A NEW CONDITION ENSURING GRADIENT CONTINUITY FOR MINIMIZERS OF NON-AUTONOMOUS FUNCTIONALS WITH MILD PHASE TRANSITION

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ABSTRACT. We show that local minimizers of the non-autonomous functional

$$\mathcal{P}_{\log}(u,\Omega) = \int_\Omega |Du|^p\big(1+a(x)\log(e+|Du|)\big)\,dx, \qquad p>1,$$

have continuous gradient provided that the function $a(\cdot)$ is (almost everywhere) non-negative and weakly differentiable, and moreover its gradient locally belongs to the Lorentz-Zygmund space $L^{n,1}\log L$. This gives a precise insight of the fact that for this type of two-phase functionals the lack of uniform ellipticity can be overcome by additional regularity of the switching coefficient $a(\cdot)$; the novelty is that the condition is not pointwise, but has integral character, and actually improves the known results ensuring regularity for minimizers of such functionals.

1. Introduction

We consider local minimizers of the functional

$$\begin{aligned} u\in W^{1,1}_{\rm loc}(\Omega)\mapsto \mathcal{P}_{\log}(u,\Omega) :&= \int_\Omega \Big(|Du|^p + a(x)|Du|^p\log(e+|Du|)\Big)\,dx \\ &= \int_\Omega H(x,Du)\,dx, \end{aligned} \tag{1.1}$$

$p>1$, where $\Omega\subseteq\mathbb{R}^n$, $n\geq 2$ is a (bounded) domain and, to start, we suppose that the function $a(\cdot)\in W^{1,1}_{\rm loc}(\Omega)$ satisfies the following assumptions:

$$a(x)\geq 0 \quad \text{for a.e. } x\in\Omega, \qquad Da\in L^{n,1}\log L \quad \text{locally in } \Omega. \tag{1.2}$$

The gradient Da belongs to the Lorentz-Zygmund space $L^{n,1}\log L(K)$, for $K\subseteq\Omega$, if its decreasing rearrangement $|Da|^*$ on K is such that

$$\int_0^{|K|} r^{\frac1n}\log\Big(\frac{|K|}{r}\Big)|Da|^*(r)\frac{dr}{r}<\infty; \tag{1.3}$$

for more details we refer to Paragraphs 1.1 and 2.3. In the latter Paragraph we will show that, thanks to the generalized Sobolev's embedding by Cianchi & Pick ([11], see also [12]), functions whose gradient belongs to $L^{n,1}\log L$ agree almost everywhere to a continuous function; therefore, it is not restrictive to speak of pointwise values (and in particular, everywhere non-negativity) of $a(\cdot)$, and we are going to do it without being afraid to be misunderstood.

The energy in (1.1) is clearly made up of two parts and the value of the switching coefficient $a(\cdot)$ determines locally if the growth properties are of standard, p-Laplacian types, of almost-polynomial type (see Paragraph 2.2) or a mixture of the two; we refer to [7, 13, 14, 16] for extensive accounts on the origin and the scopes of this functional (and also related functionals), while we focus here only on the theoretical aspects. We also refer to [1, 2, 3] for remarkable results in the negative direction, i.e., regarding constructions of counterexamples for functionals including $\mathcal{P}_{\log}$. The general concept behind the study of minimizers of $\mathcal{P}_{\log}$ is that the mild difference between the phases, i.e., the fact the difference of growths between the two parts of the energy is of logarithmic size, $\log(e+|Du|)$, allows to catch precisely, in detail, the interplay between the regularity of the non-negative switching coefficient and the subsequent regularity of local minimizers; this paper (together with the joint [6]) will be a further contribution to this analysis.

The space whose definition is in (1.3) can be seen as a logarithmic correction (done in a scaling-invariant form) to the classic, significant Lorentz space $L^{n,1}$ [27], known to be, in several conjugations, the optimal space ensuring borderline continuity results for elliptic, parabolic and variational degenerate problems with standard growth: see for instance [9, 15, 16, 21] and also more details below. This correction, exactly of the size of the phase transition (see also Paragraph 1.2), is needed in order to rebalance the non-uniform ellipticity of the functional.

Under the assumption in (1.2) we prove that the gradient of any local minimizer of the functional $\mathcal{P}_{\log}$ is continuous:

Theorem 1.1. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer, in the sense of Definition 1, to the functional $\mathcal{P}_{\log}$ in (1.1); assume that the modulating coefficient $a(\cdot)$ satisfies (1.2). Then the precise Sobolev representative of Du is locally continuous in Ω.*

Moreover, the following local boundedness estimate holds: there exists a constant c, depending on $n, p, \|H(\cdot, Du)\|_{L^1(B_{2R}(x_0))}$ and Da, such that

$$\sup_{B_R(x_0)} H(\cdot, Du) \le c \fint_{B_{2R}(x_0)} H(x, Du)\, dx \tag{1.4}$$

for every ball $B_{2R}(x_0) \Subset \Omega$.

We stress that we are not interested here in quantifying the continuity of Du in terms of moduli of continuity; moreover we stress that the dependence of the constants on Da follows only from (1.7) and the use of the second part of Corollary 2.4 and it is therefore of the type described after the same Corollary.

This result is quite unexpected, since up to the present day very few results were available where regularity could be inferred from a Sobolev-type information on the map $x \mapsto f(x, \xi)$ *without the use of any embedding*. We will be more precise on this aspect and we will list some references in Paragraph 2.4; here we only highlight that a natural condition on the coefficient $a(\cdot)$ ensuring gradient continuity for local minimizers to $\mathcal{P}_{\log}$ is its log-Dini continuity: for $\omega(\cdot)$ a modulus of continuity for $a(\cdot)$, one must have

$$\int_0 \omega(\rho) \log\left(\frac{1}{\rho}\right) \frac{d\rho}{\rho} < \infty \tag{1.5}$$

(in other words: a modulus of continuity of $a(\cdot)$ must be integrable over $(0, R)$ for some $R > 0$ with respect to the measure $d\rho/\rho$, if corrected by a logarithmic term); see [8] or simply appropriately modify the proofs in the present paper and compare for instance with the results in [5, 24, 25]. We recall that a modulus of continuity for $a(\cdot)$ is an increasing (and concave, without loss of generality) function $\omega : [0, \mathrm{diam}(\Omega)) \to [0, +\infty)$ continuous in zero, such that $\omega(0) = 0$ and

$$|a(x) - a(y)| \le c\, \omega(|x - y|) \qquad \text{for all } x, y \in \Omega$$

for some $c > 0$. Notice that is tacit in the case (1.5) holds that $a(\cdot)$ must be continuous (and therefore everywhere non-negative), compare [8, Lemma 2.2].

It is possible to explicitly compute (and we shall do it in detail in Paragraph 2.4) a modulus of continuity for $a(\cdot)$ and it will turn out that, by embedding, if (1.2) holds then the function $a(\cdot)$ is (almost everywhere equal to) a log-Hölder continuous function, that is, its modulus of continuity $\omega(\cdot)$ satisfies

$$\limsup_{\rho \searrow 0}\ \omega(\rho) \log \frac{1}{\rho} < \infty. \tag{1.6}$$

Note that as a consequence we can assume that there exists $\tilde{L} \ge 1$ such that

$$\sup_{\rho \in (0,1]} \omega(\rho) \log \frac{1}{\rho} = \tilde{L}, \tag{1.7}$$

a fact that we are going to use.

What we find interesting is that the maximal regularity one can obtain using only the continuity of $a(\cdot)$ together with the quantitative estimate (1.6), that is, only using the information obtained by *the optimal Sobolev embedding*, is the De Giorgi-Nash-Moser-Harnack theory for local minimizers of (1.1); in particular, *for what concerns the gradient*, the best one can get is its higher integrability: essentially, the fact that $Du \in L^{p(1+\delta_g)}_{\rm loc}(\Omega)$ for a small exponent $\delta_g > 0$, see Theorem 2.5. Notice that the fact that u is a local minimizer of (1.1) directly implies $Du \in L^p_{\rm loc}(\Omega)$; in other words, under the assumptions in (1.2), the minimality implies only a minimal regularity improvement for the gradient of minimizers, and only at level of its integrability.

On the other hand, using directly, in its full power, the assumption on $a(\cdot)$ in (1.2), one can prove much better results as the gradient boundedness and continuity stated in Theorem 1.1. As an example, the function

$$a(x) = \int_0^{|x|} \frac{1}{s\,\log^\alpha(1/s)}\,ds \approx \frac{1}{\log^{\alpha-1}(1/|x|)} \tag{1.8}$$

is log–Dini in $B_{1/2}(0)$ iff $\alpha - 1 > 2$ but satisfies (1.2) for $\alpha - 1 > 1$, see Paragraph 2.4; thus for $\alpha \in (2, 3]$ our result applies but that of [8] does not. This might look surprising; it however follows from a natural choice in the perturbation argument, and this is allowed by a two-steps procedure, where first gradient higher integrability is proven thanks to (1.6). This two-steps proof forces the implementation of the iteration procedure in a slightly more than usual careful way, see (3.1) and the subsequent Lemmas. Finally, notice that the Dini-log assumption (1.6) and our Lorentz-Sobolev assumption (1.2) are in general not comparable; correlate also with the content of Paragraph 2.4.

Enlarging for a moment the perspective from which we examine our results, the study of problems with coefficients satisfying assumption of Sobolev-Lorentz type is attracting more and more interest in the very recent years, even for problems satisfying classic growth assumptions; in [15] it is shown that solutions to uniformly elliptic vectorial problems of the type

$$\operatorname{div}\Big(b(x)\frac{\varphi'(|Du|)}{|Du|}Du\Big) = f \qquad \text{in } \Omega \subseteq \mathbb{R}^n \tag{1.9}$$

are locally Lipschitz (and therefore C^1, after a computation of standard flavor) regular if both f and Db locally belong to the Lorentz space $L^{n,1}$ (at least when the spatial dimension is larger or equal to three). Here the scalar non-negative function φ' has growth of Orlicz type (that is, it satisfies the assumptions stated in Paragraph 2.2) and the coefficient b is elliptic: $0 < \nu \le b \le L$.

The gradient of b belongs to the Lorentz space $L^{n,1}(K)$, for $K \subseteq \Omega$, if

$$\|Db\|_{L^{n,1}} \approx_n \int_0^{|K|} \rho^{\frac{1}{n}}\,|Db|^*(\rho)\,\frac{d\rho}{\rho} \approx_n \int_0^\infty \big|\{K \cap |Db| > \lambda\}\big|^{\frac{1}{n}}\,d\lambda < \infty, \tag{1.10}$$

compare with (1.3) and (1.12); for several examples and an exhaustive description of the literature, we refer to [9, 15, 16, 21, 24, 27].

After this introduction, we introduce some specific terminology starting by specifying what do we mean by local minimizer to $\mathcal{P}_{\log}$:

Definition 1. A function $u \in W^{1,1}_{\rm loc}(\Omega)$ is a local minimizer of the functional $\mathcal{P}_{\log}$ in (1.1) if

$$|Du|^p\big(1 + a(x)\log(e + |Du|)\big) \in L^1_{\rm loc}(\Omega)$$

and the minimality condition

$$\mathcal{P}_{\log}\big(u, \operatorname{supp}(u - v)\big) \le \mathcal{P}_{\log}\big(v, \operatorname{supp}(u - v)\big) \tag{1.11}$$

is satisfied whenever $v \in W^{1,1}_{\rm loc}(\Omega)$ is such that $\operatorname{supp}(u - v) \Subset \Omega$.

In order to avoid unessential complication, in view of the fact that all the forthcoming results have local nature, we shall assume that u is a global minimizer, i.e., u is globally integrable ($H(\cdot, Du) \in L^1(\Omega)$) and (1.11) holds for every competitor $v \in W^{1,1}(\Omega)$ with $\operatorname{supp}(u - v) \subseteq \Omega$; dependence of constant could include, as a consequence, $\|Du\|_{L^p(\Omega)}$ or $\|H(\cdot, Du)\|_{L^1(\Omega)}$. Easy, minor modifications of the current proof would lead to the results in the case of local minimizers, or with the dependence stated in Theorem 1.1 (notice indeed that, once fixing a ball $B_{2R}(x_0) \Subset \Omega$, then u is a global minimizer in $B_{2R}(x_0)$).

1.1. **O'Neil space.** It is interesting in our opinion to stress the following characterization of the Lorentz-Zygmund space in terms of decay of the measure of super-level sets. The gradient of the function $a(\cdot)$ belongs to the Lorentz-Zygmund $L^{n,1}\log L$ if and only if its belongs to the O'Neil space $K^n(\log^+ K)^n$ and this happens if

$$\int_0^\infty |\{|Da| > \lambda\}|^{\frac{1}{n}} \log^+ \lambda \, d\lambda < \infty, \tag{1.12}$$

where $\log^+$ is the positive part of the logarithm: $\log^+ \lambda = \max\{\log\lambda, 0\}$ for $\lambda > 0$; compare this condition with the definition of the Lorentz space $L^{n,1}$ in (1.10). For the equivalente of these two spaces see [10, Theorem 10.5]. Again, we see that, at least formally, exactly as it happens with the pointwise assumption, also in this setting the condition can be obtained adding a perturbation of logarithmic size to the condition ensuring gradient continuity on the coefficients in (1.9).

1.2. **Generalizations and L^s-excesses.** Theorem 1.1 will follow as a significantly important corollary of the following result, which in some respects is more interesting (see for instance the forthcoming [6]) as it captures the essence of assumption (1.2), that is, the (local) log-Dini continuity of the excess. For simplicity, we state it for minimizers with finite global energy, in particular for minimizers in $W^{1,p}(\Omega)$. Notice that it is not clear whether the assumption (1.13) implies (1.7) (which is needed to ensure that the higher integrability result of Theorem 2.5 holds true); therefore (1.7) must be supplementary assumed.

Theorem 1.2. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer to the functional $\mathcal{P}_{\log}$ in (1.1), with $a(\cdot) \in L^1_{\rm loc}(\Omega)$ almost everywhere non-negative. There exists a large constant $s \equiv s(n, p, \tilde L, \|H(\cdot, Du)\|_{L^1(\Omega)})$ such that the following holds: Suppose that for every compact set $K \subseteq \Omega$, there exists a radius R_K such that*

$$\sup_{x_0\in K} \int_0^{R_K} \log\left(\frac{1}{\rho}\right)\left(\fint_{B_\rho(x_0)} \left|a - (a)_{B_\rho(x_0)}\right|^s dx\right)^{\frac{1}{s}} \frac{d\rho}{\rho} < \infty. \tag{1.13}$$

Assume moreover that (1.7) holds for some $\tilde L \geq 1$. Then the results of Theorem 1.1 remain valid: Du is locally continuous in Ω and the local estimate (1.4) holds.

