar{v})\, dx \le \frac{1}{\nu} \int_{B_R} b(x)\, H_{\bar{a}}(D\bar{v})\, dx \\ \le \frac{1}{\nu} \int_{B_R} b(x)\, H_{\bar{a}}(Du)\, dx \le \frac{L}{\nu} \int_{B_R} H(x, Du)\, dx < \infty\,,\end{aligned} \tag{3.6}$$

thus the Direct Methods of the Calculus of Variations guarantees that problems (3.3) and (3.4) have a minimizer.

Finally, the minimality of v and (3.6) imply that

$$\int_{B_{R/2}} H_{\bar{a}}(Dv)\, dx \le \int_{B_R} H_{\bar{a}}(D\bar{v})\, dx \le \frac{L}{\nu} \int_{B_R} H(x, Du)\, dx < \infty\,, \tag{3.7}$$

so both energies of v and $\bar{v}$ can be estimated by the energy of the minimizer u.

Remark 3.2. Notice that the bound M defined in (3.1) guarantees that $M \ge \|Du\|_{L^p}$, thus the higher integrability result of Theorem 2.7 together with the local estimate (2.45) is available. Moreover, by (3.6) we get also that $M \ge \|D\bar{v}\|_{L^p(B_R)}$, thus the threshold R_0 for which the local higher integrability estimate for $D\bar{v}$ holds can be made independent of $\|D\bar{v}\|_{L^p(B_R)}$ but depending on $\|H(\cdot, Du)\|_{L^1}$.

Remark 3.3. We can show that the Euler-Lagrange equations

$$\fint_{B_{R/2}} \langle \partial_z H_{\bar{a}}(Dv), D\phi\rangle\, dx = 0\,, \tag{3.8}$$

$$\fint_{B_R} b(x) \langle \partial_z H_{\bar{a}}(D\bar{v}), D\phi\rangle\, dx = 0\,, \tag{3.9}$$

$$\fint_{B_R} b(x) \langle \partial_z H(x, Du), D\phi\rangle\, dx = 0 \tag{3.10}$$

are valid for every $\phi \in W^{1,p}_0(B_{R/2}; \mathbb{R}^N)$ with $H_{\bar{a}}(D\phi(\cdot)) \in L^1(B_{R/2})$ for (3.8), for every $\phi \in W^{1,p}_0(B_R; \mathbb{R}^N)$ with $H_{\bar{a}}(D\phi(\cdot)) \in L^1(B_R)$ for (3.9) and for every $\phi \in W^{1,p}_0(B_R; \mathbb{R}^N)$ with $H(\cdot, D\phi) \in L^1(B_R)$ for (3.10) .

Let us prove (3.10) since with exactly the same arguments we can prove (3.8) and (3.9). We argue by approximation since it is well known that the equation holds for every $\phi \in C^\infty_0(B_R; \mathbb{R}^N)$. Let $\phi \in W^{1,p}_0(B_R; \mathbb{R}^N)$ such that $H(\cdot, D\phi) \in L^1(B_R)$: by Lemma 2.6 there exists a sequence $\{\phi_k\} \subset C^\infty_0(B_R; \mathbb{R}^N)$ such that $D\phi_k \to D\phi$ a.e. and

$$H(\cdot, D\phi_k) \to H(\cdot, D\phi) \qquad \text{strongly in } L^1(B_R)\,.$$

Using the analogous of (2.32) for the function $H(\cdot,\cdot)$, with $z = Du$ and $\lambda = D\phi_k$, we estimate on B_R

$$\left| b(x) \langle \partial_z H(x, Du), D\phi_k\rangle \right| \le L\, c(p) \Big(H(x, Du) + H(x, D\phi_k) \Big)$$

and we can conclude the strong convergence in $L^1(B_R)$ of

$$b(\cdot) \langle \partial_z H(\cdot, Du), D\phi_k(\cdot)\rangle \to b(\cdot) \langle \partial_z H(\cdot, Du), D\phi(\cdot)\rangle$$

by a well-known variant of the Lebesgue's dominated convergence theorem. Therefore, since every ϕ_k satisfies (3.10) also ϕ does.

To prove the comparison Lemma 3.2, we need to test the Euler equation (3.10) with the function $\phi = u - \bar{v} \in W_0^{1,p}(B_R;\mathbb{R}^N)$, so by (2.31) again for the function $H(\cdot,\cdot)$, we must show that $H(\cdot, D\bar{v}) \in L^1(B_R)$. Indeed, from $H_{\bar{a}}(D\bar{v}) \in L^1(B_R)$ we deduce that

$$D\bar{v} \in L^p \log L(B_R;\mathbb{R}^{nN})\,, \qquad \text{thus} \qquad \int_{B_R} b(x)H(x,D\bar{v})\,dx < \infty\ :$$

this is immediate if $\bar{a} > 0$, while if $\bar{a} = 0$ the functional reduces to the classic p-Dirichlet functional, apart from the coefficient $b(\cdot)$, and the result follows for instance from the Theorem 3.1 and (2.39), since $\bar{v} \in u + W_0^{1,p}(B_R;\mathbb{R}^N)$ with $Du \in L^{p(1+\bar{\delta}_g)}(B_R;\mathbb{R}^{nN})$ by Theorem 2.7.

For the solutions $\bar{v}$ and v of the Dirichlet problems (3.3)-(3.4) the following up to the boundary higher integrability result holds; for the proof, see [33, Theorem B.1] for the scalar case and [23, Lemma 4.3] for the vectorial one. Notice that actually the result in [33] is stronger; anyway, the version we quote here will be sufficient for our purposes.

Theorem 3.1. *Let $B_r(x_0) \subset \Omega$ be a ball, $H_{\bar{a}}(\cdot)$ as in (2.18), $\bar{w} \in W^{1,p}(B_r;\mathbb{R}^N)$ with $H_{\bar{a}}(D\bar{x}) \in L^1(B_r)$ and let $w \in W^{1,p}(B_r;\mathbb{R}^N)$ be the minimizer in the Dirichlet class $\bar{w} + W_0^{1,p}(B_r;\mathbb{R}^N)$ of the functional*

$$w \longmapsto \int_{B_r} b(x)H_{\bar{a}}(Dw)\,dx$$

with $b(\cdot)$ as in (1.5). Suppose moreover that $H_{\bar{a}}(D\bar{w}) \in L^{1+\bar{\delta}_g}(B_r(x_0))$ for some $\bar{\delta}_g > 0$. Then there exists $\delta_g \equiv \delta_g(n,N,p,L/\nu) \in (0,\bar{\delta}_g)$ such that $H_{\bar{a}}(Dw) \in L^{1+\delta_g}(B_r(x_0))$ and the estimate

$$\fint_{B_r(x_0)} \big[H_{\bar{a}}(Dw)\big]^{1+\delta}\,dx \le c \fint_{B_r(x_0)} \big[H_{\bar{a}}(D\bar{w})\big]^{1+\delta}\,dx \tag{3.11}$$

holds for a constant $c \equiv c(n,N,p,L/\nu)$ and for every $\delta \in [0,\delta_g]$.

Remark 3.4. In order to simplify the proofs, we are going from now on to apply the interior estimate (2.45) to the three functions u (local minimizer of (1.4)), $\bar{v}$ (solution of (3.3)) and v (solution of (3.4)) and the boundary estimate (3.11) to both $\bar{v}$ and v with the same common value for δ: we choose δ_g appearing above in Theorem 3.1.

Now we can state and prove our comparison lemma.

Lemma 3.2 (Comparison)**.** *If $v \in W^{1,p}(B_{R/2};\mathbb{R}^N)$ is the solution of the Dirichlet problem (3.4), then there exists an exponent $q = q(n,N,p,L/\nu) < n$ such that the inequality*

$$\begin{aligned}\fint_{B_{R/2}} \big|V_{\bar{a}}(Du) - V_{\bar{a}}(Dv)\big|^2\,dx \\ \le c\Big[\big[\omega_a(R)\log\big(\frac{1}{R}\big)\big]^2 + \frac{R^2}{\nu^2}\Big(\fint_{B_R} |Db|^q\,dx\Big)^{2/q}\Big]\fint_{B_{3R/2}} H(x,Du)\,dx\end{aligned} \tag{3.12}$$

holds for a constant $c \equiv c(n,N,p,L/\nu)$.

