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The Surface Diffusion Flow
Long–Time Behavior and Asymptotic Stability

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"The future belongs to those who believe in the beauty of their dreams."
E. Roosevelt

CONTENTS

INTRODUCTION

The study of deformations of geometric structures driven by systems of nonlinear partial differential equations became very relevant in differential geometry and mathematical physics in recent decades. Concrete examples are, for instance, the analysis of the behavior in time of the interfaces surfaces in phase changes of materials or in the flows of immiscible fluids. From a mathematical point of view, the great success of this topic was the application of such deformation techniques in solving some famous long–standing open problems in geometry, notably among them, the Poincaré conjecture by Perelman, by means of the Ricci flow.

In this thesis, we deal with hypersurfaces and we study one of the most known among their geometric flows, namely the *surface diffusion flow*. We will consider the evolution in time of smooth sets E_t in the n–dimensional flat torus $\mathbb{T}^n \approx \mathbb{R}^n/\mathbb{Z}^n$, for every t in a time interval $[0, T)$, such that their boundaries ∂E_t, which are smooth hypersurfaces, move with "outer" normal velocity V_t given by

$$V_t = \Delta_t \mathrm{H}_t \qquad \text{on } \partial E_t, \tag{0.1}$$

where Δ_t and H_t are respectively the Laplacian and the mean curvature of the hypersurface ∂E_t, for all $t \in [0, T)$. Choosing as ambient space the flat torus $\mathbb{T}^n$, described as the quotient of $\mathbb{R}^n$ by a discrete group of translations generated by some n linearly independent vectors, is equivalent to consider the flow of "periodic" hypersurfaces in the Euclidean space, invariant by such group of translations. Then, it is clear that our analysis also applies to compact hypersurfaces in $\mathbb{R}^n$ or, more in general, in any (generalized) "cylinder" $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \times \mathbb{R} \times \cdots \times \mathbb{R}$ of dimension n, with a flat metric.

Such flow was first proposed by Mullins in [52] to study thermal grooving in material sciences (see also [26] for a nice presentation). Indeed, in the physically relevant case of three–dimensional space, it describes the evolution of interfaces between solid phases of a system, which are studied in a variety of physical settings including phase transitions, epitaxial deposition and grain growth (see for instance [39] and the references therein).

A very important property of this geometric flow is that it is the *gradient flow* of a functional, which clearly gives a natural "energy", decreasing in time during the evolution (the velocity V_t is minus the gradient, that is, the *Euler–Lagrange equation* of a functional). Precisely, the surface diffusion flow is the H^{-1}–gradient flow of the following *Area functional*

$$\mathcal{A}(\partial E) = \int_{\partial E} d\mu$$

that gives the *area* of the $(n-1)$–dimensional smooth boundary of any sets E, under a volume constraint (here μ is the "canonical" measure associated to the Riemannian metric on ∂E induced by the metric of $\mathbb{T}^n$ coming from the scalar product of $\mathbb{R}^n$, which coincides with the $(n-1)$–dimensional Hausdorff measure $\mathcal{H}^{n-1}$).

Parametrizing the moving smooth surfaces ∂E_t by a family of embeddings $\varphi_t : M \to \mathbb{T}^n$ such that $\varphi_t(M) = \partial E_t$, where M is a fixed smooth, compact $(n-1)$–dimensional differentiable manifold and ν_t is the outer unit normal vector to ∂E_t as above, the evolution law (0.1) can be expressed as

$$\frac{\partial \varphi_t}{\partial t} = (\Delta_t \mathrm{H}_t)\nu_t\,. \tag{0.2}$$

Then, by the general equality $\Delta\varphi = -\mathrm{H}\nu$ (equation (1.8) below), relation (0.2) can be rewritten as

$$\frac{\partial \varphi_t}{\partial t} = -\Delta\Delta\varphi_t + \text{ lower order terms} \tag{0.3}$$

hence, we have to deal with a fourth order, *quasilinear* and *degenerate*, parabolic system of PDEs. More precisely, it is quasilinear, as the coefficients (as second order partial differential operator) of the Laplacian associated to the induced metrics on the evolving hypersurfaces, depend on the first order derivatives of φ_t (and the coefficient of $\Delta\Delta$ on the third order derivatives) and the operator at the right hand side of system (0.3) is degenerate, as its symbol (that is, the symbol of its linearized operator) admits zero eigenvalues, due to the invariance of the Laplacian by diffeomorphisms.

From the evolution law (0.1), it follows easily that the volume $\mathrm{Vol}(E_t)$ of the moving sets is constant in time. However, the lack of the maximum principle, as the flow is of fourth order, implies that it does not preserve convexity (see [42]), nor the embeddedness (see [33]), indeed it also does not have a "comparison principle", while it is invariant by isometries of $\mathbb{T}^n$, reparametrizations and tangential perturbations of the velocity of the motion.

Due to the parabolic nature of this system of PDEs, it is known that for every smooth initial set E_0 in $\mathbb{T}^n$, with boundary described by $\varphi_0 : M \to \mathbb{T}^n$, the flow with such initial data exists unique and is smooth in some positive time interval $[0, T)$. The original result, proved by Escher, Mayer and Simonett in [26], deals with the evolution in the whole space $\mathbb{R}^n$ of a generic hypersurface even only immersed, hence possibly with self–intersections. It is anyway straightforward to adapt the same arguments to our case, when the ambient is a flat torus $\mathbb{T}^n$ and the hypersurfaces are boundaries of sets.

Theorem. *Let $\varphi_0 : M \to \mathbb{R}^n$ be a smooth and compact, immersed hypersurface. Then, there exists a unique smooth $\varphi : [0, T) \times M \to \mathbb{R}^n$ such that $\varphi_t = \varphi(t, \cdot)$ is the surface diffusion flow of φ_0, that is, a solution of system (0.3), for some maximal time of existence $T > 0$. Moreover, such flow and the maximal time of existence depend continuously on the $C^{2,\alpha}$–norm of the initial hypersurface φ_0.*

Actually, it is very likely, as in many geometric evolution equations, that this flow could develop singularities in finite time, even if a rigorous example is not present in literature (up to our knowledge). In [26], Escher et al. exhibited an immersed curve with a loop within a loop (namely, a limaçon) and showed that during a numerical simulation of its evolution by surface diffusion, the inner loop tightens and then contracts to a point developing a singularity. Analogously, in [25] the same authors gave numerical evidence that for an evolving dumbbell with a thin neck, a pinching–off should occur. We mention that these two situations have been instead analyzed rigorously for the mean curvature flows by Angenent in [5] and by Grayson in [35], respectively (see also [48] and [6] for alternative proofs).

Anyway, in some particular cases one can show that singularities do not appear, i.e. the flow exists smooth for all positive times. For instance, in [26] the authors showed that if the initial hypersurface is $C^{2,\alpha}$–close enough to a sphere with the same enclosed volume, then the flow exists for every time and smoothly converges to a translate of such sphere. The analogous result was obtained by Escher and Mucha in [27] for compact surfaces in $\mathbb{R}^{n+1}$ with a Besov–type condition and then by Wheeler in [62] for surfaces and in [63] for closed plane curves (see also the work of Elliott and Garcke [24]) with a weaker initial $W^{2,2}$–closedness condition. Furthermore, in [64] Wheeler showed that any surface diffusion flow of curves that exists for all time, must converge smoothly, exponentially fast to a multiply–covered circle. We also mention a work by Miura e Okabe [50] where the authors proved a global existence result provided that the initial curve is $W^{2,2}$–close to a multiply covered circle.

Later on, Acerbi, Fusco, Julin and Morini in [1] extended these results, to any *two and three–dimensional hypersurface* sufficiently "close" to the boundary of a smooth *strictly stable critical set* E for the volume constrained Area functional (as it is every ball, actually), showing that the flow exists for all positive times and asymptotically converges (in a suitable sense) to a translate of E. Our aim in this thesis is to generalize such stability conclusion to any dimension, following the lines presented by the author in [23] (in collaboration with Nicola Fusco and Carlo Mantegazza) and in [16] (in collaboration with Daniele De Gennaro, Andrea Kubin and Anna Kubin).

The notions of criticality and stability are as usual defined in terms of first and second variations of $\mathcal{A}$. We say that a smooth subset $E \subseteq \mathbb{T}^n$ is *critical* for $\mathcal{A}$ under a volume constraint, if for any smooth one–parameter family of diffeomorphisms $\Phi_t : \mathbb{T}^n \to \mathbb{T}^n$, such that $\mathrm{Vol}(\Phi_t(E)) = \mathrm{Vol}(E)$, for $t \in (-\varepsilon, \varepsilon)$ and $\Phi_0 = \mathrm{Id}$ ($E_t = \Phi_t(E)$ will be called *volume–preserving variation* of E), it follows

$$\frac{d}{dt}\mathcal{A}(\partial E_t)\Big|_{t=0} = 0\,.$$

It is easy to see that this condition is equivalent to the existence of a constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{H} = \lambda \qquad \text{on } \partial E,$$

where H is the mean curvature of ∂E, that is, ∂E is a hypersurface with *constant mean curvature*.
The second variation of $\mathcal{A}$ at a critical set E, leading to the central notion of *stability*, is more involved and, differently by other authors, we will compute it in detail with the tools of the differential/Riemannian geometry (like for the first variation). We will then see that at a critical set E, the second variation of $\mathcal{A}$ along a volume–preserving variation $E_t = \Phi_t(E)$ only depends on the normal component ψ on ∂E of the *infinitesimal generator* field $X = \frac{\partial \Phi_t}{\partial t}\big|_{t=0}$ of the variation. The volume constraint on the admissible deformations of E implies that the functions ψ must have zero integral on ∂E, hence it is natural to define a quadratic form Π_E on such space of functions which is related to the second variation of $\mathcal{A}$ by the following equality,

$$\Pi_E(\psi) = \frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0}, \tag{0.4}$$

where $E_t = \Phi_t(E)$ is a volume–preserving variation of E such that

$$\Big\langle \nu_E \,\Big|\, \frac{\partial \Phi_t}{\partial t}\Big|_{t=0}\Big\rangle = \psi$$

on ∂E, with ν_E the *outer unit normal vector* of ∂E.
Because of the obvious *translation invariance* of the functional $\mathcal{A}$, it is easy to see (by means of the formula (0.4)) that the form Π_E vanishes on the finite dimensional vector space given by the functions $\varphi = \langle \nu_E | \eta \rangle$, for every vector $\eta \in \mathbb{R}^n$. We underline that the presence of such "natural" degenerate subspace of the quadratic form Π_E (or, equivalently, the translation invariance of $\mathcal{A}$) is the main reason of several technical difficulties.
We then say that a smooth critical set $E \subseteq \mathbb{T}^n$ is *strictly stable* if

$$\Pi_E(\psi) > 0$$

for all non–zero functions $\psi : \partial E \to \mathbb{R}$, with zero integral and L^2–orthogonal to every function $\varphi = \langle \nu_E | \eta \rangle$.

The heuristic idea behind the whole thesis is that in a region around a strictly stable critical set E, we have a "potential well" for the "energy" $\mathcal{A}$ (and the set E is a local minimum) and, defining a suitable notion of "closedness", if a set starts "close enough" to E, during its evolution by (minus) the gradient of such energy, it cannot "escape" the well and eventually asymptotically converges to a set of (local) minimal energy, which must be a translate of E. This can be clearly interpreted as a kind of "dynamical stability" in a neighborhood of E (and its translates or "up to translations").
To be more precise, we will prove the following results:

Theorem (Theorem 3.3.14)**.** *Let $E \subseteq \mathbb{T}^n$, for $n \geqslant 3$, be a strictly stable critical set for the Area functional under a volume constraint. Then, there exists $\delta > 0$ such that, if E_0 is a smooth set, C^1–close to E, satisfying $\mathrm{Vol}(E_0) = \mathrm{Vol}(E)$ and*

$$\mathrm{Vol}(E_0 \triangle E) \leqslant \delta \qquad \textit{and} \qquad \int_{\partial E_0} |\nabla^{n-2}\mathrm{H}|^2\, d\mu_0 + \int_{\partial E_0} |\nabla \mathrm{H}|^2\, d\mu_0 \leqslant \delta\,,$$

the unique smooth surface diffusion flow E_t starting from E_0 is defined for all $t \geqslant 0$ and converges smoothly to $E' = E + \eta$ exponentially fast as $t \to +\infty$, for some $\eta \in \mathbb{R}^n$.

Theorem (Theorem 3.4.8)**.** *Let $E \subseteq \mathbb{T}^n$, for $n \geqslant 3$, be a strictly stable set for the Area functional under a volume constraint. Then, there exists $\delta > 0$ such that, if E_0 is a smooth set δ–close in $C^{1,1}$ to E, satisfying $\mathrm{Vol}(E_0) = \mathrm{Vol}(E)$, then the surface diffusion flow E_t starting from E_0 exists smooth for all times $t \geqslant 0$ and $E_t \to E + \tau$ as $t \to +\infty$, for some $\tau \in \mathbb{T}^n$, in C^k for every $k \in \mathbb{N}$ exponentially fast.*

In showing the first theorem we will follow the line of the proof in [1], revisited in [18] and extended to any dimension in [23], based on suitable energy estimates and compactness arguments. We underline that this was actually a completely new approach to manage the translation invariance of the functional $\mathcal{A}$, in previous literature only dealt with by means of semigroup techniques. More in detail, setting

$$\mathcal{F}(t) = \int_{\partial E_t} |\nabla^{n-2}\mathrm{H}|^2 \, d\mu_t + \int_{\partial E_t} |\nabla \mathrm{H}|^2 \, d\mu_t$$

and fixed $\delta_0 > 0$, we consider the surface diffusion flow starting from a set E_0 which is δ_0–close to E, such that $\mathrm{Vol}(E_0 \triangle E) \leqslant \delta_0$ and $\mathcal{F}(0) \leqslant \delta_0$. By means of energy estimates, we show that if δ_0 is chosen small enough, there exists $\delta > 0$ (as in the statement of the theorem) such that the maximal time of existence of E_t is actually $+\infty$. Once global–in–time existence has been established, a compactness argument yields the existence of a sequence $t_i \to +\infty$ and of a set E', critical for $\mathcal{A}$, such that $E_{t_i} \to E'$ (in a suitable sense). Since necessarily E' is close to E and $\mathrm{Vol}(E) = \mathrm{Vol}(E')$, we conclude that E' is a translate of E, then, the exponential convergence of the flow to E' follows from suitable elliptic estimates.

The proof of the second theorem is based on the gradient flow structure of the evolution, in particular, the main tool is the Alexandrov–type inequality in [17, Theorem 1.3], combined with the quantitative isoperimetric inequality in [2]. By means of an iterative procedure and higher order estimates, we extend the flow for all times. In order to do so, we need to show that the solution coming from the short–time existence and regularity result depends only on the bounds of the initial datum, which is not *a priori* clear from the existence result in [26]. More precisely, instead of using an approach by scaling (as it is done in [43]), we rely on Schauder estimates on the linearized problem solved by the flow, which is a quasilinear perturbation of the biharmonic heat equation, in spirit of [41]. After establishing global existence, we obtain the exponential convergence up to translations via a Gronwall–type argument. Finally, we prove the convergence of the flow to (a translate of) the strictly stable set, by exploiting the decay of the geometric quantities in time, as in [1, 18, 23]. We stress that this line of proof works in any dimension, without energy estimates for the high derivatives of the curvature, which is one of the main bottlenecks of the previous method.

The thesis is organized as follows:

- In Chapter 1 we first collect the necessary definitions and preliminaries about hypersurfaces. Then, we show that families of smooth hypersurfaces of $\mathbb{R}^n$ which are all C^1–close enough to a fixed compact, embedded one, have uniformly bounded constants in some relevant inequalities, like Sobolev, Gagliardo–Nirenberg and "geometric" Calderón–Zygmund inequalities.

- In Chapter 2 we introduce the Area functional and study its basic properties. In particular, we compute its first and second variations and we discuss the notions of criticality, stability and local minimality of a set and their mutual relations, in this context.

- In Chapter 3 we finally consider the surface diffusion flow and we analyze its analytic and geometric features. We prove a short–time existence result and then we show the stability of the flow along the two different lines that we described above.

Several results were partly obtained by the author in collaboration with Daniele De Gennaro, Serena Della Corte, Nicola Fusco, Andrea Kubin, Anna Kubin and Carlo Mantegazza. The thesis comprises contributions from the research papers [16, 18, 19, 23] as well as some unpublished results.

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1

SOME GEOMETRIC PRELIMINARIES

1.1 GEOMETRY OF HYPERSURFACES

We introduce the basic notations and facts about hypersurfaces that we need in the thesis, possible references are [31] or the first part of [53].

We will consider closed smooth hypersurfaces in $\mathbb{R}^n$ (or in the n–dimensional torus $\mathbb{T}^n \approx \mathbb{R}^n/\mathbb{Z}^n$), given by smooth immersions $\varphi : M \to \mathbb{R}^n$ of a smooth, $(n-1)$–dimensional, compact manifold M, representing a hypersurface $\varphi(M)$ of $\mathbb{R}^n$. Taking local coordinates around any $p \in M$, we have local bases of the tangent space T_pM, which can be identified with the $(n-1)$–dimensional hyperplane $d\varphi_p(T_pM)$ of $\mathbb{R}^n \approx T_{\varphi(p)}\mathbb{R}^n$ which is tangent to $\varphi(M)$ at $\varphi(p)$ and of the cotangent space T_p^*M, respectively given by vectors $\{\frac{\partial}{\partial x_i}\}$ and 1–forms $\{dx_j\}$. So, we will denote vectors on M by $X = X^i$, which means $X = X^i \frac{\partial}{\partial x_i}$, covectors by $Y = Y_j$, that is, $Y = Y_j dx_j$ and a general mixed tensor with $T = T^{i_1\ldots i_k}_{j_1\ldots j_l}$.

In the whole paper the convention to sum over repeated indices will be adopted.

Sometimes we will also need to consider tensors along M, viewing it as a submanifold of $\mathbb{R}^n$ (or $\mathbb{T}^n$) via the map φ, in that case we will use the Greek indices to denote the components of such tensors in the canonical basis $\{e_\alpha\}$ of $\mathbb{R}^n$, for instance, given a vector field X along M, not necessarily tangent, we will have $X = X^\alpha e_\alpha$.

The manifold M gets in a natural way a metric tensor g, pull–back via the map φ of the metric tensor of $\mathbb{R}^n$, coming from the standard scalar product $\langle \cdot \,|\, \cdot \rangle$ of $\mathbb{R}^n$, hence, turning it into a Riemannian manifold (M, g). Then, the components of g in a local chart are

$$g_{ij} = \left\langle \frac{\partial \varphi}{\partial x_i} \,\Big|\, \frac{\partial \varphi}{\partial x_j} \right\rangle$$

and the "canonical" measure μ, induced on M by the metric g is then locally described by $\mu = \sqrt{\det g_{ij}}\, \mathscr{L}^{n-1}$, where $\mathscr{L}^{n-1}$ is the standard Lebesgue measure on $\mathbb{R}^{n-1}$.

Thus, supposing that M has a *global* coordinate chart, we can write the Area functional on the hypersurface $\varphi(M)$ in the following way,

$$\mathcal{A}(\varphi(M)) = \int_M d\mu = \int_M \sqrt{\det g_{ij}(x)}\, dx\,. \tag{1.1}$$

When this is not the case (as it is usual), we need several local charts (U_k, φ_k) and a subordinated partitions of unity $f_k : M \to [0,1]$ (that is, the compact support of $f_k : M \to [0,1]$ is contained in the open set $U_k \subseteq M$, for every $k \in \mathcal{I}$), then

$$\mathcal{A}(\varphi(M)) = \int_M d\mu = \sum_{k\in\mathcal{I}} \int_M f_k\, d\mu = \sum_{k\in\mathcal{I}} \int_{U_k} f_k(x) \sqrt{\det g^k_{ij}(x)}\, dx\,,$$

where g^k_{ij} are the coefficients of the metric g in the local chart (U_k, φ_k).

In order to work with coordinates, in the computations with integrals in this section we will assume that all the hypersurfaces have a global coordinate chart, by simplicity. All the results actually hold also in the general case by using partitions of unity as above.

The inner product on M, extended to tensors, is given by

$$g(T, S) = g_{i_1 s_1} \ldots g_{i_k s_k} g^{j_1 z_1} \ldots g^{j_l z_l} T^{i_1\ldots i_k}_{j_1\ldots j_l} S^{s_1\ldots s_k}_{z_1\ldots z_l}$$

where g_{ij} is the matrix of the coefficients of the metric tensor in the local coordinates and g^{ij} is its inverse. Clearly, the norm of a tensor is then

$$|T| = \sqrt{g(T,T)}\,.$$

The induced Levi–Civita covariant derivative on (M,g) of a vector field X and of a 1–form ω are respectively given by

$$\nabla_j X^i = \frac{\partial X^i}{\partial x_j} + \Gamma^i_{jk} X^k\,, \qquad \nabla_j \omega_i = \frac{\partial \omega_i}{\partial x_j} - \Gamma^k_{ji}\omega_k\,,$$

where Γ^i_{jk} are the Christoffel symbols of the connection ∇, expressed by the formula

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}\Big(\frac{\partial}{\partial x_j} g_{kl} + \frac{\partial}{\partial x_k} g_{jl} - \frac{\partial}{\partial x_l} g_{jk}\Big)\,. \tag{1.2}$$

The covariant derivative ∇T of a tensor $T = T^{i_1\dots i_k}_{j_1\dots j_l}$ will be denoted by $\nabla_s T^{i_1\dots i_k}_{j_1\dots j_l} = (\nabla T)^{i_1\dots i_k}_{s j_1\dots j_l}$ and with $\nabla^m T$ we will mean the m–th iterated covariant derivative of a tensor T.

The gradient ∇f of a function, the divergence $\operatorname{div} X$ of a tangent vector field and the Laplacian Δf at a point $p \in M$, are defined respectively by

$$g(\nabla f(p), v) = df_p(v) \qquad \forall v \in T_pM\,,$$

$$\operatorname{div} X = \operatorname{tr}\nabla X = \nabla_i X^i = \frac{\partial X^i}{\partial x_i} + \Gamma^i_{ik} X^k$$

(in a local chart) and $\Delta f = \operatorname{div} \nabla f$. The Laplacian ΔT of a tensor T is $\Delta T = g^{ij}\nabla_i\nabla_j T$. We then recall that by the *divergence theorem* for compact manifolds (without boundary), there holds

$$\int_M \operatorname{div} X \, d\mu = 0\,, \tag{1.3}$$

for every tangent vector field X on M, which in particular implies

$$\int_M \Delta f \, d\mu = 0\,,$$

for every smooth function $f : M \to \mathbb{R}$.

Assuming that we have a globally defined unit *normal* vector field $\nu : M \to \mathbb{R}^n$ to $\varphi(M)$ (this will hold in our situation where the hypersurfaces are embedded or are boundaries of sets $E \subseteq \mathbb{T}^n$, hence we will always consider ν to be the *outer unit normal vector* at every point of ∂E), we define the *second fundamental form* B which is a symmetric bilinear form given, in local charts, by its components

$$h_{ij} = -\Big\langle \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \,\Big|\, \nu \Big\rangle$$

and whose trace is the *mean curvature* $\mathrm{H} = g^{ij} h_{ij}$ of the hypersurface (with these choices, the standard sphere of $\mathbb{R}^n$ has positive mean curvature).

Remark 1.1.1. If the hypersurface $M \subseteq \mathbb{R}^n$ is the graph of a function $f : U \to \mathbb{R}$ with U an open subset of $\mathbb{R}^{n-1}$, that is, $\varphi(x) = (x, f(x))$, then we have

$$g_{ij} = \delta_{ij} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\,, \qquad \nu = -\frac{(\nabla f, -1)}{\sqrt{1+|\nabla f|^2}} \tag{1.4}$$

$$h_{ij} = -\frac{\mathrm{Hess}_{ij} f}{\sqrt{1+|\nabla f|^2}} \tag{1.5}$$

$$\mathrm{H} = -\frac{\Delta f}{\sqrt{1+|\nabla f|^2}} + \frac{\mathrm{Hess}f(\nabla f,\nabla f)}{\left(\sqrt{1+|\nabla f|^2}\right)^3} = -\,\mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) \tag{1.6}$$

where $\mathrm{Hess}f$ is the Hessian of the function f.

Then, the following *Gauss–Weingarten relations* hold,

$$\frac{\partial^2\varphi}{\partial x_i\partial x_j} = \Gamma_{ij}^k\frac{\partial\varphi}{\partial x_k} - h_{ij}\nu \qquad\qquad \frac{\partial\nu}{\partial x_j} = h_{jl}g^{ls}\frac{\partial\varphi}{\partial x_s}\,, \tag{1.7}$$

which easily imply $|\nabla\nu| = |\mathrm{B}|$ and the identity

$$\Delta\varphi = g^{ij}\Big(\frac{\partial^2\varphi}{\partial x_i\partial x_j} - \Gamma_{ij}^k\frac{\partial\varphi}{\partial x_k}\Big) = -g^{ij}h_{ij}\nu = -\mathrm{H}\nu\,. \tag{1.8}$$

The symmetry properties of the covariant derivative of B are given by the following Codazzi equations,

$$\nabla_i h_{jk} = \nabla_j h_{ik} = \nabla_k h_{ij} \tag{1.9}$$

which imply the following *Simons' identity* (see [60]),

$$\Delta h_{ij} = \nabla_i\nabla_j\mathrm{H} + \mathrm{H}\,h_{il}g^{ls}h_{sj} - |\mathrm{B}|^2 h_{ij}\,. \tag{1.10}$$

By means of Codazzi equations (1.9), using the *normal coordinates* at a point $p \in M$ (and recalling that Γ_{ij}^k and $\frac{\partial}{\partial x_i}g^{jk}$ vanish in $p \in M$), we have

$$\begin{aligned}
\Delta\nu &= g^{ij}\Big(\frac{\partial^2\nu}{\partial x_i\partial x_j} - \Gamma_{ij}^k\frac{\partial\nu}{\partial x_k}\Big)\\
&= g^{ij}\frac{\partial}{\partial x_i}\Big(h_{jl}g^{ls}\frac{\partial\varphi}{\partial x_s}\Big)\\
&= g^{ij}\nabla_i h_{jl}g^{ls}\frac{\partial\varphi}{\partial x_s} + g^{ij}h_{jl}g^{ls}\frac{\partial^2\varphi}{\partial x_i\partial x_s}\\
&= g^{ij}\nabla_l h_{ij}g^{ls}\frac{\partial\varphi}{\partial x_s} - g^{ij}h_{jl}g^{ls}h_{is}\nu\\
&= \nabla\mathrm{H} - |B|^2\nu\,,
\end{aligned} \tag{1.11}$$

Finally, the Riemann tensor is expressed via the second fundamental form as follows (*Gauss equations*),

$$\mathrm{R}_{ijkl} \;=\; h_{ik}h_{jl} - h_{il}h_{jk} \tag{1.12}$$

hence, the formulas for the interchange of covariant derivatives, which involve the Riemann tensor, become

$$\begin{aligned}
\nabla_i\nabla_j X^s - \nabla_j\nabla_i X^s &= \mathrm{R}_{ijkl}g^{ks}X^l = \mathrm{R}_{ijl}^s X^l = \big(h_{ik}h_{jl} - h_{il}h_{jk}\big)g^{ks}X^l\\
\nabla_i\nabla_j\omega_k - \nabla_j\nabla_i\omega_k &= \mathrm{R}_{ijkl}g^{ls}\omega_s = \mathrm{R}_{ijk}^s\omega_s = \big(h_{ik}h_{jl} - h_{il}h_{jk}\big)g^{ls}\omega_s
\end{aligned}$$

for every vector field X and 1–form ω.