The statement of the previous Theorem is somewhat clumsy, as requires regularity properties for $a(\cdot)$ depending on the minimum of a functional involving the same $a(\cdot)$; clearly the assumption (1.13) can be strengtgened by requiring the L^s-excess uniformly log-Dini for any s large or, probably more transparently, requiring (1.2). We shall indeed see that, if (1.2) holds, then (1.13) and (1.7) are satisfied: see Corollary 2.4 and Paragraph 2.4. We stress now that (1.13) can be interpreted as a log-Dini condition for the L^s-excess ($s \gg 1$ large) and this actually is weaker than the log-Dini condition for the L^∞-excess (that is, for the oscillation) in (1.5).

Theorem 1.2 clarifies why the study of the relation between integral assumptions and regularity of minimizers ends - at least at the gradient level - with the results of this paper: one can imagine that an integral condition ensuring gradient Hölder continuity should be of the type

$$\sup_{x_0\in K} \log\left(\frac{1}{R}\right)\left(\fint_{B_R(x_0)} \left|a - (a)_{B_R(x_0)}\right|^s dx\right)^{\frac{1}{s}} \lesssim R^\beta, \qquad R \leq R_K,$$

for some $\beta > 0$, but this would imply the Hölder regularity of $a(\cdot)$, as

$$\sup_{x_0\in K} \left(\fint_{B_R(x_0)} \left|a - (a)_{B_R(x_0)}\right|^s dx\right)^{\frac{1}{s}} \lesssim R^{\frac{\beta}{2}}$$

and Campanato embedding could apply, ensuring that $a \in C^{0,\frac{\beta}{2}}$ locally. Now the local Hölder regularity of the gradient for minimizers would follow from [7].

Using the same approach of this paper one can show that minimizers of the functional

$$u \in W^{1,1}_{\rm loc}(\Omega) \mapsto \int_\Omega |Du|^p \big(1 + a(x)\log^\alpha(e + |Du|)\big)\, dx, \qquad p > 1, \quad \alpha > 0$$

are locally continuous if $a(x) \geq 0$ a.e. in Ω and $Da \in L^{n,1}\log^{\alpha} L$ locally in Ω while an approach similar to that in [8] would require

$$\int_0 \omega(\rho)\log^{\alpha}\left(\frac{1}{\rho}\right)\frac{d\rho}{\rho} < \infty,$$

being ω a modulus of continuity for $a(\cdot)$. Again, such continuity assumption is not guaranteed by the sharp Sobolev embeddings in [11, 12], see the comment after (2.30). The results hold exactly in the same form for vector-valued minimizers $u \in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)$, $N > 1$; see the discussion in [7, Section 6] and apply the basic elements to our proof here. A similar approach, still different in some aspects, has been developed in [5] for systems with $p(x)$-growth, compare with [19, 24].

2. Preliminaries

2.1. **Notation.** In this paper we denote by c a general positive constant, possibly varying from line to line; special occurrences will be denoted by $c_1, c_*, \bar{c}$ or the like. All such constants will always be larger or equal than one; moreover relevant dependencies on parameters will be emphasized using parentheses, i.e., $c_1 \equiv c_1(n,p,\epsilon)$ means that c_1 depends on n,p and ϵ. For S a set of parameters, the notation $A \lesssim_S B$ means that there exists a constant $c \equiv c(S) \geq 1$ such that $A \leq c(S)B$, while $A \approx_S B$ means $A \lesssim_S B$ and $B \lesssim_S A$. We write $A \lesssim B$, $A \approx B$ if the constants in play are numerical and do not depend on any of the parameters in play.

$B_r(x_0)$ is the open ball with center x_0 and radius $r > 0$; when not important, or clear from the context, we shall omit denoting the center as follows: $B_r \equiv B_r(x_0)$. Unless otherwise stated, different balls in the same context will have the same centre. With $\mathcal{B} \subseteq \mathbb{R}^n$ being a measurable set with positive, finite Lebesgue measure $|\mathcal{B}| > 0$, and with $g\colon \mathcal{B} \to \mathbb{R}^k$, $k \geq 1$, being a measurable map, we shall denote by

$$(g)_{\mathcal{B}} := \fint_{\mathcal{B}} g(x)\,dx := \frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} g(x)\,dx$$

its integral average; $\omega_n = |B_1(0)|$. For $x > 1, \gamma \in \mathbb{R}$, we shall denote by $\log^{\gamma}(x)$ the quantity $[\log(x)]^{\gamma}$. We use the agreement that $\mathbb{N}$ is the set $\{1,2,3,\dots\}$ and $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. We set, for $s \geq 1, t \in [1,n), n \in \mathbb{N}\cap[2,+\infty)$

$$s_* = \frac{ns}{n+s}, \qquad t^* = \frac{nt}{n-t};$$

as $s_* \in [1,n)$ we will always have $(s_*)^* = s$. In view of the lines after Definition 1, the Lebesgue norms of the minimizer we shall be considering are to be intended finite over the whole Ω: in short,

$$\|Du\|_{L^p} = \|Du\|_{L^p(\Omega)}, \qquad \|H(\cdot,Du)\|_{L^1} = \|H(\cdot,Du)\|_{L^1(\Omega)}.$$

Finally, we use

$$\chi_{\{p<2\}} = \begin{cases} 0 & \text{if } p \geq 2 \\ 1 & \text{if } p < 2 \end{cases}$$

as χ_A is the characteristic function of the set A.

2.2. ***N*-functions setting.** In the following we are going to introduce a general class of tools, related to the so-called general class of N-functions. Even if the study of related equations, systems and functionals has been heavily developed in the last years, see for instance [4, 18, 20, 22], the main reason of our use is that this will significantly simplify notation and will provide a unified treatment for many of the results we are going to present and use. Note that in the aforementioned papers one can find an extensive bibliography for many of the results we shall mention.

We consider a function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(0) = \varphi'(0) = 0$, such that

$$\varphi \in C^1([0,\infty)) \cap C^2((0,\infty)) \qquad \text{and} \qquad \frac{1}{c_\varphi} \leq \mathcal{O}_{\varphi'}(t) = \frac{\varphi''(t)t}{\varphi'(t)} \leq c_\varphi \tag{2.1}$$

for all $t > 0$ and some $c_\varphi \geq 1$. Note that φ turns out to be convex and integrating by parts the last double-sided inequality we also get

$$\frac{1}{c_\varphi + 1} \leq \frac{\varphi'(t)t}{\varphi(t)} \leq c_\varphi + 1.$$

The inequality in (2.1) implies that φ' satisfies both the Δ_2 and ∇_2 condition, that can be equivalently stated by saying that for any $t, \lambda \in [0, \infty)$

$$\min\{\lambda^{c_\varphi}, \lambda^{1/c_\varphi}\}\varphi'(t) \leq \varphi'(\lambda t) \leq \max\{\lambda^{c_\varphi}, \lambda^{1/c_\varphi}\}\varphi'(t);$$

we will use several times this property without explicitly stating it. Clearly the analogous inequality above for O_φ ensures that the same property, *mutatis mutandis*, holds for φ. Once given φ as above, it is well defined its *Young's conjugate*:

$$\widetilde{\varphi}(t) = \sup_{s \in [0,\infty)} st - \varphi(s) = \max_{s \in [0,\infty)} st - \varphi(s);$$

it is possible to prove that the Orlicz ratio $\mathcal{O}_{\widetilde{\varphi}}(t)$ of $\widetilde{\varphi}$ is bounded below and above if also that of φ is. Moreover the following property holds:

$$\varphi(t) \leq \widetilde{\varphi}\Big(\frac{\varphi(t)}{t}\Big) \leq 2\varphi(t) \qquad \text{for all } t \in (0, \infty)$$

(see [4, 17, 18] and references therein), so that

$$\varphi(t) \approx_{c_\varphi} \widetilde{\varphi}\big(\varphi'(t)\big) \qquad \text{for all } t \in [0, \infty)$$

and also

$$s\frac{\varphi(t)}{t} \leq \varphi(s) + 2\varphi(t) \qquad \text{for all } t \in (0, \infty), s \in [0, \infty).$$

In this paper we will use in particular three of such functions: we shall make the choices $\varphi = H_p, H_{\log}, H_{\bar{a}}$ where, for $p > 1$ and $\bar{a} \geq 0$ appropriate,

$$(2.2) \quad H_p(t) = \frac{t^p}{p}, \qquad H_{\log}(t) := \frac{t^p}{p}\log(e + t),$$
$$H_{\bar{a}}(t) := H_p(t) + \bar{a}\, H_{\log}(t) = \frac{1}{p}\Big[t^p + \bar{a}\, t^p \log(e + t)\Big]$$

and $h_p(t) = H_p'(t)$, $H_{\log}(t) = H_{\log}'(t)$, $h_{\bar{a}}(t) = H_{\bar{a}}'(t)$, so that $H(x,t) = H_{a(x)}(t)$ and $h(x,t) = h_{a(x)}(t)$. Note that these functions satisfy (2.1) for a constant c_φ depending possibly only on p; in particular $c_{\varphi_{\bar{a}}}$ *does not depend* on $\bar{a}$ (see [7, 13] for instance). We have the following nice properties, if $t \in (0, \infty)$:

$$(2.3) \qquad \begin{cases} h_{\bar{a}}(t) = H_{\bar{a}}'(t) \approx t^{p-1} + \bar{a}t^{p-1}\log(e + t) = p\dfrac{H_{\bar{a}}(t)}{t} \\ h_{\bar{a}}'(t) = H_{\bar{a}}''(t) \approx_p t^{p-2} + \bar{a}t^{p-2}\log(e + t) \end{cases},$$

see for instance [13]. We shall use these properties also when $\bar{a} = a(x)$, so that $h_{a(x)}(t) = \partial_t H_{a(x)}(t) = \partial_t H(x,t)$, etc.

2.2.1. *Two calculus facts.* We prove here that, for a fixed $\bar{a} \geq 0$, the map $t \mapsto [H_{\bar{a}}(t)]^\sigma$ is bounded above and below, up to a constant depending only on p and σ, by convex function for $\sigma \geq 1/p$ and a concave one for $\sigma \in [0, 1/[p+1])$. We set

$$\Psi(t) = \sigma \int_0^t \frac{[H_{\bar{a}}(s)]^\sigma}{s}\, ds \approx_p \int_0^t \frac{d}{ds}[H_{\bar{a}}(s)]^\sigma\, ds = [H_{\bar{a}}(t)]^\sigma;$$

the equivalence follows from (2.3). We then prove that Ψ is convex, respectively concave, for the range of exponents σ above, directly computing

$$\Psi'(t) = \sigma\frac{[H_{\bar{a}}(t)]^\sigma}{t}, \quad \Psi''(t) = \frac{\sigma[H_{\bar{a}}(t)]^\sigma}{t^2}\Big[\sigma\frac{h_{\bar{a}}(t)t}{H_{\bar{a}}(t)} - 1\Big];$$

it is easy now to make explicit the "$\approx_p$" in (2.3) since

$$p \leq \mathcal{O}_{h_{\bar{a}}}(t) = \frac{t\, h_{\bar{a}}(t)}{H_{\bar{a}}(t)} \leq p + 1$$

and this guarantees the sign to Ψ'' in the two ranges of σ considered.

2.2.2. *Vector fields related to Orlicz functions.* It will be extremely useful to define now the associated vector fields $V_\varphi, A_\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$, $\ell \in \mathbb{N}$, by

$$V_\varphi(\xi) := \sqrt{\frac{\varphi'(|\xi|)}{|\xi|}}\,\xi, \qquad A_\varphi(\xi) := \frac{\varphi'(|\xi|)}{|\xi|}\xi, \tag{2.4}$$

whenever $\xi \in \mathbb{R}^\ell$; by (2.1) one can prove that V_φ is a bijection of $\mathbb{R}^\ell$. Moreover, under the assumptions stated above, V_φ precisely describes the monotonicity properties of the vector field $A_\varphi(\cdot)$; indeed, for $\xi_1, \xi_2 \in \mathbb{R}^\ell$ it holds that

$$|V_\varphi(\xi_1) - V_\varphi(\xi_2)|^2 \approx_{c_\varphi} \langle A_\varphi(\xi_1) - A_\varphi(\xi_2), \xi_1 - \xi_2\rangle. \tag{2.5}$$

The following two relations hold for any $\xi_1, \xi_2 \in \mathbb{R}^\ell$:

$$|V_\varphi(\xi_1)|^2 \approx_{c_\varphi} \varphi\big(|\xi_1|\big), \qquad |V_\varphi(\xi_1) - V_\varphi(\xi_2)|^2 \approx_{c_\varphi} \varphi''\big(|\xi_1| + |\xi_2|\big)|\xi_1 - \xi_2|^2. \tag{2.6}$$

According to (2.4), we define in particular the vector fields

$$V_p(\xi) := |\xi|^{\frac{p-2}{2}}\xi, \qquad V_{\log}(\xi) := \left(|\xi|^{p-2}\log(e + |\xi|) + \frac{|\xi|^{p-1}}{p(e + |\xi|)}\right)^{\frac{1}{2}}\xi \tag{2.7}$$

and $V_{\bar a} := V_{\varphi_{\bar a}}$ as V_φ for the choice $\varphi = H_{\bar a}$. Similarly are defined $A_p, A_{\log}, A_{\bar a}$. We stress that for any $\xi_1, \xi_2 \in \mathbb{R}^\ell$ we have, thanks to (2.6)

$$\begin{aligned} &\big|V_p(\xi_1) - V_p(\xi_2)\big|^2 + \bar a\big|V_{\log}(\xi_1) - V_{\log}(\xi_2)\big|^2 \\ &\quad\approx_p \varphi_p''\big(|\xi_1| + |\xi_2|\big)|\xi_1 - \xi_2|^2 + \bar a\,\varphi_{\log}''\big(|\xi_1| + |\xi_2|\big)|\xi_1 - \xi_2|^2 \\ &\quad= \varphi_{\bar a}''\big(|\xi_1| + |\xi_2|\big)|\xi_1 - \xi_2|^2 \approx_p \big|V_{\bar a}(\xi_1) - V_{\bar a}(\xi_2)\big|^2. \end{aligned} \tag{2.8}$$

Note that when $p \geq 2$, for any $\xi_1, \xi_2 \in \mathbb{R}^\ell$

$$|\xi_1 - \xi_2|^p \leq c\big|V_p(\xi_1) - V_p(\xi_2)\big|^2$$

holds, again for a constant c depending on p; this is an easy consequence of both (2.6) and triangle's inequality. Moreover

$$|\xi_1 - \xi_2| \leq c\big|V_p(\xi_1) - V_p(\xi_2)\big|^{\frac{2}{p}} + c|\xi_1|^{\frac{2-p}{2}}\big|V_p(\xi_1) - V_p(\xi_2)\big| \tag{2.9}$$

if $1 < p \leq 2$, see [21, Lemma 2]. Also the constant appearing in (2.9) depends only on p. An estimate that will be useful is the next, see [8, Lemma 2.5]: for $a_1, a_2 \geq 0$ and $\xi \in \mathbb{R}^\ell$

$$\big|V_{a_1}(\xi) - V_{a_2}(\xi)\big| \leq |a_1 - a_2||\xi|^{\frac{p}{2}}\log\big(e + |\xi|\big), \tag{2.10}$$

using the Lipschitz regularity of the function $t \in [0, \infty) \mapsto \sqrt{1 + t}$.