Proof. The proof of the comparison lemma consists of two steps, by freezing the coefficients $a(\cdot)$ and $b(\cdot)$ one at a time: more precisely we consider the minimizer $\bar{v}$ of the Dirichlet problem (3.3) and we prove two comparison estimates, the first one between u and $\bar{v}$ and the second one between $\bar{v}$ and v.

Step 1. Let $\bar{v} \in W^{1,p}(B_R, \mathbb{R}^N)$ be the minimizer of the Dirichlet problem (3.3): the following comparison estimate between u and $\bar{v}$

$$\fint_{B_R} \big|V_{\bar{a}}(Du) - V_{\bar{a}}(D\bar{v})\big|^2\, dx \le c\big[\omega_a(R)\log\big(\frac{1}{R}\big)\big]^2 \fint_{B_{3R/2}} H(x, Du)\, dx \tag{3.13}$$

holds for a constant $c = c(n, N, p, L/\nu)$.

First of all, let us show the following estimate, of which we will make frequent use in the sequel. Let $\delta_g = \delta_g(n, N, p, L/\nu) \in (0, 1)$ be the higher integrability exponent of Theorem 2.7-Remark 3.4 applied to u; by Lemma 2.1, applied with $f = |Du|^p, \sigma = 1/p, \beta = 2, \theta = p, s = 1 + \delta_g$, the reverse Hölder's inequality (2.45) with $\theta = 2/3$ and $r = 3R/2$, (3.1) together with Remark 3.2 we can estimate

$$\begin{aligned} &\fint_{B_R} |Du|^p \log^2(e + |Du|)\, dx \\ &\le c(n, p, \delta_g)\big(1 + R^p\|Du\|^p_{L^p(\Omega)}\big)^2 \log^2\Big(\frac{1}{R}\Big)\Big(\fint_{B_R} |Du|^{p(1+\delta_g)}\, dx\Big)^{1/(1+\delta_g)} \\ &\le c\,\big(1 + R_0^p\|Du\|^p_{L^p(\Omega)}\big)^2 \log^2\big(\frac{1}{R}\big)\Big(\fint_{B_R} H(x, Du)^{1+\delta_g}\, dx\Big)^{1/(1+\delta_g)} \\ &\le c\log^2\big(\frac{1}{R}\big) \fint_{B_{3R/2}} H(x, Du)\, dx\,. \end{aligned} \tag{3.14}$$

for c ultimately depending only on n, N, p and L/ν. Since both u and $\bar{v}$ are minimizers, we can use the corresponding Euler-Lagrange equations (3.10) and (3.9). By Remark 3.3 we can test with $\phi = u - \bar{v} \in W^{1,p}_0(B_R; \mathbb{R}^N)$:

$$\fint_{B_R} b(x)\langle\partial_z H(x, Du), Du - D\bar{v}\rangle\, dx - \fint_{B_R} b(x)\langle\partial_z H_{\bar{a}}(D\bar{v}), Du - D\bar{v}\rangle\, dx = 0\,. \tag{3.15}$$

Using (2.32), with $z = Du(x)$ and $\lambda = Du(x) - D\bar{v}(x)$, and (2.31) we may estimate

$$\big|\langle\partial_z H_{\bar{a}}(Du), Du - D\bar{v}\rangle\big| \le c(p)\Big(H_{\bar{a}}(Du) + H_{\bar{a}}(D\bar{v})\Big)\,, \tag{3.16}$$

thus in (3.15) we can add and substract the integral

$$\fint_{B_R} b(x)\langle\partial_z H_{\bar{a}}(Du), Du - D\bar{v}\rangle\, dx\,,$$

which is finite by (3.16), (3.5) and (3.7), obtaining

$$\begin{aligned} \mathcal{D}_1 :&= \fint_{B_R} b(x)\langle\partial_z H_{\bar{a}}(Du) - \partial_z H_{\bar{a}}(D\bar{v}), Du - D\bar{v}\rangle\, dx \\ &= \fint_{B_R} b(x)\langle\partial_z H_{\bar{a}}(Du) - \partial_z H(x, Du), Du - D\bar{v}\rangle\, dx := \mathcal{D}_2\,. \end{aligned}$$

By applying (2.33) with $z_1 = Du(x)$ and $z_2 = D\bar{v}(x)$, and again (1.5), we can estimate $\mathcal{D}_1$ from below obtaining that

$$\frac{\nu}{c(p)} \fint_{B_R} |V_{\bar{a}}(Du) - V_{\bar{a}}(D\bar{v})|^2\, dx \le \mathcal{D}_1 = |\mathcal{D}_2|\,. \tag{3.17}$$

Using Cauchy-Schwarz inequality, (1.5), (2.18), (2.17) and (2.12) we can estimate $|\mathcal{D}_2|$ as

$$\begin{aligned} |\mathcal{D}_2| &\le L \fint_{B_R} |\partial_z H_{\bar{a}}(Du) - \partial_z H(x, Du)|\, |Du - D\bar{v}|\, dx \\ &= p\, L \fint_{B_R} |a(x) - \bar{a}|\, \varphi'_{\log}(|Du|)\, |Du - D\bar{v}|\, dx \end{aligned} \tag{3.18}$$

$$\leq c(p)L \fint_{B_R} |a(x)-\bar a|\,|Du|^{p-1}\log(e+|Du|)\,|Du-D\bar v|\,dx$$
$$= c(p)L \fint_{B_R} |a(x)-\bar a|\,|Du|^{\frac{p-2}{2}+\frac{p}{2}}\log(e+|Du|)\,|Du-D\bar v|\,dx \;=\; I_1\,;$$

now we need to distinguish two cases.

The case $p\geq 2$. In this case, using Young's inequality with conjugate exponents $(2,2)$ and $\varepsilon\in(0,1)$ to be chosen, we can estimate

$$I_1 \leq c(p)\,L\,\varepsilon \fint_{B_R} |Du|^{p-2}|Du-D\bar v|^2\,dx$$
$$+\frac{c(p)L}{4\varepsilon}\fint_{B_R}|a(x)-\bar a|^2|Du|^p\log^2(e+|Du|)\,dx \;=\; I_2+I_3\,.$$

As $p\geq 2$, by (2.23) and (2.25) we have

$$|Du|^{p-2}|Du-D\bar v|^2 \leq \big(|Du|+|D\bar v|\big)^{p-2}|Du-D\bar v|^2$$
$$\leq c(p)\,\big|V_p(Du)-V_p(D\bar v)\big|^2 \leq c(p)\,\big|V_{\bar a}(Du)-V_{\bar a}(D\bar v)\big|^2\,,$$

so the term I_2 can be estimated by

$$I_2 \leq c(p)L\,\varepsilon \fint_{B_R} \big|V_{\bar a}(Du)-V_{\bar a}(D\bar v)\big|^2\,dx$$

and it can be reabsorbed in the left-hand side of (3.17) for ε sufficiently small depending only on p and L/ν. Since, using monotonicity and concavity of $\omega_a(\cdot)$, it is easy to prove that

$$|a(x)-\bar a| \leq 3\,\omega_a(R)\,, \tag{3.19}$$

by (3.14) we can estimate the term I_3 as

$$I_3 \leq \frac{c(p)}{\varepsilon}\,L\big[\omega_a(R)\big]^2\fint_{B_R}|Du|^p\log^2(e+|Du|)\,dx$$
$$\leq c(n,N,p,L/\nu)\,L\big[\omega_a(R)\log\big(\frac{1}{R}\big)\big]^2\fint_{B_{3R/2}} H(x,Du)\,dx\,.$$