1.2 UNIFORM INEQUALITIES

In this section, following the line of [19], we aim to show that families of smooth hypersurfaces of $\mathbb{R}^n$ which are all C^1–close enough to a fixed compact, embedded one, have uniformly bounded constants in some relevant inequalities for the mathematical analysis, like Sobolev, Gagliardo–Nirenberg, "geometric" Calderón–Zygmund, trace and extension inequalities.

These technical results will be applied to study the behavior of the hypersurfaces close (in some norm, for instance in C^1–norm) to critical ones (possibly "stable") and the asymptotic limits of the flows existing for all times. Moreover, they are used repeatedly more or less explicitly in the works [1, 2, 18, 23], where uniform controls on the constants are necessary.

For the time being, we fix M_0 a smooth, compact, embedded hypersurface of $\mathbb{R}^n$ (or $\mathbb{T}^n$). So, it is well known (by its compactness and smoothness) that, for $\varepsilon > 0$ small enough, M_0 has a *tubular neighborhood*

$$N_\varepsilon = \{x \in \mathbb{R}^n \,:\, d(x, M_0) < \varepsilon\} \tag{1.13}$$

(where d is the Euclidean distance on $\mathbb{R}^n$) such that the *orthogonal projection map* $\pi : N_\varepsilon \to M_0$ giving the (unique) closest point on M_0, is well defined and smooth.
Then, if E is "the interior" of M_0, the *signed distance function* $d_E : N_\varepsilon \to \mathbb{R}$ from M_0

$$d_E(x) = \begin{cases} d(x, M_0) & \text{if } x \notin E \\ -d(x, M_0) & \text{if } x \in E \end{cases} \tag{1.14}$$

is well defined and smooth in N_ε (for a proof of the existence of such tubular neighborhood and of all the subsequent properties, see [49] for instance). Moreover, for every $x \in N_\varepsilon$, the projection map π is given explicitly by

$$\pi_E(x) = x - \nabla d_E^2(x)/2 = x - d_E(x)\nabla d_E(x) \tag{1.15}$$

and the unit vector $\nabla d_E(x)$ is orthogonal to M_0 at the point $\pi_E(x)$, indeed actually

$$\nabla d_E(x) = \nabla d_E(\pi_E(x)) = \nu(\pi_E(x))\,. \tag{1.16}$$

This implies that, every smooth hypersurface M which is C^1–close enough to M_0, can be written (possibly after reparametrization) as

$$M = \{x + \psi(x)\nu(x) \,:\, x \in M_0\}, \tag{1.17}$$

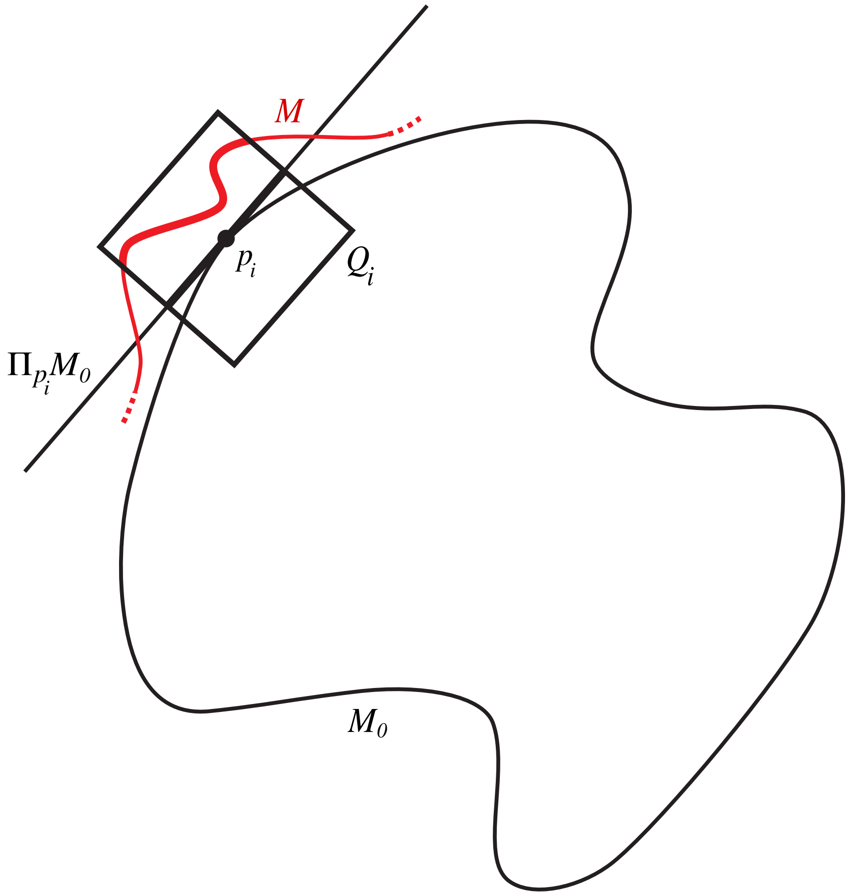
for a smooth function $\psi : M_0 \to \mathbb{R}$ with $\|\psi\|_{C^1(M_0)} < \varepsilon$. Indeed, if $\varphi_0 : \widetilde{M} \to \mathbb{R}^n$ and $\varphi : \widetilde{M} \to \mathbb{R}^n$ are two smooth immersions such that at least one of them is an embedding (φ_0, for instance) of a differentiable manifold $\widetilde{M}$, describing respectively M_0 and M, close in C^1, then the map $\pi \circ \varphi \circ \varphi_0^{-1} : M_0 \to M_0$ is a diffeomorphism, which implies that $\pi|_M : M \to M_0$ is also a diffeomorphism. Then, the map ψ above in expression (1.17), is uniquely given by $\psi(x) = d_E(\pi|_M^{-1}(x))$, which has small C^1–norm, as $\pi|_M$ gets C^1–closer and closer to the identity, as φ is C^1–close to φ_0.

Hence, from now on, we will consider families of hypersurfaces (clearly all containing M_0)

$$\mathfrak{C}^1_\delta(M_0) = \Big\{M = \{x + \psi(x)\nu(x) \,:\, x \in M_0\} \\ \text{for a smooth } \psi : M_0 \to \mathbb{R} \text{ with } \|\psi\|_{C^1(M_0)} < \delta\Big\}$$

where $\delta \in (0, \varepsilon)$. We are going to see that the constants in Sobolev, Gagliardo–Nirenberg, some geometric Calderón–Zygmund inequalities, trace and extension inequalities are uniformly bounded, depending only on M_0 and δ.

Before starting discussing that, we introduce another technical construction. We notice that, possibly choosing a smaller $\varepsilon > 0$, the tubular neighborhood N_ε of M_0 defined above, can be covered by a finite number of open hypercubes $Q_1, \dots, Q_k \subseteq \mathbb{R}^n$ respectively centered at some points $p_1, \dots, p_k \in M_0$, such that, for every $i \in \{1, \dots, k\}$ and every $M \in \mathfrak{C}^1_\delta(M_0)$, with $\delta \in (0, \varepsilon)$, the "pieces" of hypersurfaces $M \cap Q_i$ can be written as *orthogonal* graphs on the affine hyperplanes $\Pi_{p_i} M_0 = p_i + T_{p_i} M_0$, parallel to the tangent hyperplanes to M_0 at the points $p_i \in M_0$ and passing through them, as in the following figure.



Then, we let $\rho_i : \mathbb{R}^n \to [0,1]$ a smooth partition of unity (with compact support) for N_ε, associated to the open covering Q_i, hence, if $M \in \mathfrak{C}^1_\delta(M_0)$ and $u : M \to \mathbb{R}$, there holds

$$u(y) = \sum_{i=1}^{k} u(y)\rho_i(y)$$

with the compact support of $u\rho_i : M \to \mathbb{R}$ contained in the piece $M \cap Q_i$ of the hypersurface M, which is described as the graph of a smooth function $\theta_i : \Pi_{p_i} M_0 \to \mathbb{R}$, that is, $M \cap Q_i$ is the image of the map $x \mapsto \Theta(x) = x + \theta_i(x)\nu(p_i)$ on $\Pi_{p_i} M_0 \cap Q_i$. Moreover, it is easy to see that, possibly choosing an even smaller $\varepsilon > 0$, we have $\|\theta_i\|_{C^1(\Pi_{p_i} M_0)} \leqslant 2\delta$, for every $i \in \{1, \ldots, k\}$, since also M_0 can be locally written as an orthogonal graph on $\Pi_{p_i} M_0$.

We notice and underline that the family (and the number) of the hypercubes Q_i, as well as the width $\varepsilon > 0$ of the tubular neighborhood N_ε that we considered for this construction, only depend on M_0, precisely on its local and global geometry (in particular, on its second fundamental form B_0 – see [10] for more details).

We highlight to the reader that in the following, we will often denote with C a constant which may vary from a line to another.

1.2.1 *Sobolev, Poincaré and Gagliardo–Nirenberg interpolation inequalities*

We start discussing the Sobolev constants $C_S(M,p)$ of any compact $(n-1)$–dimensional hypersurface M, for every $p \in [1, n-1)$, entering in the following inequalities (which are known to hold, see [7, Chapter 2], for instance),

$$\begin{aligned}\|u\|_{L^{p^*}(M)} =&\Big(\int_M |u|^{p^*}\,d\mu\Big)^{1/p^*}\\ \leqslant& C_S(M,p)\Big(\int_M |\nabla u|^p + |u|^p\,d\mu\Big)^{1/p}\\ =& C_S(M,p)\|u\|_{W^{1,p}(M)}\end{aligned}$$

for every C^1–function $u : M \to \mathbb{R}$ (or $u \in W^{1,p}(M)$), where $p^* = \frac{(n-1)p}{n-p-1}$ is the *Sobolev conjugate exponent* of p. It is well known that a bound on $C_S(M,1)$ implies a bound on $C_S(M,p)$, for every $p \in [1, n-1)$ (see [7, Chapter 2, Section 5], for instance), hence we concentrate on the case $p = 1$, where $1^* = \frac{n-1}{n-2}$.

We first want to argue localizing things by means of the construction of the previous section. We then have a finite family of hypercubes Q_i centered at $p_i \in M_0$, the partition of unity ρ_i and a parametrization $x \mapsto \Theta(x) = x + \theta_i(x)\nu_i$ on $\Pi_{p_i}M_0 \cap Q_i$ of each piece $M \cap Q_i$ of any smooth hypersurface $M \in \mathfrak{C}^1_\delta(M_0)$, where $\nu_i = \nu(p_i)$ and the functions $\theta_i : \Pi_{p_i}M_0 \to \mathbb{R}$ satisfy $\|\theta_i\|_{C^1(\Pi_{p_i}M_0)} \leqslant 2\delta$, for every $i \in \{1, \dots, k\}$. Moreover, in dealing with any piece $M \cap Q_i$, we will assume (without clearly losing generality) that $\Pi_{p_i}M_0 = \mathbb{R}^{n-1} \subseteq \mathbb{R}^n$ and we observe that in such parametrization, by formula (1.4), the Riemannian measure μ associated to the (induced) metric g on M is given by $\mu = J\Theta\,\mathscr{L}^{n-1}$, with $\mathscr{L}^{n-1}$ the Lebesgue measure on $\Pi_{p_i}M_0 = \mathbb{R}^{n-1}$ and $J\Theta = \sqrt{1 + |\nabla^{\mathbb{R}^{n-1}}\theta_i|^2}$, which clearly satisfies $1 \leqslant J\Theta \leqslant 1 + 2\delta$.

For every C^1–function $u : M \to \mathbb{R}$, we can write

$$\Big(\int_M |u|^{\frac{n-1}{n-2}}\,d\mu\Big)^{\frac{n-2}{n-1}} = \Big(\int_M \Big|\sum_{i=1}^k u\rho_i\Big|^{\frac{n-1}{n-2}}\,d\mu\Big)^{\frac{n-2}{n-1}} \leqslant \sum_{i=1}^k \Big(\int_{M\cap Q_i} |u\rho_i|^{\frac{n-1}{n-2}}\,d\mu\Big)^{\frac{n-2}{n-1}}$$

as the compact support of $u\rho_i$ is contained in $M \cap Q_i$.

Then, for every C^1 function $v : M \to \mathbb{R}$ with compact support in $M \cap Q_i$, there holds

$$\begin{aligned}\Big(\int_{M\cap Q_i} |v(y)|^{\frac{n-1}{n-2}}\,d\mu(y)\Big)^{\frac{n-2}{n-1}} &= \Big(\int_{\mathbb{R}^{n-1}} |v(x+\theta_i(x)\nu_i)|^{\frac{n-1}{n-2}} J\Theta(x)\,dx\Big)^{\frac{n-2}{n-1}}\\ &\leqslant C(\delta)\Big(\int_{\mathbb{R}^{n-1}} |v(x+\theta_i(x)\nu_i)|^{\frac{n-1}{n-2}}\,dx\Big)^{\frac{n-2}{n-1}},\end{aligned}$$

as $J\Theta \leqslant 1+2\delta$ and applying the Sobolev inequality for functions with compact support in $\mathbb{R}^{n-1}$, we have

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^{n-1}} |v(x+\theta_i(x)\nu_i)|^{\frac{n-1}{n-2}}\,dx\Big)^{\frac{n-2}{n-1}}\\
&\leqslant C\int_{\mathbb{R}^{n-1}} |\nabla^{\mathbb{R}^{n-1}}[v(x+\theta_i(x)\nu_i)]|\,dx\\
&= C\int_{\mathbb{R}^{n-1}} \big|\nabla v(x+\theta_i(x)\nu_i)\circ\big(\mathrm{Id}+\nabla^{\mathbb{R}^{n-1}}\theta_i(x)\otimes\nu_i\big)\big|\,dx\\
&\leqslant C\int_{\mathbb{R}^{n-1}} |\nabla v(x+\theta_i(x)\nu_i)|\,\big|\mathrm{Id}+\nabla^{\mathbb{R}^{n-1}}\theta_i(x)\otimes\nu_i\big|\,dx\\
&= C\int_{\mathbb{R}^{n-1}} |\nabla v(x+\theta_i(x)\nu_i)|\,\sqrt{1+|\nabla^{\mathbb{R}^{n-1}}\theta_i|^2}\,dx\\
&= C\int_M |\nabla v(y)|\,d\mu(y)\,,
\end{aligned}\tag{1.18}$$

as $\sqrt{1+|\nabla^{\mathbb{R}^{n-1}}\theta_i|^2}=J\Theta$. Hence,

$$\Big(\int_{M\cap Q_i}|v|^{\frac{n-1}{n-2}}\,d\mu\Big)^{\frac{n-2}{n-1}}\leqslant C(\delta)\int_M|\nabla v|\,d\mu$$

and setting $v_i=u\rho_i$, after summing on $i\in\{1,\dots,k\}$, we conclude

$$\begin{aligned}
\Big(\int_M|u|^{\frac{n-1}{n-2}}\,d\mu\Big)^{\frac{n-2}{n-1}} &\leqslant \sum_{i=1}^k\Big(\int_{M\cap Q_i}|v_i|^{\frac{n-1}{n-2}}\,d\mu\Big)^{\frac{n-2}{n-1}}\\
&\leqslant C(\delta)\sum_{i=1}^k\int_M|\nabla v_i|\,d\mu\\
&= C(\delta)\sum_{i=1}^k\int_M|\nabla u|\rho_i+|u|\,|\nabla\rho_i|\,d\mu\\
&\leqslant C(\delta)\int_M|\nabla u|\,d\mu+C(M_0,\delta)\int_M|u|\,d\mu\,,
\end{aligned}\tag{1.19}$$

as $|\nabla\rho_i|\leqslant C(M_0,\delta)$, for every $i\in\{1,\dots,k\}$. This clearly gives a uniform bound on $C_S(M,1)$ for all the hypersurfaces in $\mathfrak{C}^1_\delta(M_0)$, depending only on M_0 (in particular, on its second fundamental form B_0, as we said in the previous section) and $\delta>0$.

Let now see an alternate line, based on the "global" graph representation of the hypersurfaces $M\in\mathfrak{C}^1_\delta(M_0)$ over M_0.

For every C^1 function $u:M\to\mathbb{R}$, we have

$$\Big(\int_M|u(y)|^{\frac{n-1}{n-2}}\,d\mu(y)\Big)^{\frac{n-2}{n-1}}=\Big(\int_{M_0}|u(x+\psi(x)\nu(x))|^{\frac{n-1}{n-2}}\,J\Psi(x)\,d\mu_0(x)\Big)^{\frac{n-2}{n-1}}$$

where $J\Psi$ is the Jacobian of the map $\Psi:M_0\to M$ and it is an easy check that, at every point $x\in M_0$, there holds

$$\frac{1}{C(\mathrm{B}_0,\delta)}\leqslant J\Psi\leqslant C(\mathrm{B}_0,\delta)\,,\tag{1.20}$$

for some constant $C(\mathrm{B}_0,\delta)>0$, where B_0 is the second fundamental form of M_0. Moreover, $C(\mathrm{B}_0,\delta)$ goes to 1 as $\delta\to0$. Notice that the fact that B_0 appears here can be seen from the expression of $d\Psi$, that is

$$d\Psi_x=\mathrm{Id}_{T_xM_0}+d\psi_x\otimes\nu(x)+\psi(x)d\nu_x\,,$$

as, by the Gauss–Weingarten relations (1.7), $d\nu_x$ is related to $\mathrm{B}_0(x)$.

Then, by applying the Sobolev inequality holding for M_0, we have

$$\begin{aligned}
&\Big(\int_{M_0} |u(x+\psi(x)\nu(x))|^{\frac{n-1}{n-2}}\, d\mu_0(x)\Big)^{\frac{n-2}{n-1}}\\
&\leqslant C_S(M_0,1)\int_{M_0} \big|\nabla^0[u(x+\psi(x)\nu(x))]\big|\, d\mu_0(x)\\
&\quad + C_S(M_0,1)\int_{M_0} |u(x+\psi(x)\nu(x))|\, d\mu_0(x)\\
&\leqslant C_S(M_0,1)\int_{M_0} |\nabla u(x+\psi(x)\nu(x))|\,|d\Psi(x)|\, d\mu_0(x)\\
&\quad + C_S(M_0,1)\int_{M_0} |u(x+\psi(x)\nu(x))|\, d\mu_0(x)\\
&\leqslant C(M_0,\delta)\int_{M} |\nabla u(y)|\, J\Psi^{-1}(y)\, d\mu(y)\\
&\quad + C(M_0,\delta)\int_{M} |u(y)|\, J\Psi^{-1}(y)\, d\mu(y)\\
&\leqslant C(M_0,\delta)\Big(\int_{M} |\nabla u(y)|\, d\mu(y) + \int_{M} |u(y)|\, d\mu(y)\Big)\,.
\end{aligned}$$

Hence,

$$\Big(\int_M |u|^{\frac{n-1}{n-2}}\, d\mu\Big)^{\frac{n-2}{n-1}} \leqslant C(M_0,\delta)\Big(\int_M |\nabla u|\, d\mu + \int_M |u|\, d\mu\Big)\,.$$

As before, this means that the constant $C(M_0,\delta)$ is a uniform bound on $C_S(M,1)$ for all the hypersurfaces in $\mathfrak{C}^1_\delta(M_0)$, moreover, since $C(M_0,\delta)\to 1$, as $\delta\to 0$, it also shows the continuous dependence of $C_S(M,1)$ under the C^1–convergence of the hypersurfaces.

Theorem 1.2.1. *Let $M_0 \subseteq \mathbb{R}^n$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^n$. Then, there exist uniform bounds, depending only on M_0 and δ (more precisely, on the "C^1– structure" of the immersion of M_0 in $\mathbb{R}^n$, its dimension and its second fundamental form), for all the hypersurfaces $M \in \mathfrak{C}^1_\delta(M_0)$ on:*

(i) *the volume of M from above and below away from zero,*

(ii) *the Sobolev constants for $p\in[1,n-1)$ of the embeddings $W^{1,p}(M)\hookrightarrow L^{p^*}(M)$,*

(iii) *the Sobolev constants for $p\in(n-1,+\infty]$ of the embeddings $W^{1,p}(M)\hookrightarrow C^{0,1-(n-1)/p}(M)$,*

(iv) *the constants in the Poincaré–Wirtinger inequalities on M for $p\in[1,+\infty]$,*

(v) *the constants in the embeddings of the fractional Sobolev spaces $W^{s,p}(M)$,*

(vi) *the constants in the Gagliardo–Nirenberg interpolation inequalities on M.*

Moreover, all these bounds go to the corresponding constants for M_0, as $\delta\to 0$.

Proof.

(i) This is trivial due to the C^1–closedness of M to M_0.

(ii) As explained at the beginning of the section, we can estimate the constant in the Sobolev inequality for $p\in[1,n-1)$, by means of $C_S(M,1)$, which is uniformly bounded for all the hypersurfaces $M\in\mathfrak{C}^1_\delta(M_0)$, by the above discussion.

(iii) If $p>n-1$, we show that there exists a uniform constant $C(M_0,p,\delta)$ such that

$$\|u\|_{C^{0,\alpha}(M)} \leqslant C(M_0,p,\delta)\|u\|_{W^{1,p}(M)} \tag{1.21}$$

with $\alpha = 1 - (n-1)/p$ and

$$\|u\|_{C^{0,\alpha}} = \sup_{y \in M} |u(y)| + \sup_{y,y^* \in M,\, y \neq y^*} \frac{|u(y) - u(y^*)|}{|y - y^*|^\alpha}\,,$$

for all $M \in \mathfrak{C}^1_\delta(M_0)$ and every C^1 function $u : M \to \mathbb{R}$.

In the same setting and notation at the beginning of this section, it is easy to see that we can choose a special family of hypercubes Q_i such that enlarging their edges of a small value $\sigma > 0$, we have hypercubes $\widetilde{Q}_i$ with the further property that $M \cap \widetilde{Q}_i$ can be still written as an orthogonal graph on $\Pi_{p_i} M_0 = \mathbb{R}^{n-1} \subseteq \mathbb{R}^n$.

The following holds

$$\sup_{y \in M} |u(y)| \leqslant \sum_{i=1}^k \sup_{y \in M \cap Q_i} |u(y)\rho_i(y)|$$

and for every C^1 function $v : M \to \mathbb{R}$ with compact support in $M \cap Q_i$, by applying the Sobolev inequality for $p > n-1$ in $\mathbb{R}^{n-1}$ and arguing as in obtaining estimate (1.18), we have

$$\begin{aligned}
\sup_{y \in M \cap Q_i} |v(y)| &= \sup_{x \in \mathbb{R}^{n-1}} |v(x + \theta_i(x)\nu_i)| \\
&\leqslant C \Big(\int_{\mathbb{R}^{n-1}} \big|\nabla v(x + \theta_i(x)\nu_i) \circ \big(\mathrm{Id} + \nabla^{\mathbb{R}^{n-1}} \theta_i(x) \otimes \nu_i\big)\big|^p \, dx \Big)^{1/p} \\
&\leqslant C(\delta) \Big(\int_{\mathbb{R}^{n-1}} |\nabla v(x + \theta_i(x)\nu_i)|^p \, dx \Big)^{1/p} \\
&\leqslant C(\delta) \Big(\int_{\mathbb{R}^{n-1}} |\nabla v(x + \theta_i(x)\nu_i)|^p J\Theta \, dx \Big)^{1/p} \\
&= C(\delta) \Big(\int_M |\nabla v(y)|^p \, d\mu(y) \Big)^{1/p}\,,
\end{aligned} \tag{1.22}$$

as $J\Theta \geqslant 1$. Setting $v_i = u\rho_i$ and estimating as in getting inequality (1.19), we conclude

$$\sup_M |u| \leqslant C(M_0, p, \delta) \Big(\int_M |\nabla u|^p + |u|^p \, d\mu \Big)^{1/p}. \tag{1.23}$$

Regarding the seminorm $[u]_{C^{0,\alpha}} = \sup_{y,y^* \in M,\, y \neq y^*} \frac{|u(y) - u(y^*)|}{|y - y^*|^\alpha}$, given two points $y, y^* \in M$, we have

$$|u(y) - u(y^*)| = \Big| \sum_{i=1}^k v_i(y) - v_i(y^*) \Big| \leqslant \sum_{i=1}^k |v_i(y) - v_i(y^*)|\,. \tag{1.24}$$

Then, for any C^1 function $v : M \to \mathbb{R}$ with compact support in $M \cap Q_i$, if y and y^* both belong to the intersection of M with the "enlarged" hypercube $\widetilde{Q}_i$, we can write $y = x + \theta_i(x)\nu_i$ and $y^* = x^* + \theta_i(x^*)\nu_i$ for some $x, x^* \in \widetilde{Q}_i \cap \Pi_{p_i} M_0$ (by our initial choice of the family Q_i) and there holds

$$\begin{aligned}
|v(y) - v(y^*)| &= |v(x + \theta_i(x)\nu_i) - v(x^* + \theta_i(x^*)\nu_i)| \\
&\leqslant C(M_0, p)\, |x - x^*|^\alpha \, \|\nabla^{\mathbb{R}^{n-1}} (v \circ \Theta)\|_{L^p(\mathbb{R}^{n-1})} \\
&\leqslant C(M_0, p, \delta)\, |y - y^*|^\alpha \, \|\nabla^{\mathbb{R}^{n-1}} (v \circ \Theta)\|_{L^p(\mathbb{R}^{n-1})} \\
&\leqslant C(M_0, p, \delta)\, |y - y^*|^\alpha \, \|\nabla v\|_{L^p(M)}\,,
\end{aligned}$$

where the first inequality follows as in the proof of Theorem 4 in Section 5.6.2 of [28], the second one holds since $|x - x^*| \leqslant |y - y^*|$ and the third one is obtained arguing like in estimate (1.22).

