UNIVERSITÀ DI PARMA

Università degli studi di Parma

Dottorato di ricerca in Matematica

Ciclo XXXVII

Topological and capacitary methods for Poincaré inequalities

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Anni accademici 2021/2022 - 2022/2023 - 2023/2024

Alla memoria di mia nonna, Antonietta

Acknowledgements

Every word I could say would not be enough to express my gratitude to my supervisor Lorenzo Brasco. From a professional point of view, he is one of the finest and most original mathematician I have ever met. His limpid vision of the Calculus of Variations and his amazing didactic skill inspired me greatly, and provided me with essential survival tools to explore the intricate world of mathematical research. However, these are minor features if compared to his humanity, his kind person, always available and willing to help and share its deep thoughts on the everyday reality, his friendliness and humor. It is a great honour for me having met him, and also a great and an unexpected surprise, so thank you Lorenzo!

I would like to express my deep thankfulness to Gian Paolo Leonardi. He has been the first who revealed to me the beauty and the elegance of Mathematical Analysis, the moment from which I began to cut my teeth. His support and availability over the years has been fundamental for me. I feel sincerely lucky to have started my "mathematical journey" with him. Thank you Gian Paolo for your kindness and for having shared your passion with a 19-year-old boy, almost ten years ago.

I would like to thank also Michele Miranda for his warm welcome at the University of Ferrara, Giulio Tralli for his friendliness and for his lectures given in the last spring, Francesca Prinari, Eleonora Cinti, Valeria Ruggiero and the Head of the Department Massimiliano Mella, as well as the Director of the Ph.D. program Leonardo Biliotti, and his predecessor Alessandra Lunardi, for their dedicated work over the years of my doctorate studies. A sincere thanks goes to Dorin Bucur for his welcome at the Laboratoire de Mathématiques of the Université Savoie Mont Blanc, in occasion of my one month stay in Chambéry.

An unvaluable thanks goes to Anna Chiara Zagati and Francesca Bianchi, this thesis would not be written without their advices and support, as well as to Ilaria Trombini and Ambra Catozzi, for their help and for having contribute to a friendly atmosphere in the Ph.D. students' office. A special thanks goes to Roberto Ognibene, Matteo Talluri, Giorgio Tortone and Emanuele Salato for having shared with me mathematical thoughts and beautiful moments.

Finally, my deepest gratitude goes to my family, Loredana, Antonio, Alessandro, Ilaria and Dante, for their love, for their support in my choices, for always believing in me and for the beauty they bring into my life.

Abstract

This thesis is devoted to prove characterizations of the validity of Poincaré–type inequalities on general open sets in $\mathbb{R}^N$. In the super–conformal case, i.e. when points are not removable sets, the finiteness of the inradius of an open set Ω turns out to be alone a necessary and sufficient condition for the Poincaré inequality to hold on Ω. In the planar case, this condition is sufficient for open sets with prescribed topology. A similar characterization is still valid in arbitrary dimension and for a general open set, when the points are removable sets, by using the *capacitary inradius*, in place of the usual one.

In the first two situations, we prove a geometric lower bound on the sharp Poincaré–Sobolev embedding constants associated to an open set, in terms of its inradius. In the super–conformal case, we provide an explicit constant and analyse its asymptotic behaviour, by refining a result by Maz'ya from the '70s. For planar sets with prescribed topology, we obtain an estimate which optimally depends on the topology of the sets, thus generalizing a result by Croke, Osserman and Taylor, originally devised for the first eigenvalue of the Dirichlet–Laplacian. We also consider some limit cases, like the sharp Moser–Trudinger constant and the Cheeger constant. As a byproduct of our discussion, we also obtain a Buser–type inequality for open subsets of the plane, with prescribed topology. An interesting problem on the sharp constant for this inequality is presented.

In the sub–conformal case, we prove a two–sided estimate on the sharp L^p Poincaré constant of a general open set, in terms of its capacitary inradius. This extends a result by Maz'ya and Shubin, originally proved for the case $p = 2$. We cover the whole range of p, by allowing in particular the extremal cases $p = 1$ (Cheeger constant) and $p = N$ (conformal case), as well. We also discuss the more general case of the sharp Poincaré–Sobolev embedding constants and get an analogous result. Finally, we discuss the capacitary inradius in the super–conformal regime, as well as some examples and counter–examples.

Sunto

Questa tesi è dedicata a fornire caratterizzazioni della validità di disuguaglianze di tipo Poincaré su insiemi aperti generali in $\mathbb{R}^N$. Nel caso superconforme, cioè quando i punti non sono rimovibili, la finitezza dell'*inradius* di un insieme aperto Ω risulta essere da sola una condizione necessaria e sufficiente affinchè valga la disuguaglianza di Poincaré su Ω. Nel caso planare, questa condizione è sufficiente per insiemi aperti con topologia assegnata. Una simile caratterizzazione è anche valida in dimensione arbitraria e per un insieme aperto generale, quando i punti sono insiemi rimovibili, usando l'*inradius capacitario*, al posto di quello usuale.

Nella prime due situazioni, proviamo una minorazione delle constanti ottime di immersione di Poincaré–Sobolev associate a un aperto generale, in termini del suo inradius. Nel caso superconforme, forniamo una costante esplicita and analizziamo il suo comportamento asintotico, raffinando un risultato di Maz'ya degli anni '70. Per insiemi planari con topologia assegnata, otteniamo una stima che dipende in modo ottimale dalla topologia degli insiemi. Questo generalizza un risultato di Croke, Osserman e Taylor, originariamente ideato per il primo autovalore del Laplaciano di Dirichlet. Consideriamo anche alcuni casi limite, come la costante ottima di Moser–Trudinger e la costante di Cheeger. Come sottoprodotto della nostra discussione, otteniamo anche una disuguaglianza di tipo Buser per sottoinsiemi aperti del piano, con topologia assegnata. Un problema interessante sulla costante ottima per questa disuguaglianza è presentato.

Nel caso subconforme, proviamo una stima bilatera sulla costante di Poincaré L^p ottima di un insieme aperto generale, in termini del suo *inradius capacitario*. Ciò estende un risultato di Maz'ya e Shubin, originariamente limitato al caso $p = 2$. Copriamo l'intera gamma di p, consentendo in particolare i casi limite $p = 1$ (costante di Cheeger) e $p = N$ (caso conforme). Discutiamo anche il caso più generale delle costanti di immersioni di Poincaré–Sobolev, ottenendo un risultato analogo. Infine, discutiamo l'inradius capacitario nel regime superconforme, così come alcuni esempi e controesempi.

Contents

CHAPTER 1

Introduction

1.1. Poincaré–Sobolev inequalities and principal frequencies

This thesis is devoted to the Poincaré inequality, one of the most celebrated and studied functional inequality, which naturally arises in the context of Sobolev Spaces and more in general when dealing with weakly differentiable functions. In literature, one may encounter several formulations and generalizations of this inequality: without claiming to be exhaustive we refer the reader for example to [2, Chapter 4], [18, Chapter 9], [41, Chapter 5], [50, Section 3.4], [69, Section 13.2], [76, Chapters 1-2-15], [93, Chapter 4].

In its most basic form, from a descripitive point of view, Poincaré inequality asserts that for $1 \leq p < \infty$ and for every open set $\Omega \subseteq \mathbb{R}^N$ having finite measure, the following inequality holds

$$C \int_\Omega |u|^p\,dx \leq \int_\Omega |\nabla u|^p\,dx, \qquad \forall u \in C_0^\infty(\Omega),$$

where C is a positive constant depending only on N, p and Ω (see for a proof [50, Corollary 3.1]). From a qualitative point of view, it asserts that it is possible to bound from above the L^p–norm of a *regular* function u with that of its *gradient* provided that the set $\{u(x) = 0\}$ has sufficiently *large size*. The last information can be captured through the notion of $p-$*capacity*. As we will see in the sequel, this interpretation will be the ultimate reason for which the topological and capacitary methods mentioned in the title of this thesis work. To set the scene, in this Introduction we briefly recall the definition of this key quantity, and refer the reader to Section 2.2 for an account on this topic which is sufficient for our purposes: for $1 \leq p < \infty$, for every $E \subseteq \mathbb{R}^N$ open set and every $\Sigma \subseteq E$ compact set, we define the $p-$*capacity of* Σ *relative to* E as

$$\mathrm{cap}_p(\Sigma; E) = \inf_{\varphi \in C_0^\infty(E)} \left\{ \int_E |\nabla \varphi|^p\,dx \,:\, \varphi \geq 1 \text{ on } \Sigma \right\}.$$

For a thorough study of $p-$capacity see for example [39, Chapter 8], [42, Section 4.7], [44, Chapter 2] and [76, Chapter 2].

In our treatment, we will consider a larger class of inequalities, which has been extensively used in problems arising from the Calculus of Variations, the Analysis of PDEs and the Shape Optimization, as well. This class goes under the name of *Poincaré–Sobolev inequalities*. As before, a huge literature has been devoted to these inequalities, see for example [2, Chapter 4], [18, Chapter 9], [41, Chapter 5], [50, Section 3.4], [69, Chapter 12] and [76, Chapters 1-2-15-16].

In their most basic formulation, they assert that for $1 \leq p < \infty$ and $q \geq 1$ such that[1]

$$\begin{cases} q \leq p^*, & \text{if } 1 \leq p < N, \\ q < \infty, & \text{if } p = N, \\ q \leq \infty, & \text{if } p > N, \end{cases} \tag{1.1.1}$$

and for every open set $\Omega \subseteq \mathbb{R}^N$ with[2] $|\Omega| < \infty$ one has

$$C \left(\int_\Omega |u|^p\,dx \right)^{\frac{p}{q}} \leq \int_\Omega |\nabla u|^p\,dx, \qquad \forall u \in C_0^\infty(\Omega), \tag{1.1.2}$$

[1] As usual, the number p^* denotes the exponent of the critical Sobolev embedding, defined by

$$p^* = \frac{N\,p}{N - p}.$$

[2] Here $|\cdot|$ indicates the Lebesgue measure in $\mathbb{R}^N$

for a positive constant C depending only on p, q, N and Ω (for a proof see for example [18, Chapter 9], [41, Chapter 5] or [48, Section 7.7]). More precisely, one can prove that the following inequality

$$c(N,p)\,\frac{1}{|\Omega|^{\frac{1}{N}-\frac{1}{p}+\frac{1}{q}}}\,\left(\int_\Omega |u|^p\,dx\right)^{\frac{p}{q}} \le \int_\Omega |\nabla u|^p\,dx, \qquad \forall u\in C_0^\infty(\Omega), \tag{1.1.3}$$

holds true, as long as Ω is an open subset of $\mathbb{R}^N$ with $|\Omega|<\infty$ and p, q satisfy conditions (1.1.1).

It is even worth to mention that a complete characterization of the open sets in $\mathbb{R}^N$ satisfying Poincaré–Sobolev type inequalities has been given by Vladimir Maz'ya, see for example [76, Theorems 15.4.1–15.6.1] and more in general [76, Section 16.2]. In this thesis we improve some of his results, for example [76, Theorem 15.4.1] and [76, Theorem 18.7.1]. For the moment, we prefer to introduce the key quantities analysed in it, while postponing the discussion on our contributions along this research line to Section 1.2 and Section 1.4.

The main characters studied in this work are the *sharp constants of the L^q–L^p Poincaré–Sobolev inequality* (1.1.2) associated to an open set Ω in $\mathbb{R}^N$. Each of these quantities is variationally characterized as

$$\lambda_{p,q}(\Omega) = \inf_{\varphi\in C_0^\infty(\Omega)}\left\{\int_\Omega |\nabla u|^p\,dx\,:\,\|u\|_{L^q(\Omega)}=1\right\}. \tag{1.1.4}$$

where p, q satisfy conditions (1.1.1). In the particular case $q = p$, we will use the shortcut notation

$$\lambda_p(\Omega) := \lambda_{p,p}(\Omega).$$

For the case $p = q = 2$, we will still use the distinguished notation $\lambda(\Omega)$, which is quite standard to indicate the bottom of the spectrum of the Dirichlet–Laplacian in Ω in the framework of Spectral Theory (see for example [17, Chapter 10, Section 1.1], [41, Section 6.5] or [55, Theorem 1.2.1]). Observe that if we denote by $\mathscr{D}_0^{1,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\varphi \mapsto \|\nabla\varphi\|_{L^p(\Omega)},$$

then $\lambda_{p,q}(\Omega)$ is the sharp constant for the continuous embedding $\mathscr{D}_0^{1,p}(\Omega)\hookrightarrow L^q(\Omega)$. It may happen that $\lambda_{p,q}(\Omega) = 0$: in this case, such an embedding does not hold. For a complete characterization of the open sets Ω in $\mathbb{R}^N$ for which the continuous embedding $\mathscr{D}_0^{1,p}(\Omega)\hookrightarrow L^q(\Omega)$ holds true, or equivalently $\lambda_{p,q}(\Omega) > 0$, we refer the reader to [76, Section 15.4 and 15.5]. See also [13] and [26], for an alternative characterization valid for the *sub–homogeneous case* $1 \le q \le p < \infty$, and more recently [24], for other necessary conditions.

The quantities $\lambda_{p,q}$ are sometimes called *generalized principal frequencies of the $p-$Laplacian operator with Dirichlet boundary conditions*. This name is due to the fact that, whenever a minimiser $u\in W_0^{1,p}(\Omega)$ of (1.1.4) exists[3], it satisfies in weak sense the Euler–Lagrange equation

$$-\Delta_p u = \lambda\|u\|_{L^q(\Omega)}^{p-q}|u|^{q-2}u, \qquad \text{with } \lambda=\lambda_{p,q}(\Omega),$$

where the operator $-\Delta_p v := -\mathrm{div}(|\nabla v|^{p-2}\nabla v)$, acting on functions v belonging to $\mathscr{D}_0^{1,p}(\Omega)$, is the *$p-$Laplace operator with Dirichlet boundary conditions*. In other words, the variational problem (1.1.4) can be interpreted as a *nonlinear eigenvalue problem*. For a detailed account on this topic we refer the reader to [45] and to the references therein.

By combining the definition (1.1.4) with the Poincaré–Sobolev inequality (1.1.3), under the above specified assumptions on p, q and Ω, we may infer a first lower bound on the principal frequencies $\lambda_{p,q}$ in terms of a geometric feature of the open set Ω, i.e. its volume, that is

$$\frac{c(N,p)}{|\Omega|^{\frac{1}{N}-\frac{1}{p}+\frac{1}{q}}} \le \lambda_{p,q}(\Omega),$$

where $c(N,p)$ is a universal positive constant. In other words, an information on a geometric quantity associated to Ω implies the positivity of the quantity $\lambda_{p,q}$, that is equivalent to the continuity (actually, the compactness in this case) of the embedding $\mathscr{D}_0^{1,p}(\Omega)\hookrightarrow L^q(\Omega)$.

[3]This happens, for example, if the embedding $W_0^{1,p}(\Omega)\hookrightarrow L^q(\Omega)$ is compact, thus in particular if $|\Omega|<\infty$ and p, q are as in (1.1.1).

A special mention is due to the so-called *Cheeger constant*. We recall that for an open set $\Omega \subseteq \mathbb{R}^N$ this is given by

$$h(\Omega) = \inf \left\{ \frac{\mathcal{H}^{N-1}(\partial E)}{|E|} \,:\, E \Subset \Omega \text{ has a smooth boundary} \right\}.$$

Other definitions would be possible, see for example the survey papers [68] and [84]: we refer to them for an introduction to the Cheeger constant and the interesting problems connected with it. The above definition is in the spirit of the original analogous quantity introduced by Cheeger in [31] (and especially by Buser, see [29, equation (1.5)]) in the context of Riemannian manifolds. Our choice is motivated by the fact that

$$\lambda_{1,1}(\Omega) = h(\Omega), \tag{1.1.5}$$

with this definition, i.e. $h(\Omega)$ coincides with a generalized principal frequency (see for example [76, Theorem 2.1.3]).

The primary goal of this thesis is to consider other *geometric* and *capacitary features* associated to an open set Ω than the volume, that may ensure the positivity of its generalized principal frequencies, thus getting rid of the restrictive assumption on the finiteness of $|\Omega|$. As we will see in the following sections, these *key quantities* will be the *inradius* of an open set Ω, and a capacitary–based generalization of this notion: the *capacitary inradius*. We mention that in the *sub–homogeneous case*, i.e. when $1 \leq q < p < \infty$, similar estimates cannot be true, see for example [21, Proposition 6.1] and Example A.2.1.

The inradius is the following simple geometric quantity

$$r_\Omega = \sup \Big\{ r > 0 \,:\, \exists B_r(x_0) \subseteq \Omega \Big\}, \tag{1.1.6}$$

where $B_r(x_0)$ is the $N-$dimensional open ball centered at x_0, with radius r. For every open set Ω in $\mathbb{R}^N$, its inradius is naturally linked to its principal frequencies through the following simple (yet optimal) upper bound[4]

$$\lambda_p(\Omega) \leq \frac{\lambda_p(B_1)}{r_\Omega^p}, \tag{1.1.7}$$

where $B_1 = \{x \in \mathbb{R}^N \,:\, |x| < 1\}$. This follows by observing that λ_p is monotone non-increasing with respect to set inclusion, together with its scale properties.

On the contrary, when $1 \leq p \leq N$ it is not possible to bound $\lambda_p(\Omega)$ from below in terms of r_Ω. There is a problem of "removability" in this case. In other words, the quantity $\lambda_p(\Omega)$ is not affected by the removal of compact subsets $\Sigma \subseteq \Omega$ such that their $p-$capacity relative to a ball $B_R(x_0)$

$$\mathrm{cap}_p(\Sigma; B_R(x_0)) = \inf_{\varphi \in C_0^\infty(B_R(x_0))} \left\{ \int_{B_R(x_0)} |\nabla \varphi|^p \, dx \,:\, \varphi \geq 1 \text{ on } \Sigma \right\}, \qquad \Sigma \Subset B_R(x_0),$$

is zero (see Proposition 2.2.3 below), while r_Ω is in general affected by this operation. In particular, the geometric object r_Ω is affected by the removal of *single points*, while the latter are "invisible" sets for $\lambda_p(\Omega)$, since they have null $p-$capacity in the range $1 \leq p \leq N$. The typical counterexample to the lower bound is then given by $\Omega = \mathbb{R}^N \setminus \mathbb{Z}^N$, for $N \geq 2$: this has finite inradius, but

$$\lambda_p(\mathbb{R}^N \setminus \mathbb{Z}^N) = 0.$$

Imposing to functions to vanish in an arbitrarily small neighborhood of the points of a lattice is not enough to get a L^p-Poincaré inequality, when $1 \leq p \leq N$.

To restore the situation in the case $1 \leq p \leq N$, we are left with two possible choices:

- one is to take some geometric/topological restrictions on the open sets, as we will see in the following Section 1.3;
- the other one, discussed in Section 1.4, is to "relax" the definition of inradius in a suitable sense, so that this new notion and λ_p have the same removable sets.

[4] Here we implicitly assume that $r_\Omega < \infty$. Observe also that $\lambda_p(\Omega) = 0$, whenever $r_\Omega = +\infty$.

1.2. The case $p > N$

1.2.1. Background & state of the art. The case $p > N$ is peculiar. Indeed, it is now possible to prove the lower bound

$$\lambda_p(\Omega) \geq C_{N,p}\left(\frac{1}{r_\Omega}\right)^p, \tag{1.2.1}$$

for *every* open set $\Omega \subseteq \mathbb{R}^N$.

The validity of the lower bound (1.2.1) for general open sets, in any dimension N, and under the restriction that $p > N$ is a *capacitary–type result*. Indeed, contrary to the case $1 \leq p \leq N$, under this assumption points have positive $p-$capacity, so they are not removable sets for the relevant Sobolev space: this will be sufficient to ensure (1.2.1), without imposing any additional assumption on the open set Ω. This was already obtained by Maz'ya from the '70s (see for example [**76**, Theorem 15.4.1 & Comments to Chap. 15 pag. 733] and also [**77**, Section 10.3.2, Theorem 2]), by considering in place of the inradius r_Ω the quantity

$$r_{\Omega,\infty} := \sup\{r > 0 \,:\, \exists Q_r(x_0) \subseteq \Omega\},$$

where

$$Q_r(x_0) = \prod_{i=1}^{N}(x_0^i - r, x_0^i + r), \qquad \text{for } x_0 = (x_0^1, \dots, x_0^N) \in \mathbb{R}^N, r > 0.$$

This quantity is clearly comparable to the inradius, since

$$\frac{1}{\sqrt{N}}\, r_\Omega \leq r_{\Omega,\infty} \leq \sqrt{N}\, r_\Omega.$$

Maz'ya's proof relies on a tiling argument and a Poincaré type inequality [**76**, Theorem 14.1.2] (see also [**76**, Theorem 14.2.3(2)]). Unfortunately, in this procedure the explicit form of the constant gets lost.

As a first result of this thesis, we will give a slightly different proof of (1.2.1) aimed at giving a better control on the constant $C_{N,p}$. As we will see in details in Section 1.3.4, our proof further relies on the analysis of some "punctured" Poincaré constants, that will provide explicit constants and the desired *sharp* asymptotic behaviours for $p \nearrow N$ and $p \searrow \infty$. For more details on this part, we refer the reader to Section 3.2.

Remark 1.2.1. In literature several proofs of inequality (1.2.1) are available. In [**24**, Theorem 5.4 & Remark 5.5], the authors obtained the same estimate by means of Hardy's inequality: if on the one hand the estimate in [**24**] is very simple and explicit, on the other hand it does not display the correct decay rate to 0, as p goes to N. This undesired behaviour is rectified by our proof. We mention also [**85**, Theorem 1.4.1] and [**92**, Theorem 1.1], where the authors obtained the same estimate for λ_p, without exhibiting an explicit constant. Finally, in [**81**, Corollary 2.6], the authors extend this type of result to the case of weighted eigenvalue problems, even though they do not provide exiplicit constants.

1.2.2. Main Results. Before stating our main theorem in this context, we need at first to fix some notation. We indicate by B_1 the $N-$dimensional open ball centered at the origin, with radius 1. For $p > N$, we define the "punctured" Poincaré constants

$$\Lambda_p(B_1 \setminus \{0\}) = \inf_{u \in \mathrm{Lip}(\overline{B_1})}\left\{\int_{B_1} |\nabla u|^p\, dx \,:\, \|u\|_{L^p(B_1)} = 1,\, u(0) = 0\right\},$$

and

$$\Lambda_{p,\infty}(B_1 \setminus \{0\}) = \inf_{u \in \mathrm{Lip}(\overline{B_1})}\left\{\int_{B_1} |\nabla u|^p dx \,:\, \|u\|_{L^\infty(B_1)} = 1, u(0) = 0\right\}.$$

We will prove the following

Theorem 1. *Let $1 \leq N < p$. Then, for every open set $\Omega \subseteq \mathbb{R}^N$ with finite inradius r_Ω, we have*

$$\lambda_p(\Omega) \geq \beta_{N,p}\left(\frac{1}{r_\Omega}\right)^p, \qquad \text{with } \beta_{N,p} = \max\left\{\frac{\Lambda_p(B_1 \setminus \{0\})}{(\sqrt{N}+1)^p}, \left(\frac{p-N}{p}\right)^p\right\} > 0, \tag{1.2.2}$$

and

$$\lambda_{p,\infty}(\Omega) \geq \Lambda_{p,\infty}(B_1 \setminus \{0\}) \left(\frac{1}{r_\Omega}\right)^{p-N}. \tag{1.2.3}$$

For $p < q < \infty$, we also get

$$\lambda_{p,q}(\Omega) \geq \Big(\beta_{N,p}\Big)^{\frac{p}{q}} \Big(\Lambda_{p,\infty}(B_1 \setminus \{0\})\Big)^{1-\frac{p}{q}} \left(\frac{1}{r_\Omega}\right)^{p-N+N\frac{p}{q}}. \tag{1.2.4}$$

Finally, the two constants $\beta_{N,p}$ and $\Lambda_{p,\infty}(B_1 \setminus \{0\})$ exhibit the following asymptotic behaviour

$$0 < \liminf_{p\searrow N} \frac{\beta_{N,p}}{(p-N)^{p-1}} \leq \limsup_{p\searrow N} \frac{\beta_{N,p}}{(p-N)^{p-1}} < +\infty \qquad \textit{and} \qquad \lim_{p\nearrow\infty} (\beta_{N,p})^{\frac{1}{p}} = 1,$$

$$0 < \liminf_{p\searrow N} \frac{\Lambda_{p,\infty}(B_1 \setminus \{0\})}{(p-N)^{p-1}} \leq \limsup_{p\searrow N} \frac{\Lambda_{p,\infty}(B_1 \setminus \{0\})}{(p-N)^{p-1}} < +\infty \quad \textit{and} \quad \lim_{p\nearrow\infty} (\Lambda_{p,\infty}(B_1 \setminus \{0\}))^{\frac{1}{p}} = 1,$$

Even if the constants obtained are very likely not optimal, we can prove that their asymptotic behaviour is optimal, as explained in Remark 3.3.1 below.

1.2.3. Outline of the proof: Theorem 1. We rely on a tiling argument together with precise estimates on some "punctured" Poincaré constants (see Section 3.2), in order to obtain an explicit constant with the correct asymptotic behaviour. More precisely:

(1) we tile the whole space $\mathbb{R}^N$ by cubes having inradius $r_\Omega + \varepsilon$. If $u \in C_0^\infty(\Omega)$, each of these cubes must contain at least a point outside the support of u;

(2) we use now some estimates for *ad–hoc* defined Poincaré constants on these cubes, where the Dirichlet region coincides with a point (see Lemma 3.2.2 and Lemma 3.2.3). These results used in combination with estimates on the sharp Poincaré–Wirtinger constant (see Lemma 2.5.2), yield an explicit constant which has the correct asymptotic behaviour, for $p \searrow N$ and $p \nearrow \infty$, as explained in Remark 3.3.1;

(3) for $p < q < \infty$, we use an interpolation argument to derive the desired lower bounds on the quantities $\lambda_{p,q}$, from those of the two "endpoints" λ_p and $\lambda_{p,\infty}$.

1.3. A planar case: the Croke–Osserman–Taylor inequality

1.3.1. Background & state of the art. As anticipated at the beginning of this Introduction, the possibility to reverse inequality (1.1.7) is in general forbidden, unless some geometry comes into play: for example, a lower bound of the type (1.2.1) holds for convex sets (see [56, Théorème 8.1] and [59, Theorem 2.1]). More generally, as it is clear from the proof of [59], this is still valid for open sets $\Omega \subseteq \mathbb{R}^N$ such that the *distance function*

$$d_\Omega(x) := \min_{y\in\partial\Omega} |x-y|, \qquad \text{for } x \in \Omega,$$

is weakly superharmonic in Ω (see also [25, Remark 5.8]). These are quite rigid assumptions, but it should be noticed that in general they can not be weakened too much: for example, starting from dimension $N \geq 3$, "convexity" can not be replaced by "starshapedness", as shown by a simple counterexample in [54, Section 4]. This is due to the fact that lines have zero $p-$capacity, when the ambient dimension is at least 3 and $p \leq N-1$.

On the other hand, the case $N = 2$ is special: in this case, very simple topological assumptions may lead to a positive answer. For example, a remarkable result by Makai [74] (neglected for various years and rediscovered independently by Hayman in [54, Theorem 1]) asserts that for $p = 2$ the lower bound (1.2.1) holds for every *simply connected* subset of $\mathbb{R}^2$. Actually, in this very beautiful and striking result, the topological assumption can be further relaxed. The same kind of result still holds for *multiply connected* open subsets of $\mathbb{R}^2$. Their precise definition is as follows:

Definition 1.3.1. Let us indicate by $(\mathbb{R}^2)^*$ the *one-point compactification* of $\mathbb{R}^2$, i.e. the compact space obtained by adding to $\mathbb{R}^2$ the point at infinity. We say that an open connected set $\Omega \subseteq \mathbb{R}^2$ is *multiply connected of order* k if its complement in $(\mathbb{R}^2)^*$ has k connected components. When $k = 1$, we will simply say that Ω is *simply connected*.

For this class of planar sets, Taylor in [91, Theorem 2] proved the following lower bound

$$\lambda(\Omega) \geq \frac{C}{k}\left(\frac{1}{r_\Omega}\right)^2. \tag{1.3.1}$$

The constant C can be made explicit, but its sharp value is still unknown. The best known lower bound for the case $k = 1$ is due to van den Berg and Bucur (see [12, Theorem 1] and the comment below). Their result slightly improves the previous lower bound by Bañuelos and Carroll (see [6, Corollary 1]). For the general case $k \geq 2$, a simple explicit constant has been obtained by Croke in [33], by refining the method of proof by Osserman [82].

However, it is important to notice that the dependence on the "topological index" k is optimal, i.e. one can construct sequences of open sets $\{\Omega_k\}_{k\in\mathbb{N}\setminus\{0\}}$ such that r_{Ω_k} is uniformly bounded, each Ω_k is multiply connected of order k and

$$\lambda(\Omega_k) \sim \frac{1}{k}, \qquad \text{as } k \to \infty.$$

We also refer to [52, Theorem 3] for another proof of this result, though the result in [52] is slightly worse in its dependence on k.

Our main goal in this context is to extend this kind of analysis to any Poincaré–Sobolev embedding constant, not only to the bottom of the spectrum of the Dirichlet–Laplacian.

1.3.2. Main results. We will give a *topological result*, i.e. estimates on $\lambda_{p,q}$ for *planar* sets having given topological properties, as in the Croke–Osserman–Taylor inequality. Let us present the result, while postponing some comments about comparisons with already existing results. This is taken from [B2] and contained in Chapter 4.

Theorem 2. *Let $1 \leq p < \infty$ and let $p \leq q$ be such that (1.1.1) holds, with $N = 2$. Then, there exists a constant $\Theta_{p,q} > 0$ such that for every $\Omega \subseteq \mathbb{R}^2$ open multiply connected set of order $k \in \mathbb{N} \setminus \{0\}$ with finite inradius r_Ω, we have*

$$\lambda_{p,q}(\Omega) \geq \Theta_{p,q}\left(\frac{1}{\sqrt{k}\, r_\Omega}\right)^{p-2+\frac{2\,p}{q}}. \tag{1.3.2}$$

Moreover, the constant $\Theta_{p,q}$ has the following asymptotic behaviour:

- *for* $1 \leq p < 2$
$$0 < \lim_{q\nearrow p^*} \Theta_{p,q} < +\infty;$$
- *for* $p = 2$
$$0 < \liminf_{q\nearrow\infty} \left(q\,\Theta_{2,q}\right) \leq \limsup_{q\nearrow\infty} \left(q\,\Theta_{2,q}\right) < +\infty.$$

Though not optimal, the constant $\Theta_{p,q}$ is explicit. Moreover, we show that it depends in the correct way on the parameter q, as this goes to p^* (case $p < 2 = N$) or to ∞ (case $p = 2 = N$). We also point out that the dependence on the topology k in the previous estimate is optimal. We refer to Remark 4.1.1 for these comments.

As anticipated at the beginning of this Introduction, we recall that a lower bound of the type

$$\lambda_{p,q}(\Omega) \geq C_{N,p,q}\left(\frac{1}{r_\Omega}\right)^{p-N+N\frac{p}{q}}, \tag{1.3.3}$$

can only be true in the *super–homogeneous case* $1 \leq p \leq q < \infty$, indeed in the *sub–homogeneous case* it fails even for convex sets, see Remark 4.3.1.

REMARK 1.3.2 (Comparison with previous results). The inequality of Theorem 2 is a generalization to the case of $\lambda_{p,q}$ of the classical result by Osserman, Taylor and Croke previously mentioned. For the particular case $q = p$, such a generalization has been already obtained by Poliquin in [**86**, Theorem 2]. Apart from allowing $q \neq p$, our method of proof is different: unlike Poliquin, who relies on the Osserman-Croke argument, we follow the approach by Taylor.

While producing a worse constant, Taylor's proof is extremely robust and flexible, relying only on a geometric property of multiply connected sets with finite inradius (what it is called "Taylor's fatness lemma" in [**15**]), together with some properties of $p-$capacity. The method is explained in detail in the following section. Its simplicity and intrinsically variational nature permit the whole family of $\lambda_{p,q}$ to be treated at the same time, without any distinction. In [**15**] these same ideas are applied to the case of the first eigenvalue of the *fractional* Dirichlet–Laplacian.

We point out that with this method, no a priori knowledge of the regularity properties of extremals for $\lambda_{p,q}$ is needed. On the contrary, in the proof by Osserman and Croke, the main ingredient is given by a suitable Cheeger–type inequality (see [**82**, Lemma 2]). The proof of this inequality relies on a careful analysis of the topology of the level sets of extremals. Extending this technique to the case $p \neq 2$ is quite delicate, since in this case extremals are well-known to be only $C^{1,\alpha}$ regular, a property which does not permit to apply[5] Sard's Lemma. The latter is an essential ingredient in the proof for $p = 2$ (where extremals are actually C^∞).

1.3.3. Further consequences: the Cheeger constant. By virtue of (1.1.5), the previous results imply some bounds for the Cheeger constant. Indeed, by combining this fact and Theorem 2, we immediately get the following lower bound on the Cheeger constant of a planar set, in terms of both its inradius and topology.

COROLLARY. *Let $k \in \mathbb{N} \setminus \{0\}$. For every $\Omega \subseteq \mathbb{R}^2$ open multiply connected set of order k with finite inradius r_Ω, we have*

$$h(\Omega) \geq \frac{\Theta_{1,1}}{\sqrt{k}} \frac{1}{r_\Omega}, \tag{1.3.4}$$

where $\Theta_{1,1}$ is the same constant as in Theorem 2.

REMARK 1.3.3. It is easily seen that the geometric lower bound (1.3.4) *is not possible* for the following alternative definition of Cheeger constant

$$h_{\mathrm{DG}}(\Omega) = \inf \left\{ \frac{P(E)}{|E|} \,:\, E \subseteq \Omega \text{ with } |E| > 0 \right\},$$

where $P(E)$ is the *distributional perimeter* of E, in the sense of De Giorgi. This is another possible definition of Cheeger constant, considered in many papers (in addition to the aforementioned references [**68**] and [**84**], we refer for example to [**30, 53, 61, 62, 66**] and [**67**] among others). In general, we have $h_{\mathrm{DG}}(\Omega) < h(\Omega)$, see for example [**73**, Section 3].

Since the notion of distributional perimeter is not affected by the removal of sets with zero $N-$dimensional Lebesgue measure, we can easily build a counter-example to the validity of (1.3.4) for h_{DG}. For example, by taking the following *infinite complement comb*

$$\Omega = \mathbb{R}^2 \setminus \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, |x_1| \geq 1, x_2 \in \mathbb{Z}\},$$

we see that this is a simply connected open set, such that

$$r_\Omega = \sqrt{2} \qquad \text{and} \qquad h_{\mathrm{DG}}(\Omega) \leq \lim_{n\to\infty} \frac{P((-n,n)\times(-n,n))}{|(-n,n)\times(-n,n)|} = \lim_{n\to\infty} \frac{8\,n}{4\,n^2} = 0.$$

REMARK 1.3.4. We mention that the geometric lower bound (1.3.4) has been already obtained in [**33**], by means of the isoperimetric inequality in the plane. See also [**5**, Theorem 2.1] for a generalization to the

[5] In dimension $N \geq 2$, we recall that the minimal assumption for the validity of this result is $C^{N-1,1}$ regularity (see [**7**, Theorem 1] and also [**35**]). For $C^{N-1,\alpha}$ with $\alpha < 1$, one can already build counter-examples to Sard's Lemma, see [**3**].

case when the Cheeger constant $h(\Omega)$ is replaced by

$$h_\alpha(\Omega) := \inf\left\{\frac{\mathcal{H}^{N-1}(\partial E)}{|E|^{\frac{1}{\alpha}}}\ :\ E \Subset \Omega \text{ has smooth boundary}\right\},$$

for $1 \le \alpha < N/(N-1)$.

This result, which is interesting in itself, in turn permits to give a spectral estimate relating the geometric constant h with the bottom of the spectrum λ. Indeed, by joining this lower bound with (1.1.7), one can get the following upper bound on $\lambda(\Omega)$: as simple as it is, it deserves to be explicitly stated.

THEOREM 3. *For every $\Omega \subseteq \mathbb{R}^2$ open multiply connected set of order $k \in \mathbb{N}\setminus\{0\}$, we have*

$$\lambda(\Omega) \le \left(\frac{j_{0,1}}{\Theta_{1,1}}\right)^2 k\,\Big(h(\Omega)\Big)^2, \tag{1.3.5}$$

where $\Theta_{1,1}$ is the same constant as in Theorem 2 and $j_{0,1}$ is the first zero of the Bessel function of the first kind J_0 (see for example [**55**, page 11] *for an approximate value).*

PROOF. We first observe that if $B_r(x_0) \Subset \Omega$, then by monotonicity with respect to set inclusion we have

$$\lambda(\Omega) \le \frac{\lambda(B_1)}{r^2} = \frac{(j_{0,1})^2}{r^2} \qquad \text{and} \qquad h(\Omega) \le \frac{\mathcal{H}^1(\partial B_r(x_0))}{|B_r(x_0)|} = \frac{2}{r}.$$

For the value of $\lambda(B_1)$ we refer to [**55**, Proposition 1.2.14].

Thus, if Ω has infinite inradius, from the previous upper bounds we get $\lambda(\Omega) = h(\Omega) = 0$ and the result trivially follows. In the case $r_\Omega < +\infty$, it is sufficient to combine (1.3.4) with

$$\lambda(\Omega) \le \frac{(j_{0,1})^2}{r_\Omega^2}.$$

This concludes the proof. □

Such an estimate is better appreciated by recalling the celebrated *Cheeger inequality*, i.e. the following spectral lower bound of geometric flavour

$$\left(\frac{h(\Omega)}{2}\right)^2 \le \lambda(\Omega), \tag{1.3.6}$$

which holds for every open set $\Omega \subseteq \mathbb{R}^N$ and every dimension N (see for example [**76**, Chapter 4, Section 2]). Reversing this kind of estimate in general is not possible, unless some severe geometric restrictions are taken: this is possible for convex sets (see [**83**, Proposition 4.1] and [**20**, Corollary 4.1]). On the contrary, exactly as in the case of the inradius, it fails already for starshaped sets in dimension $N \ge 3$, see [**76**, Chapter 4, Section 3]. This kind of reverse Cheeger inequality is also called *Buser inequality*, named after Buser who in [**28**] first obtained this type of estimate, in the framework of Riemannian manifolds (see also Ledoux' papers [**63, 64**]). It is also mandatory to refer to the paper [**79**].

The result of Theorem 3 can thus be regarded as *Buser inequality for multiply connected open sets in the plane*. It is quite remarkable that in dimension $N = 2$ this holds without any curvature assumption on the sets. We notice however that the estimate gets spoiled as the topology of the sets becomes more and more intricate (i.e. as k goes to ∞). We will show by means of an example that *this behaviour is "essentially" optimal*. Indeed, the factor k in (1.3.5) cannot be replaced by k^α, for $0 < \alpha < 1$ (see Proposition 4.5.1).