The case $p<2$. In this case to estimate the integral in (3.18) we use (2.24) to get

$$I_1 = c(p)L\fint_{B_R}|a(x)-\bar a|\,|Du|^{p-1}\log(e+|Du|)\,|Du-D\bar v|\,dx$$
$$\leq c(p)L\Big[\fint_{B_R}|a(x)-\bar a|\,|Du|^{p-1}\log(e+|Du|)\,|V_p(Du)-V_p(D\bar v)|^{2/p}\,dx$$
$$+\fint_{B_R}|a(x)-\bar a|\,|Du|^{p/2}\log(e+|Du|)\,|V_p(Du)-V_p(D\bar v)|\,dx\Big]\,.$$

Estimating both the integrals with Young's inequality, the first with conjugate exponents (p,p'), the second with $(2,2)$, both with $\varepsilon\in(0,1)$ to be chosen, and using (2.25) we obtain

$$I_1 \leq 2c(p)L\,\varepsilon\fint_{B_R}\big|V_{\bar a}(Du)-V_{\bar a}(D\bar v)\big|^2\,dx$$
$$+\frac{c(p)}{\varepsilon}\,L\fint_{B_R}|a(x)-\bar a|^{p'}|Du|^p\log^{p'}(e+|Du|)\,dx$$
$$+\frac{c(p)}{\varepsilon}\,L\fint_{B_R}|a(x)-\bar a|^2|Du|^p\log^2(e+|Du|)\,dx =: I_4+I_5+I_6\,.$$

The term I_4 can be reabsorbed in the left-hand side of (3.17) for ε sufficiently small depending only on p and L/ν, the term I_6 is estimated exactly as the term I_3 in the case $p \geq 2$ (note that the value of p is irrelevant). To estimate the remaining term I_5, we follow the reasoning in (3.14), using again (3.19), Lemma 2.1 with $f = |Du|^p, \sigma = 1/p, \beta = p', \theta = p, s = 1 + \delta_g$ (δ_g from Remark 3.4), assumption (2.42) together with the fact that $p' > 2$, and the reverse Hölder's inequality (2.45) with $\theta = 2/3$ and $r = 3R/2$, obtaining that

$$
\begin{aligned}
I_5 &\leq \frac{c(p)}{\varepsilon} L\big[\omega_a(R)\big]^{p'} \fint_{B_R} |Du|^p \log^{p'}(e + |Du|)\, dx \\
&\leq c(n, N, p, L/\nu) L\big[\omega_a(R)\log\big(\frac{1}{R}\big)\big]^{p'} \Big(\fint_{B_R} |Du|^{p(1+\delta_g)}\, dx\Big)^{\frac{1}{1+\delta_g}} \\
&\leq c(n, N, p, L/\nu) L\big[\omega_a(R)\log\big(\frac{1}{R}\big)\big]^{2} \fint_{B_{3R/2}} H(x, Du)\, dx\,.
\end{aligned}
$$

We conclude that in both cases ($1 < p < 2$ and $p \geq 2$) the comparison estimate (3.13) holds for a constant $c \equiv c(n, N, p, L/\nu)$.

Step 2. There exists an exponent $q = q(n, N, p, L/\nu) < n$ such that the following comparison estimate between $\bar{v}$ and v holds:

(3.20)
$$
\fint_{B_{R/2}} |V_{\bar{a}}(D\bar{v}) - V_{\bar{a}}(Dv)|^2\, dx \leq c\, \frac{R^2}{\nu^2} \Big(\fint_{B_R} |Db|^q\, dx\Big)^{2/q} \fint_{B_{3R/2}} H(x, Du)\, dx\,,
$$

for a constant c depending on n, N, p and L/ν.

Since both $\bar{v}$ and v are minimizers, we can use the corresponding Euler-Lagrange equations (3.9) and (3.8). We can test with $\phi = \bar{v} - v \in W^{1,p}_0(B_{R/2}; \mathbb{R}^N)$ (extended to 0 on $B_R \setminus B_{R/2}$) as $H_{\bar{a}}(D\phi) \in L^1(B_{R/2})$ by (3.7):

(3.21)
$$
\fint_{B_{R/2}} b(x)\langle \partial_z H_{\bar{a}}(D\bar{v}), D\bar{v} - Dv\rangle\, dx - \fint_{B_{R/2}} b_{\mathrm{av}}\langle \partial_z H_{\bar{a}}(Dv), D\bar{v} - Dv\rangle\, dx = 0\,.
$$

Using (2.32), with $z = D\bar{v}(x)$ and $\lambda = D\bar{v}(x) - Dv(x)$, and (2.31) we may estimate

$$
|\langle \partial_z H_{\bar{a}}(D\bar{v}), D\bar{v} - Dv\rangle| \leq c(p)\big(H_{\bar{a}}(D\bar{v}) + H_{\bar{a}}(Dv)\big)\,, \tag{3.22}
$$

thus in (3.21) we can add and substract the integral

$$
\fint_{B_{R/2}} b_{\mathrm{av}}\langle \partial_z H_{\bar{a}}(D\bar{v}), D\bar{v} - Dv\rangle\, dx\,,
$$

which is finite by (3.22) and (3.7), obtaining

$$
\begin{aligned}
\mathcal{D}_1 &:= \fint_{B_{R/2}} b_{\mathrm{av}}\langle \partial_z H_{\bar{a}}(D\bar{v}) - \partial_z H_{\bar{a}}(Dv), D\bar{v} - Dv\rangle\, dx \\
&= \fint_{B_{R/2}} \big(b_{\mathrm{av}} - b(x)\big)\ \langle \partial_z H_{\bar{a}}(D\bar{v}), D\bar{v} - Dv\rangle\, dx := \mathcal{D}_2\,.
\end{aligned}
$$

By applying (2.33) with $z_1 = D\bar{v}(x)$ and $z_2 = Dv(x)$, and noticing that $\nu \leq b_{\mathrm{av}} \leq L$, we can estimate $\mathcal{D}_1$ from below obtaining that

$$
\frac{\nu}{c(p)} \fint_{B_{R/2}} \big|V_{\bar{a}}(D\bar{v}) - V_{\bar{a}}(Dv)\big|^2\, dx \leq \mathcal{D}_1 = |\mathcal{D}_2|\,. \tag{3.23}
$$

By Cauchy-Schwarz inequality, (2.34) with $z = D\bar{v}(x)$, and (2.37) with $z_1 = D\bar{v}(x)$ and $z_2 = Dv(x)$, we can estimate $|\mathcal{D}_2|$ as

$$
|\mathcal{D}_2| \leq \fint_{B_{R/2}} |b(x) - b_{\mathrm{av}}||\partial_z H_{\bar{a}}(D\bar{v})|\, |D\bar{v} - Dv|\, dx
$$

$$
\begin{aligned}
&\le c(p) \fint_{B_{R/2}} |b(x)-b_{\rm av}|\, h_{\bar a}(|D\bar v|)|D\bar v - Dv|\,dx \\
&\le c(p) \fint_{B_{R/2}} |b(x)-b_{\rm av}|\big|V_{\bar a}(D\bar v)-V_{\bar a}(Dv)\big|\sqrt{H_{\bar a}\big(D\bar v\big)}\,dx \\
&+ c(p) \fint_{B_{R/2}} |b(x)-b_{\rm av}|\big|V_{\bar a}(D\bar v)-V_{\bar a}(Dv)\big|\sqrt{H_{\bar a}\big(Dv\big)}\,dx = I_1+I_2\,.
\end{aligned} \tag{3.24}
$$

Using Young's inequality with conjugate exponents $(2,2)$ and $\varepsilon \in (0,1)$ to be chosen, we can estimate

$$
\begin{aligned}
I_1 &\le c(p)\,\varepsilon \fint_{B_{R/2}} \big|V_{\bar a}(D\bar v)-V_{\bar a}(Dv)\big|^2\,dx \\
&\qquad + \frac{c(p)}{4\varepsilon}\fint_{B_{R/2}} |b(x)-b_{\rm av}|^2 H_{\bar a}(D\bar v)\,dx = I_3+I_4\,,
\end{aligned} \tag{3.25}
$$

where the term I_3 can be reabsorbed in the left-hand side of (3.23) for ε sufficiently small depending only on p,ν.