If both y^* and y do not belong to $M \cap \widetilde{Q}_i$ clearly $|v(y) - v(y^*)| = 0$, while if $y \in M \cap \widetilde{Q}_i$ with $v(y) \neq 0$ but $y^* \notin M \cap \widetilde{Q}_i$, then $y \in M \cap Q_i$, hence $|y - y^*| \geqslant \sigma$ and

$$\frac{|v(y) - v(y^*)|}{|y - y^*|^\alpha} \leqslant \frac{|v(y)|}{\sigma^\alpha} \leqslant C(M_0, p, \delta) \frac{\|\nabla v\|_{L^p(M)}}{\sigma^\alpha},$$

by estimate (1.22).

It follows that, for every y and y^* in M, we have

$$\frac{|v(y) - v(y^*)|}{|y - y^*|^\alpha} \leqslant C(M_0, p, \delta)(1 + \sigma^{-\alpha}) \|\nabla v\|_{L^p(M)}.$$

Then, putting together this and inequality (1.24), we conclude, for every y and y^* in M,

$$|u(y) - u(y^*)| \leqslant \sum_{i=1}^{k} |v_i(y) - v_i(y^*)| \leqslant C(M_0, p, \delta)\, |y - y^*|^\alpha\, \|\nabla u\|_{W^{1,p}(M)}$$

which, with inequality (1.23) gives the desired estimate (1.21).

(iv) In order to obtain the conclusion for the Poincaré–Wirtinger inequality, for any $p \in [1, +\infty]$ and all $M \in \mathfrak{C}^1_\delta(M_0)$,

$$\|u - \widetilde{u}\|_{L^p(M)} \leqslant C(M_0, p, \delta) \|\nabla u\|_{L^p(M)},$$

where $\widetilde{u} = \fint_M u\, d\mu$, we argue by contradiction assuming this uniform estimate is false. Then, for each $k \in \mathbb{N}$, there would exist a graph hypersurface $M_k \in \mathfrak{C}^1_\delta(M_0)$ and a function $u_k \in W^{1,p}(M_k)$ such that

$$\|u_k - \widetilde{u}_k\|_{L^p(M_k)} \geqslant k \|\nabla u_k\|_{L^p(M_k)}.$$

where $\widetilde{u}_k = \fint_{M_k} u_k\, d\mu_k$. We renormalize these function as

$$v_k = \frac{u_k - \widetilde{u}_k}{\|u_k - \widetilde{u}_k\|_{L^p(M_k)}},$$

then, $\int_{M_k} v_k\, d\mu_k = 0$, $\|v_k\|_{L^p(M_k)} = 1$ and $\|\nabla v_k\|_{L^p(M_k)} \leqslant 1/k$.

If we consider the functions $w_k = v_k \circ \Psi_k : M_0 \to \mathbb{R}$, where $\Psi_k : M_0 \to M_k$ is given by $\Psi_k(x) = x + \psi_k(x)\nu(x)$ (as in the second way to deal with $C_S(M, 1)$, at the beginning of this section), we have

$$0 < C'(M_0, p, \delta) \leqslant \|w_k\|_{L^p(M_0)} \leqslant C(M_0, p, \delta) \tag{1.25}$$

and

$$\|\nabla w_k\|_{L^p(M_0)} \leqslant C(M_0, p, \delta)/k. \tag{1.26}$$

In particular, the functions w_k are equibounded in $W^{1,p}(M_0)$, hence by the Rellich–Kondrachov embedding theorem and the estimate (1.26), there exists a subsequence (not relabeled) converging in $L^p(M_0)$ to a constant function equal to some $\lambda \in \mathbb{R}$ which cannot be zero, by the estimate (1.25). Moreover, there holds

$$\int_{M_0} w_k(x)\, J\Psi_k(x)\, d\mu_0(x) = \int_{M_k} w_k \circ \Psi_k^{-1}(y)\, d\mu_k(y) = \int_{M_k} v_k(y)\, d\mu_k(y) = 0,$$

hence, since $J\Psi_k$ are equibounded (formula (1.20)) and assuming, possibly passing again to a subsequence, that $\mathrm{Vol}(M_k) \to V > 0$, by means of point (i), we conclude

$$0 = \int_{M_0} (w_k(x) - \lambda)\, J\Psi_k(x)\, d\mu_0(x) + \lambda \int_{M_0} J\Psi_k(x)\, d\mu_0(x) \to \lambda V,$$

as $k \to \infty$, being $\int_{M_0} J\Psi_k(x)\, d\mu_0(x) = \mathrm{Vol}(M_k)$. This is clearly a contradiction, as $\lambda, V \neq 0$ and we are done.

The case $p = +\infty$ is analogous.

(v) As for the "usual" (with integer order) Sobolev spaces, all the constants in the embeddings of the fractional Sobolev spaces are also uniform for the family $\mathfrak{C}^1_\delta(M_0)$. The proof is along the same line, localizing with a partition of unity and using the inequalities holding in $\mathbb{R}^{n-1}$ (see [22] and [57]).

(vi) Finally, we want to show that for any q, r real numbers $1 \leqslant q \leqslant +\infty, 1 \leqslant r \leqslant +\infty$ and j, m integers $0 \leqslant j < m$, there exists a constant C depending on j, m, r, q, θ, M_0 and δ such that the following interpolation inequalities hold

$$\|\nabla^j u\|_{L^p(M)} \leqslant C\big(\|\nabla^m u\|_{L^r(M)} + \|u\|_{L^r(M)}\big)^\theta \|u\|^{1-\theta}_{L^q(M)}, \tag{1.27}$$

for all $M \in \mathfrak{C}^1_\delta(M_0)$, where

$$\frac{1}{p} = \frac{j}{n-1} + \theta\Big(\frac{1}{r} - \frac{m}{n-1}\Big) + \frac{1-\theta}{q}$$

for every $\theta \in [j/m, 1]$ such that p is nonnegative, with the exception of the case $r = \frac{n-1}{m-j} \neq 1$ for which the inequality is not valid for $\theta = 1$.

Moreover, if $u : M \to \mathbb{R}$ is a smooth function with $\fint_M u\, d\mu = 0$, inequality (1.27) simplifies to

$$\|\nabla^j u\|_{L^p(M)} \leqslant C\|\nabla^m u\|^\theta_{L^r(M)} \|u\|^{1-\theta}_{L^q(M)}. \tag{1.28}$$

We can obtain inequality (1.27) arguing as in Proposition 5.1 of [47], essentially following the line of the proof of Theorem 3.70 in [7], but substituting the *Sobolev–Poincarè inequality* (41) in the argument there with its version where the constant is uniform for all $M \in \mathfrak{C}^1_\delta(M_0)$. Indeed, the other "ingredients" in such proof are a bound on the volume (uniform, by point (i)) and some "universal" inequalities in which the constants do not depend on the hypersurfaces at all [7, Theorem 3.69].

Such Sobolev–Poincarè inequality (41) in Theorem 3.70 of [7] reads

$$\|u\|_{L^{p^*}(M)} \leqslant C_{SP}(M, p)\|\nabla u\|_{L^p(M)}, \tag{1.29}$$

for every C^1–function $u : M \to \mathbb{R}$ (or $u \in W^{1,p}(M)$) with $\int_M u\, d\mu = 0$, (here, as before, $p^* = \frac{(n-1)p}{n-p-1}$ is the Sobolev conjugate exponent) and we actually need it with a uniform constant, in order to get inequality (1.28), by the very same proof of such theorem.

This inequality actually follows by points (ii) and (iv). Indeed, for every $u \in W^{1,p}(M)$, by Sobolev inequality, we have

$$\|u\|_{L^{p^*}(M)} \leqslant C(M_0, p, \delta)\big(\|\nabla u\|_{L^p(M)} + \|u\|_{L^p(M)}\big)$$

and, by Poincarè–Wirtinger inequality, as $\widetilde{u} = \int_M u\, d\mu = 0$,

$$\|u\|_{L^p(M)} \leqslant C(M_0, p, \delta)\|\nabla u\|_{L^p(M)}$$

hence, we obtain inequality (1.29) with $C_{SP}(M, p)$ bounded by a uniform constant $C(M_0, p, \delta)$, for every $M \in \mathfrak{C}^1_\delta(M_0)$. □

Remark 1.2.2 (The fractional Sobolev spaces $W^{s,p}(M)$).
At point (v) of the theorem above we considered the fractional Sobolev space $W^{s,p}$ on the hypersurfaces $M \in \mathfrak{C}^1_\delta(M_0)$, which are usually defined via local charts for M and partitions of unity, that is, getting back to the definition with the Gagliardo $W^{s,p}$–seminorms in $\mathbb{R}^{n-1}$ (we refer to [3, 21, 22, 57], for details). They can be also defined equivalently by considering directly on M the Gagliardo $W^{s,p}$–seminorm of a function $f \in L^p(M)$, for $s \in (0, 1)$, as follows

$$[f]^p_{W^{s,p}(M)} = \int_M \int_M \frac{|f(x) - f(y)|^p}{|x - y|^{2+sp}}\, d\mu(x) d\mu(y)$$

and setting $\|f\|_{W^{s,p}(M)} = \|f\|_{L^p(M)} + [f]_{W^{s,p}(M)}$. Moreover, the constants giving the equivalence of the two norms obtained by localization or by this direct definition are uniform for all $M \in \mathfrak{C}^1_\delta(M_0)$. Indeed, the localization method is "uniform" for all $M \in \mathfrak{C}^1_\delta(M_0)$, meaning that the number of necessary local charts is fixed and the diffeomorphisms between $\mathbb{R}^{n-1}$ and "corresponding" (associated to correlated local charts, that is, being a graph on the same piece of M_0, as in our construction) local "pieces" of any different hypersurfaces $M \in \mathfrak{C}^1_\delta(M_0)$, are uniformly close each other in C^1–norm.

1.2.2 *Geometric Calderón–Zygmund inequalities*

Theorem 1.2.3. *Let $M_0 \subseteq \mathbb{R}^n$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^n$ and $p \in (1, +\infty)$. Then, if $\delta > 0$ is small enough, there exists a constant $C(M_0, p, \delta)$ such that the following geometric Calderón–Zygmund inequality holds,*

$$\|\mathrm{B}\|_{L^p(M)} \leqslant C(M_0, p, \delta)\big(1 + \|\mathrm{H}\|_{L^p(M)}\big)$$

for every $M \in \mathfrak{C}^1_\delta(M_0)$.

Proof. We recall the local representation as graphs of the hypersurfaces $M \in \mathfrak{C}^1_\delta(M_0)$ over M_0, as at the beginning of the previous section. We have a finite family of hypercubes Q_i centered at $p_i \in M_0$, the partition of unity ρ_i and a parametrization $x \mapsto \Theta(x) = x + \theta_i(x)\nu_i$ on $\Pi_{p_i} M_0 \cap Q_i$ of each piece $M \cap Q_i$ of any smooth hypersurface $M \in \mathfrak{C}^1_\delta(M_0)$, where $\nu_i = \nu(p_i)$ and the functions $\theta_i : \Pi_{p_i} M_0 \to \mathbb{R}$ satisfy $\|\theta_i\|_{C^1(\Pi_{p_i} M_0)} \leqslant 2\delta$, for every $i \in \{1, \dots, k\}$. Moreover, in dealing with any piece $M \cap Q_i$, we will assume (clearly without losing generality) that $\Pi_{p_i} M_0 = \mathbb{R}^{n-1} \subseteq \mathbb{R}^n$ and that $Q_i \cap \Pi_{p_i} M_0$ is the hypercube $Q_{2R} \subseteq \Pi_{p_i} M_0 = \mathbb{R}^{n-1}$ with edges of length $2R > 0$, centered at the origin. Finally, we can also ask that the family of hypercubes $Q'_i \subseteq \mathbb{R}^{n-1}$ with edges parallel to the ones of Q_i and of length R (half of the one of Q_i), centered at p_i, covers any hypersurface $M \in \mathfrak{C}^1_\delta(M_0)$.

By formulas (1.5) and (1.6), in the parametrization of $M \cap Q_i$ given by Θ, the second fundamental form B and mean curvature H of M are then expressed by

$$\mathrm{B} \circ \Theta = -\frac{\mathrm{Hess}^{\mathbb{R}^{n-1}} \theta_i}{\sqrt{1 + |\nabla^{\mathbb{R}^{n-1}} \theta_i|^2}} \tag{1.30}$$

and

$$\mathrm{H} \circ \Theta = -\frac{\Delta^{\mathbb{R}^{n-1}} \theta_i}{\sqrt{1 + |\nabla^{\mathbb{R}^{n-1}} \theta_i|^2}} + \frac{\mathrm{Hess}^{\mathbb{R}^{n-1}} \theta_i(\nabla^{\mathbb{R}^{n-1}} \theta_i, \nabla^{\mathbb{R}^{n-1}} \theta_i)}{\big(\sqrt{1 + |\nabla^{\mathbb{R}^{n-1}} \theta_i|^2}\big)^3}\,.$$

Letting and $\rho : \mathbb{R}^{n-1} \to [0,1]$ a cut–off function with compact support in Q_{2R} and equal to 1 on $Q_R = Q'_i \cap \Pi_{p_i} M_0$ and setting $A_R = \{(x, \theta_i(x)) \, : \, x \in Q_R\}$, $A_{2R} = \{(x, \theta_i(x)) \, : \, x \in Q_{2R}\}$, we have

$$\|\mathrm{B}\|^p_{L^p(A_R)} = \int_{Q_R} |\mathrm{B} \circ \Theta|^p J\Theta \, dx \leqslant \int_{Q_R} \rho^p |\mathrm{Hess}^{\mathbb{R}^{n-1}} \theta_i|^p \, dx = \int_{\mathbb{R}^{n-1}} |\rho \mathrm{Hess}^{\mathbb{R}^{n-1}} \theta_i|^p \, dx\,, \tag{1.31}$$

as $\mu = J\Theta \mathscr{L}^{n-1}$ and $J\Theta = \sqrt{1 + |\nabla^{\mathbb{R}^{n-1}} \theta_i|^2}$. Then, we estimate

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} |\rho \mathrm{Hess}^{\mathbb{R}^{n-1}} \theta_i|^p \, dx &\leqslant C \int_{\mathbb{R}^{n-1}} |\mathrm{Hess}^{\mathbb{R}^{n-1}} (\rho\theta_i)|^p \, dx + C \int_{\mathbb{R}^{n-1}} |2\nabla^{\mathbb{R}^{n-1}} \rho \otimes \nabla^{\mathbb{R}^{n-1}} \theta_i|^p \, dx \\
&\quad + C \int_{\mathbb{R}^{n-1}} |\theta_i \mathrm{Hess}^{\mathbb{R}^{n-1}} \rho|^p \, dx \\
&\leqslant C \int_{\mathbb{R}^{n-1}} |\mathrm{Hess}^{\mathbb{R}^{n-1}} (\rho\theta_i)|^p \, dx + C\,,
\end{aligned}$$

where $C = C(M_0, p, \delta)$, as the last two integrals in the first line are clearly bounded by a constant $C = C(M_0, p, \delta)$.

Hence, applying the standard Calderón–Zygmund estimates in $\mathbb{R}^{n-1}$ (see [34], for instance) to the last term above, we get

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} |\rho \mathrm{Hess}^{\mathbb{R}^{n-1}}\theta_i|^p\,dx \\
&\leqslant C\int_{\mathbb{R}^{n-1}} |\Delta^{\mathbb{R}^{n-1}}(\rho\theta_i)|^p\,dx + C \\
&\leqslant C\int_{\mathbb{R}^{n-1}} |\rho\Delta^{\mathbb{R}^{n-1}}\theta_i|^p\,dx + C\int_{\mathbb{R}^{n-1}} |2\langle\nabla^{\mathbb{R}^{n-1}}\rho\,|\,\nabla^{\mathbb{R}^{n-1}}\theta_i\rangle|^p\,dx + C\int_{\mathbb{R}^{n-1}} |\theta_i\Delta^{\mathbb{R}^{n-1}}\rho|^p\,dx \\
&\leqslant C\int_{\mathbb{R}^{n-1}} \Big|-\rho(\mathrm{H}\circ\Theta)\sqrt{1+|\nabla^{\mathbb{R}^{n-1}}\theta_i|^2} + \frac{\rho\mathrm{Hess}^{\mathbb{R}^{n-1}}\theta_i(\nabla^{\mathbb{R}^{n-1}}\theta_i,\nabla^{\mathbb{R}^{n-1}}\theta_i)}{1+|\nabla^{\mathbb{R}^{n-1}}\theta_i|^2}\Big|^p\,dx + C \\
&\leqslant C\int_{\mathbb{R}^{n-1}} |\rho(\mathrm{H}\circ\Theta)|^p\,dx + C\int_{\mathbb{R}^{n-1}} |\rho\mathrm{Hess}^{\mathbb{R}^{n-1}}\theta_i(\nabla^{\mathbb{R}^{n-1}}\theta_i,\nabla^{\mathbb{R}^{n-1}}\theta_i)|^p\,dx + C \\
&\leqslant C\int_{\mathbb{R}^{n-1}} |\rho(\mathrm{H}\circ\Theta)|^p\,dx + C\int_{\mathbb{R}^{n-1}} |\nabla^{\mathbb{R}^{n-1}}\theta_i|^{2p}|\rho\mathrm{Hess}^{\mathbb{R}^{n-1}}\theta_i|^p\,dx + C
\end{aligned}$$

where the constant C depends only on M_0, p and δ (we estimated the last two integrals in the second line with such a constant, as we did above for the Hessian).

If $\delta > 0$ is small enough, then $C|\nabla^{\mathbb{R}^{n-1}}\theta_i|^{2p} < 1/2$ and we get

$$\int_{\mathbb{R}^{n-1}} |\rho\mathrm{Hess}^{\mathbb{R}^{n-1}}\theta_i|^p\,dx \leqslant 2C\int_{\mathbb{R}^{n-1}} |\rho(\mathrm{H}\circ\Theta)|^p\,dx + 2C \leqslant 2C\int_{Q_{2R}} |(\mathrm{H}\circ\Theta)|^p\,dx + 2C$$

which clearly implies, by formula (1.31),

$$\begin{aligned}
\|\mathrm{B}\|_{L^p(A_R)} \leqslant& C\int_{Q_{2R}} |(\mathrm{H}\circ\Theta)|^p\,dx + C \leqslant C\int_{Q_{2R}} |(\mathrm{H}\circ\Theta)|^p J\Theta\,dx + C \\
\leqslant& C\big(1+\|\mathrm{H}\|^p_{L^p(A_{2R})}\big)\,,
\end{aligned}$$

with $C = C(M_0, p, \delta)$.

Hence, by construction and invariance by isometry,

$$\|\mathrm{B}\|_{L^p(M\cap Q_i')} \leqslant C\big(1+\|\mathrm{H}\|^p_{L^p(M\cap Q_i)}\big) \leqslant C\big(1+\|\mathrm{H}\|^p_{L^p(M)}\big)\,.$$

Since the number of hypercubes Q_i' covering M is fixed and $C = C(M_0, p, \delta)$, we obtain the thesis of the theorem. □

We have an analogous theorem for Schauder estimates, after defining appropriately the Hölder $C^{0,\alpha}$–norm of a tensor T on M, that is,

$$\|T\|_{C^{0,\alpha}(M)} = \sup_M |T| + [T]_{C^{0,\alpha}(M)}$$

where we need to give a meaning to the seminorm $[T]_{C^{0,\alpha}(M)}$.

If T is an m–form (hence, a covariant m–tensor), one possibility is to "extend the action" of the tensor T from the bundle $\oplus^m TM$ of covariant m—tensors on M to the one of the whole "ambient" $\mathbb{R}^n$ by means of the orthogonal projection on the tangent bundle TM (as we identify T_xM with a vector subspace of $T_x\mathbb{R}^n \approx \mathbb{R}^n$, for every $x \in M$). To give an example, if $T = \mathrm{B}$, letting $\pi_x : \mathbb{R}^n \to T_xM$ be the orthogonal projection on the tangent space of M, for every $x \in M$, we can define the "extension" of B (without relabeling it) by considering at every $x \in M$ the bilinear form $\mathrm{B}_x : \oplus^2 T_x\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as $\mathrm{B}_x(v,w) = \mathrm{B}_x(\pi_x(v), \pi_x(w))$. Extending analogously a general m–form T from operating on $\oplus^m TM$ to $\oplus^m T\mathbb{R}^n$, its norm as a multilinear functional is unchanged at every point $x \in M$ and we can then consider its components $T_{j_1\dots j_m}$ in the canonical basis of $\mathbb{R}^n$ to define

$$[T]_{C^{0,\alpha}(M)} = \sum_{j_1,\dots,j_m=1}^{n} [T_{j_1\dots j_m}]_{C^{0,\alpha}(M)} = \sum_{j_1,\dots,j_m=1}^{n} \sup_{\substack{x,y\in M\\ x\neq y}} \frac{|T_{j_1\dots j_m}(x) - T_{j_1\dots j_m}(y)|}{|x-y|^\alpha}.$$

Finally, if the tensor is of general type (it has also contravariant components), we "transform" it in a covariant one by means of the musical isomorphisms (see [31], for instance) and then proceed as above. Anyway, in the following all the tensors will be covariant.

Remark 1.2.4. This "global", partially coordinate–free definition (only the canonical coordinates of $\mathbb{R}^n$ are involved, not any coordinate chart for M) is useful in general, but in our special case of families of hypersurfaces which are representable as graphs on a fixed one, we can also consider an *equivalent* Hölder seminorm by means of the local description of M with the hypercubes Q_i, which is more convenient for our computations. For any m–form T on M, we set (in the notation of the proof of Theorem 1.2.3)

$$[T]_{C^{0,\alpha}(V)} = \sum_{j_1,\dots,j_m=1}^{n-1} [T_{j_1\dots j_m} \circ \Theta]_{C^{0,\alpha}(\Theta^{-1}(V))} = \sum_{j_1,\dots,j_m=1}^{n-1} \sup_{\substack{x,y\in\Theta^{-1}(V)\\ x\neq y}} \frac{|T_{j_1\dots j_m}(\Theta(x)) - T_{j_1\dots j_m}(\Theta(y))|}{|x-y|^\alpha},$$

for every open set $V \subseteq M \cap Q_i$, where $T_{j_1\dots j_m}$ are the components of T in the parametrization $x \mapsto \Theta(x) = x + \theta_i(x)e_{n+1}$. Then, we define

$$[T]_{C^{0,\alpha}(M)} = \sum_{i=1}^{k} [T]_{C^{0,\alpha}(A_R)},$$

by means of the finite family of sets A_R (whose number is fixed) covering $M \in \mathfrak{C}^1_\delta(M_0)$.

Theorem 1.2.5. *Let $M_0 \subseteq \mathbb{R}^n$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^n$ and $\alpha \in (0,1]$. Then, if $\delta > 0$ is small enough, there exists a constant $C(M_0,\alpha,\delta)$ such that the following geometric Schauder estimate holds,*

$$\|\mathrm{B}\|_{C^{0,\alpha}(M)} \leqslant C(M_0,\alpha,\delta)\left(1 + \|\mathrm{H}\|_{C^{0,\alpha}(M)}\right)$$

for every $M \in \mathfrak{C}^{1,\alpha}_\delta(M_0)$.

Proof. In the same setting and notation of the proof of Theorem 1.2.3, for every hypercube Q_i, the function θ_i belongs to $C^{1,\alpha}(Q_{2R})$, with $\|\theta_i\|_{C^{1,\alpha}(Q_{2R})} \leqslant 2\delta$. Then, keeping into account Remark 1.2.4, we deal with $\|\mathrm{B}\|_{C^{0,\alpha}(A_R)}$, which satisfies

$$\|\mathrm{B}\|_{C^{0,\alpha}(A_R)} = \|\mathrm{B}\circ\Theta\|_{C^{0,\alpha}(Q_R)} = \left\| \frac{\mathrm{Hess}^{\mathbb{R}^{n-1}}\theta_i}{\sqrt{1+|\nabla^{\mathbb{R}^{n-1}}\theta_i|^2}} \right\|_{C^{0,\alpha}(Q_R)} \leqslant C\,\|\theta_i\|_{C^{2,\alpha}(Q_R)}, \tag{1.32}$$

by equality (1.30) and since $Q_R = \Theta^{-1}(A_R)$, by construction.

Hence, by the standard Schauder estimates in $Q_{2R} = \Theta^{-1}(A_{2R})$ (see [34], for instance), we get

$$\begin{aligned}
&\|\theta_i\|_{C^{2,\alpha}(Q_R)}\\
&\leqslant C\,\|\Delta^{\mathbb{R}^{n-1}}\theta_i\|_{C^{0,\alpha}(Q_{2R})} + C\|\theta_i\|_{C^{1,\alpha}(Q_{2R})}\\
&\leqslant C\left\|-(\mathrm{H}\circ\Theta)\sqrt{1+|\nabla^{\mathbb{R}^{n-1}}\theta_i|^2} + \frac{\mathrm{Hess}^{\mathbb{R}^{n-1}}\theta_i(\nabla^{\mathbb{R}^{n-1}}\theta_i, \nabla^{\mathbb{R}^{n-1}}\theta_i)}{1+|\nabla^{\mathbb{R}^{n-1}}\theta_i|^2}\right\|_{C^{0,\alpha}(Q_{2R})} + C\\
&\leqslant C\,\|\mathrm{H}\circ\Theta\|_{C^{0,\alpha}(Q_{2R})} + C\,\|\nabla^{\mathbb{R}^{n-1}}\theta_i\|^2_{C^{0,\alpha}(Q_{2R})}\|\mathrm{Hess}^{\mathbb{R}^{n-1}}\theta_i\|_{C^{0,\alpha}(Q_{2R})} + C\\
&\leqslant C\,\|\mathrm{H}\circ\Theta\|_{C^{0,\alpha}(Q_{2R})} + C\delta^2\|\theta_i\|_{C^{2,\alpha}(Q_{2R})} + C\,,
\end{aligned}$$

where the constant C depends only on M_0, α and δ, as $\|\theta_i\|_{C^{1,\alpha}(Q_{2R})} \leqslant 2\delta$. This estimate clearly implies, by formula (1.32) and equality (1.30),

$$\|\mathrm{B}\|_{C^{0,\alpha}(A_R)} \leqslant C\,\|\mathrm{H}\|_{C^{0,\alpha}(M)} + C\delta^2\|\mathrm{B}\|_{C^{0,\alpha}(M)} + C$$

and since the family of sets A_R covering $M \in \mathfrak{C}^1_\delta(M_0)$ is finite and its number is fixed, we conclude

$$\|\mathrm{B}\|_{C^{0,\alpha}(M)} \leqslant C\,\|\mathrm{H}\|_{C^{0,\alpha}(M)} + C\delta^2\|\mathrm{B}\|_{C^{0,\alpha}(M)} + C\,,$$

with a constant C depending only on M_0, α and δ (and we can clearly choose C to be monotonically increasing with δ).

