RELAXATION AND CONVEXITY OF FUNCTIONALS WITH POINTWISE NONLOCALITY

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Abstract. It is shown that the relaxation of the integral functional involving argument deviations

$$I(u) := \int_\Omega f(x, \{u_i(g_{ij}(x))\}_{i,j=1}^{k,l})\, d\mu_\Omega(x),$$

in weak topology of a Lebesgue space $(L^p(\Theta,\mu_\Theta))^k$, where $(\Omega,\Sigma(\Omega),\mu_\Omega)$ and $(\Theta,\Sigma(\Theta),\mu_\Theta)$ are standard measure spaces, the latter with nonatomic measure, coincides with its convexification whenever the matrix of measurable functions $g_{ij}\colon \Omega\to\Theta$, satisfies the special condition, called unifiability, which can be regarded as collective nonergodicity or commensurability property, and is automatically satisfied only if $k = l = 1$. If, however, either $k > 1$ or $l > 1$, then it is shown that as opposed to the classical case without argument deviations, for nonunifiable function matrix $\{g_{ij}\}$ one can always construct an integrand f so that already the functional I itself be weakly lower semicontinuous but not convex.

1. Introduction

Let $(\Omega,\Sigma(\Omega),\mu_\Omega)$ and $(\Theta,\Sigma(\Theta),\mu_\Theta)$ be *standard measure spaces* (i.e. Ω and Θ Polish spaces, $\Sigma(\Omega)$ and $\Sigma(\Theta)$ completions of the respective Borel σ-algebrae with respect to the finite Borel measures μ_Ω and μ_Θ). The reference to the particular σ-algebra and measure will further on be omitted for the sake of brevity, whenever this cannot lead to a confusion. The measure μ_Θ is further supposed to be nonatomic.

Define the functional $I\colon L^p_k \equiv (L^p(\Theta))^k \to \mathbb{R}\cup\{+\infty\} \equiv \bar{\mathbb{R}}$ by the relationship

$$I(u) := \int_\Omega f(x, \{u_i(g_{ij}(x))\}_{i,j=1}^{k,l})\, d\mu_\Omega(x), \tag{1}$$

where $u := (u_1, \dots, u_k)$ is a vector with components $u_i \in L^p(\Theta)$, $i = 1,\dots,k$, $f\colon \Omega\times\mathbb{R}^{kl}\to\bar{\mathbb{R}}$ is an integrand (i.e. a $\Sigma(\Omega)\otimes\mathcal{B}^{kl}$ measurable function, $\mathcal{B}^{kl}$ standing for the Borel σ-algebra of $\mathbb{R}^{kl}$), while $g_{ij}\colon \Omega\to\Theta$, $i = 1,\dots,k$, $j = 1,\dots,l$ are measurable functions satisfying

$$e\subset\Theta, \mu_\Theta(e) = 0 \Rightarrow \mu_\Omega(g_{ij}^{-1}(e)) = 0. \tag{2}$$

Note that the latter condition is necessary to define correctly the compositions $u_i\circ g_{ij}$ for $u_i\in L^p(\Theta)$ (or just for u_i being classes of μ_Θ-a.e. equal measurable functions). Thus we will always assume it satisfied.

The primary source of such functionals is provided by optimal control theory. In fact, an optimal control problem for differential equations with deviating argument can be easily reduced by introducing auxiliary variables as in [1, 2] to the purely variational problem of minimizing a functional of the type (1) with $k > 1$. It is

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worth noting however that there are also other applications of such functionals even in the "scalar" case $k = 1$, mainly related to functional differential equations with variational structure (see [3] and references therein).