In order to estimate I_4 in (3.25), let $\delta_g \in (0,1)$ be the higher integrability exponent from Theorem 2.7-Remarks 3.2 & 3.4 which holds for both $\bar v$ and v, depending on n,N,p and L/ν, and let us choose $q=q(n,\delta_g)<n$ such that

$$
q^* = \frac{nq}{n-q} = 2\Big(1+\frac{1}{\delta_g}\Big) \quad \Longleftrightarrow \quad q = \frac{q^*}{n+q^*}\,n = \frac{2(1+\delta_g)}{n\delta_g + 2(1+\delta_g)}\,n\,. \tag{3.26}
$$

Now, by applying first Hölder's inequality with conjugate exponents $(1+1/\delta_g, 1+\delta_g)$, then the Sobolev-Poincaré and the reverse Hölder's (2.45) inequalities with $\theta = 1/2$ and $r=R$, and finally (3.7), we obtain

$$
\begin{aligned}
I_4 &\le \frac{c(p)}{\varepsilon}\Big(\fint_{B_{R/2}} |b(x)-b_{\rm av}|^{2(1+\frac{1}{\delta_g})}\,dx\Big)^{\frac{\delta_g}{1+\delta_g}} \Big(\fint_{B_{R/2}} \big[H_{\bar a}(D\bar v)\big]^{1+\delta_g}\,dx\Big)^{\frac{1}{1+\delta_g}} \\
&= \frac{c(p)}{\nu}\Big[\Big(\fint_{B_{R/2}} |b(x)-b_{\rm av}|^{q^*}\,dx\Big)^{1/q^*}\Big]^2 \Big(\fint_{B_{R/2}} \big[H_{\bar a}(D\bar v)\big]^{1+\delta_g}\,dx\Big)^{\frac{1}{1+\delta_g}} \\
&\le \frac{c(n,N,p,L/\nu)}{\nu}\Big[R\Big(\fint_{B_{R/2}} |Db|^q\,dx\Big)^{1/q}\Big]^2 \fint_{B_R} H_{\bar a}(D\bar v)\,dx \\
&\le \frac{c(n,N,p,L/\nu)}{\nu} R^2 \Big(\fint_{B_R} |Db|^q\,dx\Big)^{2/q} \fint_{B_{3R/2}} H(x,Du)\,dx\,.
\end{aligned} \tag{3.27}
$$

It remains to estimate the term I_2 in (3.24): arguing as in (3.25) we obtain first

$$
\begin{aligned}
I_2 &\le c(p)\,\varepsilon \fint_{B_{R/2}} |V_{\bar a}(D\bar v)-V_{\bar a}(Dv)|^2\,dx \\
&\qquad + \frac{c(p)}{4\varepsilon}\fint_{B_{R/2}} |b(x)-b_{\rm av}|^2 H_{\bar a}(Dv)\,dx = I_5+I_6\,,
\end{aligned}
$$

with the term I_5 which can be reabsorbed in the left-hand side of (3.23) for ε sufficiently small depending only on p,ν. Then, with the same exponent q defined in (3.26) and the same computations in (3.27), using (3.11) we can estimate the term I_6 as

$$
I_6 \le c\;,\frac{R^2}{\nu}\Big(\fint_{B_R} |Db|^q\,dx\Big)^{2/q} \fint_{B_{3R/2}} H(x,Du)\,dx\,,
$$

for a constant $c \equiv c(n,N,p,L/\nu)$. Putting together all the estimates into (3.23) and (3.24), we obtain (3.20).

From (3.13) and (3.20), we deduce immediately the comparison estimate (3.12). □

Next, an inequality allowing to replace the energy on the right-hand side with a quantity more appropriate for the forthcoming iteration proof.

Proposition 3.3. *There exists a constant $c \equiv c(n, N, p, L/\nu)$ such that*

$$\fint_{B_{3R/2}} H(x, Du)\,dx \le c \fint_{B_{2R}} \left|V_{a(x_0)}(Du)\right|^2 dx$$

holds for every ball $B_{2R} \equiv B_{2R}(x_0) \subset \Omega$ with R smaller than $R_0/2$.

Proof. The self-improving character of reverse-Hölder inequality yields that, as a consequence of the higher integrability estimate (2.45), for every $\sigma > 0$ there holds

$$\fint_{B_{3R/2}} H(x, Du)\,dx \le c(n, N, p, L/\nu, \sigma)\Big(\fint_{B_{2R}} \left[H(x, Du)\right]^\sigma dx\Big)^{1/\sigma}. \tag{3.28}$$

Observing that for every $x_0 \in \Omega$ the function $H(x_0, Du(\cdot)) \in L^1(B_{2R})$ due to the higher integrability result of Theorem 2.7 and (2.39), we choose $\sigma = 1/2$ and we use sub-additivity to estimate the right-hand side:

$$\begin{aligned}\fint_{B_{2R}} \left[H(x, Du)\right]^\sigma dx \le{}& \fint_{B_{2R}} \left[H(x_0, Du)\right]^{\frac12} dx\\ &+ \fint_{B_{2R}} \left|a(x) - a(x_0)\right|^{\frac12} |Du|^{\frac{p}{2}} \log^{\frac12}(e + |Du|)\,dx\,.\end{aligned}$$

Since the first integral is bounded by $\Big(\fint_{B_{2R}} \left|V_{a(x_0)}(Du)\right|^2 dx\Big)^{1/2}$, using Hölder's inequality and (2.22), we focus on the second one. In order to estimate it, we use first Lemma 2.1 with $f = |Du|^{\frac{p}{2}}, \sigma = 2/p, \beta = 1/2, \theta = p, s = 2$, then (2.42) and the fact that $R \le R_0 < 1$, and finally (2.22):

$$\begin{aligned}&\fint_{B_{2R}} \left|a(x) - a(x_0)\right|^{\frac12} |Du|^{\frac{p}{2}} \log^{\frac12}(e + |Du|)\,dx\\
&\quad\le \left[\omega_a(2R)\right]^{\frac12} \fint_{B_{2R}} |Du|^{\frac{p}{2}} \log^{\frac12}(e + |Du|)\,dx\\
&\quad\le c(n, p)\left[\omega_a(R)\log\big(\frac{1}{2R}\big)\right]^{\frac12} \left[1 + R^p \|Du\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(B_{2R})}\right)^{\frac12} \Big(\fint_{B_{2R}} |Du|^p\,dx\Big)^{\frac12}\\
&\quad\le c\Big[\omega_a(R)\log\big(\frac{1}{R}\big)\Big]^{\frac12} \Big[1 + R^p \int_{B_{2R}} (1 + |Du|^p)\,dx\Big]^{\frac12} \Big(\fint_{B_{2R}} |Du|^p\,dx\Big)^{\frac12}\\
&\quad\le c\big[\omega_a(R)\log\big(\frac{1}{R}\big)\big]^{\frac12} \big[1 + c(n) R_0^{n+p} + R_0^p \|Du\|^p_{L^p(\Omega)}\big]^{\frac12} \Big(\fint_{B_{2R}} |Du|^p\,dx\Big)^{\frac12}\\
&\quad\le c\Big(\fint_{B_{2R}} H(x_0, Du)\,dx\Big)^{1/2} \le c(n, p)\Big(\fint_{B_{2R}} \left|V_{a(x_0)}(Du)\right|^2 dx\Big)^{\frac12}.\end{aligned}$$

The proof is concluded in view of (3.28) with $\sigma = 1/2$. □

Remark 3.5. Notice that we can prove in a similar way that if $B_{4R}(x_0) \subset \Omega$ has radius $R \le R_0/4$, then

$$\fint_{B_{2R}} \left|V_{a(x_0)}(Du)\right|^2 dx \le c(n, N, p, L/\nu) \fint_{B_{4R}} H(x, Du)\,dx\,.$$