Finally, from (2.6) and (2.3) we have for every $\tilde\omega \in [0, 1]$

$$\begin{aligned} \big|V_{\log}(\xi_1) - V_{\log}(\xi_2)\big|^2 &\approx_p (|\xi_1| + |\xi_2|)^{p-2}\log(e + |\xi_1| + |\xi_2|)|\xi_1 - \xi_2|^2 \\ &\lesssim_p |V_p(\xi_1) - V_p(\xi_2)|^{2(1-\tilde\omega)}\big(|\xi_1| + |\xi_2|\big)^{p\tilde\omega}\log\big(e + |\xi_1| + |\xi_2|\big). \end{aligned} \tag{2.11}$$

2.2.3. *Logarithms.* We have the following useful properties of the logarithm:

$$\begin{cases} \log\left(\dfrac{x}{\ell}\right) \leq (1 + |\log \ell|)\log x & \text{for every } x \geq e \text{ and for all } \ell \in (0, 1]; \\ \log(e + x^\sigma) \leq 1 + c(\sigma)\log(e + x) & \text{for all } x \geq 0 \text{ and } \sigma > 0; \\ \log(e + xy) \leq \log(e + x) + \log(e + y) & \text{for all } x, y \geq 0 \\ \log(e + Ax) \leq A\log(e + x) & \text{for all } x \geq 0 \text{ and } A \geq 1. \end{cases} \tag{2.12}$$

The proofs are very simple, we only highlight for the second one that distinguishing the cases $\sigma < 1$, where $\log(e + x^\sigma) \leq \log(2(e + x))$, and $\sigma \geq 1$ where $\log(e + x^\sigma) \leq \sigma\log(e + x)$ leads to the result.

As a consequence, as proven in [8, Lemma 2.1], we have

Lemma 2.1. *Let $\tilde{\varsigma} > 1$, $\sigma, \beta, \theta \geq 0$ and let $f \in L^{\tilde{\varsigma}}(B_r)$ for some ball $B_r(x_0)$ with radius $r \leq e^{-1}$. Then there exists a constant c depending on n, β, σ, θ and $\tilde{\varsigma}$ such that*

$$\fint_{B_r} |f| \log^{\beta}\left(e + |f|^{\sigma}\right) dx \leq c\big(1 + r^{\theta}\|f\|_{L^1(B_r)}\big) \log^{\beta}\left(\frac{1}{r}\right)\left(\fint_{B_r} |f|^{\tilde{\varsigma}}\, dx\right)^{1/\tilde{\varsigma}}.$$

2.2.4. *Excesses.* We recall a standard property of the excess: for any $\mathcal{B}$ measurable set with positive and finite measure, for any $F \in L^s(\mathcal{B}; \mathbb{R}^{\ell})$, $\ell \in \mathbb{N}$, $s \geq 1$ it holds

$$\fint_{\mathcal{B}} \left|F - (F)_{\mathcal{B}}\right|^s dx \leq 2^s \fint_{\mathcal{B}} |F - \xi|^s\, dx \qquad \text{for each } \xi \in \mathbb{R}^{\ell}. \tag{2.13}$$

It will be useful to consider the following form of the classic excess functional: for $B \subseteq \Omega$ a ball, φ as in (2.1) (with particular emphasis on the choices in (2.2)) and V_{φ} as in (2.4), we set

$$E_{\varphi}(Du, B) := \left(\fint_B \left|V_{\varphi}(Du) - \big(V_{\varphi}(Du)\big)_B\right|^2 dx\right)^{1/2}.$$

We also notice the following useful equivalence:

$$E_{\varphi}(Du, B) \approx_{c_{\varphi}} \left(\fint_B \left|V_{\varphi}(Du) - V_{\varphi}\big((Du)_B\big)\right|^2 dx\right)^{1/2}, \tag{2.14}$$

see the Appendix in [18]; as a consequence, being $\bar{a}$ non-negative, we have by (2.8)

$$\begin{aligned}
&\fint_B \left|V_{\bar{a}}(Du) - \big(V_{\bar{a}}(Du)\big)_B\right|^2 dx \\
&\qquad \approx_p \fint_B \left|V_p(Du) - V_p\big((Du)_B\big)\right|^2 dx + \bar{a}\fint_B \left|V_{\log}(Du) - V_{\log}\big((Du)_B\big)\right|^2 dx \\
&\qquad \approx_p \fint_B \left|V_p(Du) - \big(V_p(Du)\big)_B\right|^2 dx + \bar{a}\fint_B \left|V_{\log}(Du) - \big(V_{\log}(Du)\big)_B\right|^2 dx.
\end{aligned} \tag{2.15}$$

2.3. **Rearrangements, Lorentz and Lorentz-Zygmund spaces.** Good references for most of the following facts are [23, 26]. Being $E \subseteq \mathbb{R}^n$ and $f : E \to \mathbb{R}^{\ell}, \ell \in \mathbb{N}$ both measurable, we define the distribution function of f as $\mu_f : [0, +\infty) \to [0, +\infty)$

$$\mu_f(\lambda) = \big|\{x \in E : |f(x)| > \lambda\}\big|$$

and the (non-increasing) rearrangement of f as the map $f^* : [0, \infty) \to [0, \infty]$ given by

$$f^*(\rho) = \inf\left\{\lambda \geq 0 : \mu_f(\lambda) \leq \rho\right\}. \tag{2.16}$$

Notice that f^* is non-increasing $\operatorname{supp}(f^*) \subseteq [0, |E|]$ and f and f^* have the same distribution, that is $\mu_{f^*} = \mu_f$. Moreover, if $E = B_R(x_0)$ is a ball, f is decreasingly radial (that is, $f(x) = g(|x - x_0|)$ with g a non-increasing function) we have the simple expression for its rearrangement

$$f^*(\rho) = g\Big(\Big(\frac{\rho}{\omega_n}\Big)^{1/n}\Big). \tag{2.17}$$

We shall focus on the definition of Lorentz-Zygmund space: the more common Lorentz spaces can be retrieved simply by our treatment by putting $\alpha = 0$, that is, $L^{\gamma,q}(E)$, for exponents $1 \leq \gamma < \infty$, is by definition $L^{\gamma,q} \log^{\alpha} L(E)$ if $\alpha = 0$. Given $\gamma \in [1, +\infty), q \in (0, \infty]$ and $\alpha \in \mathbb{R}$, the Lorentz–Zygmund spaces $L^{\gamma,q} \log^{\alpha} L(E)$ are defined in terms of the quasi-norms (they indeed lack of sub-additivity when $\gamma < q$ - employ here a well-established abuse of notation)

$$\begin{aligned}
\|f\|_{L^{\gamma,q}\log^{\alpha} L(E)} &= \left\|(\cdot)^{1/\gamma}\big(1 + |\log(\cdot)|\big)^{\alpha} f^*(\cdot)\right\|_{L^q((0,|E|), dt/t)} \\
&= \begin{cases} \left(\displaystyle\int_0^{|E|} \left[t^{1/\gamma}\left(1 + |\log t|\right)^{\alpha} f^*(t)\right]^q \frac{dt}{t}\right)^{1/q} & \text{if } q < \infty \\[2ex] \displaystyle\sup_{t \in (0,|E|)} \left[t^{1/\gamma}\left(1 + |\log t|\right)^{\alpha} f^*(t)\right] & \text{if } q = \infty \end{cases},
\end{aligned} \tag{2.18}$$

for a measurable function f over E, see [26, Chapter 9]. It is easy to show that when $\alpha = 0$ the norm defined in (2.18) agrees with the more classic quasi-norm for Lorentz spaces defined by

$$\|f\|_{L^{\gamma,q}(E)} = \begin{cases} \left(\gamma \int_0^\infty \left[\lambda^\gamma \mu_f(\lambda)\right]^{\frac{q}{\gamma}} \frac{d\lambda}{\lambda}\right)^{1/q} & \text{if } q < \infty \\ \sup_{\lambda>0} \lambda\left[\mu_f(\lambda)\right]^{\frac{1}{\gamma}} & \text{if } q = \infty \end{cases}$$

and that in the case $|E| < \infty$, it is possible to simply require in our case (1.3). It is useful now to introduce the maximal rearrangement of f as

$$f^{**}(\rho) = \frac{1}{\rho}\int_0^\rho f^*(\sigma)\, d\sigma \qquad \text{for } \rho \in (0,\infty);$$

notice that, being f^* decreasing, also f^{**} is and moreover $f^*(\rho) \le f^{**}(\rho)$ for all $\rho \in (0,\infty)$. The maximal rearrangement allows to define a useful norm, equivalent to the quasi-norm in (2.18), when $\gamma > 1$ and $q < \infty$:

$$\|f\|_{L^{\gamma,q}\log^\alpha L(E)} \approx_{\gamma,q,\alpha} \left(\int_0^{|E|} \left[t^{1/\gamma}\left(1 + |\log t|\right)^\alpha f^{**}(t)\right]^q \frac{dt}{t}\right)^{1/q} \tag{2.19}$$

see [26, Theorem 9.5.1]. To conclude, note that for all $0 < \gamma, \eta < \infty$, $0 < q \le \infty$ and $\alpha \in \mathbb{R}$ we have

$$\big\||f|^\eta\big\|_{L^{\gamma,q}\log^\alpha L(E)} = \|f\|^\eta_{L^{\gamma\eta,q\eta}\log^{\frac{\alpha}{\eta}} L(E)} \tag{2.20}$$

since $(f^\eta)^* = (f^*)^\eta$ for $\eta > 0$.

In the setting of Lorentz-Zygmund spaces, for a ball $B_{2R}(x_0) \subseteq \mathbb{R}^n$, a scalar function $f : B_{2R}(x_0) \to \mathbb{R}$ and a vectorial one $F : B_{2R}(x_0) \to \mathbb{R}^\ell$, $\ell \in \mathbb{N}$, we define the sums

$$E^\delta_{s,\alpha}(f)(x_0) := \sum_{k=0}^\infty \log^\alpha\left(\frac{1}{R_k}\right)\left(\fint_{B_k(x_0)} |f - (f)_{B_k(x_0)}|^s\, dx\right)^{\frac{1}{s}}, \tag{2.21}$$

and

$$M^\delta_{q,\alpha}(f)(x_0) := \sum_{k=0}^\infty R_k \log^\alpha\left(\frac{1}{R_k}\right)\left(\fint_{B_k(x_0)} |F|^q\, dx\right)^{\frac{1}{q}}, \quad q \in (1,n), \alpha \in \mathbb{R}, \tag{2.22}$$

where

$$R \in (0, [2e]^{-1}), \quad \delta \in (0, 1/2), \quad R_k = \delta^k R, \quad B_k(x_0) = B_{R_k}(x_0), \qquad k \in \mathbb{N}_0,$$

and $s \in [1, +\infty), q \in (1,n), \alpha \in \mathbb{R}$; it is clear that, if f has weak gradient sufficiently integrable, by Poincaré's inequality

$$E^\delta_{s,\alpha}(f)(x_0) \le c(n,s) M^\delta_{s_*,\alpha}(Df)(x_0). \tag{2.23}$$

These dyadic sums will take on great importance in view of the following lemmas, where f, F and R, δ are as before; compare the second with [21, Lemma 1].

Lemma 2.2. *Let $f \in L^1_{\mathrm{loc}}(\Omega)$ and $K \Subset \Omega$. There exists a constant, depending on n, δ and α such that*

$$\int_0^R \log^\alpha\left(\frac{1}{\rho}\right)\left(\fint_{B_\rho(x_0)} |f - (f)_{B_\rho(x_0)}|^s\, dx\right)^{\frac{1}{s}} \frac{d\rho}{\rho} \le c\, E^\delta_{s,\alpha}(f)(x_0),$$

$$E^\delta_{s,\alpha}(f)(x_0) \le c\int_0^{2R} \log^\alpha\left(\frac{1}{\rho}\right)\left(\fint_{B_\rho(x_0)} |f - (f)_{B_\rho(x_0)}|^s\, dx\right)^{\frac{1}{s}} \frac{d\rho}{\rho}$$

for any $R \le \mathrm{dist}(K, \partial\Omega)/2$.