This paper studies the relationship between convexity and weak lower semicontinuity of the above functionals. In particular, we study when the relaxation of the latter in the weak topology $sc^-(w-L^p_k)I$ (i.e. the greatest weakly lower semicontinuous functional less than I) coincides with its convexification I^{**} (i.e. the greatest convex and strongly lower semicontinuous functional less than I). It is shown that the situation in general changes drastically with respect to the well-studied case $k = l = 1$, $\Omega = \Theta$ with $g_{11}(x) = x$, sometimes referred to as "local". The well-known results of A. Ioffe show that in this case the convexity of the functional, (and even, in an appropriate sense, that of the integrand), is *necessary* for the functional to be weakly lower semicontinuous (see, e.g., [5, 6, 1]). This is however strictly related to the fact that in the definition of the functional one has then the *local* (namely, Nemytskiĭ) operator of u under the integration sign. In the presence of just "pointwise" nonlocality given by the functions g_{ij}, as we show, convexity in general is no more necessary for the functional studied to be weakly lower semicontinuous. We provide, moreover, a precise condition on the functions g_{ij} in order that the relaxation of these functionals coincide with their convexifications. This condition is satisfied automatically only if $k = l = 1$ (whatever the function g_{11} is), while in general it is a special extension to function matrices of the notion of nonergodicity of a single function. It provides very strong relationship between functions in each row and rather weak relationship between functions in each column.

2. Unifiability of functions and orbital conditions

We will call a measurable (with respect to $\Sigma(\Theta)$) function $\gamma\colon \Theta \to \mathbb{R}$ *unifier* of the vector row of functions $(g_1, \dots, g_l)$, $g_i\colon \Omega \to \Theta$, $i = 1, \dots, l$, if

$$\gamma(g_1(x)) = \ldots = \gamma(g_l(x))$$

for μ_Ω-a.e. $x \in \Omega$. Note that the set of unifiers for a given vector row of functions is closed with respect to convergence in measure, hence also with respect to pointwise convergence.

Definition 1. *The vector row of functions $(g_1, \dots, g_l)$, $g_i\colon \Omega \to \Theta$, $i = 1, \dots, l$ is called unifiable, if it admits a unifier $\gamma\colon \Theta \to \mathbb{R}$ satisfying*

$$\mu_\Theta(\gamma^{-1}(y)) = 0 \tag{3}$$

for all $y \in \mathbb{R}$.

We emphasize that we are deliberately speaking of vector rows of functions (and not just of sets). In fact, although it changes nothing in the scalar case (the order of functions in the row has clearly no importance), the distinction will be obvious in the vectorial case.

It is easy to see that not every vector row of functions is unifiable.

Example 1. Let $(\Theta, \Sigma(\Theta), \mu_\Theta) = (\Omega, \Sigma(\Omega), \mu_\Omega)$. Then the simplest example of a nonunifiable row of functions is given by a pair $g_1 := \mathrm{id}_\Omega$ (the identity map) and g_2: $\Omega \to \Omega$ any *ergodic* map. In fact, the ergodicity of g_2 means that the only function γ satisfying $\gamma(x) = \gamma(g_2(x))$ for μ_Ω-a.e. $x \in \Omega$ is a constant [4]. A classical particular example of the ergodic map is the rotation of the circle $\Omega = S^1$ (equipped with the one-dimensional Hausdorff measure) by the angle $2\pi\alpha$ where α is irrational. It can also be stated in terms of transformation of the interval $\Omega = (0,1)$ (with the usual Lebesgue measure) as $g(x) = \{x+\alpha\}$ where $\{\cdot\}$ stands for the fractional part function.

In view of the above example the unifiability can be understood as the natural extension of *nonergodicity* property to collections of functions. Note for instance, that the vector row of rotations of the circle S^1 by the angles $2\pi\alpha_i$, $i = 1, \dots, l$ is unifiable, if and only if the angles are commensurable (i.e. they relate with each other as rational numbers).

Now we try to study the unifiability condition from another point of view. For this purpose for each $x \in \Omega$ define the *orbit* $O(x) \subset \Theta$ by

$$O(x) := \bigcup_{j\in\mathbb{N}} O_j(x),$$

where

$$\begin{array}{rcl} O_0(x) & := & \{x\}, \\ O_1(x) & := & \{y \in \Theta \,:\, g_i^{-1}(y) \cap g_j^{-1}(x) \neq \emptyset \text{ for some } i \neq j\}, \\ O_j(x) & := & O_1(O_{j-1}(x)). \end{array}$$

Observe that O defines an equivalence relation $x \sim y \Leftrightarrow x \in O(y)$.

We will say that a set $e \subset \Theta$ is O-invariant, if $O(e) = e$. It is easy to note that O-invariant subsets form a σ-algebra. By Σ_O we denote the σ-algebra of O-invariant sets belonging to $\Sigma(\Theta)$. A real valued function on Θ will be called O-measurable, if it is measurable with respect to Σ_O.