Indeed, using (2.22), Lemma 2.5, (3.14) from B_{2R} to B_{4R} (in the higher integrability estimate (2.45) choose $\theta = 1/2$ and $r = 4R$), the concavity of $\omega_a(\cdot)$ together with the monotonicity of the logarithm function, and (2.42) we have

$$\begin{aligned}
&\fint_{B_{2R}} \big|V_{a(x_0)}(Du)\big|^2\,dx \\
&\quad\le 2\fint_{B_{2R}} \big|V_{a(x)}(Du)\big|^2\,dx + 2\fint_{B_{2R}} \big|V_{a(x)}(Du)-V_{a(x_0)}(Du)\big|^2\,dx \\
&\quad\le 4\fint_{B_{2R}} H(x,Du)\,dx + 2\fint_{B_{2R}} |a(x)-a(x_0)|^2|Du|^p\log^2(e+|Du|)\,dx \\
&\quad\le c(n)\fint_{B_{4R}} H(x,Du)\,dx + c(n,N,p,L/\nu)\Big[\omega_a(R)\log\Big(\frac{1}{R}\Big)\Big]^2\fint_{B_{4R}} H(x,Du)\,dx\,.
\end{aligned}$$

3.2. **Excess decay estimate.** From now on, let us denote for radii $r\le R$ and for the exponent $q\in(1,n)$ defined in Lemma 3.2

$$\mathcal{B}_{r,q}\equiv\mathcal{B}_{r,q}(x_0):=\frac{r}{\nu}\Big(\fint_{B_r(x_0)}|Db|^q\,dx\Big)^{1/q}$$

and we recall that $\omega_{\log}$ has been defined in (1.6). Also in this paragraph the center of all the balls will be x_0 and therefore we shall omit it.

The forthcoming Lemma is a preliminary decay estimate for the L^2-excess of a certain nonlinear function of the gradient. The function $\xi\mapsto V_{\bar a}(\xi)$ with $\bar a$ as in (3.2) reflects the growth of the comparison problems but it is not appropriate for the iteration procedures we are going to perform; it will be replaced later in order to get the final excess-decay estimate (3.30).

Lemma 3.4. *There exist an exponent $\alpha\in(0,1)$ depending on n,N and p, and a constant $c\equiv c(n,N,p,L/\nu)$, such that for every pair of concentric balls $B_\rho\equiv B_\rho(x_0)\subset B_{2R}\equiv B_{2R}(x_0)\subset\Omega$ with $R\le R_0/2$, it holds*

(3.29)
$$\begin{aligned}
\fint_{B_\rho}\big|V_{\bar a}(Du)-\big(V_{\bar a}(Du)\big)_{B_\rho}\big|^2\,dx &\le c\Big(\frac{\rho}{R}\Big)^{2\alpha}\fint_{B_{2R}}\big|V_{\bar a}(Du)-\big(V_{\bar a}(Du)\big)_{B_{2R}}\big|^2\,dx \\
&\quad + c\Big[\Big(\frac{R}{\rho}\Big)^n+\Big(\frac{\rho}{R}\Big)^{2\alpha}\Big]\Big[\omega_{\log}^2(R)+\mathcal{B}_{R,q}^2\Big]\fint_{B_{2R}}\big|V_{a(x_0)}(Du)\big|^2\,dx\,.
\end{aligned}$$

Proof. It suffices to prove the lemma for $0<\rho\le R/2$, indeed for $R/2<\rho\le 2R$ the estimate follows immediately using (2.1), enlarging the integral, and observing that $2(\rho/R)\ge 1$. Now, noticing that the function v is also a local minimizer of the functional $\mathcal{P}_{\bar a}$ defined in (2.47) on $A=B_{R/2}$, thus Theorem 2.8 applies to the minimizer v, and using also (2.1), we can estimate

$$\begin{aligned}
&\fint_{B_\rho}\big|V_{\bar a}(Du)-\big(V_{\bar a}(Du)\big)_{B_\rho}\big|^2\,dx \\
&\quad\le 8\Big[\fint_{B_\rho}\big|V_{\bar a}(Du)-V_{\bar a}(Dv)\big|^2\,dx+\fint_{B_\rho}\big|V_{\bar a}(Dv)-\big(V_{\bar a}(Dv)\big)_{B_\rho}\big|^2\,dx\Big] \\
&\quad\le c(n,N,p)\Big[\Big(\frac{R}{\rho}\Big)^n\fint_{B_{R/2}}\big|V_{\bar a}(Du)-V_{\bar a}(Dv)\big|^2\,dx \\
&\qquad\qquad + \Big(\frac{\rho}{R}\Big)^{2\alpha}\fint_{B_{R/2}}\big|V_{\bar a}(Dv)-\big(V_{\bar a}(Dv)\big)_{B_{R/2}}\big|^2\,dx\Big]\,.
\end{aligned}$$

As

$$\fint_{B_{R/2}}\big|V_{\bar a}(Dv)-\big(V_{\bar a}(Dv)\big)_{B_{R/2}}\big|^2\,dx\le c(n)\Big[\fint_{B_{R/2}}\big|V_{\bar a}(Du)-V_{\bar a}(Dv)\big|^2\,dx$$

$$+ \fint_{B_{2R}} \big|V_{\bar a}(Du) - \big(V_{\bar a}(Du)\big)_{B_{2R}}\big|^2\,dx\Big]\,,$$

by the comparison Lemma 3.2 and Proposition 3.3 we conclude that

$$\begin{aligned}
&\fint_{B_\rho} \big|V_{\bar a}(Du) - \big(V_{\bar a}(Du)\big)_{B_\rho}\big|^2\,dx\\
&\quad\le c(n,N,p)\Big[\big(\frac{R}{\rho}\big)^n + \big(\frac{\rho}{R}\big)^{2\alpha}\Big] \fint_{B_{R/2}} \big|V_{\bar a}(Du) - V_{\bar a}(Dv)\big|^2\,dx\\
&\qquad\qquad + c(n,N,p)\big(\frac{\rho}{R}\big)^{2\alpha} \fint_{B_{2R}} \big|V_{\bar a}(Du) - \big(V_{\bar a}(Du)\big)_{B_{2R}}\big|^2\,dx\\
&\quad\le c\big(\frac{\rho}{R}\big)^{2\alpha} \fint_{B_{2R}} \big|V_{\bar a}(Du) - \big(V_{\bar a}(Du)\big)_{B_{2R}}\big|^2\,dx\\
&\qquad\qquad + c\Big[\big(\frac{R}{\rho}\big)^n + \big(\frac{\rho}{R}\big)^{2\alpha}\Big]\Big[\omega_{\log}^2(R) + \mathcal{B}_{R,q}^2\Big] \fint_{B_{2R}} |V_{a(x_0)}(Du)|^2\,dx
\end{aligned}$$

with $c \equiv c(n,N,p,L/\nu)$. □

The next Lemma is necessary in order to perform the final iteration more smoothly; in particular we need to uniformize the nonlinear expression of the gradient appearing in the left- and right-hand sides of (3.29), replacing $V_{\bar a}(\cdot)$ with $V_{a(x_0)}(\cdot)$.

Lemma 3.5. *There exists a constant $c \equiv c(n,N,p,L/\nu)$ such that for every pair of concentric balls $B_\rho \equiv B_\rho(x_0) \subset B_{2R} \equiv B_{2R}(x_0) \subset \Omega$ with $R \le R_0/2$, it holds*

$$\fint_{B_\rho} \big|V_{\bar a}(Du) - V_{a(x_0)}(Du)\big|^2\,dx \le c\big(\frac{R}{\rho}\big)^n \big[\omega_{\log}(R)\big]^2 \fint_{B_{2R}} \big|V_{a(x_0)}(Du)\big|^2\,dx\,,$$

where $\bar a$ is the infimum of the continuous function $a(\cdot)$ on B_{2R} as defined in (3.2).

Proof. Using Lemma 2.5, with $a_1 = \bar a$ and $a_2 = a(x_0)$, the fact that $|\bar a - a(x_0)| \le 2\omega_a(R)$, (3.14) and Proposition 3.3, we can estimate

$$\begin{aligned}
&\fint_{B_\rho} \big|V_{\bar a}(Du) - V_{a(x_0)}(Du)\big|^2\,dx\\
&\qquad\le \big(\frac{R}{\rho}\big)^n \fint_{B_R} |\bar a - a(x_0)|^2 |Du|^p \log^2(e+|Du|)\,dx\\
&\qquad\le c(n,N,p,L/\nu)\big(\frac{R}{\rho}\big)^n \big[\omega_{\log}(R)\big]^2 \fint_{B_{2R}} |V_{a(x_0)}(Du)|^2\,dx\,.
\end{aligned}$$

□

Finally, the decay estimate we were looking for.