Then, if $\delta > 0$ is small enough, we have $C\delta^2\,\|\mathrm{B}\|^2_{C^{0,\alpha}(M)} < \|\mathrm{B}\|^2_{C^{0,\alpha}(M)}/2$, hence we get

$$\|\mathrm{B}\|_{C^{0,\alpha}(M)} \leqslant 2C\,\|\mathrm{H}\|_{C^{0,\alpha}(M)} + 2C\,,$$

that is,

$$\|\mathrm{B}\|_{C^{0,\alpha}(M)} \leqslant C\left(1 + \|\mathrm{H}\|_{C^{0,\alpha}(M)}\right),$$

where the constant C depends only on M_0, α and δ, which is the thesis of the theorem. □

We now consider families of $(n-1)$–dimensional graph hypersurfaces in $M \in \mathfrak{C}^1_\delta(M_0)$ over M_0 as above, with a uniform bound $\|\mathrm{H}\|_{L^p(M)} \leqslant C_{\mathrm{H}}$ with $p \geqslant n-1$, for every M in such family (by Theorem 1.2.3, if $\delta > 0$ is small enough, this implies $\|\mathrm{B}\|_{L^p(M)} \leqslant C_{\mathrm{B}}$) or $\|\mathrm{B}\|_{L^\infty(M)} \leqslant C_{\mathrm{B}}$.

Arguing again in the same setting and notation of the proof of Theorem 1.2.3, for $p \in (1, +\infty)$ and any C^2–function $u : M \to \mathbb{R}$ (or $u \in W^{2,p}(M)$), we have

$$\|\nabla^2 u\|_{L^p(M)} \leqslant C\sum_{i=1}^{k}\|\nabla^2(u\rho_i)\|_{L^p(M\cap Q_i)} \tag{1.33}$$

(here ∇ is the Levi–Civita connection of M) and, for every C^2 function $v : M \to \mathbb{R}$, with compact support in $M \cap Q_i$, there holds

$$\begin{aligned}
\int_{M\cap Q_i}|\nabla^2 v(y)|^p\,d\mu(y) &= \int_{\mathbb{R}^{n-1}}\left|(\nabla^2 v)(x+\theta_i(x)\nu_i)\right|^p J\Theta(x)\,dx\\
&\leqslant C(\delta)\int_{\mathbb{R}^{n-1}}\left|(\nabla^2 v)(x+\theta_i(x)\nu_i)\right|^p dx\,,
\end{aligned} \tag{1.34}$$

as $J\Theta = \sqrt{1+|\nabla^{\mathbb{R}^{n-1}}\theta_i|^2} \leqslant 1+2\delta$.

In the coordinates given by the parametrization Θ, the coefficients of the metric g of M (induced by $\mathbb{R}^n$) in $M \cap Q_i$ are

$$g_{\ell m}(\Theta(x)) = \delta_{\ell m} + \frac{\partial\theta_i}{\partial x_\ell}(x)\frac{\partial\theta_i}{\partial x_m}(x)\,,$$

hence, they and the ones of the inverse matrix are bounded by a constant depending only on M_0 and δ. By formula (1.2), the Christoffel symbols of the Levi–Civita connection ∇ satisfy

$$|\Gamma^s_{\ell m}(\Theta(x))| \leqslant C \sum_{p,q,r=1}^{n-1} \Big|\frac{\partial(g_{pq}\circ\Theta)}{\partial x_r}(x)\Big| = C \sum_{p,q,r=1}^{n-1} \Big|\frac{\partial^2\theta_i}{\partial x_r \partial x_p}(x)\frac{\partial\theta_i}{\partial x_q}(x)\Big|. \tag{1.35}$$

Then, recalling the first formula (1.7),

$$\begin{aligned}\Big|\frac{\partial^2\theta_i}{\partial x_\ell \partial x_m}(x)\Big| &= \Big|\frac{\partial^2\Theta}{\partial x_\ell \partial x_m}(x)\Big| \\ &= \Big|\Gamma^s_{\ell m}(\Theta(x))\frac{\partial\Theta}{\partial x_s}(x) - \mathrm{B}_{\ell m}(\Theta(x))\nu(\Theta(x))\Big| \\ &\leqslant C|\Gamma^s_{\ell m}(\Theta(x))|\,\Big|\frac{\partial\Theta}{\partial x_s}(x)\Big| + |\mathrm{B}_{\ell m}(\Theta(x))| \\ &\leqslant C|\mathrm{Hess}^{\mathbb{R}^{n-1}}\theta_i(x)|\,|\nabla^{\mathbb{R}^{n-1}}\theta_i(x)|\,(1+|\nabla^{\mathbb{R}^{n-1}}\theta_i(x)|) + |\mathrm{B}(\Theta(x))|\,,\end{aligned}$$

where in the last passage we estimated the Christoffel symbols by means of inequality (1.35). As $|\nabla^{\mathbb{R}^{n-1}}\theta_i| \leqslant 2\delta$, we conclude

$$\begin{aligned}|\mathrm{Hess}^{\mathbb{R}^{n-1}}\theta_i(x)| \leqslant& C|\mathrm{Hess}^{\mathbb{R}^{n-1}}\theta_i(x)|\,|\nabla^{\mathbb{R}^{n-1}}\theta_i(x)| + C|\mathrm{B}(\Theta(x))| \\ \leqslant& C|\mathrm{Hess}^{\mathbb{R}^{n-1}}\theta_i(x)|\delta + C|\mathrm{B}(\Theta(x))|\end{aligned}$$

with a constant C depending only on δ, which implies, if δ is smaller than $1/2C$, the estimate

$$|\mathrm{Hess}^{\mathbb{R}^{n-1}}\theta_i(x)| \leqslant 2C(M_0,\delta)|\mathrm{B}(\Theta(x))|\,,$$

for every $x \in Q_i \cap \Pi_{p_i} M \subseteq \mathbb{R}^{n-1}$.

By the first formula (1.35), it follows

$$|\Gamma^s_{\ell m}(\Theta(x))| \leqslant C|\mathrm{Hess}^{\mathbb{R}^{n-1}}\theta_i(x)|\,|\nabla^{\mathbb{R}^{n-1}}\theta_i| \leqslant C\delta|\mathrm{B}(\Theta(x))|$$

with $C = C(\delta)$, then computing schematically, we have

$$(\nabla^2 v)(\Theta(x)) = \mathrm{Hess}^{\mathbb{R}^{n-1}}(v\circ\Theta)(x) - \Gamma(\Theta(x)) * \nabla^{\mathbb{R}^{n-1}}(v\circ\Theta)(x)\,, \tag{1.36}$$

hence,

$$|(\nabla^2 v)(\Theta(x))| \leqslant C|\mathrm{Hess}^{\mathbb{R}^{n-1}}(v\circ\Theta)(x)| + C\delta|\mathrm{B}(\Theta(x))|\,|\nabla^{\mathbb{R}^{n-1}}(v\circ\Theta)(x)|\,.$$

Applying the Calderón–Zygmund inequality in $\mathbb{R}^{n-1}$, we get

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \big|(\nabla^2 v)(x+\theta_i(x)\nu_i)\big|^p\,dx \leqslant& C\int_{\mathbb{R}^{n-1}} |\mathrm{Hess}^{\mathbb{R}^{n-1}}[v(x+\theta_i(x)\nu_i)]|^p\,dx \\ &+ C\delta\int_{\mathbb{R}^{n-1}} |\mathrm{B}(\Theta(x))|^p\,|\nabla^{\mathbb{R}^{n-1}}[v(x+\theta_i(x)\nu_i])|^p\,dx \\ \leqslant& C\int_{\mathbb{R}^{n-1}} |\Delta^{\mathbb{R}^{n-1}}[v(x+\theta_i(x)\nu_i)]|^p\,dx \\ &+ C(\delta)\int_{\mathbb{R}^{n-1}} |\mathrm{B}(\Theta(x))|^p\,|\nabla v(\Theta(x))|^p\,dx\,. \\ \leqslant& C\int_{\mathbb{R}^{n-1}} |\Delta^{\mathbb{R}^{n-1}}[v(x+\theta_i(x)\nu_i)]|^p\,dx \\ &+ C(\delta)\int_{M\cap Q_i} |\mathrm{B}(y)|^p|\nabla v(y)|^p\,d\mu(y)\,,\end{aligned} \tag{1.37}$$

arguing as in estimate (1.22) to get the last inequality.

Contracting equation (1.36) with the inverse of the metric and estimating, we have

$$|\Delta^{\mathbb{R}^{n-1}}(v\circ\Theta)(x)| \leqslant C|(\Delta v)(\Theta(x))| + C\delta|(\mathrm{B}\circ\Theta)(x)|\,|\nabla^{\mathbb{R}^{n-1}}(v\circ\Theta)(x)|$$

thus, by inequalities (1.34) and (1.37), we obtain

$$\begin{aligned}\int_{M\cap Q_i}|\nabla^2 v(y)|^p\,d\mu(y) &\leqslant C\int_{\mathbb{R}^{n-1}}|(\Delta v)(x+\theta_i(x)\nu_i)|^p\,dx\\ &\quad + C\int_{M\cap Q_i}|\mathrm{B}(y)|^p|\nabla v(y)|^p\,d\mu(y)\\ &\leqslant C\int_{M\cap Q_i}|\Delta v(y)|^p\,d\mu(y)\\ &\quad + C\int_{M\cap Q_i}|\mathrm{B}(y)|^p|\nabla v(y)|^p\,d\mu(y)\,,\end{aligned}$$

with $C = C(M_0, p, \delta)$, arguing again as above.

Getting back to inequality (1.33), we conclude

$$\begin{aligned}\|\nabla^2 u\|^p_{L^p(M)} &\leqslant C\sum_{i=1}^k\|\nabla^2(u\rho_i)\|^p_{L^p(M\cap Q_i)}\\ &\leqslant C\sum_{i=1}^k\int_{M\cap Q_i}|\Delta(u\rho_i)|^p\,d\mu + C\int_{M\cap Q_i}|\mathrm{B}|^p|\nabla(u\rho_i)|^p\,d\mu\\ &\leqslant C\sum_{i=1}^k\int_{M\cap Q_i}|\Delta u|^p\,d\mu + C\int_{M\cap Q_i}\big(|u|^p+|\nabla u|^p\big)\,d\mu\\ &\leqslant C\int_M|\Delta u|^p\,d\mu + C\int_M\big(|u|^p+|\nabla u|^p\big)\,d\mu\,,\end{aligned}\tag{1.38}$$

with $C = C(M_0, p, \delta, \|\mathrm{B}\|_{L^\infty(M)})$. Interpolating the integral of $|\nabla u|^p$ between $\|\nabla^2 u\|_{L^p(M)}$ and $\|u\|_{L^p(M)}$ by means of the uniform Gagliardo–Nirenberg inequalities of the previous section, we obtain the following theorem.

Theorem 1.2.6. *Let $M_0 \subseteq \mathbb{R}^n$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^n$ and $p \in (1, +\infty)$. Then, if $\delta > 0$ is small enough, there exists a constant C which depends only on M_0, p, δ and $\|\mathrm{B}\|_{L^\infty(M)}$ such that the following Calderón–Zygmund inequality holds,*

$$\|\nabla^2 u\|_{L^p(M)} \leqslant C\|\Delta u\|_{L^p(M)} + C\|u\|_{L^p(M)}\tag{1.39}$$

hence,

$$\|u\|_{W^{2,p}(M)} \leqslant C\|\Delta u\|_{L^p(M)} + C\|u\|_{L^p(M)}\,,\tag{1.40}$$

for every hypersurface $M \in \mathfrak{C}^1_\delta(M_0)$ and $u \in W^{2,p}(M)$.

Remark 1.2.7. Notice that if $p < n-1$, we can modify the chain of inequalities (1.38) as follows:

$$\begin{aligned}
\|\nabla^2 u\|^p_{L^p(M)} &\leqslant C\sum_{i=1}^k \|\nabla^2(u\rho_i)\|^p_{L^p(M\cap Q_i)} \\
&\leqslant C\sum_{i=1}^k \int_{M\cap Q_i} |\Delta(u\rho_i)|^p\,d\mu + C\int_{M\cap Q_i} |\mathrm{B}|^p|\nabla(u\rho_i)|^p\,d\mu \\
&\leqslant C\sum_{i=1}^k \int_{M\cap Q_i} |\Delta(u\rho_i)|^p\,d\mu \\
&\quad + C\Big(\int_{M\cap Q_i} |\mathrm{B}|^{n-1}\,d\mu\Big)^{\frac{p}{(n-1)}}\Big(\int_{M\cap Q_i} |\nabla(u\rho_i)|^{\frac{(n-1)p}{(n-p-1)}}\,d\mu\Big)^{\frac{(n-p-1)}{(n-1)}} \\
&\leqslant C\sum_{i=1}^k \int_{M\cap Q_i} |\Delta(u\rho_i)|^p\,d\mu + C\|\mathrm{B}\|^p_{L^{n-1}(M\cap Q_i)}\|\nabla^2(u\rho_i)\|^p_{L^p(M\cap Q_i)}\,.
\end{aligned}$$

Hence, arguing as before, it is easy to conclude that inequalities (1.39) and (1.40) hold with a constant $C = C(M_0, p, \delta, \|\mathrm{B}\|_{L^{n-1}(M)})$, if $\delta > 0$ is small enough. Moreover, since we have seen in Theorem 1.2.3 that a control on $\|\mathrm{H}\|_{L^{n-1}(M)}$ implies a control on $\|\mathrm{B}\|_{L^{n-1}(M)}$, we have uniform Calderón–Zygmund inequalities for families of $(n-1)$–dimensional graph hypersurfaces over M_0, with mean curvature uniformly bounded in $L^{n-1}(M)$.

With a similar argument, computing as in Theorem 1.2.5, we have analogous Schauder estimates for $C^{2,\alpha}$ functions $u : M \to \mathbb{R}$, with $M \in \mathfrak{C}^{1,\alpha}_\delta(M_0)$ and $\delta > 0$ small enough,

$$\|u\|_{C^{2,\alpha}(M)} \leqslant C\|\Delta u\|_{C^{0,\alpha}(M)} + C\|u\|_{C^{0,\alpha}(M)}\,, \tag{1.41}$$

where the constant C depends only on M_0, $\alpha \in (0,1]$, δ and $\|\mathrm{B}\|_{C^{0,\alpha}(M)}$ (or $\|\mathrm{H}\|_{C^{0,\alpha}(M)}$, by Theorem 1.2.5).

Remark 1.2.8. Localizing and computing in coordinates (see Remark 1.2.4), it is easy to generalize estimates (1.39), (1.40) and (1.41) also to tensors, under the same hypotheses. The same holds also for all the estimates of the previous section (see [47] for an example of how this can be done).

1.2.3 *Geometric higher order Calderón–Zygmund estimates*

We let M_0 as above and $p > 1$, we want to deal with $\|\nabla^k\mathrm{B}\|_{L^p(M)}$, assuming that we have a uniform bound $\|\mathrm{H}\|_{L^q(M)} \leqslant C_\mathrm{H}$ with $q > n-1$, where M is an $(n-1)$–dimensional graph hypersurfaces over M_0 in $\mathfrak{C}^1_\delta(M_0)$ as above, if $\delta > 0$ is small enough, which implies $\|\mathrm{B}\|_{L^q(M)} \leqslant C_\mathrm{B}$, by Theorem (1.2.3).

Theorem 1.2.9. *Let $M_0 \subseteq \mathbb{R}^n$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^n$. Then, for any $q > n-1$, if $\delta > 0$ is small enough, there exists a constant C which depends only on M_0, p, q, δ and $\|\mathrm{H}\|_{L^q(M)}$, such that the following geometric higher order Calderón–Zygmund inequality holds, for $p \in (1, n-1)$,*

$$\|\nabla^k\mathrm{B}\|_{L^p(M)} \leqslant C\big(1 + \|\nabla^k\mathrm{H}\|_{L^p(M)}\big)$$

hence,

$$\|\mathrm{B}\|_{W^{k,p}(M)} \leqslant C\big(1 + \|\mathrm{H}\|_{W^{k,p}(M)}\big)\,, \tag{1.42}$$

for any hypersurface $M \in \mathfrak{C}^1_\delta(M_0)$ and $k \in \mathbb{N}$.
Moreover, the same inequalities hold for any $p \in (1, +\infty)$ with a constant C depending only on M_0, p, δ and $\|\mathrm{B}\|_{L^\infty(M)}$.

Proof. We first deal with the case $p \in (1, n-1)$. Fixed $k \in \mathbb{N}$, by means of inequality (1.39), which holds with a constant $C = C(M_0, p, \delta, \|\mathrm{B}\|_{L^{n-1}(M)})$, by Remark 1.2.7 and taking into account Remark 1.2.8, we have

$$
\begin{aligned}
\|\nabla^k \mathrm{B}\|_{L^p(M)} &= \|\nabla_{i_1}\nabla_{i_2}(\nabla_{i_3}\cdots\nabla_{i_k}\mathrm{B})\|_{L^p(M)} \\
&\leqslant C\|\Delta(\nabla_{i_3}\cdots\nabla_{i_k}\mathrm{B})\|_{L^p(M)} + C\|\nabla_{i_3}\cdots\nabla_{i_k}\mathrm{B}\|_{L^p(M)} \\
&= C\|g^{\ell m}\nabla_\ell\nabla_m\nabla_{i_3}\cdots\nabla_{i_k}\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} \\
&\leqslant C\|g^{\ell m}\nabla_\ell\nabla_{i_3}\nabla_m\cdots\nabla_{i_k}\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} \\
&\quad + C\|\mathrm{Riem} * \nabla^{k-2}\mathrm{B}\|_{L^p(M)} + C\|\nabla\mathrm{Riem} * \nabla^{k-3}\mathrm{B}\|_{L^p(M)} \\
&\leqslant C\|g^{\ell m}\nabla_\ell\nabla_{i_3}\nabla_{i_4}\nabla_m\cdots\nabla_{i_k}\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} \\
&\quad + C\|\mathrm{Riem} * \nabla^{k-2}\mathrm{B}\|_{L^p(M)} + C\|\nabla\mathrm{Riem} * \nabla^{k-3}\mathrm{B}\|_{L^p(M)} \\
&\quad + C\|\nabla^2\mathrm{Riem} * \nabla^{k-4}\mathrm{B}\|_{L^p(M)} \\
&\qquad\qquad \cdots \\
&\leqslant C\|g^{\ell m}\nabla_\ell\nabla_{i_3}\nabla_{i_4}\cdots\nabla_{i_k}\nabla_m\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} \\
&\quad + C\sum_{s=0}^{k-2}\|\nabla^s\mathrm{Riem} * \nabla^{k-2-s}\mathrm{B}\|_{L^p(M)} \\
&\leqslant C\|g^{\ell m}\nabla_{i_3}\nabla_\ell\nabla_{i_4}\cdots\nabla_{i_k}\nabla_m\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} \\
&\quad + C\sum_{s=0}^{k-2}\|\nabla^s\mathrm{Riem} * \nabla^{k-2-s}\mathrm{B}\|_{L^p(M)} \\
&\qquad\qquad \cdots \\
&\leqslant C\|g^{\ell m}\nabla_{i_3}\nabla_{i_4}\cdots\nabla_{i_k}\nabla_\ell\nabla_m\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} \\
&\quad + C\sum_{s=0}^{k-2}\|\nabla^s\mathrm{Riem} * \nabla^{k-2-s}\mathrm{B}\|_{L^p(M)} \\
&= C\|\nabla^{k-2}\Delta\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} \\
&\quad + C\sum_{s=0}^{k-2}\|\nabla^s\mathrm{Riem} * \nabla^{k-2-s}\mathrm{B}\|_{L^p(M)}
\end{aligned}
$$

where the symbol $T * S$ (following Hamilton [40]) denotes a tensor formed by a sum of terms each one given by some contraction of the pair T, S with the inverse of the metric g^{ij}. A very useful property of such $*$ product is that $|T * S| \leqslant C|T||S|$ where the constant C depends only on the "algebraic structure" of $T * S$, moreover, it clearly holds $\nabla T * S = \nabla T * S + T * \nabla S$.

By formula (1.12) for the Riemann tensor, we can then write $\mathrm{Riem} = \mathrm{B} * \mathrm{B}$, hence

$$
\begin{aligned}
\|\nabla^k\mathrm{B}\|_{L^p(M)} &\leqslant C\|\nabla^{k-2}\Delta\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} \\
&\quad + C\sum_{s=0}^{k-2}\|\nabla^s(\mathrm{B} * \mathrm{B}) * \nabla^{k-2-s}\mathrm{B}\|_{L^p(M)} \\
&\leqslant C\|\nabla^{k-2}\Delta\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} \\
&\quad + C\sum_{\substack{s,r,t\in\mathbb{N} \\ s+r+t=k-2}}\|\nabla^s\mathrm{B} * \nabla^r\mathrm{B} * \nabla^t\mathrm{B}\|_{L^p(M)}\,.
\end{aligned} \tag{1.43}
$$

Now, by Simons' identity (1.10), we have

$$
\nabla^{k-2}\Delta\mathrm{B} = \nabla^k\mathrm{H} + \nabla^{k-2}(\mathrm{H}\mathrm{B}^2) - \nabla^{k-2}(|\mathrm{B}|^2\mathrm{B})\,,
$$

hence,

$$\|\nabla^{k-2}\Delta \mathrm{B}\|_{L^p(M)} \leqslant \|\nabla^k \mathrm{H}\|_{L^p(M)} + C \sum_{\substack{s,r,t\in\mathbb{N}\\ s+r+t=k-2}} \|\nabla^s \mathrm{B} * \nabla^r \mathrm{B} * \nabla^t \mathrm{B}\|_{L^p(M)}\,.$$

Using this estimate in inequality (1.43), we conclude

$$\begin{aligned}\|\nabla^k \mathrm{B}\|_{L^p(M)} \leqslant\, & C\|\nabla^k \mathrm{H}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\ & + C \sum_{\substack{s,r,t\in\mathbb{N}\\ s+r+t=k-2}} \|\nabla^s \mathrm{B} * \nabla^r \mathrm{B} * \nabla^t \mathrm{B}\|_{L^p(M)}\,.\end{aligned}$$

We now estimate any of the terms in the last sum as follows: we have

$$\|\nabla^s \mathrm{B} * \nabla^r \mathrm{B} * \nabla^t \mathrm{B}\|_{L^p(M)} \leqslant\, C\|\nabla^s \mathrm{B}\|_{L^{\alpha p}(M)}\|\nabla^r \mathrm{B}\|_{L^{\beta p}(M)}\|\nabla^t \mathrm{B}\|_{L^{\gamma p}(M)}\,, \tag{1.44}$$

with

$$\alpha = \frac{k+1}{s+1}\,, \qquad \beta = \frac{k+1}{r+1}\,, \qquad \gamma = \frac{k+1}{t+1}\,,$$

hence, $1/\alpha + 1/\beta + 1/\gamma = 1$, as $s + r + t = k - 2$. Moreover, using the interpolation estimates (1.27) (extended to tensors – see Remark 1.2.8), there hold

$$\begin{aligned}\|\nabla^s \mathrm{B}\|_{L^{p\alpha}(M)} &\leqslant C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\overline{\theta}_\alpha}\|\mathrm{B}\|^{1-\overline{\theta}_\alpha}_{L^{n-1}(M)}\\ \|\nabla^r \mathrm{B}\|_{L^{p\beta}(M)} &\leqslant C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\overline{\theta}_\beta}\|\mathrm{B}\|^{1-\overline{\theta}_\beta}_{L^{n-1}(M)}\\ \|\nabla^t \mathrm{B}\|_{L^{p\gamma}(M)} &\leqslant C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\overline{\theta}_\gamma}\|\mathrm{B}\|^{1-\overline{\theta}_\gamma}_{L^{n-1}(M)}\end{aligned}$$

with $\overline{\theta}_\alpha = \frac{s+1}{k+1}$, $\overline{\theta}_\beta = \frac{r+1}{k+1}$ and $\overline{\theta}_\gamma = \frac{t+1}{k+1}$, determined by

$$\begin{aligned}\frac{1}{p\alpha} &= \frac{s}{n-1} + \overline{\theta}_\alpha\Big(\frac{1}{p} - \frac{k}{n-1}\Big) + \frac{1-\overline{\theta}_\alpha}{n-1}\\ \frac{1}{p\beta} &= \frac{r}{n-1} + \overline{\theta}_\beta\Big(\frac{1}{p} - \frac{k}{n-1}\Big) + \frac{1-\overline{\theta}_\beta}{n-1}\\ \frac{1}{p\gamma} &= \frac{t}{n-1} + \overline{\theta}_\gamma\Big(\frac{1}{p} - \frac{k}{n-1}\Big) + \frac{1-\overline{\theta}_\gamma}{n-1}.\end{aligned}$$

Noticing that $\overline{\theta}_\alpha \in (s/k, 1)$, $\overline{\theta}_\beta \in (r/k, 1)$ and $\overline{\theta}_\gamma \in (t/k, 1)$, if we choose θ_α, θ_β and θ_γ such that

$$\frac{s}{k} < \theta_\alpha < \overline{\theta}_\alpha = \frac{s+1}{k+1}, \quad \frac{r}{k} < \theta_\beta < \overline{\theta}_\beta = \frac{r+1}{k+1} \quad \text{and} \quad \frac{t}{k} < \theta_\gamma < \overline{\theta}_\gamma = \frac{t+1}{k+1}\,,$$

respectively close to $\overline{\theta}_\alpha$, $\overline{\theta}_\beta$ and $\overline{\theta}_\gamma$, the uniquely determined values q_α, q_β and q_γ satisfying

$$\begin{aligned}\frac{1}{p\alpha} &= \frac{s}{n-1} + \theta_\alpha\Big(\frac{1}{p} - \frac{k}{n-1}\Big) + \frac{1-\theta_\alpha}{q_\alpha}\\ \frac{1}{p\beta} &= \frac{r}{n-1} + \theta_\beta\Big(\frac{1}{p} - \frac{k}{n-1}\Big) + \frac{1-\theta_\beta}{q_\beta}\\ \frac{1}{p\gamma} &= \frac{t}{n-1} + \theta_\gamma\Big(\frac{1}{p} - \frac{k}{n-1}\Big) + \frac{1-\theta_\gamma}{q_\gamma}\end{aligned}$$

must be close to n, thus properly choosing θ_α, θ_β and θ_γ, as above, we have that q_α, q_β and q_γ are smaller than $q > n-1$. Hence, by the interpolation estimates again, we have

$$\begin{aligned}\|\nabla^s \mathrm{B}\|_{L^{p\alpha}(M)} &\leqslant C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\theta_\alpha}\|\mathrm{B}\|^{1-\theta_\alpha}_{L^{q_\gamma}(M)}\\ \|\nabla^r \mathrm{B}\|_{L^{p\beta}(M)} &\leqslant C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\theta_\beta}\|\mathrm{B}\|^{1-\theta_\beta}_{L^{q_\beta}(M)}\\ \|\nabla^t \mathrm{B}\|_{L^{p\gamma}(M)} &\leqslant C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\theta_\gamma}\|\mathrm{B}\|^{1-\theta_\gamma}_{L^{q_\gamma}(M)}\,.\end{aligned}$$

Then, since $\|B\|_{L^{q_\alpha}(M)}$, $\|B\|_{L^{q_\beta}(M)}$ and $\|B\|_{L^{q_\gamma}(M)}$ are bounded by $C\|B\|_{L^q(M)}$, being the three exponents smaller that q (the volumes are equibounded for all $M \in \mathfrak{C}^1_\delta(M_0)$), we get

$$\begin{aligned}
\|\nabla^s B\|_{L^{p\alpha}(M)} &\leqslant C\big(\|\nabla^k B\|_{L^p(M)} + \|B\|_{L^p(M)}\big)^{\theta_\alpha}\|B\|^{1-\theta_\alpha}_{L^q(M)} \\
\|\nabla^r B\|_{L^{p\beta}(M)} &\leqslant C\big(\|\nabla^k B\|_{L^p(M)} + \|B\|_{L^p(M)}\big)^{\theta_\beta}\|B\|^{1-\theta_\beta}_{L^q(M)} \\
\|\nabla^t B\|_{L^{p\gamma}(M)} &\leqslant C\big(\|\nabla^k B\|_{L^p(M)} + \|B\|_{L^p(M)}\big)^{\theta_\gamma}\|B\|^{1-\theta_\gamma}_{L^q(M)}\,,
\end{aligned}$$

Letting

$$\Theta = (\theta_\alpha + \theta_\beta + \theta_\gamma) < \frac{s+1}{k+1} + \frac{r+1}{k+1} + \frac{t+1}{k+1} = 1\,,$$

as $s + r + t = k - 2$, putting these estimates in inequality (1.44) and recalling Theorem 1.2.3, we conclude

$$\begin{aligned}
\|\nabla^s B * \nabla^r B * \nabla^t B\|_{L^p(M)} &\leqslant C\big(\|\nabla^k B\|_{L^p(M)} + \|B\|_{L^p(M)}\big)^{\Theta}\|B\|^{3-\Theta}_{L^q(M)} \\
&\leqslant C\big(\|\nabla^k B\|_{L^p(M)} + \|B\|_{L^p(M)}\big)^{\Theta}\big(1 + \|H\|_{L^q(M)}\big)^{3-\Theta} \\
&\leqslant C\big(\|\nabla^k B\|_{L^p(M)} + \|B\|_{L^p(M)}\big)^{\Theta}\,,
\end{aligned} \tag{1.45}$$

with $C = C(M_0, p, \delta, \|H\|_{L^{n-1}(M)}, \|H\|_{L^q(M)}) = C(M_0, p, \delta, \|H\|_{L^q(M)})$, as $q > n - 1$.