Proof. We prove the two inequalities assuming that their right-hand sides are finite. Setting $R_{-1} = 2R$ we have that, for $\rho \in (R_k, R_{k-1}]$, $k \in \mathbb{N}_0$,

$$\fint_{B_k(x_0)} |f - (f)_{B_k(x_0)}|^s\, dx \le \frac{2^s}{\delta^n}\fint_{B_\rho(x_0)} |f - (f)_{B_\rho(x_0)}|^s\, dx$$

and

$$\fint_{B_\rho(x_0)} \left|f-(f)_{B_\rho(x_0)}\right|^s dx \le \frac{2^s}{\delta^n}\fint_{B_{k-1}(x_0)} \left|f-(f)_{B_{k-1}(x_0)}\right|^s dx,$$

using (2.13). Using the first property in (2.12),

$$\frac{1}{1+|\log\delta|}\log\left(\frac{1}{R_k}\right) \le \log\left(\frac{1}{R_{k-1}}\right) \le \log\left(\frac{1}{\rho}\right) \le \log\left(\frac{1}{R_k}\right) \tag{2.24}$$

for all $k \in \mathbb{N}_0$; thus, setting

$$\mathfrak{E}(\rho) = \log^\alpha\left(\frac{1}{\rho}\right)\left(\fint_{B_\rho(x_0)} \left|f-(f)_{B_\rho(x_0)}\right|^s dx\right)^{\frac{1}{s}}, \qquad \mathfrak{E}_k = \mathfrak{E}(R_k), \tag{2.25}$$

we have $\mathfrak{E}_k/c \le \mathfrak{E}(\rho) \le c\,\mathfrak{E}_{k-1}$ if $\rho \in (R_k, R_{k-1}]$ for any $k \in \mathbb{N}_0$, with $c \equiv c(n,s,\delta,\alpha)$. Hence, performing a standard computation,

$$\int_0^R \mathfrak{E}(\rho)\frac{d\rho}{\rho} = \sum_{k=1}^\infty \int_{R_k}^{R_{k-1}} \mathfrak{E}(\rho)\,\frac{d\rho}{\rho} \le c\sum_{k=0}^\infty |\log\delta|\,\mathfrak{E}_k = c\,E^\delta_{s,\alpha}(f)(x_0)$$

and

$$E^\delta_{s,\alpha}(f)(x_0) = \sum_{k=0}^\infty \mathfrak{E}_k \le \sum_{k=0}^\infty \frac{1}{\log 2}\int_{R_k}^{R_{k-1}} \mathfrak{E}_k\frac{d\rho}{\rho} \le c\int_0^{2R} \mathfrak{E}(\rho)\frac{d\rho}{\rho}. \tag{2.26}$$

□

Lemma 2.3. *Let $F \in L^{n,1}\log^\alpha L(B_{2R}(x_0))$ for some $\alpha \in \mathbb{R}$; then*

$$M^\delta_{q,\alpha}(F)(x_0) \le c(n,q,\alpha,\delta)\|F\|_{L^{n,1}\log^\alpha(B_{2R}(x_0))}.$$

Proof. First we consider the case $\alpha \ge 0$. If we set $g = |F|^q$ for ease of notation, we have

$$\fint_{B_k(x_0)} |F|^q\,dx = \frac{1}{|B_k(x_0)|}\int_0^{|B_k(x_0)|} g^*(\rho)\,d\rho = g^{**}(|B_k(x_0)|);$$

if then $R_k \le \rho < R_{k-1}$, setting $R_{-1} = 2R$, then by monotonicity (see also (2.24))

$$\begin{aligned}\mathfrak{M}_k &:= R_k\log^\alpha\left(\frac{1}{R_k}\right)\left(\fint_{B_k(x_0)} |F|^q\,dx\right)^{\frac{1}{q}} \\ &\le \delta^{-\frac{n}{q}}\left(1+|\log\delta|\right)^\alpha\left[\rho^q\log^{\alpha q}\left(\frac{1}{\rho}\right)g^{**}(\omega_n\rho^n)\right]^{\frac{1}{q}} =: c(n,q,\alpha,\delta)\mathfrak{M}(\rho)\end{aligned}$$

as $2 \le \delta^{-1}$; therefore, estimating as in (2.26) and using (2.19)

$$\begin{aligned}\sum_{k=0}^\infty \mathfrak{M}_k &\le c\sum_{k=0}^\infty \int_{R_k}^{R_{k-1}} \mathfrak{M}(\rho)\,\frac{d\rho}{\rho} = c\int_0^{2R}\left[\rho^q\log^{\alpha q}\left(\frac{1}{\rho}\right)g^{**}(\omega_n\rho^n)\right]^{\frac{1}{q}}\frac{d\rho}{\rho} \\ &= c\int_0^{|B_{2R}(x_0)|}\left[\varrho^{\frac{q}{n}}\left[\log\omega_n + |\log\varrho|\right]^{\alpha q}g^{**}(\varrho)\right]^{\frac{1}{q}}\frac{d\varrho}{\varrho} \\ &\le c(n,q,\alpha,\delta)\|g\|^{\frac{1}{q}}_{L^{\frac{n}{q},\frac{1}{q}}\log^{\alpha q}L(B_{2R}(x_0))}\end{aligned}$$

(notice that $n/q > 1$), that is, recalling the definition of g and (2.20),

$$M^\delta_{q,\alpha}(F)(x_0) \le c\left\||F|^q\right\|^{\frac{1}{q}}_{L^{\frac{n}{q},\frac{1}{q}}\log^{\alpha q}L(B_{2R}(x_0))} \le c\,\|F\|_{L^{n,1}\log^\alpha L(B_{2R}(x_0))}$$

for $c \equiv c(n,q,\alpha,\delta)$. We conclude noticing that the case $\alpha < 0$ is completely analogous, even simpler.

□

As a consequence of the previous results, since the Lorentz norm $L^{n,1}\log^\alpha L$ is defined in terms of an integral, by absolute continuity (cf. [26, Paragraph 9.9]) it follows that we can make, taking the initial radius R sufficiently small, the sums $E^\delta_{s,\alpha}(f)$ and $M^\delta_{q,\alpha}(F)(x_0)$ uniformly small.

Corollary 2.4. *Let $\Omega \subseteq \mathbb{R}^n$, $f \in L^1_{\mathrm{loc}}(\Omega)$ and suppose that for every compact set $K \Subset \Omega$, there exists a radius R_K such that*

$$\sup_{x_0 \in K} \int_0^{R_K} \log^\alpha\left(\frac{1}{\rho}\right)\left(\fint_{B_\rho(x_0)} \left|f-(f)_{B_\rho(x_0)}\right|^s dx\right)^{\frac{1}{s}} \frac{d\rho}{\rho} < \infty \tag{2.27}$$

for some $s \geq 1$ and $\alpha \in \mathbb{R}$. Set $\delta \in (0,1)$ and fix $K \Subset \Omega$; for every $\varepsilon \in (0,1)$, there exists a radius $R_\varepsilon > 0$, depending on $n, \alpha, \delta, f, \operatorname{dist}(\partial\Omega, K)$ and ε such that if $R \in (0, R_\varepsilon]$, then defining $E^\delta_{s,\alpha}(f)$ as in (2.21), one has

$$\sup_{x_0 \in K} E^\delta_{s,\alpha}(f)(x_0) \leq \varepsilon. \tag{2.28}$$

Moreover, suppose that $F \in L^{n,1}\log^\alpha L(\Omega)$ locally for $\alpha \in \mathbb{R}$; again fix $\delta \in (0,1)$, $q \in (1,n)$ and $K \Subset \Omega$. For every $\varepsilon > 0$, there exists a radius $R_\varepsilon > 0$ depending on $n, q, \alpha, \delta, F, \operatorname{dist}(\partial\Omega, K)$ and ε such that if $R \in (0, R_\varepsilon]$, then defining $M^\delta_{q,\alpha}(f)$ as in (2.22), one has

$$\sup_{x_0 \in K} M^\delta_{q,\alpha}(F)(x_0) \leq \varepsilon. \tag{2.29}$$

We stress now that explicit dependences on the function $a(\cdot)$ and/or on its gradient $Da(\cdot)$, in particular in the definition of several threshold radii, are uniquely derived from the use of the previous Corollary; that is, on the rate of blow-up of the integrand in (2.27) and the relation $\varepsilon - \delta$ in the absolute continuity of the Lorentz-Zygmund norm of $|Da|$.

2.4. **Embeddings and associate spaces.** Lorentz-Zygmund spaces are a special instance of *rearrangement invariant* (r.i. for short) *spaces*, that is, linear spaces of measurable functions equipped with a norm satisfying some natural properties (see [12, Definition 2.1]), the most significant being the fact that two functions having the same rearrangement (as defined in (2.16)) must have the same norm. It is clear that this last property holds for Lorentz and Lorentz-Zygmund spaces by the very definitions of the norm; one can also check that these are r.i. spaces. A r.i. space is a Banach space and a significant role in the theory is played by its associate space. If $X(E)$ is a r.i. space of functions $f : E \to \mathbb{R}^\ell$, $\ell \in \mathbb{N}$, its associate (rearrangement invariant) space $X'(E)$ is the space of all measurable function $g : E \to \mathbb{R}^\ell$ such that the (r.i.) norm

$$\|g\|_{X'(E)} = \sup_{f \not\equiv 0} \frac{1}{\|f\|_{X(E)}} \int_E |fg|\, dx$$

is finite. Given a r.i. space $X(E)$, another important related space is its *representation space* $\bar{X}(0,\infty)$, that is a space such that

$$\|f\|_{X(E)} = \|f^*\|_{\bar{X}(0,\infty)} = \|f^*\|_{\bar{X}([0,|E|])} \qquad \text{for all } f \in X(E);$$

representation spaces are not uniquely determined in general. Finally, given $X(E)$ r.i. space, $W^1X(E)$ is the r.i. invariant space of weakly differentiable functions whose partial derivatives belong to $X(E)$.

Now we recall some results from [11, 12] with the purpose to justify the assertion of quantitative continuity of $a(\cdot)$, that is, the estimate in (1.6). Given a r.i. invariant space X over a domain with "nice" boundary (for instance, Lipschitz regular) we define the quantity

$$\omega_X(\rho) = \left\|s^{-\frac{n-1}{n}}\chi_{(0,\rho^n)}(s)\right\|_{\bar{X}'(0,\infty)},$$

where $\chi_{(0,\rho^n)}$ is the characteristic function of the interval $(0,\rho^n)$ and $\bar{X}'(0,\infty)$ is the associate space of a representation space for X. It can be shown that different choices of the representation space result in functions ω_X equivalent, up to multiplicative constants, near zero and therefore the choice of a representation spaces becomes immaterial for what we concern, see [12, Proposition 5.1]. Theorem 1.3 of [11] or Theorem 3.4 of [12] state that, under these assumptions, if $\omega_X(\rho) \to 0$ as $\rho \searrow 0^+$, then a function belonging to W^1X is almost everywhere equal to a function having modulus of continuity ω_X; in other words, $W^1X(E) \hookrightarrow C^{\omega_X}(E)$ (in the sense of precise representatives). Note that local version of those embedding do not require any regularity on the boundary of the domain E.

In the case of our interst, $X = L^{n,1}\log L$ and (cf. [26, Theorem 9.6.8 (i)] with $\alpha_0 = \alpha_\infty = 1$, $\beta_0 = \beta_\infty = 0$), we have $X' = \bar{X}' = L^{n',\infty}\log^{-1} L$. Therefore, for $\rho \le 1$

$$\begin{aligned}(2.30)\qquad \omega(\rho) = \|s^{-\frac{n-1}{n}}\chi_{(0,\rho^n)}(s)\|_{\bar{X}'(0,\infty)} &= \|s^{-\frac{1}{n'}}\chi_{(0,\rho^n)}(s)\|_{L^{n',\infty}\log^{-1}L(0,\infty)} \\ &= \Big\|\frac{\chi_{(0,\rho^n)}(s)}{1+\log(1/s)}\Big\|_{L^\infty(0,\infty)} = \frac{1}{1+\log(1/\rho^n)} \le \frac{1}{\log(1/\rho)}\end{aligned}$$

and (1.6) is proven. Notice that in the case $Da \in L^{n,1}\log^\alpha L(\Omega)$ locally, exactly the same calculation above shows that if $\alpha > 2$, then a is almost everywhere equal to a log-Dini continuous function as $(L^{n,1}\log^\alpha L)' = L^{n',\infty}\log^{-\alpha} L$.

For $\varepsilon_0 \in (0,1)$ fixed and $a : B_{\varepsilon_0} \to \mathbb{R}$ radial, we take

$$a(x) = \int_0^{|x|} \frac{1}{s\log(1/s)\eta(s)}\,ds = \tilde{a}(|x|)$$

for $\eta : (0,\varepsilon_0) \to \mathbb{R}$ regular to be chosen, such that

$$s \mapsto \frac{1}{s\log(1/s)\eta(s)} \quad \text{is decreasing,} \qquad \int_0^{\varepsilon_0} \frac{1}{s\log(1/s)\eta(s)}\,ds < +\infty$$

so that $a(0) = 0$. Under these assumption it turns out that $Da \in L^{n,1}\log L(B_{\varepsilon_0})$, using (2.17) and changing variable, if and only if

$$\int_0^{\varepsilon_0} \frac{1}{s\log(1/s)\eta(s)}\log\left(\frac{1}{s}\right)ds = \int_0^{\varepsilon_0}\frac{1}{s\eta(s)}\,ds < +\infty.$$

On the other hand, by radiality and concavity, a modulus of continuity for $a(\cdot)$ is

$$\omega(\rho) = \int_0^\rho \frac{1}{s\log(1/s)\eta(s)}\,ds$$

and $a(\cdot)$ *is not* log–*Dini* if and only if

$$\int_0^{\varepsilon_0}\int_0^\rho \frac{1}{s\log(1/s)\eta(s)}\,ds\,\log\left(\frac{1}{\rho}\right)\frac{d\rho}{\rho} = +\infty.$$

This is equivalent to

$$\int_0^{\varepsilon_0}\frac{1}{s\eta(s)}\log\left(\frac{1}{s}\right)ds = +\infty.$$

As an example, therefore, taking $\eta(s) = \log^{\alpha-1}(1/s)$, $\alpha \in (2,3]$, a function log-Dini continuous but not in $W^1L^{n,1}\log L(B_{\varepsilon_0})$ is the function in (1.8).

2.5. **Known estimates for local minimizers of** (1.1)**.** The fact stated in (1.6) allows to make use of some estimates from [7]. In particular we have the following local higher integrability theorem, which will be fundamental in order to properly handle the logarithmic part of the functional as a perturbative one; it follows from [7, Theorem 4.1] once observing that (1.7) holds.

Theorem 2.5. *Let $u \in W^{1,H(\cdot)}(\Omega)$ be a minimizer of $\mathcal{P}_{\log}$ as in* (1.1)*, where $a(\cdot) \ge 0$ is continuous and its modulus of continuity satisfies* (1.7)*. Then there exists an exponent $\delta_g > 0$, depending only on $n, p, \tilde{L}$ and $\|Du\|_{L^p}$ such that*

$$H(\cdot, Du) \in L^{1+\delta_g}_{\rm loc}(\Omega).$$

Moreover it holds the following local estimate: there exists a constant $c \ge 1$, depending on $n, p, \tilde{L}$ and $\|Du\|_{L^p}$, such that

$$(2.31)\qquad \fint_{B_{\vartheta R}} \big[H(x,Du)\big]^{1+\delta_g}\,dx \le c\bigg(\fint_{B_R} H(x,Du)\,dx\bigg)^{1+\delta_g}$$

for every ball $B_R \equiv B_R(x_0) \subseteq \Omega$ with radius $R \le e^{-1}$ and every $\vartheta \in [1/2, 3/4]$.

As a consequence of the previous theorem and Lemma 2.1 we have the perturbative result we were mentioning before.