Proposition 3.6. *There exist an exponent $\alpha \in (0,1)$ depending on n, N and p, and a constant $\bar c \equiv \bar c(n,N,p,L/\nu)$, such that for every pair of concentric balls $B_\rho \equiv B_\rho(x_0) \subset B_{2R} \equiv B_{2R}(x_0) \subset \Omega$ with $R \le R_0/2$, it holds*

$$\begin{aligned}
&\fint_{B_\rho} \big|V_{a(x_0)}(Du) - \big(V_{a(x_0)}(Du)\big)_{B_\rho}\big|^2\,dx\\
&\qquad\le \bar c\big(\frac{\rho}{R}\big)^{2\alpha} \fint_{B_{2R}} \big|V_{a(x_0)}(Du) - \big(V_{a(x_0)}(Du)\big)_{B_{2R}}\big|^2\,dx\\
&\qquad\quad + \bar c\Big[\big(\frac{R}{\rho}\big)^n + \big(\frac{\rho}{R}\big)^{2\alpha}\Big]\Big[\omega_{\log}^2(2R) + \mathcal{B}_{2R,q}^2\Big] \fint_{B_{2R}} \big|V_{a(x_0)}(Du)\big|^2\,dx\,.
\end{aligned} \tag{3.30}$$

Proof. Putting together Lemmata 3.5 and 3.4, we obtain for a constant c depending on n, N, p and L/ν:

$$
\begin{aligned}
&\fint_{B_\rho} \left|V_{a(x_0)}(Du) - \big(V_{a(x_0)}(Du)\big)_{B_\rho}\right|^2 dx \\
&\qquad\qquad \le c\Big(\frac{\rho}{R}\Big)^{2\alpha} \fint_{B_{2R}} \left|V_{\bar a}(Du) - \big(V_{\bar a}(Du)\big)_{B_{2R}}\right|^2 dx \\
&\qquad\qquad\quad + c\Big[\Big(\frac{R}{\rho}\Big)^n + \Big(\frac{\rho}{R}\Big)^{2\alpha}\Big]\Big[\omega_{\log}^2(R) + \mathcal{B}_{R,q}^2\Big] \fint_{B_{2R}} \left|V_{a(x_0)}(Du)\right|^2 dx\,.
\end{aligned}
$$

Using again Lemma 3.5 we estimate

$$
\begin{aligned}
&\fint_{B_{2R}} \left|V_{\bar a}(Du) - \big(V_{\bar a}(Du)\big)_{B_{2R}}\right|^2 \\
&\le 8\Big[\fint_{B_{2R}} \left|V_{\bar a}(Du) - V_{a(x_0)}(Du)\right|^2 dx + \fint_{B_{2R}} \left|V_{a(x_0)}(Du) - \big(V_{a(x_0)}(Du)\big)_{B_{2R}}\right|^2 dx\Big] \\
&\le c\Big[\fint_{B_{2R}} \left|V_{a(x_0)}(Du) - \big(V_{a(x_0)}(Du)\big)_{B_{2R}}\right|^2 dx + \omega_{\log}^2(R) \fint_{B_{2R}} \left|V_{a(x_0)}(Du)\right|^2 dx\Big]
\end{aligned}
$$

for a constant $c \equiv c(n, N, p, L/\nu)$; the conclusion follows from $\mathcal{B}_{R,q} \le c(n,q)\,\mathcal{B}_{2R,q}$ and $\omega_{\log}(R) \le \omega_{\log}(2R)$, with the second one obtained by the monotonicity of $\omega_a(\cdot)$ and the inequality $\log\big(\frac{1}{R}\big) \le 4\,\log\big(\frac{1}{2R}\big)$ which holds for every $R \le 1/e$. □

4. Iteration and conclusion

Once having at hand the excess decay estimate of Proposition 3.6, the conclusion is quite standard (see [38, 2, 3, 40] for instance). We sketch the proof for the reader's convenience. We take $x_0 \in \Omega$ and a radius R such that $B_{2R}(x_0) \subset \Omega$ and $2R$ is smaller than the threshold R_0 as defined in (3.1); we shall further reduce the value of R_0. We define the sequence of radii and corresponding balls

$$
\tilde R_j = 2R\delta^j\,, \qquad \ell B_j = \ell B_j(x_0) = B_{\ell \tilde R_j}(x_0)\,, \qquad \ell > 0,\ j \in \mathbb{N}_0\,, \tag{4.1}
$$

for $\delta \in (0,1)$ that will be chosen later. The fundamental result all the forthcoming proofs are based upon is the following, whose proof can be found in [38, Lemma 1] or [3].

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ and $f \in L^{n,1}_{\rm loc}(\Omega;\mathbb{R}^n)$; let moreover $\delta \in (0,1)$ and $q \in (1,n)$ be fixed. For every $K \Subset \Omega$ and $\varepsilon > 0$, there exists a radius $R_\varepsilon > 0$ depending on $n, q, \delta, |f(\cdot)|$ and ε such that if $R \in (0, R_\varepsilon]$ and $R < \operatorname{dist}(\partial\Omega, K)$, then*

$$
\sup_{x \in K} \sum_{j=0}^{\infty} \tilde R_j \Big(\fint_{B_j(x)} |f|^q\, dy\Big)^{1/q} \le \varepsilon
$$

with $\tilde R_j$ and $B_j(x)$ as in (4.1) *(with x replacing x_0).*

The analogous result regarding $\omega_{\log}$ is based on a simple computation (see [13, Eq. (4.6)]) and ensures

$$
\sum_{j=0}^{\infty} \omega_{\log}(\tilde R_j) \le \delta^{-1} \int_0^{2R_0} \omega_{\log}(\rho)\,\frac{d\rho}{\rho}\,; \tag{4.2}
$$

note that the right-hand side of the previous inequality tends to zero as $R_0 \searrow 0^+$.

We immediately choose $\delta \equiv \delta(n, N, p, L/\nu) \in (0, 1/4)$ as the constant satisfying $\sqrt{\bar c}\,(2\delta)^\alpha = 1/4$, with $\bar c$ and α the constants from Proposition 3.6. We then define the radii and the balls as in (4.1), set for $j \in \mathbb{N}_0$ $\omega_j = \omega_{\log}(\tilde R_j)$, $\mathcal{B}_j = \mathcal{B}_{\tilde R_j, q}$ and

(4.3)

$$
a_j := \Big|\fint_{B_j} V_{a(x_0)}(Du)\,dx\Big|, \quad E_j := \Big(\fint_{B_j} \left|V_{a(x_0)}(Du) - \big(V_{a(x_0)}(Du)\big)_{B_j}\right|^2 dx\Big)^{1/2}.
$$

At this point we have for every $j \in \mathbb{N}_0$, using (3.30) with $2R = \tilde{R}_j, \rho = \tilde{R}_{j+1}$

$$\begin{aligned} E_{j+1} &\le \frac{1}{4} E_j + \tilde{c}\big[\omega_j + \mathcal{B}_j\big] \Big(\fint_{B_j} |V_{a(x_0)}(Du)|^2 \, dx \Big)^{\frac{1}{2}} \\ &\le \frac{1}{4} E_j + \tilde{c}\big[\omega_j + \mathcal{B}_j\big] E_j + \tilde{c}\big[\omega_j + \mathcal{B}_j\big] \Big| \fint_{B_j} V_{a(x_0)}(Du) \, dx \Big| \end{aligned}$$

with the constant $\tilde{c}$ depending only on n, N, p and L/ν; in the last line we used triangle's inequality. Now we reduce, in view of (2.41) and Lemma 4.1, the value of R_0 so that

$$\sup_{R \le R_0} \omega_{\log}(R) + \sup_{R \le R_0} \mathcal{B}_{R,q} \le \frac{1}{4\tilde{c}} \qquad \Longrightarrow \qquad \tilde{c}\big[\omega_j + \mathcal{B}_j\big] \le \frac{1}{4} \, ;$$

R_0 at this point depends also on $\omega_a(\cdot)$ and $|Db(\cdot)|/\nu$; this yields

$$E_{j+1} \le \frac{1}{2} E_j + \tilde{c}\big[\omega_j + \mathcal{B}_j\big] a_j$$

and in turn, summing for $j \in \{\ell, \dots, k\}$ with $\ell, k \in \mathbb{N}_0, \ell \le k$ fixed,

$$\sum_{j=\ell+1}^{k+1} E_j \le \frac{1}{2} \sum_{j=\ell}^{k} E_j + \tilde{c} \sum_{j=\ell}^{k} \big[\omega_j + \mathcal{B}_j\big] a_j \;\Rightarrow\; \sum_{j=\ell}^{k+1} E_j \le 2E_\ell + 2\tilde{c} \sum_{j=\ell}^{k} \big[\omega_j + \mathcal{B}_j\big] a_j \tag{4.4}$$

for $\tilde{c}$ depending on n, N, p and L/ν.