REMARK 1.3.5. With exactly the same proof of Theorem 3, one can obtain the following Buser–type inequality, for the whole family of generalized principal frequencies: for every $\Omega \subseteq \mathbb{R}^2$ open multiply connected set of order $k \in \mathbb{N}\setminus\{0\}$ and every $1 \le q$ which satisfies (1.1.1), we have

$$\lambda_{p,q}(\Omega) \le C\,k^{\frac{p-2}{2}+\frac{p}{q}}\,\Big(h(\Omega)\Big)^{p-2+2\frac{p}{q}},$$

with the constant C given by

$$\frac{\lambda_{p,q}(B_1)}{(\Theta_{1,1})^{p-2+2\frac{p}{q}}}.$$

Observe that this is now valid for the sub-homogeneous regime $1 \le q < p$, as well. In particular, by recalling that $\lambda_{2,1}(\Omega)$ coincides with the reciprocal of the so-called *torsional rigidity* $T(\Omega)$, we get the following inequality

$$\frac{1}{C\,k^2} \le \Big(h(\Omega)\Big)^4\, T(\Omega), \qquad \text{with } C = (\Theta_{1,1})^4\, \frac{\pi}{8}.$$

We also used that $T(B_1) = 1/\lambda_{2,1}(B_1) = \pi/8$, in dimension $N = 2$. We refer to [73] for a study of this inequality, sometimes called *Cheeger–Kohler-Jobin inequality*.

1.3.4. Outline of the proof: Theorem 2. Our strategy is based on that of Taylor's proof of [91, Theorem 1.1], and it is similar to that of Theorem 1 since we rely on a tiling argument, but now we need a *geometric lemma* in order to circumvent the fact that points may have null $p-$capacity:

(1) we tile the plane by a collection of squares $\{Q_{ij}\}_{(i,j)\in\mathbb{Z}^2}$, all equivalent to each other, and having side–length d equal to the inradius r_Ω, up to a multiplicative factor which depends on k. Then, for every[6] $u \in C_0^\infty(\Omega)$ we split the L^p–norm of its gradient over this "grid", and restrict ourselves to consider a single square Q_{ij} on which we look for a Poincaré–type inequality;

(2) the choice of the square's side–length is made on purpose, in order to exploit a topological argument due to Taylor (see "Taylor's fatness Lemma" 4.2.1). Basically, under our assumption on Ω, it guarantees the existence of a "fat" compact set in the (relative) complement $\overline{Q} \setminus \Omega$, such that the length of at least one of its orthogonal projections is bounded from below only in terms of r_Ω and k;

(3) the foregoing Taylor's geometric lemma combined with a simple *analytic–geometric* estimate between capacity and one-dimensional Hausdorff measure (see Lemma 4.2.3) implies that we can apply a *Maz'ya–Poincaré type inequality* (see Theorem 2.6.1), valid for smooth functions on a square that vanish on a compact set with positive capacity. We emphasize that this is the point where the main difference between Taylor's proof and ours emerges. Indeed, Taylor's argument relies on heat kernel estimates aimed at providing a lower bound on the first eigenvalue of the Lapacian with mixed boundary conditions, Dirichlet and Neumann, in terms of the Dirichlet region. While ours relies on Theorem 2.6.1, whose proof, adapted from [76, Theorem 14.1.2(1)], is genuinely variational. It is based on a classical *cut–off* argument and an extension lemma devised for Sobolev Spaces (see Section 2.4);

(4) the last step is now reconstructing the norm of the function and of its gradient by summing up over all the squares tiling the plane.

1.4. A capacitary criterion for Poincaré–Sobolev inequalities

1.4.1. Background & state of the art. As anticipated in Section 1.1, another possible way to restore inequality (1.2.1), without imposing any additional condition on the open sets, would be that of extending the notion of inradius, in such way that, just like λ_p, it is no longer sensitive to the removal of sets with zero $p-$capacity. This is the content of Chapter 5, which is based on results taken from [B1], and it will be dedicated to establish a two–sided estimate of $\lambda_p(\Omega)$ in terms of a capacitary variant of the inradius, the *capacitary inradius*.

A natural idea to achieve this goal would be that of replacing the inradius r_Ω with a "relaxed" version, which allows the balls to be contained in Ω only in a "capacitary sense". Thus, a first naive attempt would be that of replacing the usual inradius r_Ω with the following capacitary variant

$$\mathfrak{R}_\Omega := \sup\Big\{r > 0\,:\,\exists x_0 \in \mathbb{R}^N \text{ such that } \mathrm{cap}_p\Big(\overline{B_r(x_0)} \setminus \Omega; B_{2r}(x_0)\Big) = 0\Big\}. \tag{1.4.1}$$

However, even by using the inradius defined by (1.4.1), one could show that a lower bound on $\lambda_p(\Omega)$ is not possible, without any further assumption on the open set Ω. We refer to Example A.1.1 for a counter-example. The main problem in the definition (1.4.1) is the lack of some "uniformity" in the portion of complement $\mathbb{R}^N \setminus \Omega$ that this capacitary variant of the inradius can detect.

[6] Here we implicitly assume to extend u by zero on the whole plane.

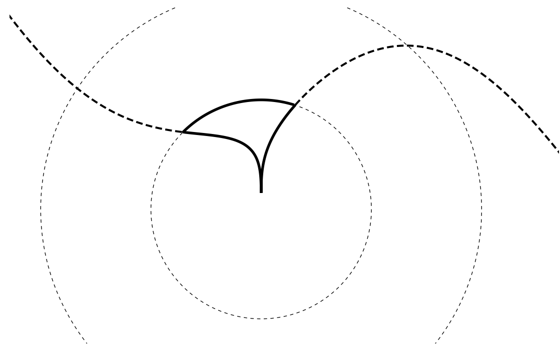

FIGURE 1. Contoured by the bold line, the set $\overline{B_r(x_0)}\setminus\Omega$. For γ small enough, its $p-$capacity is smaller than γ times the capacity of the whole ball (the smaller one, in dashed line). Accordingly, this radius r is a feasible competitor in the definition of the capacitary inradius. The largest ball in dashed line corresponds to the "box" $B_{2r}(x_0)$ which is used to compute the relative capacity.

In order to circumvent this problem, in [78] Maz'ya and Shubin proposed to work with the concept of *negligible set* (in the sense of Molchanov), for a fixed parameter $0 < \gamma < 1$ (see also [76, Sections 16.6-18.7]). This leads us to the following

DEFINITION 1.4.1. Let $1 \le p < \infty$ and $0 < \gamma < 1$, we say that a compact set $\Sigma \subseteq \overline{B_r(x_0)}$ is $(p,\gamma)-$*negligible* if

$$\mathrm{cap}_p(\Sigma; B_{2r}(x_0)) \le \gamma\,\mathrm{cap}_p\left(\overline{B_r(x_0)}; B_{2r}(x_0)\right).$$

Accordingly, we consider the *capacitary inradius of* Ω, defined as follows[7]

$$R_{p,\gamma}(\Omega) := \sup\left\{r > 0 \,:\, \exists x_0 \in \mathbb{R}^N \text{ such that } \overline{B_r(x_0)} \setminus \Omega \text{ is } (p,\gamma)-\text{negligible}\right\}. \tag{1.4.2}$$

see Figure 1. From its definition, we can immediately record the following two properties

$$r_\Omega \le R_{p,\gamma}(\Omega), \text{ for every } 0 < \gamma < 1, \qquad \text{and} \qquad \gamma \mapsto R_{p,\gamma}(\Omega) \text{ is monotone non-decreasing.}$$

REMARK 1.4.2. The analysis of the paper [78] was confined to the case $p = 2$. Moreover, the definition of capacitary inradius there contained is slightly different from ours (1.4.2), since the authors use the *absolute* $2-$*capacity*

$$\mathrm{cap}_2(\Sigma) := \inf_{\varphi \in C_0^\infty(\mathbb{R}^N)} \left\{\int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx \,:\, \varphi \ge 1 \text{ on } \Sigma\right\}.$$

Observe that it is necessary to use the concept of *relative* capacity, in order to include in the discussion the conformal case $p = N$, as well. Indeed, we recall that for every compact set $\Sigma \subseteq \mathbb{R}^N$ its *absolute* $N-$*capacity*, defined by

$$\mathrm{cap}_N(\Sigma) := \inf_{\varphi \in C_0^\infty(\mathbb{R}^N)} \left\{\int_{\mathbb{R}^N} |\nabla\varphi|^N\,dx \,:\, \varphi \ge 1 \text{ on } \Sigma\right\},$$

is always zero, due to the scale invariance of the $N-$Dirichlet integral (see [76, pages 148–149]). For this reason, the case $p = N = 2$ is not explicitly treated in [78]. We will come back on a comparison between our result and those of [78] in a while.

[7] For ease of simplicity, we prefer to simply call it *capacitary inradius*, rather than $(p,\gamma)-$*capacitary inradius* or something similar.

1.4.2. Main results. The following two–sided estimate is the main result of the second part of this thesis. This can be seen as an extension of [**78**, Theorem 1.1], to the case $p \neq 2$. We refer to Remark 5.4.1 for a comment on the constant $C_{N,p,\gamma}$.

THEOREM 4. *Let $1 \leq p \leq N$, $0 < \gamma < 1$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set. Then we have*

$$\sigma_{N,p}\,\gamma\left(\frac{1}{R_{p,\gamma}(\Omega)}\right)^p \leq \lambda_p(\Omega) \leq C_{N,p,\gamma}\left(\frac{1}{R_{p,\gamma}(\Omega)}\right)^p,$$

with the constant $C_{N,p,\gamma}$ which diverges to $+\infty$, as γ goes to 1. In particular, we have

$$\lambda_p(\Omega) > 0 \qquad \Longleftrightarrow \qquad R_{p,\gamma}(\Omega) < +\infty,$$

and the last condition does not depend on $0 < \gamma < 1$.

As in [**78**], the proof of this result is constructive and thus the constants $\sigma_{N,p}$ and $C_{N,p,\gamma}$ are computable, in principle. However, since they are very likely not sharp and their explicit expression is not particularly pleasant, we prefer to avoid writing them in the statement above.

Before going further, we wish to highlight a couple of consequences: the first one is a simple rewriting of the statement, in the case $p = 1$. Indeed, as already seen, for $p = 1$ the quantity $\lambda_p(\Omega)$ actually coincides with the Cheeger constant of Ω.

We get the following two–sided estimate, which deserves to be explicitly written.

COROLLARY. *Let $0 < \gamma < 1$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set. Then we have*

$$\sigma_{N,1}\,\gamma\,\frac{1}{R_{1,\gamma}(\Omega)} \leq h(\Omega) \leq C_{N,1,\gamma}\,\frac{1}{R_{1,\gamma}(\Omega)},$$

with the constant $C_{N,1,\gamma}$ which diverges to $+\infty$, as γ goes to 1. In particular, we have

$$h(\Omega) > 0 \qquad \Longleftrightarrow \qquad R_{1,\gamma}(\Omega) < +\infty,$$

and the last condition does not depend on $0 < \gamma < 1$.

A second consequence concerns the so-called *$p-$torsion function* of an open set Ω. This function, denoted by w_Ω, is informally defined as the solution of

$$-\Delta_p w_\Omega = 1, \qquad \text{in } \Omega,$$

with homogeneous Dirichlet boundary conditions on $\partial\Omega$. For the precise definition in the case of a general open set, we refer to [**26**, Section 2], for example. The importance of this function in the context of the theory of Sobolev spaces is encoded in the following equivalence

$$\lambda_p(\Omega) > 0 \qquad \Longleftrightarrow \qquad w_\Omega \in L^\infty(\Omega).$$

Actually, this equivalence can be made "quantitative". Indeed, from [**26**, Theorem 1.3] and [**16**, Theorem 9], we know that

$$1 \leq \lambda_p(\Omega)\,\|w_\Omega\|_{L^\infty(\Omega)}^{p-1} \leq \mathbf{D}_{N,p}. \tag{1.4.3}$$

We also refer to [**27**, Proposition 6] and [**49**, Lemma 4.1] for the leftmost estimate, in the case of smooth bounded domains.

By joining this two–sided estimate with that of Theorem 4, we get the following

COROLLARY. *Let $1 < p \leq N$, $0 < \gamma < 1$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set. Then we have*

$$\left(\frac{1}{C_{N,p,\gamma}}\right)^{\frac{1}{p-1}}\Big(R_{p,\gamma}(\Omega)\Big)^{\frac{p}{p-1}} \leq \|w_\Omega\|_{L^\infty(\Omega)} \leq \left(\frac{\mathbf{D}_{N,p}}{\gamma\,\sigma_{N,p}}\right)^{\frac{1}{p-1}}\Big(R_{p,\gamma}(\Omega)\Big)^{\frac{p}{p-1}},$$

where $\sigma_{N,p}$ and $C_{N,p,\gamma}$ are the same constants as in Theorem 4.

REMARK 1.4.3. For completeness, let us discuss the counterpart of the previous result, with the classical inradius r_Ω in place of $R_{p,\gamma}(\Omega)$. The lower bound holds for every open set. Indeed, by the comparison principle for the $p-$Laplacian, for every ball $B_r(x_0) \subseteq \Omega$ we have

$$w_\Omega(x) \geq w_{B_r(x_0)}(x) = \frac{p-1}{p}\,\frac{1}{N^{\frac{1}{p-1}}}\,\left(r^{\frac{p}{p-1}} - |x-x_0|^{\frac{p}{p-1}}\right)_+.$$

By passing to the essential supremum and using the arbitrariness of the ball, we obtain the sharp lower bound

$$\|w_\Omega\|_{L^\infty(\Omega)} \geq \frac{p-1}{p}\,\frac{1}{N^{\frac{1}{p-1}}}\,\left(r_\Omega\right)^{\frac{p}{p-1}}.$$

The upper bound on the contrary is not always true, unless some restrictions are imposed on the open sets. Once again, removability issues can be held responsible for the failure. It is known to be true for *convex sets* and for *planar multiply connected sets*, for example. In the first case, this is contained in [**36**, Theorem 1.2] (see also [**25**, Corollary 5.3]). In the second case, it can be obtained by combining the rightmost inequality in (1.4.3), with the lower bound on $\lambda_p(\Omega)$ given by Theorem 2. The special case $p = 2$ for an open simply connected subset of the plane was contained in [**6**, Corollary 1, equation (0.6)].

1.4.3. Some comments on Theorem 4. We fairly admit that the ideas here adopted are very much inspired to [**78**]. Indeed, it was our original intention to expand the analysis of [**78**], shed some light on the methods therein used and extend the results to the general case of the L^p Poincaré inequality (and more generally to $L^q - L^p$ Poincaré–Sobolev inequalities, see Section 5.5).

We remark at first that a two–sided estimate like that of Theorem 4, still valid for every $1 \leq p \leq N$, was already contained in the old version of Maz'ya's book [**77**]: with a brave and careful inspection, one could trace it back to [**77**, Theorem 11.4.1] (this is [**76**, Theorem 15.4.1] in the new version). To be more precise, the latter is concerned with a slight variant of the capacitary inradius $R_{p,\gamma}(\Omega)$ introduced above, defined by replacing balls with cubes. In the notation and terminology of [**77**, Theorem 11.4.1] and [**76**, Theorem 15.4.1], this is the quantity $D_{p,l}(\Omega)$ with $l = 1$, called $(p,l)-$*inner diameter* (see [**77**, Definition 10.2.2] or [**76**, Definition 14.2.2], by taking $Q_d = \mathbb{R}^n$, with the notation there). In the aforementioned result, the author proved that

$$D_{p,1}(\Omega) \lesssim C \lesssim D_{p,1}(\Omega),$$

where the constant C in [**76, 77**] coincides with $(\lambda_p(\Omega))^{-1/p}$, in our notation. For the equivalence between the notions of $(p,1)$–inner diameter and that of (p,γ)–capacitary inradius see Proposition 2.2.5.

Apart for the use of cubes in place of balls, the key point which marks the big difference with both [**78**, Theorem 1.1] and our result, is that [**77**, Theorem 11.4.1] is proved under a *restriction on the negligibility parameter* γ. In other words, for the arguments used in [**76, 77**] it is needed that

$$0 < \gamma \leq \gamma_{N,p} < 1,$$

with $\gamma_{N,p}$ explicit and exponentially decaying to 0, as N goes to ∞ (see [**77**, equation (10.1.2)] or [**76**, equation (14.1.2)]).

Maz'ya and Shubin in their paper [**78**] dropped this restriction, at least in the quadratic case $p = 2$. Our main result then permits to overcome this limitation on γ *for the whole range of* p, as well. Moreover, at the same price, we can get the same type of two–sided estimate for the quantities $\lambda_{p,q}(\Omega)$, for every subcritical exponent $q > p$.

Indeed, the main interest of both [**78**, Theorem 1.1] and our Theorem 4 lies in the fact that the results hold *for every* $0 < \gamma < 1$. This is quite remarkable, since as γ gets closer and closer to 1, a ball which is $(p,\gamma)-$negligible is admitted to catch more and more portion of the complement of Ω. This means that $R_{p,\gamma}(\Omega)$ starts to keep less and less memory of Ω: nevertheless, as far as $\gamma < 1$, it carries an information which is still enough to assure the validity of the L^p Poincaré inequality (and even of the $L^q - L^p$ Poincaré–Sobolev inequalities).

Even if we follow quite closely [**78**], this does not mean that the proof of Theorem 4 is just a straightforward transposition of that of [**78**, Theorem 1.1]. For example, in the proof of the upper bound, Maz'ya and Shubin rely very much on the representation formula for the *capacitary potential*, i.e. the function attaining the minimum value

$$\mathrm{cap}_2\left(\overline{B_r(x_0)} \setminus \Omega; B_{2r}(x_0)\right).$$

Such a potential can be expressed in terms of the fundamental solution of the Laplacian (more precisely, in terms of the Green function, at least in our case which uses the relative capacity). It is probably superfluous to mention that this is not possible for $p \neq 2$, due to the nonlinearity of the relevant equation. Whenever possible, we also tried to simplify certain technical points of the original paper and add some explanations.

REMARK 1.4.4. Vladimir Bobkov informed us of some recent related results by A.-K. Gallagher, see [46, 47]. In these papers, the author introduces an alternative notion of capacitary inradius, slightly different from the one used here and in [78], and characterizes the validity of the L^p Poincaré inequality (for $1 < p < \infty$) in terms of the finiteness of such a capacitary inradius.

1.4.4. Outline of the proof: Theorem 4. We now wish to make some comments on the proofs.

- *Lower bound*: we proceed quite similarly to Maz'ya and Shubin. As in Maz'ya's proof of the lower bound (1.2.1) (see Section 1.2.1), the key point is the use of a Maz'ya–Poincaré inequality for functions in a cube or a ball, vanishing on a Dirichlet region with positive capacity (the prototype of this type of results is [76, Theorem 14.1.2]). We partially amend this strategy, by using a variant of such an inequality for functions on cubes, but with the capacity of the Dirichlet region computed with respect to a ball, Theorem 2.6.1. This is a sort of "mixed" strategy taken from [B2], also used in the proof of Theorem 2 (see Section 1.3.4). This permits to get the result by a *tiling argument* with cubes, rather than by a *covering argument* with balls as in [78]. This simplifies the argument, to a certain extent (it is not necessary to take into account the dimensional-dependent multiplicity factor of the covering). This gives a constant which is quantitatively rougher than that of [78], but it is qualitatively comparable in terms of γ, i.e. our lower bound still decays to 0 linearly with γ, when this goes to 0 (compare it with [78, equation (3.19)]).
- *Upper bound*: this is the point that requires greater care, in order to allow the parameter γ to be arbitrarily close to 1. Here as well we follow Maz'ya and Shubin, but as remarked above a nonlinear approach is now needed to get (or to judiciously estimate) the sharp constant in a subtle $L^1 - L^p$ Poincaré–type inequality, for $p \neq 2$. Even if we are not able to get the explicit expression for the extremals of this inequality, by suitably using some integral identities we can determine the optimal constant. The expression of such a constant is a bit involved and difficult to handle: nevertheless, by using a monotonicity property of the relative $p-$capacity of balls, which can be seen as a weaker version of *Grötzsch's lemma* (see Lemma 2.3.2 and Remark 2.3.3), we can finally get an estimate of the sharp constant which is handy and good enough for our purposes. All this part is the content of Section 5.2.

  At a technical level, we also avoid the delicate approximation argument used in [78], to replace $\overline{B_r(x_0)} \setminus \Omega$ (which may be very rough) with a smoother set. This is needed in [78] so to work with a capacitary potential which is sufficiently smooth and exploit the fact that this is harmonic. Here, on the contrary, we work directly with $\overline{B_r(x_0)} \setminus \Omega$ and show that, in place of a capacitary potential of this set, it is sufficient to take any "almost" minimiser of the relative $p-$capacity (and by density, this can be taken as smooth as we wish). Thus, we can be dispensed with the use of the PDE and simply use the minimality (or almost minimality) property of the function. This simplifies the argument, at the price of a slight quantitative worsening of the constant. This is not a big deal, since in any case the constants involved in the two–sided estimate are not sharp, both in [78] and in our result. On the contrary, at a qualitative level, our final estimate in terms of γ is as good as that of Maz'ya and Shubin (see Remark 5.4.1 and compare with [78, equation (4.16)]).

1.5. Plan of the thesis

Chapter 2 is devoted to set up the notation and give the basic definitions, which will be needed troughout the whole thesis. Particular attention is paid to introduce the notion of $p-$capacity and prove some of its properties: this is the content of Sections 2.2 - 2.3. In Section 2.4, we construct an extension operator devised for Sobolev functions defined on open bounded convex subsets of $\mathbb{R}^N$, with an explicit control on the extension constants. Next, we still consider this class of sets, and analyse the behaviour of the sharp constants of Poincaré–Wirtinger type inequalities, for them. In Section 2.6, we prove one of the cornerstone of our main results, a so-called *Maz'ya–Poincaré inequality* (Theorem 2.6.1). This is nothing else than a Poincaré–type inequality devised for smooth functions defined over a closed cube, which contains a common "Dirichlet region" of positive $p-$capacity. The proof of this last result basically use all the tools previously introduced. At last, Section 2.7 contains an application to the evaluation of Poincaré–type constants of a convexity principle for the $p-$Dirichlet energy due to Benguria.

Chapter 3 is aimed at giving the proof of Theorem 1. As specified in Section 1.2.1, this is a refinement of an inequality already obtained by Maz'ya in the *super–conformal case* $p > N$. In addition to this, Section 3.2 contains a deep analysis of some *punctured Poincaré constants* and related consequences, of indipendent interest.

Chapter 4 is aimed at giving the proof of Theorem 2: this is an extension, to the case of the principal frequencies of the $p-$Dirichlet Laplacian λ_p, of a classical inequality, established in the late $'70$s/beginning $'80$s by Ossermann, Taylor and Croke, between the first eigenvalue of the Dirichlet–Laplacian and the *inradius* of a planar multiply connected open set. Section 4.1 also contains three technical facts that reveal why the case of the dimension $N = 2$ is peculiar. In Section 4.4, we discuss some consequences of Theorem 2, as well. At last, in Section 4.5 we discuss the sharpness of the constant obtained in the *Buser–type inequality* (Theorem 3), which can be derived from Theorem 2, and present an interesting open problem connected to it.

Chapter 5 is devoted to the proof of Theorem 4. It concerns a two–sided estimate for λ_p, valid for general open sets of $\mathbb{R}^N$, in terms of the *capacitary inradius*. By virtue of Theorem 5 which is still contained in this chapter, Theorem 4 can be seen as the natural counterpart of Theorem 1, in the range $1 \leq p \leq N$. A key ingredient of the proof is given by a careful analysis of a $L^1 - L^p$ Poincaré–type constant on balls: this is the content of Section 5.1. As a byproduct, we also obtain the explicit value of a Cheeger–type constant and identify the optimal shape for the related Cheeger–type problem (see Remark 5.2.5). Next, we give the proof of Theorem 4 and, in Section 5.5, we extend the result obtained to the case of the general Poincaré–Sobolev constants $\lambda_{p,q}$. The chapter ends with a comparison between the notion of inradius and that of capacitary inradius in the super–conformal case, i.e. for $p > N$.

Appendix A concludes this manuscript and contains the analysis of some degenerate beahviours concerning the notion of capacitary inradius.

CHAPTER 2

Some facts from the theory of Sobolev Spaces

2.1. Notation and basic definitions

We will use the usual standard notations for $N-$dimensional balls and hypercubes, that is

$$B_R(x_0) = \Big\{x \in \mathbb{R}^N \,:\, |x - x_0| < R\Big\}, \qquad \text{for } x_0 \in \mathbb{R}^N,\ R > 0,$$

and

$$Q_R(x_0) = \prod_{i=1}^{N}(x_0^i - R, x_0^i + R), \qquad \text{for } x_0 = (x_0^1, \dots, x_0^N) \in \mathbb{R}^N,\ R > 0.$$

When the center x_0 coincides with the origin, we will simply write B_R and Q_R, respectively. For every $k \in \mathbb{N}$, by the symbol $\mathcal{H}^k$ we will denote the $k-$dimensional Hausdorff measure.

For $1 \le p \le \infty$ and for an open set $\Omega \subseteq \mathbb{R}^N$, we will denote by $W^{1,p}(\Omega)$ the standard Sobolev space

$$W^{1,p}(\Omega) = \Big\{u \in L^p(\Omega) \,:\, \nabla u \in L^p(\Omega;\mathbb{R}^N)\Big\},$$

endowed with its natural norm

$$\|u\|_{W^{1,p}(\Omega)} = \Big(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)}\Big)^{\frac{1}{p}}, \qquad \text{for } u \in W^{1,p}(\Omega),$$

where we used the notation

$$\|\Phi\|_{L^p(\Omega)} = \left(\int_\Omega |\Phi(x)|^p\,dx\right)^{\frac{1}{p}}, \qquad \text{for } \Phi \in L^p(\Omega;\mathbb{R}^N).$$

Here, as usual in the literature, the symbol ∇ stands for the *weak* or *distributional gradient*. We refer the reader to [**69**, Chapter 11] for its precise definition and the main properties of weak derivatives. Moreover, we will denote with the same symbol weak and classical derivatives of a function.

For $1 \le p < \infty$, we indicate the *homogeneous Sobolev space* with the symbol $\mathscr{D}^{1,p}_0(\Omega)$, i.e. the completion of $C^\infty_0(\Omega)$ with respect to the norm

$$\varphi \mapsto \|\nabla\varphi\|_{L^p(\Omega)}.$$

Occasionally, we will need the space $W^{1,p}_0(\Omega)$: this is the closure of $C^\infty_0(\Omega)$ in $W^{1,p}(\Omega)$ with respect to $\|\cdot\|_{W^{1,p}(\Omega)}$.

Remark 2.1.1. By recalling (1.1.4), note that the spaces $W^{1,p}_0(\Omega)$ and $\mathscr{D}^{1,p}_0(\Omega)$ coincide, whenever $\lambda_{p,q}(\Omega) > 0$ for some $1 \le q \le p$ (see for example [**24**, Proposition 2.4]). We also recall that the value $\lambda_{p,q}(\Omega)$ is unchanged, if we replace $C^\infty_0(\Omega)$ by its closure $W^{1,p}_0(\Omega)$ (see [**24**, Lemma 2.6]).

2.2. Capacity

For a thorough study of the properties of $p-$capacity, we refer the reader to [**76**, Chapter 2, Section 2] (see also [**42**, Section 4.7], [**39**, Chapter 8] and [**44**]). Here, without claiming to be exhaustive, we give the definition and collect some of its basic properties that will be instrumental for our scopes.

Definition 2.2.1. Let $1 \le p < \infty$, for every $E \subseteq \mathbb{R}^N$ open set and every $\Sigma \subseteq E$ compact set, we define the $p-$*capacity of* Σ *relative to* E through the following minimization problem

$$\mathrm{cap}_p(\Sigma; E) = \inf_{\varphi \in C^\infty_0(E)} \left\{\int_E |\nabla\varphi|^p\,dx \,:\, \varphi \ge 1 \text{ on } \Sigma\right\}.$$

REMARK 2.2.2. By using standard approximation methods, it is not difficult to see that the infimum above does not change, if we replace $C_0^\infty(E)$ by the space of Lipschitz functions, compactly supported in E. We indicate this space by $\mathrm{Lip}_0(E)$. We observe that, for every $\varphi \in \mathrm{Lip}_0(E)$ with $\varphi \geq 1$ on Σ, the new function

$$\widetilde{\varphi} := \min\{|\varphi|, 1\},$$

still belongs to $\mathrm{Lip}_0(E)$ and is such that

$$\int_E |\nabla \widetilde{\varphi}|^p\,dx \leq \int_E |\nabla \varphi|^p\,dx, \qquad 0 \leq \widetilde{\varphi} \leq 1 \qquad \text{and} \qquad \widetilde{\varphi} = 1 \text{ on } \Sigma.$$

This shows that we also have the following equivalent characterization

$$\mathrm{cap}_p(\Sigma; E) = \inf_{\varphi \in \mathrm{Lip}_0(E)} \left\{ \int_E |\nabla \varphi|^p\,dx \,:\, 0 \leq \varphi \leq 1,\ \varphi = 1 \text{ on } \Sigma \right\}. \tag{2.2.1}$$

A first link between capacity and principal frequencies is that $\lambda_p(\Omega)$ remains unchanged under the removal of sets of null $p-$capacity: this is the content of the following proposition.

PROPOSITION 2.2.3. *Let $1 \leq p < \infty$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set. Let $K \subseteq \Omega$ be a compact set such that there exists a ball $B_R(x_0)$ with $K \Subset B_R(x_0)$ and*

$$\mathrm{cap}_p(K; B_R(x_0)) = 0.$$

Then, we have

$$W_0^{1,p}(\Omega) = W_0^{1,p}(\Omega \setminus K).$$

In particular

$$\lambda_{p,q}(\Omega) = \lambda_{p,q}(\Omega \setminus K). \tag{2.2.2}$$

PROOF. The inclusion $W_0^{1,p}(\Omega \setminus K) \subseteq W_0^{1,p}(\Omega)$ is trivial. On the other hand, if $u \in W_0^{1,p}(\Omega)$ then there exists a sequence of functions $\{u_n\}_{n\in\mathbb{N}} \subseteq C_0^\infty(\Omega)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{W^{1,p}(\Omega)} = 0. \tag{2.2.3}$$

By Remark 2.2.2, for every $m \in \mathbb{N}$, we also have a function $\varphi_m \in \mathrm{Lip}_0(B_R(x_0))$ with $0 \leq \varphi_m \leq 1$ and $\varphi_m = 1$ on K such that

$$\lim_{m\to\infty} \|\nabla \varphi_m\|_{L^p(B_R(x_0))} = 0. \tag{2.2.4}$$

Then, for every $n \in \mathbb{N}$ fixed, we consider the following sequence

$$v_m^{(n)} := (1 - \varphi_m)\,u_n, \qquad \text{for every } m \in \mathbb{N}.$$

Since, for every $1 \leq p < \infty$, $v_m^{(n)} \in W^{1,p}(\Omega \setminus K) \cap C(\mathbb{R}^N)$ and $v_m^{(n)} = 0$ on $\partial(\Omega \setminus K)$, by [18, Theorem 9.17 & Remark 19] we get that

$$v_m^{(n)} \in W_0^{1,p}(\Omega \setminus K), \qquad \text{for every } m, n \in \mathbb{N}.$$

Furthermore, we have that

$$\lim_{m\to\infty} \int_{\Omega\setminus K} |v_m^{(n)} - u_n|^p\,dx = 0, \qquad \text{for every } n \in \mathbb{N}. \tag{2.2.5}$$

Indeed, by the Hölder inequality and the Poincaré inequality we obtain

$$\begin{aligned}
\int_{\Omega\setminus K} |v_m^{(n)} - u_n|^p\,dx &= \int_{\Omega\setminus K} |\varphi_m|^p |u_n|^p\,dx \\
&\leq \|u_n\|_{L^\infty(\Omega)}^p \int_{B_R(x_0)} |\varphi_m|^p\,dx \\
&\leq \frac{\|u_n\|_{L^\infty(\Omega)}^p}{\lambda_p(B_R(x_0))} \int_{B_R(x_0)} |\nabla \varphi_m|^p\,dx,
\end{aligned}$$

and, by (2.2.4), the last term tends to zero as $m \nearrow \infty$. Moreover, by convexity and the Poincaré inequality we also get that

$$\begin{aligned}\int_{\Omega\setminus K} |\nabla(v_m^{(n)} - u_n)|^p\,dx &= \int_{\Omega\setminus K} |\varphi_m \nabla u_n + u_n\nabla\varphi_m|^p\,dx \\ &\le 2^{p-1}\int_{\Omega\setminus K} |\nabla u_n|^p|\varphi_m|^p\,dx + 2^{p-1}\int_{\Omega\setminus K} |\nabla\varphi_m|^p|u_n|^p\,dx \\ &\le 2^{p-1}\frac{\|\nabla u_n\|^p_{L^\infty(\Omega)}}{\lambda_p(B_R(x_0))}\int_{B_R(x_0)} |\nabla\varphi_m|^p\,dx + 2^{p-1}\|u_n\|^p_{L^\infty(\Omega)}\int_{B_R(x_0)} |\nabla\varphi_m|^p\,dx.\end{aligned}$$

By using again (2.2.4), this implies that

$$\lim_{m\to\infty}\int_{\Omega\setminus K} |\nabla(v_m^{(n)} - u_n)|^p\,dx = 0, \qquad \text{for every } n\in\mathbb{N}. \tag{2.2.6}$$

Thanks to (2.2.5) and (2.2.6), in particular we have

$$u_n \in W_0^{1,p}(\Omega\setminus K), \qquad \text{for every } n\in\mathbb{N}.$$

Finally, since

$$\|u_n - u\|_{W^{1,p}(\Omega\setminus K)} \le \|u - u_n\|_{W^{1,p}(\Omega)}, \qquad \text{for every } n\in\mathbb{N},$$

by (2.2.3) we can conclude that $u \in W_0^{1,p}(\Omega\setminus K)$. In light of Remark 2.1.1, this also implies (2.2.2). □

The following standard property of the capacity is a particular case of [76, Chapter 13, Proposition 1, page 658]. This will be used in Section 5.1 in the proof of the lower bound of Theorem 4. We report the proof for the reader's convenience.

Proposition 2.2.4. *Let $1 \le p < \infty$ and let $\Sigma \subseteq B_r(x_0)$ be a compact set. Then, for every $R > r$ we have*

$$\mathrm{cap}_p(\Sigma; B_R(x_0)) \le \mathrm{cap}_p(\Sigma; B_r(x_0)) \le \left(\frac{1}{\lambda_p(B_1)^{\frac{1}{p}}}\frac{R}{d} + 1\right)^p \mathrm{cap}_p(\Sigma; B_R(x_0)), \tag{2.2.7}$$

where $d := \mathrm{dist}(\Sigma, \partial B_r(x_0)) > 0$.

Proof. The leftmost inequality is straightforward, we thus focus on proving the rightmost one. Without loss of generality, we can assume that $x_0 = 0$. Let $u \in C_0^\infty(B_R)$ be such that $u \ge 1$ on Σ. For every $0 < \varepsilon < d/2$ we take the Lipschitz cut-off function, compactly supported in B_r, given by

$$\eta(x) = \min\left\{\left(\frac{(r-\varepsilon) - |x|}{(r-\varepsilon) - (r-d)}\right)_+, 1\right\}.$$

Observe that by construction the function $\psi = \eta\, u$ is a Lipschitz function, compactly supported in B_r and such that $\psi \ge 1$ on Σ. Thus, this is an admissible function to test the definition of relative $p-$capacity, thanks to Remark 2.2.2. By using Minkowski's inequality and the properties of η, we get

$$\begin{aligned}\Big(\mathrm{cap}_p(\Sigma; B_r)\Big)^{\frac{1}{p}} &\le \frac{1}{d-\varepsilon}\,\|u\|_{L^p(B_r)} + \|\nabla u\|_{L^p(B_r)} \\ &\le \frac{1}{d-\varepsilon}\,\|u\|_{L^p(B_R)} + \|\nabla u\|_{L^p(B_R)} \\ &\le \left(\frac{1}{d-\varepsilon}\,\frac{R}{\lambda_p(B_1)^{\frac{1}{p}}} + 1\right)\|\nabla u\|_{L^p(B_R)}.\end{aligned} \tag{2.2.8}$$

In the third inequality we also used Poincaré's inequality for the set B_R. By taking the limit as ε goes to 0 and using the arbitrariness of u, we get the claimed estimate. □

According to [76, Definition 14.2.2], we recall that for every $0<\gamma<1$ and for every open set $\Omega\subseteq\mathbb{R}^N$ its $(p,1)$–*inner (cubic) diameter* is defined as

$$D_{p,1}(\Omega) := \sup\{r>0\,:\,\exists x_0\in\mathbb{R}^N \text{ such that } \mathrm{cap}_p(\overline{Q_r(x_0)}\setminus\Omega;Q_{2r}(x_0))\leq\gamma\,\mathrm{cap}_p(\overline{Q_r(x_0)};Q_{2r}(x_0))\}. \tag{2.2.9}$$

Here, in order to stress the dependence on γ of this definition, we set

$$\mathfrak{D}_{p,\gamma}(\Omega) := D_{p,1}(\Omega).$$

In the proposition below, we are going to prove that the notion of (p,γ)-capacitary inradius $R_{p,\gamma}(\Omega)$, given in Definition 1.4.1, and that of inner cubic diameter $\mathfrak{D}_{p,\beta}(\Omega)$ are equivalent. For convenience sake, coherently with (2.2.9) and Definition 1.4.1, we also set

$$R_{p,\gamma}(\Omega)=\mathfrak{D}_{p,\gamma}(\Omega):=+\infty,$$

whenever $\gamma\geq 1$.

PROPOSITION 2.2.5. *Let $1\leq p\leq N$, $0<\gamma<1$ and let $\Omega\subseteq\mathbb{R}^N$ be an open set. Then, there exist constants $0<c\leq 1$ and $d>0$, both depending only on N and p, such that we have*

$$\mathfrak{D}_{p,c\cdot\gamma}(\Omega)\leq R_{p,\gamma}(\Omega)\leq\sqrt{N}\,\mathfrak{D}_{p,d\cdot\gamma}(\Omega). \tag{2.2.10}$$

PROOF. We can suppose that $R_{p,\gamma}(\Omega)<\infty$, otherwise the leftmost inequality in (2.2.10) is trivial. Let $r>R_{p,\gamma}(\Omega)$ so that

$$\mathrm{cap}_p(\overline{B_r(x_0)}\setminus\Omega;B_{2r}(x_0))>\gamma\,\mathrm{cap}_p(\overline{B_r(x_0)};B_{2r}(x_0)),\qquad \text{for every } x_0\in\mathbb{R}^N, \tag{2.2.11}$$

and let $u\in C_0^\infty(Q_{2r}(x_0))$ such that $0\leq u\leq 1$ and $u=1$ on $\overline{B_r(x_0)}\setminus\Omega$. Without loss of generality, we can assume $x_0=0$. We preliminarily observe that, by the scaling properties of the relative capacity, we have

$$\mathrm{cap}_p(\overline{B_r};B_{2r})=\left(\frac{\mathrm{cap}_p(\overline{B_1};B_2)}{\mathrm{cap}_p(\overline{Q_1};Q_2)}\right)\mathrm{cap}_p(\overline{Q_r};Q_{2r}). \tag{2.2.12}$$

For every $0<\delta<r$, we define the cut-off function

$$\eta_\delta(x):=\min\left\{\left(\frac{(2r-\delta)-|x|}{(2r-\delta)-r}\right)_+,\ 1\right\}.$$

In particular, $\eta_\delta\in\mathrm{Lip}_0(B_{2r})$ and

$$\eta_\delta=1\quad\text{on }\overline{B_r},\qquad |\nabla\eta_\delta|\leq\frac{1}{r-\delta},\qquad \eta_\delta=0\quad\text{on }\mathbb{R}^N\setminus B_{2r-\delta}.$$

Then, by the definition of relative capacity and Minkowski inequality, we have

$$\begin{aligned}\Big(\mathrm{cap}_p(\overline{B_r}\setminus\Omega;B_{2r})\Big)^{\frac{1}{p}}&\leq\|\nabla(u\,\eta_\delta)\|_{L^p(Q_{2r})}\\&\leq\frac{1}{r-\delta}\,\|u\|_{L^p(Q_{2r})}+\|\nabla u\|_{L^p(Q_{2r})}\\&\leq\left(\frac{1}{r-\delta}\,\frac{1}{\lambda_p(Q_{2r})^{\frac{1}{p}}}+1\right)\|\nabla u\|_{L^p(Q_{2r})},\end{aligned}$$

where in the last line we used Poincaré inequality. By sending $\delta\to 0$ and by the arbitrariness of u, we then obtain

$$\begin{aligned}\mathrm{cap}_p(\overline{B_r}\setminus\Omega;B_{2r})&\leq\left(\frac{2}{\lambda_p(Q_1)^{\frac{1}{p}}}+1\right)^p\mathrm{cap}_p(\overline{B_r}\setminus\Omega;Q_{2r}),\\&\leq\left(\frac{2}{\lambda_p(Q_1)^{\frac{1}{p}}}+1\right)^p\mathrm{cap}_p(\overline{Q_r}\setminus\Omega;Q_{2r}).\end{aligned}$$

This, combined with (2.2.11) and (2.2.12), gives that

$$c\cdot\gamma\,\mathrm{cap}_p(\overline{Q_r};Q_{2r})\leq\mathrm{cap}_p(\overline{Q_r}\setminus\Omega;Q_{2r}),$$

where we set

$$c = c(N,p) = \left(\frac{\mathrm{cap}_p(\overline{B_1}; B_2)}{\mathrm{cap}_p(\overline{Q_1}; Q_2)}\right) \left(\frac{2}{\lambda_p(Q_1)^{\frac{1}{p}}} + 1\right)^{-p}. \tag{2.2.13}$$

Thus, by the definition of inner cubic diameter and by the arbitrariness of r, we infer that

$$R_{p,\gamma}(\Omega) \geq \mathfrak{D}_{p,c\cdot\gamma}(\Omega), \tag{2.2.14}$$

which gives the leftmost inequality in (2.2.10).