Recall that an atom of a σ-algebra $\Sigma' \subset \Sigma(\Theta)$ is is a set $E \in \Sigma'$ with $\mu_\Theta(E) > 0$ such that for every $e \subset E$, $e \in \Sigma'$ one has either $\mu_\Theta(e) = 0$ or $\mu_\Theta(e) = \mu_\Theta(E)$. Σ' is said to be nonatomic, if it contains no atoms. With these notions at hand we are able to prove the following statement about the relationship between unifiability of $(g_1, \dots, g_l)$ and nonatomicity of Σ_O.

Lemma 1. *If the vector row of functions $(g_1, \dots, g_l)$ is unifiable, then Σ_O contains a nonatomic σ-subalgebra.*

PROOF: Let the vector row $(g_1, \dots, g_l)$ be unifiable by means of a unifier γ: $\Theta \to \mathbb{R}$ satisfying (3). We will show that $\gamma^{-1}(\mathcal{B}) \subset \Sigma_O$, where $\mathcal{B}$ stands for the σ-algebra of Borel subsets of Ω, is nonatomic. Suppose the contrary, namely that the latter has an atom E. The set $\gamma(E) \in \mathcal{B}$ is necessarily more than countable, and hence by the theorem of Kuratowski [7] there is a measurable one-to-one map j: $\gamma(E) \to [0,1)$. Now consider the map $\gamma' := j \circ \gamma$: $E \to [0,1)$. For each $\nu \in \mathbb{N}$ let

$$B^i_\nu := [(i-1)/2^\nu, i/2^\nu), \qquad i = 1, \dots, 2^\nu.$$

Since $\sqcup_j B^j_\nu = [0,1)$, then there is a $i' = i'(\nu)$ such that $\mu_\Theta(\gamma'^{-1}(B^{i'(\nu)}_\nu)) = \mu_\Theta(E)$. Clearly, $B^{i'(\nu+1)}_{\nu+1} \subset B^{i'(\nu)}_\nu$ and

$$\omega_0 := \bigcap_\nu B^{i'(\nu)}_\nu$$

consists of no more than one point. Therefore, we have an obvious contradiction

$$0 = \mu_\Theta(\gamma'^{-1}(\omega_0)) = \mu_\Theta\left(\bigcap_\nu \gamma'^{-1}\left(B^{i'(\nu)}_\nu\right)\right) = \mu_\Theta(E),$$

where the first equality comes from (3). □

3. Relaxation and convexity: scalar case

We first concentrate on the scalar case $k = 1$, and show that the phenomenon announced in the introduction reveals itself already in this setting, and hence does not have any special "vectorial" nature. To abbreviate the notation, we omit everywhere the unnecessary index 1, i.e. we write g_i instead of g_{1i}, u instead of u_1 and L^p instead of L^p_1. The functional $I\colon L^p \to \bar{\mathbb{R}}$ defined by (1) is then reduced to

$$I(u) = \int_\Omega f(x, u(g_1(x)), \ldots, u(g_l(x)))\, dx. \tag{4}$$

We are now at a position to prove that its relaxation in the weak topology of L^p is convex, and hence coincides with its convexification, provided the vector row $(g_1, \ldots, g_l)$ is unifiable.

Theorem 1. *Let $1 \leq p \leq +\infty$. If the vector row of functions $(g_1, \ldots, g_l)$ is unifiable, then*

$$sc^-(w - L^p)I(u) = I^{**}(u),$$

where in the case $p = +\infty$ the relaxation is meant in the $*$-weak topology.

We emphasize that according to the definition of unifiability in the case $k = 1$ this theorem implies no extra condition on g_1.

PROOF: The proof is identical for $1 < p < +\infty$ and $p = +\infty$ modulo changing the words "weak topology" to "$*$-weak topology" in the latter case. To show the claim, it is enough to prove that $sc^-(w - L^p)I$ is convex. Thus consider the arbitrary $\{u_1, u_2\} \subset L^p$ and let the sequences $\{u^1_\nu\}$, $\{u^2_\nu\}$ be such that

$$u^i_\nu \rightharpoonup u_i \text{ weakly in } L^p, \text{ while } sc^-(w - L^p)I(u_i) = \lim_\nu I(u^i_\nu), \qquad i = 1, 2.$$

Pick up a countable dense set $\{p_j\} \subset L^{p'}$, $1/p' + 1/p = 1$, and fix an arbitrary $\lambda \in (0, 1)$. Since the vector row $(g_1, \ldots, g_l)$ is unifiable, according to the lemma 1 there is a nonatomic σ-algebra $\Sigma' \subset \Sigma_O$.