Lemma 2.6. *Let $u \in W^{1,H(\cdot)}(\Omega)$ be as in Theorem 2.5 and let $B_{2R} \equiv B_{2R}(x_0) \subseteq \Omega$ with $R \le e^{-1}$; then, for every $\gamma \in (0, 1+\delta_g)$ and $\beta > 0$, there exists a constant c depending on $n, p, \tilde{L}, \gamma, \beta$ and $\|H(\cdot, Du)\|_{L^1}$ such that*

$$\fint_{B_{\vartheta R}} |Du|^{p\gamma} \log^{\beta}\big(e + |Du|\big)\, dx \le c \log^{\beta}\Big(\frac{1}{R}\Big)\Big(\fint_{B_R} H(x, Du)\, dx\Big)^{\gamma} \tag{2.32}$$

for any $\vartheta \in [1/2, 3/4]$.

Proof. We use Lemma 2.1 with the choice $r = \vartheta R$, $f = |Du|^{p\gamma}$, $\sigma = 1/(p\gamma)$, $\theta = n\delta_g$ and $\varsigma = (1+\delta_g)/\gamma > 1$ and the previous (2.31) twice, getting

$$\begin{aligned}
&\fint_{B_{\vartheta R}} |Du|^{p\gamma} \log^{\beta}(e + |Du|)\, dx \\
&\le c\Big(1 + R^{n\delta_g} \int_{B_{\vartheta R}} \big(1 + |Du|^{p(1+\delta_g)}\big)\, dx\Big) \log^{\beta}\Big(\frac{1}{\vartheta R}\Big)\Big(\fint_{B_{\vartheta R}} |Du|^{p(1+\delta_g)}\, dx\Big)^{\frac{\gamma}{1+\delta_g}} \\
&\le c\Big(1 + \Big(\int_{B_R} H(x, Du)\, dx\Big)^{1+\delta_g}\Big) \log^{\beta}\Big(\frac{1}{R}\Big)\Big(\fint_{B_R} H(x, Du)\, dx\Big)^{\gamma}.
\end{aligned}$$

□

Proposition 2.7 (Reverse Hölder's inequality)**.** *Let u be a minimizer to $\mathcal{P}_{\log}$ as in Theorem 2.5. There exists a constant depending on $n, p, \tilde{L}$ and $\|Du\|_{L^p}$ such that*

$$\fint_{B_{3R/2}} H(x, Du)\, dx \le c\, H_{(a)_{B_{2R}}}\Big(\fint_{B_{2R}} |Du|\, dx\Big) \le c \fint_{B_{2R}} \big|V_{(a)_{B_{2R}}}(Du)\big|^2\, dx \tag{2.33}$$

holds for every ball $B_{2R} \equiv B_{2R}(x_0) \subseteq \Omega$ with R smaller than e^{-1}. Moreover

$$\fint_{B_R} \big|V_{(a)_{B_R}}(Du)\big|^2\, dx \le c \fint_{B_{2R}} H(x, Du)\, dx \tag{2.34}$$

for a constant c depending on the same quantities.

Proof. Using the self-improving character of reverse-Hölder inequalities, a standard consequence of Proposition 2.5 is that for every $\sigma > 0$ it holds

$$\fint_{B_{3R/2}} H(x, Du)\, dx \le c\Big(\fint_{B_{2R}} \big[H(x, Du)\big]^{\sigma}\, dx\Big)^{\frac{1}{\sigma}}$$

for a constant depending on $n, p, \tilde{L}, \|Du\|_{L^p}$ and σ. We choose $\sigma = 1/[2p]$ and we use sub-additivity to estimate the right-hand side in the following way:

$$\begin{aligned}
\fint_{B_{2R}} \big[H(x, Du)\big]^{\sigma}\, dx \le \fint_{B_{2R}} &\big[H_{(a)_{B_{2R}}}(Du)\big]^{\frac{1}{2p}}\, dx \\
&+ \fint_{B_{2R}} \big|a(\cdot) - (a)_{B_{2R}}\big|^{\frac{1}{2p}} |Du|^{\frac{1}{2}} \log^{\frac{1}{2p}}(e + |Du|)\, dx.
\end{aligned}$$

To get a bound for the first integral we observe that the concavity-type property of Paragraph 2.2.1 implies via Jensen's inequality that

$$\fint_{B_{2R}} \big[H_{(a)_{B_{2R}}}(Du)\big]^{\frac{1}{2p}}\, dx \le c(p)\Big[H_{(a)_{B_{2R}}}\Big(\fint_{B_{2R}} |Du|\, dx\Big)\Big]^{\frac{1}{2p}}.$$

To estimate the second integral, on the other hand, we use the first estimate of Lemma 2.1 with $f = |Du|^{\frac{1}{2}}, \sigma = 2, \beta = 1/(2p)$ and the fact that, as explained, $a(\cdot)$ is log-Hölder continuous, that is, inequality (1.7):

$$\begin{aligned}
&\fint_{B_{2R}} \big|a(\cdot) - (a)_{B_{2R}}\big|^{\frac{1}{2p}} |Du|^{\frac{1}{2}} \log^{\frac{1}{2p}}(e + |Du|)\, dx \\
&\qquad\qquad \le [\omega(2R)]^{\frac{1}{2p}} \fint_{B_{2R}} |Du|^{\frac{1}{2}} \log^{\frac{1}{2p}}(e + |Du|)\, dx \\
&\qquad\qquad \le c\Big[\omega(R) \log\Big(\frac{1}{2R}\Big)\Big]^{\frac{1}{2p}}\Big(\fint_{B_{2R}} |Du|\, dx\Big)^{p\cdot\frac{1}{2p}}
\end{aligned}$$

$$\leq c\left[H_{(a)_{B_{2R}}}\left(\fint_{B_{2R}}|Du|\,dx\right)\right]^{\frac{1}{2p}};$$

with $c \equiv c(n,p,\tilde{L},\|Du\|_{L^p})$; the proof of the first inequality of (2.33) is concluded. The second follows from the convexity of $t \mapsto H_{(a)_{B_{2R}}}(t)$ (see again Paragraph 2.2.1) together with (2.6) for $\varphi = H_{(a)_{B_{2R}}}$. The proof of (2.34) is similar, see for instance [8, Remark 3.8]. □

2.6. **Estimates for frozen functionals.** We collect here some results for minimizers of a reference functionals, obtained by freezing the switching coefficient $a(\cdot)$ in $\mathcal{P}_{\log}$, defined in (1.1). For these basic results basic references are the important paper of Lieberman [22], as long as the scalar case is concerned, and [20] for the vectorial case. We consider, for $E \subseteq \mathbb{R}^n$ bounded domain (that in our case will always be a ball inside Ω), minimizers of functionals of the type

$$\mathcal{P}_{\bar{a}}(w,E) := \int_E \left[|Dw|^p + \bar{a}|Dw|^p\log(e+|Dw|)\right]dx = \int_E H_{\bar{a}}(Dw)\,dx, \tag{2.35}$$

where $\bar{a} \geq 0$ is a constant.

The first result we want to recall is the following excess decay estimate, which encodes the local $C^{1,\beta}$ regularity of minimizers; it can be found in [20, Theorem 6.4], for similar results see also [4, Lemma 4.1] and [7, Theorem 3.1].

Theorem 2.8. *Let $v \in W^{1,p}(E)$ be a minimizer of the functional $\mathcal{P}_{\bar{a}}$ defined in (2.35) and let $B_R \equiv B_R(x_0) \subseteq E$. The excess-decay estimate*

$$\fint_{B_\varrho}\left|V_{\bar{a}}(Dv)-\big(V_{\bar{a}}(Dv)\big)_{B_\rho}\right|^2dx \leq c\left(\frac{\rho}{R}\right)^{2\beta}\fint_{B_R}\left|V_{\bar{a}}(Dv)-\big(V_{\bar{a}}(Dv)\big)_{B_R}\right|^2dx$$

holds for every couple of concentric balls $B_\rho \subseteq B_R$ for a constant $c \geq 1$ and an exponent $\beta \in (0,1)$ both depending only on n and p.

3. Various comparison results

In this section $u \in W^{1,H}(\Omega)$ will always be a minimizer of the functional $\mathcal{P}_{\log}$ in (1.1); we will work with a ball $B_{2R} \equiv B_{2R}(x_0) \subseteq \Omega$ fixed, with radius $2R \leq [2e]^{-1}$, and all the balls in play will have center x_0, therefore being concentric to B_{2R}.

For convenience, we are going to denote for $s > 1$

$$\mathfrak{E}_{r,s} = \log\left(\frac{1}{r}\right)\left(\fint_{B_r(x_0)}\left|a-(a)_{B_r(x_0)}\right|^s dx\right)^{\frac{1}{s}}$$

and

$$\bar{a}_r := (a)_{B_r(x_0)} = \fint_{B_r(x_0)} a(x)\,dx,$$

both for radii $r \leq 2R$. We consider

$$v = \operatorname*{arg\,min}_{w\in u+W_0^{1,\varphi_{\bar{a}_R}}(B_R)}\int_{B_R}\Big(|Dw|^p + \bar{a}_R|Dw|^p\log(e+|Dw|)\Big)\,dx. \tag{3.1}$$

The existence of the minimizer of the problem above follows from the higher integrability Proposition 2.5 and we refer to the discussion in [7, Chapter 5] for more details. For similar reasons, the minimizer satisfies the Euler-Lagrange equation in its weak formulation

$$\int_{B_R}\langle h_{\bar{a}_R}(Dv), D\varphi\rangle\,dx = 0$$

that is valid for every $\varphi \in W_0^{1,p}(B_R)$ with, moreover, $D\varphi \in L^p\log L(B_R)$ if $\bar{a}_R > 0$.

We start by deriving a comparison estimate.

Lemma 3.1 (Comparison)**.** *Let $v \in W^{1,p}(B_R)$ be the minimizer to the comparison Dirichlet problem (3.1); there exists an exponent*

$$s = s(n,p,\tilde{L},\|H(\cdot,Du)\|_{L^1}) \gg 1 \tag{3.2}$$

such that

$$\fint_{B_R} \big|V_{\bar a_R}(Du)-V_{\bar a_R}(Dv)\big|^2\,dx \le c\Big[\mathfrak{E}^2_{R,s}+\chi_{\{p<2\}}\mathfrak{E}^{p'}_{R,s}\Big]\fint_{B_{3R/2}} H(x,Du)\,dx \tag{3.3}$$

holds true for a constant c depending only on $n, p, \tilde L$ and $\|H(\cdot,Du)\|_{L^1}$.

Proof. We subtract the Euler-Lagrange equations for both u and v and we test such difference with the function $\varphi = u-v \in W^{1,p}_0(B_R)$: notice that, if $\bar a_R>0$, $Du-Dv \in L^p\log L(B_R)$ as $Du \in L^{p(1+\delta_g)}(B_R) \subseteq L^p\log L(B_R)$ by Theorem 2.5 and $Dv \in L^p\log L(B_R)$ as its energy is finite; thus φ is allowed in the Euler equation for Dv. On the other hand, $H(\cdot,Dv)\in L^1$ as $Dv\in L^p\log L(B_R)$ and $a(\cdot)$ is bounded on B_R, hence φ is also allowed in the Euler equation for Du. We compute

$$\begin{aligned} I &= \fint_{B_R} \big\langle h_{\bar a_R}(Du)-h_{\bar a_R}(Dv), Du-Dv\big\rangle\,dx = \fint_{B_R} \big\langle h_{\bar a_R}(Du), Du-Dv\big\rangle\,dx \\ &= \fint_{B_R} \big[\bar a_R - a(x)\big]\big\langle h_{\log}(Du), Du-Dv\big\rangle\,dx = II. \end{aligned}$$

Then (2.5) applied to V_p and $V_{\log}$ yields

$$I \ge \frac{1}{c(p)}\fint_{B_R} |V_{\bar a_R}(Du)-V_{\bar a_R}(Dv)|^2\,dx;$$

on the other hand we are going to estimate II, for $\varepsilon\in(0,1)$ to be chosen, using Young's inequality:

$$|II| \le c(p)\fint_{B_R} \big|a(x)-\bar a_R\big||Du|^{\frac{p-2}{2}+\frac{p}{2}}\log(e+|Du|)|Du-Dv|\,dx =: III \tag{3.4}$$

Now we need to distinguish two cases.

The case $p\ge 2$. In this case we can estimate, using Young's inequality

$$\begin{aligned} III \le c(p)\varepsilon\fint_{B_R} |V_{\bar a_R}(Du)-V_{\bar a_R}(Dv)|^2\,dx \\ + c(p,\varepsilon)\fint_{B_R} \big|a(x)-\bar a_R\big|^2|Du|^p\log^2(e+|Du|)\,dx. \end{aligned} \tag{3.5}$$

Note indeed that thanks to (2.6) applied to $\varphi(t)=t^p$ and the fact that $p\ge 2$ we have

$$|Du|^{\frac{p-2}{2}}|Du-Dv| \le \big[|Du|+|Dv|\big]^{\frac{p-2}{2}}|Du-Dv| \le c\big|V_p(Du)-V_p(Dv)\big|.$$

At this point, for $\delta_g\in(0,1)$ the higher integrability exponent of Proposition 2.5,

$$\begin{aligned} &\fint_{B_R} \big|a(x)-\bar a_R\big|^2|Du|^p\log^2(e+|Du|)\,dx \\ &\le \Big(\fint_{B_R} \big|a(x)-\bar a_R\big|^{2(1+\frac{2}{\delta_g})}\,dx\Big)^{\frac{\delta_g}{2+\delta_g}}\times \\ &\times\Big(\fint_{B_R} |Du|^{p(1+\frac{\delta_g}{2})}\log^{2(1+\frac{\delta_g}{2})}(e+|Du|)\,dx\Big)^{\frac{2}{2+\delta_g}}; \end{aligned} \tag{3.6}$$

we choose

$$s_1 = s_1(n,\delta_g) = 2\Big(1+\frac{2}{\delta_g}\Big).$$

For the second term we apply (2.32) with $r=3R/2, \beta=2(1+\delta_g/2), \gamma=1+\delta_g/2, \vartheta=2/3$ to get

$$\begin{aligned} &\fint_{B_R} |Du|^{p(1+\frac{\delta_g}{2})}\log^{2(1+\frac{\delta_g}{2})}(e+|Du|)\,dx \\ &\le c\log^{2(1+\frac{\delta_g}{2})}\Big(\frac{1}{R}\Big)\Big(\fint_{B_{3R/2}} H(x,Du)\,dx\Big)^{1+\frac{\delta_g}{2}}; \end{aligned} \tag{3.7}$$

notice that we also used (2.12). Taking into account these estimates, we get

$$\begin{aligned}
&\fint_{B_R} \big|a(x) - (a)_{B_R}\big|^2 |Du|^p \log^2(e+|Du|)\,dx \\
&\qquad\qquad \le c\log^2\Big(\frac{1}{R}\Big)\Big(\fint_{B_R} \big|a(x) - \bar a_R\big|^{s_1}\,dx\Big)^{\frac{2}{s_1}} \fint_{B_{3R/2}} H(x,Du)\,dx;
\end{aligned}$$

inserting this into (3.5), choosing ε sufficiently small and reabsorbing gives (3.3).