On the other hand, suppose that for a constant $d > 0$, which will be suitably choosen later, we have $\mathfrak{D}_{p,d\cdot\gamma}(\Omega) < \infty$ and take any $r > \mathfrak{D}_{p,d\cdot\gamma}(\Omega)$, so that

$$\mathrm{cap}_p(\overline{Q_r(x_0)} \setminus \Omega; Q_{2r}(x_0)) > d\cdot\gamma\, \mathrm{cap}_p(\overline{Q_r(x_0)}; Q_{2r}(x_0)), \qquad \text{for every } x_0 \in \mathbb{R}^N. \tag{2.2.15}$$

Without loss of generality, we assume $x_0 = 0$. For the leftmost term, by using the same argument as before, we have

$$\mathrm{cap}_p(\overline{Q_r} \setminus \Omega; Q_{2r}) \leq \left(\frac{2\sqrt{N}}{\lambda_p(B_1)^{\frac{1}{p}}} + 1\right)^p \mathrm{cap}_p(\overline{B_{\sqrt{N}r}} \setminus \Omega; B_{2\sqrt{N}r}). \tag{2.2.16}$$

Indeed, let $u \in \mathrm{Lip}_0(B_{2r\sqrt{N}})$ be such that $0 \leq u \leq 1$ and $u = 1$ on $\overline{Q_r} \setminus \Omega$. For every $0 < \delta < r$, we introduce a Lipschitz cut-off function $\xi_\delta \in \mathrm{Lip}_0(Q_{2r})$ given by

$$\xi_\delta(x) = \min\left\{\frac{1}{r-\delta}\,\mathrm{dist}\Big(x;\ \partial Q_{2r-\delta}\Big),\ 1\right\}, \qquad \text{for } x \in Q_{2r-\delta},$$

and extended by zero over the whole $\mathbb{R}^N$. In particular, it satisfies

$$\xi_\delta = 1 \quad \text{on } \overline{Q_r}, \qquad |\nabla \xi_\delta| \leq \frac{1}{r-\delta}, \qquad \xi_\delta = 0 \quad \text{on } \mathbb{R}^N \setminus Q_{2r-\delta}.$$

Thus, by the definition of relative capacity and Minkowski inequality, we obtain

$$\begin{aligned}
\Big(\mathrm{cap}_p(\overline{Q_r} \setminus \Omega; Q_{2r})\Big)^{\frac{1}{p}} &\leq \|\nabla\,(u\,\xi_\delta)\,\|_{L^p(B_{2\sqrt{N}r})} \\
&\leq \frac{1}{r-\delta}\,\|u\|_{L^p(B_{2\sqrt{N}r})} + \|\nabla u\|_{L^p(B_{2\sqrt{N}r})} \\
&\leq \left(\frac{r}{r-\delta}\,\frac{2\sqrt{N}}{\lambda_p(B_1)^{\frac{1}{p}}} + 1\right) \|\nabla u\|_{L^p(B_{2\sqrt{N}r})},
\end{aligned}$$

where in the last line we used the Poincaré inequality. By the arbitrariness of u and δ, and by monotonicity we then have

$$\begin{aligned}
\mathrm{cap}_p(\overline{Q_r} \setminus \Omega; Q_{2r}) &\leq \left(\frac{2\sqrt{N}}{\lambda_p(B_1)^{\frac{1}{p}}} + 1\right)^p \mathrm{cap}_p(\overline{Q_r} \setminus \Omega; B_{2\sqrt{N}r}) \\
&\leq \left(\frac{2\sqrt{N}}{\lambda_p(B_1)^{\frac{1}{p}}} + 1\right)^p \mathrm{cap}_p(\overline{B_{\sqrt{N}r}} \setminus \Omega; B_{2\sqrt{N}r}).
\end{aligned}$$

For the rightmost term in (2.2.15), we observe that by choosing

$$d = d(N,p) = N^{\frac{N-p}{2}} \left(\frac{\mathrm{cap}_p(\overline{B_1}; B_2)}{\mathrm{cap}_p(\overline{Q_1}; Q_2)}\right) \left(\frac{2\sqrt{N}}{\lambda_p(B_1)^{\frac{1}{p}}} + 1\right)^p,$$

we have

$$d\cdot\gamma\, \mathrm{cap}_p(\overline{Q_r}; Q_{2r}) = \gamma\, \left(\frac{2\sqrt{N}}{\lambda_p(B_1)^{\frac{1}{p}}} + 1\right)^p \mathrm{cap}_p\Big(\overline{B_{\sqrt{N}r}}; B_{2\sqrt{N}r}\Big).$$

Together with (2.2.15) and (2.2.16), and by recalling our assumption $x_0 = 0$, this implies that

$$\mathrm{cap}_p\Big(\overline{B_{\sqrt{N}r}(x_0)} \setminus \Omega; B_{2\sqrt{N}r}(x_0)\Big) > \gamma\, \mathrm{cap}_p\Big(\overline{B_{\sqrt{N}r}(x_0)}; B_{2\sqrt{N}r}(x_0)\Big), \qquad \text{for every } x_0 \in \mathbb{R}^N.$$

Therefore, by the definition of (p,γ)-capacitary inradius and by arbitrariness of r we have

$$\sqrt{N}\,\mathfrak{D}_{p,d\cdot\gamma}(\Omega) \geq R_{p,\gamma}(\Omega),$$

which gives the rightmost inequality in (2.2.10). □

Remark 2.2.6. More precisely, with the same notation as in the previous proposition, we can choose

$$c = \frac{\mathrm{cap}_p(\overline{B_1};B_2)}{\mathrm{cap}_p(\overline{Q_1};Q_2)}\left(\frac{2}{\lambda_p(Q_1)^{\frac{1}{p}}}+1\right)^{-p},$$

and

$$d = N^{\frac{N-p}{2}}\,\frac{\mathrm{cap}_p(\overline{B_1};B_2)}{\mathrm{cap}_p(\overline{Q_1};Q_2)}\left(\frac{2\sqrt{N}}{\lambda_p(B_1)^{\frac{1}{p}}}+1\right)^{p}\left(\frac{2}{\lambda_p(Q_1)^{\frac{1}{p}}}+1\right)^{p}.$$

Moreover, with a similar argument to that of the foregoing proof, it is possible to show that

$$\left(\frac{2}{\lambda_p(Q_1)^{\frac{1}{p}}}+1\right)^{-p}\mathrm{cap}_p(\overline{B_1};B_2)\leq \mathrm{cap}_p(\overline{Q_1};Q_2)\leq \left(\frac{2\sqrt{N}}{\lambda_p(B_1)^{\frac{1}{p}}}+1\right)^{p} N^{\frac{N-p}{2}}\,\mathrm{cap}_p(\overline{B_1};B_2). \tag{2.2.17}$$

In particular, the leftmost inequality implies that $c \leq 1$.

2.3. Capacity of balls

For $N \geq 2$, we recall the expression for the $p-$capacity of a ball relative to a concentric ball (see [76, page 148]). Due to translation invariance, we can suppose that all the balls are centered at the origin. We have to distinguish the cases $p = 1$, $p \in (1,N)\cup(N,\infty)$ or $p = N$. For every $0 < r < R$, this is given by[1]

$$\mathrm{cap}_1\left(\overline{B_r};B_R\right) = N\,\omega_N\,r^{N-1}, \tag{2.3.1}$$

$$\mathrm{cap}_p\left(\overline{B_r};B_R\right) = N\,\omega_N\left|\frac{N-p}{p-1}\right|^{p-1}\frac{r^{N-p}}{\left|1-\left(\dfrac{r}{R}\right)^{\frac{N-p}{p-1}}\right|^{p-1}},\qquad \text{if } p\in(1,N)\cup(N,\infty), \tag{2.3.2}$$

and

$$\mathrm{cap}_N\left(\overline{B_r};B_R\right) = N\,\omega_N\left(\log\left(\frac{R}{r}\right)\right)^{1-N}. \tag{2.3.3}$$

For $p > N$, we can even take the limit as r goes to 0 and get

$$\mathrm{cap}_p\left(\{0\};B_R\right) = N\,\omega_N\left(\frac{p-N}{p-1}\right)^{p-1}R^{N-p},\qquad \text{if } p>N. \tag{2.3.4}$$

Remark 2.3.1. We observe in particular that we have the following scaling relations

$$\mathrm{cap}_p\left(\overline{B_r};B_R\right) = r^{N-p}\,\mathrm{cap}_p\left(\overline{B_1};B_{R/r}\right),\qquad \text{if } 1\leq p<\infty,$$

The following technical result will be useful. It is a sort of “quantified” monotonicity inequality for the relative $p-$capacity of balls, with a geometric remainder term.

Lemma 2.3.2. *Let $N \geq 2$ and $1 < p \leq N$, for every $0 < r_1 < r_2 < R$ we have*

$$\frac{|B_{r_2}\setminus B_{r_1}|}{(\mathcal{H}^{N-1}(\partial B_{r_2}))^{\frac{p}{p-1}}}+\left(\frac{1}{\mathrm{cap}_p(\overline{B_{r_2}};B_R)}\right)^{\frac{1}{p-1}}\leq\left(\frac{1}{\mathrm{cap}_p(\overline{B_{r_1}};B_R)}\right)^{\frac{1}{p-1}}.$$

[1]The reader should keep in mind that in [76] the constant ω_N stands for the perimeter of $B_1(0)$, rather than for its volume. This explains the apparent difference with the formulas here given.

Proof. The proof is just based on writing explicitly all the involved quantities and then using a convexity inequality. We start from the case $1 < p < N$: by using (2.3.2), the claimed inequality is equivalent to

$$\frac{1}{(N\,\omega_N)^{\frac{1}{p-1}}}\,\left(\frac{p-1}{N-p}\right)\,\frac{1-\left(\dfrac{r_1}{R}\right)^{\frac{N-p}{p-1}}}{\left(\dfrac{r_1}{R}\right)^{\frac{N-p}{p-1}}}\,\frac{1}{R^{\frac{N-p}{p-1}}}$$

$$\geq \frac{1}{(N\,\omega_N)^{\frac{1}{p-1}}}\,\left(\frac{p-1}{N-p}\right)\,\frac{1-\left(\dfrac{r_2}{R}\right)^{\frac{N-p}{p-1}}}{\left(\dfrac{r_2}{R}\right)^{\frac{N-p}{p-1}}}\,\frac{1}{R^{\frac{N-p}{p-1}}}+\frac{1}{N\,(N\,\omega_N)^{\frac{1}{p-1}}}\,\frac{1}{r_2^{\frac{N-p}{p-1}}}\,\left(1-\left(\frac{r_1}{r_2}\right)^N\right).$$

In turn, this is equivalent to the following inequality

$$\frac{1-\left(\dfrac{r_1}{R}\right)^{\frac{N-p}{p-1}}}{\left(\dfrac{r_1}{R}\right)^{\frac{N-p}{p-1}}}\geq\frac{1-\left(\dfrac{r_2}{R}\right)^{\frac{N-p}{p-1}}}{\left(\dfrac{r_2}{R}\right)^{\frac{N-p}{p-1}}}+\frac{N-p}{N\,(p-1)}\,\left(\frac{R}{r_2}\right)^{\frac{N-p}{p-1}}\,\left(1-\left(\frac{r_1}{r_2}\right)^N\right),$$

which is the same as

$$\left(\frac{R}{r_1}\right)^{\frac{N-p}{p-1}}\geq\left(\frac{R}{r_2}\right)^{\frac{N-p}{p-1}}+\frac{N-p}{N\,(p-1)}\,\left(\frac{R}{r_2}\right)^{\frac{N-p}{p-1}}\,\left(1-\left(\frac{r_1}{r_2}\right)^N\right).$$

This can be further rewritten as follows

$$\left(\frac{R}{r_1}\right)^{\frac{N-p}{p-1}}\geq\left(\frac{N-p}{N\,(p-1)}+1\right)\,\left(\frac{R}{r_2}\right)^{\frac{N-p}{p-1}}-\frac{N-p}{N\,(p-1)}\,\left(\frac{R}{r_2}\right)^{\frac{N-p}{p-1}}\,\left(\frac{r_1}{r_2}\right)^N \tag{2.3.5}$$

We now introduce the following notation

$$t=\left(\frac{R}{r_1}\right)^N,\qquad s=\left(\frac{R}{r_2}\right)^N,\qquad \alpha=\frac{N-p}{N\,(p-1)}+1.$$

In light of this notation, the above inequality (2.3.5) can be written as

$$t^{\alpha-1}\geq\alpha\,s^{\alpha-1}-(\alpha-1)\,s^{\alpha-1}\,\frac{s}{t}.$$

By multiplying both sides by the positive number t, the latter is equivalent to

$$t^\alpha\geq\alpha\,s^{\alpha-1}\,t-(\alpha-1)\,s^\alpha,$$

that is

$$t^\alpha\geq s^\alpha+\alpha\,s^{\alpha-1}\,(t-s).$$

We finally observe that this last inequality holds true for every $t,s\geq 0$, since this is nothing but the "above tangent" property of the convex function $\tau\mapsto\tau^\alpha$ (recall that $\alpha>1$, by definition). This concludes the proof for the case $1<p<N$.

For the case $p=N$, one could simply observe that for every $0<r<R$, we have

$$\lim_{p\nearrow N}\left(\frac{1}{\mathrm{cap}_p(\overline{B_r};B_R)}\right)^{\frac{1}{p-1}}=\lim_{p\nearrow N}\frac{1}{(N\,\omega_N)^{\frac{1}{p-1}}}\,\left(\frac{p-1}{N-p}\right)\,\frac{1-\left(\dfrac{r}{R}\right)^{\frac{N-p}{p-1}}}{\left(\dfrac{r}{R}\right)^{\frac{N-p}{p-1}}}\,\frac{1}{R^{\frac{N-p}{p-1}}}$$

$$=\frac{1}{(N\,\omega_N)^{\frac{1}{N-1}}}\,\lim_{p\nearrow N}\left(\frac{p-1}{N-p}\right)\,\left[\left(\frac{R}{r}\right)^{\frac{N-p}{p-1}}-1\right]$$

$$=\frac{1}{(N\,\omega_N)^{\frac{1}{N-1}}}\,\left(\log\left(\frac{R}{r}\right)\right)=\left(\frac{1}{\mathrm{cap}_N(\overline{B_r};B_R)}\right)^{\frac{1}{N-1}}.$$

Thus, it is sufficient to take the limit in the inequality for the case $1<p<N$, in order to conclude. □

REMARK 2.3.3. If $1 < p < \infty$ and Σ is a compact subset of the open set $E \subseteq \mathbb{R}^N$, the quantity

$$\left(\frac{1}{\mathrm{cap}_p(\Sigma; E)}\right)^{\frac{1}{p-1}},$$

is sometimes called the $p-$*modulus of* Σ *relative to* E, see for example [44, Chapter 2]. In light of this, the estimate of Lemma 2.3.2 could also be seen as a consequence of the subadditivity of the $p-$modulus, which in the case of concentric balls reads as follows

$$\left(\frac{1}{\mathrm{cap}_p(\overline{B_{r_1}}; B_{r_2})}\right)^{\frac{1}{p-1}} + \left(\frac{1}{\mathrm{cap}_p(\overline{B_{r_2}}; B_R)}\right)^{\frac{1}{p-1}} \le \left(\frac{1}{\mathrm{cap}_p(\overline{B_{r_1}}; B_R)}\right)^{\frac{1}{p-1}},$$

see [44, Lemma 2.1]. Indeed, by using the explicit expression of the quantities involved, it is not too difficult to see that

$$\left(\frac{1}{\mathrm{cap}_p(\overline{B_{r_1}}; B_{r_2})}\right)^{\frac{1}{p-1}} \ge \frac{|B_{r_2} \setminus B_{r_1}|}{\left(\mathcal{H}^{N-1}(\partial B_{r_2})\right)^{\frac{p}{p-1}}}.$$

We preferred to give here an elementary proof of the estimate which is needed for our purposes.

For $p = 2$, the previous subadditivity property of the $p-$modulus is also known as *Grötzsch's lemma* (see for example [37, Lemma 1.2] and [87, page 52, equation (8)]).

2.4. An extension operator

In the next lemma we construct an extension operator for Sobolev functions defined on a ball, with a precise control on the extension constants. This is taken from [19, Proposition 3.8.1]. For sake of completeness, we report the complete proof below. A variant of the following lemma is also available for fractional Sobolev Spaces, see [15, Proposition 3.1].

The extension operator is obtained by simply composing functions with the *inversion with respect to* $\mathbb{S}^{N-1}$, i.e. the C^1 invertible mapping $\mathcal{K} : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^N \setminus \{0\}$, given by

$$\mathcal{K}(x) = \frac{x}{|x|^2}, \qquad \text{for every } x \in \mathbb{R}^N \setminus \{0\}.$$

LEMMA 2.4.1. *Let $x_0 \in \mathbb{R}^N$ and $r > 0$. There exists a linear extension operator*

$$\mathcal{E}_r : L^1(B_r(x_0)) \to L^1_{\mathrm{loc}}(\mathbb{R}^N),$$

such that, for every $1 \le p \le \infty$, it maps $W^{1,p}(B_r(x_0))$ to $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, for every $u \in W^{1,p}(B_r(x_0))$ and every $R > r$, it holds

$$\big\|\mathcal{E}_r[u]\big\|_{L^p(B_R(x_0))} \le 2^{\frac{1}{p}} \left(\frac{R}{r}\right)^{\frac{2N}{p}} \|u\|_{L^p(B_r(x_0))}, \tag{2.4.1}$$

$$\big\|\nabla\mathcal{E}_r[u]\big\|_{L^p(B_R(x_0))} \le 2^{\frac{1}{p}} \left(\frac{R}{r}\right)^{\frac{2N}{p}} \|\nabla u\|_{L^p(B_r(x_0))}. \tag{2.4.2}$$

PROOF. Without loss of generality, we can suppose that x_0 coincides with the origin and that $r = 1$. For every $u \in L^1(B_1)$, the extension $\mathcal{E}_1[u]$ is given by

$$\mathcal{E}_1[u](x) = \begin{cases} u(x), & \text{if } x \in B_1, \\ u(\mathcal{K}(x)), & \text{if } x \in \mathbb{R}^N \setminus B_1, \end{cases} \tag{2.4.3}$$

It is easily seen that if $x \in B_R \setminus B_1$, then $\mathcal{K}(x) \in B_1 \setminus B_{1/R}$. Moreover, we have

$$\mathcal{K}^{-1}(x) = \mathcal{K}(x) \quad \text{and} \quad |\det(D\mathcal{K}(x))| = \frac{1}{|x|^{2\,N}}, \qquad \text{for every } x \in \mathbb{R}^N \setminus \{0\}, \tag{2.4.4}$$

and

$$\frac{\partial}{\partial x_j}\mathcal{K}_i(x) = \frac{\delta_{i,j}}{|x|^2} - 2\frac{x_i x_j}{|x|^4} = \frac{\partial}{\partial x_i}\mathcal{K}_j(x).$$

This in particular implies that

$$\frac{\partial \mathcal{K}_i}{\partial x_j}(\mathcal{K}(x)) = \frac{\delta_{i,j}}{|\mathcal{K}(x)|^2} - 2\,\frac{\mathcal{K}_i(x)\mathcal{K}_j(x)}{|\mathcal{K}(x)|^4} = \delta_{i,j}\,|x|^2 - 2x_ix_j. \tag{2.4.5}$$

For later reference, we also notice that

$$\big(D\mathcal{K}(\mathcal{K}(x))\big)^2 = |x|^4\,\mathrm{Id}_N, \qquad \text{for every } x\in\mathbb{R}^N\setminus\{0\}. \tag{2.4.6}$$

Indeed, by using (2.4.5) we can compute that the (i,j) entry of this matrix

$$\begin{aligned} m_{i,j} &= \sum_{k=1}^N \big(\delta_{i,k}\,|x|^2 - 2x_i\,x_k\big)\big(\delta_{k,i}\,|x|^2 - 2x_k\,x_j\big)\\ &= \delta_{i,j}\,|x|^4 - 2|x|^2\,x_i\,x_j - 2|x|^2\,x_j\,x_i + 4x_i\,x_j\sum_{k=1}^N x_k^2 = \delta_{i,j}\,|x|^4. \end{aligned}$$

From (2.4.6) we get also the following identity

$$|D\mathcal{K}(\mathcal{K}(x))\cdot\xi| = |x|^4\,|\xi|^2, \qquad \text{for every } x\in\mathbb{R}^N\setminus\{0\},\ \xi\in\mathbb{R}^N. \tag{2.4.7}$$

Indeed, by the symmetry of $D\mathcal{K}(\mathcal{K}(x))$ and (2.4.6), we have

$$\begin{aligned} |D\mathcal{K}(\mathcal{K}(x))\cdot\xi|^2 &= \langle D\mathcal{K}(\mathcal{K}(x))\cdot\xi, D\mathcal{K}(\mathcal{K}(x))\cdot\xi\rangle\\ &= \langle\big(D\mathcal{K}(\mathcal{K}(x))\big)^2\cdot\xi,\xi\rangle\\ &= |x|^4\,\langle\xi,\xi\rangle. \end{aligned}$$

The estimate (2.4.1) for the L^p norm is readily obtained: for every $R>1$, thanks to the properties of $\mathcal{K}$ we have

$$\begin{aligned} \big\|\mathcal{E}_1[u]\big\|^p_{L^p(B_R)} &= \int_{B_R\setminus B_1} |u(\mathcal{K}(x))|^p\,dx + \int_{B_1}|u|^p\,dx\\ &= \int_{B_1\setminus B_{1/R}} |u(y)|^p\,|\det(D\mathcal{K}^{-1}(y))|\,dy + \int_{B_1}|u|^p\,dx\\ &\le (R^{2\,N}+1)\int_{B_1}|u|^p\,dx \le 2\,R^{2\,N}\int_{B_1}|u|^p\,dx. \end{aligned} \tag{2.4.8}$$

thus in particular $\mathcal{E}_1[u]\in L^p(B_R)$ and (2.4.1) holds true.

We now have to show that, for every $1\le p\le\infty$, if $u\in W^{1,p}(B_1)$, then $\mathcal{E}_1[u]\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ and the estimate (2.4.2) holds. More precisely, we will show that

$$\phi_{\mathcal{K}} = \nabla\mathcal{E}_1[u], \qquad \text{a.e. in } B_R,$$

where $\phi_{\mathcal{K}}$ is the vector-field

$$\phi_{\mathcal{K}}(x) = \begin{cases} \nabla u(x), & \text{if } x\in B_1,\\ D\mathcal{K}(x)\cdot\nabla u(\mathcal{K}(x)), & \text{if } x\in B_R\setminus B_1. \end{cases} \tag{2.4.9}$$

The standard notation $D\mathcal{K}\cdot\nabla u$ identifies the vector

$$D\mathcal{K}\cdot\nabla u = \left(\sum_{j=1}^N \frac{\partial u}{\partial x_j}\frac{\partial \mathcal{K}_j}{\partial x_1},\dots,\sum_{j=1}^N \frac{\partial u}{\partial x_j}\frac{\partial \mathcal{K}_j}{\partial x_N}\right).$$

Note that, by virtue of the classical change of variables formula for Sobolev functions (see for example [18, Proposition 9.6]), we already know that $\mathcal{E}_1[u]$ is a Sobolev function when restricted to the open sets B_1 and $B_R\setminus B_1$. In order to prove that $\mathcal{E}_1[u]$ is a Sobolev function on the whole B_R, we have to verify that $\phi_{\mathcal{K}}$ verifies the integration by parts formulas given by

$$\int_{B_R}\mathcal{E}_1[u]\,\frac{\partial\psi}{\partial x_i}\,dx = -\int_{B_R}(\phi_{\mathcal{K}})_i\,\psi\,dx, \qquad \text{for every } \psi\in C^\infty_0(B_R),$$

for all $i = 1, \dots, N$. For every $\varepsilon > 0$, we construct a *cut-off* function in the following way: let $\psi_\varepsilon \in C^\infty([0,1])$ be such that

$$0 \le \psi_\varepsilon \le 1, \qquad \psi_\varepsilon \equiv 1 \text{ on } [0, 1-2\varepsilon], \qquad \psi_\varepsilon \equiv 0 \text{ on } [1-\varepsilon, 1),$$

and

$$|\psi_\varepsilon'(t)| \le \frac{C}{\varepsilon}, \qquad \text{for } t \in [0,1]. \tag{2.4.10}$$

Hence we set

$$\eta_\varepsilon(x) = \psi_\varepsilon(|x|) \in C_0^\infty(B_1), \tag{2.4.11}$$

and define

$$\Psi_\varepsilon(x) = \begin{cases} \eta_\varepsilon(x), & \text{if } x \in B_1, \\ \eta_\varepsilon(\mathcal{K}(x)), & \text{if } x \in B_R \setminus B_1. \end{cases}$$

Thus, by construction we have that $\Psi_\varepsilon(x) \in C_0^\infty(B_R)$, it is radially symmetric, it satisfies $0 \le \Psi_\varepsilon \le 1$ and

$$\Psi_\varepsilon \equiv 0 \text{ on the annular set } B_{\frac{1}{1-\varepsilon}} \setminus B_{1-\varepsilon}, \tag{2.4.12}$$

$$\Psi_\varepsilon \equiv 1 \text{ on } B_{1-2\varepsilon} \cup \left(B_R \setminus B_{\frac{1}{1-2\varepsilon}} \right). \tag{2.4.13}$$

In particular

$$\lim_{\varepsilon \to 0} \Psi_\varepsilon(x) = 0, \qquad \text{for a.e. } x \in B_R.$$

Then, by the Dominated Convergence Theorem and the properties of Ψ_ε we infer that

$$\int_{B_R} \mathcal{E}_1[u] \, \frac{\partial \psi}{\partial x_i} \, dx = \lim_{\varepsilon \to 0} \int_{B_R} \mathcal{E}_1[u] \, \frac{\partial \psi}{\partial x_i} \, \Psi_\varepsilon \, dx, \tag{2.4.14}$$

for every $i = 1, \dots, N$. By using (2.4.12), (2.4.13) and the fact that $\mathcal{E}_1[u] = u$ on B_1, the last integral can be rewritten in the following way

$$\begin{aligned} \int_{B_R} \mathcal{E}_1[u] \frac{\partial \psi}{\partial x_i} \Psi_\varepsilon \, dx &= \int_{B_R} \mathcal{E}_1[u] \frac{\partial}{\partial x_i} (\psi \Psi_\varepsilon) \, dx - \int_{B_R} \mathcal{E}_1[u] \frac{\partial \Psi_\varepsilon}{\partial x_i} \psi \, dx \\ &= \int_{B_R \setminus B_1} \mathcal{E}_1[u] \frac{\partial}{\partial x_i} (\psi \Psi_\varepsilon) \, dx + \int_{B_1} u \frac{\partial}{\partial x_i} (\psi \Psi_\varepsilon) \, dx \\ &\quad - \int_{B_{\frac{1}{1-2\varepsilon}} \setminus B_{\frac{1}{1-\varepsilon}}} \mathcal{E}_1[u] \frac{\partial \Psi_\varepsilon}{\partial x_i} \psi \, dx - \int_{B_{1-\varepsilon} \setminus B_{1-2\varepsilon}} u \frac{\partial \Psi_\varepsilon}{\partial x_i} \psi \, dx. \end{aligned}$$

We discuss separately these four integrals. For the first integral, we note that

$$\Psi_\varepsilon \psi \in C_0^\infty(B_R \setminus \overline{B_1}).$$

Thus, by the definition of weak derivative and by [18, Proposition 9.6] applied to $\mathcal{E}_1[u] = u \circ \mathcal{K}$ on $B_R \setminus \overline{B_1}$, we get

$$\begin{aligned} \int_{B_R \setminus B_1} \mathcal{E}_1[u] \frac{\partial}{\partial x_i} (\psi \Psi_\varepsilon) \, dx &= - \int_{B_R \setminus B_1} \frac{\partial \mathcal{E}_1[u]}{\partial x_i} \, \psi \, \Psi_\varepsilon \, dx \\ &= - \int_{B_R \setminus B_{\frac{1}{1-\varepsilon}}} \sum_{j=1}^N \frac{\partial u}{\partial x_j} \frac{\partial \mathcal{K}_j}{\partial x_i} \, \psi \, \Psi_\varepsilon \, dx \\ &= - \int_{B_R \setminus B_{\frac{1}{1-\varepsilon}}} (\phi_\mathcal{K})_i \, \psi \, \Psi_\varepsilon \, dx, \end{aligned} \tag{2.4.15}$$

where $(\phi_\mathcal{K})_i$ stands for the i–th component of the vector field $\phi_\mathcal{K}$ defined in (2.4.9). By using the Dominated Convergence Theorem, then we have

$$\lim_{\varepsilon \to 0^+} \int_{B_R \setminus B_{\frac{1}{1-\varepsilon}}} \mathcal{E}_1[u] \frac{\partial}{\partial x_i} (\psi \Psi_\varepsilon) \, dx = - \int_{B_R \setminus B_1} (\phi_\mathcal{K})_i \, \psi \, dx.$$

The second integral can be treated in the same way, once it is observed that

$$\Psi_\varepsilon\,\psi \in C_0^\infty(B_1),$$

thanks to (2.4.12). Thus, by the definition of weak derivative for u we get

$$\int_{B_1} u\,\frac{\partial}{\partial x_i}(\psi\Psi_\varepsilon)\,dx = -\int_{B_1}\frac{\partial u}{\partial x_i}\,\psi\,\Psi_\varepsilon\,dx = -\int_{B_1}(\phi_{\mathcal{K}})_i\,\psi\,\Psi_\varepsilon\,dx,$$

having denoted with $(\phi_{\mathcal{K}})_i$ the i–th component of the vector field $\phi_{\mathcal{K}}$ defined in (2.4.9). Then, by the Dominated Convergence Theorem we also get

$$\lim_{\varepsilon\to 0^+}\int_{B_1} u\frac{\partial}{\partial x_i}(\psi\Psi_\varepsilon)\,dx = -\int_{B_1}(\phi_{\mathcal{K}})_i\,\psi\,dx.$$

Up to now, by recalling (2.4.14) and (2.4.15), we have obtained that

$$\int_{B_R}\mathcal{E}_1[u]\,\frac{\partial\psi}{\partial x_i}\,dx = -\int_{B_R\setminus B_1}(\phi_{\mathcal{K}})_i\,\psi\,dx - \int_{B_1}(\phi_{\mathcal{K}})_i\,\psi\,dx - \lim_{\varepsilon\to 0^+} I(\varepsilon), \tag{2.4.16}$$

where we set

$$I(\varepsilon) = \int_{B_{\frac{1}{1-2\varepsilon}}}\mathcal{E}_1[u]\,\frac{\partial\Psi_\varepsilon}{\partial x_i}\,\psi\,dx + \int_{B_{1-\varepsilon}\setminus B_{1-2\varepsilon}} u\frac{\partial\Psi_\varepsilon}{\partial x_i}\,\psi\,dx.$$

The desired result then follows by showing that

$$\lim_{\varepsilon\to 0^+} I(\varepsilon) = 0.$$

By using the definition of both $\mathcal{E}_1[u]$ and Ψ_ε and the change of variables $y = \mathcal{K}(x)$, we get

$$\begin{aligned}
I(\varepsilon) &= \int_{B_{\frac{1}{1-2\varepsilon}}\setminus B_{\frac{1}{1-\varepsilon}}} u(\mathcal{K}(x))\,\sum_{j=1}^N \frac{\partial\eta_\varepsilon}{\partial x_j}(\mathcal{K}(x))\frac{\partial\mathcal{K}_j}{\partial x_i}(x)\,\psi(x)\,dx \\
&\quad + \int_{B_{1-\varepsilon}\setminus B_{1-2\varepsilon}} u\,\frac{\partial\eta_\varepsilon}{\partial x_i}\,\psi\,dx \\
&= \int_{B_{1-\varepsilon}\setminus B_{1-2\varepsilon}} u(y)\sum_{j=1}^N \frac{\partial\eta_\varepsilon}{\partial x_j}(y)\frac{\mathcal{K}_j}{\partial x_i}(\mathcal{K}(y))\,\psi(\mathcal{K}(y))\,|\mathrm{det}\mathcal{K}(y)|\,dy \\
&\quad + \int_{B_{1-\varepsilon}\setminus B_{1-2\varepsilon}} u\,\frac{\partial\eta_\varepsilon}{\partial x_i}\,\psi\,dx.
\end{aligned}$$

By recalling (2.4.11) and (2.4.5), we have

$$\begin{aligned}
\sum_{j=1}^N \frac{\partial\eta_\varepsilon}{\partial x_j}\frac{\partial\mathcal{K}_j}{\partial x_i}(\mathcal{K}(y)) &= \sum_{j=1}^N \frac{\partial\eta_\varepsilon}{\partial x_j}\frac{\partial\eta_\varepsilon}{\partial x_i}(y)\,(\delta_{j,i}|y|^2 - 2y_j y_i) \\
&= |y|^2\frac{\partial\eta_\varepsilon}{\partial x_i}(y) - 2\langle\nabla\eta_\varepsilon(y), y\rangle y_i \\
&= |y|^2\left\langle \psi_\varepsilon^{'}(|y|)\,\frac{y}{|y|}, \mathbf{e}_i\right\rangle - 2\left\langle \psi_\varepsilon^{'}(|y|)\,\frac{y}{|y|}, y\right\rangle y_i \\
&= -|y|\,\psi_\varepsilon^{'}(|y|)\,y_i.
\end{aligned}$$

Thus, we have

$$I(\varepsilon) = -\int_{B_{1-\varepsilon}\setminus B_{1-2\varepsilon}} u(x)\,|x|\,\psi_\varepsilon^{'}(|x|)\,x_j\,\psi(\mathcal{K}(x))\,J(x)dx,$$

where

$$J(x) = -|x|\,\psi(\mathcal{K}(x))\,|\mathrm{det}\mathcal{K}(x)| + \frac{\psi(x)}{|x|} = -|x|^{1-2N}\,\psi(\mathcal{K}(x)) + \frac{\psi(x)}{|x|},$$

thanks to (2.4.4). In order to estimate the last term, we add and substract $\dfrac{\psi(x)}{|x|^{2N-1}}$, take the absolute value and use the fact that $1-2\varepsilon<|x|<1-\varepsilon$, thus elementary estimates leads to

$$|J(x)|\le c\,\varepsilon,\qquad \text{for } x\in B_{1-\varepsilon}\setminus B_{1-2\varepsilon},$$

where we can take

$$c=c(\psi,N)=2N\,\max\{\|\nabla\psi\|_{L^\infty(\mathbb{R}^N)},\|\psi\|_{L^\infty(\mathbb{R}^N)}\},\tag{2.4.17}$$

for $\varepsilon>0$ small enough. By using this fact, we have

$$|I(\varepsilon)|\le \widetilde{C}\int_{B_{1-\varepsilon}\setminus B_{1-2\varepsilon}}|u|\,|x_i|\,dx,$$

where, by recalling (2.4.10) and (2.4.17), we set

$$\widetilde{C}=c\cdot C.$$

Then, since $u\in L^p(B_1)$, we can newly apply the Dominated Convergence Theorem to infer that

$$\lim_{\varepsilon\to 0^+}I(\varepsilon)=0.$$

In conclusion, from (2.4.16) we have obtained that for every $\psi\in C^\infty_0(B_R)$ and every $i\in\{1,\dots,N\}$ we have

$$\int_{B_R}\mathcal{E}_1[u]\,\frac{\partial\psi}{\partial x_i}\,dx=-\int_{B_R}(\phi_{\mathcal{K}})_i\,\psi\,dx,$$

thus showing that $\mathcal{E}_1[u]$ has a weak gradient, given by

$$\nabla\mathcal{E}_1[u]=\phi_{\mathcal{K}}=\begin{cases}\nabla u(x), & \text{if } x\in B_1,\\ D\mathcal{K}(x)\cdot\nabla u(\mathcal{K}(x)), & \text{if } x\in B_R\setminus B_1.\end{cases}$$

We also observe that, by using the change of variables $\mathcal{K}(x)=y$ and by recalling (2.4.4) and (2.4.6) we get

$$\begin{aligned}\int_{B_R}|\nabla\mathcal{E}_1[u]|^p\,dx&=\int_{B_1}|\nabla u|^p\,dx+\int_{B_R\setminus B_1}|D\mathcal{K}(x)\cdot\nabla u(\mathcal{K}(x))|^p\,dx\\&=\int_{B_1}|\nabla u|^p\,dx+\int_{B_1\setminus B_{\frac{1}{R}}}|D\mathcal{K}(\mathcal{K}(y))\cdot\nabla u(y)|^p\,|\mathrm{det}D\mathcal{K}(y)|\,dy\\&=\int_{B_1}|\nabla u|^p\,dx+\int_{B_1\setminus B_{\frac{1}{R}}}|y|^{2p}\,|\nabla u(y)|^p\,\frac{1}{|y|^{2N}}\,dy\\&\le\left(1+R^{2N}\right)\int_{B_1}|\nabla u|^p\,dx.\end{aligned}$$

this yields that $\nabla\mathcal{E}_1[u]\in L^p(B_R;\mathbb{R}^N)$ and proves (2.4.2). In particular, by combining the last estimate with (2.4.8) we obtain that

$$\|\mathcal{E}_1[u]\|_{W^{1,p}(B_R)}\le 2^{\frac{1}{p}}R^{\frac{2N}{p}}\|u\|_{W^{1,p}(B_1)},$$

for every $R>0$. □

By joining the previous result and the fact that each open bounded convex set $K\subseteq\mathbb{R}^N$ is bi–Lipschitz homeomorphic to a ball, we can obtain an extension operator for functions defined on K. This is taken from [**15**, Section 3].