Consider for each fixed $\nu \in \mathbb{N}$ the following systems of equations with respect to $e_\nu \in \Sigma'$:

$$\left\{\begin{array}{rcl}
\displaystyle\int_{e_\nu} p_j(y) u^1_\nu(y)\, d\mu_\Theta(y) & = & \displaystyle\lambda \int_\Theta p_j(y) u^1_\nu(y)\, d\mu_\Theta(y), \\
\displaystyle\int_{\Theta\setminus e_\nu} p_j(y) u^2_\nu(y)\, d\mu_\Theta(y) & = & \displaystyle(1-\lambda) \int_\Theta p_j(y) u^2_\nu(y)\, d\mu_\Theta(y), \\
\multicolumn{3}{l}{\displaystyle\int_{g_1^{-1}(e_\nu)} f(x, u^1_\nu(g_1(x)), \ldots, u^1_\nu(g_l(x)))\, d\mu_\Omega(x) =} \\
\multicolumn{3}{r}{\displaystyle\lambda \int_\Omega f(x, u^1_\nu(g_1(x)), \ldots, u^1_\nu(g_l(x)))\, d\mu_\Omega(x),} \\
\multicolumn{3}{l}{\displaystyle\int_{\Omega\setminus g_1^{-1}(e_\nu)} f(x, u^2_\nu(g_1(x)), \ldots, u^2_\nu(g_l(x)))\, d\mu_\Omega(x) =} \\
\multicolumn{3}{r}{\displaystyle(1-\lambda) \int_\Omega f(x, u^2_\nu(g_1(x)), \ldots, u^2_\nu(g_l(x)))\, d\mu_\Omega(x),}
\end{array}\right. \tag{5}$$

where $j = 1, \dots, \nu$. To solve this system, introduce the vector measure $\mu^\nu\colon \Sigma' \to \mathbb{R}^{2\nu+2}$ by the following formulae:

$$\begin{aligned}
\mu_j^\nu(e) &:= \int_e p_j(y) u_\nu^1(y)\, d\mu_\Theta(y), \qquad j = 1, \dots, \nu,\\
\mu_{\nu+j}^\nu(e) &:= \int_e p_j(y) u_\nu^2(y)\, d\mu_\Theta(y), \qquad j = 1, \dots, \nu,\\
\mu_{2\nu+1}^\nu(e) &:= \int_{g_1^{-1}(e)} f(x, u_\nu^1(g_1(x)), \dots, u_\nu^1(g_l(x)))\, d\mu_\Omega(x),\\
\mu_{2\nu+2}^\nu(e) &:= \int_{g_1^{-1}(e)} f(x, u_\nu^2(g_1(x)), \dots, u_\nu^2(g_l(x)))\, d\mu_\Omega(x).
\end{aligned}$$

The system of equations (5) can then be written as

$$\mu^\nu(e_\nu) = \lambda \mu^\nu(\Theta).$$

It is clear that it admits a solution thanks to the Lyapunov theorem on convexity of the range of a vector measure (note that $\mu_{2\nu+1}^\nu << \mu_\Theta$, $\mu_{2\nu+2}^\nu << \mu_\Theta$ and hence are nonatomic). Therefore the first and second equations in (5) imply

$$\chi_{e_\nu} u_\nu^1 + \chi_{\Theta\setminus e_\nu} u_\nu^2 \rightharpoonup \lambda u_1 + (1-\lambda) u_2 \tag{6}$$

weakly in L^p, and hence, by definition of $sc^-(w - L^p)I$ with the help of the third and the fourth equations of (5) we obtain