The case $p < 2$. In this case to estimate the integral in (3.4) we use (2.9) in order to get

$$\begin{aligned}
III &\le c\fint_{B_R} \big|a(x)-\bar a_R\big||Du|^{p-1}\log(e+|Du|)\big|V_p(Du)-V_p(Dv)\big|^{\frac{2}{p}}\,dx \\
&\qquad + c\fint_{B_R} \big|a(x)-\bar a_R\big||Du|^{\frac{p}{2}}\log(e+|Du|)\big|V_p(Du)-V_p(Dv)\big|\,dx \\
&\le 2\varepsilon\fint_{B_R} \big|V_p(Du)-V_p(Dv)\big|^2\,dx \\
&\qquad + c_\varepsilon\fint_{B_R} \big|a(x)-\bar a_R\big|^{p'}|Du|^p\log^{p'}(e+|Du|)\,dx \\
&\qquad + c_\varepsilon\fint_{B_R} \big|a(x)-\bar a_R\big|^2|Du|^p\log^2(e+|Du|)\,dx =: IV + V + VI;
\end{aligned}$$

we estimated both the integrals with Young's inequality, the first with conjugate exponents (p,p'), the second with $(2,2)$, both with $\varepsilon \in (0,1)$ to be chosen and $c_\varepsilon = c_\varepsilon(p,\varepsilon)$. The term IV will be reabsorbed in the left-hand side for ε sufficiently small, the term VI is estimated exactly as in (3.6) and subsequent lines (notice that the fact that $p \ge 2$ is there irrelevant) while we focus our attention on the remaining term (even if the estimate is very similar to the previous one):

$$\begin{aligned}
V &\le \Big(\fint_{B_R} \big|a(x)-(a)_{B_R}\big|^{p'(1+\frac{2}{\delta_g})}\,dx\Big)^{\frac{\delta_g}{2+\delta_g}} \times \\
&\qquad\qquad \times\Big(\fint_{B_R} |Du|^{p(1+\frac{\delta_g}{2})}\log^{p'(1+\frac{\delta_g}{2})}(e+|Du|)\,dx\Big)^{\frac{2}{2+\delta_g}}
\end{aligned}$$

by Hölder's inequality, where $\delta_g \in (0,1)$ is again the exponent appearing in Proposition 2.5. Now we choose

$$s = s(n,\delta_g) = s\big(n,p,\tilde L,\|H(\cdot,Du)\|_{L^1}\big) = p'\Big(1+\frac{2}{\delta_g}\Big).$$

Notice that since $p' > 2$, then $s > s_1$ - therefore we can, up to using Hölder's inequality, use the exponent s also in the case $p \ge 2$. For the second integral we again use (2.32) with evident changes with respect to (3.7):

$$\begin{aligned}
&\fint_{B_R} |Du|^{p(1+\frac{\delta_g}{2})}\log^{p'(1+\frac{\delta_g}{2})}(e+|Du|)\,dx \\
&\qquad\qquad \le c\log^{p'(1+\frac{\delta_g}{2})}\Big(\frac{1}{R}\Big)\Big(\fint_{B_{3R/2}} H(x,Du)\,dx\Big)^{1+\frac{\delta_g}{2}};
\end{aligned}$$

inserting this estimate into the bound for V completes the proof also in this case.

□

The comparison Lemma above allows to prove the following excess-like decay estimate for our minimizer u. We stress two aspects here: (3.8) is not a true excess decay estimate due to the presence of the second term on the right-hand side; nonetheless it will allow to prove gradient boundedness and continuity, since that term is stable under the operation of summation along a sequence of dyadic radii (see Lemma 2.3). Moreover, notice that the Orlicz function dictating the behaviour of the left-hand side is $H_{\bar a_R}$, that is, we are still considering the growth of the functional in (3.1); this is not suitable for iteration procedures, and this will require a further effort (see Lemma 3.3).

Proposition 3.2. *Let $u \in W^{1,H(\cdot)}(\Omega)$ be a minimizer of $\mathcal{P}_{\log}$ as in Theorem 2.5. There exists a constant c such that for every pair of concentric balls $B_\rho \equiv B_\rho(x_0) \subseteq B_{2R} \equiv B_{2R}(x_0) \subseteq \Omega$ with $R \le [2e]^{-1}$, it holds*

(3.8)
$$\begin{aligned}\fint_{B_\rho} \left|V_{\bar a_R}(Du) - \big(V_{\bar a_R}(Du)\big)_{B_\rho}\right|^2 dx \le c_1 \Big(\frac{\rho}{R}\Big)^{2\beta} \fint_{B_R} \left|V_{\bar a_R}(Du) - \big(V_{\bar a_R}(Du)\big)_{B_R}\right|^2 dx \\ + c_2 \Big(\frac{R}{\rho}\Big)^n \Big[\mathfrak{E}^2_{R,s} + \chi_{\{p<2\}} \mathfrak{E}^{p'}_{R,s}\Big] \fint_{B_{2R}} \left|V_{\bar a_{2R}}(Du)\right|^2 dx;\end{aligned}$$

$\beta \in (0,1)$ is the exponent appearing in Theorem 2.8, s in (3.2), the constant c_1 depends on n, p and $\tilde L$ and c_2 on the same quantities but also $\|H(\cdot, Du)\|_{L^1}$.

Proof. Let v be the solution to the comparison problem (3.1) over B_R. We have, using basic properties as (2.13)

$$\begin{aligned}\fint_{B_\rho} \left|V_{\bar a_R}(Du) - (V_{\bar a_R}(Du))_{B_\rho}\right|^2 dx &\le 4 \fint_{B_\rho} \left|V_{\bar a_R}(Du) - (V_{\bar a_R}(Dv))_{B_\rho}\right|^2 dx \\ &\le 8 \fint_{B_\rho} \left|V_{\bar a_R}(Du) - V_{\bar a_R}(Dv)\right|^2 dx + 8 \fint_{B_\rho} \left|V_{\bar a_R}(Dv) - (V_{\bar a_R}(Dv))_{B_\rho}\right|^2 dx.\end{aligned}$$

The first term is simply estimated using (3.3) and Proposition 2.7, as

$$\fint_{B_\rho} \left|V_{\bar a_R}(Du) - V_{\bar a_R}(Dv)\right|^2 dx \le \Big(\frac{R}{\rho}\Big)^n \fint_{B_R} \left|V_{\bar a_R}(Du) - V_{\bar a_R}(Dv)\right|^2 dx;$$

for the second, we use Theorem 2.8 together with the fact that, similarly to above, we can further estimate the right-hand side as follows:

$$\begin{aligned}\Big(\frac{\rho}{R}\Big)^{2\beta} \fint_{B_R} \left|V_{\bar a_R}(Dv) - \big(V_{\bar a_R}(Dv)\big)_{B_R}\right|^2 dx \le 8 \fint_{B_R} \left|V_{\bar a_R}(Dv) - V_{\bar a_R}(Du)\right|^2 dx \\ + 8 \Big(\frac{\rho}{R}\Big)^{2\beta} \fint_{B_R} \left|V_{\bar a_R}(Du) - \big(V_{\bar a_R}(Du)\big)_{B_R}\right|^2 dx.\end{aligned}$$

□

Next lemma is the localization estimate for the excess of the map u we were mentioning before. It allows to replace the function $V_{\bar a_R}$ with $V_{\bar a_\rho}$, for $\rho < R$, and, since we are considering the excess, it has a particularly clean form.

Lemma 3.3. *Let $u \in W^{1,H(\cdot)}(\Omega)$ and the balls $B_\rho \subseteq B_{2R} \subseteq \Omega$ be concentric as in Proposition 3.2, with $\rho \le R \le [2e]^{-1}$. Then*

$$\begin{aligned}\fint_{B_\rho} \left|V_{\bar a_\rho}(Du) - \big(V_{\bar a_\rho}(Du)\big)_{B_\rho}\right|^2 dx \le c_1 \fint_{B_\rho} \left|V_{\bar a_R}(Du) - \big(V_{\bar a_R}(Du)\big)_{B_\rho}\right|^2 dx \\ + c_2\, \mathfrak{E}^2_{R,s} \fint_{B_{2R}} \left|V_{\bar a_{2R}}(Du)\right|^2 dx,\end{aligned} \tag{3.9}$$

where c_1 depends on $n, p, \tilde L$ and $\|H(\cdot, Du)\|_{L^1}$ and c_2 on the same quantities but also on R/ρ.

Proof. Using (2.14)–(2.15) we see that

$$\begin{aligned}&\fint_{B_\rho} \left|V_{\bar a_\rho}(Du) - \big(V_{\bar a_\rho}(Du)\big)_{B_\rho}\right|^2 dx \\ &\le c(p) \fint_{B_\rho} \left|V_p(Du) - \big(V_p(Du)\big)_{B_\rho}\right|^2 dx \\ &\qquad + c(p)\, \bar a_\rho \fint_{B_\rho} \left|V_{\log}(Du) - \big(V_{\log}(Du)\big)_{B_\rho}\right|^2 dx \\ &\le c(p) \fint_{B_\rho} \left|V_{\bar a_R}(Du) - \big(V_{\bar a_R}(Du)\big)_{B_\rho}\right|^2 dx\end{aligned} \tag{3.10}$$

$$+c(p)\left(\bar a_\rho-\bar a_R\right)\fint_{B_\rho}\left|V_{\log}(Du)-V_{\log}\left((Du)_{B_\rho}\right)\right|^2dx.$$

Now we estimate separately, using (2.11) with $\tilde\omega=1/2$ and again (2.14), and then Hölder's inequality

$$\begin{aligned}
&\left(\bar a_\rho-\bar a_R\right)\fint_{B_\rho}\left|V_{\log}(Du)-V_{\log}\left((Du)_{B_\rho}\right)\right|^2dx\\
&\quad\le c\fint_{B_\rho}|a-\bar a_R|\,dx\fint_{B_\rho}\left|V_p(Du)-V_p\left((Du)_{B_\rho}\right)\right|\times\\
&\qquad\qquad\times\left(|Du|+\left|(Du)_{B_\rho}\right|\right)^{\frac p2}\log\left(e+|Du|+\left|(Du)_{B_\rho}\right|\right)dx\\
&\quad\le c\left(\frac R\rho\right)^n\left(\fint_{B_R}|a-\bar a_R|^s\,dx\right)^{\frac1s}\left(\fint_{B_\rho}\left|V_p(Du)-\left(V_p(Du)\right)_{B_\rho}\right|^2dx\right)^{\frac12}\times\\
&\qquad\qquad\times\left(\fint_{B_\rho}\left(|Du|+\left|(Du)_{B_\rho}\right|\right)^p\log^2\left(e+|Du|+\left|(Du)_{B_\rho}\right|\right)dx\right)^{\frac12}
\end{aligned}$$

for $s>1$ as in (3.2). By Lemma 2.1 with $f=(|Du|+|(Du)_{B_\rho}|)^p$, $\beta=2$, $\sigma=1/p$, $\theta=0$, $\tilde\varsigma=1+\delta_g$, using also the higher integrability estimate (2.31) and the first property in (2.12), we can estimate

$$\begin{aligned}
&\fint_{B_\rho}\left(|Du|+\left|(Du)_{B_\rho}\right|\right)^p\log^2\left(e+|Du|+\left|(Du)_{B_\rho}\right|\right)dx\\
&\qquad\le c\log^2\left(\frac1\rho\right)\left(\fint_{B_\rho}\left(|Du|+\left|(Du)_{B_\rho}\right|\right)^{p(1+\delta_g)}dx\right)^{\frac{1}{1+\delta_g}}\\
&\qquad\le c\left(1+\log\left(\frac R\rho\right)\right)^2\log^2\left(\frac1R\right)\left(\fint_{B_\rho}|Du|^{p(1+\delta_g)}\,dx\right)^{\frac{1}{1+\delta_g}}\\
&\qquad\le c\log^2\left(\frac1R\right)\fint_{B_{3R/2}}H(x,Du)\,dx;
\end{aligned}$$

the constant c depends on $n,p,\tilde L,\|H(\cdot,Du)\|_{L^1}$ and R/ρ. Now using Young's inequality, reabsorbing the term

$$\fint_{B_\rho}\left|V_p(Du)-\left(V_p(Du)\right)_{B_\rho}\right|^2dx$$

and then using the reverse Hölder inequality (2.33) give (3.9). □

A similar one, dealing with the right-hand side: this estimate will allow to perform the final iteration in a more transparent form.

Lemma 3.4. *Let $u\in W^{1,H(\cdot)}(\Omega)$ be a minimizer to $\mathcal P_{\log}$ as in Theorem 2.5 and $B_{2R}\subseteq\Omega$ with $2R\le[2e]^{-1}$. Then*

$$\begin{aligned}
\fint_{B_R}\left|V_{\bar a_R}(Du)-\left(V_{\bar a_R}(Du)\right)_{B_R}\right|^2dx&\le c\fint_{B_{2R}}\left|V_{\bar a_{2R}}(Du)-\left(V_{\bar a_{2R}}(Du)\right)_{B_{2R}}\right|^2dx\\
&\quad+c\,\mathfrak E^2_{2R,s}\fint_{B_{2R}}\left|V_{\bar a_{2R}}(Du)\right|^2dx
\end{aligned}$$

where c depends on $n,p,\tilde L$ and $\|H(\cdot,Du)\|_{L^1}$.