Hence, by means of Young inequality, as $\Theta < 1$, we estimate

$$\begin{aligned}
\|\nabla^k B\|_{L^p(M)} &\leqslant C\|\nabla^k H\|_{L^p(M)} \\
&\quad + C\|\nabla^{k-2} B\|_{L^p(M)} + C\big(\|\nabla^k B\|_{L^p(M)} + \|B\|_{L^p(M)}\big)^{\Theta} \\
&\leqslant C\|\nabla^k H\|_{L^p(M)} \\
&\quad + C\|\nabla^{k-2} B\|_{L^p(M)} + C\varepsilon\|\nabla^k B\|_{L^p(M)} + C\|B\|_{L^p(M)} + C\,,
\end{aligned}$$

then choosing $\varepsilon > 0$ such that $C\varepsilon < 1/2$, after "absorbing" in the left hand side the term $C\varepsilon\|\nabla^k B\|_{L^p(M)}$ and estimating $\|B\|_{L^p(M)}$ with $C(1 + \|H\|_{L^p(M)})$, we obtain

$$\|\nabla^k B\|_{L^p(M)} \leqslant C\|\nabla^k H\|_{L^p(M)} + C\|\nabla^{k-2} B\|_{L^p(M)} + C\|H\|_{L^p(M)} + C\,.$$

The term $\|\nabla^{k-2} B\|_{L^p(M)}$ can be treated analogously, by interpolation between $\|\nabla^k B\|_{L^p(M)}$ and $\|B\|_{L^p(M)}$ (it is actually easier to deal with it) and $\|H\|_{L^p(M)} \leqslant C(M_0, p, q, \delta)\|H\|_{L^q(M)}$, hence we finally have the desired estimate

$$\|\nabla^k B\|_{L^p(M)} \leqslant C\|\nabla^k H\|_{L^p(M)} + C\,,$$

with $C = C(M_0, p, q, \delta, \|H\|_{L^q(M)})$, for any $M \in \mathfrak{C}^1_\delta(M_0)$ with $\delta > 0$ small enough.

If $p \in (1, +\infty)$, we can argue as before, but using directly inequality (1.39), which holds with a constant $C = C(M_0, p, \delta, \|B\|_{L^\infty(M)})$ and getting inequality (1.45) with a constant $C = C(M_0, p, \delta, \|B\|_{L^\infty(M)})$, by simply choosing a suitably large $q > n - 1$ and estimating $\|B\|_{L^q(M)}$ with $C\|B\|_{L^\infty(M)}$. The rest of the proof goes in the same way, estimating all the terms $\|B\|_{L^q(M)}$ and $\|H\|_{L^q(M)}$ with $C\|B\|_{L^\infty(M)}$. □

1.2.4 *Other inequalities*

For the sake of completeness, we recall some other inequalities that hold uniformly in our setting, even if we will not use them in the sequel.

Let M_0 be a smooth and compact hypersurface embedded in $\mathbb{R}^{n-1}$, bounding a domain E_0 and $\varepsilon > 0$ the width of a tubular neighborhood N_ε of M_0. For any $\delta \in (0, \varepsilon)$, we consider the family $\mathcal{C}^1_\delta(E_0)$, defined as

$$\left\{ E = \Psi(E_0) \, : \, \begin{array}{l} \Psi : \overline{E}_0 \to \overline{E} \text{ is a diffeomorphism with } \|\Psi - \mathrm{Id}\|_{C^1(E_0)} < \delta \\ \Psi(x) = x + \psi(x)\nu_0(x) \text{ for every } x \in M_0 \text{ and } \|\psi\|_{C^1(M_0)} < \delta \end{array} \right\}$$

where ν_0 is the unit normal vector field pointing outward of M_0.

Then, the Jacobian of the map $\Psi : \overline{E}_0 \to \overline{E}$ (and also the tangential one of its restriction to M_0) is bounded from above and from below by some constants which depend only on δ and the second fundamental form of M_0 (see Section 1.2.1 for details).

It clearly follows that if $E \in \mathcal{C}^1_\delta(E_0)$, then $M = \partial E = \Psi(M_0) \in \mathfrak{C}^1_\delta(M_0)$. Moreover, if $M \in \mathfrak{C}^1_{\delta'}(M_0)$, then there exists a smooth function $\psi : M_0 \to \mathbb{R}$ with $\|\psi\|_{C^1(M_0)} < \delta'$, such that $M = \{x + \psi(x)\nu(x) \, : \, x \in M_0\}$, then we can construct a smooth diffeomorphism $\Psi : \overline{E}_0 \to \overline{E}$ as follows (E is the domain bounded by M):

$$\Psi(x) = \begin{cases} x & \text{if } x \in E_0 \setminus N_\varepsilon \\ x + \zeta(d_0(x)/\varepsilon)\psi(\pi_0(x))\nabla^{\mathbb{R}^{n-1}} d_0(x) & \text{if } x \in \overline{E}_0 \cap N_\varepsilon \end{cases}$$

where d_0 is the signed distance function from M_0 (which is negative in E_0) and $t \mapsto \zeta(t)$ is a smooth monotone non–decreasing function, defined on $\mathbb{R}$, such that it is equal to 1 if $t \geqslant 0$ and to 0 if $t \leqslant -1/2$, with $|\zeta'(t)| \leqslant 3$, for every $t \in \mathbb{R}$. So, it follows

$$\begin{aligned} \|\Psi - \mathrm{Id}\|_{C^1(E_0)} &= \|\zeta(d_0(\cdot)/\varepsilon)\psi(\pi_0(\cdot))\nabla^{\mathbb{R}^{n-1}} d_0(\cdot)\|_{C^1(\overline{E}_0 \cap N_\varepsilon)} \\ &\leqslant C(M_0, \varepsilon)\|\psi\|_{C^1(M_0)} \, . \end{aligned}$$

Hence, fixed any $\delta \in (0, \varepsilon)$, depending the constant C only on M_0 and ε, possibly choosing δ' small enough, the set E belongs to $\mathcal{C}^1_\delta(E_0)$.

We now discuss some uniform inequalities involving also the domains which are bounded by the hypersurfaces.

Trace inequalities

Letting E_0, M_0, $\varepsilon > 0$ and $\delta > 0$ as above and any $E \in \mathcal{C}^1_\delta(E_0)$ (with associated smooth diffeomorphism $\Psi : \overline{E}_0 \to \overline{E}$), it is well known that the *trace* of any function $u \in H^1(E)$ (a real function on $M = \partial E$, which we still simply denote by u, that coincides with the restriction of u to M, if $u \in C^0(\overline{E})$) is well defined and that the following *trace inequality* holds (see [61, Chapter 4, Proposition 4.5]),

$$\|u\|^2_{H^{1/2}(M)} \leqslant C_E \int_E u^2 + |\nabla u|^2 \, dx \, , \tag{1.46}$$

which implies

$$\|u - \widetilde{u}\|^2_{H^{1/2}(M)} \leqslant C_E \int_E |\nabla u|^2 \, dx \, ,$$

where $\widetilde{u} = \fint_E u \, dx$ (see also [28, 46]). We want to show that these inequalities hold with uniform constants $C(M_0, \delta)$, for every $E \in \mathcal{C}^1_\delta(E_0)$.

Expressing $\|u\|^2_{H^{1/2}(M)}$ by means of the Gagliardo $W^{1/2,2}$–seminorm of a function $u \in L^2(M)$ and setting $\Phi = \Psi|_{M_0} : M_0 \to M$, we have

$$\begin{aligned}
\|u\|^2_{H^{1/2}(M)} &= \|u\|_{L^2(M)} + [u]^2_{W^{1/2,2}(M)}\\
&= \|u\|_{L^2(M)} + \int_M \int_M \frac{|u(y) - u(y^*)|^2}{|y - y^*|^n}\, d\mu(y)\, d\mu(y^*)\\
&\leqslant C\|u \circ \Phi\|_{L^2(M_0)}\\
&+ \int_{M_0} \int_{M_0} \frac{|u(\Phi(x)) - u(\Phi(x^*))|^2}{|\Phi(x) - \Phi(x^*)|^n}\, J\Phi(x) J\Phi(x^*)\, d\mu_0(x) d\mu_0(x^*)\\
&\leqslant C\|u \circ \Psi\|_{L^2(M_0)}\\
&+ C\int_{M_0} \int_{M_0} \frac{|u(\Psi(x)) - u(\Psi(x^*))|^2}{|x - x^*|^n}\, d\mu_0(x) d\mu_0(x^*)\\
&\leqslant C_{E_0} \int_{E_0} |u(\Psi(x))|^2 + |\nabla^0(u \circ \Psi(x))|^2\, dx\\
&\leqslant C \int_E u^2 + |\nabla u|^2\, dx = C\|u\|^2_{H^1(E)}\,, \qquad (1.47)
\end{aligned}$$

where the constant C depends only on E_0 (we applied inequality (1.46) for E_0 in passing from the fourth to the fifth line) and δ (in bounding $|d\Psi|$, $|d\Phi|$, $J\Psi$ and $J\Phi$ above and below away from zero).

Remark 1.2.10. With a similar argument, we can show the following generalization of this inequality, with a uniform constant

$$\|u\|_{H^{s-1/2}(M)} \leqslant C(E_0, s, \delta)\|u\|_{H^s(E)}$$

(see again [61, Chapter 4, Proposition 4.5]), for $s \in (1/2, 3/2)$.

Inequalities for harmonic extensions

We let E_0, M_0, $\varepsilon > 0$ and $\delta > 0$ as above and $E \in \mathcal{C}^1_\delta(E_0)$ (with associated smooth diffeomorphism $\Psi : \overline{E}_0 \to \overline{E}$), with $M = \partial E \in \mathfrak{C}^1_\delta(M_0)$.

We denote by $u : E \to \mathbb{R}$ the harmonic extension of a function $f : M \to \mathbb{R}$ in $H^{1/2}(M)$ to E. We aim to show that the following inequality (see [61, Chapter 5, Proposition 1.7])

$$\|u\|_{H^1(E)} \leqslant C_E \|f\|_{H^{1/2}(M)}\,, \qquad (1.48)$$

which implies

$$\int_E |\nabla u|^2\, dx \leqslant C_E \|f\|^2_{H^{1/2}(M)}\,,$$

for every $E \in \mathcal{C}^1_\delta(E_0)$, with uniform constants $C = (E_0, \delta)$.

Arguing as above, in formula (1.47), we end up with the following inequalities:

$$\begin{aligned}
\|u\|_{H^1(E)} &\leqslant C(E_0, \delta)\|u \circ \Psi\|_{H^1(E_0)}\\
\|u \circ \Psi\|_{H^1(E_0)} &\leqslant C_{E_0}\|f \circ \Psi\|_{H^{1/2}(M_0)} = C_{E_0}\|f \circ \Phi\|_{H^{1/2}(M_0)}\\
\|f \circ \Phi\|_{H^{1/2}(M_0)} &\leqslant C(M_0, \delta)\|f\|_{H^{1/2}(M)}
\end{aligned}$$

where the second estimate is given by inequality (1.48) for E_0. Putting them together, we have the conclusion.

Remark 1.2.11. As above, we also have the following generalization, for $s \in [1/2, 3/2)$,

$$\|u\|_{H^{s+1/2}(E)} \leqslant C(E_0, s, \delta)\|f\|_{H^s(M)}$$

(see again [61, Chapter 5, Proposition 1.7]).

1.2.5 *Some remarks*

- All the previous uniform constants depend on the geometric properties of M_0, in particular on the maximal width of a tubular neighborhood, its volume and its second fundamental form. Hence, uniformly controlling such quantities gives uniform estimates for larger families of hypersurfaces, see [8, 9, 10, 20, 45] for a deeper and detailed discussion).
- Notice that for Sobolev, Poincaré, interpolation, trace and "harmonic extension" inequalities, we do not ask $\delta > 0$ to be small, but just $\delta < \varepsilon$, while for the Calderón–Zygmund–type inequalities, that we worked out in Section 1.2.2, a smallness condition on δ is necessary for the conclusions.
- All the inequalities hold uniformly also for families of immersed–only hypersurfaces (non necessarily embedded), if they can be expressed as graphs on a fixed compact, smooth hypersurface, possibly immersed–only too.
- It is easy to see that everything we did still works also if the ambient is a *flat*, complete Riemannian manifold, in particular in any flat torus $\mathbb{T}^n$ (as it is in the rest of this thesis). With some effort, the results can be generalized to graph hypersurfaces in any complete Riemannian manifold, then the constants also depend on the geometry (in particular, on the curvature) of such an ambient space.

1.3 HYPERSURFACES IN THE n−DIMENSIONAL FLAT TORUS

In all the following $\mathbb{T}^n \approx \mathbb{R}^n/\mathbb{Z}^n$ is a flat n–dimensional torus, quotient of $\mathbb{R}^n$ by a discrete group of translations generated by some n linearly independent vectors.

Since, in the next chapters, we will deal with embedded smooth hypersurfaces which are boundaries of smooth sets, we give the following definitions.

We say that a set $E \subseteq \mathbb{T}^n$ is a *smooth set* if it is the closure of an open subset of $\mathbb{T}^n$ and its boundary ∂E is a smooth embedded hypersurface (unless otherwise stated all the sets we are going to consider will be smooth). Then, for a smooth set $E \subseteq \mathbb{T}^n$ and $\varepsilon > 0$ small enough, we define the tubular neighborhood N_ε of ∂E, the orthogonal projection map π_E and the signed distance function d_E from ∂E, as in (1.13), (1.15) and (1.14), respectively, replacing M_0 with ∂E.

This clearly implies that the map

$$\partial E \times (-\varepsilon, \varepsilon) \ni (y,t) \mapsto L(y,t) = y + t\nabla d_E(y) = y + t\nu(y) \in N_\varepsilon \tag{1.49}$$

is a smooth diffeomorphism with inverse

$$N_\varepsilon \ni x \mapsto L^{-1}(x) = (\pi_E(x), d_E(x)) \in \partial E \times (-\varepsilon, \varepsilon)\,.$$

Moreover, denoting with JL its Jacobian (relative to the hypersurface ∂E), there holds

$$0 < C_1 \leqslant JL(y,t) \leqslant C_2$$

on $\partial E \times (-\varepsilon, \varepsilon)$, for a couple of constants C_1, C_2, depending on E and ε.

From now on, in all the rest of the work, with N_ε we will always denote a suitable tubular neighborhood of a smooth set, with the above properties.

By means of such tubular neighborhoods of smooth sets $E \subseteq \mathbb{T}^n$, we can speak of "$W^{k,p}$–closedness" (or of "C^k–closedness" and "$C^{k,\alpha}$–closedness") of sets. Indeed, fixed a smooth set E, we say that $F, F' \subseteq \mathbb{T}^n$ are δ–close in $W^{k,p}$ (or in C^k), for some $\delta > 0$ "small enough", if we have $\mathrm{Vol}(F\triangle F') < \delta$ and that $\partial F, \partial F'$ are contained in a tubular neighborhood N_ε of E as above, described by

$$\partial F = \{y + \psi(y)\nu_E(y) \,:\, y \in \partial E\} \qquad \text{and} \qquad \partial F' = \{y + \psi'(y)\nu_E(y) \,:\, y \in \partial E\},$$

for two functions $\psi : \partial E \to \mathbb{R}$ with $\|\psi - \psi'\|_{W^{k,p}(\partial E)} < \delta$ (respectively, $\|\psi - \psi'\|_{C^k(\partial E)} < \delta$ and $\|\psi - \psi'\|_{C^{k,\alpha}(\partial E)} < \delta$). That is, we are asking that the two sets F and F' differ by a set of small Lebesgue measure and that their boundaries are "close" in $W^{k,p}$ (or C^k and $C^{k,\alpha}$) as graphs on ∂E.

Definition 1.3.1. Given a smooth set $E \subseteq \mathbb{T}^n$ and a smooth function $\psi : \partial E \to \mathbb{R}$ such that $\|\psi\|_{C^0(\partial E)}$ is sufficiently small, we define the *normal deformation of E induced by ψ* to be the set E_ψ having as boundary

$$\partial E_\psi = \{x + \psi(x)\nu_E(x) \,:\, x \in \partial E\}\,.$$

Definition 1.3.2. Given a smooth set $E \subseteq \mathbb{T}^n$ and a tubular neighborhood N_ε of ∂E, for any $M < \varepsilon$, we denote by $\mathfrak{C}^1_M(E)$, the class of all sets $F \subseteq E \cup N_\varepsilon$ such that $\mathrm{Vol}(F\triangle E) \leqslant M$ and F is a normal deformation of E induced by some function $\psi_F \in C^1(\partial E)$, that is

$$\partial F = \{y + \psi_F(y)\nu_E(y) \,:\, y \in \partial E\}\,,$$

with $\|\psi_F\|_{C^1(\partial E)} \leqslant M$ (hence, $\partial F \subseteq N_\varepsilon$).
Analogously, we define $\mathfrak{C}^{1,1}_M(E)$ to be the class of all sets F as above, with the associate function ψ_F belongings to $C^{1,1}(\partial E)$ and $\|\psi_F\|_{C^{1,1}(\partial E)} \leqslant M$.

Definition 1.3.3. Given a sequence of smooth sets $F_i \in \mathfrak{C}^1_M(E)$, for some smooth set $E \subseteq \mathbb{T}^n$, we will write $F_i \to F$ in $W^{k,p}$ if there exists $F \in \mathfrak{C}^1_M(E)$ such that for every $\delta > 0$, if $i \in \mathbb{N}$ is large enough there holds $\mathrm{Vol}(F_i\triangle F) < \delta$ and, describing the boundaries of F_i, F as

$$\partial F_i = \{y + \psi_i(y)\nu_E(y) \,:\, y \in \partial E\} \qquad \text{and} \qquad \partial F = \{y + \psi(y)\nu_E(y) \,:\, y \in \partial E\},$$

for some smooth function $\psi_i, \psi : \partial E \to \mathbb{R}$, we have $\|\psi_i - \psi\|_{W^{k,p}(\partial E)} < \delta$.

2 THE AREA FUNCTIONAL

In this sections we discuss the *Area functional* and its basic properties.

Definition 2.0.1 (Area functional)**.** For every smooth set $E \subseteq \mathbb{T}^n$ we define the Area functional

$$\mathcal{A}(\partial E) = \int_{\partial E} d\mu\,,$$

where μ is the "canonical" measure associated to the Riemannian metric on ∂E induced by the metric tensor of $\mathbb{T}^n$, coming from the scalar product of $\mathbb{R}^n$ (it is easy to see that μ coincides with the $(n-1)$–dimensional Hausdorff measure restricted to ∂E).

2.1 FIRST AND SECOND VARIATION

We start by computing the *first variation* of the functional $\mathcal{A}$.

Definition 2.1.1. Let $E \subseteq \mathbb{T}^n$ be a smooth set. Given a smooth map $\Phi : (-\varepsilon, \varepsilon) \times \mathbb{T}^n \to \mathbb{T}^n$, for $\varepsilon > 0$, such that $\Phi_t = \Phi(t, \cdot) : \mathbb{T}^n \to \mathbb{T}^n$ is a one–parameter family of diffeomorphism with $\Phi_0 = \mathrm{Id}$, we say that $E_t = \Phi_t(E)$ is the *variation* of E associated to Φ (or to Φ_t). If moreover there holds $\mathrm{Vol}(E_t) = \mathrm{Vol}(E)$ for every $t \in (-\varepsilon, \varepsilon)$, we call E_t a *volume–preserving* variation of E.
The vector field $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ defined as $X = \frac{\partial \Phi_t}{\partial t}\big|_{t=0}$, is called the *infinitesimal generator* of the variation E_t.

Remark 2.1.2. As we are going to consider only smooth sets E, it is easy to see that this definition of variation is equivalent to have a family of diffeomorphisms Φ_t of E only, indeed these latter can always be extended to the whole $\mathbb{T}^n$.

Moreover, as the relevant objects are actually the boundaries of the sets E and in view of the sequel, we could even consider only smooth "deformations" of ∂E. We then give the following definition since it is easier and more convenient for the computations.

Definition 2.1.3. Given a smooth one parameter family of immersions $\varphi_t : \partial E \to \mathbb{T}^n$, with $t \in (-\varepsilon, \varepsilon)$, we say that φ_t is the "deformation" of ∂E induced by the variation E_t in Definition 2.1.1, if $\varphi_0 = \mathrm{Id}$, $\varphi_t(\partial E) = \partial E_t$ and $\frac{\partial \varphi_t}{\partial t}\big|_{t=0} = X$ along ∂E, where the field X is the *infinitesimal generator* of the variation E_t.

Definition 2.1.4. Given a variation E_t of E, coming from the one–parameter family of diffeomorphism Φ_t, the *first variation of $\mathcal{A}$ at E with respect to Φ_t* is given by

$$\frac{d}{dt}\mathcal{A}(\partial E_t)\Big|_{t=0}.$$

We say that E is a *critical set* for $\mathcal{A}$, if all the first variations relative to variations E_t of E are zero. We say that E is a *critical set* for $\mathcal{A}$ under a volume constraint, if all the first variations relative to volume–preserving variations E_t of E are zero.

It is clear that if E is a minimum for $\mathcal{A}$ (under a volume constraint), then it is a critical set for $\mathcal{A}$ (under a volume constraint). We are now going to compute the first variation of $\mathcal{A}$ and see that it depends only on the restriction to ∂E of the infinitesimal generator X of the variation E_t of E.

Theorem 2.1.5 (First variation of the functional $\mathcal{A}$). *Let $E \subseteq \mathbb{T}^n$ a smooth set and $\Phi : (-\varepsilon, \varepsilon) \times \mathbb{T}^n \to \mathbb{T}^n$ a smooth map giving a variation $E_t = \Phi_t(E)$ with infinitesimal generator $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$. Then,*

$$\frac{d}{dt}\mathcal{A}(\partial E_t)\Big|_{t=0} = \int_{\partial E} \mathrm{H}\langle X|\nu_E\rangle \, d\mu$$

where ν_E is the outer unit normal vector and H the mean curvature of ∂E.
In particular, the first variation of the functional $\mathcal{A}$ depends only on the normal component of the restriction of the infinitesimal generator X to ∂E.