$$\begin{aligned}
sc^-(w - L^p)I(\lambda u_1 + (1-\lambda)u_2) &\le \liminf\nolimits_\nu I(\chi_{e_\nu} u_\nu^1 + \chi_{\Theta\setminus e_\nu} u_\nu^2) =\\
&\lambda \lim\nolimits_\nu I(u_\nu^1) + (1-\lambda) \lim\nolimits_\nu I(u_\nu^2) =\\
&\lambda\, sc^-(w - L^p)I(u_1) + (1-\lambda)\, sc^-(w - L^p)I(u_2),
\end{aligned}$$

concluding the proof for the case $p \neq 1$. If $p = 1$, one should reiterate the above arguments taking $\{p_j\}$ to be a countable dense subset in the space of continuous functions, which gives (6) in the weak sense of measures. It is then enough to observe that in fact, (6) holds also in the sense of weak convergence in L^1, because the sequence $\{\chi_{e_\nu} u_\nu^1 + \chi_{\Theta\setminus e_\nu} u_\nu^2\}$ is equiintegrable, since so are the sequences $\{u_\nu^i\}$, $i = 1, 2$. □

Corollary 1. *If the vector row of functions $(g_1, \dots, g_l)$, $g_i\colon \Omega \to \Theta$, $i = 1, \dots, l$, is unifiable, then for the functional I defined by (4) to be weakly lower semicontinuous over $L^p(\Theta)$, $1 \le p < +\infty$ (resp. $*$-weakly lower semicontinuous in L^∞), it is necessary and sufficient that it be convex and lower semicontinuous in the strong topology.*

Now we pass to the more difficult question, namely, what happens when the vector row $(g_1, \dots, g_l)$ is not unifiable. We show that in this case in general convexity has nothing to do with weak lower semicontinuity of the respective functionals. The basic idea for this proof is provided by the example below.

Example 2. Let (g_1, g_2) be a nonunifiable function pair given by the example 1, namely, $g_1 := \mathrm{id}_\Omega$ and $g_2\colon \Omega \to \Omega$ any ergodic map. Define $f\colon \mathbb{R}^2 \to \bar{\mathbb{R}}$ by

$$f(y_1, y_2) := \begin{cases} 0, & y_1 = y_2 \ge 1 \text{ or } y_1 = y_2 \le 0,\\ +\infty, & \text{elsewhere.} \end{cases}$$

Then for the functional

$$I(u) := \int_\Omega f(u(g_1(x)), u(g_2(x)))\, d\mu_\Omega(x)$$

one has $I(u)=0$, only if $u\circ g_1 = u\circ g_2$ μ_Ω-a.e. in Ω, but this can happen only if $u=$ const in view of ergodicity of g_2. Hence

$$I(u) = \begin{cases} 0, & u = \text{const} \geq 1 \text{ or } u = \text{const} \leq 0, \\ +\infty, & \text{elsewhere,} \end{cases}$$

which implies that I is weakly lower semicontinuous, i.e. $I = sc^-(w-L^p)I$, but not convex, namely, $I \neq I^{**}$.

We start now with the following technical auxiliary statement which will be of utmost importance for our construction.

Lemma 2. *If the vector row of functions $(g_1,\ldots,g_l)$, $g_i\colon \Omega\to\Theta$, $i=1,\ldots,l$ is not unifiable, then there exist a measurable set $\hat{\Theta}\subset\Theta$ such that every its unifier is constant over $\hat{\Theta}$.*

PROOF:
Step 1. For each measurable $\gamma\colon \Theta\to\mathbb{R}$ we set

$$M_\gamma := \sup_{s\in\mathbb{R}} \mu_\Theta(\gamma^{-1}(s)).$$

Clearly, in view of nonunifiability of the functions $g_1,\ldots,g_l$ one has $M_\gamma > 0$ for each their unifier γ. We prove that

$$M := \inf_\gamma M_\gamma > 0, \tag{7}$$

where inf is taken over all the unifiers γ of $(g_1,\ldots,g_l)$.