Proof. Estimating similarly as in (3.10) after enlarging the domain of integration yields

$$\begin{aligned}
&\fint_{B_R}\left|V_{\bar a_R}(Du)-\left(V_{\bar a_R}(Du)\right)_{B_R}\right|^2dx\\
&\qquad\le c(n,p)\fint_{B_{2R}}\left|V_{\bar a_{2R}}(Du)-\left(V_{\bar a_{2R}}(Du)\right)_{B_{2R}}\right|^2dx\\
&\qquad\qquad+c(p)\left(\bar a_R-\bar a_{2R}\right)\fint_{B_R}\left|V_{\log}(Du)-\left(V_{\log}(Du)\right)_{B_R}\right|^2dx
\end{aligned}$$

$$\leq c_\varepsilon \Big(\fint_{B_{2R}} |a - \bar{a}_{2R}|^s \, dx \Big)^{\frac{2}{s}} \log^2 \Big(\frac{1}{2R}\Big) \fint_{B_{2R}} |V_{\bar{a}_{2R}}(Du)|^2 \, dx$$
$$+ \varepsilon \fint_{B_R} \big|V_{\bar{a}_R}(Du) - \big(V_{\bar{a}_R}(Du)\big)_{B_R}\big|^2 \, dx$$

for every $\varepsilon \in (0,1)$, with c_ε depending on $n, p, \tilde{L}, \|H(\cdot, Du)\|_{L^1}$ and ε, and this completes the proof. □

The following Corollary is the goal of this section: simply merging Lemmas 3.2, 3.3 and 3.4 leads to the following corollary:

Corollary 3.5. *Let $u \in W^{1,H(\cdot)}(\Omega)$ be a minimizer to $\mathcal{P}_{\log}$ as in Theorem 2.5; let $B_R \equiv B_R(x_0) \subseteq \Omega$ with $R \leq [2e]^{-1}$. There exists $s > 1$ as in (3.2) such that for every $\varepsilon \in (0,1)$ there exists a constant $\delta \in (0, 1/2)$, depending on $n, p, \tilde{L}, \|H(\cdot, Du)\|_{L^1}$ and ε, such that*

$$\fint_{B_{\delta R}} \big|V_{\bar{a}_{\delta R}}(Du) - \big(V_{\bar{a}_{\delta R}}(Du)\big)_{B_{\delta R}}\big|^2 \, dx \leq \varepsilon \fint_{B_R} \big|V_{\bar{a}_R}(Du) - (V_{\bar{a}_R}(Du))_{B_R}\big|^2 \, dx$$
$$+ c_\varepsilon \Big[\mathfrak{E}_{R,s}^2 + \chi_{\{p<2\}} \mathfrak{E}_{R,s}^{p'}\Big] \fint_{B_R} |V_{\bar{a}_R}(Du)|^2 \, dx, \tag{3.11}$$

where the constant c_ε depends on $n, p, \tilde{L}, \|H(\cdot, Du)\|_{L^1}$ and ε.

Proof. As previously stated, merging the three lemmas and estimating $\mathfrak{E}_{R,s} \leq c\,\mathfrak{E}_{2R,s}$, then renaming $2R$ to R, gives

$$\fint_{B_\rho} \big|V_{\bar{a}_\rho}(Du) - \big(V_{\bar{a}_\rho}(Du)\big)_{B_\rho}\big|^2 \, dx \leq c_1 \Big(\frac{\rho}{R}\Big)^{2\beta} \fint_{B_R} \big|V_{\bar{a}_R}(Du) - \big(V_{\bar{a}_R}(Du)\big)_{B_R}\big|^2 \, dx$$
$$+ c_2 \Big[\mathfrak{E}_{R,s}^2 + \chi_{\{p<2\}} \mathfrak{E}_{R,s}^{p'}\Big] \fint_{B_R} |V_{\bar{a}_R}(Du)|^2 \, dx;$$

for every $\rho \leq R/2$ and constants c_1 depending on $n, p, \tilde{L}$ and $\|H(\cdot, Du)\|_{L^1}$, c_2 depending on $n, p, \tilde{L}, \|H(\cdot, Du)\|_{L^1}$ and R/ρ. Now we simply take $\rho = \delta R$ for

$$\delta = \min\Big\{ \Big(\frac{\varepsilon}{c_1}\Big)^{1/[2\beta]}, \frac{1}{2} \Big\}.$$

□

4. Iteration procedures

The conclusion follows the arguments of [21] but we propose the proof in detail both for the reader's convenience and also to highlight the various modifications needed to adapt it to our case. We prove the results under the assumptions of Theorem 1.2; if (1.2) holds, then (1.6)-(1.7) are in force due to (2.30) and (1.13) due to Lemma 2.2, (2.23) and finally (2.29).

We start by fixing a compact subset $K_1 \Subset \Omega$ and a ball $B_R(x_0) \subseteq \Omega$ with center $x_0 \in K_1$ and radius R smaller than $\tilde{R} = \min\{[2e]^{-1}, \operatorname{dist}(K_1, \partial\Omega)/2\}$; we will further reduce the value of $\tilde{R}$ several times.

We fix then $\delta \in (0, 1/2)$ as the constant, depending on $n, p, \tilde{L}$ and $\|H(\cdot, Du)\|_{L^1}$, given by Corollary 3.2 and corresponding to the choice $\varepsilon = 1/16$; accordingly, we set for $k \in \mathbb{N}_0$

$$R_k = \delta^k R, \qquad B_k = B_{R_k}(x_0), \qquad \bar{a}_k = (a)_{B_k} \tag{4.1}$$

and

$$v_k = \fint_{B_k} V_{\bar{a}_k}(Du) \, dx, \qquad E_k = \Big(\fint_{B_k} \big|V_{\bar{a}_k}(Du) - (V_{\bar{a}_k}(Du))_{B_k}\big|^2 \, dx \Big)^{\frac{1}{2}}, \tag{4.2}$$
$$d_k = |v_k| = \Big| \fint_{B_k} V_{\bar{a}_k}(Du) \, dx \Big|, \qquad \mathfrak{E}_k = \log\Big(\frac{1}{R_k}\Big) \Big(\fint_{B_k} |a - (a)_{B_k}|^s \, dx \Big)^{\frac{1}{s}} = \mathfrak{E}_{R_k,s},$$

for $s > 1$ fixed in Corollary 3.5. Using this compact notation (3.11), after taking square roots and performing standard manipulations, implies that for every $k \in \mathbb{N}_0$ we have

$$\begin{aligned} E_{k+1} &\le \frac{1}{4} E_k + \tilde{c}\left[\mathfrak{E}_k + \chi_{\{p<2\}}\mathfrak{E}_k^{\frac{p'}{2}}\right]\left(\fint_{B_k} \left|V_{\bar{a}_k}(Du)\right|^2 dx\right)^{\frac{1}{2}} \\ &\le \frac{1}{4} E_k + 2\tilde{c}\,\mathfrak{E}_k\left(\fint_{B_k} \left|V_{\bar{a}_k}(Du)\right|^2 dx\right)^{\frac{1}{2}}; \end{aligned} \tag{4.3}$$

the constant $\tilde{c} \ge 1$ depends on $n, p, \tilde{L}$ and $\|H(\cdot, Du)\|_{L^1}$. Notice indeed that if $p < 2$ then for any $j \in \mathbb{N}_0$

$$\mathfrak{E}_j^{\frac{p'}{2}} \le \Big[\sum_{k=0}^{\infty} \mathfrak{E}_k\Big]^{\frac{p'}{2}-1} \mathfrak{E}_j \le \mathfrak{E}_j \tag{4.4}$$

if we take $R \le \tilde{R}$ and $\tilde{R}$, depending on $n, p, \tilde{L}, \|H(\cdot, Du)\|_{L^1}, a(\cdot), \operatorname{dist}(\partial\Omega, K_1)$ *but not on* $x_0 \in K_1$, small enough so that (2.28) is satisfied with $\varepsilon = 1$.

4.1. **Gradient boundedness.** First we prove by induction that

$$\begin{aligned} \fint_{B_k} |Du|^p\big(1 + (a)_{B_k} \log(e + |Du|)\big)\, dx &= \fint_{B_k} H_{\bar{a}_k}(Du)\, dx = \fint_{B_k} \left|V_{\bar{a}_k}(Du)\right|^2 dx \\ &\le c_1 \fint_{B_R(x_0)} \left|V_{(a)_{B_R}}(Du)\right|^2 dx = c_1 \fint_{B_R(x_0)} H_{(a)_{B_R}}(Du)\, dx =: \lambda_0^2 \end{aligned}$$

for every $k \in \mathbb{N}_0$, with $c_1 = [10(1+\delta^{-n})]^2$, so that c_1 depends on $n, p, \tilde{L}$ and $\|H(\cdot, Du)\|_{L^1}$; this would yield the local boundedness result thanks to Lebesgue's differentiation theorem. The local estimate (1.4) in Theorem 1.2 will follow from the reverse Hölder inequality (2.34) together with a standard covering argument.

Observe that, using (2.8), triangle inequality and enlarging the domain of integration

$$\begin{aligned} d_{k+1} - d_k &= \Big|\fint_{B_{k+1}} V_{\bar{a}_{k+1}}(Du)\, dx\Big| - \Big|\fint_{B_k} V_{\bar{a}_k}(Du)\, dx\Big| \\ &\le |v_{k+1} - v_k| \le \fint_{B_{k+1}} \big|V_{\bar{a}_{k+1}}(Du) - \big(V_{\bar{a}_k}(Du)\big)_{B_k}\big|\, dx \\ &\le \fint_{B_{k+1}} \big|V_{\bar{a}_k}(Du) - \big(V_{\bar{a}_k}(Du)\big)_{B_k}\big|\, dx + \fint_{B_{k+1}} \big|V_{\bar{a}_{k+1}}(Du) - V_{\bar{a}_k}(Du)\big|\, dx \\ &\le \delta^{-n}\, E_k + \fint_{B_{k+1}} \big|V_{\bar{a}_{k+1}}(Du) - V_{\bar{a}_k}(Du)\big|\, dx; \end{aligned} \tag{4.5}$$

using telescopic summations gives

$$\begin{aligned} d_{j+1} &= d_0 + \sum_{k=0}^{j} \big(d_{k+1} - d_k\big) \\ &\le d_0 + \delta^{-n} \sum_{k=0}^{j} E_k + \sum_{k=0}^{j} \fint_{B_{k+1}} \big|V_{\bar{a}_{k+1}}(Du) - V_{\bar{a}_k}(Du)\big|\, dx. \end{aligned} \tag{4.6}$$

Now we separately estimate the last two terms of the right-hand side. For the first one, notice that if we estimate

$$\left(\fint_{B_k} \left|V_{\bar{a}_k}(Du)\right|^2 dx\right)^{\frac{1}{2}} \le E_k + d_k; \tag{4.7}$$

we see that if we take $R \le \tilde{R}$ with $\tilde{R} \equiv \tilde{R}(n, p, \tilde{L}, \|H(\cdot, Du)\|_{L^1}, a(\cdot), \operatorname{dist}(K_1, \partial\Omega))$ such that, being $\tilde{c} = \tilde{c}(n, p, \tilde{L}, \|H(\cdot, Du)\|_{L^1})$ the constant in (4.3)

$$\mathfrak{E}_k \le \sum_{k=0}^{\infty} \log\Big(\frac{1}{R_k}\Big)\Big(\fint_{B_k} \left|a - (a)_{B_k}\right|^s dx\Big)^{\frac{1}{s}} \le \frac{1}{8\tilde{c}} \le 1, \qquad k \in \mathbb{N}_0 \tag{4.8}$$

(and this is possible again thanks to Corollary 2.4) we can improve (4.3) to

$$E_{k+1} \le \frac{1}{2} E_k + c\, \mathfrak{E}_k d_k.$$

Summing up this sequence of inequalities for $k = 0$ to j, $j \in \mathbb{N}_0$ given, and reabsorbing gives

$$\sum_{k=0}^{j+1} E_k \le 2E_0 + c \sum_{k=0}^{j} \mathfrak{E}_k d_k. \tag{4.9}$$

To estimate the second term in (4.6) we single out the integrand and we recall the definition of $V_{\bar{a}}$ after (2.7); taking into account (2.10)

$$\left|V_{\bar{a}_{k+1}}(Du) - V_{\bar{a}_k}(Du)\right| \le c(p)\, |\bar{a}_{k+1} - \bar{a}_k| |Du|^{\frac{p}{2}} \log(e + |Du|); \tag{4.10}$$

averaging (4.10) over B_{k+1} and using Lemma 2.1

$$\begin{aligned}
&\fint_{B_{k+1}} \left|V_{\bar{a}_{k+1}}(Du) - V_{\bar{a}_k}(Du)\right| dx \\
&\quad \le \fint_{B_k} \left|a - (a)_{B_k}\right| dx \fint_{B_{k+1}} |Du|^{\frac{p}{2}} \log(e + |Du|)\, dx \\
&\quad \le c \log\Big(\frac{1}{R_{k+1}}\Big) \Big(\fint_{B_k} \left|a - (a)_{B_k}\right|^s dx \Big)^{\frac{1}{s}} \Big(\fint_{B_{k+1}} |Du|^p\, dx \Big)^{\frac{1}{2}} \\
&\quad \le c \log\Big(\frac{1}{R_k}\Big) \Big(\fint_{B_k} \left|a - (a)_{B_k}\right|^s dx \Big)^{\frac{1}{s}} \Big(\fint_{B_k} \left|V_{\bar{a}_k}(Du)\right|^2 dx \Big)^{\frac{1}{2}} \\
&\quad \le c\, \mathfrak{E}_k \big(E_k + d_k\big) \le E_k + c\, \mathfrak{E}_k d_k,
\end{aligned}$$

for s as in (3.2) and c depending on $n, p, \tilde{L}$ and $\|H(\cdot, Du)\|_{L^1}$, up to possibly reducing again the value of $\tilde{R}$ as done in (4.8); we also used (2.12) and (4.7). Estimating similarly as above and then summing up yields

$$\sum_{k=0}^{j} \fint_{B_{k+1}} \left|V_{\bar{a}_{k+1}}(Du) - V_{\bar{a}_k}(Du)\right| dx \le \sum_{k=0}^{j} E_k + c \sum_{k=0}^{j} \mathfrak{E}_k\, d_k,$$

with $c \equiv c(n, p, \tilde{L}, \|H(\cdot, Du)\|_{L^1})$. Inserting all these informations into (4.6) and then using (4.9) finally leads to

$$\begin{aligned}
d_{j+1} &\le d_0 + (1 + \delta^{-n}) \sum_{k=0}^{j} E_k + c \sum_{k=0}^{j} \mathfrak{E}_k\, d_k \le d_0 + 2(1 + \delta^{-n}) E_0 + c \sum_{k=0}^{j} \mathfrak{E}_k\, d_k \\
&\le \big[1 + 4(1 + \delta^{-n})\big] \Big(\fint_{B_R(x_0)} \left|V_{\bar{a}_0}(Du)\right|^2 dx \Big)^{\frac{1}{2}} + c \sum_{k=0}^{j} \mathfrak{E}_k\, d_k \le \frac{\lambda_0}{2} + c \sum_{k=0}^{j} \mathfrak{E}_k\, d_k
\end{aligned} \tag{4.11}$$

with c depending on $n, p, \tilde{L}$ and $\|H(\cdot, Du)\|_{L^1}$. It is easy now to show by induction that

$$d_j \le \lambda_0 \qquad \text{for all } j \in \mathbb{N}_0,$$

and this will conclude the proof of gradient boundedness as stated above: for $j = 0$ this is immediate. Suppose now that it holds for all $k \in \{0, 1, \ldots, j\}$ for some fixed $j \in \mathbb{N}_0$; by (4.11) and the inductive hypothesis we have

$$d_{j+1} \le \frac{\lambda_0}{2} + \bar{c}\, \lambda_0 \sum_{k=0}^{j} \mathfrak{E}_k \le \frac{\lambda_0}{2} + \bar{c}\, \lambda_0 \sum_{k=0}^{\infty} \mathfrak{E}_k \le \frac{\lambda_0}{2} + \bar{c}\, \lambda_0 \sup_{x_0 \in K_1} E^{\delta}_{s,1}(a)(x_0).$$

Now we further reduce the value of $\tilde{R}$ in order to have

$$\sup_{x_0 \in K_1} E^{\delta}_{s,1}(a)(x_0) \le \frac{\bar{c}}{2} \qquad \text{for every } R \le \tilde{R}, \tag{4.12}$$

and this is possible in view of Corollary 2.4, with a value of $\tilde{R}$ depending on $n, p, \tilde{L}, \|H(\cdot, Du)\|_{L^1}$, $a(\cdot)$ and $\mathrm{dist}(K_1, \partial\Omega)$; the proof of the local Lipschitz character of minimizers is concluded.