Proof. Let φ_t be the deformation of ∂E induced by the variation E_t, as in Definition 2.1.3.
Denoting by $g_{ij} = g_{ij}(t)$ the induced metrics (via φ_t, as above) on the smooth hypersurfaces ∂E_t and setting $\varphi = \varphi_0$, in a local chart we have

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij}\Big|_{t=0} &= \frac{\partial}{\partial t}\left\langle \frac{\partial \varphi_t}{\partial x_i} \,\Big|\, \frac{\partial \varphi_t}{\partial x_j}\right\rangle\Big|_{t=0} \\
&= \left\langle \frac{\partial X}{\partial x_i} \,\Big|\, \frac{\partial \varphi}{\partial x_j}\right\rangle + \left\langle \frac{\partial X}{\partial x_j} \,\Big|\, \frac{\partial \varphi}{\partial x_i}\right\rangle \\
&= \frac{\partial}{\partial x_i}\left\langle X \,\Big|\, \frac{\partial \varphi}{\partial x_j}\right\rangle + \frac{\partial}{\partial x_j}\left\langle X \,\Big|\, \frac{\partial \varphi}{\partial x_i}\right\rangle - 2\left\langle X \,\Big|\, \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right\rangle \\
&= \frac{\partial}{\partial x_i}\left\langle X_\tau \,\Big|\, \frac{\partial \varphi}{\partial x_j}\right\rangle + \frac{\partial}{\partial x_j}\left\langle X_\tau \,\Big|\, \frac{\partial \varphi}{\partial x_i}\right\rangle - 2\Gamma_{ij}^k\left\langle X_\tau \,\Big|\, \frac{\partial \varphi}{\partial x_k}\right\rangle + 2h_{ij}\langle X \,|\, \nu_E\rangle\,,
\end{aligned}$$

where we used the Gauss–Weingarten relations (1.7) in the last step and we denoted with $X_\tau = X - \langle X|\nu_E\rangle\nu_E$ the "tangential part" of the vector field X along the hypersurface ∂E (seeing $T_x\partial E$ as a hyperplane of $\mathbb{R}^n \approx T_x\mathbb{T}^n$).
Letting ω be the 1–form defined by $\omega(Y) = g(X_\tau, Y)$, this formula can be rewritten as

$$\frac{\partial}{\partial t} g_{ij}\Big|_{t=0} = \frac{\partial \omega_j}{\partial x_i} + \frac{\partial \omega_i}{\partial x_j} - 2\Gamma_{ij}^k \omega_k + 2h_{ij}\langle X|\nu_E\rangle = \nabla_i\omega_j + \nabla_j\omega_i + 2h_{ij}\langle X|\nu_E\rangle\,. \tag{2.1}$$

Hence, by the formula

$$\frac{d}{dt}\det A(t) = \det A(t)\,\mathrm{tr}\,[A^{-1}(t)\circ A'(t)]\,, \tag{2.2}$$

holding for any $n \times n$ squared matrix $A(t)$ dependent on t, we get

$$\begin{aligned}
\frac{\partial}{\partial t}\sqrt{\det g_{ij}}\,\Big|_{t=0} &= \frac{\sqrt{\det g_{ij}}\, g^{ij}\frac{\partial}{\partial t} g_{ij}\Big|_{t=0}}{2} \\
&= \frac{\sqrt{\det g_{ij}}\, g^{ij}\big(\nabla_i\omega_j + \nabla_j\omega_i + 2h_{ij}\langle X \,|\, \nu_E\rangle\big)}{2} \\
&= \sqrt{\det g_{ij}}\big(\mathrm{div} X_\tau + \mathrm{H}\langle X \,|\, \nu_E\rangle\big)\,,
\end{aligned} \tag{2.3}$$

where the divergence is the (Riemannian) one relative to the hypersurface ∂E. Then, we conclude (recalling the discussion after formula (1.1))

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{A}(\partial E_t)\Big|_{t=0} = \frac{\partial}{\partial t}\mathcal{A}(\varphi_t(\partial E))\Big|_{t=0} &= \frac{\partial}{\partial t}\int_{\partial E} d\mu_t\Big|_{t=0}\\
&= \frac{\partial}{\partial t}\int_{\partial E}\sqrt{\det g_{ij}}\,dx\Big|_{t=0}\\
&= \int_{\partial E}\frac{\partial}{\partial t}\sqrt{\det g_{ij}}\Big|_{t=0}dx\\
&= \int_{\partial E}\big(\operatorname{div}X_\tau + \mathrm{H}\langle X\,|\,\nu_E\rangle\big)\sqrt{\det g_{ij}}\,dx\\
&= \int_{\partial E}\big(\operatorname{div}X_\tau + \mathrm{H}\langle X\,|\,\nu_E\rangle\big)\,d\mu\\
&= \int_{\partial E}\mathrm{H}\langle X\,|\,\nu_E\rangle\,d\mu
\end{aligned}$$

where in the last step we applied the divergence theorem, that is, formula (1.3), on ∂E. □

Given a smooth set E and any vector field $X \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$, considering the associated smooth flow $\Phi : (-\varepsilon,\varepsilon)\times\mathbb{T}^n \to \mathbb{T}^n$, defined by the system

$$\begin{cases}\frac{\partial\Phi}{\partial t}(t,x) = X(\Phi(t,x)),\\ \Phi(0,x) = x\end{cases} \tag{2.4}$$

for every $x \in \mathbb{T}^n$ and $t \in (-\varepsilon,\varepsilon)$, for some $\varepsilon > 0$, we have a variation $E_t = \Phi_t(E)$ with infinitesimal generator X. We call this variation the *special variation* associated to X. Moreover, given any smooth vector field $\overline{X} \in C^\infty(\partial E;\mathbb{R}^n)$, it can be extended easily to a smooth vector field $X \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$ with $X|_{\partial E} = \overline{X}$.

Hence, if E is a critical set for $\mathcal{A}$ there holds

$$\int_{\partial E}\mathrm{H}\langle X|\nu_E\rangle\,d\mu = 0\,,$$

for every $X \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$. Choosing a smooth vector field $X \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$ with $X|_{\partial E} = \mathrm{H}\nu_E$, we then obtain the following corollary.

Corollary 2.1.6. *A smooth set $E \subseteq \mathbb{T}^n$ is a critical set for $\mathcal{A}$ if and only if $\mathrm{H} = 0$ on ∂E, that is the condition of a* minimal surface *holds.*

It is less easy to characterize the infinitesimal generators of the volume–preserving variations of E, in order to find an analogous criticality condition on a set E, for the functional $\mathcal{A}$ under a volume constraint.
Given $\Phi : (-\varepsilon,\varepsilon)\times\mathbb{T}^n \to \mathbb{T}^n$ such that $\mathrm{Vol}(\Phi_t(E)) = \mathrm{Vol}(E_t) = \mathrm{Vol}(E)$ for all $t \in (-\varepsilon,\varepsilon)$, we let $X_t \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$ be the family of the vector fields (well) defined by the formula

$$X_t(\Phi(t,z)) = \frac{\partial\Phi}{\partial t}(t,z),$$

for every $t \in (-\varepsilon,\varepsilon)$ and $z \in \mathbb{T}^n$, hence, if $t = 0$, the vector field $X = X_0$ is the infinitesimal generator of the volume–preserving variation E_t. Then, by changing variables, we have

$$0 = \frac{d}{dt}\mathrm{Vol}(E_t) = \frac{d}{dt}\int_{E_t}dx = \frac{d}{dt}\int_E J\Phi(t,z)\,dz = \int_E\frac{\partial}{\partial t}J\Phi(t,z)\,dz\,. \tag{2.5}$$

As $J\Phi(t,z) = \det[d\Phi(t,z)]$, by means of formula (2.2), we obtain

$$\frac{\partial}{\partial t}J\Phi(t,z) = J\Phi(t,z)\,\mathrm{tr}\,[d\Phi(t,z)^{-1}\circ dX_t(\Phi(t,z))\circ d\Phi(t,z)],$$

since, by the definition of X_t above,

$$\frac{\partial}{\partial t} d\Phi(t,z) = d\,\frac{\partial\Phi}{\partial t}(t,z) = d[X_t(\Phi(t,z))] = dX_t(\Phi(t,z)) \circ d\Phi(t,z).$$

Being the trace of a matrix invariant by conjugation, we conclude

$$\frac{\partial}{\partial t} J\Phi(t,z) = J\Phi(t,z)\,\mathrm{tr}\,[dX_t(\Phi(t,z))] = J\Phi(t,z)\,\mathrm{div}X_t(\Phi(t,z)),$$

hence, by equality (2.5) and the divergence theorem (in $\mathbb{T}^n$), it follows

$$0 = \int_E \mathrm{div}X_t(\Phi(t,z))J\Phi(t,z)\,dz = \int_{E_t} \mathrm{div}X_t(x)\,dx = \int_{\partial E} \langle X_t \circ \Phi_t | \nu_{E_t}\rangle\,d\mu_t\,, \tag{2.6}$$

where ν_{E_t} is the outer unit normal vector and μ_t the canonical Riemannian measure of the smooth hypersurface ∂E_t, given by the embedding $\varphi_t = \Phi_t : \partial E \to \mathbb{T}^n$. Thus, letting $t = 0$,

$$\frac{d}{dt}\mathrm{Vol}(E_t)\Big|_{t=0} = \int_{\partial E} \langle X|\nu_E\rangle\,d\mu = 0 \tag{2.7}$$

and we conclude that if $X \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$ is the infinitesimal generator of a volume–preserving variation for E, its normal component $\psi = \langle X|\nu_E\rangle$ on ∂E has zero integral (with respect to the measure μ).

Conversely, we have the following lemma whose proof is postponed after Lemma 2.2.13, since the arguments in the two proofs are very similar.

Lemma 2.1.7. *Let $\psi : \partial E \to \mathbb{R}$ a smooth function with zero integral with respect to the measure μ on ∂E. Then, there exists a smooth vector field $X \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$ such that $\psi = \langle X|\nu_E\rangle$, $\mathrm{div}X = 0$ in a neighborhood of ∂E and the flow Φ defined by system (2.4) having X as infinitesimal generator, gives a volume–preserving variation $E_t = \Phi_t(E)$ of E.*

Hence, with this characterization of the infinitesimal generators of the volume–preserving variations for E, by Theorem 2.1.5 we have that E is a critical set for the functional $\mathcal{A}$ under a volume constraint if and only if

$$\int_{\partial E} \mathrm{H}\langle X|\nu_E\rangle\,d\mu = 0\,,$$

for every $X \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$ such that $\langle X|\nu_E\rangle$ has zero integral on ∂E. By Lemma 2.1.7, this is similarly to say that

$$\int_{\partial E} \mathrm{H}\psi\,d\mu = 0\,,$$

for all $\psi \in C^\infty(\partial E)$ such that $\int_{\partial E}\psi\,d\mu = 0$, which is equivalent to the existence of a constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{H} = \lambda \qquad \text{on } \partial E.$$

That is, ∂E is a smooth hypersurface with *constant mean curvature*.

This motivates the following proposition.

Proposition 2.1.8. *A smooth set $E \subseteq \mathbb{T}^n$ is a critical set for the Area functional $\mathcal{A}$ under a volume constraint, if there exists a constant $\lambda \in \mathbb{R}$ such that*

$$\mathrm{H} = \lambda \qquad \text{on } \partial E.$$

Remark 2.1.9. Clearly, the critical sets for the *unconstrained* Area functional must satisfy

$$\int_{\partial E} \mathrm{H}\langle X, \nu_E\rangle\,d\mu = 0$$

for *every* $X \in C^\infty(\mathbb{T}^n,\mathbb{R}^n)$, which easily implies the *minimal surface* equation $\mathrm{H} = 0$ on ∂E.

Now we deal with the *second variation* of the functional $\mathcal{A}$.

Definition 2.1.10. Given a variation E_t of E, coming from the one–parameter family of diffeomorphism Φ_t, the *second variation of $\mathcal{A}$ at E with respect to Φ_t* is given by

$$\frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0}.$$

In the following proposition we compute the second variation of the Area functional.

Proposition 2.1.11. *Let $E \subseteq \mathbb{T}^n$ a smooth set and $\Phi : (-\varepsilon, \varepsilon) \times \mathbb{T}^n \to \mathbb{T}^n$ a smooth map giving a variation $E_t = \Phi_t(E)$ with infinitesimal generator $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$. Then,*

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0} &= \int_{\partial E}\big(|\nabla\langle X|\nu_E\rangle|^2 - \langle X|\nu_E\rangle^2|B|^2\big)\,d\mu \\ &\quad+ \int_{\partial E}\mathrm{H}\big(\mathrm{H}\langle X|\nu_E\rangle^2 + \langle Z|\nu_E\rangle - 2\langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle + B(X_\tau, X_\tau)\big)\,d\mu\,, \qquad (2.8)\end{aligned}$$

where $X_\tau = X - \langle X|\nu_E\rangle\nu_E$ is the tangential part *of X on ∂E, B and H are respectively the second fundamental form and the mean curvature of ∂E and*

$$Z = \frac{\partial^2\Phi}{\partial t^2}(0,\cdot) = \frac{\partial}{\partial t}[X_t(\Phi(t,\cdot))]\Big|_{t=0} = \frac{\partial X_t}{\partial t}\Big|_{t=0} + dX(X)\,, \qquad (2.9)$$

where, for every $t \in (-\varepsilon, \varepsilon)$, the vector field $X_t \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ is defined by the formula

$$X_t(\Phi(t,z)) = \frac{\partial\Phi}{\partial t}(t,z),$$

for every $z \in \mathbb{T}^n$, hence, $X_0 = X$.

Proof. Let φ_t be the deformation of ∂E induced by the variation E_t, as in Definition 2.1.3.
By arguing as in the first part of the proof of Theorem 2.1.5 (without taking $t = 0$), we have

$$\frac{d}{dt}\mathcal{A}(\partial E_t) = \int_{\partial E}\mathrm{H}_t\langle X_t \circ \Phi_t|\nu_{E_t}\rangle\,d\mu_t,$$

where H_t is the mean curvature of ∂E_t. Consequently, we have

$$\frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0} = \frac{d}{dt}\int_{\partial E}\mathrm{H}_t\langle X_t\circ\Phi_t|\nu_{E_t}\rangle\sqrt{\det g_{ij}}\,dx\,\Big|_{t=0}$$

where $g_{ij} = g_{ij}(t)$.
In order to simplify the notation in the following computations, we drop the subscripts, that is, we let $\mathrm{H}(t,\cdot) = \mathrm{H}_t$, $\nu(t,\cdot) = \nu_{E_t}$, $\psi(t,\cdot) = \langle X_t \circ \Phi_t|\nu_{E_t}\rangle$, $\varphi(t,\cdot) = \varphi_t$ and $X(t,\cdot) = X_t \circ \Phi_t$ (by a little abuse of notation, since X is already the infinitesimal generator of the variation).
We then need to compute the derivatives

$$\frac{\partial \mathrm{H}}{\partial t}\Big|_{t=0} \qquad \text{and} \qquad \frac{\partial}{\partial t}\langle X|\nu\rangle\Big|_{t=0} \qquad (2.10)$$

since we already know, by formula (2.3), that

$$\frac{\partial}{\partial t}\sqrt{\det g_{ij}}\,\Big|_{t=0} = \big(\mathrm{div}X_\tau + \mathrm{H}\psi\big)\sqrt{\det g_{ij}}\,\Big|_{t=0},$$

hence, this derivative gives the following contribution to the second variation,

$$\int_{\partial E}(\psi\mathrm{H}\,\mathrm{div}X_\tau + \psi^2\mathrm{H}^2)\,d\mu\,.$$

Then, we compute (recalling formula (2.9))

$$\frac{\partial\langle X|\nu\rangle}{\partial t}\Big|_{t=0} = \Big\langle \frac{\partial X}{\partial t}\Big|\nu\Big\rangle\Big|_{t=0} + \Big\langle X\Big|\frac{\partial \nu}{\partial t}\Big\rangle\Big|_{t=0} = \langle Z|\nu\rangle + \Big\langle X\Big|\frac{\partial \nu}{\partial t}\Big\rangle\Big|_{t=0}$$

and using the fact that $\frac{\partial\nu}{\partial t}\big|_{t=0}$ is tangent to ∂E, in a local coordinate chart we obtain

$$\Big\langle X\Big|\frac{\partial \nu}{\partial t}\Big\rangle\Big|_{t=0} = X^p_\tau \Big\langle \frac{\partial\varphi}{\partial x_p}\Big|\frac{\partial \nu}{\partial t}\Big\rangle\Big|_{t=0},$$

where in the last inequality we used the notation $X_\tau = X^p_\tau \frac{\partial\varphi}{\partial x_p}$. Notice that, $\langle \frac{\partial\varphi}{\partial x_p}|\nu\rangle = 0$ for every $p \in \{1,\dots,n-1\}$ and $t \in (-\varepsilon,\varepsilon)$, hence, using the Gauss–Weingarten relations (1.7),

$$\begin{aligned}
0 = \frac{\partial}{\partial t}\Big\langle \frac{\partial\varphi}{\partial x_p}\Big|\nu\Big\rangle\Big|_{t=0} &= \Big\langle \frac{\partial X}{\partial x_p}\Big|\nu\Big\rangle + \Big\langle \frac{\partial\varphi}{\partial x_p}\Big|\frac{\partial \nu}{\partial t}\Big\rangle\Big|_{t=0}\\
&= \frac{\partial}{\partial x_p}\langle X|\nu\rangle - \Big\langle X\Big|\frac{\partial \nu}{\partial x_p}\Big\rangle + \Big\langle \frac{\partial\varphi}{\partial x_p}\Big|\frac{\partial \nu}{\partial t}\Big\rangle\Big|_{t=0}\\
&= \frac{\partial\psi}{\partial x_p} - \Big\langle X_\tau\Big|\frac{\partial \nu}{\partial x_p}\Big\rangle + \Big\langle \frac{\partial\varphi}{\partial x_p}\Big|\frac{\partial \nu}{\partial t}\Big\rangle\Big|_{t=0}\\
&= \frac{\partial\psi}{\partial x_p} - X^q_\tau\Big\langle \frac{\partial\varphi}{\partial x_q}\Big|\frac{\partial \nu}{\partial x_p}\Big\rangle + \Big\langle \frac{\partial\varphi}{\partial x_p}\Big|\frac{\partial \nu}{\partial t}\Big\rangle\Big|_{t=0}\\
&= \frac{\partial\psi}{\partial x_p} - X^q_\tau\Big\langle \frac{\partial\varphi}{\partial x_q}\Big|h_{pl}g^{li}\frac{\partial \varphi}{\partial x_i}\Big\rangle + \Big\langle \frac{\partial\varphi}{\partial x_p}\Big|\frac{\partial \nu}{\partial t}\Big\rangle\Big|_{t=0}\\
&= \frac{\partial\psi}{\partial x_p} - X^q_\tau h_{pl}g^{li}g_{qi} + \Big\langle \frac{\partial\varphi}{\partial x_p}\Big|\frac{\partial \nu}{\partial t}\Big\rangle\Big|_{t=0}
\end{aligned}$$

and we can conclude that

$$\Big\langle \frac{\partial\varphi}{\partial x_p}\Big|\frac{\partial \nu}{\partial t}\Big\rangle\Big|_{t=0} = -\frac{\partial\psi}{\partial x_p} + X^q_\tau h_{pq}\,, \tag{2.11}$$

where h_{pq} are the components of the second fundamental form B of ∂E in the local chart. Thus, we obtain the following identity

$$\begin{aligned}
\frac{\partial}{\partial t}\langle X|\nu\rangle\Big|_{t=0} &= \langle Z|\nu\rangle + X^p_\tau\Big\langle \frac{\partial\varphi}{\partial x_p}\Big|\frac{\partial \nu}{\partial t}\Big\rangle\Big|_{t=0}\\
&= \langle Z|\nu\rangle - \frac{\partial\psi}{\partial x_p}X^p_\tau + X^p_\tau X^q_\tau h_{pq}\\
&= \langle Z|\nu\rangle - \langle X_\tau|\nabla\langle X|\nu\rangle\rangle + B(X_\tau,X_\tau)
\end{aligned} \tag{2.12}$$

and the relative contribution to the second variation is given by

$$\int_{\partial E} \mathrm{H}\big(\langle Z|\nu\rangle - \langle X_\tau|\nabla\langle X|\nu\rangle\rangle + B(X_\tau,X_\tau)\big)\,d\mu\,.$$

Now we conclude by computing the first derivative in (2.10). To this aim, we note that

$$\mathrm{H} = -\Big\langle \frac{\partial^2\varphi}{\partial x_i\partial x_j}\Big|\nu\Big\rangle g^{ij}$$

hence, we need the following terms

$$\frac{\partial g^{ij}}{\partial t}\Big|_{t=0} \tag{2.13}$$

$$\left\langle \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \middle| \frac{\partial \nu}{\partial t} \right\rangle \Big|_{t=0} \tag{2.14}$$

$$\left\langle \frac{\partial}{\partial t} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \middle| \nu \right\rangle \Big|_{t=0} . \tag{2.15}$$

We start with the term (2.13), recalling that

$$\frac{\partial g_{ij}}{\partial t} \Big|_{t=0} = \nabla_i \omega_j + \nabla_j \omega_i + 2h_{ij} \langle X | \nu \rangle$$

by equation (2.1), where ω is the 1–form defined by $\omega(Y) = g(X_\tau, Y)$.
Using the fact that $g_{ij} g^{jk} = 0$, we obtain

$$0 = \frac{\partial g_{ij}}{\partial t} \Big|_{t=0} g^{jk} + g_{ij} \frac{\partial g^{jk}}{\partial t} \Big|_{t=0} = g^{jk} (\nabla_i \omega_j + \nabla_j \omega_i + 2h_{ij} \langle X | \nu \rangle) + g_{ij} \frac{\partial g^{jk}}{\partial t} \Big|_{t=0}$$

then,

$$\frac{\partial g^{pk}}{\partial t} \Big|_{t=0} = -g^{jp} g^{ik} \Big(\nabla_i \omega_j + \nabla_j \omega_i + 2h_{ij} \langle X | \nu \rangle \Big) = -\nabla^p X^k_\tau - \nabla^k X^p_\tau - 2h^{pk} \psi \, .$$

We then proceed with the computation of the term (2.14), by means of equation (2.11),

$$\left\langle \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \middle| \frac{\partial \nu}{\partial t} \right\rangle \Big|_{t=0} = \Gamma^k_{ij} \left\langle \frac{\partial \varphi}{\partial x_k} \middle| \frac{\partial \nu}{\partial t} \right\rangle \Big|_{t=0} = \Gamma^k_{ij} \big(- \frac{\partial \psi}{\partial x_k} + X^q_\tau h_{qk} \big)$$

and finally we compute the term (2.15),

$$\left\langle \frac{\partial}{\partial t} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \middle| \nu \right\rangle = \left\langle \frac{\partial^2 X}{\partial x_i \partial x_j} \middle| \nu \right\rangle \Big|_{t=0} = \left\langle \frac{\partial^2 (\psi \nu)}{\partial x_i \partial x_j} \middle| \nu \right\rangle + \left\langle \frac{\partial^2 X_\tau}{\partial x_i \partial x_j} \middle| \nu \right\rangle .$$

We have

$$\begin{aligned}
\left\langle \frac{\partial^2 (\psi \nu)}{\partial x_i \partial x_j} \middle| \nu \right\rangle &= \frac{\partial^2 \psi}{\partial x_i \partial x_j} + \left\langle \frac{\partial^2 \nu}{\partial x_i \partial x_j} \middle| \nu \right\rangle \psi \\
&= \frac{\partial^2 \psi}{\partial x_i \partial x_j} + \left\langle \frac{\partial}{\partial x_i} \big(h_{jl} g^{lp} \frac{\partial \varphi}{\partial x_p} \big) \middle| \nu \right\rangle \psi \\
&= \frac{\partial^2 \psi}{\partial x_i \partial x_j} + h_{jl} g^{lp} \left\langle \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \middle| \nu \right\rangle \psi \\
&= \frac{\partial^2 \psi}{\partial x_i \partial x_j} + \psi h_{jl} g^{lp} h_{ip}
\end{aligned}$$

and

$$
\begin{aligned}
\left\langle \frac{\partial^2 X_\tau}{\partial x_i \partial x_j} \middle| \nu \right\rangle &= \frac{\partial}{\partial x_i} \left\langle \frac{\partial X_\tau}{\partial x_j} \middle| \nu \right\rangle - \left\langle \frac{\partial X_\tau}{\partial x_j} \middle| \frac{\partial \nu}{\partial x_i} \right\rangle \\
&= \frac{\partial}{\partial x_i} \left\langle \frac{\partial}{\partial x_j} \Big(X_\tau^p \frac{\partial \varphi}{\partial x_p} \Big) \middle| \nu \right\rangle - \left\langle \frac{\partial X_\tau}{\partial x_j} \middle| \frac{\partial \nu}{\partial x_i} \right\rangle \\
&= \frac{\partial}{\partial x_i} \left[X_\tau^p \left\langle \frac{\partial^2 \varphi}{\partial x_j \partial x_p} \middle| \nu \right\rangle \right] - \left\langle \frac{\partial X_\tau}{\partial x_j} \middle| \frac{\partial \nu}{\partial x_i} \right\rangle \\
&= -\frac{\partial}{\partial x_i} (X_\tau^p h_{pj}) - \left\langle \frac{\partial X_\tau}{\partial x_j} \middle| \frac{\partial \nu}{\partial x_i} \right\rangle \\
&= -\frac{\partial}{\partial x_i} (X_\tau^p h_{pj}) - \left\langle \frac{\partial}{\partial x_j} \Big(X_\tau^p \frac{\partial \varphi}{\partial x_p} \Big) \middle| \frac{\partial \nu}{\partial x_i} \right\rangle \\
&= -\frac{\partial}{\partial x_i} (X_\tau^p h_{pj}) - X_\tau^p \left\langle \frac{\partial^2 \varphi}{\partial x_j \partial x_p} \middle| \frac{\partial \nu}{\partial x_i} \right\rangle - \frac{\partial X_\tau^p}{\partial x_j} \left\langle \frac{\partial \varphi}{\partial x_p} \middle| \frac{\partial \nu}{\partial x_i} \right\rangle \\
&= -\frac{\partial}{\partial x_i} (X_\tau^p h_{pj}) - X_\tau^p \Gamma_{jp}^k \left\langle \frac{\partial \varphi}{\partial x_k} \middle| \frac{\partial \nu}{\partial x_i} \right\rangle - \frac{\partial X_\tau^p}{\partial x_j} \left\langle \frac{\partial \varphi}{\partial x_p} \middle| \frac{\partial \nu}{\partial x_i} \right\rangle \\
&= -\frac{\partial}{\partial x_i} (X_\tau^p h_{pj}) - X_\tau^p \Gamma_{jp}^k h_{il} g^{lq} g_{kq} - \frac{\partial X^p}{\partial x_j} h_{il} g^{lq} g_{pq} \\
&= -\frac{\partial}{\partial x_i} (X_\tau^p h_{pj}) - X_\tau^p \Gamma_{jp}^k h_{ik} - \frac{\partial X^k}{\partial x_j} h_{ik}.
\end{aligned}
$$

Hence, we finally get

$$
\begin{aligned}
\frac{\partial \mathrm{H}}{\partial t} \Big|_{t=0} &= -2 h_{ij} \nabla^i X_\tau^j - 2\langle X|\nu\rangle |B|^2 - g^{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} + g^{ij} \Gamma_{ij}^k \frac{\partial \psi}{\partial x_k} \\
&\quad + |B|^2 \langle X|\nu\rangle - g^{ij} \Gamma_{ij}^k h_{kq} X_\tau^q + g^{ij} \frac{\partial}{\partial x_i} (X_\tau^p h_{pj}) + h_{ij} \nabla^i X_j^j \\
&= -|B|^2 \langle X|\nu\rangle - h_{ij} \nabla^i X_\tau^j - \Delta\psi \\
&\quad + g^{ij} \Big[\frac{\partial}{\partial x_i} \Big(X_\tau^p h_{pj} \Big) - \Gamma_{ij}^k \Big(X_\tau^p h_{pk} \Big) \Big] \\
&= -\psi |B|^2 - \Delta\psi - h_{ij} \nabla^i X_\tau^j + g^{ij} \nabla_i (X_\tau^p h_{pj}) \\
&= -\psi |B|^2 - \Delta\psi - h_{ij} \nabla^i X_\tau^j + \operatorname{div}(X_\tau^p h_{pj}) \\
&= -\psi |B|^2 - \Delta\psi + \langle X_\tau | \operatorname{div} B \rangle \\
&= -\psi |B|^2 - \Delta\psi + \langle X_\tau | \nabla \mathrm{H} \rangle \,,
\end{aligned}
$$

where in the last equality we used the Codazzi–Mainardi equations (see [48]). We conclude that the contribution of the first term in (2.10) is then

$$
\int_{\partial E} \psi \big(-\psi |B|^2 - \Delta\psi + \langle X_\tau | \nabla \mathrm{H} \rangle \big) \, d\mu.
$$

Putting all these contributions together, we obtain the second variation of the Area functional,

$$
\begin{aligned}
\frac{d^2}{dt^2} \mathcal{A}(\partial E_t) \Big|_{t=0} = \int_{\partial E} \Big[&-\psi\Delta\psi - \psi^2 |B|^2 + \psi \langle X_\tau | \nabla \mathrm{H} \rangle + \psi \mathrm{H} \operatorname{div} X_\tau + \psi^2 \mathrm{H}^2 \\
&+ \mathrm{H} \big(\langle Z|\nu\rangle - \langle X_\tau | \nabla \psi \rangle + B(X_\tau, X_\tau) \big) \Big] \, d\mu \,.
\end{aligned}
$$

Integrating by parts, we have

$$
\int_{\partial E} \psi \langle X_\tau | \nabla \mathrm{H} \rangle \, d\mu = -\int_{\partial E} \big[\mathrm{H} \langle X_\tau | \nabla \psi \rangle + \mathrm{H} \psi \operatorname{div} X_\tau \big] \, d\mu
$$

and we can conclude

$$\frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0} = \int_{\partial E}\Big[|\nabla\psi|^2 - \psi^2|B|^2 + \psi^2\mathrm{H}^2 + \mathrm{H}(\langle Z|\nu\rangle - 2\langle X_\tau|\nabla\psi\rangle + B(X_\tau, X_\tau))\Big]\,d\mu\,,$$

which is the formula we wanted. □

In the following proposition we rewrite explicitly formula (2.8) and we notice that the second variation of $\mathcal{A}$ only depends on the normal component of X on ∂E, that is, on $\langle X, \nu_E\rangle$.