4.1. **Gradient boundedness by induction.** In this paragraph we suppose that $x_0 \in \Omega$ is a Lebesgue's point for Du and we finally are in the position to prove by induction that

$$a_j = \Big| \fint_{B_j} V_{a(x_0)}(Du) \, dx \Big| \le 12 \delta^{-n} \Big(\fint_{B_{2R}(x_0)} \big|V_{a(x_0)}(Du)\big|^2 \, dx \Big)^{\frac{1}{2}} = \lambda_0$$

for all $j \in \mathbb{N}_0$, with δ defined just after (4.2): thanks to the choice of x_0, this leads to (1.10), using also Remark 3.5, after renaming R. The base case $j = 0$ is trivial by Hölder's inequality; notice also that

$$a_0 + E_0 \le 3 \Big(\fint_{B_{2R}(x_0)} \big|V_{a(x_0)}(Du)\big|^2 \, dx \Big)^{\frac{1}{2}} \le \frac{\delta^n}{4} \lambda_0 \, .$$

Suppose now that $a_j \le \lambda_0$ holds for all $j \in \{0, 1, \dots, k\}$ for some $k \in \mathbb{N}_0$ and further reduce R_0, in a way depending on $n, N, p, L/\nu, \omega_a(\cdot)$ and $|Db(\cdot)|/\nu$, so that

$$\sum_{j=0}^{\infty} \big[\omega_j + \mathcal{B}_j\big] \le \frac{\delta^n}{8\tilde{c}} \, ,$$

this being possible thanks to Lemma 4.1 and (4.2). Since

$$\begin{aligned} a_{j+1} - a_j &= \Big| \fint_{B_{j+1}} V_{a(x_0)}(Du) \, dx \Big| - \Big| \fint_{B_j} V_{a(x_0)}(Du) \, dx \Big| \\ &\le \Big| \fint_{B_{j+1}} V_{a(x_0)}(Du) \, dx - \fint_{B_j} V_{a(x_0)}(Du) \, dx \Big| \\ &\le \delta^{-n} \fint_{B_j} \big| V_{a(x_0)}(Du) - \big(V_{a(x_0)}(Du)\big)_{B_j} \big| \, dx \le \delta^{-n} E_j \, , \end{aligned}$$

we have by telescopic summation

$$a_{k+1} = a_0 + \sum_{j=0}^{k} \big[a_{j+1} - a_j\big] \le a_0 + \delta^{-n} \sum_{j=0}^{k} E_j$$

and using (4.4) for $\ell = 0$ and our inductive assumption

$$a_{k+1} \le a_0 + 2\delta^{-n} E_0 + 2\tilde{c}\,\delta^{-n} \sum_{j=0}^{\infty} \big[\omega_j + \mathcal{B}_j\big] \lambda_0 \le \frac{\lambda_0}{4} + \frac{\lambda_0}{2} + \frac{\lambda_0}{4} = \lambda_0\,;$$

the boundedness proof is thus concluded.

4.2. **Quantitative locally uniform VMO-type estimate.** In order to be able to prove the gradient continuity in the next paragraph, we need as intermediate step a qualitative result of VMO-type. From the result of the previous paragraph, we know that the gradient is locally bounded in Ω and therefore, for $K \Subset \Omega$ we fix an intermediate compact set $K \Subset \tilde{K} \Subset \Omega$ such that $\operatorname{dist}(K, \partial\tilde{K}) = \operatorname{dist}(K, \partial\Omega)/2$ and we set

$$\lambda_1^2 := \|H(\cdot, Du)\|_{L^\infty(\tilde{K})}\,.$$

We prove here that for every $\varepsilon > 0$, there exists a radius R_ε, depending on $n, N, p, L/\nu, \omega_a(\cdot), |Db(\cdot)|/\nu, \|H(\cdot, Du)\|_{L^1}$ and ε, such that if $R_\varepsilon \le \operatorname{dist}(K, \partial\Omega)/4$ then

$$\sup_{R\in(0,R_\varepsilon]} \sup_{x\in K} \Big(\fint_{B_R(x)} \big|V_{a(x)}(Du) - \big(V_{a(x)}(Du)\big)_{B_R(x)}\big|^2 \, dy\Big)^{\frac{1}{2}} \le \varepsilon\, \lambda_1\,. \tag{4.5}$$

We fix $\delta \equiv \delta(n, N, p, L/\nu, \varepsilon)$ such that $2\sqrt{\bar{c}}\,(2\delta)^\alpha = \varepsilon/2$, where $\bar{c}$ and α are the constants from Proposition 3.6, and we define, for a starting radius $R \le R_1/2$, with $R_1 \le R_0/2$ a threshold to be chosen appropriately, $\tilde{R}_j, B_j(x)$ as in (4.1) and $E_j(x)$ the excess over $B_j(x)$ as in (4.3). Using Proposition 3.6 and Remark 3.5 we have for $x \in K$, if $B_{4R}(x) \subset \tilde{K}$

$$\begin{aligned} E_{j+1}(x) &\le \sqrt{\bar{c}}\,(2\delta)^\alpha E_j(x) + \sqrt{\bar{c}}\,2(2\delta)^{-\frac{n}{2}} \big[\omega_{\log}(\tilde{R}_j) + \mathcal{B}_{\tilde{R}_j,q}\big] \Big(\fint_{2B_j(x)} H(y, Du)\, dy\Big)^{\frac{1}{2}} \\ &\le \sqrt{\bar{c}}\Big[2(2\delta)^\alpha + 2(2\delta)^{-\frac{n}{2}} \big[\omega_{\log}(\tilde{R}_j) + \mathcal{B}_{\tilde{R}_j,q}\big]\Big]\lambda_1 \\ &\le \frac{\varepsilon}{2}\lambda_1 + \sqrt{\bar{c}}\Big[\delta^{-\frac{n}{2}} \big[\omega_{\log}(\tilde{R}_j) + \mathcal{B}_{\tilde{R}_j,q}\big]\Big]\lambda_1 \end{aligned}$$

for all $j \in \mathbb{N}_0$, due to our choice of δ. Now we can reduce the value of R_1 so that $\sup_{R\in(0,R_1]} \omega_{\log}(R) \le \delta^{\frac{n}{2}}\varepsilon/[4\sqrt{\bar{c}}]$ (and this is possible in view of (2.41)) and so that $\sup_{R\in(0,R_1]} \mathcal{B}_{R,q} \le \delta^{\frac{n}{2}}\varepsilon/[4\sqrt{\bar{c}}]$ (and this is possible in view of Lemma 4.1, uniformly in K) so that $\sup_{x\in K} E_j(x) \le \varepsilon\lambda_1$ for all $j \in \mathbb{N}$. We conclude the proof noticing that if we take $R_\varepsilon = \delta R_1$, then for every $R \le R_\varepsilon$ there exists a radius $r \in (\delta R_1, R_1]$ such that $R = \delta^j r$ for some $j \in \mathbb{N}$ and thus $B_j(x) = B_{\delta^j r}(x) = B_R(x)$; this concludes the proof of (4.5).