Corollary 2.4.2. *Let $K\subseteq\mathbb{R}^N$ be an open bounded convex set and $x_0\in K$. There exists a linear extension operator*

$$\mathcal{E}_K:L^1(K)\to L^1_{\rm loc}(\mathbb{R}^N),$$

such that, for every $1\le p\le\infty$, it maps $W^{1,p}(K)$ to $W^{1,p}_{\rm loc}(\mathbb{R}^N)$. Moreover, if we define the following scaled copy of K

$$K_R(x_0):=R\,(K-x_0)+x_0=\Big\{R\,(x-x_0)+x_0\,:\,x\in K\Big\},$$

for every $u \in W^{1,p}(K)$ *and every* $R > 1$ *we have*

$$\big\|\nabla \mathcal{E}_K[u]\big\|_{L^p(K_R(x_0))} \le \mathcal{A}\, R^{\frac{2\,N}{p}}\, \|\nabla u\|_{L^p(K)}, \tag{2.4.18}$$

and

$$\big\|\mathcal{E}_K[u]\big\|_{L^p(K_R(x_0))} \le \mathcal{B}\, R^{\frac{2\,N}{p}}\, \|u\|_{L^p(K)}. \tag{2.4.19}$$

The constants $\mathcal{A} = \mathcal{A}(N,p,K,x_0) > 0$ *and* $\mathcal{B} = \mathcal{B}(N,p,K,x_0) > 0$ *are given by*

$$\mathcal{A} = \left(4\cdot 6^{3\,N+p}\right)^{\frac{1}{p}} \left(\frac{D_K(x_0)}{d_K(x_0)}\right)^{\frac{6\,N}{p}+2} \qquad \textit{and} \qquad \mathcal{B} = \left(2\cdot 6^{N}\right)^{\frac{1}{p}} \left(\frac{D_K(x_0)}{d_K(x_0)}\right)^{\frac{2\,N}{p}},$$

where

$$d_K(x_0) = \min_{x\in\partial K} |x - x_0|, \qquad D_K(x_0) = \max_{x\in\partial K} |x-x_0|.$$

2.5. Poincaré–Wirtinger inequalities

We will occasionally need also the sharp constants for some Poincaré–Wirtinger–type inequalities. More precisely, for an open bounded Lipschitz set $\Omega \subseteq \mathbb{R}^N$, for $1 \le p, q < \infty$ such that (1.1.1) holds, we introduce the quantity

$$\mu_{p,q}(\Omega) = \inf\left\{ \frac{\|\nabla u\|^p_{L^p(\Omega)}}{\min\limits_{t\in\mathbb{R}} \|u-t\|^p_{L^q(\Omega)}} \,:\, u \in \mathrm{Lip}(\overline{\Omega}) \text{ is not constant} \right\}. \tag{2.5.1}$$

In the case $q = p$, we will simply use the symbol $\mu_p(\Omega)$.

We notice that $\mu_{p,q}(\Omega) > 0$ if and only if Ω supports a Poincaré–Wirtinger inequality of the form

$$C \min_{t\in\mathbb{R}} \|u-t\|^p_{L^q(\Omega)} \le \|\nabla u\|^p_{L^p(\Omega)}, \qquad \text{for every } u \in \mathrm{Lip}(\overline{\Omega}),$$

for some $C > 0$. In this case, we have $\mu_{p,q}(\Omega) \ge C$ and $\mu_{p,q}(\Omega)$ is the sharp constant in such an inequality.

REMARK 2.5.1. We recall that for every $u \in \mathrm{Lip}(\overline{\Omega})$ and every $1 \le q \le \infty$, there exists a unique minimiser t_u of the function

$$t \mapsto \|u-t\|_{L^q(\Omega)}.$$

For $1 < q < \infty$, this is characterized by the following optimality condition

$$\int_\Omega |u-t_u|^{q-2}\,(u-t_u)\,dx = 0.$$

In the limit case $q = \infty$, this is given by

$$t_u = \frac{1}{2}\sup_{\overline{\Omega}} u + \frac{1}{2}\inf_{\overline{\Omega}} u.$$

Finally, in the limit case $q = 1$, the optimal t_u coincides with the unique value[2] t such that

$$\Big|\Big\{x \in \overline{\Omega} \,:\, u(x) \ge t\Big\}\Big| = \Big|\Big\{x\in\overline{\Omega} \,:\, u(x) \le t\Big\}\Big|.$$

We refer to [**57**, Theorem 2.1] for these facts.

Accordingly, in the case $1 < q < \infty$ the constant $\mu_{p,q}(\Omega)$ can be equivalently rewritten as

$$\mu_{p,q}(\Omega) = \inf\left\{ \frac{\|\nabla u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^q(\Omega)}} \,:\, \int_\Omega |u|^{q-2}\,u\,dx = 0,\ u \in \mathrm{Lip}(\overline{\Omega})\setminus\{0\}\right\}.$$

In the sequel, we will need the following geometric lower bound on $\mu_{p,q}$ for convex sets, which is quite classical. In general, this estimate is not sharp, but it will be largely sufficient for our purposes. We refer to [**23, 34, 40**] and [**43**] for some finer estimates.

[2] In this case, t_u is called the *median* of u. Its uniqueness is due to the continuity of u: for a discontinuous function, it is easily seen that medians may not be unique.

Lemma 2.5.2. *Let $1 \le p < \infty$ and $q \ge p$ be such that (1.1.1) holds. For every $\Omega \subseteq \mathbb{R}^N$ open bounded convex set, we have*

$$\mu_{p,q}(\Omega) \ge \left(N\,\omega_N^{\frac{1}{N}}\right)^p \left(\frac{|\Omega|}{\mathrm{diam}(\Omega)^N}\right)^p \left(\frac{\dfrac{1}{N}-\dfrac{1}{p}+\dfrac{1}{q}}{1-\dfrac{1}{p}+\dfrac{1}{q}}\right)^{p-1+\frac{p}{q}} |\Omega|^{1-\frac{p}{N}-\frac{p}{q}}. \tag{2.5.2}$$

Proof. With $u \in \mathrm{Lip}(\overline{\Omega})$, it is sufficient to combine [**48**, Lemma 7.12] and [**48**, Lemma 7.16]. This leads to

$$\left\|u-\frac{1}{|\Omega|}\int_\Omega u\,dy\right\|_{L^q(\Omega)} \le \frac{1}{N\,\omega_N^{1/N}}\,\frac{\mathrm{diam}(\Omega)^N}{|\Omega|}\left(\frac{1-\dfrac{1}{p}+\dfrac{1}{q}}{\dfrac{1}{N}-\dfrac{1}{p}+\dfrac{1}{q}}\right)^{1-\frac{1}{p}+\frac{1}{q}} |\Omega|^{\frac{1}{N}-\frac{1}{p}+\frac{1}{q}}\,\|\nabla u\|_{L^p(\Omega)}.$$

By simply noticing that

$$\min_{t\in\mathbb{R}}\|u-t\|_{L^q(\Omega)} \le \left\|u-\frac{1}{|\Omega|}\int_\Omega u\,dy\right\|_{L^q(\Omega)},$$

we obtain the claimed lower bound. □

2.6. A Maz'ya–Poincaré type inequality

The first cornerstone of our main results is the following Maz'ya–type inequality, for functions defined on a closed cube and vanishing in a (relative) neighborhood of a compact subset. For the proof of such result, we closely follow [**76**, Chapter 14, Section 1.2], up to some minor modifications. We will also give an explicit value for the constant appearing in the estimate (see Remark 2.6.2 below).

Theorem 2.6.1. *Let $1 \le p \le q$ such that (1.1.1) holds and let $\Sigma \subseteq \overline{Q_d(x_0)}$ be a compact set. Then, for every $D > \sqrt{N}d$ there exists a constant $\mathscr{C} = \mathscr{C}(N,p,q,D/d) > 0$ such that*

$$\frac{\mathscr{C}}{d^{\frac{N}{q}}}\left(\mathrm{cap}_p(\Sigma;B_D(x_0))\right)^{\frac{1}{p}}\|u\|_{L^q(Q_d(x_0))} \le \|\nabla u\|_{L^p(Q_d(x_0))},$$

for every $u \in C^\infty(\overline{Q_d(x_0)})$ with $\mathrm{dist}(\mathrm{supp}\,u,\Sigma) > 0$.

Proof. We can assume that $x_0 = 0$. Let $u \in C^\infty(\overline{Q_d})$ be as in the statement, without loss of generality we can also suppose that

$$\|u\|_{L^q(Q_d)} = |Q_d|^{\frac{1}{q}} = (2\,d)^{\frac{N}{q}}. \tag{2.6.1}$$

We use the standard convention that the right-hand side is 1, in the limit case $q = \infty$. Hence, we consider the function

$$\widetilde{u} := \mathcal{E}_{Q_d}[u],$$

i.e. the extended function provided by Corollary 2.4.2, with $K = Q_d$ and $x_0 = 0$. For every $D > d$, by applying formula (2.4.19) with $R = D/d$, we get

$$\|\nabla\widetilde{u}\|_{L^p(B_D)} \le \|\nabla\widetilde{u}\|_{L^p(Q_D)} \le \mathcal{A}\left(\frac{D}{d}\right)^{\frac{2N}{p}}\|\nabla u\|_{L^p(Q_d)}. \tag{2.6.2}$$

We observe that with this choice for K and x_0, we have $D_K(x_0)/d_K(x_0) = \sqrt{N}$, thus the constant $\mathcal{A}$ only depends on N and p. More precisely, it is given by

$$\mathcal{A} = \left(4\cdot 6^{3N+p}\right)^{\frac{1}{p}}\left(\sqrt{N}\right)^{\frac{6N}{p}+2} =: \alpha_{N,p}. \tag{2.6.3}$$

We now fix $D > \sqrt{N}\,d$ as in the statement and let η be a Lipschitz continuous cut-off function compactly supported in B_D, such that

$$0 \le \eta \le 1, \qquad \eta \equiv 1 \text{ on } B_{\sqrt{N}\,d}, \qquad \eta \equiv 0 \text{ on } B_D \setminus B_{\frac{\sqrt{N}\,d+D}{2}}, \quad |\nabla\eta| \le \frac{2}{D-\sqrt{N}d}. \tag{2.6.4}$$

Then, the function

$$\psi := \eta\,(1-\widetilde{u}),$$

is Lipschitz continuous, compactly supported in B_D and such that $\psi \geq 1_\Sigma$, by construction. Thus, it is an admissible function to test the Definition 2.2.1 of relative $p-$capacity. By the triangle inequality and the properties (2.6.4) of η, this yields

$$\Big(\mathrm{cap}_p(\Sigma;B_D)\Big)^{\frac{1}{p}} \leq \|\nabla\psi\|_{L^p(B_D)} \leq \|\nabla\widetilde{u}\|_{L^p(B_D)} + \frac{2}{D-\sqrt{N}d}\,\|1-\widetilde{u}\|_{L^p(B_D)}.$$

We now denote by $\widetilde{t}$ the unique real number (recall Remark 2.5.1 above) such that

$$\|\widetilde{u}-\widetilde{t}\|_{L^q(B_D)} = \min_{t\in\mathbb{R}}\|\widetilde{u}-t\|_{L^q(B_D)}.$$

Without loss of generality, we can suppose that

$$\widetilde{t} \geq 0. \tag{2.6.5}$$

By a further application of the triangle inequality, we obtain

$$\Big(\mathrm{cap}_p(\Sigma;B_D)\Big)^{\frac{1}{p}} \leq \|\nabla\widetilde{u}\|_{L^p(B_D)} + \frac{2}{D-\sqrt{N}d}\,\|1-\widetilde{t}\|_{L^p(B_D)} + \frac{2}{D-\sqrt{N}d}\,\|\widetilde{t}-\widetilde{u}\|_{L^p(B_D)}.$$

We have to estimate the last two L^p norms. Actually, the first one can be estimated in terms of the second one. Indeed, by using (2.6.5) and (2.6.1), we get

$$\begin{aligned}\|1-\widetilde{t}\|_{L^p(B_D)} = |1-\widetilde{t}|\,|B_D|^{\frac{1}{p}} &= \big|\|u\|_{L^q(Q_d)} - \|\widetilde{t}\|_{L^q(Q_d)}\big|\,\frac{|B_D|^{\frac{1}{p}}}{|Q_d|^{\frac{1}{q}}}\\ &\leq \|u-\widetilde{t}\|_{L^q(Q_d)}\,\frac{|B_D|^{\frac{1}{p}}}{|Q_d|^{\frac{1}{q}}} \leq \|\widetilde{u}-\widetilde{t}\|_{L^q(B_D)}\,\frac{|B_D|^{\frac{1}{p}}}{|Q_d|^{\frac{1}{q}}}.\end{aligned}$$

By inserting this estimate in the inequality above, we get

$$\Big(\mathrm{cap}_p(\Sigma;B_D)\Big)^{\frac{1}{p}} \leq \|\nabla\widetilde{u}\|_{L^p(B_D)} + \frac{2\,|B_D|^{\frac{1}{p}}}{D-\sqrt{N}d}\left(\frac{1}{|Q_d|^{\frac{1}{q}}}+\frac{1}{|B_D|^{\frac{1}{q}}}\right)\|\widetilde{t}-\widetilde{u}\|_{L^q(B_D)}.$$

Moreover, by recalling the definition of $\mu_{p,q}(B_D)$ and the definition of $\widetilde{t}$, we have

$$\|\widetilde{t}-\widetilde{u}\|_{L^q(B_D)} \leq \left(\frac{1}{\mu_{p,q}(B_D)}\right)^{\frac{1}{p}}\,\|\nabla\widetilde{u}\|_{L^p(B_D)}.$$

We thus obtain

$$\Big(\mathrm{cap}_p(\Sigma;B_D)\Big)^{\frac{1}{p}} \leq \left[1+\frac{2\,\omega_N^{\frac{1}{p}}}{D-\sqrt{N}d}\left(\frac{1}{|B_D|^{\frac{1}{q}}}+\frac{1}{|Q_d|^{\frac{1}{q}}}\right)\left(\frac{D^{p+\frac{N}{q}p}}{\mu_{p,q}(B_1)}\right)^{\frac{1}{p}}\right]\|\nabla\widetilde{u}\|_{L^p(B_D)}.$$

We make some small manipulations, in order to simplify the expression of the constant: we have

$$\begin{aligned}1+&\frac{2\,\omega_N^{\frac{1}{p}}}{D-\sqrt{N}d}\left(\frac{1}{|B_D|^{\frac{1}{q}}}+\frac{1}{|Q_d|^{\frac{1}{q}}}\right)\left(\frac{D^{p+\frac{N}{q}p}}{\mu_{p,q}(B_1)}\right)^{\frac{1}{p}}\\ &= \frac{1}{|Q_d|^{\frac{1}{q}}}\left[|Q_d|^{\frac{1}{q}} + \frac{2\,\omega_N^{\frac{1}{p}}}{D-\sqrt{N}d}\left(\frac{|Q_d|^{\frac{1}{q}}}{|B_D|^{\frac{1}{q}}}+1\right)\left(\frac{D^{p+\frac{N}{q}p}}{\mu_{p,q}(B_1)}\right)^{\frac{1}{p}}\right]\\ &\leq \frac{1}{(2\,d)^{\frac{N}{q}}}\left[\omega_N^{\frac{1}{q}}\,D^{\frac{N}{q}} + \frac{4\,\omega_N^{\frac{1}{p}}}{D-\sqrt{N}d}\left(\frac{D^{p+\frac{N}{q}p}}{\mu_{p,q}(B_1)}\right)^{\frac{1}{p}}\right]\\ &= \left(\frac{D}{2\,d}\right)^{\frac{N}{q}}\left[\omega_N^{\frac{1}{q}} + \frac{4\,\omega_N^{\frac{1}{p}}}{1-(\sqrt{N}d)/D}\left(\frac{1}{\mu_{p,q}(B_1)}\right)^{\frac{1}{p}}\right].\end{aligned}$$

Thus, by recalling the normalization condition (2.6.1), we have obtained

$$\frac{1}{d^{\frac{N}{q}}}\,\Big(\mathrm{cap}_p(\Sigma;B_D)\Big)^{\frac{1}{p}}\,\|u\|_{L^q(Q_d)}\le\left(\frac{D}{d}\right)^{\frac{N}{q}}\,\left[\omega_N^{\frac{1}{q}}+\frac{4\,\omega_N^{\frac{1}{p}}}{1-(\sqrt{N}d)/D}\,\left(\frac{1}{\mu_{p,q}(B_1)}\right)^{\frac{1}{p}}\right]\,\|\nabla\widetilde{u}\|_{L^p(B_D)}.$$

At last, by using (2.6.2) in the right-hand side, we get the desired conclusion. □

REMARK 2.6.2. By inspecting the proof, we see that the constant $\mathscr{C}$ obtained in the previous theorem has the following explicit expression

$$\mathscr{C}=\frac{1}{\alpha_{N,p}}\,\left(\frac{d}{D}\right)^{\frac{2\,N}{p}+\frac{N}{q}}\,\left[\omega_N^{\frac{1}{q}}+\frac{4\,\omega_N^{\frac{1}{p}}}{1-\dfrac{\sqrt{N}d}{D}}\,\left(\frac{1}{\mu_{p,q}(B_1)}\right)^{\frac{1}{p}}\right]^{-1}.$$

The constant $\alpha_{N,p}$ is given in (2.6.3) and it comes from the extension operator. Actually, the constant $\mu_{p,q}(B_1)$ may not look so explicit: however, it can be conveniently estimated from below by Lemma 2.5.2, in terms of quantities only depending on N, p and q.

2.7. Benguria's hidden convexity and perforated cubes

To set the scene for the proof of the subsequent Lemma 3.2.4, we need a technical result of indipendent interest which concerns the minimization problem

$$\Lambda_p(Q_R(x_0)\setminus\overline{B_r(x_0)})=\inf_{u\in\mathrm{Lip}(\overline{Q_R(x_0)})}\left\{\int_{Q_R(x_0)}|\nabla u|^p\,dx\,:\,\|u\|_{L^p(Q_R(x_0))}=1,\,u(x_0)=0\text{ on }\overline{B_r(x_0)}\right\},$$

with $0\le r<R$. As we will see, this result will derive from the so-called *Benguria's hidden convexity*, a property of the $p-$Dirichlet integral originally devised in [10, 11] for $p=2$, and extended to $1<p<\infty$ in [9, 38, 60, 90], see also [25, Theorem 2.9]. For sake of completeness, we report here its statement in the simplest case: this will be enough for our scopes.

THEOREM 2.7.1 (Benguria's hidden convexity). *Let $1\le p<\infty$ and let $\Omega\subseteq\mathbb{R}^N$ be an open set. For every pair of non-negative functions $u,v\in W^{1,p}(\Omega)$ we set*

$$\sigma^t:=\big((1-t)u^p+tv^p\big)^{\frac{1}{p}},\qquad \text{for every } t\in[0,1].$$

Then $\sigma^t\in W^{1,p}(\Omega)$ and

$$\int_\Omega|\nabla\sigma^t|^p\,dx\le(1-t)\int_\Omega|\nabla u|^p\,dx+t\int_\Omega|\nabla v|^p\,dx,\qquad \text{for every } t\in[0,1].$$

As an application of Benguria's hidden convexity principle we obtain the following technical lemma. For simplicity, we state it with $x_0=0$.

LEMMA 2.7.2. *Let $0\le r<R$ and $1\le p<\infty$. We set*

$$\mathrm{Lip}^{\mathrm{S}}_+(\overline{Q_R})=\Big\{u\in\mathrm{Lip}(\overline{Q_R})\,:\,u\ge0,\;u\circ\mathcal{R}_i=u\text{ for }i=1,\dots,N\Big\},$$

where $\mathcal{R}_i:\mathbb{R}^N\to\mathbb{R}^N$ is the reflection with respect to the hyperplane $\{x\in\mathbb{R}^N\,:\,\langle x,\mathbf{e}_i\rangle=0\}$. Then, for every every $1\le p<\infty$ we have

$$\inf_{u\in\mathrm{Lip}(\overline{Q_R})}\left\{\int_{Q_R}|\nabla u|^p\,dx\,:\,\|u\|_{L^p(Q_R)}=1,\,u=0\text{ on }\overline{B_r}\right\}$$
$$=\inf_{u\in\mathrm{Lip}^{\mathrm{S}}_+(\overline{Q_R})}\left\{\int_{Q_R}|\nabla u|^p\,dx\,:\,\|u\|_{L^p(Q_R)}=1,\,u=0\text{ on }\overline{B_r}\right\}.$$

Proof. Obviously, we have

$$\inf_{u\in \mathrm{Lip}(\overline{Q_R})} \left\{\int_{Q_R} |\nabla u|^p\,dx \,:\, \|u\|_{L^p(Q_R)}=1,\, u=0 \text{ on } \overline{B_r}\right\} \\ \le \inf_{u\in \mathrm{Lip}^{\mathrm{S}}_+(\overline{Q_R})} \left\{\int_{Q_R} |\nabla u|^p\,dx \,:\, \|u\|_{L^p(Q_R)}=1,\, u=0 \text{ on } \overline{B_r}\right\}.$$

In order to prove the reverse inequality, we take $u\in \mathrm{Lip}(\overline{Q_R})$ to be admissible for the variational problem on the left-hand side. Then, we define recursively the non-negative Lipschitz functions

$$\sigma_1=\left(\frac{1}{2}\,|u|^p+\frac{1}{2}\,|u\circ \mathcal{R}_1|^p\right)^{\frac{1}{p}},$$

and

$$\sigma_{i+1}=\left(\frac{1}{2}\,(\sigma_i)^p+\frac{1}{2}\,(\sigma_i\circ \mathcal{R}_{i+1})^p\right)^{\frac{1}{p}},\qquad \text{for } i=1,\dots,N-1.$$

We claim that for every $i=1,\dots,N$:

(i) $\sigma_i\circ\mathcal{R}_j=\sigma_i$, for every $1\le j\le i$;

(ii) $\|\nabla\sigma_i\|_{L^p(Q_R)}\le\|\nabla u\|_{L^p(Q_R)}$;

(iii) $\|\sigma_i\|_{L^p(Q_R)}=1$ and $\sigma_i=0$ on $\overline{B_r}$.

In particular, by taking $i=N$, we would get that σ_N is admissible for the variational problem on $\mathrm{Lip}^{\mathrm{S}}_+(\overline{Q_R})$ and

$$\int_{Q_R}|\nabla\sigma_N|^p\,dx\le\int_{Q_R}|\nabla u|^p\,dx.$$

This would be enough to conclude the proof.

We are left with proving that σ_i has the claimed properties. We proceed by induction: for $i=1$, properties (i) and (iii) are straightforward. As for property (ii), this follows from Benguria's hidden convexity principle, which gives

$$\int_{Q_R}|\nabla\sigma_1|^p\,dx\le\frac{1}{2}\int_{Q_R}|\nabla|u||^p\,dx+\frac{1}{2}\int_{Q_R}|\nabla|u\circ\mathcal{R}_1||^p\,dx=\int_{Q_R}|\nabla u|^p\,dx,$$

where we used that $\mathcal{R}_1(Q_R)=Q_R$ and that $\mathcal{R}_1$ is a linear isometry, together with the fact that $|\nabla|u||=|\nabla u|$ almost everywhere.

We now take $1\le\ell\le N-1$ and suppose that (i), (ii) and (iii) hold for every $\sigma_1,\dots,\sigma_\ell$. We need to prove that these properties hold for $\sigma_{\ell+1}$, as well. Again, property (iii) is immediate by construction and by the inductive assumption. For point (i), we have

$$\sigma_{\ell+1}=\left(\frac{1}{2}\,(\sigma_\ell)^p+\frac{1}{2}\,(\sigma_\ell\circ\mathcal{R}_{\ell+1})^p\right)^{\frac{1}{p}},$$

thus for $1\le j\le\ell$

$$\begin{aligned}\sigma_{\ell+1}\circ\mathcal{R}_j&=\left(\frac{1}{2}\,(\sigma_\ell\circ\mathcal{R}_j)^p+\frac{1}{2}\,(\sigma_\ell\circ\mathcal{R}_{\ell+1}\circ\mathcal{R}_j)^p\right)^{\frac{1}{p}}\\&=\left(\frac{1}{2}\,(\sigma_\ell)^p+\frac{1}{2}\,(\sigma_\ell\circ\mathcal{R}_j\circ\mathcal{R}_{\ell+1})^p\right)^{\frac{1}{p}}=\left(\frac{1}{2}\,\sigma_\ell^p+\frac{1}{2}\,(\sigma_\ell\circ\mathcal{R}_{\ell+1})^p\right)^{\frac{1}{p}}=\sigma_{\ell+1},\end{aligned}$$

where we exploited the validity of (i) for $1\le j\le\ell$. As for the composition with $\mathcal{R}_{\ell+1}$, we also have

$$\begin{aligned}\sigma_{\ell+1}\circ\mathcal{R}_{\ell+1}&=\left(\frac{1}{2}\,(\sigma_\ell\circ\mathcal{R}_{\ell+1})^p+\frac{1}{2}\,(\sigma_\ell\circ\mathcal{R}_{\ell+1}\circ\mathcal{R}_{\ell+1})^p\right)^{\frac{1}{p}}\\&=\left(\frac{1}{2}\,(\sigma_\ell\circ\mathcal{R}_{\ell+1})^p+\frac{1}{2}\,(\sigma_\ell)^p\right)^{\frac{1}{p}}=\sigma_{\ell+1},\end{aligned}$$

thanks to the fact that $\mathcal{R}_{\ell+1}\circ\mathcal{R}_{\ell+1}$ is the identity map. This establishes the validity of (i) for $\ell+1$, as well. We still need to verify property (ii): by using again Benguria's hidden convexity, we get

$$\int_{Q_R} |\nabla\sigma_{\ell+1}|^p\,dx \le \frac{1}{2}\int_{Q_R} |\nabla\sigma_\ell|^p\,dx + \frac{1}{2}\int_{Q_R} |\nabla(\sigma_\ell\circ\mathcal{R}_{\ell+1})|^p\,dx = \int_{Q_R} |\nabla\sigma_\ell|^p\,dx,$$

thanks to the fact that $\mathcal{R}_{\ell+1}(Q_R)=Q_R$. By using that (ii) holds for σ_ℓ, we get the desired conclusion. □

CHAPTER 3

Inradius and Poincaré–Sobolev inequalities: the case $p > N$

3.1. The case $p > N$

This chapter is devoted to prove that in the *super–conformal case* we can get a lower bound on the principal frequencies of a general open set Ω in $\mathbb{R}^N$ in terms of its inradius, without taking any restriction on the geometry/topology of Ω. This result was originally proved by Vladimir Maz'ya in the '70s (see [**76**, Theorem 15.4.1 & Comments to Chap. 15 pag. 733]), and after that by other several authors (see Section 1.2.1 for more details).

Here, we give a slightly different proof taken from [**B2**], based on the analysis of some particular "punctured" Poincaré constants (see Section 3.2). It improves the already existing proofs by providing an explicit constant, and contains a detailed analysis of its asymptotic behaviour as $p \searrow N$ and $p \nearrow \infty$. In particular, in these regimes we get a sharp asymptotic behaviour.

Finally, by means of an interpolation argument, we extend the result to the case of $\lambda_{p,q}$, for $N < p < q \le \infty$. The main theorem of this chapter reads as follows

THEOREM 1. *Let $1 \le N < p$. Then, for every open set $\Omega \subseteq \mathbb{R}^N$ with finite inradius r_Ω, we have*

$$\lambda_p(\Omega) \ge \beta_{N,p} \left(\frac{1}{r_\Omega}\right)^p, \qquad \text{with } \beta_{N,p} = \max\left\{\frac{\Lambda_p(B_1 \setminus \{0\})}{(\sqrt{N}+1)^p}, \left(\frac{p-N}{p}\right)^p\right\} > 0, \tag{3.1.1}$$

and

$$\lambda_{p,\infty}(\Omega) \ge \Lambda_{p,\infty}(B_1 \setminus \{0\}) \left(\frac{1}{r_\Omega}\right)^{p-N}. \tag{3.1.2}$$

For $p < q < \infty$, we also get

$$\lambda_{p,q}(\Omega) \ge \Big(\beta_{N,p}\Big)^{\frac{p}{q}} \Big(\Lambda_{p,\infty}(B_1 \setminus \{0\})\Big)^{1-\frac{p}{q}} \left(\frac{1}{r_\Omega}\right)^{p-N+N\frac{p}{q}}. \tag{3.1.3}$$

Finally, the two constants $\beta_{N,p}$ and $\Lambda_{p,\infty}(B_1 \setminus \{0\})$ exhibit the following asymptotic behaviour

$$0 < \liminf_{p\searrow N} \frac{\beta_{N,p}}{(p-N)^{p-1}} \le \limsup_{p\searrow N} \frac{\beta_{N,p}}{(p-N)^{p-1}} < +\infty \qquad \text{and} \qquad \lim_{p\nearrow\infty} (\beta_{N,p})^{\frac{1}{p}} = 1,$$

$$0 < \liminf_{p\searrow N} \frac{\Lambda_{p,\infty}(B_1 \setminus \{0\})}{(p-N)^{p-1}} \le \limsup_{p\searrow N} \frac{\Lambda_{p,\infty}(B_1 \setminus \{0\})}{(p-N)^{p-1}} < +\infty \quad \text{and} \quad \lim_{p\nearrow\infty} (\Lambda_{p,\infty}(B_1 \setminus \{0\}))^{\frac{1}{p}} = 1.$$

3.2. Punctured Poincaré constants

To begin with, we set up some technical results of independent interest. Let $p > N \ge 1$ and let $K \subseteq \mathbb{R}^N$ be an open bounded convex set. For every $x_0 \in \overline{K}$, we define the following Poincaré constants

$$\Lambda_p(K \setminus \{x_0\}) = \inf_{u \in \mathrm{Lip}(\overline{K})} \left\{\int_K |\nabla u|^p\, dx \,:\, \|u\|_{L^p(K)} = 1,\, u(x_0) = 0\right\},$$

and

$$\Lambda_{p,\infty}(K \setminus \{x_0\}) = \inf_{u \in \mathrm{Lip}(\overline{K})} \left\{\int_K |\nabla u|^p\, dx \,:\, \|u\|_{L^\infty(K)} = 1, u(x_0) = 0\right\}.$$

We observe that in the particular case $K = B_R(x_0)$, we have

$$\Lambda_p(B_R(x_0) \setminus \{x_0\}) = \frac{\Lambda_p(B_1 \setminus \{0\})}{R^p} \qquad \text{and} \qquad \Lambda_{p,\infty}(B_R(x_0) \setminus \{x_0\}) = \frac{\Lambda_{p,\infty}(B_1 \setminus \{0\})}{R^{p-N}}. \tag{3.2.1}$$

The following simple result is instrumental to get a lower bound on this constant.

LEMMA 3.2.1. *Let $1 \le N < p$ and $R > 0$. For every $u \in \mathrm{Lip}([0, R]) \setminus \{0\}$ such that $u(0) = 0$, we have*

$$\frac{\Lambda_p(B_1 \setminus \{0\})}{R^p} \int_0^R |u(t)|^p \, t^{N-1} \, dt \le \int_0^R |u'(t)|^p \, t^{N-1} \, dt.$$

PROOF. For every u as in the statement, we define

$$U(x) = u(|x - x_0|), \qquad \text{for every } x \in B_R(x_0).$$

By definition of $\Lambda_p(B_R(x_0) \setminus \{x_0\})$, we have

$$\Lambda_p(B_R(x_0) \setminus \{x_0\}) \int_{B_R(x_0)} |U|^p \, dx \le \int_{B_R(x_0)} |\nabla U|^p \, dx.$$

By using spherical coordinates centered at x_0 and taking (3.2.1) into account, we get the desired conclusion. □

Now we can prove the following sharp inequality, which is interesting in itself.

LEMMA 3.2.2. *Let $1 \le N < p$ and let $K \subseteq \mathbb{R}^N$ be an open bounded convex set. For every $x_0 \in \overline{K}$, we have*

$$\Lambda_p(K \setminus \{x_0\}) \ge \frac{\Lambda_p(B_1 \setminus \{0\})}{D_K(x_0)^p}, \qquad \text{where } D_K(x_0) = \max_{y \in \partial K} |x_0 - y|.$$

Moreover, we have equality for $K = B_R(x_0)$.

PROOF. Let u be an admissible function for the problem which defines $\Lambda_p(K \setminus \{x_0\})$. By using spherical coordinates centered at x_0, we get

$$\begin{aligned}\int_K |\nabla u|^p \, dx &= \int_{\mathbb{S}^{N-1}} \int_0^{r(\omega)} \left[\left(\frac{\partial u}{\partial \varrho} \right)^2 + \frac{1}{\varrho^2} |\nabla_\tau u|^2 \right]^{\frac{p}{2}} \varrho^{N-1} \, d\varrho \, d\mathcal{H}^{N-1}(\omega) \\ &\ge \int_{\mathbb{S}^{N-1}} \int_0^{r(\omega)} \left| \frac{\partial u}{\partial \varrho} \right|^p \varrho^{N-1} \, d\varrho \, d\mathcal{H}^{N-1}(\omega).\end{aligned}$$

By using Lemma 3.2.1 in the innermost integral, we obtain

$$\int_K |\nabla u|^p \, dx \ge \Lambda_p(B_1 \setminus \{0\}) \int_{\mathbb{S}^{N-1}} \frac{1}{r(\omega)^p} \int_0^{r(\omega)} |u|^p \, \varrho^{N-1} \, d\varrho \, d\mathcal{H}^{N-1}(\omega).$$

Finally, by noticing that

$$r(\omega) \le D_K(x_0), \qquad \text{for every } \omega \in \mathbb{S}^{N-1},$$

we get the desired conclusion. □

The following estimate on the quantities Λ_p and $\Lambda_{p,\infty}$ will be useful in the sequel, in the particular case $K = B_1$ and $x_0 = 0$.

LEMMA 3.2.3. *Let $1 \le N < p$ and let $K \subseteq \mathbb{R}^N$ be an open bounded convex set. For every $x_0 \in \overline{K}$, we have the following estimates*

$$|K| \, \Lambda_p(K \setminus \{x_0\}) \ge \Lambda_{p,\infty}(K \setminus \{x_0\}) \ge \frac{\mu_{p,\infty}(K)}{2^p}, \tag{3.2.2}$$

where we recall that $\mu_{p,\infty}(K)$ is defined in (2.5.1). Moreover, we have

$$\lim_{p \to \infty} \Big(\Lambda_p(K \setminus \{x_0\}) \Big)^{\frac{1}{p}} = \lim_{p \to \infty} \Big(\Lambda_{p,\infty}(K \setminus \{x_0\}) \Big)^{\frac{1}{p}} = \frac{1}{D_K(x_0)},$$

where as above $D_K(x_0) = \max_{x \in \partial K} |x - x_0|$.

PROOF. The leftmost inequality in (3.2.2) easily follows from Hölder's inequality. In order to prove the rightmost one, let u be a Lipschitz function on $\overline{K}$, such that $u(x_0) = 0$ and $\|u\|_{L^\infty(K)} = 1$. Let t_u be such that

$$\|u - t_u\|_{L^\infty(K)} = \min_{t\in\mathbb{R}} \|u - t\|_{L^\infty(K)}.$$

By definition of $\mu_{p,\infty}(K)$, we have

$$|u(x) - t_u| \leq \left(\frac{1}{\mu_{p,\infty}(K)}\right)^{\frac{1}{p}} \left(\int_K |\nabla u|^p\, dx\right)^{\frac{1}{p}}, \qquad \text{for every } x \in K.$$

Thus, we obtain

$$\begin{aligned}|u(x)| = |u(x) - u(x_0)| &\leq |u(x) - t_u| + |u(x_0) - t_u| \\ &\leq 2 \left(\frac{1}{\mu_{p,\infty}(K)}\right)^{\frac{1}{p}} \left(\int_K |\nabla u|^p\, dx\right)^{\frac{1}{p}}, \quad \text{for every } x \in K.\end{aligned}$$

By taking the supremum over $x \in K$ and recalling the normalization on u, we get

$$\frac{\mu_{p,\infty}(K)}{2^p} \leq \Lambda_{p,\infty}(K \setminus \{0\}),$$

as desired.

For the second part of the statement, we first observe that if $N < p_1 < p_2$, then

$$\left(\frac{\Lambda_{p_1,\infty}(K \setminus \{x_0\})}{|K|}\right)^{\frac{1}{p_1}} \leq \left(\frac{\Lambda_{p_2,\infty}(K \setminus \{x_0\})}{|K|}\right)^{\frac{1}{p_2}},$$

by Hölder's inequality. Thus, the limit

$$\lim_{p\to\infty} \left(\frac{\Lambda_{p,\infty}(K \setminus \{x_0\})}{|K|}\right)^{\frac{1}{p}},$$

exists, by monotonicity. This in turn implies that $\lim_{p\to\infty}(\Lambda_{p,\infty}(K \setminus \{x_0\}))^{1/p}$ exists, as well. In order to estimate this limit from above, we notice that the function

$$u(x) = \left(\int_K |x - x_0|^p\, dx\right)^{-\frac{1}{p}} |x - x_0|,$$

is admissible for $\Lambda_p(K \setminus \{x_0\})$. Thus, we get

$$\begin{aligned}\lim_{p\nearrow\infty} \Big(\Lambda_{p,\infty}(K \setminus \{x_0\})\Big)^{\frac{1}{p}} &\leq \lim_{p\nearrow\infty} \Big(\Lambda_p(K \setminus \{x_0\})\Big)^{\frac{1}{p}} \\ &\leq \lim_{p\nearrow\infty} |K|^{\frac{1}{p}} \left(\int_K |x - x_0|^p\, dx\right)^{-\frac{1}{p}} = \frac{1}{D_K(x_0)}.\end{aligned} \tag{3.2.3}$$

Observe that we used that

$$D_K(x_0) = \max_{x\in\partial K} |x - x_0| = \max_{x\in\overline{K}} |x - x_0|,$$

thanks to the convexity of K.