Theorem 2.1.12 (Second variation of $\mathcal{A}$). *Let $E \subseteq \mathbb{T}^n$ a smooth set and $\Phi : (-\varepsilon, \varepsilon) \times \mathbb{T}^n \to \mathbb{T}^n$ a smooth map giving a variation E_t with infinitesimal generator $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$. Then,*

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0} = &\int_{\partial E}\big(|\nabla\langle X|\nu_E\rangle|^2 - \langle X|\nu_E\rangle^2|B|^2\big)\,d\mu\\ &+\int_{\partial E}\mathrm{H}\Big[\langle X|\nu_E\rangle\,\mathrm{div}^{\mathbb{T}^n}X - \mathrm{div}\big(\langle X|\nu_E\rangle X_\tau\big) + \Big\langle\frac{\partial X_t}{\partial t}\Big|_{t=0}\Big|\nu_E\Big\rangle\Big]\,d\mu \qquad (2.16)\end{aligned}$$

where ν_E is the outer unit normal vector to ∂E, $X_\tau = X - \langle X|\nu_E\rangle\nu_E$ is the tangential part *of X on ∂E, B and H are respectively the second fundamental form and the mean curvature of ∂E, the vector field $X_t \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ is defined by the formula $X_t(\Phi(t,z)) = \frac{\partial\Phi}{\partial t}(t,z)$ for every $t \in (-\varepsilon, \varepsilon)$ and $z \in \mathbb{T}^n$.*

Proof. We claim that

$$\begin{aligned}&\mathrm{H}\langle X|\,\nu_E\rangle^2 + \langle Z|\nu_E\rangle - 2\langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle + B(X_\tau, X_\tau)\\ &\quad= \langle X|\nu_E\rangle\,\mathrm{div}^{\mathbb{T}^n}X - \mathrm{div}(\langle X|\nu_E\rangle X_\tau) + \Big\langle\frac{\partial X_t}{\partial t}\Big|_{t=0}\Big|\nu_E\Big\rangle\,. \qquad (2.17)\end{aligned}$$

In order to show the claim in (3.62) we notice that, being every derivative of ν_E a tangent vector field,

$$\begin{aligned}\langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle &= \langle\nu_E|dX(X_\tau)\rangle + \langle X|\langle X_\tau|\nabla\nu_E\rangle\rangle\\ &= \langle\nu_E|dX(X_\tau)\rangle + \langle X_\tau|\langle X_\tau|\nabla\nu_E\rangle\rangle\\ &= \langle\nu_E|dX(X_\tau)\rangle + B(X_\tau, X_\tau)\,,\end{aligned}$$

by the Gauss–Weingarten relations (1.7).
Therefore, since $Z - \frac{\partial X_t}{\partial t}\big|_{t=0} = dX(X)$, we have

$$\begin{aligned}&\mathrm{H}\langle X|\nu_E\rangle^2 + \langle Z|\nu_E\rangle - 2\langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle + B(X_\tau, X_\tau) - \Big\langle\frac{\partial X_t}{\partial t}\Big|_{t=0}\Big|\nu_E\Big\rangle\\ &= \mathrm{H}\langle X|\nu_E\rangle^2 + \langle\nu_E|dX(X)\rangle - \langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle - \langle\nu_E|dX(X_\tau)\rangle\\ &= \mathrm{H}\langle X|\nu_E\rangle^2 + \langle\nu_E|dX(\langle X|\nu_E\rangle\nu_E)\rangle - \langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle\\ &= \mathrm{H}\langle X|\nu_E\rangle^2 + \langle X|\nu_E\rangle\langle\nu_E|dX(\nu_E)\rangle + \langle X|\nu_E\rangle\,\mathrm{div}X_\tau - \mathrm{div}(\langle X|\nu_E\rangle X_\tau)\,. \qquad (2.18)\end{aligned}$$

We also notice that, choosing an orthonormal basis $e_1, \ldots, e_{n-1}, e_n = \nu_E$ of $\mathbb{R}^n$ at a point $p \in \partial E$ and letting $X = X^i e_i$, we have

$$\langle e_i|\nabla^\top X^i\rangle = \big\langle e_i\big|\nabla^{\mathbb{T}^n}X^i - \langle\nabla^{\mathbb{T}^n}X^i|\nu_E\rangle\nu_E\big\rangle = \mathrm{div}^{\mathbb{T}^n}X - \langle\nu_E|dX(\nu_E)\rangle\,,$$

where the symbol $\nabla^\top f$ denotes the projection on the tangent space to ∂E of the gradient $\nabla^{\mathbb{T}^n}f$ of a function, called *tangential gradient* of f and coincident with the gradient operator of ∂E applied to the restriction of f to the hypersurface, while $\langle e_i|\nabla^\top X^i\rangle$ is called *tangential divergence* of X, usually denoted with $\mathrm{div}^\top X$ and coincident with the (Riemannian) divergence of ∂E if X is a tangent vector field, as we will see below (see [59]). Moreover, if we choose a local parametrization of ∂E such that $\frac{\partial\varphi}{\partial x_i}(p) = e_i$, for $i \in \{1, \dots, n-1\}$, we have $e_i^j = \frac{\partial \varphi^j}{\partial x_i} = g^{ij} = \delta_{ij}$ at p and

$$\begin{aligned}\langle e_i | \nabla^\top X^i \rangle = \operatorname{div}^\top X &= \langle e_i | \nabla^\top X^i_\tau \rangle + \langle e_i | \nabla^\top (\langle X | \nu_E \rangle \nu_E^i) \rangle \\ &= \langle e_i | \nabla X^i_\tau \rangle + \langle X | \nu_E \rangle \langle e_i | \nabla^{\mathbb{T}^n} \nu_E^i \rangle \\ &= \langle e_i | \nabla X^i_\tau \rangle + \langle X | \nu_E \rangle \frac{\partial \varphi^j}{\partial x_i} h_{jl} g^{ls} \frac{\partial \varphi^i}{\partial x_s} \\ &= \nabla_{e_i} X^i_\tau + \langle X | \nu_E \rangle h_{ii} \\ &= \operatorname{div} X_\tau + \langle X | \nu_E \rangle \mathrm{H}\,,\end{aligned}$$

where we used again the Gauss–Weingarten relations (1.7) and the fact that the covariant derivative of a tangent vector field along a hypersurface of $\mathbb{R}^n$ can be obtained by differentiating in $\mathbb{R}^n$ (a local extension of) the vector field and projecting the result on the tangent space to the hypersurface (see [31], for instance). Hence, we get

$$\langle \nu_E | dX(\nu_E) \rangle = \operatorname{div}^{\mathbb{T}^n} X - \langle e_i | \nabla^\top X^i \rangle = \operatorname{div}^{\mathbb{T}^n} X - \operatorname{div} X_\tau - \langle X | \nu_E \rangle \mathrm{H}$$

and claim (2.17) follows by substituting this left term in formula (2.18). □

Remark 2.1.13. We are not aware of the presence in literature of this “geometric” line in deriving the (first and) second variation of $\mathcal{A}$, moreover, in [14, Theorem 2.6, Step 3, equation 2.67], this latter is obtained only at a critical set, while in [11, Theorem 3.6] the methods are strongly “analytic” and in our opinion less straightforward. These two papers are actually the ones on which is based the computation in [2, Theorem 3.1] of the second variation of the (nonlocal) Area functional at a general smooth set $E \subseteq \mathbb{T}^n$. Anyway, in this last paper, the variations of E are all *special* variations, that is, they are given by the flows in system (2.4), indeed, the term with the time derivative of X_t is missing (see formulas 3.1 and 7.2 in [2]).

Notice that the second variation in general does not depend only on the normal component $\langle X | \nu_E \rangle$ of the restriction to ∂E of the infinitesimal generator X of a variation Φ (this will anyway be true at a critical set E, see below), due to the presence of the Z–term and of $B(X_\tau, X_\tau)$ depending also on the tangential component of X and of its behavior around ∂E. Even if we restrict ourselves to the special variations coming from system (2.4), with a *normal* infinitesimal generator X, which imply that all the vector fields X_t are the same and coinciding with X, hence $Z = dX(X)$ and $X_\tau = 0$, the second variation still depends also on the behavior of X in a neighborhood of ∂E (as Z). However, there are very particular case in which it depend only on $\langle X | \nu_E \rangle$, for instance when the variation is special and X is normal with zero divergence (of $\mathbb{T}^n$) on ∂E (in particular, if $\operatorname{div}^{\mathbb{T}^n} X = 0$ in a neighborhood of ∂E or in the whole $\mathbb{T}^n$), as it can be seen easily in the above theorem.

It follows that if we have a critical set E for the *unconstrained* Area functional, hence $\mathrm{H} = 0$ on ∂E (see Remark 2.1.9), the second variation of $\mathcal{A}$ is simply given by

$$\frac{d^2}{dt^2} \mathcal{A}(\partial E_t) \Big|_{t=0} = \int_{\partial E} \big(|\nabla \langle X, \nu_E \rangle|^2 - \langle X, \nu_E \rangle^2 |B|^2 \big)\, d\mu\,.$$

However, we see that the second variation has the same form also for $\mathcal{A}$ under a volume constraint, at a critical set.

Proposition 2.1.14. *If $E \subseteq \mathbb{T}^n$ is a critical set for $\mathcal{A}$ under a volume constraint, there holds*

$$\frac{d^2}{dt^2} \mathcal{A}(\partial E_t) \Big|_{t=0} = \int_{\partial E} \big(|\nabla \langle X | \nu_E \rangle|^2 - \langle X | \nu_E \rangle^2 |B|^2 \big)\, d\mu$$

for every volume–preserving variation E_t of E.
Hence, the second variation of $\mathcal{A}$ at E depends only on the normal component of the restriction of the infinitesimal generator X to ∂E, that is, on $\langle X | \nu_E \rangle$.

Proof. Computing the second derivative of the (constant) volume of E_t, by equations (2.5)–(2.6) we have (recalling formulas (2.3), (2.12) and using the divergence theorem)

$$\begin{aligned}
0 = \frac{d^2}{dt^2}\mathrm{Vol}(E_t)\Big|_{t=0} &= \frac{d}{dt}\int_{E_t}\mathrm{div}X_t(x)\,dx\,\Big|_{t=0} = \frac{d}{dt}\int_{\partial E}\langle X|\nu_{E_t}\rangle\,d\mu_t\,\Big|_{t=0}\\
&= \int_{\partial E}\Big[\mathrm{div}X_\tau\langle X\,|\,\nu_E\rangle + \mathrm{H}\langle X\,|\,\nu_E\rangle^2 + \langle Z|\nu_E\rangle - \langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle + B(X_\tau,X_\tau)\Big]\,d\mu\\
&= \int_{\partial E}\Big[\mathrm{H}\langle X\,|\,\nu_E\rangle^2 + \langle Z|\nu_E\rangle - 2\langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle + B(X_\tau,X_\tau)\Big]\,d\mu\,, \qquad (2.19)
\end{aligned}$$

hence, being H constant on ∂E, we are done. □

Remark 2.1.15. Notice that by the previous computation and relation (2.17), it follows

$$\frac{d^2}{dt^2}\mathrm{Vol}(E_t)\Big|_{t=0} = \int_{\partial E}\Big[\langle X|\nu_E\rangle\,\mathrm{div}^{\mathbb{T}^n}X + \Big\langle\frac{\partial X_t}{\partial t}\Big|_{t=0}\Big|\nu\Big\rangle\Big]\,d\mu = 0\,, \qquad (2.20)$$

for every volume–preserving variation E_t of E. Hence, if we restrict ourselves to the special (volume–preserving) variations coming from system (2.4), as in [2], we have

$$\frac{d^2}{dt^2}\mathrm{Vol}(E_t)\Big|_{t=0} = \int_{\partial E}\langle X|\nu_E\rangle\,\mathrm{div}^{\mathbb{T}^n}X\,d\mu = 0\,,$$

indeed, for such variations we have $X_t = X$, for every $t \in (-\varepsilon,\varepsilon)$. Thus, one can clearly use equality (2.20) to show the above proposition.
Moreover, we see that if we have a special variation generated by a vector field X such that $\mathrm{div}^{\mathbb{T}^n}X = 0$ on ∂E, then $\frac{d^2}{dt^2}\mathrm{Vol}(E_t)\big|_{t=0} = 0$ and if E is a critical set, the second integral in formula (2.16) vanishes. This is then true for the special volume–preserving variations coming from Lemma 2.1.7 and when X is a constant vector field, hence the associated special variation E_t is simply a translation of E (clearly, in this case $\mathcal{A}(\partial E_t)$ is constant and the first and second variations are zero).

2.2 STABILITY AND $W^{2,p}$–LOCAL MINIMALITY

By Proposition 2.1.14, the second variation of the Area functional under a volume constraint at a smooth critical set E is a quadratic form in the normal component on ∂E of the infinitesimal generator $X \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$ of a volume–preserving variation, that is, on $\psi = \langle X|\nu_E\rangle$. This and the fact that the infinitesimal generators of the volume–preserving variations are "characterized" by having zero integral of such normal component on ∂E, by Lemma 2.1.7 and the discussion immediately before, motivate the following definition.

Definition 2.2.1. Given any smooth open set $E \subseteq \mathbb{T}^n$ we define the space of (Sobolev) functions (see [7])

$$\widetilde{H}^1(\partial E) = \Big\{\psi : \partial E \to \mathbb{R}\,:\,\psi \in H^1(\partial E)\ \text{ and }\ \int_{\partial E}\psi\,d\mu = 0\Big\},$$

and the quadratic form $\Pi_E : \widetilde{H}^1(\partial E) \to \mathbb{R}$ as

$$\Pi_E(\psi) = \int_{\partial E}\big(|\nabla\psi|^2 - \psi^2|B|^2\big)\,d\mu \qquad (2.21)$$

with the notations of Theorem 2.1.12.

Definition 2.2.2. Given any smooth open set $E \subseteq \mathbb{T}^n$, we say that a smooth vector field $X \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$ is *admissible for* E if the function $\psi : \partial E \to \mathbb{R}$ given by $\psi = \langle X|\nu_E\rangle$ belongs to $\widetilde{H}^1(\partial E)$, that is, has zero integral on ∂E.

Remark 2.2.3. Clearly, if $X \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$ is the infinitesimal generator of a volume–preserving variation for E, then X is admissible, by the discussion after Corollary 2.1.6.

Remark 2.2.4. By what we said above, if E is a smooth critical set for $\mathcal{A}$ under a volume constraint, we can from now on consider only the special variations $E_t = \Phi_t(E)$ associated to admissible vector fields X, given by the flow Φ defined by system (2.4), hence

$$\frac{d}{dt}\mathcal{A}(\partial E_t)\Big|_{t=0} = \int_{\partial E} \langle X|\nu_E\rangle\, d\mu = 0$$

and

$$\frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0} = \Pi_E(\langle X|\nu_E\rangle)$$

where Π_E is the quadratic form defined by formula (2.21).

We notice that every constant vector field $X = \eta \in \mathbb{R}^n$ is clearly admissible, as

$$\int_{\partial E} \langle \eta\,|\nu_E\rangle\, d\mu = \int_E \operatorname{div}\eta\, dx = 0$$

and the associated flow is given by $\Phi(t,x) = x + t\eta$, then, by the translation invariance of the functional $\mathcal{A}$, we have $\mathcal{A}(\partial E_t) = \mathcal{A}(\partial E)$ and

$$0 = \frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0} = \Pi_E(\langle \eta\,|\nu_E\rangle)\,,$$

that is, the form Π_E is zero on the vector subspace

$$T(\partial E) = \big\{\langle \eta\,|\nu_E\rangle\,:\,\eta \in \mathbb{R}^n\big\} \subseteq \widetilde{H}^1(\partial E)$$

of dimension clearly less than or equal to n. We split

$$\widetilde{H}^1(\partial E) = T(\partial E) \oplus T^\perp(\partial E)\,,$$

where $T^\perp(\partial E) \subseteq \widetilde{H}^1(\partial E)$ is the vector subspace L^2–orthogonal to $T(\partial E)$ (with respect to the measure μ on ∂E), that is,

$$\begin{aligned} T^\perp(\partial E) &= \Big\{\psi \in \widetilde{H}^1(\partial E)\,:\, \int_{\partial E}\psi\nu_E\, d\mu = 0\Big\} \\ &= \Big\{\psi \in H^1(\partial E)\,:\, \int_{\partial E}\psi\, d\mu = 0 \;\text{ and }\; \int_{\partial E}\psi\nu_E\, d\mu = 0\Big\} \end{aligned}$$

and we give the following "stability" conditions.

Definition 2.2.5 (Stability)**.** We say that a critical set $E \subseteq \mathbb{T}^n$ for $\mathcal{A}$ under a volume constraint is *stable* if

$$\Pi_E(\psi) \geqslant 0 \qquad \text{for all } \psi \in \widetilde{H}^1(\partial E)$$

and *strictly stable* if moreover

$$\Pi_E(\psi) > 0 \qquad \text{for all } \psi \in T^\perp(\partial E)\setminus\{0\}.$$

We postpone a quite detailed discussion about the classification of stable and strictly stable critical sets for the volume constrained Area functional (see Section 3.5).

Remark 2.2.6. Introducing the symmetric bilinear form associated (by polarization) to Π_E on $\widetilde{H}^1(\partial E)$,

$$b_E(\psi,\varphi) = \frac{\Pi_E(\psi+\varphi) - \Pi_E(\psi-\varphi)}{4}$$

at a critical set $E \subseteq \mathbb{T}^n$, it can be seen that actually $T(\partial E)$ is a degenerate vector subspace of $\widetilde{H}^1(\partial E)$ for b_E, that is, $b_E(\psi,\varphi) = 0$ for every $\psi \in \widetilde{H}^1(\partial E)$ and $\varphi \in T(\partial E)$. By means of formula (1.11), since E (being critical) satisfies $\mathrm{H} = \lambda$ for some constant $\lambda \in \mathbb{R}$, we have

$$-\Delta\nu_E - |\mathrm{B}|^2\nu_E = 0$$

on ∂E. This equation can be written as $L(\nu_i) = 0$, for every $i \in \{1,\dots,n\}$, where L is the self–adjoint, linear operator defined as

$$L(\psi) = -\Delta\psi - |\mathrm{B}|^2\psi\,,$$

which clearly satisfies

$$b_E(\psi,\varphi) = \int_{\partial E} \langle L(\psi)|\varphi\rangle\,d\mu \qquad \text{and} \qquad \Pi_E(\psi) = \int_{\partial E} \langle L(\psi)|\psi\rangle\,d\mu\,.$$

Then, if we "decompose" a smooth function $\psi \in \widetilde{H}^1(\partial E)$ as $\psi = \varphi + \langle \eta|\nu_E\rangle$, for some $\eta \in \mathbb{R}^n$ and $\varphi \in T^\perp(\partial E)$, we have (recalling formula (2.21))

$$\begin{aligned}\Pi_E(\psi) &= \int_{\partial E} \langle L(\psi)|\psi\rangle\,d\mu \\ &= \int_{\partial E} \langle L(\varphi)|\varphi\rangle\,d\mu + 2\int_{\partial E} \langle L(\langle\eta|\nu_E\rangle)|\varphi\rangle\,d\mu + \int_{\partial E} \langle L(\langle\eta|\nu_E\rangle)|\langle\eta|\nu_E\rangle\rangle\,d\mu \\ &= \Pi_E(\varphi)\,.\end{aligned}$$

By approximation with smooth functions, we conclude that this equality holds for every function in $\widetilde{H}^1(\partial E)$.
The initial claim about the form b_E then easily follows by its definition. Moreover, if E is a strictly stable critical set there holds

$$\Pi_E(\psi) > 0 \qquad \text{for every } \psi \in \widetilde{H}^1(\partial E) \setminus T(\partial E). \tag{2.22}$$

Remark 2.2.7. We observe that there exists an orthonormal frame $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$ such that

$$\int_{\partial E} \langle\nu_E|e_i\rangle\langle\nu_E|e_j\rangle\,d\mu = 0, \tag{2.23}$$

for all $i \neq j$, indeed, considering the symmetric $n \times n$–matrix $A = (a_{ij})$ with components $a_{ij} = \int_{\partial E} \nu_E^i \nu_E^j\,d\mu$, where $\nu_E^i = \langle\nu_E|\varepsilon_i\rangle$ for some basis $\{\varepsilon_1,\dots,\varepsilon_n\}$ of $\mathbb{R}^n$, we have

$$\int_{\partial E} (O\nu_E)_i(O\nu_E)_j\,d\mu = (OAO^{-1})_{ij},$$

for every $O \in SO(n)$. Choosing O such that OAO^{-1} is diagonal and setting $e_i = O^{-1}\varepsilon_i$, relations (2.23) are clearly satisfied.
Hence, the functions $\langle\nu_E|e_i\rangle$ which are not identically zero are an orthogonal basis of $T(\partial E)$. We set

$$\mathrm{I}_E = \big\{i \in \{1,\dots,n\} \,:\, \langle\nu_E|e_i\rangle \text{ is not identically zero}\big\} \tag{2.24}$$

and

$$\mathrm{O}_E = \mathrm{Span}\{e_i \,:\, i \in \mathrm{I}_E\}, \tag{2.25}$$

then, given any $\psi \in \widetilde{H}^1(\partial E)$, its projection on $T^\perp(\partial E)$ is

$$\pi(\psi) = \psi - \sum_{i \in \mathrm{I}_E} \frac{\int_{\partial E} \psi\langle\nu_E|e_i\rangle\,d\mu}{\|\langle\nu_E|e_i\rangle\|^2_{L^2(\partial E)}}\langle\nu_E|e_i\rangle\,. \tag{2.26}$$

From now on we will extensively use Sobolev spaces on smooth hypersurfaces. Most of their properties hold as in $\mathbb{R}^n$, standard references are [3] in the Euclidean space and [7] when the ambient is a manifold.

Definition 2.2.8. We say that a smooth set $E \subseteq \mathbb{T}^n$ is a *local minimizer* for the Area functional $\mathcal{A}$ if there exists $\delta > 0$ such that

$$\mathcal{A}(\partial F) \geqslant \mathcal{A}(\partial E)$$

for all smooth sets $F \subseteq \mathbb{T}^n$ with $\mathrm{Vol}(F) = \mathrm{Vol}(E)$ and $\mathrm{Vol}(E \triangle F) < \delta$.

We say that a smooth set $E \subseteq \mathbb{T}^n$ is a $W^{2,p}$*–local minimizer* if there exists $\delta > 0$ and a tubular neighborhood N_ε of E, such that

$$\mathcal{A}(\partial F) \geqslant \mathcal{A}(\partial E)$$

for all smooth sets $F \subseteq \mathbb{T}^n$ with $\mathrm{Vol}(F) = \mathrm{Vol}(E)$, $\mathrm{Vol}(E \triangle F) < \delta$ and ∂F contained in N_ε, described by

$$\partial F = \{y + \psi(y)\nu_E(y) \, : \, y \in \partial E\},$$

for a smooth function $\psi : \partial E \to \mathbb{R}$ with $\|\psi\|_{W^{2,p}(\partial E)} < \delta$.
Clearly, any local minimizer is a $W^{2,p}$–local minimizer.

We immediately show a *necessary* condition for $W^{2,p}$–local minimizers.

Proposition 2.2.9. *Let the smooth set $E \subseteq \mathbb{T}^n$ be a $W^{2,p}$–local minimizer of $\mathcal{A}$, then E is a critical set and*

$$\Pi_E(\psi) \geqslant 0 \qquad \text{for all } \psi \in \widetilde{H}^1(\partial E),$$

in particular, E is stable.

Proof. If E is a $W^{2,p}$–local minimizer of $\mathcal{A}$, given any $\psi \in C^\infty(\partial E) \cap \widetilde{H}^1(\partial E)$, we consider the admissible vector field $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ given by Lemma 2.1.7 and the associated flow Φ. Then, the variation $E_t = \Phi_t(E)$ of E is volume–preserving, that is, $\mathrm{Vol}(E_t) = \mathrm{Vol}(E)$ and for every $\delta > 0$, there clearly exists a tubular neighborhood N_ε of E and $\overline{\varepsilon} > 0$ such that for $t \in (-\overline{\varepsilon}, \overline{\varepsilon})$ we have

$$\mathrm{Vol}(E \triangle E_t) < \delta$$

and

$$\partial E_t = \{y + \psi_{E_t}(y)\nu_E(y) \, : \, y \in \partial E\} \subseteq N_\varepsilon$$

for a smooth function $\psi_{E_t} : \partial E \to \mathbb{R}$ with $\|\psi_{E_t}\|_{W^{2,p}(\partial E)} < \delta$. Hence, the $W^{2,p}$–local minimality of E implies

$$\mathcal{A}(\partial E) \leqslant \mathcal{A}(\partial E_t),$$

for every $t \in (-\overline{\varepsilon}, \overline{\varepsilon})$. It follows

$$0 = \frac{d}{dt}\mathcal{A}(\partial E_t)\Big|_{t=0} = \int_{\partial E} \mathrm{H}\psi \, d\mu,$$

by Theorem 2.1.5, which implies that E is a critical set, by the subsequent discussion and

$$0 \leqslant \frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0} = \Pi_E(\psi),$$

by Proposition 2.1.14 and Remark 2.2.4.
Then, the thesis easily follows by the density of $C^\infty(\partial E) \cap \widetilde{H}^1(\partial E)$ in $\widetilde{H}^1(\partial E)$ (see [7], for instance) and the definition of Π_E, formula (2.21). ☐

The rest of this section will be devoted to showing that the strict stability (see Definition 2.2.5) is a *sufficient* condition for the $W^{2,p}$–local minimality. Precisely, we will prove the following theorem, which is [2, Theorem 3.9].