Suppose the contrary, i.e. that there exists a sequence of unifiers γ^ν such that $M_{\gamma^\nu}\to 0$ when $\nu\to\infty$. Obviously, for every γ^ν there exist no more than a countable number of points $s_j^\nu\in\mathbb{R}$ such that $\mu_\Theta(\gamma^{-1}(s_j^\nu))>0$. Denote

$$\begin{array}{rclrcl} C_j^\nu & := & (\gamma^\nu)^{-1}(s_j^\nu), & V^\nu & := & \Theta\setminus\bigcup_j C_j^\nu, \\ V & := & \bigcup_\nu V^\nu, & C & := & \Theta\setminus V \;= \cap_\nu \cup_j C_j^\nu, \\ W^\nu & := & V^\nu\setminus\bigcup_{l=1}^{\nu-1} V^l, & C^{j_1 j_2\cdots j_\nu} & := & C\cap\left(\bigcap_\nu C_{j_\nu}^\nu\right). \end{array}$$

For every $\nu\in\mathbb{N}$ we define the function $\hat{\gamma}^\nu\colon \Theta\to\mathbb{R}$ by setting

$$\hat{\gamma}^\nu(x) \quad := \quad \begin{cases} \gamma^l(x), & x\in W^l, \qquad l\in\mathbb{N}, \\ c^\nu(x), & x\in C, \end{cases}$$

where the function $c^\nu\colon C\to\mathbb{R}$ is defined by

$$c^\nu(x) := 1 + 1/2^{j_1} + 1/2^{j_1+j_2} + \ldots + 1/2^{j_1+j_2+\ldots+j_\nu}, \qquad x\in C^{j_1 j_2\cdots j_\nu}\cap C.$$

It is clear that $\hat{\gamma}^\nu$ is a unifier, since all the sets introduced above are O-invariant.

Now we notice that the sequence $\hat{\gamma}^\nu$ is nondecreasing and bounded, so it has a pointwise limit $\hat{\gamma}$, which is also a unifier. We prove that $\hat{\gamma}$ satisfies (3), thus contradicting the nonunifiability assumption and proving (7). Indeed, if it is not so, then there exists a point $s\in\mathbb{R}$ such that $\mu_\Theta(\hat{\gamma}^{-1}(s))>0$. Note that then by construction $\hat{\gamma}^{-1}(s)\subset C$, and for some $\nu\in\mathbb{N}$ such that $M_{\gamma^\nu}<\mu_\Theta(\hat{\gamma}^{-1}(s))$ we have $\hat{\gamma}^{-1}(s)$ covering several $C_j^\nu\cap C$. That means that $\hat{\gamma}$ cannot take a single value over $\hat{\gamma}^{-1}(s)$ and leads to a contradiction.

Step 2. Consider now a sequence of unifiers γ^ν such that $M_{\gamma^\nu}\to M$ when $\nu\to\infty$. We preserve the notation from the Step 1. For every $\nu\in\mathbb{N}$ we have

$$\mu_\Theta(C^{j_1 j_2\cdots j_\nu}) \geq M \tag{8}$$

for some $j_1, j_2, \ldots, j_\nu \in \mathbb{N}$. Indeed, otherwise we could have constructed a unifier $\hat\gamma^\nu$ with $M_{\hat\gamma^\nu} < M$. Moreover, for every $\varepsilon > 0$ one has

$$\mu_\Theta(C^{j_1 j_2 \ldots j_\nu}) \le M + \varepsilon \tag{9}$$

for all sufficiently large $\nu \in \mathbb{N}$. Combining (8) and (9), one concludes the existence of at least one and no more than a finite number of measurable O-invariant sets $\hat\Theta_j$, $\mu_\Theta(\hat\Theta_j) = M$, such that for every $\nu \in \mathbb{N}$ one has $\hat\Theta_j \subset C^{j_1 j_2 \ldots j_\nu}$ for some $j_1, j_2, \ldots, j_\nu \in \mathbb{N}$. Besides, since all Θ_j are O-invariant, we can construct a unifier $\hat\gamma$ such that $\hat\Theta_j = \hat\gamma^{-1}(s_j)$ for some points $s_j \in \mathbb{R}$.

Note that among the sets $\hat\Theta_j$ there exists at least one set $\hat\Theta$ such that every unifier γ takes a single value over $\hat\Theta$. Indeed, otherwise we could have divided each $\hat\Theta_j$ into several parts of positive measure and construct a unifier $\tilde\gamma$ with $M_{\tilde\gamma} < M$. □

Now we are able to prove that in fact the unifiability condition is unavoidable both in the theorem 1 and in the corollary 1. Namely, the following assertion is valid.