The estimate from below is more elaborated, but the argument is nowadays quite standard (see for example [14, Section 2]). For every $m \in \mathbb{N}$ such that $m \geq N + 1$, let us take $u_m \in \mathrm{Lip}(\overline{K})$ such that

$$\|u_m\|_{L^\infty(K)} = 1, \qquad u_m(x_0) = 0, \qquad \int_K |\nabla u_m|^m\, dx < 2\,\Lambda_{m,\infty}(K \setminus \{x_0\}).$$

By Hölder's inequality, for every $m \geq N + 1$ we have

$$\int_K |\nabla u_m|^{N+1}\, dx \leq |K|^{1-\frac{N+1}{m}} \left(\int_K |\nabla u_m|^m\, dx\right)^{\frac{N+1}{m}} \leq |K|^{1-\frac{N+1}{m}} \Big(2\,\Lambda_{m,\infty}(K \setminus \{x_0\})\Big)^{\frac{N+1}{m}}.$$

In light of (3.2.3), this shows that $\{u_m\}_{m\geq N+1}$ is a bounded sequence in $W^{1,N+1}(K)$. By the Morrey-Sobolev compact embedding (see [69, Theorem 12.61]), we have that there exists a subsequence $\{u_{m_n^1}\}_{n\in\mathbb{N}} \subseteq$

$\{u_m\}_{m\geq N+1}$ and a limit function $u_\infty \in W^{1,N+1}(K)\cap C(\overline{K})$, such that $u_{m_n^1}$ converges weakly in $W^{1,N+1}(K)$ and uniformly on $\overline{K}$ to u_∞. Thus, we still have

$$\|u_\infty\|_{L^\infty(K)} = 1, \qquad u_\infty(x_0) = 0.$$

Moreover, by lower semicontinuity and (3.2.3), we have

$$\left(\int_K |\nabla u_\infty|^{N+1}\,dx\right)^{\frac{1}{N+1}} \leq \frac{|K|^{\frac{1}{N+1}}}{D_K(x_0)}.$$

We can now recursively repeat the previous argument: we take $N+\ell+1$ for $\ell \in \mathbb{N}\setminus\{0\}$ and extract a subsequence $\{u_{m_n^{\ell+1}}\}_{n\in\mathbb{N}}$ from the previous one $\{u_{m_n^\ell}\}_{n\in\mathbb{N}}$. Indeed, at each step, we have

$$\int_K |\nabla u_{m_n^\ell}|^{N+\ell+1}\,dx \leq |K|^{1-\frac{N+\ell+1}{m_n^\ell}} \left(2\,\Lambda_{m_n^\ell,\infty}(K\setminus\{x_0\})\right)^{\frac{N+\ell+1}{m_n^\ell}}, \tag{3.2.4}$$

which shows that $\{u_{m_n^\ell}\}_{n\in\mathbb{N}}$ is a bounded sequence in $W^{1,N+\ell+1}(K)$. As before, there exists a subsequence $\{u_{m_n^{\ell+1}}\}_{n\in\mathbb{N}}$ which converges weakly in $W^{1,N+\ell+1}(K)$ and uniformly on $\overline{K}$. By construction, the limit function must still coincide with the original limit function u_∞. This shows that $u_\infty \in W^{1,N+\ell+1}$ for every $\ell\in\mathbb{N}$ and that

$$\begin{aligned}\left(\int_K |\nabla u_\infty|^{N+\ell+1}\,dx\right)^{\frac{1}{N+\ell+1}} &\leq \lim_{n\to\infty} |K|^{\frac{1}{N+\ell+1}-\frac{1}{m_n^\ell}} \left(2\,\Lambda_{m_n^\ell,\infty}(K\setminus\{x_0\})\right)^{\frac{1}{m_n^\ell}} \\ &\leq \frac{|K|^{\frac{1}{N+\ell+1}}}{D_K(x_0)}.\end{aligned} \tag{3.2.5}$$

By taking the limit as ℓ goes to ∞, we get that $u_\infty \in \mathrm{Lip}(\overline{K})$, with

$$\|\nabla u_\infty\|_{L^\infty(K)} \leq \frac{1}{D_K(x_0)}, \qquad \|u_\infty\|_{L^\infty(K)} = 1, \qquad u_\infty(x_0) = 0.$$

Actually, the last two properties show that the first one can be improved. Indeed, let $\overline{x}\in\overline{K}$ be a maximum point of $|u_\infty|$. We then have[1]

$$1 = |u_\infty(\overline{x})| = |u_\infty(\overline{x}) - u_\infty(x_0)| \leq \|\nabla u_\infty\|_{L^\infty(K)}\,|\overline{x}-x_0| \leq \frac{|\overline{x}-x_0|}{D_K(x_0)} \leq 1.$$

This implies that equality must hold everywhere. In particular, we get

$$\|\nabla u_\infty\|_{L^\infty(K)} = \frac{1}{D_K(x_0)}. \tag{3.2.6}$$

With this information at hand, we can now conclude: we go back to (3.2.5) and observe that

$$\lim_{n\to\infty} |K|^{\frac{1}{N+\ell+1}-\frac{1}{m_n^\ell}} \left(2\,\Lambda_{m_n^\ell,\infty}(K\setminus\{0\})\right)^{\frac{1}{m_n^\ell}} = |K|^{\frac{1}{N+\ell+1}} \lim_{m\to\infty} \left(\Lambda_{m,\infty}(K\setminus\{0\})\right)^{\frac{1}{m}}.$$

Thus, we obtain

$$\lim_{m\to\infty} \left(\Lambda_{m,\infty}(K\setminus\{0\})\right)^{\frac{1}{m}} \geq |K|^{-\frac{1}{N+\ell+1}} \left(\int_K |\nabla u_\infty|^{N+\ell+1}\,dx\right)^{\frac{1}{N+\ell+1}}.$$

By taking the limit as ℓ goes to ∞ and using (3.2.6), we conclude. □

We conclude this section, by observing that $\Lambda_p(Q_1\setminus\{0\})$ actually coincides with λ_p of a suitable "pepper" set[2]. More precisely, we have the following

[1] As already observed, by convexity of K, we have

$$D_K(x_0) = \max_{x\in\partial K} |x-x_0| = \max_{x\in\overline{K}} |x-x_0|.$$

[2] We borrow this fancy terminology from Adams, see for example [1].

LEMMA 3.2.4. *For $1 \leq N < p$, we have*

$$\lambda_p(\mathbb{R}^N \setminus \mathbb{Z}^N) = \Lambda_p(Q_{1/2} \setminus \{0\}) = 2^p\, \Lambda_p(Q_1 \setminus \{0\}).$$

PROOF. The rightmost equality simply follows by scaling. Let us prove the leftmost one.
Let $u \in C_0^\infty(\mathbb{R}^N \setminus \mathbb{Z}^N)$. By tiling the space with the cubes

$$Q_{1/2}(\mathbf{i}), \qquad \text{with } \mathbf{i} \in \mathbb{Z}^N,$$

we easily see that u is admissible for the variational problem which defines $\Lambda_p(Q_{1/2}(\mathbf{i}) \setminus \{\mathbf{i}\})$. Thus, we get

$$\int_\Omega |\nabla u|^p\, dx = \sum_{\mathbf{i}\in\mathbb{Z}^N} \int_{Q_{1/2}(\mathbf{i})} |\nabla u|^p\, dx \geq \sum_{\mathbf{i}\in\mathbb{Z}^N} \Lambda_p(Q_{1/2}(\mathbf{i}) \setminus \{\mathbf{i}\}) \int_{Q_{1/2}(\mathbf{i})} |u|^p\, dx.$$

Since we have

$$\Lambda_p(Q_{1/2}(\mathbf{i}) \setminus \{\mathbf{i}\}) = \Lambda_p(Q_{1/2} \setminus \{0\}), \qquad \text{for every } \mathbf{i} \in \mathbb{Z}^N,$$

we can infer

$$\int_\Omega |\nabla u|^p\, dx \geq \Lambda_p(Q_{1/2} \setminus \{0\}) \int_\Omega |u|^p\, dx.$$

By the arbitrariness of u, this yields $\lambda_p(\mathbb{R}^N \setminus \mathbb{Z}^N) \geq \Lambda_p(Q_{1/2} \setminus \{0\})$.

In order to prove the reverse inequality, we take $u \in \mathrm{Lip}(\overline{Q_{1/2}})$ such that $u(0) = 0$ and $\|u\|_{L^p(Q_{1/2})} = 1$. According to Lemma 2.7.2, we can further suppose that u is non-negative and symmetric with respect to each variable. For every $m \in \mathbb{N}$, we define

$$\mathbb{Z}^N_m = \Big\{\mathbf{i} = (i_1, \dots, i_N) \in \mathbb{Z}^N \,:\, |\mathbf{i}|_{\ell^\infty} := \max_{k=1,\dots,N} |i_k| \leq m\Big\},$$

and then we set

$$U_m(x) = \sum_{\mathbf{i}\in\mathbb{Z}^N_m} u(x + \mathbf{i}).$$

Observe that this is Lipschitz function on the cube $Q_{m+1/2}$ (thanks to the symmetries of u), vanishing at each point $\mathbf{i} \in \mathbb{Z}^N_m$. We then take η_m a 1-Lipschitz cut-off function, such that

$$0 \leq \eta_m \leq 1, \qquad \eta_m \equiv 1 \text{ on } Q_{m-1/2}, \qquad \eta_m = 0 \text{ on } \partial Q_{m+1/2}.$$

By construction, we get that $\eta_m\, U_m \in W^{1,p}_0(\mathbb{R}^N \setminus \mathbb{Z}^N)$, where we extend it by 0 outside $Q_{m+1/2}$. Thus, we get

$$\begin{aligned}
\Big(\lambda_p(\mathbb{R}^N \setminus \mathbb{Z}^N)\Big)^{\frac{1}{p}} &\leq \frac{\left(\displaystyle\int_{\mathbb{R}^N} |\nabla(\eta_m\, U_m)|^p\, dx\right)^{\frac{1}{p}}}{\left(\displaystyle\int_{\mathbb{R}^N} |\eta_m\, U_m|^p\, dx\right)^{\frac{1}{p}}} \\
&\leq \frac{\left(\displaystyle\sum_{\mathbf{i}\in\mathbb{Z}^N_m} \int_{Q_{1/2}(\mathbf{i})} |\nabla \eta_m|^p\, |U_m|^p\, dx\right)^{\frac{1}{p}}}{\left(\displaystyle\sum_{\mathbf{i}\in\mathbb{Z}^N_m} \int_{Q_{1/2}(\mathbf{i})} |\eta_m\, U_m|^p\, dx\right)^{\frac{1}{p}}} + \frac{\left(\displaystyle\sum_{\mathbf{i}\in\mathbb{Z}^N_m} \int_{Q_{1/2}(\mathbf{i})} |\nabla U_m|^p\, |\eta_m|^p\, dx\right)^{\frac{1}{p}}}{\left(\displaystyle\sum_{\mathbf{i}\in\mathbb{Z}^N_m} \int_{Q_{1/2}(\mathbf{i})} |\eta_m\, U_m|^p\, dx\right)^{\frac{1}{p}}}.
\end{aligned}$$

We now observe that, thanks to the properties of η_m, we have

$$\begin{aligned}
\sum_{\mathbf{i}\in\mathbb{Z}^N_m} \int_{Q_{1/2}(\mathbf{i})} |\eta_m\, U_m|^p\, dx \geq \sum_{\mathbf{i}\in\mathbb{Z}^N_{m-1}} \int_{Q_{1/2}(\mathbf{i})} |\eta_m\, U_m|^p\, dx &= \sum_{\mathbf{i}\in\mathbb{Z}^N_{m-1}} \int_{Q_{1/2}(\mathbf{i})} |U_m|^p\, dx \\
&= (2\,m-1)^N \int_{Q_{1/2}} |u|^p\, dx.
\end{aligned}$$

We also used that U_m coincides with a translated copy of the original function u defined on $Q_{1/2}$. As for the first integral at the numerator, since η_m is $1-$Lipschitz and is constant on $Q_{m-1/2}$, we get

$$\sum_{\mathbf{i}\in\mathbb{Z}^N_m}\int_{Q_{1/2}(\mathbf{i})}|\nabla\eta_m|^p\,|U_m|^p\,dx\le\sum_{|\mathbf{i}|_{\ell^\infty}=m}\int_{Q_{1/2}(\mathbf{i})}|U_m|^p\,dx=\Big[(2\,m+1)^N-(2\,m-1)^N\Big]\int_{Q_{1/2}}|u|^p\,dx.$$

Finally, by using that $|\eta_m|\le 1$, we have

$$\sum_{\mathbf{i}\in\mathbb{Z}^N_m}\int_{Q_{1/2}(\mathbf{i})}|\nabla U_m|^p\,|\eta_m|^p\,dx\le\sum_{\mathbf{i}\in\mathbb{Z}^N_m}\int_{Q_{1/2}(\mathbf{i})}|\nabla U_m|^p\,dx=(2\,m+1)^N\int_{Q_{1/2}}|\nabla u|^p\,dx.$$

By using these estimates, we get

$$\Big(\lambda_p(\mathbb{R}^N\setminus\mathbb{Z}^N)\Big)^{\frac{1}{p}}\le\left(\left(\frac{2\,m+1}{2\,m-1}\right)^N-1\right)^{\frac{1}{p}}+\left(\frac{2\,m+1}{2\,m-1}\right)^{\frac{N}{p}}\frac{\|\nabla u\|_{L^p(Q_{1/2})}}{\|u\|_{L^p(Q_{1/2})}}.$$

If we now take the limit as m goes to ∞, this yields

$$\Big(\lambda_p(\mathbb{R}^N\setminus\mathbb{Z}^N)\Big)^{\frac{1}{p}}\le\frac{\|\nabla u\|_{L^p(Q_{1/2})}}{\|u\|_{L^p(Q_{1/2})}}.$$

Since u is arbitrary, we get $\lambda_p(\mathbb{R}^N\setminus\mathbb{Z}^N)\le\Lambda_p(Q_{1/2}\setminus\{0\})$, as well. □

3.3. Proof of Theorem 1

By means of the analysis performed in the previous section, we can derive the main result of this chapter: Theorem 1.

PROOF OF THEOREM 1. We divide the proof in five parts.

Part 1: inequality for $q = p$. From [24, Theorem 5.4 & Remark 5.5], we already have

$$\lambda_p(\Omega)\ge\left(\frac{p-N}{p}\right)^p\frac{1}{r_\Omega^p}. \tag{3.3.1}$$

This is a plain consequence of the Hardy inequality contained in [51, Theorem 1.1]. Unfortunately, the constant obtained in this way has a sub-optimal behaviour as $p\searrow N$. In order to rectify this fact, we give a different proof, based on Taylor's idea of tiling the space with cubes "large enough". We will see that for $p > N$, the situation is simpler.

Without loss of generality, we can assume $r_\Omega = 1$. We fix $\varepsilon > 0$ and consider the tiling of $\mathbb{R}^N$ made by the cubes

$$\mathcal{Q}_{\mathbf{i},\varepsilon}:=Q_{1+\varepsilon}((2+2\,\varepsilon)\,\mathbf{i}),\qquad\text{for }\mathbf{i}\in\mathbb{Z}^N.$$

We also consider the set of indices

$$\mathbb{Z}^N_{\Omega,\varepsilon}=\Big\{\mathbf{i}=(i_1,\dots,i_N)\in\mathbb{Z}^N\,:\,\mathcal{Q}_{\mathbf{i},\varepsilon}\cap\Omega\neq\emptyset\Big\}.$$

Let $u\in C^\infty_0(\Omega)$, we observe that for every $\mathbf{i}\in\mathbb{Z}^N_{\Omega,\varepsilon}$ there must exist

$$x_{\mathbf{i},\varepsilon}\in B_{1+\varepsilon}((2+2\,\varepsilon)\,\mathbf{i})\setminus\Omega,$$

thanks to the fact that $r_\Omega = 1$: this implies that a ball of radius $1+\varepsilon$ cannot be entirely contained in Ω. Then, by the tiling property of the collection $\{\mathcal{Q}_{\mathbf{i},\varepsilon}\}_{\mathbf{i}\in\mathbb{Z}^N}$, the definition of $\Lambda_p(\mathcal{Q}_{\mathbf{i},\varepsilon}\setminus\{x_{\mathbf{i},\varepsilon}\})$ and Lemma 3.2.2 applied to each cube of this collection, we get

$$\begin{aligned}\int_\Omega|\nabla u|^p dx&=\sum_{\mathbf{i}\in\mathbb{Z}^N_{\Omega,\varepsilon}}\int_{\mathcal{Q}_{\mathbf{i},\varepsilon}}|\nabla u|^p dx\\&\ge\sum_{\mathbf{i}\in\mathbb{Z}^N_{\Omega,\varepsilon}}\Lambda_p(\mathcal{Q}_{\mathbf{i},\varepsilon}\setminus\{x_{\mathbf{i},\varepsilon}\})\,\|u\|^p_{L^p(\mathcal{Q}_{\mathbf{i},\varepsilon})}\ge\frac{\Lambda_p(B_1\setminus\{0\})}{\Big((1+\varepsilon)\,\sqrt{N}+1+\varepsilon\Big)^p}\,\|u\|^p_{L^p(\Omega)}.\end{aligned}$$

Observe that we used that $x_{\mathbf{i},\varepsilon} \in B_{1+\varepsilon}((2+2\,\varepsilon)\,\mathbf{i})$, to infer that

$$\max_{y\in\partial\mathcal{Q}_{\mathbf{i},\varepsilon}} |x_0-y| \le (1+\varepsilon)\,\sqrt{N}+1+\varepsilon.$$

By taking the limit as ε goes to 0 in the estimate above, we obtain

$$\|\nabla u\|^p_{L^p(\Omega)} \ge \frac{\Lambda_p(B_1\setminus\{0\})}{(\sqrt{N}+1)^p}\,\|u\|^p_{L^p(\Omega)}. \tag{3.3.2}$$

Finally, by joining the two estimates (3.3.2) and (3.3.1) we obtain

$$\lambda_p(\Omega) \ge \frac{\beta_{N,p}}{r_\Omega^p}, \qquad \text{with } \beta_{N,p} = \max\left\{\frac{\Lambda_p(B_1\setminus\{0\})}{(\sqrt{N}+1)^p}, \left(\frac{p-N}{p}\right)^p\right\} > 0,$$

as desired.

Part 2: inequality for $q=\infty$. We give the counterpart of (3.1.1), for the endpoint case $q=\infty$. The argument is extremely simple, based on the properties of the L^∞ norm and on a basic geometric fact. As before, up to scaling, we can assume that $r_\Omega = 1$. For every $\varepsilon>0$, we consider the family of balls

$$\Big\{B_{1+\varepsilon}(y)\,:\,y\in\partial\Omega\Big\}.$$

It is not difficult to see that this is a covering of Ω. Indeed, by definition of inradius, for every $x\in\Omega$, there exists $y\in\partial\Omega$ such that

$$|x-y| = d_\Omega(x) \le r_\Omega = 1.$$

In particular, this implies that $x\in B_{1+\varepsilon}(y)$. By arbitrariness of $x\in\Omega$, we get the claimed covering property.

We now take $u\in C_0^\infty(\Omega)$. Thus, this is a continuous compactly supported function. Hence, there exists $\overline{x}\in\Omega$ such that

$$|u(\overline{x})| = \|u\|_{L^\infty(\Omega)}.$$

Thanks to the previous discussion, there exists $\overline{y}\in\partial\Omega$ such that $\overline{x}\in B_{1+\varepsilon}(\overline{y})$. Thus, we obtain

$$\begin{aligned}\|u\|^p_{L^\infty(\Omega)} = |u(\overline{x})|^p = \|u\|^p_{L^\infty(B_{1+\varepsilon}(\overline{y}))}\\ \le \frac{1}{\Lambda_{p,\infty}(B_{1+\varepsilon}(\overline{y})\setminus\{\overline{y}\})}\int_\Omega |\nabla u|^p\,dx = \frac{(1+\varepsilon)^{p-N}}{\Lambda_{p,\infty}(B_1\setminus\{0\})}\int_\Omega |\nabla u|^p\,dx.\end{aligned}$$

In the last equality we used (3.2.1). By letting ε go to 0, we obtain (3.1.2).

Part 3: inequality for $p<q<\infty$. By a simple interpolation argument, we can now fill the gap and prove the result for the whole range $p\le q\le\infty$. Indeed, for every $u\in C_0^\infty(\Omega)\setminus\{0\}$ and $p<q<\infty$, we have

$$\|u\|^p_{L^q(\Omega)} \le \Big(\|u\|^p_{L^\infty(\Omega)}\Big)^{1-\frac{p}{q}}\,\Big(\|u\|^p_{L^p(\Omega)}\Big)^{\frac{p}{q}},$$

then

$$\frac{\|\nabla u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^q(\Omega)}} \ge \left(\frac{\|\nabla u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^\infty(\Omega)}}\right)^{1-\frac{p}{q}}\left(\frac{\|\nabla u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega)}}\right)^{\frac{p}{q}}.$$

This entails that

$$\lambda_{p,q}(\Omega) \ge \Big(\lambda_{p,\infty}(\Omega)\Big)^{1-\frac{p}{q}}\,\Big(\lambda_p(\Omega)\Big)^{\frac{p}{q}}.$$

Hence, the thesis follows by combining (3.1.1) and (3.1.2).

Part 4: asymptotics for $\Lambda_{p,\infty}(B_1\setminus\{0\})$. By Lemma 3.2.3, we know that

$$\Lambda_{p,\infty}(B_1\setminus\{0\}) \ge \frac{\mu_{p,\infty}(B_1)}{2^p}.$$

In turn, the right-hand side can be bounded from below thanks to (2.5.2). Thus, we obtain

$$\Lambda_{p,\infty}(B_1\setminus\{0\}) \ge \frac{N}{2^p}\,\left(\frac{\omega_N}{2^N}\right)^p\left(\frac{p-N}{p-1}\right)^{p-1}\omega_N,$$

which gives the claimed asymptotic behaviour from below, as p goes to N.

On the other hand, by testing the definition of $\Lambda_{p,\infty}(B_1 \setminus \{0\})$ with

$$u_\varepsilon(x) := \left(\varepsilon^2 + |x|^2\right)^{\frac{p-N}{2\,(p-1)}} - \varepsilon^{\frac{p-N}{p-1}}, \qquad \text{with } \varepsilon > 0, \tag{3.3.3}$$

we get

$$\Lambda_{p,\infty}(B_1 \setminus \{0\}) \leq \lim_{\varepsilon \searrow 0} \frac{\displaystyle\int_{B_1} |\nabla u_\varepsilon|^p\,dx}{\|u_\varepsilon\|^p_{L^\infty(B_1)}} = N\,\omega_N\,\left(\frac{p-N}{p-1}\right)^{p-1}.$$

This gives the desired asymptotic behaviour from above, as well. Finally, for the limit $p \nearrow \infty$ it is sufficient to use Lemma 3.2.3 with $K = B_1$ and $x_0 = 0$.

Part 5: asymptotics for $\beta_{N,p}$. We recall that this is given by

$$\beta_{N,p} = \max\left\{\frac{\Lambda_p(B_1 \setminus \{0\})}{(\sqrt{N}+1)^p}, \left(\frac{p-N}{p}\right)^p\right\}.$$

In particular, by Lemma 3.2.3 we have

$$\beta_{N,p} \geq \max\left\{\frac{\Lambda_{p,\infty}(B_1 \setminus \{0\})}{\omega_N\,(\sqrt{N}+1)^p}, \left(\frac{p-N}{p}\right)^p\right\}.$$

Thus, the information

$$0 < \liminf_{p \searrow N} \frac{\beta_{N,p}}{(p-N)^{p-1}},$$

comes from **Part 4** and the behaviour of $\Lambda_{p,\infty}(B_1 \setminus \{0\})$. The related upper bound can be proved as before, by using (3.3.3) as a test function and taking the limit as ε goes to 0.

Finally, as for the limit $p \nearrow \infty$, we observe that by its definition

$$\liminf_{p \nearrow \infty} \left(\beta_{N,p}\right)^{\frac{1}{p}} \geq \liminf_{p \nearrow \infty} \frac{p-N}{p} = 1.$$

On the other hand, by using (3.1.1) with $\Omega = B_1$, we get

$$\limsup_{p \nearrow \infty} \left(\beta_{N,p}\right)^{\frac{1}{p}} \leq \lim_{p \nearrow \infty} \left(\lambda_p(B_1)\right)^{\frac{1}{p}} = 1,$$

thanks to [58, Lemma 1.5]. This concludes the proof. □

Remark 3.3.1 (Asymptotic optimality). We recall that for *every* open set $\Omega \subseteq \mathbb{R}^N$, we have

$$\lim_{p\to\infty} \left(\lambda_p(\Omega)\right)^{\frac{1}{p}} = \lim_{p\to\infty} \left(\lambda_{p,\infty}(\Omega)\right)^{\frac{1}{p}} = \frac{1}{r_\Omega},$$

see [24, Corollary 6.1 and Corollary 6.4]. Thus, the estimates (3.1.1) and (3.1.2) becomes identities in the limit as p goes to ∞.

As for the case when p goes to N: from Lemma 3.2.4 we know that

$$\lambda_p(\mathbb{R}^N \setminus \mathbb{Z}^N) = \Lambda_p(Q_{1/2} \setminus \{0\}).$$

The last quantity can be estimated from above by using the test function (3.3.3), as before. This gives

$$\limsup_{p \searrow N} \frac{\lambda_p(\mathbb{R}^N \setminus \mathbb{Z}^N)}{(p-N)^{p-1}} < +\infty,$$

and thus the constant $\beta_{N,p}$ in (3.1.1) vanishes with the sharp decay rate. Finally, from both the definition of $\lambda_{p,\infty}$ and that of $p-$capacity, we have

$$\lambda_{p,\infty}(B_1) \leq \mathrm{cap}_p(\{0\}; B_1) = N\,\omega_N\,\left(\frac{p-N}{p-1}\right)^{p-1}.$$

Thus, also the constant in (3.1.2) has the sharp decay rate to 0.

CHAPTER 4

Inradius and Poincaré–Sobolev inequalities: a planar case

4.1. The Croke–Osserman–Taylor inequality for Poincaré–Sobolev constants

This chapter is taken from [**B2**]. Our ambient space is now $\mathbb{R}^2$ and we are going to extend the *Croke–Osserman–Taylor inequality* (1.3.1), which is known to be true for planar multiply connected open sets and $p = 2$ (see Section 1.3.1), to the case $p \neq 2$. This is the content of the main theorem of this chapter: Theorem 2.

The same argument even works for the more general case of $\lambda_{p,q}$ whenever $1 \le p < q < p^*$, where p^* indicates the critical Sobolev exponent. The restriction taken on the exponent q is not by chance: as discussed in Remark 4.3.1, the same results cannot hold in the sub–homogeneous case $q < p$.

In the proof of Theorem 2, we pay due attention to the dependence of the constant on the parameter q as $q \nearrow p^*$ in the case $p < 2$, and as $q \nearrow \infty$ in the case $p = 2$. It reads as follows

THEOREM 2. *Let $1 \le p < \infty$ and let $p \le q$ be such that*

$$\begin{cases} q < p^*, & \text{if } 1 \le p < 2, \\ q < \infty, & \text{if } p = 2, \\ q \le \infty, & \text{if } p > 2. \end{cases} \tag{4.1.1}$$

Then, there exists a constant $\Theta_{p,q} > 0$ such that for every $\Omega \subseteq \mathbb{R}^2$ open multiply connected set of order $k \in \mathbb{N} \setminus \{0\}$ with finite inradius r_Ω, we have

$$\lambda_{p,q}(\Omega) \ge \Theta_{p,q} \left(\frac{1}{\sqrt{k}\, r_\Omega} \right)^{p-2+\frac{2p}{q}}. \tag{4.1.2}$$

Moreover, the constant $\Theta_{p,q}$ has the following asymptotic behaviours:

- *for $1 \le p < 2$*
$$0 < \lim_{q \nearrow p^*} \Theta_{p,q} < +\infty;$$
- *for $p = 2$*
$$0 < \liminf_{q \nearrow \infty} \left(q\, \Theta_{2,q} \right) \le \limsup_{q \nearrow \infty} \left(q\, \Theta_{2,q} \right) < +\infty.$$

REMARK 4.1.1 (Asymptotic optimality). Let $1 \le p \le 2$ and let $p \le q$ satisfy (1.1.1). By proceeding as in [**15**, Theorem 1.2, point (2)], we can construct a sequence $\{\Omega_k\}_{k \in \mathbb{N}\setminus\{0\}} \subseteq \mathbb{R}^2$ of open sets such that Ω_k is multiply connected of order k

$$r_{\Omega_k} \le C \qquad \text{and} \qquad \limsup_{k \to \infty} k^{\frac{p-2}{2}+\frac{2p}{q}}\, \lambda_{p,q}(\Omega_k) < +\infty.$$

This shows that the lower bound (4.1.2) is sharp in its dependence on k, as k goes to ∞. For $p > 2$, we will see in the next section that this estimate can be considerably improved, by removing the dependence on k.

We also recall that for $1 \le p < 2$ we have

$$\lim_{q \nearrow p^*} \lambda_{p,q}(\Omega) = \lambda_{p,p^*}(\Omega),$$

and the latter is actually independent of the set Ω: it simply coincides with the sharp constant in the Sobolev inequality for the whole space $\mathbb{R}^2$ (see for example [**89**, Chapter I, Section 4.5]). The asymptotic behaviour of the constant $\Theta_{p,q}$ in (4.1.2) is perfectly consistent with this fact.

Finally, for $p=2$ we have that for a multiply connected planar set with finite inradius, it holds

$$\lim_{q\nearrow\infty} q\,\lambda_{2,q}(\Omega)=8\,\pi\,e,$$

see Corollary 4.4.2 below. Thus, here as well, the asymptotic behaviour of the constant $\Theta_{2,q}$ is consistent with this limit.

4.2. Three technical facts

We start by collecting three technical Lemmas that will be needed for the proof of Theorem 2. We recall the following geometric result due to Taylor (see [91, proof of Theorem 2]). In the form below, this can be found in [15, Lemma 2.1]. For every $\alpha\in\mathbb{R}$, we denote by

$$\lfloor\alpha\rfloor=\max\Big\{n\in\mathbb{Z}\,:\,\alpha\ge n\Big\},$$

its *integer part*. For every direction $\omega\in\mathbb{S}^{N-1}$, we also use the notation Π_ω for the orthogonal projection onto the space $\langle\omega\rangle^\perp:=\{x\in\mathbb{R}^N\,:\,\langle x,\omega\rangle=0\}$.

LEMMA 4.2.1 (Taylor's fatness Lemma). *Let $k\in\mathbb{N}\setminus\{0\}$ and let $\Omega\subseteq\mathbb{R}^2$ be an open multiply connected set of order k, with finite inradius. Let Q be an open square with side length $10\,(\lfloor\sqrt{k}\rfloor+1)\,r_\Omega$, whose sides are parallel to the coordinate axes. Then, there exists a compact set $\Sigma\subseteq\overline{Q}\setminus\Omega$ such that*

$$\max\Big\{\mathcal{H}^1(\Pi_{\mathbf{e}_1}(\Sigma)),\,\mathcal{H}^1(\Pi_{\mathbf{e}_2}(\Sigma))\Big\}\ge\frac{\sqrt{k}}{4}\,r_\Omega,$$

where $\mathbf{e}_1=(1,0)$ and $\mathbf{e}_2=(0,1)$.

We need also the following simple result.

LEMMA 4.2.2. *Let $(a,b)\subseteq\mathbb{R}$ and $a<x_0<b$. Then for every $p\ge1$ we have*

$$\mathrm{cap}_p(\{x_0\};(a,b))\ge\frac{2^p}{(b-a)^{p-1}}.$$

PROOF. Let $\psi\in C^\infty_0((a,b))$ such that $\psi(x_0)\ge1$, then

$$\int_a^b|\psi'|\,dx=\int_a^{x_0}|\psi'|\,dx+\int_{x_0}^b|\psi'|\,dx\ge|\psi(x_0)-\psi(a)|+|\psi(b)-\psi(x_0)|\ge2.$$

By Jensen's inequality, we obtain

$$\frac{1}{b-a}\int_a^b|\psi'|^p dx\ge\left(\frac{1}{b-a}\int_a^b|\psi'|\,dx\right)^p\ge\frac{2^p}{(b-a)^p}.$$

By recalling the Definition 2.2.1, the claimed inequality easily follows. □

As a last ingredient, we need a geometric lower bound for $\mathrm{cap}_p(\Sigma;B_r(x_0))$, in the plane. This is the content of the following result, which can be proved along the lines of [76, Chapter 13, Section 1.2, Proposition 1].

LEMMA 4.2.3 (Capacity and projections). *Let $\Sigma\Subset B_r(x_0)\subseteq\mathbb{R}^2$ be a compact set. Then, for every $1\le p<\infty$ and every $\omega\in\mathbb{S}^1$ it holds*

$$\mathrm{cap}_p(\Sigma;B_r(x_0))\ge\frac{2}{r^{p-1}}\,\mathcal{H}^1(\Pi_\omega(\Sigma)),$$

where, as above, Π_ω is the orthogonal projection onto $\langle\omega\rangle^\perp=\{x\in\mathbb{R}^2\,:\,\langle x,\omega\rangle=0\}$.

PROOF. It is not restrictive to suppose that $x_0=0$. We fix $\omega\in\mathbb{S}^1$ and choose $\omega^\perp\in\mathbb{S}^1$ to be orthogonal to it. We can also assume that $\mathcal{H}^1(\Pi_\omega(\Sigma))>0$, otherwise there is nothing to prove.

Fix $p\ge1$ and take any function $u\in C^\infty_0(B_r)$ such that $u\ge1_\Sigma$. Let Q be the square centered at the origin, with side length $2\,r$ and whose sides are parallel to ω and $\omega^\perp$. By Fubini's Theorem and writing every $x\in Q$ as follows

$$x=z_1\,\omega+z_2\,\omega^\perp,\qquad\text{for }(z_1,z_2)\in(-r,r)\times(-r,r),$$

we have

$$\int_{B_r} |\nabla u|^p\,dx = \int_Q |\nabla u|^p\,dx \geq \int_Q |\partial_\omega u|^p\,dx = \int_{-r}^{r}\int_{-r}^{r} |\partial_{z_1} u(z_1,z_2)|^p\,dz_1 dz_2$$
$$\geq \int_{\Pi_\omega(\Sigma)} \|\partial_{z_1} u(\cdot,z_2)\|^p_{L^p((-r,r))}\,dz_2.$$

By using that for every $z_2 \in \Pi_\omega(\Sigma)$, the function $z_1 \mapsto u(z_1,z_2)$ is admissible for the definition of the $p-$capacity of a point relative to the interval $(-r,r)$, from Lemma 4.2.2, we get

$$\int_{\Pi_\omega(\Sigma)} \|\partial_{z_1} u(\cdot,z_2)\|^p_{L^p((-r,r))}\,dz_2 \geq \frac{2^p}{(2\,r)^{p-1}}\mathcal{H}^1(\Pi_\omega(\Sigma)) = \frac{2}{r^{p-1}}\mathcal{H}^1(\Pi_\omega(\Sigma)).$$

This conclues the proof. □

4.3. Proof of Theorem 2

We are ready to adapt Taylor's proof and prove the announced lower bound for multiply connected open sets in the plane. We can cover the case of any generalized principal frequency with the same effort.

Proof of Theorem 2. We first prove the inequality (4.1.2). Then by using the explicit expression of the constant $\Theta_{p,q}$, we will prove the second part of the statement.

Part 1: inequality. Up to a scaling, we can suppose that $r_\Omega = 1$. We take $\delta = \lfloor\sqrt{k}\rfloor + 1 \in \mathbb{N}$ and consider the family of squares

$$Q_{ij} := Q_{5\delta}(10\,\delta\,i, 10\,\delta\,j), \qquad \text{for every } (i,j) \in \mathbb{Z}^2.$$

We introduce the set of indices

$$\mathbb{Z}^2_\Omega = \{(i,j) \in \mathbb{Z}^2 \,:\, Q_{ij} \cap \Omega \neq \emptyset\},$$

and for every $(i,j) \in \mathbb{Z}^2_\Omega$ we take $\Sigma_{ij} \subseteq \overline{Q_{ij}} \setminus \Omega$ to be the compact set provided by Lemma 4.2.1. Let $u \in C^\infty_0(\Omega)$, then by Theorem 2.6.1 with $d = 5\,\delta$ and $D = 2\,d = 10\,\delta$, we have

$$\int_\Omega |\nabla u|^p dx = \sum_{(i,j)\in\mathbb{Z}^2} \int_{Q_{ij}} |\nabla u|^p dx \geq \frac{\mathscr{C}^p}{(5\,\delta)^{\frac{2\,p}{q}}} \sum_{(i,j)\in\mathbb{Z}^2_\Omega} \mathrm{cap}_p(\Sigma_{ij};\widetilde{B}_{ij})\,\|u\|^p_{L^q(Q_{ij})},$$

where we denoted with $\widetilde{B}_{ij}$ the ball with radius $D = 2\,d = 10\,\delta$, concentric with Q_{ij}. The key point now is to give a uniform bound from below on the capacity of the sets Σ_{ij}: by relying on Lemma 4.2.3 and Lemma 4.2.1, we can infer

$$\mathrm{cap}_p(\Sigma_{ij};\widetilde{B}_{ij}) \geq \frac{2}{(10\,\delta)^{p-1}}\,\max\Big\{\mathcal{H}^1(\Pi_{\mathbf{e}_1}(\Sigma_{ij})), \mathcal{H}^1(\Pi_{\mathbf{e}_2}(\Sigma_{ij}))\Big\} \geq \frac{\sqrt{k}}{2\cdot(10\,\delta)^{p-1}}.$$

By collecting these estimates, we get

$$\int_\Omega |\nabla u|^p dx \geq \frac{\mathscr{C}^p\,\sqrt{k}}{2^p\cdot(5\,\delta)^{p-1+\frac{2\,p}{q}}} \sum_{(i,j)\in\mathbb{Z}^2_\Omega} \|u\|^p_{L^q(Q_{ij})}.$$

and $\mathscr{C}$ is the same constant as in Theorem 2.6.1. Since $k \geq 1$, we have

$$\frac{\sqrt{k}}{\delta^{p-1+\frac{2\,p}{q}}} = \frac{\sqrt{k}}{(\lfloor\sqrt{k}\rfloor+1)^{p-1+\frac{2\,p}{q}}} \geq \frac{\sqrt{k}}{(2\,\sqrt{k})^{p-1+\frac{2\,p}{q}}} = \frac{1}{2^{p-1+\frac{2\,p}{q}}}\,\left(\frac{1}{\sqrt{k}}\right)^{p-2+\frac{2\,p}{q}}.$$

In order to conclude the proof, we are only left to observe that $q \geq p$, thus the power $\tau \mapsto \tau^{p/q}$ is sub-additive. This implies that[1]

$$\sum_{(i,j)\in\mathbb{Z}^2_\Omega} \|u\|^p_{L^q(Q_{ij})} \geq \left(\sum_{(i,j)\in\mathbb{Z}^2_\Omega} \|u\|^q_{L^q(Q_{ij})}\right)^{\frac{p}{q}} = \|u\|^p_{L^q(\Omega)}, \tag{4.3.1}$$

Then, we get

$$\int_\Omega |\nabla u|^p dx \geq \frac{\mathscr{C}^p}{2^p \cdot 10^{p-1+\frac{2\,p}{q}}} \left(\frac{1}{\sqrt{k}}\right)^{p-2+\frac{2\,p}{q}} \|u\|^p_{L^q(\Omega)},$$

and (4.1.2) follows by definition of $\lambda_{p,q}(\Omega)$.

Part 2: asymptotics for $\Theta_{p,q}$. In **Part 1** we have obtained the following constant

$$\Theta_{p,q} = \frac{\mathscr{C}^p}{2^p \cdot 10^{p-1+\frac{2\,p}{q}}},$$

with $\mathscr{C}$ as in Theorem 2.6.1. Thus, in order to understand the asymptotic behaviour of $\Theta_{p,q}$ as q goes to p^* or to ∞, it is sufficient to focus on the same issue for the constant $\mathscr{C}^p$. By Remark 2.6.2 and taking $N = 2$, $d/D = 1/2$, this is given by

$$\mathscr{C} = \frac{1}{\alpha_{2,p}} \left(\frac{1}{2}\right)^{\frac{8}{p}+\frac{2}{q}} \left[\pi^{\frac{1}{q}} + \frac{4\,\pi^{\frac{1}{p}}}{1-\dfrac{\sqrt{2}}{2}} \left(\frac{1}{\mu_{p,q}(B_1)}\right)^{\frac{1}{p}}\right]^{-1}.$$

For $1 \leq p < 2$, we have that (see [23, Lemma 6.2])

$$\lim_{q\nearrow p^*} \mu_{p,q}(B_1) = \mu_{p,p^*}(B_1) > 0.$$

For a lower bound on the last constant, see for example [34, Proposition 3.1].