4.2. **VMO-type gradient regularity.** Given again a compact subset $K_1 \Subset \Omega$, we have as a consequence of the result in the previous paragraph that $Du \in L^\infty(K_2)$ where $K_1 \Subset K_2 \Subset \Omega$ (we can choose, for instance, $K_2 = \{x \in \Omega : \mathrm{dist}\,(x, K_1) \le \mathrm{dist}\,(\partial\Omega, K_1)/2\}$) and hence set

$$\lambda_1 := \left\| V_{a(\cdot)}(Du) \right\|_{L^\infty(K_2)} < +\infty. \tag{4.13}$$

In this intermediate technical step, needed for the forthcoming continuity proof, we will show the following property of VMO-regularity type: setting

$$\omega(\rho) = \sup_{r\le\rho} \sup_{x_0\in K_1} \fint_{B_r(x_0)} \left| V_{\bar a_r}(Du) - \left(V_{\bar a_r}(Du)\right)_{B_r(x_0)} \right|^2 dx$$

for $\rho \le \min\{[2e]^{-1}, \mathrm{dist}\,(\partial\Omega, K_1)/2\}$, we have

$$\lim_{\rho \searrow 0} \omega(\rho) = 0.$$

We take therefore $\tilde\varepsilon > 0$ and we show the existence of a threshold $R_{\tilde\varepsilon}$, depending on $n, p, \tilde L, \|H(\cdot, Du)\|_{L^1}$, $a(\cdot)$, $\mathrm{dist}(\partial\Omega, K_1)$ and $\tilde\varepsilon$, such that that it holds

$$\sup_{x_0\in K_1} \fint_{B_r(x_0)} \left| V_{\bar a_r}(Du) - \left(V_{\bar a_r}(Du)\right)_{B_r(x_0)} \right|^2 dx \le \tilde\varepsilon \lambda_1^2 \qquad \text{for every } r \le R_{\tilde\varepsilon}. \tag{4.14}$$

For $R \in (0, \min\{[2e]^{-1}, \mathrm{dist}\,(\partial\Omega, K_1)/4\}]$ fixed, we take $\delta \in (0, 1/2)$ as the constant given in Corollary 3.5 for $\varepsilon = \tilde\varepsilon/4$, so that δ depends on on $n, p, \tilde L$, $\|H(\cdot, Du)\|_{L^1}$ and $\tilde\varepsilon$. Then we choose $\bar R$, depending on $n, p, \tilde L, \|H(\cdot, Du)\|_{L^1}$, $a(\cdot)$ and $\mathrm{dist}(\partial\Omega, K_1)$ smaller than $\min\{[2e]^{-1}, \mathrm{dist}\,(\partial\Omega, K_1)/2\}$ but also so small that

$$\sup_{x\in K_1} E^\delta_{s,1}(a)(x) \le 1 \qquad \Longrightarrow \qquad \mathfrak{E}^2_{R,s} + \chi_{\{p<2\}} \mathfrak{E}^{p'}_{R,s} \le 2\mathfrak{E}^2_{R,s}$$

for all $R \le \bar R$, as in (4.4). Then we further possibly reduce its value so that, being c_δ the constant from Corollary 3.5 corresponding to the choice of δ made above,

$$\sup_{x\in K_1} E^\delta_{s,1}(a)(x) \le \frac{\tilde\varepsilon}{4c_\delta} \qquad \Longrightarrow \qquad c_\delta \mathfrak{E}^2_{R,s} \le \frac{\tilde\varepsilon}{4}$$

holds for every $R \le \bar R$; now $\bar R$ also depends on $\tilde\varepsilon$. By Corollary 3.5 we have, using triangle's inequality and (2.34)

$$\begin{aligned} \fint_{B_{\delta R}} \left| V_{\bar a_{\delta R}}(Du) - \left(V_{\bar a_{\delta R}}(Du)\right)_{B_{\delta R}} \right|^2 dx &\le \left[2\varepsilon + 2c_\delta \mathfrak{E}^2_{R,s}\right] \fint_{B_R} \left| V_{\bar a_R}(Du) \right|^2 dx \\ &\le \left[2\varepsilon + 2c_\delta \mathfrak{E}^2_{R,s}\right] \lambda_1^2 \le \tilde\varepsilon \lambda_1^2 \end{aligned} \tag{4.15}$$

for every radius R smaller than $\bar R$; therefore we get what wanted if we take $R_{\tilde\varepsilon} = \delta \bar R$. Notice indeed that the estimate in (4.15) is clearly uniform with respect to $x_0 \in K_1$ and if $r \le R_{\tilde\varepsilon}$, then there exists $R \le \bar R$ such that $r = \delta R$: the estimate in (4.15) is exactly (4.14).

4.3. **Gradient continuity.** As Lipschitz regularity has been proven, the non-uniform ellipticity of the functional becomes immaterial, see [15, 16], and gradient continuity follows from the regularity theory of functionals with standard growth and $WL^{n,1}$ dependence on the x variable; we anyway provide a short proof in the spirit of the previous ones. Given a compact set K_1 we find an intermediate one $K_1 \Subset K_2 \Subset \Omega$ and we again fix λ_1 as in (4.13).

We prove now that the gradient of our minimizer u is continuous in K_1 by showing that Du is the uniform limit of a sequence of continuous functions, its averages on small balls. More precisely, starting here from a generic but fixed radius $R = \min\{\tilde R, \mathrm{dist}\,(\partial\Omega, K_1)/2\}$, $\tilde R$ as chosen in Paragraph 4.1 so that (4.4)-(4.12) hold, and a point x in K_1, we define the quantities $v_j, d_j, E_j, \mathfrak{E}_j$ as in (4.1)–(4.2) starting from the radius R. $\delta \in (0, 1)$ is again the constant, depending on $n, p, \tilde L$ and $\|H(\cdot, Du)\|_{L^1}$, given by Corollary 3.5 for $\varepsilon = 1/16$. Notice that all the quantities just defined depend on the point x, center of the ball considered, but we shall keep this in the notation implicit - and somehow ambiguous - for simplicity, avoiding to make the dependences on x explicit. We stress, however, that all our estimates will be uniform with respect to x. We also highlight that we are going to further reduce the value of R. Thanks to Lebesgue's differentiation theorem, we know that

$$\lim_{j\to+\infty} v_j = \lim_{j\to+\infty} v_j(x) = V_{a(x)}\big(Du(x)\big) \qquad \text{for a.e. } x \in K_1$$

and as we will prove it is a uniform limit of continuous functions, it will follow that $V_{a(x)}(Du)$ is also a.e. equal in K_1 to a continuous function; being the set of Lebesgue's points dense, $V_{a(\cdot)}(Du)$ will turn out to be continuous (therefore it will satisfy (4.13) everywhere) and the continuity of Du will follow. Indeed by triangle's inequality, for every $x, y \in K_1$ we have

$$\begin{aligned}\big|V_{a(x)}\big(Du(x)\big) - V_{a(x)}\big(Du(y)\big)\big| \le \big|V_{a(x)}\big(Du(x)\big) - V_{a(y)}\big(Du(y)\big)\big| \\ + \big|V_{a(x)}\big(Du(y)\big) - V_{a(y)}\big(Du(y)\big)\big|;\end{aligned}$$

from (2.8) and the fact that $a(\cdot) \ge 0$ it follows

$$\big|V_{a(x)}\big(Du(x)\big) - V_{a(x)}\big(Du(y)\big)\big| \ge c(p)\big|V_p\big(Du(x)\big) - V_p\big(Du(y)\big)\big|$$

and using again (2.10) and then (4.13)

$$|V_{a(x)}\big(Du(y)\big) - V_{a(y)}\big(Du(y)\big)\big| \le c(p)|a(x) - a(y)|\lambda_1 \log(e + \lambda_1^{\frac{2}{p}})$$

as $|Du|^p \le |V_{a(\cdot)}(Du)|^2$. Thus from the continuity of $V_{a(\cdot)}(Du)$ and that of $a(\cdot)$ it follows the continuity of $V_p(Du)$ and it is well known that V_p is a locally bi-Lipschitz bijection; therefore Du will be continuous if also $V_{a(\cdot)}(Du)$ will be.

Therefore now we are going to prove that for every $\varepsilon \in (0,1)$ there exists an index $\bar{\jmath}$ depending on $n, p, \tilde{L}$, $\|H(\cdot, Du)\|_{L^1}$, $a(\cdot)$, $\mathrm{dist}(\partial\Omega, K_1)$ and ε *but not on* $x \in K_1$ such that

$$\big|V_{a(x)}(Du(x)) - v_{\bar{\jmath}}(x)\big| \le \varepsilon\lambda_1;$$

this will prove the uniform convergence of $v_j(\cdot)$. We start noticing that working similarly as how done to prove (4.9) (but this time summing the previous inequalities for $k = \bar{\jmath}$ to m, $\bar{\jmath} \ge 1$ to be chosen and $m > \bar{\jmath}$, reabsorbing and then passing to the limit for $m \to +\infty$), we get

$$\sum_{k=\bar{\jmath}}^{\infty} E_k \le 2E_{\bar{\jmath}} + c\sum_{k=\bar{\jmath}}^{\infty} \mathfrak{E}_k d_k \le 2E_{\bar{\jmath}} + c\lambda_1 \sum_{k=\bar{\jmath}}^{\infty} \mathfrak{E}_k; \tag{4.16}$$

notice indeed that from our choice of $R \le \tilde{R}$ and the fact stated in (4.13), it follows that $d_k \le \lambda_1$ for all $k \in \mathbb{N}_0$. Note that the previous estimate is uniform in K_1, being the dependence on the point $x \in K_1$ (center of the balls B_k defining in turn the excess) implicit for simplicity of the notation.

Now for almost every $x \in K_1$ and for $\bar{\jmath} \ge 1$ we have, similarly as in (4.5)–(4.6)

$$\begin{aligned}\big|V_{a(x)}(Du(x)) - v_{\bar{\jmath}}(x)\big| &\le \sum_{k=\bar{\jmath}}^{\infty} |v_{k+1}(x) - v_k(x)| = \sum_{k=\bar{\jmath}}^{\infty} |v_{k+1} - v_k| \\ &\le \delta^{-n}\sum_{k=\bar{\jmath}}^{\infty} E_k + \sum_{k=\bar{\jmath}}^{\infty} \fint_{B_{k+1}} \big|V_{\bar{a}_{k+1}}(Du) - V_{\bar{a}_k}(Du)\big|\,dx\end{aligned}$$

and due to

$$\sum_{k=\bar{\jmath}}^{\infty} \fint_{B_{k+1}} \big|V_{\bar{a}_{k+1}}(Du) - V_{\bar{a}_k}(Du)\big|\,dx \le \sum_{k=\bar{\jmath}}^{\infty} E_k + c\,\lambda_1 \sum_{k=\bar{\jmath}}^{\infty} \mathfrak{E}_k$$

(see before (4.11)), merging the previous estimates and (4.16) leads to

$$\big|V_{a(x)}(Du(x)) - v_{\bar{\jmath}}\big| \le c_1 E_{\bar{\jmath}} + c_2\lambda_1 \sum_{k=0}^{\infty} \mathfrak{E}_k$$

being both c_1 and c_2 constants depending on $n, p, \tilde{L}$ and $\|H(\cdot, Du)\|_{L^1}$. Now, given $\varepsilon > 0$, we reduce the value of R so that

$$c_2 \sum_{k=0}^{\infty} \mathfrak{E}_k = c_2 \sum_{k=0}^{\infty} \mathfrak{E}_k(x) \le c_2 \sup_{x\in K_1} E^{\delta}_{1,s}(a)(x) \le \frac{\varepsilon}{2}$$

for all $x \in K_1$ in view of Corollary 2.4. R, at this point, will depend on $n, p, \tilde{L}, \|H(\cdot, Du)\|_{L^1}$, $a(\cdot)$, $\operatorname{dist}(\partial\Omega, K_1)$ and ε. Finally, being now R_ε the radius corresponding to the choice $\tilde{\varepsilon} = (\varepsilon/[2c_1])^2$ in (4.14), we choose $\bar{\jmath}$ so large, depending on $n, p, \tilde{L}, \|H(\cdot, Du)\|_{L^1}, a(\cdot), \operatorname{dist}(\partial\Omega, K_1)$ and ε, so that $\delta^{\bar{\jmath}} R \le R_\varepsilon$. With this choice we have, recalling that $B_{\bar{\jmath}} = B_{\delta^{\bar{\jmath}} R}(x)$,

$$c_1 E_{\bar{\jmath}} = c_1 E_{\bar{\jmath}}(x) \le c_1 \left(\fint_{B_{\bar{\jmath}}} \left| V_{\bar{a}_j}(Du) - (V_{\bar{a}_j}(Du))_{B_j} \right|^2 dx \right)^{\frac{1}{2}} \le \frac{\varepsilon}{2} \lambda_1;$$

inserting the information in the last two displays into (4.16) the proof is concluded.

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