4.3. **Gradient continuity by locally uniform convergence.** Here we prove that the gradient Du is continuous in Ω by taking a generic but fixed compact set $K \Subset \Omega$ and proving that Du is continuous in K; to do that, we show that the family of continuous maps

$$\mathcal{M}_R : x \in K \longmapsto \fint_{B_R(x)} V_{a(x)}\big(Du(y)\big)\, dy, \qquad R \le \frac{1}{4}\operatorname{dist}(K, \partial\Omega)\,,$$

satisfy the Cauchy's criterion uniformly in K. Since their limit coincides almost everywhere with $V_{a(\cdot)}(Du)$, the continuity of $V_{a(\cdot)}(Du)$ is hence proven. Gradient continuity follows easily thanks to triangle's inequality, the estimate in Lemma 2.5 and the fact that Du is bounded, see [3, Last section]. We take $\tilde{K}, \lambda_1$ as in the previous Paragraph 4.2 and we show that for every $\varepsilon > 0$ there exists a radius $R_2 \le R_0/4$ and a constant $\delta \in (0, 1)$, the first depending on $n, N, p, L/\nu, \omega_a(\cdot), |Db(\cdot)|/\nu, \operatorname{dist}(K, \partial\Omega)$, $\|H(\cdot, Du)\|_{L^1}$ and ε, the latter on n, N, p and L/ν such that, setting again as in the previous Paragraph $\tilde{R}_j = \delta^j(2R_2), \omega_j = \omega_{\log}(\tilde{R}_j), \mathcal{B}_j(x) = \mathcal{B}_{\tilde{R}_j,q}(x)$, it holds $B_j \subset B_{4R_2}(x) \subset \tilde{K}$ for all $x \in K$ and

$$\sup_{x\in K}\left|\big(V_{a(x)}(Du)\big)_{B_k(x)} - \big(V_{a(x)}(Du)\big)_{B_\ell(x)}\right| < \varepsilon\lambda_1 \qquad \text{for all } 1 \le \ell < k\,; \tag{4.6}$$

to see how this does lead to the conclusion we refer for instance to [2, Paragraph 4.2], [13, After *Step 3* in the proof of Theorem 4.3], [38, After eq. (124)], [39, After eq. (179)]. (4.6) indeed allows to prove that for any $\varepsilon > 0$ there exists a threshold R_ε, depending on $n, N, p, L/\nu, \omega_a(\cdot), |Db(\cdot)|/\nu, \mathrm{dist}(K, \partial\Omega), \|H(\cdot, Du)\|_{L^1}$ and ε such that

$$\sup_{x\in K}\left|\big(V_{a(x)}(Du)\big)_{B_{r_1}(x)} - \big(V_{a(x)}(Du)\big)_{B_{r_2}(x)}\right| < \varepsilon\lambda_1 \qquad \text{for all } 0 < r_1 \le r_2 \le R_\varepsilon\,.$$

To prove (4.6) we define $\delta \equiv \delta(n, N, p, L/\nu) \in (0, 1/4)$ as after (4.1) and we notice that from (4.4) and for $x \in K$

$$\begin{aligned}
&\left|\big(V_{a(x)}(Du)\big)_{B_k(x)} - \big(V_{a(x)}(Du)\big)_{B_\ell(x)}\right| \\
&\qquad\qquad \le \delta^{-n}\sum_{j=\ell}^{k-1} E_j(x) \le 2\delta^{-n}E_\ell(x) + 2\tilde{c}\delta^{-n}\sum_{j=0}^{\infty}\big[\omega_j + \mathcal{B}_j(x)\big]\lambda_1
\end{aligned}$$

since now $a_j \le \lambda_1$ for all $j \in \mathbb{N}_0$. For R_2 sufficiently small, depending on $n, N, p, L/\nu$, $\omega_a(\cdot), |Db(\cdot)|/\nu$ and ε, the second term is smaller than $\lambda_1\varepsilon/2$, uniformly in $x \in K$, thanks to Lemma 4.1 and (4.2). The first one is also smaller than $\lambda_1\varepsilon/2$ thanks to (4.5) again for R_2 sufficiently small. The proof is concluded.

5. Dini continuous coefficients

The proofs for $b(\cdot)$ Dini continuous are actually much simpler; therefore we are going only to highlight the necessary changes. We start defining the comparison maps $\bar{v} \in W^{1,p}(B_R; \mathbb{R}^N)$ and $v \in W^{1,p}(B_{R/2}; \mathbb{R}^N)$ exactly as in (3.3) and (3.4). Comparison estimate (3.12) is now replaced by

$$\fint_{B_{R/2}} \big|V_{\bar{a}}(Du) - V_{\bar{a}}(Dv)\big|^2\,dx \le c\Big[\big[\omega_{\log}(R)\big]^2 + \big[\omega_b(R)\big]^2\Big]\fint_{B_{3R/2}} H(x, Du)\,dx\,, \tag{5.1}$$

the constant having the same dependencies and $\omega_b(\cdot)$ as defined in (1.8). The only change in the proof is in the estimate for $\mathcal{D}_2$ (3.24): now we can simply estimate

$$\begin{aligned}
|\mathcal{D}_2| &\le c(p)\,\omega_b(R)\fint_{B_{R/2}}\big|V_{\bar{a}}(D\bar{v}) - V_{\bar{a}}(Dv)\big|\sqrt{H_{\bar{a}}\big(D\bar{v}\big)}\,dx \\
&\quad + c(p)\,\omega_b(R)\fint_{B_{R/2}}\big|V_{\bar{a}}(D\bar{v}) - V_{\bar{a}}(Dv)\big|\sqrt{H_{\bar{a}}\big(Dv\big)}\,dx \\
&\le 2c(p)\,\varepsilon\fint_{B_{R/2}}\big|V_{\bar{a}}(D\bar{v}) - V_{\bar{a}}(Dv)\big|^2\,dx \\
&\qquad\qquad + \frac{c(p)}{\varepsilon}\big[\omega_b(R)\big]^2\left[\fint_{B_{R/2}} H_{\bar{a}}(D\bar{v})\,dx + \fint_{B_{R/2}} H_{\bar{a}}(Dv)\,dx\right]
\end{aligned}$$

for $\varepsilon \in (0, 1)$ to be chosen, using Young's inequality. At this point we use (3.6)-(3.7) and reabsorb the integral of $|V_{\bar{a}}(D\bar{v}) - V_{\bar{a}}(Dv)|^2$ and the proof is concluded. The subsequent results have the same form except for the fact that $\omega_b(R)$ replaces $\mathcal{B}_{R,q}$ and the same with $2R$. Finally, also the results of Section 4 are formally identical only replacing $\mathcal{B}_j$ with $\tilde{\mathcal{B}}_j = \omega_b(\tilde{R}_j)$; the final result is however the same, since again one can make both $\tilde{\mathcal{B}}_j$ and $\sum_{j=0}^{\infty} \tilde{\mathcal{B}}_j$ arbitrarily small by choosing an initial radius $\tilde{R}$ small in view of the fact

$$\sum_{j=0}^{\infty} \omega_b(\tilde{R}_j) \leq \delta^{-1} \int_0^{2\tilde{R}} \omega_b(\rho)\, \frac{d\rho}{\rho} \qquad \text{if } \tilde{R}_j = \delta^j R \text{ with } R \leq \tilde{R} \tag{5.2}$$

(see [13, Eq. (46) and subsequent lines] for instance); observe that thanks to the first assumption in (1.9), that is the Dini continuity of $b(\cdot)$, the integral on the right-hand side vanishes as $\tilde{R} \searrow 0^+$.

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Paolo Baroni, Department of Mathematical, Physical and Computer Sciences, University of Parma, I-43124 Parma, Italy
Email address: `paolo.baroni@unipr.it`

Alessandra Coscia, Department of Mathematical, Physical and Computer Sciences, University of Parma, I-43124 Parma, Italy
Email address: `alessandra.coscia@unipr.it`