The case $p = 2$ is slightly more delicate. In this case, we have

$$\lim_{q\nearrow\infty} \mu_{2,q}(B_1) = 0.$$

More precisely, one can prove that

$$4\,\pi\,e \leq \liminf_{q\to\infty} \big(q\,\mu_{2,q}(B_1)\big) \leq \limsup_{q\to\infty} \big(q\,\mu_{2,q}(B_1)\big) \leq 8\,\pi\,e,$$

see [23, Proposition 6.5]. In light of the expression of $\mathscr{C}$, this is enough to deduce the asymptotic behaviour of $\Theta_{2,q}$ as q goes to ∞. □

Remark 4.3.1 (The case $1 \leq q < p$). We observe that the proof of Theorem 2 does not for work for $q < p$: the main obstruction is the sub–additivity inequality (4.3.1). This is not a mere technicality: in the case $q < p$, *inequality* (4.1.2) *cannot hold*. Indeed, it already fails for convex sets. The typical counter-example is given by the infinite strip $\Omega = \mathbb{R} \times (-1, 1)$, for which we have

$$r_\Omega = 1 \qquad \text{and} \qquad \lambda_{p,q}(\Omega) = 0, \text{ for } 1 \leq q < p.$$

We refer to [21, Proposition 6.1] for more details.

[1] In the limit case $q = \infty$, we just use that

$$\sum_{(i,j)\in\mathbb{Z}^2_\Omega} \|u\|^p_{L^\infty(Q_{ij})} \geq \|u\|^p_{L^\infty(\Omega)}.$$

4.4. Embeddings for homogeneous spaces

In this section, we briefly discuss some consequences of Theorem 2 for the embedding properties of the homogeneous Sobolev space $\mathscr{D}^{1,p}_0$. We recall that the latter is the completion of $C^\infty_0(\Omega)$, with respect to the norm

$$\varphi \mapsto \|\nabla\varphi\|_{L^p(\Omega)}, \qquad \text{for every } \varphi \in C^\infty_0(\Omega).$$

COROLLARY 4.4.1. *Let $k \in \mathbb{N}\setminus\{0\}$ and let $\Omega \subseteq \mathbb{R}^2$ be an open multiply connected set of order k. Let $1 \le p \le 2$ and let $p \le q$ satisfy (4.1.1). Then we have*

$$\mathscr{D}^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega) \qquad \Longleftrightarrow \qquad r_\Omega < +\infty.$$

PROOF. The validity of the continuous embedding $\mathscr{D}^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ is equivalent to the fact that $\lambda_{p,q}(\Omega) > 0$. Thus, the implication $\Longleftarrow$ is a direct consequence of (4.1.2). For the converse implication, it is sufficient to observe that for every disk $B_r(x_0) \subseteq \Omega$, we have

$$\lambda_{p,q}(\Omega) \le \lambda_{p,q}(B_r(x_0)) = \frac{\lambda_{p,q}(B_1)}{r^{p-2+\frac{2\,p}{q}}}.$$

By taking the supremum over the disks contained in Ω, we get

$$\lambda_{p,q}(\Omega) \le \frac{\lambda_{p,q}(B_1)}{r_\Omega^{p-2+\frac{2\,p}{q}}},$$

and thus the conclusion. □

We now focus on the case $p = 2$. In this case, there is no limit Sobolev exponent, i.e. the exponent q may become arbitrary large, but it can not attain ∞. In general, the limit embedding for $\mathscr{D}^{1,2}_0(\Omega)$ is on the scale of Orlicz spaces of exponential type. For example, for open planar sets with finite area, the Moser–Trudinger inequality asserts that

$$\sup_{u\in C^\infty_0(\Omega)} \left\{ \int_\Omega \big(\exp(4\,\pi\,u^2) - 1\big)\,dx \,:\, \int_\Omega |\nabla u|^2\,dx = 1 \right\} < +\infty,$$

see [80, Theorem 1]. In [75, Theorem 1.2], the authors proved that for an open simply connected set $\Omega \subseteq \mathbb{R}^2$, we have

$$\sup_{u\in C^\infty_0(\Omega)} \left\{ \int_\Omega \big(\exp(4\,\pi\,u^2) - 1\big)\,dx \,:\, \int_\Omega |\nabla u|^2\,dx = 1 \right\} < +\infty \qquad \Longleftrightarrow \qquad r_\Omega < +\infty.$$

In the next result, we extend this characterization to planar sets with non-trivial topology.

COROLLARY 4.4.2 (Moser–Trudinger). *Let $k \in \mathbb{N}\setminus\{0\}$ and let $\Omega \subseteq \mathbb{R}^2$ be an open multiply connected set of order k. Then, we have*

$$\sup_{u\in C^\infty_0(\Omega)} \left\{ \int_\Omega \big(\exp(4\,\pi\,u^2) - 1\big)\,dx \,:\, \int_\Omega |\nabla u|^2\,dx = 1 \right\} < +\infty \qquad \Longleftrightarrow \qquad r_\Omega < +\infty.$$

Moreover, if $r_\Omega < +\infty$ we have

$$\lim_{q\nearrow\infty} q\,\lambda_{2,q}(\Omega) = 8\,\pi\,e.$$

PROOF. According to [8, Theorem 2.2], for an open connected set $\Omega \subseteq \mathbb{R}^2$ we have that

$$\sup_{u\in C^\infty_0(\Omega)} \left\{ \int_\Omega \big(\exp(4\,\pi\,u^2) - 1\big)\,dx \,:\, \int_\Omega |\nabla u|^2\,dx = 1 \right\} < +\infty \qquad \Longleftrightarrow \qquad \lambda(\Omega) < +\infty.$$

If Ω is multiply connected of order k, the last condition is equivalent to $r_\Omega < +\infty$, thanks to Corollary 4.4.1 with $p = q = 2$.

The second statement now follows by reproducing verbatim the argument of [88, Lemma 2.2]: the first part of the proof assures that we have the Moser–Trudinger inequality at our disposal, which is sufficient to reproduce the argument in [88]. □

4.5. Sharpness: the case of Buser inequality and an open problem

A further consequence of Theorem 2 is the following Buser inequality for planar $k-$connected open sets. We report below its statement without proof, having been anticipated in the Introduction, and refer the reader to Section 1.3.3 for a more detailed discussion.

THEOREM 3. *For every $\Omega \subseteq \mathbb{R}^2$ open multiply connected set of order $k \in \mathbb{N}\setminus\{0\}$, we have*

$$\lambda(\Omega) \le \left(\frac{j_{0,1}}{\Theta_{1,1}}\right)^2 k \left(h(\Omega)\right)^2,$$

where $\Theta_{1,1}$ is the same constant as in Theorem 2 and $j_{0,1}$ is the first zero of the Bessel function of the first kind J_0 (see for example [55, page 11] *for an approximate value).*

In this section, we analyse the asymptotic behaviour of the sharp constant in the previous Buser inequality with respect to the dependence on the parameter k as $k \nearrow \infty$. Next, we compare its asymptotic behaviour with that of the constant appearing in the rightmost term of (1.3.5), and present a related open problem.

For every $k \in \mathbb{N} \setminus \{0\}$, we now define the sharp constant for the Buser inequality proved in Theorem 3, i.e. we set

$$\mathcal{C}_{\rm B}(k) := \sup\left\{\frac{\lambda(\Omega)}{\left(h(\Omega)\right)^2}\ :\ \Omega \subseteq \mathbb{R}^2 \text{ multiply connected of order } k \text{ with } r_\Omega < +\infty\right\}.$$

Its precise value is known for $k = 1$ and $k = 2$ only, see the recent paper [32]. In light of Theorem 3, we know that such a constant is finite for every k and grows at most like k, as this diverges to ∞. We are going to show that this growth is "essentially" sharp. This is the main result of this section.

PROPOSITION 4.5.1. *The quantity $k \mapsto \mathcal{C}_{\rm B}(k)$ is monotone non-decreasing. Moreover, for every $0 < \alpha < 1$, we have*

$$\lim_{k\to\infty} \frac{\mathcal{C}_{\rm B}(k)}{k^\alpha} = +\infty.$$

PROOF. For the monotonicity part, it is sufficient to proceed as follows: if $\Omega \subseteq \mathbb{R}^2$ is admissible for $\mathcal{C}_{\rm B}(k)$, then the set $\widetilde{\Omega} = \Omega \setminus \{x_0\}$ with $x_0 \in \Omega$ is admissible for $\mathcal{C}_{\rm B}(k+1)$ and we have

$$\frac{\lambda(\Omega)}{\left(h(\Omega)\right)^2} = \frac{\lambda(\widetilde{\Omega})}{\left(h(\widetilde{\Omega})\right)^2}.$$

This is due to the fact that points in dimension $N = 2$ have zero $p-$capacity, for every $1 \le p \le 2$.

For the second part of the statement, we are going to exhibit a sequence of open sets $\{\Omega_k\}_{k\ge2}$ such that each Ω_k is multiply connected of order $k+1$, it has finite inradius and

$$\lim_{k\to\infty} \frac{\lambda(\Omega_k)}{k^\alpha \left(h(\Omega_k)\right)^2} = +\infty, \qquad \text{for every } 0 < \alpha < 1. \tag{4.5.1}$$

At this aim, we will slightly modify the construction of [15, Theorem 1.2, point (2)]. We will produce a sequence of enlarging periodically perforated sets, such that the radius of the perforation shrinks "not too fast" as the sets grow.

Let $k \ge 2$ be a natural number and let $\varepsilon_k = k^{-\beta}$ for some fixed $\beta > 1/2$, we indicate by

$$\mathring{Q}_k := \Big([0,1]\times[0,1]\Big) \setminus B_{\varepsilon_k}\left(\frac{1}{2},\frac{1}{2}\right). \tag{4.5.2}$$

The parameter ε_k will be the shrinking radius of the perforation. If we set

$$\mathcal{I}_k = \Big\{\mathbf{i} = (i_1, i_2) \in \mathbb{N}^2\ :\ \max\{i_1,\, i_2\} \le \lfloor\sqrt{k}\rfloor - 1\Big\},$$

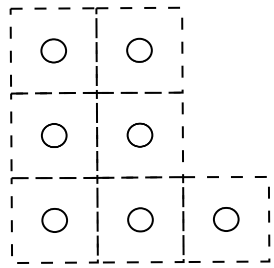

FIGURE 1. The set Ω_k for $k = 7$

we define

$$\mathcal{Q}_k = \bigcup_{\mathbf{i} \in \mathcal{I}_k} (\mathring{Q}_k + \mathbf{i}).$$

Observe that this is a square with side length $\lfloor \sqrt{k} \rfloor$, containing $(\lfloor \sqrt{k} \rfloor)^2$ equally spaced circular holes of radius ε_k. To this set, whenever $\sqrt{k} \notin \mathbb{N}$, we attach the perforated horizontal strip

$$\mathcal{S}_k = \bigcup_{j=0}^{k - \lfloor \sqrt{k} \rfloor^2 - 1} (\mathring{Q}_k - \mathbf{e}_2 + j\, \mathbf{e}_1),$$

where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. At last, we define

$$\Omega_k := \mathrm{int}(\mathcal{Q}_k \cup \mathcal{S}_k),$$

i.e. the interior of this union (see Figure 1). By construction, this is an open multiply connected set of order $k + 1$. Also observe that the inradius r_{Ω_k} is uniformly bounded, with respect to k.

Estimate for $\lambda(\Omega_k)$. For every $u \in C_0^\infty(\Omega_k)$, by applying Theorem 2.6.1 with $d = 1/2$ and $D = 1$, we get

$$\begin{aligned}
\int_{\Omega_k} |\nabla u|^2\, dx &= \int_{\mathcal{Q}_k} |\nabla u|^2\, dx + \int_{\mathcal{S}_k} |\nabla u|^2\, dx \\
&= \sum_{\mathbf{i} \in \mathcal{I}_k} \int_{\mathring{Q}_k + \mathbf{i}} |\nabla u|^2\, dx + \sum_{j=0}^{k - \lfloor \sqrt{k} \rfloor^2 - 1} \int_{\mathring{Q}_k - \mathbf{e}_2 + j\, \mathbf{e}_1} |\nabla u|^2\, dx \\
&\geq C\, \mathrm{cap}_2(B_{\varepsilon_k}; B_1) \left(\sum_{\mathbf{i} \in \mathcal{I}_k} \int_{\mathring{Q}_k + \mathbf{i}} |u|^2\, dx + \sum_{j=0}^{k - \lfloor \sqrt{k} \rfloor^2 - 1} \int_{\mathring{Q}_k - \mathbf{e}_2 + j\, \mathbf{e}_1} |u|^2\, dx \right) \\
&= C\, \mathrm{cap}_2(B_{\varepsilon_k}; B_1) \int_{\Omega_k} |u|^2\, dx.
\end{aligned}$$

By arbitrariness of u and by using [76, formula (2.2.14)] for the relative capacity of a disk, we can infer existence of a constant $C_0 > 0$ such that

$$\lambda(\Omega_k) \geq \frac{C_0}{|\log \varepsilon_k|} = \frac{C_0}{\beta\, (\log k)}, \tag{4.5.3}$$

since $\varepsilon_k = k^{-\beta}$. We now also prove a similar upper bound for $\lambda(\Omega_k)$. We proceed similarly as in the proof of Lemma 3.2.4 above. We first observe that

$$\lambda(\Omega_k) \leq \lambda(\mathrm{int}(\mathcal{Q}_k)).$$

We take the following Lipschitz function defined on $\mathring{Q}_k$ by

$$u_k(x) = \left(\log \left(\frac{1}{2\, \varepsilon_k} \right) \right)^{-1} \min \left\{ \log \left(\frac{1}{2\, \varepsilon_k} \right), \log \left(\frac{1}{\varepsilon_k} \sqrt{\left(x_1 - \frac{1}{2} \right)^2 + \left(x_2 - \frac{1}{2} \right)^2} \right) \right\}.$$

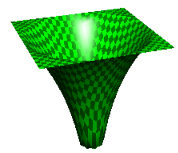

FIGURE 2. The graph of the funnel–type function u_k.

Observe that this identically vanishes on $\partial B_{\varepsilon_k}(1/2, 1/2)$ and coincides with 1 on $((0,1)\times(0,1))\setminus B_{1/2}(1/2, 1/2)$, see Figure 2. Then, we periodically repeat it, i.e. we consider

$$U_k(x) = \sum_{\mathbf{i}\in\mathcal{I}_k} u_k(x + \mathbf{i}).$$

Finally, we take η_k a $1-$Lipschitz cut-off function such that

$$0 \le \eta_k \le 1, \qquad \eta_k \equiv 1 \text{ on } \widetilde{\mathcal{Q}_k}, \qquad \eta_k \equiv 0 \text{ on } \partial\mathcal{Q}_k,$$

where[2]

$$\widetilde{\mathcal{Q}_k} = \bigcup_{\mathbf{i}\in\widetilde{\mathcal{I}}_k} (\mathring{Q}_k + \mathbf{i}), \qquad \text{with } \widetilde{\mathcal{I}}_k = \left\{\mathbf{i} = (i_1, i_2) \in \mathbb{N}^2 \,:\, 1 \le \max\{i_1,\, i_2\} \le \lfloor\sqrt{k}\rfloor - 2\right\},$$

see Figure 3. It is easy to see that $\varphi = \eta_k\, U_k \in W^{1,2}_0(\mathrm{int}(\mathcal{Q}_k))$. Thus, by definition of λ, we have

$$\sqrt{\lambda(\Omega_k)} \le \sqrt{\lambda(\mathrm{int}(\mathcal{Q}_k))} \le \frac{\left(\displaystyle\int_{\mathcal{Q}_k} |\nabla\eta_k|^2\,|U_k|^2\,dx\right)^{\frac12} + \left(\displaystyle\int_{\mathcal{Q}_k} |\nabla U_k|^2\,|\eta_k|^2\,dx\right)^{\frac12}}{\left(\displaystyle\int_{\mathcal{Q}_k} |\eta_k\, U_k|^2\,dx\right)^{\frac12}} \tag{4.5.4}$$

By using the properties of both U_k and η_k, we have

$$\int_{\mathcal{Q}_k} |\eta_k\, U_k|^2\,dx \ge \int_{\widetilde{\mathcal{Q}}_k} |U_k|^2\,dx = \left(\lfloor\sqrt{k}\rfloor - 2\right)^2 \int_{\mathring{Q}_k} |u_k|^2\,dx$$

and

$$\int_{\mathcal{Q}_k} |\nabla U_k|^2\,|\eta_k|^2\,dx \le \int_{\mathcal{Q}_k} |\nabla U_k|^2\,dx = \left(\lfloor\sqrt{k}\rfloor\right)^2 \int_{\mathring{Q}_k} |\nabla u_k|^2\,dx.$$

We recall that $\mathring{Q}_k$ has been defined in (4.5.2). Similarly, by recalling that η_k is constant on $\widetilde{\mathcal{Q}_k}$, we have

$$\begin{aligned}\int_{\mathcal{Q}_k} |\nabla\eta_k|^2\,|U_k|^2\,dx &= \int_{\mathcal{Q}_k\setminus\widetilde{\mathcal{Q}_k}} |\nabla\eta_k|^2\,|U_k|^2\,dx \le \int_{\mathcal{Q}_k\setminus\widetilde{\mathcal{Q}_k}} |U_k|^2\,dx \\ &= \left[\left(\lfloor\sqrt{k}\rfloor\right)^2 - \left(\lfloor\sqrt{k}\rfloor - 2\right)^2\right] \int_{\mathring{Q}_k} |u_k|^2\,dx.\end{aligned}$$

[2]In what follows, we suppose that $k \ge 9$. In view of our scopes, this is not restrictive.

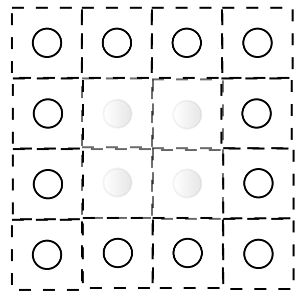

FIGURE 3. The set $\widetilde{\mathcal{Q}}_k$ for $k=16$: it is made of the "internal" perforated squares in grey.

We still need to compute the $W^{1,2}$ norm of u_k. From its definition, there exists a constant $C_1>0$ such that we have

$$\begin{aligned}\int_{\mathring{Q}_k} |u_k|^2\,dx &\geq \frac{1}{\big(\log(2\,\varepsilon_k)\big)^2}\int_{B_{1/2}\setminus B_{\varepsilon_k}} \left(\log\left(\frac{|x|}{\varepsilon_k}\right)\right)^2 dx\\ &= \frac{\varepsilon_k^2}{\big(\log(2\,\varepsilon_k)\big)^2}\int_{B_{1/2\varepsilon_k}\setminus B_1} \log^2|y|\,dy = \frac{2\,\pi\,\varepsilon_k^2}{\big(\log(2\,\varepsilon_k)\big)^2}\int_1^{\frac{1}{2\,\varepsilon_k}} \varrho\,\log^2\varrho\,d\varrho \geq C_1,\end{aligned}$$

for k large enough. As for its gradient, we have

$$\begin{aligned}\int_{\mathring{Q}_k} |\nabla u_k|^2\,dx &= \frac{1}{\big(\log(2\,\varepsilon_k)\big)^2}\int_{B_{1/2}\setminus B_{\varepsilon_k}} \frac{1}{|x|^2}\,dx\\ &= \frac{2\,\pi}{\big(\log(2\,\varepsilon_k)\big)^2}\int_{\varepsilon_k}^{\frac{1}{2}} \frac{1}{\varrho}\,d\varrho\\ &= \frac{2\,\pi}{\big(\log(2\,\varepsilon_k)\big)^2}\,|\log(2\,\varepsilon_k)| = \frac{2\,\pi}{|\log(2\,\varepsilon_k)|} \leq \frac{4\,\pi}{\beta\,(\log k)}.\end{aligned}$$

By spending all these informations in (4.5.4), we get

$$\sqrt{\lambda(\Omega_k)} \leq \left[\frac{\big(\lfloor\sqrt{k}\rfloor\big)^2}{\big(\lfloor\sqrt{k}\rfloor-2\big)^2}-1\right]^{\frac{1}{2}} + \frac{\lfloor\sqrt{k}\rfloor}{\lfloor\sqrt{k}\rfloor-2}\sqrt{\frac{4\,\pi}{\beta\,C_1}\,\frac{1}{(\log k)}}. \tag{4.5.5}$$

By using that

$$\lim_{k\to\infty}\frac{\lfloor\sqrt{k}\rfloor}{\lfloor\sqrt{k}\rfloor-2}=1 \qquad \text{and} \qquad \left[\frac{\big(\lfloor\sqrt{k}\rfloor\big)^2}{\big(\lfloor\sqrt{k}\rfloor-2\big)^2}-1\right]^{\frac{1}{2}} \leq \left(\frac{64}{\lfloor\sqrt{k}\rfloor}\right)^{\frac{1}{2}} \leq \frac{8\cdot\sqrt{2}}{\sqrt[4]{k}},\ \text{for } k\geq 9,$$

from (4.5.5) we finally get that there exists a constant $C_2>0$ such that

$$\lambda(\Omega_k) \leq \frac{C_2}{(\log k)}, \tag{4.5.6}$$

for k sufficiently large.

Estimate for $h(\Omega_k)$. By a standard approximation argument (see for example [83, Proposition 3.3]), we can use $\mathcal{Q}_k$ as an admissible set in the definition of $h(\Omega_k)$. This gives

$$h(\Omega_k) \leq \frac{\mathcal{H}^{N-1}(\partial\mathcal{Q}_k)}{|\mathcal{Q}_k|} = \frac{4\,\lfloor\sqrt{k}\rfloor + 2\,\pi\,\big(\lfloor\sqrt{k}\rfloor\big)^2\,\varepsilon_k}{\big(\lfloor\sqrt{k}\rfloor\big)^2(1-\pi\,\varepsilon_k^2)}.$$

Since by definition we have $\varepsilon_k^2 = k^{-2\beta} = o(1/k)$ (recall that $\beta > 1/2$), there exists a constant $C_3 > 0$ such that

$$h(\Omega_k) \leq \frac{C_3}{\sqrt{k}}, \tag{4.5.7}$$

for k large enough. Moreover, for any $k \geq 2$, we have that

$$\Omega_k \subseteq \mathbb{R} \times \big(-1, \lfloor\sqrt{k}\rfloor\big).$$

Thus, by monotonicity with respect to set inclusion and the scaling property of the Cheeger constant, we get

$$h(\Omega_k) \geq \frac{1}{1+\lfloor\sqrt{k}\rfloor}\, h(\mathbb{R}\times(0,1)) = \frac{1}{1+\lfloor\sqrt{k}\rfloor}. \tag{4.5.8}$$

In the last equality, we used [**62**, Theorem 3.1].

Conclusion. By gathering together the estimates (4.5.3), (4.5.6), (4.5.7) and (4.5.8), we finally obtain

$$\frac{1}{C}\,\frac{k}{\log k} \leq \frac{\lambda(\Omega_k)}{\Big(h(\Omega_k)\Big)^2} \leq C\,\frac{k}{\log k}, \qquad \text{for } k \text{ large enough.}$$

This is enough to establish (4.5.1) and conclude the proof. □

As the reader may easily realize, the previous perforated set does not permit to show that

$$\mathcal{C}_{\mathrm{B}}(k) \sim k, \qquad \text{for } k \nearrow \infty.$$

Such an example may suggest that the sharp growth of $\mathcal{C}_{\mathrm{B}}(k)$ could be $k/\log k$, as k goes to ∞. In other words, the estimate of Theorem 3 might perhaps be improved by a logarithmic factor. We leave the following open problem, that we think to be quite interesting.

Open problem. Prove or disprove that

$$\mathcal{C}_{\mathrm{B}}(k) \sim \frac{k}{\log k} \qquad \text{for } k \nearrow \infty.$$

CHAPTER 5

Capacitary inradius and Poincaré–Sobolev inequalities

5.1. The case $1 \leq p \leq N$

The main theorem of this chapter is a Maz'ya–type characterization for the validity of the continuous embedding between the spaces

$$\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

within the range $1 \leq p \leq N$, when Ω is a general open subset of $\mathbb{R}^N$. In other words, we are going to exhibit a necessary and sufficient condition for the L^p-Poincaré inequality to hold in Ω, or equivalently for the positivity of $\lambda_p(\Omega)$. We will also cover the case of the Poincaré–Sobolev embedding constants $\lambda_{p,q}(\Omega)$, extending the previous characterization to the validity of the continuous embedding between the spaces

$$\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

within the range of parameters $1 \leq p \leq N$ and $q \geq p$ satisfying (strictly) the *subcriticality condition* (1.1.1). These results are taken from [**B1**]

To be more precise, we will prove a two–sided estimate for $\lambda_p(\Omega)$ of a general open set Ω in $\mathbb{R}^N$ in terms of negative powers of its capacitary inradius $R_{p,\gamma}(\Omega)$ (see Definition 1.4.1), for all values of $0 < \gamma < 1$ and for all $1 \leq p \leq N$. The main novelty of our result is to cover the whole range of $0 < \gamma < 1$, thus extending to the case $p \neq 2$ a result due to Maz'ya and Shubin, [**78**, Theorem 1.1]. It reads as follows

THEOREM 4. *Let $1 \leq p \leq N$, $0 < \gamma < 1$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set. Then we have*

$$\sigma_{N,p}\,\gamma\,\left(\frac{1}{R_{p,\gamma}(\Omega)}\right)^p \leq \lambda_p(\Omega) \leq C_{N,p,\gamma}\,\left(\frac{1}{R_{p,\gamma}(\Omega)}\right)^p,$$

with the constant $C_{N,p,\gamma}$ which diverges to $+\infty$, as γ goes to 1. In particular, we have

$$\lambda_p(\Omega) > 0 \qquad \Longleftrightarrow \qquad R_{p,\gamma}(\Omega) < +\infty,$$

and the last condition does not depend on $0 < \gamma < 1$.

5.2. Analysis of a Poincaré–type constant in a ball

The following result will be expedient in order to get the upper bound of Theorem 4. The main point is the identity (5.2.2) below.

LEMMA 5.2.1. *Let $N \geq 2$ and $1 < p \leq N$. For $0 < r_1 < r_2 < R$, we set*

$$S_{r_1,r_2} := B_{r_2} \setminus \overline{B_{r_1}} = \Big\{x \in \mathbb{R}^N \,:\, r_1 < |x| < r_2\Big\}.$$

Let V be the unique minimiser of the following problem

$$\min_{\varphi \in W_0^{1,p}(B_R)} \left\{\frac{1}{p}\int_{B_R} |\nabla\varphi|^p\,dx - \int_{S_{r_1,r_2}} \varphi\,dx\right\}.$$

Then V is a $C^1(\overline{B_R})$ radially symmetric non-increasing function and it satisfies

$$\int_{B_R} \langle|\nabla V|^{p-2}\,\nabla V, \nabla\varphi\rangle\,dx = \int_{S_{r_1,r_2}} \varphi\,dx, \qquad \textit{for every } \varphi \in W_0^{1,p}(B_R). \tag{5.2.1}$$

Moreover, we have

$$\int_{B_R} |\nabla V|^p\,dx = \int_{S_{r_1,r_2}} \left(\frac{|x|}{N}\left(1-\left(\frac{r_1}{|x|}\right)^N\right)\right)^{\frac{p}{p-1}} dx + |S_{r_1,r_2}|^{\frac{p}{p-1}} \left(\frac{1}{\mathrm{cap}_p(\overline{B_{r_2}};B_R)}\right)^{\frac{1}{p-1}}. \tag{5.2.2}$$

Proof. Existence of a minimiser follows from the Direct Method in the Calculus of Variations. Uniqueness is a consequence of the strict convexity of the functional which is minimised. Finally, we can observe that (5.2.1) is the Euler-Lagrange equation of this minimization problem, thus V satisfies it just by minimality. We can also infer that $V \in C^{1,\alpha}(\overline{B_R})$ for some $0<\alpha<1$, thanks to the classical regularity result [70, Theorem 1].

The radial symmetry of V follows from its uniqueness and the fact that the data of the problem are rotationally invariant. Thus, we must have

$$V(x) = v(|x|), \qquad \text{for } x \in B_R,$$

where v is a function of one variable. We want to prove that v is non-increasing: at this aim, we set

$$\widetilde{v}(t) = \int_t^R |v'(\tau)|\,d\tau, \qquad \text{for } t \in (0,R).$$

Thus, by definition $\widetilde{v}$ is non-increasing. Moreover, we have

$$\widetilde{v}'(t) = -|v'(t)|, \qquad \text{for } t \in (0,R),$$

and

$$\widetilde{v}(t) = \int_t^R |v'(\tau)|\,d\tau \geq \left|\int_t^R v'(\tau)\,d\tau\right| = |v(t)| \geq v(t).$$

These facts show that if we set $\widetilde{V}(x) = \widetilde{v}(|x|)$, then

$$\frac{1}{p}\int_{B_R} |\nabla \widetilde{V}|^p\,dx - \int_{S_{r_1,r_2}} \widetilde{V}\,dx \leq \frac{1}{p}\int_{B_R} |\nabla V|^p\,dx - \int_{S_{r_1,r_2}} V\,dx.$$

By minimality of V, we must have $V = \widetilde{V}$ and thus the claimed monotonicity follows.

We now need to prove formula (5.2.2). We observe at first that by testing (5.2.1) with $\varphi = V$, we obtain

$$\int_{B_R} |\nabla V|^p\,dx = \int_{S_{r_1,r_2}} V\,dx. \tag{5.2.3}$$

Still from (5.2.1), we get in particular

$$\int_{B_R} \langle |\nabla V|^{p-2}\,\nabla V, \nabla\varphi\rangle\,dx = 0, \qquad \text{for every } \varphi \in W^{1,p}_0(B_{r_1}).$$

Thus, the function V is weakly $p-$harmonic in the ball B_{r_1}. Moreover, thanks to its radial symmetry, it is constant on ∂B_{r_1}. By uniqueness of the Dirichlet problem for the $p-$Laplacian, we obtain that V must be constant on the whole B_{r_1}. Thus, we obtain

$$\int_{B_R} |\nabla V|^p\,dx = \int_{B_R\setminus B_{r_1}} |\nabla V|^p\,dx. \tag{5.2.4}$$

In turn, we split the last integral in two parts

$$\int_{B_R\setminus B_{r_1}} |\nabla V|^p\,dx = \int_{S_{r_1,r_2}} |\nabla V|^p\,dx + \int_{B_R\setminus B_{r_2}} |\nabla V|^p\,dx. \tag{5.2.5}$$

In order to determine the first integral on the right-hand side, we take $h \in C^\infty_0((r_1,r_2))$ and use (5.2.1) with test function $\varphi(x) = h(|x|)$. By using spherical coordinates and recalling the notation $V(x) = v(|x|)$, we get

$$\int_{r_1}^{r_2} |v'|^{p-2}\,v'\,h'\,\varrho^{N-1}\,d\varrho = \int_{r_1}^{r_2} h\,\varrho^{N-1}\,d\varrho.$$

We integrate by parts the last term, so to obtain

$$\int_{r_1}^{r_2} \left[|v'|^{p-2}\, v'\, \varrho^{N-1} + \frac{\varrho^N}{N}\right] h'\, d\varrho = 0, \qquad \text{for every } h \in C_0^\infty((r_1,r_2)).$$

This implies that there exists a constant C such that

$$|v'|^{p-2}\, v'\, \varrho^{N-1} + \frac{\varrho^N}{N} = C, \qquad \text{on } (r_1,r_2).$$

By recalling that $v' \le 0$, from this identity we get

$$(-v'(\varrho))^{p-1} = \frac{\varrho}{N} - \frac{C}{\varrho^{N-1}}, \qquad \text{for } \varrho \in (r_1,r_2).$$

The constant C can be determined, by recalling that v is C^1 and that v is constant on $[0,r_1]$, from the above discussion. Thus, it must result

$$0 = (-v'(r_1))^{p-1} = \frac{r_1}{N} - \frac{C}{r_1^{N-1}} \qquad \text{that is} \qquad C = \frac{r_1^N}{N}.$$

In conclusion, we get that

$$|\nabla V(x)|^p = (-v'(|x|))^p = \left(\frac{|x|}{N} - \frac{r_1}{N}\left(\frac{r_1}{|x|}\right)^{N-1}\right)^{\frac{p}{p-1}} = \left(\frac{|x|}{N}\left(1 - \left(\frac{r_1}{|x|}\right)^N\right)\right)^{\frac{p}{p-1}}.$$

By integrating it over S_{r_1,r_2}, we get

$$\int_{S_{r_1,r_2}} |\nabla V|^p\, dx = \int_{S_{r_1,r_2}} \left(\frac{|x|}{N}\left(1 - \left(\frac{r_1}{|x|}\right)^N\right)\right)^{\frac{p}{p-1}} dx. \tag{5.2.6}$$

We still need to determine the second integral in (5.2.5). To this aim, we take for every $n \in \mathbb{N}$ sufficiently large, the following function

$$\varphi_n = 1_{S_{r_1,r_2}} * \rho_n,$$

where $\{\rho_n\}_{n\in\mathbb{N}\setminus\{0\}}$ is the usual family of radial smoothing kernels. By using (5.2.1) with $\varphi = V\,\varphi_n$, we get

$$\int_{B_R} |\nabla V|^p\, \varphi_n\, dx + \int_{B_R} \langle |\nabla V|^{p-2}\, \nabla V, \nabla \varphi_n\rangle\, V\, dx = \int_{S_{r_1,r_2}} V\, \varphi_n\, dx.$$

By using the properties of convolutions, the regularity of V and the radial symmetry of both V and φ_n, the limit as n goes to ∞ yields

$$\int_{S_{r_1,r_2}} |\nabla V|^p\, dx + v(r_2)\,(-v'(r_2))^{p-1}\,\mathcal{H}^{N-1}(\partial B_{r_2}) = \int_{S_{r_1,r_2}} V\, dx.$$

Observe that we also used that $v'(r_1) = 0$, as explained above. By using (5.2.3), (5.2.4) and (5.2.5), this in turn implies

$$\cancel{\int_{S_{r_1,r_2}} |\nabla V|^p\, dx} + v(r_2)\,(-v'(r_2))^{p-1}\,\mathcal{H}^{N-1}(\partial B_{r_2}) = \cancel{\int_{S_{r_1,r_2}} |\nabla V|^p\, dx} + \int_{B_R\setminus B_{r_2}} |\nabla V|^p\, dx.$$

That is

$$\int_{B_R\setminus B_{r_2}} |\nabla V|^p\, dx = v(r_2)\,(-v'(r_2))^{p-1}\,\mathcal{H}^{N-1}(\partial B_{r_2}).$$

The term $v'(r_2)$ can be computed, thanks to the fact that v is C^1 and to the exact determination of v' on the interval (r_1,r_2). We must have

$$(-v'(r_2))^{p-1} = \frac{r_2}{N} - \frac{r_1}{N}\left(\frac{r_1}{r_2}\right)^{N-1} = \frac{r_2}{N}\left(1 - \left(\frac{r_1}{r_2}\right)^N\right).$$

Thus, up to now we have obtained

$$\int_{B_R\setminus B_{r_2}} |\nabla V|^p\,dx = v(r_2)\,\frac{r_2}{N}\left(1-\left(\frac{r_1}{r_2}\right)^N\right)\mathcal{H}^{N-1}(\partial B_{r_2}) = v(r_2)\,|S_{r_1,r_2}|. \tag{5.2.7}$$

We still need to determine $v(r_2)$. To this aim, it is sufficient to observe that, thanks to both the monotonicity and the $p-$harmonicity of V, the function

$$W = \min\left\{\frac{V}{v(r_2)},\,1\right\},$$

is a weakly $p-$harmonic function in $B_R\setminus\overline{B_{r_2}}$, vanishing on ∂B_R and is equal to 1 on $\overline{B_{r_2}}$. Thus, it must be the $p-$capacitary potential of $\overline{B_{r_2}}$, relative to B_R, i.e. we have[1]

$$\int_{B_R\setminus B_{r_2}} |\nabla W|^p = \mathrm{cap}_p(\overline{B_{r_2}};B_R).$$

This is the same as

$$\int_{B_R\setminus B_{r_2}} |\nabla V|^p\,dx = v(r_2)^p\,\mathrm{cap}_p(\overline{B_{r_2}};B_R).$$

By comparing the previous two expressions for $\int_{B_R\setminus B_{r_2}} |\nabla V|^p\,dx$, we get

$$v(r_2)\,|S_{r_1,r_2}| = v(r_2)^p\,\mathrm{cap}_p(\overline{B_{r_2}};B_R).$$

This finally gives

$$v(r_2) = \left(\frac{|S_{r_1,r_2}|}{\mathrm{cap}_p(\overline{B_{r_2}};B_R)}\right)^{\frac{1}{p-1}}.$$

By using this expression in (5.2.7), we end up with

$$\int_{B_R\setminus B_{r_2}} |\nabla V|^p\,dx = |S_{r_1,r_2}|^{\frac{p}{p-1}}\left(\frac{1}{\mathrm{cap}_p(\overline{B_{r_2}};B_R)}\right)^{\frac{1}{p-1}}. \tag{5.2.8}$$

By using (5.2.6) and (5.2.8) in (5.2.5) and recalling (5.2.4), we finally obtain the desired formula. □

As a consequence of the properties of the function V, we can estimate a suitable Poincaré–type constant. This is the main result of this section, contained in the following

Proposition 5.2.2. *Let $N\ge 2$ and $1<p\le N$. With the same notation of Proposition 5.2.1, we have*

$$\sup_{\varphi\in W^{1,p}_0(B_R)\setminus\{0\}} \frac{\left(\displaystyle\int_{S_{r_1,r_2}} |\varphi|\,dx\right)^p}{\displaystyle\int_{B_R} |\nabla\varphi|^p\,dx} = \left(\int_{B_R} |\nabla V|^p\,dx\right)^{p-1}. \tag{5.2.9}$$

---

[1] It is not difficult to see that there exists a sequence $\{W_n\}_{n\in\mathbb{N}}\subseteq \mathrm{Lip}_0(B_R)$ such that $0\le W_n\le 1$, $W_n\equiv 1$ on $\overline{B_{r_2}}$ and

$$\lim_{n\to\infty}\|\nabla W_n\|_{L^p(B_R)} = \|\nabla W\|_{L^p(B_R)}.$$

Thus, in light of (2.2.1), we have

$$\mathrm{cap}_p(\overline{B_{r_2}};B_R)\le \lim_{n\to\infty}\int_{B_R}|\nabla W_n|^p\,dx = \int_{B_R}|\nabla W|^p\,dx = \int_{B_R\setminus B_{r_2}}|\nabla W|^p\,dx.$$

On the other hand, for every $\varphi\in \mathrm{Lip}_0(B_R)$ such that $0\le\varphi\le 1$ and $\varphi\equiv 1$ on $\overline{B_{r_2}}$, we have

$$\int_{B_R}|\nabla\varphi|^p\,dx = \int_{B_R\setminus B_{r_2}}|\nabla\varphi|^p\,dx \ge \int_{B_R\setminus B_{r_2}}|\nabla W|^p\,dx + p\int_{B_R\setminus B_{r_2}}\langle|\nabla W|^{p-2}\,\nabla W,\nabla\varphi-\nabla W\rangle\,dx.$$

By using (5.2.1), the fact that $\varphi - W\in W^{1,p}_0(B_R)$ and $\varphi-W\equiv 0$ on $\overline{B_{r_2}}$, we get that the rightmost integral vanishes. By arbitrariness of φ and using again (2.2.1), we obtain the claimed identity.