Theorem 2. *If the vector row of functions $(g_1, \ldots, g_l)$, $g_i \colon \Omega \to \Theta$, $i = 1, \ldots, l$, is not unifiable, then there exists an integrand $f \colon \Theta \times \mathbb{R}^l \to \bar{\mathbb{R}}$ such that the respective functional I defined by (4) is weakly lower semicontinuous over L^p (resp. $*$-weakly lower semicontinuous in L^∞) and not convex.*

PROOF: According to the lemma 2 there exists a set $\hat\Theta \in \Sigma(\Theta)$ such that every unifier of $(g_1, \ldots, g_l)$ takes a single value over $\hat\Theta$. Consider the set G of all unifiers $\gamma \in L^p$ such that $|\gamma(x)| \ge 1$ for μ_Θ-a.e. $x \in \hat\Theta$. We prove that G is closed in the weak topology of L^p. In fact, consider an arbitrary sequence $\gamma^\nu \in G$ such that $\gamma^\nu \to \gamma$ weakly. Clearly, γ is still a unifier, since it is O-measurable. It remains to show that $|\gamma(x)| \ge 1$ for μ_Θ-a.e. $x \in \hat\Theta$. Since $\gamma^\nu(x) \equiv c^\nu$ and $\gamma(x) \equiv c$ for μ_Θ-a.e. $x \in \hat\Theta$, while $|c^\nu| \ge 1$, we have $c^\nu \to c$, and hence $|c| \ge 1$ as well. Thus G is closed in the weak topology of L^p, but is clearly nonconvex.

Now we construct an integrand f by setting

$$f(x,y) := \begin{cases} 0, & g_i(x) \in \hat\Theta \text{ and there exists } \alpha \in \mathbb{R} : \text{ all } y_i = \alpha, \text{ while } |\alpha| > 1, \\ +\infty, & \text{elsewhere.} \end{cases}$$

It is clear that the functional I defined by (4) is then

$$I(u) = \begin{cases} 0, & u \in G, \\ +\infty, & u \notin G, \end{cases}$$

and hence is lower semicontinuous in the weak topology of $L^p(\Theta)$ but not convex. □

Note that the assertions we proved tell only that unifiability condition for the row $(g_1, \ldots, g_l)$ is necessary for the relaxation of the functional (4) in weak topology of L^p to coincide with its convexification for a *generic* integrand f. Nevertheless, for a particular integrand it may happen that the respective relaxation coincides with the convexification of the functional even without the unifiability requirement. For instance, by modifying slightly of the proof of the theorem 1 one concludes that when

$$f(x, y_1, \ldots, y_l) = f_1(x, y_1, \ldots, y_m) + f_2(x, y_{m+1}, \ldots, y_l)$$

for some $m \in \{1, \ldots, l\}$, then the statement of the theorem 1 remains valid under only the condition that each of the two vector rows of functions $(g_1, \ldots, g_m)$

and $(g_{m+1},\dots,g_l)$ be unifiable, which is obviously much less restrictive than the unifiability of the whole row $(g_1,\dots,g_l)$.

4. Relaxation and convexity: vectorial case

Consider now the general vectorial setting (1). We extend now the definition 1 to the case of function matrices $\{g_{ij}\}_{i,j=1}^{k,l}$.

Definition 2. *A vector row of measurable (with respect to $\Sigma(\Theta)$) functions*

$$\gamma=(\gamma_1,\dots,\gamma_k),\qquad \gamma_i:\Theta\to\mathbb{R},\ i=1,\dots,k$$

is said to be unifier *of the function matrix $\{g_{ij}\}_{i,j=1}^{k,l}$, if all the functions $\gamma_i\circ g_{ij}$: $\Omega\to\mathbb{R}$ coincide μ_Ω-a.e. over Ω.*

Definition 3. *The function matrix $\{g_{ij}\}_{i,j=1}^{k,l}$ is called unifiable, if it admits such a unifier $(\gamma_1,\dots,\gamma_k)$ that*

$$\mu_\Theta(\gamma_i^{-1}(y))=0,\qquad i=1,\dots,k \tag{10}$$

for all $y\in\mathbb{R}$.