In particular, for every $\varphi \in W^{1,p}_0(B_R)$ we get

$$\left(\fint_{S_{r_1,r_2}} |\varphi|\, dx \right)^p \le \left[\frac{|S_{r_1,r_2}|}{(\mathcal{H}^{N-1}(\partial B_{r_2}))^{\frac{p}{p-1}}} + \left(\frac{1}{\mathrm{cap}_p(\overline{B_{r_2}}; B_R)} \right)^{\frac{1}{p-1}} \right]^{p-1} \int_{B_R} |\nabla \varphi|^p\, dx. \tag{5.2.10}$$

Proof. By using V as a test function, we have

$$\sup_{\varphi \in W^{1,p}_0(B_R)\setminus\{0\}} \frac{\left(\displaystyle\int_{S_{r_1,r_2}} |\varphi|\, dx \right)^p}{\displaystyle\int_{B_R} |\nabla \varphi|^p\, dx} \ge \frac{\left(\displaystyle\int_{S_{r_1,r_2}} V\, dx \right)^p}{\displaystyle\int_{B_R} |\nabla V|^p\, dx} = \left(\int_{B_R} |\nabla V|^p\, dx \right)^{p-1}.$$

We also used the identity (5.2.3). On the other hand, by taking $\varphi \in W^{1,p}_0(B_R)$ and testing the equation (5.2.1) with $|\varphi| \in W^{1,p}_0(B_R)$, we get

$$\int_{S_{r_1,r_2}} |\varphi|\, dx = \int_{B_R} \langle |\nabla V|^{p-2}\, \nabla V, \nabla |\varphi| \rangle\, dx \le \left(\int_{B_R} |\nabla V|^p\, dx \right)^{\frac{p-1}{p}} \left(\int_{B_R} |\nabla \varphi|^p\, dx \right)^{\frac{1}{p}}.$$

The desired conclusion (5.2.9) now follows, thanks to the arbitrariness of $\varphi \in W^{1,p}_0(B_R)$.

The estimate (5.2.10) will simply follow from (5.2.9), once we recall the expression (5.2.2) for the L^p norm of ∇V. We estimate from above the latter: observe that the function

$$t \mapsto \left(\frac{t}{N} \right)^{\frac{p}{p-1}} \left(1 - \left(\frac{r_1}{t} \right)^N \right)^{\frac{p}{p-1}},$$

is monotone increasing. Thus, the estimate (5.2.2) implies

$$\int_{B_R} |\nabla V|^p\, dx \le \left(\frac{r_2}{N} \left(1 - \left(\frac{r_1}{r_2} \right)^N \right) \right)^{\frac{p}{p-1}} |S_{r_1,r_2}| + \left(\frac{|S_{r_1,r_2}|^p}{\mathrm{cap}_p(\overline{B_{r_2}}; B_R)} \right)^{\frac{1}{p-1}}.$$

We then observe that

$$\left(\frac{r_2}{N} \left(1 - \left(\frac{r_1}{r_2} \right)^N \right) \right)^{\frac{p}{p-1}} = \left(\frac{|B_{r_2}|}{\mathcal{H}^{N-1}(\partial B_{r_2})} \frac{|S_{r_1,r_2}|}{|B_{r_2}|} \right)^{\frac{p}{p-1}}.$$

Thus, we get

$$\frac{\left(\displaystyle\int_{B_R} |\nabla V|^p\, dx \right)^{p-1}}{|S_{r_1,r_2}|^p} \le \left[\frac{|S_{r_1,r_2}|}{(\mathcal{H}^{N-1}(\partial B_{r_2}))^{\frac{p}{p-1}}} + \left(\frac{1}{\mathrm{cap}_p(\overline{B_{r_2}}; B_R)} \right)^{\frac{1}{p-1}} \right]^{p-1}.$$

This concludes the proof. □

Remark 5.2.3. We observe that, by using the geometric estimate of Lemma 2.3.2 in (5.2.10), one can also get the slightly rougher (but definitely handier) estimate

$$\left(\fint_{S_{r_1,r_2}} |\varphi|\, dx \right)^p \le \frac{1}{\mathrm{cap}_p(\overline{B_{r_1}}; B_R)} \int_{B_R} |\nabla \varphi|^p\, dx, \qquad \text{for every } \varphi \in W^{1,p}_0(B_R). \tag{5.2.11}$$

By a limiting argument, we can cover the case $p = 1$, as well. In this case, the sharp constant has a simpler and nicer expression.

Corollary 5.2.4. *Let $N \ge 2$. With the same notation of Proposition 5.2.1, we have*

$$\sup_{\varphi \in W^{1,1}_0(B_R)\setminus\{0\}} \frac{\displaystyle\int_{S_{r_1,r_2}} |\varphi|\, dx}{\displaystyle\int_{B_R} |\nabla \varphi|\, dx} = \frac{|S_{r_1,r_2}|}{\mathrm{cap}_1(\overline{B_{r_2}}; B_R)}. \tag{5.2.12}$$

PROOF. As above, we take for every $n \in \mathbb{N}$ the following function

$$\varphi_n = 1_{B_{r_2}} * \rho_n,$$

where $\{\rho_n\}_{n\in\mathbb{N}\setminus\{0\}}$ is the usual family of radial smoothing kernels. Since $B_{r_2} \Subset B_R$, for n sufficiently large we have that $\varphi_n \in C_0^\infty(B_R)$. By the properties of convolutions and by [4, page 121], we have

$$\sup_{\varphi\in W_0^{1,1}(B_R)\setminus\{0\}} \frac{\displaystyle\int_{S_{r_1,r_2}} |\varphi|\,dx}{\displaystyle\int_{B_R} |\nabla\varphi|\,dx} \geq \lim_{n\to\infty} \frac{\displaystyle\int_{S_{r_1,r_2}} |\varphi_n|\,dx}{\displaystyle\int_{B_R} |\nabla\varphi_n|\,dx} = \frac{|S_{r_1,r_2}|}{\mathcal{H}^{N-1}(\partial B_{r_2})} = \frac{|S_{r_1,r_2}|}{\mathrm{cap}_1(\overline{B_{r_2}};B_R)}.$$

In order to prove the reverse inequality, we first observe that

$$\lim_{p\searrow 1} \frac{1}{\mathrm{cap}_p(\overline{B_{r_2}};B_R)} = \frac{1}{\mathrm{cap}_1(\overline{B_{r_2}};B_R)}.$$

This simply follows by recalling the expressions (2.3.2) and (2.3.1). We also claim that

$$\lim_{p\searrow 1} \left[\frac{|S_{r_1,r_2}|}{\mathcal{H}^{N-1}(\partial B_{r_2})} \left(\frac{\mathrm{cap}_p(\overline{B_{r_2}};B_R)}{\mathcal{H}^{N-1}(\partial B_{r_2})}\right)^{\frac{1}{p-1}} + 1\right]^{p-1} = 1.$$

Indeed, we have

$$\frac{|S_{r_1,r_2}|}{\mathcal{H}^{N-1}(\partial B_{r_2})} \left(\frac{\mathrm{cap}_p(\overline{B_{r_2}};B_R)}{\mathcal{H}^{N-1}(\partial B_{r_2})}\right)^{\frac{1}{p-1}} = \frac{r_2^N - r_1^N}{N\,r_2^N} \left(\frac{N-p}{p-1}\right) \frac{1}{\left(1-\left(\frac{r_2}{R}\right)^{\frac{N-p}{p-1}}\right)}.$$

Thus, for p converging to 1

$$\begin{aligned}(p-1)\,\log&\left(\frac{|S_{r_1,r_2}|}{\mathcal{H}^{N-1}(\partial B_{r_2})} \left(\frac{\mathrm{cap}_p(\overline{B_{r_2}};B_R)}{\mathcal{H}^{N-1}(\partial B_{r_2})}\right)^{\frac{1}{p-1}} + 1\right)\\ &= (p-1)\,\log\left(\frac{r_2^N - r_1^N}{N\,r_2^N} \left(\frac{N-p}{p-1}\right) \frac{1}{\left(1-\left(\frac{r_2}{R}\right)^{\frac{N-p}{p-1}}\right)} + 1\right) \sim (p-1)\,\log\left(\frac{N-p}{p-1}\right).\end{aligned}$$

Since the last quantity is infinitesimal, we get the claimed limit.

We now take $\varphi \in C_0^\infty(B_R)\setminus\{0\}$. By taking the limit as p goes to 1 in (5.2.10) and using the previous results, we get

$$\frac{\displaystyle\int_{S_{r_1,r_2}} |\varphi|\,dx}{\displaystyle\int_{B_R} |\nabla\varphi|\,dx} \leq \frac{|S_{r_1,r_2}|}{\mathrm{cap}_1(\overline{B_{r_2}};B_R)}.$$

By arbitrariness of φ and by density of $C_0^\infty(B_R)$ in $W_0^{1,1}(B_R)$, we get the conclusion. □

REMARK 5.2.5 (A Cheeger–type constant). It is not difficult to see that

$$\inf_{\varphi\in C_0^\infty(B_R)\setminus\{0\}} \frac{\displaystyle\int_{B_R} |\nabla\varphi|\,dx}{\displaystyle\int_{S_{r_1,r_2}} |\varphi|\,dx} = \inf\left\{\frac{\mathcal{H}^{N-1}(\partial E)}{|E\cap S_{r_1,r_2}|}\,:\,E \Subset B_R \text{ has smooth boundary}\right\},$$

see for example [76, Theorem 2.1.3]. We tacitly assume that the last ratio is $+\infty$, for those sets E such that $|E\cap S_{r_1,r_2}|=0$. In light of Corollary 5.2.4, we thus have

$$\inf\left\{\frac{\mathcal{H}^{N-1}(\partial E)}{|E\cap S_{r_1,r_2}|}\,:\,E\Subset B_R \text{ has smooth boundary}\right\}=\frac{\mathrm{cap}_1(\overline{B_{r_2}};B_R)}{|S_{r_1,r_2}|}=\frac{\mathcal{H}^{N-1}(\partial B_{r_2})}{|S_{r_1,r_2}|}.$$

In particular, we have that $E=B_{r_2}$ is an optimal shape, for this Cheeger–type constant.

5.3. Proof of Theorem 4: lower bound

We split the proof of Theorem 4, proving separately lower and upper bound. We start with the lower bound. This can be derived by using a tiling argument of the ambient space in combination with a Maz'ya–Poincaré type inequality, Theorem 2.6.1.

More precisely, by assuming the finiteness of the capacitary inradius of an open set $\Omega\subseteq\mathbb{R}^N$, it is possible to tile the whole space with translated copies of a cube, having side–length large enough to contain a "fat" compact set outside Ω. In light of the definition of capacitary inradius and Proposition 2.2.4, the "fatness" condition is quantified in terms of the parameter γ and the relative capacity of a ball having radius half the side–legth of the cube, with respect to a concentric ball having doubled radius. Then, by applying Theorem 2.6.1 we eventually obtain the claimed lower bound.

Proof of Theorem 4: lower bound. We first observe that if $R_{p,\gamma}(\Omega)=+\infty$, then there is nothing to prove. Thus, let us assume that $R_{p,\gamma}(\Omega)<+\infty$. Let $r>R_{p,\gamma}(\Omega)$ and let $u\in C^\infty_0(\Omega)$, extended by 0 to the complement $\mathbb{R}^N\setminus\Omega$. For every $x_0\in\mathbb{R}^N$, by definition of capacitary inradius we have

$$\mathrm{cap}_p(\overline{B_r(x_0)}\setminus\Omega;B_{2r}(x_0))>\gamma\,\mathrm{cap}_p(\overline{B_r(x_0)};B_{2r}(x_0))=\gamma\,\mathrm{cap}_p(\overline{B_1};B_2)\,r^{N-p}. \tag{5.3.1}$$

The last identity simply follows from the scaling properties of the relative $p-$capacity (see Remark 2.3.1). We now consider the cube $Q_r(x_0)$ concentric with $B_r(x_0)$. We observe that u is a C^∞ function on $\overline{Q_r(x_0)}$, which vanishes on the compact subset $\overline{B_r(x_0)}\setminus\Omega\subseteq\overline{Q_r(x_0)}$. Thus, we can use the Maz'ya-Poincaré inequality of [76, Theorem 14.1.2] (more precisely, we use its slight variant of Theorem 2.6.1) to infer that

$$\frac{\mathscr{C}}{r^{\frac{N}{p}}}\,\mathrm{cap}_p\left(\overline{B_r(x_0)}\setminus\Omega;B_{2\sqrt{N}r}(x_0)\right)^{\frac{1}{p}}\,\|u\|_{L^p(Q_r(x_0))}\le\|\nabla u\|_{L^p(Q_r(x_0))},$$

where $\mathscr{C}=\mathscr{C}(N,p)$ is the same constant as in Theorem 2.6.1. Furthermore, by applying (2.2.7) we get

$$\frac{\mathscr{C}}{\left(\dfrac{2\sqrt{N}}{\lambda_p(B_1)^{\frac{1}{p}}}+1\right)^{\frac{1}{p}}}\,\frac{1}{r^{\frac{N}{p}}}\,\mathrm{cap}_p\left(\overline{B_r(x_0)}\setminus\Omega;B_{2r}(x_0)\right)^{\frac{1}{p}}\,\|u\|_{L^p(Q_r(x_0))}\le\|\nabla u\|_{L^p(Q_r(x_0))}.$$

We can further apply (5.3.1), in order to estimate from below the left-hand side. By raising to the power p, this gives

$$\frac{\mathscr{C}^p\,\mathrm{cap}_p(\overline{B_1};B_2)}{\dfrac{2\sqrt{N}}{\lambda_p(B_1)^{\frac{1}{p}}}+1}\,\frac{\gamma}{r^p}\,\|u\|^p_{L^p(Q_r(x_0))}\le\|\nabla u\|^p_{L^p(Q_r(x_0))}. \tag{5.3.2}$$

By using this estimate for a family of disjoint cubes having inradius r and tiling the whole space, summing up we get

$$\frac{\mathscr{C}^p\,\mathrm{cap}_p(\overline{B_1};B_2)}{\dfrac{2\sqrt{N}}{\lambda_p(B_1)^{\frac{1}{p}}}+1}\,\frac{\gamma}{r^p}\,\|u\|^p_{L^p(\Omega)}\le\|\nabla u\|^p_{L^p(\Omega)}.$$

This concludes the proof by arbitrariness of u. □

REMARK 5.3.1. By inspecting the proof above, we see that we get the following constant

$$\sigma_{N,p} = \frac{\mathscr{C}^p \operatorname{cap}_p(\overline{B_1}; B_2)}{\left(\dfrac{2\sqrt{N}}{\lambda_p(B_1)^{\frac{1}{p}}} + 1\right)}.$$

A possible value for the constant $\mathscr{C} = \mathscr{C}(N,p) > 0$ can be found in Remark 2.6.2, by taking $q = p$ there and $D/d = 2\sqrt{N}$.

5.4. Proof of Theorem 4: upper bound

Armed with the results contained in Section 5.2, we can establish the upper bound of Theorem 4, as well. Contrary to what one may think, in spite of the variational nature of λ_p in terms of an infimum problem, the proof of the upper bound of Theorem 4 is more involved than that of the lower bound. This is due to the request to cover the *whole range* of the attainable values of the parameter $0 < \gamma < 1$. This remarkably improves [76, Theorem 15.4.1], where the author proved a similar result assuming a restriction on the attainable values of the negligibility parameter γ, i.e.

$$0 < \gamma \le \gamma_{N,p} < 1$$

with $\gamma_{N,p} = 4^{-pN}$. For this reason, our result is more in the spirit of Maz'ya and Shubin's one, [78, Theorem 1.1].

The proof is based on testing the definition of λ_p with a sequence of admissible functions, defined *ad hoc* for the definition of capacitary inradius. This, in combination with (5.2.11) and (5.2.12), leads to the claimed upper bound, respectively for $1 < p \le N$ and for $p = 1$.

PROOF OF THEOREM 4 : UPPER BOUND. Let $\gamma \in (0,1)$ be fixed, we take a ball $B_r(x_0)$ such that

$$\operatorname{cap}_p\left(\overline{B_r(x_0)} \setminus \Omega; B_{2r}(x_0)\right) \le \gamma \operatorname{cap}_p\left(\overline{B_r(x_0)}; B_{2r}(x_0)\right). \tag{5.4.1}$$

We will show that for every such r, we can bound

$$\lambda_p(\Omega) \le \frac{C_{N,p,\gamma}}{r^p}. \tag{5.4.2}$$

By taking the supremum over the admissible r, we will eventually get the result. In particular, if $R_{p,\gamma}(\Omega) = +\infty$ the previous upper bound will prove that $\lambda_p(\Omega) = 0$.

We set for simplicity $F = \overline{B_r(x_0)} \setminus \Omega$. We preliminary observe that if $F = \emptyset$, we have $\overline{B_r(x_0)} \setminus \Omega = \emptyset$, that is $\overline{B_r(x_0)} \subseteq \Omega$. From the monotonicity of λ_p with respect to set inclusion, we then obtain our claim (5.4.2) for every constant $C_{N,p,\gamma} \ge \lambda_p(B_1)$.

Let F be nonempty. For every $\delta > 0$, we take $\varphi_\delta \in \operatorname{Lip}_0(B_{2r}(x_0))$ such that

$$0 \le \varphi_\delta \le 1, \qquad \varphi_\delta = 1 \text{ on } F,$$

$$\int_{B_{2r}(x_0)} |\nabla \varphi_\delta|^p \, dx \le \delta \operatorname{cap}_p\left(\overline{B_r(x_0)}; B_{2r}(x_0)\right) + \operatorname{cap}_p\left(F; B_{2r}(x_0)\right). \tag{5.4.3}$$

Such a function exists, by recalling (2.2.1). Without loss of generality, we can suppose that x_0 coincides with the origin.

We observe at first that by density we have

$$\lambda_p(\Omega) = \inf_{\varphi \in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla \varphi|^p \, dx \,:\, \|\varphi\|_{L^p(\Omega)} = 1 \right\} = \inf_{\varphi \in W_0^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla \varphi|^p \, dx \,:\, \|\varphi\|_{L^p(\Omega)} = 1 \right\}.$$

We fix $0 < \varepsilon < 1/2$ and take a Lipschitz cut-off function η defined on $\overline{B_r}$, such that

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ on } B_{(1-\varepsilon)\,r}, \quad \eta \equiv 0 \text{ on } \partial B_r, \quad \|\nabla \eta\|_{L^\infty} = \frac{1}{\varepsilon\, r}.$$

We use the test function $\psi = (1-\varphi_\delta)\,\eta/\|(1-\varphi_\delta)\,\eta\|_{L^p(\Omega)}$ in the definition of $\lambda_p(\Omega)$. Observe that this is a feasible test function: indeed, by construction we have that ψ is a Lipschitz function on the whole $\mathbb{R}^N$. Moreover, we have that

$$\psi \equiv 0 \qquad \text{on } \partial(B_r\cap\Omega) \subseteq (\partial B_r \cap \overline{\Omega}) \cup (\partial\Omega \cap \overline{B_r}).$$

More precisely, we have

$$(1-\varphi_\delta) \equiv 0 \qquad \text{on } F = \overline{B_r}\setminus\Omega \supseteq \partial\Omega\cap\overline{B_r},$$

and

$$\eta \equiv 0 \qquad \text{on } \partial B_r \supseteq \partial B_r \cap \overline{\Omega}.$$

Thus, by [18, Theorem 9.17 & Remark 19] we get that

$$\psi = (1-\varphi_\delta)\,\eta \in W^{1,p}_0(B_r\cap\Omega) \subseteq W^{1,p}_0(\Omega).$$

This gives

$$\lambda_p(\Omega) \le \frac{\displaystyle\int_\Omega |(1-\varphi_\delta)\,\nabla\eta - \eta\,\nabla\varphi_\delta|^p\,dx}{\displaystyle\int_\Omega (1-\varphi_\delta)^p\,\eta^p\,dx} \le 2^{p-1}\,\frac{\displaystyle\int_\Omega (1-\varphi_\delta)^p\,|\nabla\eta|^p\,dx + \int_\Omega \eta^p\,|\nabla\varphi_\delta|^p\,dx}{\displaystyle\int_\Omega (1-\varphi_\delta)^p\,\eta^p\,dx}.$$

By using the properties of η, we get

$$\lambda_p(\Omega)\int_{B_{(1-\varepsilon)\,r}} (1-\varphi_\delta)^p\,dx \le 2^{p-1}\left[\frac{1}{\varepsilon^p\,r^p}\int_{B_r\setminus B_{(1-\varepsilon)\,r}} (1-\varphi_\delta)^p\,dx + \int_{B_r} |\nabla\varphi_\delta|^p\,dx\right].$$

We also use (5.4.3) and (5.4.1) on the right-hand side: this leads to

$$\lambda_p(\Omega)\int_{B_{(1-\varepsilon)\,r}} (1-\varphi_\delta)^p\,dx \le 2^{p-1}\left[\frac{\omega_N\,r^N}{\varepsilon^p\,r^p}\,(1-(1-\varepsilon)^N) + (\delta+\gamma)\,\mathrm{cap}_p\Big(\overline{B_r};B_{2\,r}\Big)\right].$$

We also observe that by convexity of the map $\tau\mapsto\tau^N$ we have

$$t^N \ge 1 + N\,(t-1), \qquad \text{for } t\ge 0,$$

and thus

$$(1-\varepsilon)^N \ge 1 - N\,\varepsilon \qquad \text{that is} \qquad 1-(1-\varepsilon)^N \le N\,\varepsilon.$$

This leads us to

$$\lambda_p(\Omega)\int_{B_{(1-\varepsilon)\,r}} (1-\varphi_\delta)^p\,dx \le \frac{2^{p-1}}{r^p}\,r^N\left[\frac{N\,\omega_N}{\varepsilon^{p-1}} + (\delta+\gamma)\,\mathrm{cap}_p\Big(\overline{B_1};B_2\Big)\right]. \tag{5.4.4}$$

Observe that we also used the scaling properties of the $p-$capacity. We now wish to give a lower bound for the leftmost integral. To this aim, we set

$$r_1 = (1-\ell)\,r, \qquad r_2 = (1-\varepsilon)\,r,$$

with $1>\ell>\varepsilon>0$ and $0<\varepsilon<1/2$. By defining the spherical shell

$$S_{r_1,r_2} := B_{r_2}\setminus\overline{B_{r_1}} = \left\{x\in\mathbb{R}^N\,:\, r_1<|x|<r_2\right\},$$

from the estimate above we obviously get

$$\lambda_p(\Omega)\fint_{S_{r_1,r_2}} (1-\varphi_\delta)^p\,dx \le \frac{2^{p-1}}{r^p}\,\frac{r^N}{|S_{r_1,r_2}|}\left[\frac{N\,\omega_N}{\varepsilon^{p-1}} + (\delta+\gamma)\,\mathrm{cap}_p\Big(\overline{B_1};B_2\Big)\right]. \tag{5.4.5}$$

By Jensen's inequality we have[2]

$$\fint_{S_{r_1,r_2}} (1-\varphi_\delta)^p\,dx \ge \left(\fint_{S_{r_1,r_2}} (1-\varphi_\delta)\,dx\right)^p.$$

[2]For $p=1$, this is not needed, of course.

By inserting this estimate in (5.4.5), we get

$$\lambda_p(\Omega)\,\left(1-\fint_{S_{r_1,r_2}}\varphi_\delta\,dx\right)^p\le\frac{2^{p-1}}{r^p}\,\frac{r^N}{|S_{r_1,r_2}|}\,\left[\frac{N\,\omega_N}{\varepsilon^{p-1}}+(\delta+\gamma)\,\mathrm{cap}_p\Big(\overline{B_1};B_2\Big)\right]. \tag{5.4.6}$$

In order to conclude, it would be sufficient to show that

$$1-\fint_{S_{r_1,r_2}}\varphi_\delta\,dx\ge\frac{1}{C},$$

for some positive constant C, depending only on N,p and γ. This is the key point, where the results of Section 5.2 will be crucial. We now distinguish the case $1<p\le N$ and the case $p=1$.

1. Case $1<p\le N$**.** To this aim, we can use the Poincaré–type inequality of (5.2.11), with $R=2\,r$. This yields

$$\left(\fint_{S_{r_1,r_2}}\varphi_\delta\,dx\right)^p\le\frac{1}{\mathrm{cap}_p(\overline{B_{r_1}};B_{2\,r})}\int_{B_{2\,r}}|\nabla\varphi_\delta|^p\,dx.$$

By using again (5.4.3) and (5.4.1) in order to estimate the rightmost integral, we then obtain

$$\fint_{S_{r_1,r_2}}\varphi_\delta\,dx\le\left(\frac{\mathrm{cap}_p\left(\overline{B_r};B_{2\,r}\right)}{\mathrm{cap}_p\Big(\overline{B_{r_1}};B_{2\,r}\Big)}\right)^{\frac{1}{p}}\,(\delta+\gamma)^{\frac{1}{p}}. \tag{5.4.7}$$

Now, we make the choice $\ell=2\,\varepsilon$, so that

$$r_1=(1-2\,\varepsilon)\,r,\qquad r_2=(1-\varepsilon)\,r,$$

and observe that

$$\left(\frac{\mathrm{cap}_p\left(\overline{B_r};B_{2\,r}\right)}{\mathrm{cap}_p\Big(\overline{B_{r_1}};B_{2\,r}\Big)}\right)^{\frac{1}{p}}>1\qquad\text{and}\qquad\lim_{\varepsilon\to0}\left(\frac{\mathrm{cap}_p\left(\overline{B_r};B_{2\,r}\right)}{\mathrm{cap}_p\Big(\overline{B_{r_1}};B_{2\,r}\Big)}\right)^{\frac{1}{p}}=1.$$

Thanks to the presence of the factor $\gamma<1$ and to the arbitrariness of $\delta>0$, this implies that up to choosing $\varepsilon>0$ small enough (depending on γ), we could uniformly bound from above the right-hand side of (5.4.7), by a factor strictly smaller than 1. Of course, the smaller we will choose ε, the larger the right-hand side of (5.4.6) will be (because of the factor $|S_{r_1,r_2}|$).

In particular, we claim that we can choose $0<\varepsilon<1/2$ so that

$$\left(\frac{\mathrm{cap}_p\left(\overline{B_r};B_{2\,r}\right)}{\mathrm{cap}_p\Big(\overline{B_{r_1}};B_{2\,r}\Big)}\right)^{\frac{1}{p}}\,\gamma^{\frac{1}{p}}\le\frac{1+\gamma^{\frac{1}{p}}}{2}, \tag{5.4.8}$$

the latter being smaller than 1. In particular, for every $\delta>0$ small enough, the right-hand side of (5.4.7) is strictly smaller than 1. By using this estimate in (5.4.6), we obtain

$$\lambda_p(\Omega)\,\left(1-\left(\frac{\mathrm{cap}_p\left(\overline{B_r};B_{2\,r}\right)}{\mathrm{cap}_p\Big(\overline{B_{r_1}};B_{2\,r}\Big)}\right)^{\frac{1}{p}}\,(\delta+\gamma)^{\frac{1}{p}}\right)^p\le\frac{2^{p-1}}{r^p}\,\frac{r^N}{|S_{r_1,r_2}|}\,\left[\frac{N\,\omega_N}{\varepsilon^{p-1}}+(\delta+\gamma)\,\mathrm{cap}_p\Big(\overline{B_1};B_2\Big)\right].$$

This is valid for every $\delta>0$ small enough, thus we can eliminate it by taking the limit as δ goes to 0. We thus obtain

$$\lambda_p(\Omega)\,\left(\frac{1-\gamma^{\frac{1}{p}}}{2}\right)^p\le\frac{2^{p-1}}{r^p}\,\frac{r^N}{|S_{r_1,r_2}|}\,\left[\frac{N\,\omega_N}{\varepsilon^{p-1}}+\gamma\,\mathrm{cap}_p\Big(\overline{B_1};B_2\Big)\right], \tag{5.4.9}$$

in light of (5.4.8).

We still have to show that the choice (5.4.8) is feasible. This is the same as

$$\left(\frac{1+\gamma^{\frac{1}{p}}}{2\,\gamma^{\frac{1}{p}}}\right)^{\frac{p}{p-1}} \geq \frac{1}{(1-2\,\varepsilon)^{\frac{N-p}{p-1}}}\,\left(\frac{\mathrm{cap}_p\left(\overline{B_1};B_2\right)}{\mathrm{cap}_p\left(\overline{B_1};B_{2/(1-2\,\varepsilon)}\right)}\right)^{\frac{1}{p-1}}, \tag{5.4.10}$$

where we also used Remark 2.3.1. We now need to further distinguish the case $p<N$ and $p=N$.

1.A Case $1<p<N$. By recalling (2.3.2), the condition (5.4.10) is equivalent to

$$\frac{1}{(1-2\,\varepsilon)^{\frac{N-p}{p-1}}}\,\frac{1}{1-\left(\frac{1}{2}\right)^{\frac{N-p}{p-1}}}\,\left(1-\left(\frac{1-2\,\varepsilon}{2}\right)^{\frac{N-p}{p-1}}\right) \leq \left(\frac{1+\gamma^{\frac{1}{p}}}{2\,\gamma^{\frac{1}{p}}}\right)^{\frac{p}{p-1}}.$$

By simplifying a bit the expression, this is equivalent to

$$\frac{2^{\frac{N-p}{p-1}}}{2^{\frac{N-p}{p-1}}-1}\,\frac{1}{(1-2\,\varepsilon)^{\frac{N-p}{p-1}}} - \frac{1}{2^{\frac{N-p}{p-1}}-1} \leq \left(\frac{1+\gamma^{\frac{1}{p}}}{2\,\gamma^{\frac{1}{p}}}\right)^{\frac{p}{p-1}}.$$

In turn, this can be recast as follows

$$(1-2\,\varepsilon)^{\frac{N-p}{p-1}}\,\left[\left(\frac{1+\gamma^{\frac{1}{p}}}{2\,\gamma^{\frac{1}{p}}}\right)^{\frac{p}{p-1}} + \frac{1}{2^{\frac{N-p}{p-1}}-1}\right] \geq \frac{2^{\frac{N-p}{p-1}}}{2^{\frac{N-p}{p-1}}-1}.$$

After some simple (yet tedious) computations, we get that it is sufficient to choose

$$\varepsilon \leq \varepsilon_0 := \min\left\{\frac{1}{4\,(N-1)},\,\frac{1}{2}-\left[\left(2^{\frac{N-p}{p-1}}-1\right)\,\left(\frac{1+\gamma^{\frac{1}{p}}}{2\,\gamma^{\frac{1}{p}}}\right)^{\frac{p}{p-1}}+1\right]^{-\frac{p-1}{N-p}}\right\}. \tag{5.4.11}$$

We observe that $\varepsilon_0 \leq 1/4$, in particular. Finally, by making this choice for ε, we get from (5.4.9) and (5.4.8)

$$\lambda_p(\Omega) \leq \frac{1}{r^p}\,\left(\frac{2}{1-\gamma^{\frac{1}{p}}}\right)^p\,\frac{2^{p-1}}{\omega_N\,\left((1-\varepsilon_0)^N-(1-2\,\varepsilon_0)^N\right)}\,\left[\frac{N\,\omega_N}{\varepsilon_0^{p-1}}+\gamma\,\mathrm{cap}_p\left(\overline{B_1};B_2\right)\right].$$

We eventually get the claimed estimate (5.4.2) with

$$C_{N,p,\gamma} = \left(\frac{2}{1-\gamma^{\frac{1}{p}}}\right)^p\,2^p\,\left[\frac{1}{\varepsilon_0^p}+\frac{\gamma}{\varepsilon_0}\,\frac{\mathrm{cap}_p\left(\overline{B_1};B_2\right)}{N\,\omega_N}\right],$$

once observed that[3]

$$(1-\varepsilon_0)^N-(1-2\,\varepsilon_0)^N \geq N\,(1-2\,\varepsilon_0)^{N-1}\,\varepsilon_0 \geq \frac{N}{2}\,\varepsilon_0.$$

1.B Case $p=N$. By recalling (2.3.3), the first condition (5.4.10) is equivalent to

$$\log\frac{2}{1-\varepsilon} \leq \left(\frac{1+\gamma^{\frac{1}{N}}}{2\,\gamma^{\frac{1}{N}}}\right)^{\frac{N}{N-1}}\,\log 2.$$

This is equivalent to

$$\log(1-\varepsilon) \geq \left[1-\left(\frac{1+\gamma^{\frac{1}{N}}}{2\,\gamma^{\frac{1}{N}}}\right)^{\frac{N}{N-1}}\right]\,\log 2.$$

[3]For the first inequality, use that

$$(1-\varepsilon_0)^N-(1-2\,\varepsilon_0)^N = N\int_{1-2\,\varepsilon_0}^{1-\varepsilon_0}\tau^{N-1}\,d\tau \geq N\,(1-2\,\varepsilon_0)^{N-1}\,\varepsilon_0.$$

In the second inequality, use Bernoulli's inequality and the fact that $\varepsilon_0 \leq 1/(4\,(N-1))$.

By exponentiating, we obtain

$$1-\varepsilon \geq 2^{1-\alpha_{N,\gamma}}, \qquad \text{with } \alpha_{N,\gamma} := \left(\frac{1+\gamma^{\frac{1}{N}}}{2\,\gamma^{\frac{1}{N}}}\right)^{\frac{N}{N-1}}.$$

If we choose

$$\varepsilon \leq \varepsilon_0 := \min\left\{1-2^{1-\alpha_{N,\gamma}}, \frac{1}{4\,(N-1)}\right\}, \tag{5.4.12}$$

we then obtain the desired property. The conclusion now follows as before.

2. Case $p=1$**.** We go back to (5.4.6). As done before, by combining (5.2.12) with assumptions (5.4.3) and (5.4.1), we infer that

$$\fint_{S_{r_1,r_2}} \varphi_\delta\,dx \leq \frac{1}{\mathrm{cap}_1(\overline{B_{r_2}}; B_{2r})} \int_{B_{2r}} |\nabla \varphi_\delta|\,dx \leq \frac{\mathrm{cap}_1(\overline{B_r}; B_{2r})}{\mathrm{cap}_1(\overline{B_{r_2}}; B_{2r})}\,(\delta+\gamma).$$

We use this upper bound in the left-hand side of (5.4.6). This yields

$$\lambda_1(\Omega)\,\left(1-\frac{\mathrm{cap}_1(\overline{B_r}; B_{2r})}{\mathrm{cap}_1(\overline{B_{r_2}}; B_{2r})}\,(\delta+\gamma)\right) \leq \frac{1}{r}\,\frac{r^N}{|S_{r_1,r_2}|}\,\left[N\,\omega_N + (\delta+\gamma)\,\mathrm{cap}_1\left(\overline{B_1}; B_2\right)\right].$$

We remove again the useless parameter δ, by taking the limit as this goes to 0. We then obtain

$$\lambda_1(\Omega)\,\left(1-\frac{\mathrm{cap}_1(\overline{B_r}; B_{2r})}{\mathrm{cap}_1(\overline{B_{r_2}}; B_{2r})}\,\gamma\right) \leq \frac{1}{r}\,\frac{r^N}{|S_{r_1,r_2}|}\,\left[N\,\omega_N + \gamma\,\mathrm{cap}_1\left(\overline{B_1}; B_2\right)\right].$$

The choice of the parameter ε and ℓ is now simpler: observe in particular that the role of parameter ℓ *is now immaterial*, thanks to the fact that the left-hand side in the previous estimate *only depends on* r_2*, and not on* r_1. We can thus take the limit as ℓ goes to 1 (that is, r_1 goes to 0) and obtain

$$\lambda_1(\Omega)\,\left(1-\frac{\mathrm{cap}_1(\overline{B_r}; B_{2r})}{\mathrm{cap}_1(\overline{B_{r_2}}; B_{2r})}\,\gamma\right) \leq \frac{1}{r}\,\frac{r^N}{|B_{r_2}|}\,\left[N\,\omega_N + \gamma\,\mathrm{cap}_1\left(\overline{B_1}; B_2\right)\right]. \tag{5.4.13}$$

Finally, we choose $\varepsilon>0$ in such a way that

$$1-\frac{\mathrm{cap}_1\left(\overline{B_r}; B_{2\,r}\right)}{\mathrm{cap}_1\left(\overline{B_{r_2}}; B_{2\,r}\right)}\,\gamma \geq \frac{1-\gamma}{2}.$$

This is the same as

$$\mathrm{cap}_1(\overline{B_{r_2}}; B_{2\,r}) \geq \mathrm{cap}_1(\overline{B_r}; B_{2\,r})\,\frac{2\,\gamma}{1+\gamma}.$$

By recalling the expression (2.3.1), we want

$$(1-\varepsilon)^{N-1} \geq \frac{2\,\gamma}{1+\gamma}.$$

Thus, by taking

$$\varepsilon_0 = \min\left\{1-\left(\frac{2\,\gamma}{1+\gamma}\right)^{\frac{1}{N-1}}, \frac{1}{2\,N}\right\},$$

we get from (5.4.13)

$$\lambda_1(\Omega) \leq \frac{1}{r}\,\frac{2}{1-\gamma}\,\frac{2}{\omega_N}\,\left[N\,\omega_N + \gamma\,\mathrm{cap}_1\left(\overline{B_1}; B_2\right)\right].$$

Observe that we also used that

$$|B_{r_2}| = \omega_N\,(1-\varepsilon_0)^N\,r^N \geq \omega_N\,(1-N\,\varepsilon_0)\,r^N \geq \frac{\omega_N}{2}\,r^N,$$

thanks to the choice of ε_0. Thus, we get (5.4.2), as desired. □

Remark 5.4.1 (Quality of the constant). We discuss the asymptotic behaviour of the constant $C_{N,p,\gamma}$ obtained in the previous result, as γ goes to 1. We distinguish two cases.

- *Case* $p = 1$: this is easier, in this case we have obtained
$$C_{N,1,\gamma} = 4\,N\,\frac{1+\gamma}{1-\gamma},$$
where we also used the explicit expression of the relative $1-$capacity of $\overline{B_1}$. Hence, we have the following asymptotic behaviour
$$0 < \lim_{\gamma\nearrow 1}(1-\gamma)\,C_{N,1,\gamma} < +\infty.$$
- *Case* $1 < p \le N$: we first observe that as γ goes to 1, from (5.4.11) and (5.4.12) we have
$$\varepsilon_0 = \begin{cases} (1-\gamma)\,\dfrac{1}{N-p}\,\dfrac{2^{\frac{N-p}{p-1}}-1}{2\cdot 2^{\frac{N-1}{p-1}}} + o(1-\gamma), & \text{if } 1<p<N,\\[2ex] (1-\gamma)\,\dfrac{1}{N-1}\,\dfrac{\log 2}{2} + o(1-\gamma), & \text{if } p=N. \end{cases}$$
Thus, by inspecting the proof above, we have

$$0 < \lim_{\gamma\nearrow 1}(1-\gamma)^{2\,p}\,C_{N,p,\gamma} < +\infty. \tag{5.4.14}$$

We see in particular that the constant $C_{N,p,\gamma}$ blows-up as γ goes to 1. While not claiming that the behaviour above is optimal, we point out that the upper bound of Theorem 4 can not be true with a constant which stays finite as γ goes to 1. We refer to Example A.3.1 for a counter-example.

5.5. Extension to Poincaré–Sobolev embedding constants

In this section, we briefly discuss how Theorem 4 can be extended to the more general case of the generalized principal frequencies associated to a general open set Ω, $\lambda_{p,q}(\Omega)$, whenever $1 \le p \le N$ and the exponent $q \ge 1$ is (strictly) subcritical, that is

$$\begin{cases} q < p^*, & \text{if } 1 \le p < N,\\ q < \infty, & \text{if } p = N, \end{cases} \tag{5.5.1}$$

being p^* the Sobolev conjugate exponent of p.

With some minor modifications of the proofs contained in the previous Sections 5.3-5.4, we infer a two–sided estimate for $\lambda_{p,q}(\Omega)$ in terms of the capacitary inradius $R_{p,\gamma}(\Omega)$, as well. We point out that for the lower bound *the additional restriction $q \ge p$ is mandatory* (see Remark 5.5.2 below). We have the following

Theorem 5.5.1. *Let $1 \le p \le N$ and let $q \ge 1$ satisfy (5.5.1). Let $0 < \gamma < 1$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set. Then, we have*

$$\lambda_{p,q}(\Omega) \le C_{N,\gamma,p,q}\left(\frac{1}{R_{p,\gamma}(\Omega)}\right)^{p-N+N\frac{p}{q}}, \tag{5.5.2}$$

where it is intended that $\lambda_{p,q}(\Omega) = 0$, whenever $R_{p,\gamma}(\Omega) = +\infty$.

Furthermore, if $q \ge p$ we also have

$$\gamma\,\sigma_{N,p,q}\left(\frac{1}{R_{p,\gamma}(\Omega)}\right)^{p-N+N\frac{p}{q}} \le \lambda_{p,q}(\Omega). \tag{5.5.3}$$

Proof. We prove (5.5.2) and (5.5.3) separately.

Upper bound. We proceed along the same lines as the proof of the upper bound of Theorem 4. In particular, by using the same notation as in Section 5.4, we now get

$$\lambda_{p,q}(\Omega)\left(\fint_{S_{r_1,r_2}}(1-\varphi_\delta)^q\,dx\right)^{\frac{p}{q}} \le \frac{2^{p-1}}{r^{p-N+N\frac{p}{q}}}\left(\frac{r^N}{|S_{r_1,r_2}|}\right)^{\frac{p}{q}}\left[\frac{N\omega_N}{\varepsilon^{p-1}} + (\delta+\gamma)\,\mathrm{cap}_p(\overline{B}_r;B_{2r})\right],$$

in place of (5.4.5). We use Jensen's inequality[4] to estimate from below the leftmost term. This gives

$$\lambda_{p,q}(\Omega)\left(1-\fint_{S_{r_1,r_2}}\varphi_\delta\,dx\right)^p\le\frac{2^{p-1}}{r^{p-N+N\frac{p}{q}}}\left(\frac{r^N}{|S_{r_1,r_2}|}\right)^{\frac{p}{q}}\left[\frac{N\omega_N}{\varepsilon^{p-1}}+(\delta+\gamma)\,\mathrm{cap}_p(\overline{B}_r;B_{2r})\right],\tag{5.5.4}$$

in place of (5.4.6). We distinguish again the case $1<p\le N$ and the case $p=1$.

A. Case $1<p\le N$. As done before, by applying (5.2.11) with $R=2r$, choosing ε appropriately and then taking the limit as δ goes to 0, we obtain

$$\lambda_{p,q}(\Omega)\left(\frac{1-\gamma^{\frac{1}{p}}}{2}\right)^p\le\frac{2^{p-1}}{r^{p-N+N\frac{p}{q}}}\left(\frac{r^N}{|S_{r_1,r_2}|}\right)^{\frac{p}{q}}\left[\frac{N\omega_N}{\varepsilon^{p-1}}+\gamma\,\mathrm{cap}_p(\overline{B_1};B_2)\right].$$

Observe that this is the same as (5.4.9), except for the presence of the correct scaling power on r and the power p/q on the term $r^N/|S_{r_1,r_2}|$, in the right-hand side. Then one concludes as in the case $p=q$ previously treated.

B. Case $p=1$. In (5.5.4), we use this time (5.2.12). By taking the limit as δ goes to 0 again, we infer that

$$\lambda_{1,q}(\Omega)\left(1-\frac{\mathrm{cap}_1(\overline{B_r};B_{2r})}{\mathrm{cap}_1(\overline{B_{r_2}};B_{2r})}\,\gamma\right)\le\frac{1}{r^{1-N+\frac{N}{q}}}\left(\frac{r^N}{|S_{r_1,r_2}|}\right)^{\frac{1}{q}}\left[N\,\omega_N+\gamma\,\mathrm{cap}_1\left(\overline{B_1};B_2\right)\right].$$

We can take again the limit as r_1 goes to 0 and obtain

$$\lambda_{1,q}(\Omega)\left(1-\frac{\mathrm{cap}_1(\overline{B_r};B_{2r})}{\mathrm{cap}_1(\overline{B_{r_2}};B_{2r})},\gamma\right)\le\frac{1}{r^{1-N+\frac{N}{q}}}\left(\frac{r^N}{|S_{r_1,r_2}|}\right)^{\frac{1}{q}}\left[N\,\omega_N+\gamma\,\mathrm{cap}_1\left(\overline{B_1};B_2\right)\right],$$

in place of (5.4.13). The conclusion then follows as in the case $q=p=1$.

Lower bound. We can assume $R_{p,\gamma}(\Omega)<+\infty$, otherwise there is nothing to prove. Let $r>R_{p,\gamma}(\Omega)$ and let $u\in C_0^\infty(\Omega)$. As in the proof of the lower bound of Theorem 4 (Section 5.3), for $p<q$ satisfying (5.5.1), we can still apply the Maz'ya–Poincaré inequality Theorem 2.6.1 and get this time

$$\frac{\mathscr{C}}{r^{\frac{N}{q}}}\,\mathrm{cap}_p\left(\overline{B_r(x_0)}\setminus\Omega;B_{2\sqrt{N}r}(x_0)\right)^{\frac{1}{p}}\,\|u\|_{L^q(Q_r(x_0))}\le\|\nabla u\|_{L^p(Q_r(x_0))},$$

where $\mathscr{C}$ is the same constant as in Theorem 2.6.1. Observe that now it depends on q, as well. The relative $p-$capacity on the left-hand side can be estimated from below as well, so to get

$$\frac{\mathscr{C}^p\,\mathrm{cap}_p(\overline{B_1};B_2)}{\dfrac{2\sqrt{N}}{\lambda_p(B_1)^{\frac{1}{p}}}+1}\left(\frac{1}{r}\right)^{p-N+p\,\frac{N}{q}}\gamma\,\|u\|^p_{L^q(Q_r(x_0))}\le\|\nabla u\|^p_{L^p(Q_r(x_0))},$$

in place of (5.3.2). In order to conclude, we want to use again a tiling of $\mathbb{R}^N$, made of a countable family of disjoint cubes with inradius r. A slight difference now arises, which explains the restriction on q: indeed, if $\{\mathcal{Q}_\alpha\}_{\alpha\in\mathbb{N}}$ is such a family of cubes, we have this time

$$\sum_{\alpha\in\mathbb{N}}\|\nabla u\|^p_{L^p(\mathcal{Q}_\alpha)}=\|\nabla u\|^p_{L^p(\Omega)}\qquad\text{but}\qquad\sum_{\alpha\in\mathbb{N}}\|u\|^p_{L^q(\mathcal{Q}_\alpha)}\ne\|u\|^p_{L^q(\Omega)}.$$

However, the choice $q>p$ entails that the function $\delta\mapsto\delta^{p/q}$ is subadditive. Thus, in particular

$$\sum_{\alpha\in\mathbb{N}}\|u\|^p_{L^q(\mathcal{Q}_\alpha)}\ge\left(\sum_{\alpha\in\mathbb{N}}\|u\|^q_{L^q(\mathcal{Q}_\alpha)}\right)^{\frac{p}{q}}=\|u\|^p_{L^q(\Omega)}.$$

We can now get the desired conclusion, as in the case $q=p$. □

Remark 5.5.2. The previous proof for the lower bound *does not* work if $1\le q<p$. This is not by chance: in Example A.2.1 we construct a counter-example to the validity of the lower bound in this case.

[4] For $q=1$ this is not needed.

5.6. Capacitary inradius VS inradius: the case $p > N$

In the previous section we excluded the range $p > N$, since in this case, as discussed in Section 1.2.1 and Chapter 3, we already know that $\lambda_p(\Omega)$ admits a two–sided estimate in terms of the usual inradius r_Ω.

We now compare the notion of capacitary inradius with that of inradius, when $p > N$. In the following theorem, we are going to prove that they coincide, at least for γ smaller than a certain (optimal) threshold. It reads as follows

THEOREM 5. *Let $p > N$ and*

$$\gamma_0 := \frac{\mathrm{cap}_p\left(\{0\}; B_2\right)}{\mathrm{cap}_p\left(\overline{B_1}; B_2\right)} = \frac{1}{2^{p-N}}\,\left(2^{\frac{p-N}{p-1}}-1\right)^{p-1}. \tag{5.6.1}$$

For every open set $\Omega \subseteq \mathbb{R}^N$ we have

$$R_{p,\gamma}(\Omega) = r_\Omega, \qquad \textit{for every } 0 \le \gamma < \gamma_0.$$

Moreover, for the punctured ball $\dot{B}_R := B_R \setminus \{0\}$ we have

$$R_{p,\gamma}(\dot{B}_R) > r_{\dot{B}_R}, \qquad \textit{for every } 1 > \gamma \ge \gamma_0.$$

This result is certainly not surprising, but it requires some work and some precise estimates on the capacity of points. At this aim, we start by pointing out that for a compact set $\Sigma \Subset B_R$, the definition of relative $p-$capacity can be also written as

$$\mathrm{cap}_p(\Sigma; B_R) = \inf_{\varphi \in W^{1,p}_0(B_R)} \left\{\int_{B_R} |\nabla\varphi|^p\,dx\,:\,\varphi \ge 1 \text{ on } \Sigma\right\},$$

for $p > N$. Observe that the pointwise requirement on the test functions make sense, in light of Morrey's inequality, i.e. $W^{1,p}_0(B_R)$ is embedded in a space of continuous functions on $\overline{B_R}$. Moreover, by a standard application of the Direct Method, the previous infimum is actually (uniquely) attained, by a function u_Σ called $p-$capacitary potential. By minimality and uniqueness, it is not difficult to see that this is a $p-$harmonic function in $B_R \setminus \Sigma$, such that

$$0 \le u_\Sigma \le 1 \qquad \text{and} \qquad u_\Sigma = 1 \text{ on } \Sigma.$$

LEMMA 5.6.1. *Let $p > N \ge 2$ and let $R > 0$. We choose a set of distinct points $\{x_1, \dots, x_k\} \in B_R$ and set*

$$D := \min\left\{|x_i - x_j|,\, \mathrm{dist}(x_i; \partial B_R)\,:\, i, j \in \{1, \dots, k\},\, i \ne j\right\} > 0.$$

There exists a constant $c_p > 0$, depending on p only, such that for every $\delta < D$ we have

$$\mathrm{cap}_p\big(\{x_1, \dots, x_{k-1}\}; B_R\big) + c_p \int_{B_\delta(x_k)} |\nabla u - \nabla H_u|^p\,dx \le \mathrm{cap}_p\big(\{x_1, \dots, x_k\}; B_R\big).$$

Here u is the $p-$capacitary potential of the set $\{x_1, \dots, x_k\}$ relative to B_R, while H_u is the $p-$harmonic function in $B_\delta(x_k)$ such that $u - H_u \in W^{1,p}_0(B_\delta(x_k))$. In particular, we have

$$\mathrm{cap}_p\big(\{x_1, \dots, x_{k-1}\}; B_R\big) < \mathrm{cap}_p\big(\{x_1, \dots, x_k\}; B_R\big).$$

PROOF. We take $u \in W^{1,p}_0(B_R)$ to be an optimal function for $\mathrm{cap}_p\big(\{x_1, \dots, x_k\}; B_R\big)$. This means that $0 \le u \le 1$ and

$$\int_{B_R} |\nabla u|^p\,dx = \mathrm{cap}_p\big(\{x_1, \dots, x_k\}; B_R\big), \qquad u(x_i) = 1, \text{ for } i = 1, \dots, k.$$

Observe that by minimality, the function u is weakly $p-$harmonic in the open connected set $B_R \setminus \{x_1, \dots, x_k\}$. Thus, by the minimum and maximum principles (see for example [72, Corollary 2.22]), we get that

$$0 < u(x) < 1 \qquad \text{in } B_R \setminus \{x_1, \dots, x_k\}.$$

We will use a "$p-$harmonic replacement trick" in order to modify u and produce a trial function, which is admissible for the $p-$capacity of $\{x_1,\dots,x_{k-1}\}$. Namely, we introduce the new function

$$U(x)=\begin{cases} u(x), & \text{if } x\in B_R\setminus B_\delta(x_k),\\ H_u(x), & \text{if } x\in B_\delta(x_k),\end{cases}$$

where $H_u\in W^{1,p}(B_\delta(x_k))$ is the unique minimiser of

$$\min_{\varphi\in W^{1,p}(B_\delta(x_k))}\left\{\int_{B_\delta(x_k)}|\nabla\varphi|^p\,dx\,:\,\varphi-u\in W^{1,p}_0(B_\delta(x_k))\right\}.$$

Observe that by minimality, the function H_u satisfies

$$\int_{B_\delta(x_k)}\langle|\nabla H_u|^{p-2}\,\nabla H_u,\nabla\varphi\rangle\,dx=0,\qquad \text{for every } \varphi\in W^{1,p}_0(B_\delta(x_k)).$$

Thus, in particular, we have

$$\int_{B_\delta(x_k)}\langle|\nabla H_u|^{p-2}\,\nabla H_u,\nabla u-\nabla H_u\rangle\,dx=0. \tag{5.6.2}$$

It is not difficult to see that the function U is admissible for the $p-$capacity of $\{x_1,\dots,x_{k-1}\}$. This gives

$$\begin{aligned}\mathrm{cap}_p\big(\{x_1,\dots,x_{k-1}\};B_R\big)&\le\int_{B_R}|\nabla U|^p\,dx\\ &=\int_{B_R\setminus B_\delta(x_k)}|\nabla u|^p\,dx+\int_{B_\delta(x_k)}|\nabla H_u|^p\,dx\\ &=\int_{B_R}|\nabla u|^p\,dx+\left(\int_{B_\delta(x_k)}|\nabla H_u|^p\,dx-\int_{B_\delta(x_k)}|\nabla u|^p\,dx\right)\\ &=\mathrm{cap}_p\big(\{x_1,\dots,x_k\};B_R\big)+\left(\int_{B_\delta(x_k)}|\nabla H_u|^p\,dx-\int_{B_\delta(x_k)}|\nabla u|^p\,dx\right).\end{aligned}$$

In order to conclude, we just need to estimate the rightmost term into parentheses. To this aim, we need to recall the following convexity inequality, which is valid for $p>2$ (see [71, Lemma 4.2, equation (4.3)]):

$$|z|^p\ge|w|^p+p\,\langle|w|^{p-2}\,w,z-w\rangle+c_p\,|z-w|^p,\qquad \text{for every } z,w\in\mathbb{R}^N.$$

From this inequality, we get

$$\begin{aligned}\int_{B_\delta(x_k)}|\nabla u|^p\,dx&\ge\int_{B_\delta(x_k)}|\nabla H_u|^p\,dx+\int_{B_\delta(x_k)}\langle|\nabla H_u|^{p-2}\,\nabla H_u,\nabla u-\nabla H_u\rangle\,dx\\ &+c_p\int_{B_\delta(x_k)}|\nabla u-\nabla H_u|^p\,dx\\ &=\int_{B_\delta(x_k)}|\nabla H_u|^p\,dx+c_p\int_{B_\delta(x_k)}|\nabla u-\nabla H_u|^p\,dx.\end{aligned}$$

In the last identity, we used (5.6.2). This implies that we have

$$\mathrm{cap}_p\big(\{x_1,\dots,x_{k-1}\};B_R\big)\le\mathrm{cap}_p\big(\{x_1,\dots,x_k\};B_R\big)-c_p\int_{B_\delta(x_k)}|\nabla u-\nabla H_u|^p\,dx.$$

Finally, we observe that the last quantity can not vanish, otherwise u would be weakly $p-$harmonic on $B_\delta(x_k)$ and would attain its maximum at the center of the ball, thus violating the maximum principle. □

The following Lemma, which essentially relies on a symmetrization argument, will be needed in the sequel to obtain the equality case between $R_{p,\gamma}(\Omega)$ and r_Ω, for small values of the negligibility parameter γ.

LEMMA 5.6.2. *Let $p>N$, for every $x_0\in B_R(y_0)$, we have*

$$\mathrm{cap}_p\left(\{x_0\};B_R(y_0)\right)\ge\mathrm{cap}_p\left(\{y_0\};B_R(y_0)\right).$$

PROOF. We can suppose that y_0 coincides with the origin. It is sufficient to use [76, (2.2.10)] with $F = \{x_0\}$. This gives

$$\mathrm{cap}_p(\{x_0\}; B_R) \geq (N\,\omega_N)^{\frac{p}{N}}\, N^{\frac{N-p}{N}} \left(\frac{p-N}{p-1}\right)^{p-1} |B_R|^{\frac{N-p}{N}}.$$

By recalling (2.3.4), we easily see that the right-hand side coincides with the capacity of the center of the ball. □

We are now in position to compare the usual notion of inradius r_Ω with its capacitary variant $R_{p,\gamma}(\Omega)$, in the case $p > N$. We will prove that for γ smaller than a universal sharp constant, they actually coincide.

PROOF OF THEOREM 5. We prove the two claims separately.

We have already observed that

$$r_\Omega \leq R_{p,\gamma}(\Omega).$$

In particular, if $r_\Omega = +\infty$, then the conclusion trivially follows. Let us suppose that $r_\Omega < +\infty$. For every ball $B_r(y_0)$ with $r > r_\Omega$, we then must have

$$\overline{B_r(y_0)} \setminus \Omega \neq \emptyset.$$

In particular, there exists a point $x_0 \in \overline{B_r(y_0)} \setminus \Omega$. By monotonicity of the $p-$capacity with respect to the set inclusion, we get

$$\mathrm{cap}_p\left(\overline{B_r(y_0)} \setminus \Omega; B_{2\,r}(y_0)\right) \geq \mathrm{cap}_p(\{x_0\}; B_{2\,r}(y_0)) \geq \mathrm{cap}_p(\{y_0\}; B_{2\,r}(y_0)).$$

In the second inequality, we used Lemma 5.6.2. In particular, by recalling the definition of γ_0, we get

$$\mathrm{cap}_p\left(\overline{B_r(y_0)} \setminus \Omega; B_{2\,r}(y_0)\right) \geq \gamma_0\, r^{N-p}\, \mathrm{cap}_p\left(\overline{B_1}; B_2\right) = \gamma_0\, \mathrm{cap}_p\left(\overline{B_r(y_0)}; B_{2\,r}(y_0)\right).$$

This implies that if $\gamma < \gamma_0$, then $\overline{B_r(y_0)} \setminus \Omega$ is not $(p,\gamma)-$negligible, for $r > r_\Omega$. This gives the desired conclusion, in light of the definition of $R_{p,\gamma}(\Omega)$.

We now show the optimality of the previous result. We consider the punctured ball $\dot{B}_R = B_R \setminus \{0\}$. We clearly have $r_{\dot{B}_R} = R/2$. On the other hand, it is not difficult to show that

$$R_{p,\gamma}(\dot{B}_R) \geq R, \qquad \text{for every } \gamma \geq \gamma_0,$$

where γ_0 is still defined by (5.6.1). Indeed, we may notice that if $r < R$ we have

$$\mathrm{cap}_p(\overline{B_r} \setminus \dot{B}_R; B_{2r}) = \mathrm{cap}_p(\{0\}; B_{2r}) = r^{N-p}\,\gamma_0\, \mathrm{cap}_p(\overline{B_r}; B_{2r}),$$

which shows that $\overline{B_r} \setminus \dot{B}_R$ is $(p,\gamma_0)-$negligible. Thus, this already shows that for $\gamma \geq \gamma_0$

$$R_{p,\gamma}(\dot{B}_R) \geq R_{p,\gamma_0}(\dot{B}_R) \geq r, \qquad \text{for every } r < R.$$

Actually, we can show that $R_{p,\gamma_0}(\dot{B}_R) = R$. It is sufficient to observe that for every $r > R$ and every $x_0 \in \mathbb{R}^N$, the set $\overline{B_r(x_0)} \setminus \dot{B}_R$ contains at least two distinct points. By Lemma 5.6.1, this implies that

$$\begin{aligned}\mathrm{cap}_p(\overline{B_r(x_0)} \setminus \dot{B}_R; B_{2r}(x_0)) &> \mathrm{cap}_p(\{0\}; B_{2r}) \\ &= \gamma_0\, \mathrm{cap}_p(\overline{B_r}; B_{2r}) = \gamma_0\, \mathrm{cap}_p(\overline{B_r(x_0)}; B_{2r}(x_0)),\end{aligned}$$

that is any ball with radius $r > R$ is not $(p,\gamma_0)-$negligible. This gives $R_{p,\gamma_0}(\dot{B}_R) = R$, as claimed. □

APPENDIX A

Capacitary inradius: some counter-examples

This appendix mainly concerns the capacitary inradius and some degenerate behaviour connected to it. More precisely, we are going to discuss the extremal cases when γ tends to 0 or 1, and the failure of the lower bound (5.5.3) in the *sub–homogeneous case*.

A.1. Failure for $\gamma = 0$

EXAMPLE A.1.1. For $1 \leq p \leq N$ and an open set $\Omega \subseteq \mathbb{R}^N$, we introduce the quantity

$$\mathfrak{R}_\Omega := R_{p,0}(\Omega) = \sup\left\{r > 0 \,:\, \exists x_0 \in \mathbb{R}^N \text{ such that } \mathrm{cap}_p\left(\overline{B_r(x_0)} \setminus \Omega; B_{2r}(x_0)\right) = 0\right\}.$$

This may appear as the natural capacitary extension of the usual inradius. However, in this section, we will give an example showing that this notion is not strong enough to permit having the uniform lower bound

$$\lambda_p(\Omega) \geq C \left(\frac{1}{\mathfrak{R}_\Omega}\right)^p,$$

for every $\Omega \subseteq \mathbb{R}^N$ open set. Indeed, for every $0 < \varepsilon < 1/4$, we introduce the periodically perforated set

$$\Omega_\varepsilon = \mathbb{R}^N \setminus \bigcup_{\mathbf{i} \in \mathbb{Z}^N} \overline{B_\varepsilon(\mathbf{i})}.$$

We claim that

$$\lim_{\varepsilon \searrow 0} \lambda_p(\Omega_\varepsilon) = 0 \qquad \text{while} \qquad \mathfrak{R}_{\Omega_\varepsilon} \leq \frac{\sqrt{N}}{2}, \qquad \text{for every } 0 < \varepsilon < \frac{1}{4}. \tag{A.1.1}$$

We first observe that the usual inradius of Ω_ε is uniformly bounded, that is

$$r_{\Omega_\varepsilon} \leq \frac{\sqrt{N}}{2}, \qquad \text{for every } 0 < \varepsilon < \frac{1}{4}.$$

In particular, for every ball $B_r(x_0)$ with $r > \sqrt{N}/2$ we have

$$B_r(x_0) \cap \left(\bigcup_{\mathbf{i} \in \mathbb{Z}^N} \overline{B_\varepsilon(\mathbf{i})}\right) \neq \emptyset.$$

More precisely, let $\mathbf{i}_0 \in \mathbb{Z}^N$ be such that

$$|x_0 - \mathbf{i}_0| = \mathrm{dist}(x_0, \mathbb{Z}^N),$$

this distance does not exceed $\sqrt{N}/2$. Consequently, we have $\mathbf{i}_0 \in B_r(x_0)$ and thus

$$|B_r(x_0) \setminus \Omega_\varepsilon| \geq |B_r(x_0) \cap \overline{B_\varepsilon(\mathbf{i}_0)}| > 0.$$

By the properties of capacity (see equations [76, (2.2.10) & (2.2.11) pag. 148]), we can infer that

$$\mathrm{cap}_p\left(\overline{B_r(x_0)} \setminus \Omega_\varepsilon; B_{2r}(x_0)\right) > 0.$$

Thus, for every $r > \sqrt{N}/2$, we have that $\overline{B_r(x_0)} \setminus \Omega$ has positive relative $p-$capacity and, according to the definition, we obtain

$$\mathfrak{R}_{\Omega_\varepsilon} \leq \frac{\sqrt{N}}{2}, \qquad \text{for every } 0 < \varepsilon < \frac{1}{4},$$

as well. In order to conclude, we need to prove the first property in (A.1.1). It is not difficult to see that

$$\lambda_p(\Omega_\varepsilon) = \inf_{u\in \mathrm{Lip}(\overline{Q_{1/2}})} \left\{ \int_{Q_{1/2}} |\nabla u|^p\, dx \,:\, \|u\|_{L^p(Q_{1/2})} = 1,\, u = 0 \text{ on } \overline{B_\varepsilon} \right\}.$$

It is sufficient to proceed as in the proof of Lemma 3.2.4, for example. In particular, we take $\varphi \in \mathrm{Lip}_0(B_{1/2})$ such that $\varphi = 1$ on $\overline{B_\varepsilon}$ and $0 \le \varphi \le 1$, extended by 0 to $\overline{Q_{1/2}} \setminus B_{1/2}$. By using the test function $u = (1-\varphi)/\|1-\varphi\|_{L^q(Q_{1/2})}$, we get

$$\lambda_p(\Omega_\varepsilon) \le \frac{\displaystyle\int_{B_{1/2}} |\nabla \varphi|^p\, dx}{\displaystyle\int_{Q_{1/2}} (1-\varphi)^p\, dx} \le \frac{\displaystyle\int_{B_{1/2}} |\nabla \varphi|^p\, dx}{|Q_{1/2}\setminus B_{1/2}|}.$$

Thanks to the arbitrariness of φ and recalling formula (2.2.1) from Remark 2.2.2, we obtain

$$\lambda_p(\Omega_\varepsilon) \le \frac{\mathrm{cap}_p(\overline{B_\varepsilon}; B_{1/2})}{|Q_{1/2}\setminus B_{1/2}|}.$$

By using (2.3.1), (2.3.2) and (2.3.3), the previous estimate finally implies (A.1.1).

A.2. Failure of the lower bound for $q < p$

Example A.2.1. We exhibit an open set $\Omega \subseteq \mathbb{R}^N$ such that for $1 \le q < p \le N$

$$\lambda_{p,q}(\Omega) = 0 \qquad \text{and} \qquad R_{p,\gamma}(\Omega) < +\infty, \text{ for every } 0 < \gamma < 1.$$

This implies that the lower bound (5.5.3) cannot be true in this case. We stick for simplicity to the case $1 < p < N$, the case $p = N$ can be treated with minor modifications. We take the slab

$$\Omega = \mathbb{R}^{N-1} \times (-1, 1),$$

for which we have $\lambda_{p,q}(\Omega) = 0$ for every $1 \le q < p$ (see for example [21, Proposition 6.1]). We need to prove that its capacitary inradius is finite, for every $0 < \gamma < 1$. At this aim, we fix $0 < \gamma < 1$ and take a ball $B_r(x_0)$ such that $r > 1$ and

$$\mathrm{cap}_p(\overline{B_r(x_0)} \setminus \Omega; B_{2r}(x_0)) \le \gamma\, \mathrm{cap}_p(\overline{B_r(x_0)}; B_{2r}(x_0)).$$

Thanks to the invariance of Ω by translations in directions belonging to $\{x_N = 0\}$, we can suppose without loss of generality that $x_0 = t\,\mathbf{e}_N$, for some $t \in \mathbb{R}$. Thus, we have

$$\mathrm{cap}_p(\overline{B_r(t\,\mathbf{e}_N)} \setminus \Omega; B_{2r}(t\,\mathbf{e}_N)) \le \gamma\, r^{N-p}\, \mathrm{cap}_p(\overline{B_1}; B_2), \tag{A.2.1}$$

where we also used Remark 2.3.1. By using [76, (2.2.10)], we get

$$\begin{aligned}\mathrm{cap}_p(\overline{B_r(t\,\mathbf{e}_N)} \setminus \Omega; B_{2r}(t\,\mathbf{e}_N)) \ge (N\,\omega_N)^{\frac{p}{N}}\, N^{\frac{N-p}{N}} \left(\frac{N-p}{p-1}\right)^{p-1} \\ \times \left| |B_{2r}(t\,\mathbf{e}_N)|^{\frac{p-N}{N\,(p-1)}} - |B_r(t\,\mathbf{e}_N)\setminus\Omega|^{\frac{p-N}{N\,(p-1)}} \right|^{1-p}.\end{aligned}$$

The expression on the right-hand side can be simplified: indeed, if we introduce r^* the radius such that

$$|B_{r^*}(t\,\mathbf{e}_N)| = |B_r(t\,\mathbf{e}_N)\setminus\Omega| \qquad \text{that is} \qquad r^* = \left(\frac{|B_r(t\,\mathbf{e}_N)\setminus\Omega|}{\omega_N}\right)^{\frac{1}{N}}.$$

and recall (2.3.2), one can see that it coincides with

$$\mathrm{cap}_p(\overline{B_{r^*}(t\,\mathbf{e}_N)}; B_{2r}(t\,\mathbf{e}_N)).$$

Accordingly, we obtain

$$\mathrm{cap}_p(\overline{B_r(t\,\mathbf{e}_N)} \setminus \Omega; B_{2r}(t\,\mathbf{e}_N)) \ge \mathrm{cap}_p(\overline{B_{r^*}(t\,\mathbf{e}_N)}; B_{2r}(t\,\mathbf{e}_N)). \tag{A.2.2}$$

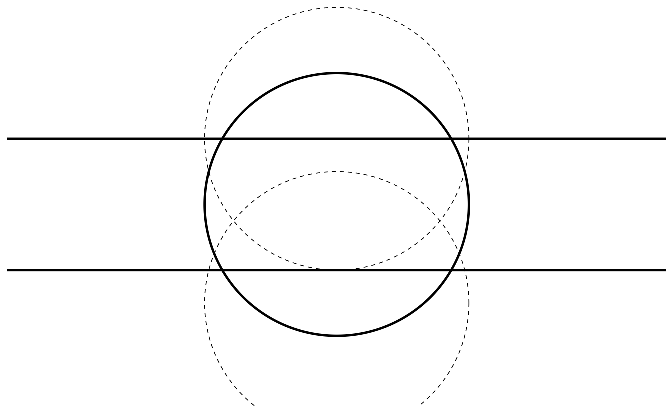

FIGURE 1. The ball in bold line maximizes the volume of the intersection with the slab Ω.

The volume of $B_r(t\,\mathbf{e}_N)\setminus\Omega$ can be uniformly bounded from below. Indeed, observe at first that if we set

$$\Omega_m=\prod_{i=1}^{N-1}(-m,m)\times(-1,1),$$

then

$$\overline{B_r(t\,\mathbf{e}_N)}\setminus\Omega=\overline{B_r(t\,\mathbf{e}_N)}\setminus\Omega_m,\qquad \text{for every } m>r.$$

Then, as a consequence of [22, Lemma 3.13], for every $m>r$ the function

$$t\mapsto|B_r(t\,\mathbf{e}_N)\setminus\Omega_m|=|B_r(t\,\mathbf{e}_N)|-|B_r(t\,\mathbf{e}_N)\cap\Omega_m|,$$

attains its minimum for $t=0$ (see Figure 1). In other words, this volume is minimal if the ball and Ω_m are concentric. We also observe that such a minimal value is given by

$$\begin{aligned}|B_r\setminus\Omega_m|=|B_r\setminus\Omega|&=2\,\omega_{N-1}\int_1^r(r^2-z^2)^{\frac{N-1}{2}}\,dz\\&=2\,\omega_{N-1}\,r^N\int_{\arcsin\frac{1}{r}}^{\frac{\pi}{2}}\cos^N t\,dt:=r^N\,\varphi_N(r).\end{aligned}$$

In conclusion, we get that

$$r^*\geq r\left(\frac{\varphi_N(r)}{\omega_N}\right)^{\frac{1}{N}}=:r\,\Phi_N(r). \tag{A.2.3}$$

From (A.2.1), (A.2.2), the monotonicity of the capacity with respect to the set inclusion, the lower bound on r^* and again the scaling relations for the capacity, we get

$$\big(\Phi_N(r)\big)^{N-p}\operatorname{cap}_p(\overline{B_1};B_{2/\Phi_N(r)})\leq\gamma\operatorname{cap}_p(\overline{B_1};B_2). \tag{A.2.4}$$

This relation must be satisfied by every radius $r>1$, such that $\overline{B_r(x_0)}\setminus\Omega$ is $(p,\gamma)-$negligible.

By recalling (2.3.2), the previous inequality is equivalent to

$$\big(\Phi_N(r)\big)^{N-p}\left(1-\left(\frac{1}{2}\right)^{\frac{N-p}{p-1}}\right)^{p-1}\leq\gamma\left(1-\left(\frac{\Phi_N(r)}{2}\right)^{\frac{N-p}{p-1}}\right)^{p-1}.$$

With simple algebraic manipulations, we get that for every admissible radius r, we must have

$$\Phi_N(r)\leq\left(\frac{2^{\frac{N-p}{p-1}}\,\gamma^{\frac{1}{p-1}}}{2^{\frac{N-p}{p-1}}-1+\gamma^{\frac{1}{p-1}}}\right)^{\frac{p-1}{N-p}}.$$

Observe that the right-hand side is strictly smaller than 1. Moreover, by construction the function $r \mapsto \Phi_N(r)$ is continuous monotone increasing, with

$$\lim_{r\searrow 1} \Phi_N(r) = 0 \qquad \text{and} \qquad \lim_{r\nearrow +\infty} \Phi_N(r) = 1. \tag{A.2.5}$$

This implies that there exists a finite radius $r_\gamma > 1$ such that

$$\Phi_N(r_\gamma) = \left(\frac{2^{\frac{N-p}{p-1}}\,\gamma^{\frac{1}{p-1}}}{2^{\frac{N-p}{p-1}} - 1 + \gamma^{\frac{1}{p-1}}} \right)^{\frac{p-1}{N-p}},$$

and that every ball with radius $r > r_\gamma$ violates the previous conditions, i.e. it *is not* $(p,\gamma)-$negligibile. This finally proves that

$$R_{p,\gamma}(\Omega) \le r_\gamma < +\infty,$$

as claimed.

A.3. Degeneration for $\gamma \nearrow 1$

EXAMPLE A.3.1. We maintain the same notation as in Example A.2.1 and take again $\Omega = \mathbb{R}^{N-1} \times (-1,1)$. Since this set is bounded in the direction $\mathbf{e}_N$, we have $\lambda_p(\Omega) > 0$. We claim that

$$\lim_{\gamma\nearrow 1} R_{p,\gamma}(\Omega) = +\infty. \tag{A.3.1}$$

This proves that an upper bound of the type

$$\lambda_p(\Omega) \le \widetilde{C}_{N,p,\gamma} \left(\frac{1}{R_{p,\gamma}(\Omega)} \right)^p,$$

with $\widetilde{C}_{N,p,\gamma}$ staying bounded for γ converging to 1, *can not be true*.

In order to show (A.3.1), we first recall that the function

$$\gamma \mapsto R_{p,\gamma}(\Omega), \qquad \text{with } \gamma \in [0,1),$$

is monotone non-decreasing. Thus, the limit in (A.3.1) exists. For every $r > 1$, we set

$$\gamma_r = \frac{\mathrm{cap}_p(\overline{B_r} \setminus \Omega; B_{2r})}{\mathrm{cap}_p(\overline{B_r}; B_{2r})} < 1,$$

thus $\overline{B_r} \setminus \Omega$ is γ_r-negligible, obviously. Accordingly, we get from (A.2.4)

$$\big(\Phi_N(r)\big)^{N-p} \frac{\mathrm{cap}_p(\overline{B_1}; B_{2/\Phi_N(r)})}{\mathrm{cap}_p(\overline{B_1}; B_2)} \le \gamma_r.$$

By recalling (A.2.5), from the previous inequality we get

$$\lim_{r\nearrow +\infty} \gamma_r = 1.$$

In particular, by monotonicity we have

$$\lim_{\gamma\nearrow 1} R_{p,\gamma}(\Omega) = \lim_{r\nearrow +\infty} R_{p,\gamma_r}(\Omega).$$

On the other hand, since the set $\overline{B_r} \setminus \Omega$ is γ_r-negligible, we must have

$$R_{p,\gamma_r}(\Omega) \ge r.$$

By joining the last two facts, we finally obtain (A.3.1).

List of symbols

Listed below, we collect some basic notations used throughout this thesis. In the following, Ω indicates a general open subset of $\mathbb{R}^N$.

$A \Subset B$	A has compact closure in B
1_A	characteristic function of A, i.e. $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ otherwise
$\lfloor \alpha \rfloor$	integer part of a real number α
$B_R(x_0)$	$N-$dimensional open ball centered at x_0 with radius $R > 0$
B_R	$N-$dimensional open ball centered at the origin with radius $R > 0$
$\mathbb{S}^{N-1}$	$N-$dimensional unit sphere
$\lvert \cdot \rvert$	$N-$dimensional Lebesgue measure
$\mathcal{H}^k$	$k-$dimensional Hausdorff measure
ω_N	$\lvert B_1 \rvert$, the $N-$dimensional Lebesgue measure of the unit ball in $\mathbb{R}^N$
$C(\Omega)$	continuous functions on Ω
$C^m(\Omega)$	continuous functions together with their derivatives up to order $m \in \mathbb{N}$
$C_0^\infty(\Omega)$	infinitely differentiable functions whose support is a compact subset of Ω
$\mathrm{Lip}(\Omega)$	Lipschitz continuous functions on Ω
$\mathrm{Lip}_0(\Omega)$	Lipschitz continuous functions whose support is a compact subset of Ω
$L^p(\Omega)$	$p-$integrable Lebesgue measurable functions on Ω
$L^p_{\mathrm{loc}}(\Omega)$	measurable functions $u : \Omega' \to \mathbb{R}$ such that $u \in L^p(\Omega')$, for every open set $\Omega' \Subset \Omega$
$W^{1,p}(\Omega)$	standard Sobolev space, functions in $L^p(\Omega)$ with distributional gradient in $L^p(\Omega; \mathbb{R}^N)$
$W^{1,p}_{\mathrm{loc}}(\Omega)$	functions in $L^p_{\mathrm{loc}}(\Omega)$ with distributional gradient in $L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$
$W_0^{1,p}(\Omega)$	closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$
$\mathscr{D}_0^{1,p}(\Omega)$	homogeneous Sobolev space, the completion of $C_0^\infty(\Omega)$ with respect to $\lVert \nabla \cdot \rVert_{L^p(\Omega)}$
div	divergence operator, if $\phi : \mathbb{R}^N \to \mathbb{R}^N$ then $\mathrm{div}\phi = \displaystyle\sum_{i=1}^N \frac{\partial \phi_i}{\partial x_i}$
∇	gradient operator, if $u : \mathbb{R}^N \to \mathbb{R}^N$ then $\nabla u = \left(\dfrac{\partial u}{\partial x_1}, \dots, \dfrac{\partial u}{\partial x_N} \right)$
Δ	Laplace operator, if $u : \mathbb{R}^N \to \mathbb{R}^N$ then $\Delta u = \displaystyle\sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$
Δ_p	$p-$Laplace operator, if $u : \mathbb{R}^N \to \mathbb{R}^N$ then $\Delta_p u = \displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\lvert \nabla u \rvert^{p-2} \frac{\partial u}{\partial x_i} \right)$

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