Let us make some remarks. First, if $k=1$, then the function matrix $\{g_{ij}\}_{i,j=1}^{k,l}$ reduces to a row and we are in the case of the definition 1. But if instead $l=1$, $k>1$, then the above function matrix is a vector column. The unifiability condition in this case is actually much less restrictive than for vector rows. In fact, just injectivity of each of the functions in the column would suffice for the unifiability of the whole column, which is clearly not the case for unifiability of rows. Namely, the following trivial statement holds.

Proposition 1. *A vector column of injective functions $(g_1,\dots,g_k)^T$ (where T stands for transposition, to distinguish rows from columns) is unifiable.*

Proof: Since all the functions g_i satisfy (2), up to restricting them to the Borel subset of full measure in Ω, we may assume that the respective inverses g_i^{-1} satisfy (2). Assuming without loss of generality $\mu_\Omega(\Omega)=1$, we consider the measure preserving one-to-one map θ: $\Omega\to[0,1]$ (assuming $[0,1]$ is equipped with the standard Lebesgue measure). Hence it is enough to take $\gamma_i:=\theta\circ g_i^{-1}$, $i=1,\dots,k$. □

However, still not all vector columns of functions are unifiable, as the example below shows.

Example 3. Let $\Theta=\Omega=(0,1)$. Set

$$\begin{aligned} g_1(x) &:= 1/2+1/4\cos 2\pi x,\\ g_2(x) &:= 1/2+1/4\cos 2\pi(x-\alpha/2), \end{aligned}$$

where α is some irrational number. Suppose that there exists a unifier (γ_1,γ_2) of the vector column $(g_1,g_2)^T$ and consider the function $g=\gamma_1\circ g_1=\gamma_2\circ g_2$. Let us take an arbitrary point $x\in\Omega$. First, since $g_1(x)=g_1(1-x)$, we have $g(x)=g(1-x)$. Second, since $g_2(1-x)=g_2(\{1-(1-x)+\alpha\})=g_2(\{x+\alpha\})$, we have $g(1-x)=g(\{x+\alpha\})$. Summing up, we obtain $g(x)=g(\{x+\alpha\})$, which means that g is a constant function, and hence γ_1 and γ_2 are constant over $g_1(\Omega)$ and $g_1(\Omega)$ respectively.

At last is seems worth mentioning that according to our definition a matrix consisting of a single function (i.e. $k = l = 1$) is always unifiable.

Now we are at a position to extend the theorems 1 and 2 to the general vectorial case.

Theorem 3. *Let $1 \leq p \leq +\infty$. If the function matrix $\{g_{ij}\}_{i,j=1}^{k,l}$ is unifiable, then for the functional I defined by (1) one has*

$$sc^-(w - L_k^p)I(u) = I^{**}(u),$$

where in the case $p = +\infty$ the relaxation is meant in the $*$-weak topology.

If the function matrix $\{g_{ij}\}_{i,j=1}^{k,l}$ is not unifiable, then there is an integrand $f\colon \Omega \times \mathbb{R}^{kl} \to \bar{\mathbb{R}}$ such that the respective functional I defined by (1) is weakly lower semicontinuous over L_k^p (resp. $*$-weakly lower semicontinuous in L_k^∞) and not convex.

PROOF: Let $\Theta' := \{1, \dots, k\} \times \Theta$ equipped with the product σ-algebra $2^{\{1,\dots,k\}} \otimes \Sigma(\Theta)$, the measure being the product of the uniform measure in $\{1, \dots, k\}$ and μ_Θ. Obviously, the linear operator $T\colon L_k^p \to L^p(\Theta')$ defined by

$$(Tu)(i, x)) = u_i(x), \qquad i = 1, \dots, k,\ x \in \Theta,$$

is an isomorphism. Thus the weak lower semicontinuity of I over L_k^p is equivalent of that of $I \circ T^{-1}$ over $L^p(\Theta')$. But the latter functional is of the type (4) (i.e. we are again in the scalar case) involving the set of deviations $\bar{g}_{ij}\colon \Omega \to \Theta'$ defined by

$$\bar{g}_{ij}(x) := (i, g_{ij}(x)).$$

Assemble them in a single row of $k \times l$ elements. Clearly, unifiability of this row is equivalent to that of the original function matrix $\{g_{ij}\}_{i,j=1}^{k,l}$, and therefore the conclusion follows from the results of the previous section. □

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