Theorem 2.2.10. *Let $p > \max\{2, n-1\}$ and $E \subseteq \mathbb{T}^n$ a smooth strictly stable critical set for the Area functional $\mathcal{A}$ (under a volume constraint), with N_ε a tubular neighborhood of ∂E. Then, there exist constants $\delta, C > 0$ such that*

$$\mathcal{A}(\partial F) \geqslant \mathcal{A}(\partial E) + C[\alpha(E, F)]^2,$$

for all smooth sets $F \subseteq \mathbb{T}^n$ such that $\mathrm{Vol}(F) = \mathrm{Vol}(E)$, $\mathrm{Vol}(F \triangle E) < \delta$, $\partial F \subseteq N_\varepsilon$ and

$$\partial F = \{y + \psi_F(y)\nu_E(y) \,:\, y \in \partial E\},$$

for a smooth function ψ_F with $\|\psi_F\|_{W^{2,p}(\partial E)} < \delta$, where the "distance" $\alpha(E, F)$ is defined as

$$\alpha(E, F) = \min_{\eta \in \mathbb{R}^n} \mathrm{Vol}(E \triangle (F + \eta)).$$

As a consequence, E is a $W^{2,p}$–local minimizer of $\mathcal{A}$. Moreover, if F is $W^{2,p}$–close enough to E and $\mathcal{A}(\partial F) = \mathcal{A}(\partial E)$, then F is a translate of E, that is, E is locally the unique $W^{2,p}$–local minimizer, up to translations.

Remark 2.2.11. We could have introduced the definitions of *strict* local minimizer or *strict* $W^{2,p}$–local minimizer for the Area functional, by asking that the inequalities $\mathcal{A}(\partial F) \leqslant \mathcal{A}(\partial E)$ in Definition 2.2.8 are equalities if and only if F is a translate of E. With such notion, the conclusion of this theorem is that E is actually a strict $W^{2,p}$–local minimizer (with a "quantitative" estimate of its minimality).

Remark 2.2.12. With a non trivial extra effort, by using some fine results from the regularity theory for minimal surfaces, it can be proved that in the same hypotheses of this theorem, the set E is actually a local minimizer (see [2]).

For the proof, we need some technical lemmas. We underline that most of the difficulties are due to the presence of the degenerate subspace $T(\partial E)$ of the form Π_E (where it is zero), related to the translation invariance of the Area functional (recall the discussion after Remark 2.2.4).

In the next key lemma we are going to show how to construct volume–preserving variations (hence, admissible smooth vector fields) "deforming" a set E to any other smooth set with the same volume, which is $W^{2,p}$–close enough. By the same technique we will also prove Lemma 2.1.7 immediately after, whose proof was postponed.

Lemma 2.2.13. *Let $E \subseteq \mathbb{T}^n$ be a smooth set and N_ε a tubular neighborhood of ∂E. For all $p > n-1$, there exist constants $\delta, C > 0$ such that if $\psi \in C^\infty(\partial E)$ and $\|\psi\|_{W^{2,p}(\partial E)} \leqslant \delta$, then there exists a vector field $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ with $\mathrm{div} X = 0$ in N_ε and the associated flow Φ, defined by system (2.4), satisfies*

$$\Phi(1, y) = y + \psi(y)\nu_E(y)\,, \qquad \text{for all } y \in \partial E. \tag{2.27}$$

Moreover, for every $t \in [0,1]$

$$\|\Phi(t, \cdot) - \mathrm{Id}\|_{W^{2,p}(\partial E)} \leqslant C\|\psi\|_{W^{2,p}(\partial E)}\,. \tag{2.28}$$

Finally, if $\mathrm{Vol}(E_1) = \mathrm{Vol}(E)$, then the variation $E_t = \Phi_t(E)$ is volume–preserving, that is, $\mathrm{Vol}(E_t) = \mathrm{Vol}(E)$ for all $t \in [-1, 1]$ and the vector field X is admissible.

Proof. We start considering the vector field $\widetilde{X} \in C^\infty(N_\varepsilon; \mathbb{R}^n)$ defined as

$$\widetilde{X}(x) = \xi(x)\nabla d_E(x) \tag{2.29}$$

for every $x \in N_\varepsilon$, where $d_E : N_\varepsilon \to \mathbb{R}$ is the signed distance function from E and $\xi : N_\varepsilon \to \mathbb{R}$ is the function defined as follows: for all $y \in \partial E$, we let $f_y : (-\varepsilon, \varepsilon) \to \mathbb{R}$ to be the unique solution of the ODE

$$\begin{cases} f_y'(t) + f_y(t)\Delta d_E(y + t\nu_E(y)) = 0 \\ f_y(0) = 1 \end{cases}$$

and we set

$$\xi(x) = \xi(y + t\nu_E(y)) = f_y(t) = \exp\Big(-\int_0^t \Delta d_E(y + s\nu_E(y))\, ds\Big),$$

recalling that the map $(y, t) \mapsto x = y + t\nu_E(y)$ is a smooth diffeomorphism between $\partial E \times (-\varepsilon, \varepsilon)$ and N_ε (with inverse $x \mapsto (\pi_E(x), d_E(x))$, where π_E is the orthogonal projection map on E, defined by formula (1.15)). Notice that the function f is always positive, thus the same holds for ξ and $\xi = 1$, $\nabla d_E = \nu_E$, hence $\widetilde{X} = \nu_E$ on ∂E.

Our aim is then to prove that the smooth vector field X defined by

$$X(x) = \int_0^{\psi(\pi_E(x))} \frac{ds}{\xi(\pi_E(x) + s\nu_E(\pi_E(x)))}\, \widetilde{X}(x) \tag{2.30}$$

for every $x \in N_\varepsilon$ and extended smoothly to all $\mathbb{T}^n$, satisfies all the properties of the statement of the lemma.

Step 1. We saw that $\widetilde{X}|_{\partial E} = \nu_E$, now we show that $\mathrm{div}\widetilde{X} = 0$ and analogously $\mathrm{div}X = 0$ in N_ε. Given any $x = y + t\nu_E(y) \in N_\varepsilon$, with $y \in \partial E$, we have

$$\begin{aligned}
\mathrm{div}\widetilde{X}(x) &= \mathrm{div}[\xi(x)\nabla d_E(x)] \\
&= \langle \nabla\xi(x) | \nabla d_E(x)\rangle + \xi(x)\Delta d_E(x) \\
&= \frac{\partial}{\partial t}[\xi(y + t\nu_E(y))] + \xi(y + t\nu_E(y))\Delta d_E(y + t\nu_E(y)) \\
&= f_y'(t) + f_y(t)\Delta d_E(y + t\nu_E(y)) \\
&= 0,
\end{aligned}$$

where we used the fact that $f_y'(t) = \langle \nabla\xi(y + t\nu_E(y)) | \nu_E(y)\rangle$ and $\nabla d_E(y + t\nu_E(y)) = \nu_E(y)$, by formula (1.16).

Since the function

$$x \mapsto \theta(x) = \int_0^{\psi(\pi_E(x))} \frac{ds}{\xi(\pi_E(x) + s\nu(\pi_E(x))}$$

is clearly constant along the segments $t \mapsto x + t\nabla d_E(x)$, for every $x \in N_\varepsilon$, it follows that

$$0 = \frac{\partial}{\partial t}\big[\theta(x + t\nabla d_E(x))\big]\Big|_{t=0} = \langle \nabla\theta(x) | \nabla d_E(x)\rangle,$$

hence,

$$\mathrm{div}X = \langle \nabla\theta | \nabla d_E\rangle\xi + \theta\, \mathrm{div}\widetilde{X} = 0.$$

Step 2. Recalling that $\psi \in C^\infty(\partial E)$ and $p > n - 1$, we have

$$\|\psi\|_{L^\infty(\partial E)} \leqslant \|\psi\|_{C^1(\partial E)} \leqslant C_E \|\psi\|_{W^{2,p}(\partial E)},$$

by Sobolev embeddings (see [7]). Then, we can choose $\delta < \varepsilon / C_E$ such that for all $x \in \partial E$ we have that $x \pm \psi(x)\nu_E(x) \in N_\varepsilon$.

To check that equation (2.27) holds, we observe that

$$\theta(x) = \int_0^{\psi(\pi_E(x))} \frac{ds}{\xi(\pi_E(x) + s\nu_E(\pi_E(x)))}$$

represents the time needed to go from $\pi_E(x)$ to $\pi_E(x) + \psi(\pi_E(x))\nu_E(\pi_E(x))$ along the trajectory of the vector field $\widetilde{X}$, which is the segment connecting $\pi_E(x)$ and $\pi_E(x) + \psi(\pi_E(x))\nu_E(\pi_E(x))$, of length $\psi(\pi_E(x))$, parametrized as

$$s \mapsto \pi_E(x) + s\psi(\pi_E(x))\nu_E(\pi_E(x)),$$

for $s \in [0,1]$ and which is traveled with velocity $\xi(\pi_E(x) + s\nu_E(\pi_E(x))) = f_{\pi_E(x)}(s)$. Therefore, by the above definition of $X = \theta\widetilde{X}$ and the fact that the function θ is constant along such segments, we conclude that

$$\Phi(1,y) - \Phi(0,y) = \psi(y)\nu_E(y)\,,$$

that is, $\Phi(1,y) = y + \psi(y)\nu_E(y)$, for all $y \in \partial E$.

Step 3. To establish inequality (2.28), we first show that

$$\|X\|_{W^{2,p}(N_\varepsilon)} \leqslant C\|\psi\|_{W^{2,p}(\partial E)} \tag{2.31}$$

for a constant $C > 0$ depending only on E and ε. This estimate will follow from the definition of X in equation (2.30) and the definition of $W^{2,p}$–norm, that is,

$$\|X\|_{W^{2,p}(N_\varepsilon)} = \|X\|_{L^p(N_\varepsilon)} + \|\nabla X\|_{L^p(N_\varepsilon)} + \|\nabla^2 X\|_{L^p(N_\varepsilon)}\,.$$

As $|\nabla d_E| = 1$ everywhere and the positive function ξ satisfies $0 < C_1 \leqslant \xi \leqslant C_2$ in N_ε, for a pair of constants C_1 and C_2, we have

$$\begin{aligned}
\|X\|^p_{L^p(N_\varepsilon)} &= \int_{N_\varepsilon}\left|\int_0^{\psi(\pi_E(x))}\frac{ds}{\xi(\pi_E(x) + s\nu_E(\pi_E(x)))}\,\xi(x)\nabla d_E(x)\right|^p dx\\
&\leqslant \|\xi\|^p_{L^\infty(N_\varepsilon)}\int_{N_\varepsilon}\left|\int_0^{\psi(\pi_E(x))}\frac{ds}{\xi(\pi_E(x) + s\nu_E(\pi_E(x)))}\right|^p dx\\
&\leqslant \frac{C_2^p}{C_1^p}\int_{N_\varepsilon}|\psi(\pi_E(x))|^p\,dx\\
&= \frac{C_2^p}{C_1^p}\int_{\partial E}\int_{-\varepsilon}^{\varepsilon}|\psi(\pi_E(y + t\nu_E(y)))|^p JL(y,t)\,dt\,d\mu(y)\\
&= \frac{C_2^p}{C_1^p}\int_{\partial E}|\psi(y)|^p\int_{-\varepsilon}^{\varepsilon}JL(y,t)\,dt\,d\mu(y)\\
&\leqslant C\int_{\partial E}|\psi(y)|^p\,d\mu(y)\\
&= C\|\psi\|^p_{L^p(\partial E)}\,,
\end{aligned}$$

where $L : \partial E \times (-\varepsilon, \varepsilon) \to N_\varepsilon$ the smooth diffeomorphism defined in formula (1.49) and JL its Jacobian. Notice that the constant C depends only on E and ε.

Now we estimate the L^p–norm of ∇X. We compute

$$\begin{aligned}
\nabla X = &\frac{\nabla\psi(\pi_E(x))d\pi_E(x)}{\xi(\pi_E(x) + \psi(\pi_E(x))\nu_E(\pi_E(x)))}\,\xi(x)\nabla d_E(x)\\
&- \left[\int_0^{\psi(\pi_E(x))}\frac{\nabla\xi(\pi_E(x) + s\nu_E(\pi_E(x)))}{\xi^2(\pi_E(x) + s\nu_E(\pi_E(x)))}d\pi_E(x)\,\mathrm{Id}\,ds\right]\xi(x)\nabla d_E(x)\\
&- \left[\int_0^{\psi(\pi_E(x))}\frac{\nabla\xi(\pi_E(x) + s\nu_E(\pi_E(x)))}{\xi^2(\pi_E(x) + s\nu_E(\pi_E(x)))}d\pi_E(x)s\,d\nu_E(\pi_E(x))\,ds\right]\xi(x)\nabla d_E(x)\\
&+ \int_0^{\psi(\pi_E(x))}\frac{ds}{\xi(\pi_E(x) + s\nu_E(\pi_E(x)))}\big(\nabla\xi(x)\nabla d_E(x) + \xi(x)\nabla^2 d_E(x)\big)
\end{aligned}$$

and we deal with the integrals in the three terms as before, changing variable by means of the function L. That is, since all the functions $d\pi_E$, $d\nu_E$, $\nabla^2 d_E$, ξ, $1/\xi$, $\nabla\xi$ are bounded by some constants depending only on E and ε, we easily get (the constant C could vary from line to line)

$$\begin{aligned}\|\nabla X\|^p_{L^p(N_\varepsilon)} &\leqslant C\int_{N_\varepsilon}|\nabla\psi(\pi_E(x))|^p\,dx + C\int_{N_\varepsilon}|\psi(\pi_E(x))|^p\,dx\\ &= C\int_{\partial E}\int_{-\varepsilon}^{\varepsilon}|\nabla\psi(\pi_E(y+t\nu_E(y)))|^p\,JL(y,t)\,dt\,d\mu(y)\\ &\quad + C\int_{\partial E}\int_{-\varepsilon}^{\varepsilon}|\psi(\pi_E(y+t\nu_E(y)))|^p\,JL(y,t)\,dt\,d\mu(y)\\ &= C\int_{\partial E}\big(|\psi(y)|^p+|\nabla\psi(y)|^p\big)\int_{-\varepsilon}^{\varepsilon}JL(y,t)\,dt\,d\mu(y)\\ &\leqslant C\|\psi\|^p_{L^p(\partial E)} + C\|\nabla\psi\|^p_{L^p(\partial E)}\\ &\leqslant C\|\psi\|^p_{W^{1,p}(\partial E)}\,.\end{aligned}$$

A very analogous estimate works for $\|\nabla^2 X\|^p_{L^p(N_\varepsilon)}$ and we obtain also

$$\|\nabla^2 X\|^p_{L^p(N_\varepsilon)} \leqslant C\|\psi\|^p_{W^{2,p}(\partial E)}\,,$$

hence, inequality (2.31) follows with $C = C(E,\varepsilon)$.

Applying now Lagrange theorem to every component of $\Phi(\cdot,y)$ for any $y\in\partial E$ and $t\in[0,1]$, we have

$$\Phi_i(t,y) - y_i = \Phi_i(t,y) - \Phi_i(0,y) = tX^i(\Phi(s,y))\,,$$

for every $i\in\{1,\dots,n\}$, where $s = s(y,t)$ is a suitable value in $(0,1)$. Then, it clearly follows

$$\|\Phi(t,\cdot) - \mathrm{Id}\|_{L^\infty(\partial E)} \leqslant C\|X\|_{L^\infty(N_\varepsilon)} \leqslant C\|X\|_{W^{2,p}(N_\varepsilon)} \leqslant C\|\psi\|_{W^{2,p}(\partial E)} \tag{2.32}$$

by estimate (2.31), with $C = C(E,\varepsilon)$ (notice that we used Sobolev embeddings, being $p > n-1$, the dimension of ∂E).

Differentiating the equations in system (2.4), we have (recall that we use the convention of summing over the repeated indices)

$$\begin{cases}\frac{\partial}{\partial t}\nabla^i\Phi_j(t,y) = \nabla^k X^j(\Phi(t,y))\nabla^i\Phi_k(t,y)\\ \nabla^i\Phi_j(0,y) = \delta_{ij}\end{cases} \tag{2.33}$$

for every $i,j\in\{1,\dots,n\}$. It follows,

$$\begin{aligned}\frac{\partial}{\partial t}\big|\nabla^i\Phi_j(t,y)-\delta_{ij}\big|^2 &\leqslant 2\big|(\nabla^i\Phi_j(t,y)-\delta_{ij})\nabla^k X^j(\Phi(t,y))\nabla^i\Phi_k(t,y)\big|\\ &\leqslant 2\|\nabla X\|_{L^\infty(N_\varepsilon)}\big|\nabla^i\Phi_j(t,y)-\delta_{ij}\big|^2 + 2\|\nabla X\|_{L^\infty(N_\varepsilon)}\big|\nabla^i\Phi_j(t,y)-\delta_{ij}\big|\end{aligned}$$

hence, for almost every $t\in[0,1]$, where the following derivative exists,

$$\frac{\partial}{\partial t}\big|\nabla^i\Phi_j(t,y)-\delta_{ij}\big| \leqslant C\|\nabla X\|_{L^\infty(N_\varepsilon)}\big(\big|\nabla^i\Phi_j(t,y)-\delta_{ij}\big|+1\big)\,.$$

Integrating this differential inequality, we get

$$\big|\nabla^i\Phi_j(t,y)-\delta_{ij}\big| \leqslant e^{tC\|\nabla X\|_{L^\infty(N_\varepsilon)}} - 1 \leqslant e^{C\|X\|_{W^{2,p}(N_\varepsilon)}} - 1,$$

as $t\in[0,1]$, where we used Sobolev embeddings again. Then, by inequality (2.31), we estimate

$$\sum_{1\leqslant i,j\leqslant n}\|\nabla^i\Phi_j(t,\cdot)-\delta_{ij}\|_{L^\infty(\partial E)} \leqslant C\big(e^{C\|\psi\|_{W^{2,p}(\partial E)}}-1\big) \leqslant C\|\psi\|_{W^{2,p}(\partial E)}, \tag{2.34}$$

as $\|\psi\|_{W^{2,p}(\partial E)} \leqslant \delta$, for any $t \in [0,1]$ and $y \in \partial E$, with $C = C(E, \varepsilon, \delta)$.
Differentiating equations (2.33), we obtain

$$\begin{cases} \frac{\partial}{\partial t}\nabla^\ell\nabla^i\Phi_j(t,y) = \nabla^s\nabla^k X^j(\Phi(t,y))\nabla^i\Phi_k(t,y)\nabla^\ell\Phi_s(t,y) \\ \qquad\qquad\qquad\qquad + \nabla^k X^j(\Phi(t,y))\nabla^\ell\nabla^i\Phi_k(t,y) \\ \nabla^\ell\nabla^i\Phi(0,y) = 0 \end{cases}$$

(where we sum over s and k), for every $t \in [0,1]$, $y \in \partial E$ and $i, j, \ell \in \{1, \dots, n\}$.
This is a linear *non–homogeneous* system of ODEs such that, if we control $C\|\psi\|_{W^{2,p}(\partial E)}$, the smooth coefficients in the right hand side multiplying the solutions $\nabla^\ell\nabla^i\Phi_j(\cdot, y)$ are uniformly bounded (as in estimate (2.34), the Sobolev embeddings imply that ∇X is bounded in L^∞ by $C\|\psi\|_{W^{2,p}(\partial E)}$). Hence, arguing as before, for almost every $t \in [0,1]$ where the following derivative exists, there holds

$$\begin{aligned} \frac{\partial}{\partial t}\big|\nabla^2\Phi(t,y)\big| &\leqslant C\|\nabla X\|_{L^\infty(N_\varepsilon)}\big|\nabla^2\Phi(t,y)\big| + C|\nabla^2 X(\Phi(t,y))| \\ &\leqslant C\delta\big|\nabla^2\Phi(t,y)\big| + C|\nabla^2 X(\Phi(t,y))|\,, \end{aligned}$$

by inequality (2.31) (notice that inequality (2.34) gives an L^∞–bound on $\nabla\Phi$, *not only* in L^p, which is crucial). Thus, by means of Gronwall's lemma (see [54], for instance), we obtain the estimate

$$\big|\nabla^2\Phi(t,y)\big| \leqslant C\int_0^t |\nabla^2 X(\Phi(s,y))| e^{C\delta(t-s)}\,ds \leqslant C\int_0^t |\nabla^2 X(\Phi(s,y))|\,ds\,,$$

hence,

$$\begin{aligned} \|\nabla^2\Phi(t,\cdot)\|^p_{L^p(\partial E)} &\leqslant C\int_{\partial E}\Big(\int_0^t |\nabla^2 X(\Phi(s,y))|\,ds\Big)^p\,d\mu(y) \\ &\leqslant C\int_0^t\int_{\partial E} |\nabla^2 X(\Phi(s,y))|^p\,d\mu(y)ds \\ &= C\int_{N_\varepsilon} |\nabla^2 X(x)|^p JL^{-1}(x)\,dx \\ &\leqslant C\|\nabla^2 X\|^p_{L^p(N_\varepsilon)} \\ &\leqslant C\|X\|^p_{W^{2,p}(N_\varepsilon)} \\ &\leqslant C\|\psi\|^p_{W^{2,p}(\partial E)}\,, \end{aligned} \tag{2.35}$$

by estimate (2.31), for every $t \in [0,1]$, with $C = C(E, \varepsilon, \delta)$.
Clearly, putting together inequalities (2.32), (2.34) and (2.35), we get the estimate (2.28) in the statement of the lemma.
Step 4. Finally, computing as in formula (2.19) and Remark 2.1.15, we have

$$\frac{d^2}{dt^2}\mathrm{Vol}(E_t) = \int_{\partial E} \langle X|\nu_{E_t}\rangle\,\mathrm{div}^{\mathbb{T}^n} X\,d\mu_t,$$

for every $t \in [-1,1]$, hence, since by Step 1 we know that $\mathrm{div}^{\mathbb{T}^n} X = 0$ in N_ε (which contains each ∂E_t), we conclude that $\frac{d^2}{dt^2}\mathrm{Vol}(E_t) = 0$ for all $t \in [-1,1]$, that is, the function $t \mapsto \mathrm{Vol}(E_t)$ is linear. If then $\mathrm{Vol}(E_1) = \mathrm{Vol}(E) = \mathrm{Vol}(E_0)$, it follows that $\mathrm{Vol}(E_t) = \mathrm{Vol}(E)$, for all $t \in [-1,1]$ which implies that X is admissible, by Remark 2.2.3. □

With an argument similar to the one of this proof, we now prove Lemma 2.1.7.

Proof of Lemma 2.1.7. Let $\psi : \partial E \to \mathbb{R}$ a C^∞ function with zero integral, then we define the following smooth vector field in N_ε,

$$X(x) = \psi(\pi_E(x))\widetilde{X}(x),$$

where $\widetilde{X}$ is the smooth vector field defined by formula (2.29) and we extend it to a smooth vector field $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ on the whole $\mathbb{T}^n$. Clearly, by the properties of $\widetilde{X}$ seen above,

$$\langle X(y)|\nu_E(y)\rangle = \psi(y)\langle \widetilde{X}(y)|\nu_E(y)\rangle = \psi(y)$$

for every $y \in \partial E$.
As the function $x \mapsto \psi(\pi_E(x))$ is constant along the segments $t \mapsto x + t\nabla d_E(x)$, for every $x \in N_\varepsilon$, it follows, as in Step 1 of the previous proof, that $\mathrm{div}X = 0$ in N_ε. Then, arguing as in Step 4, the flow Φ defined by system (2.4) having X as infinitesimal generator, gives a variation $E_t = \Phi_t(E)$ of E such that the function $t \mapsto \mathrm{Vol}(E_t)$ is linear, for t in some interval $(-\delta, \delta)$. Since, by equation (2.7), there holds

$$\frac{d}{dt}\mathrm{Vol}(E_t)\Big|_{t=0} = \int_{\partial E} \langle X|\nu_E\rangle \, d\mu = \int_{\partial E} \psi \, d\mu = 0,$$

such function $t \mapsto \mathrm{Vol}(E_t)$ must actually be constant.
Hence, $\mathrm{Vol}(E_t) = \mathrm{Vol}(E)$, for all $t \in (-\delta, \delta)$ and the variation E_t is volume–preserving. □

Lemma 2.2.14. *Let $p > \max\{2, n-1\}$ and $E \subseteq \mathbb{T}^n$ a strictly stable critical set for the Area functional $\mathcal{A}$ (under a volume constraint). Then, in the hypotheses and notation of Lemma 2.2.13, there exist constants $\delta, C > 0$ such that if $\|\psi\|_{W^{2,p}(\partial E)} \leqslant \delta$ then $|X| \leqslant C|\langle X|\nu_{E_t}\rangle|$ on ∂E_t and*

$$\|\nabla X\|_{L^2(\partial E_t)} \leqslant C\|\langle X|\nu_{E_t}\rangle\|_{H^1(\partial E_t)} \tag{2.36}$$

(here ∇ is the covariant derivative along E_t), for all $t \in [0, 1]$, where $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ is the smooth vector field defined in formula (2.30).

Proof. Fixed $\varepsilon > 0$, from inequality (2.28) it follows that there exist $\delta > 0$ such that if $\|\psi\|_{W^{2,p}(\partial E)} \leqslant \delta$ there holds

$$|\nu_{E_t}(\Phi(t, y)) - \nu_E(y)| \leqslant \varepsilon$$

for every $y \in \partial E$, hence, as $\nabla d_E = \nu_E$ on ∂E, we have

$$|\nabla d_E(\Phi^{-1}(t, x)) - \nu_{E_t}(x)| = |\nu_E(\Phi^{-1}(t, x)) - \nu_{E_t}(x)| \leqslant \varepsilon$$

for every $x \in \partial E_t$. Then, if $\|\psi\|_{W^{2,p}(\partial E)}$ is small enough, $\Phi^{-1}(t, \cdot)$ is close to the identity, thus

$$|\nabla d_E(\Phi^{-1}(t, x)) - \nabla d_E(x)| \leqslant \varepsilon$$

on ∂E_t and we conclude

$$\|\nabla d_E - \nu_{E_t}\|_{L^\infty(\partial E_t)} \leqslant 2\varepsilon\,.$$

Moreover, using again the inequality (2.28) and following the same argument above, we also obtain

$$\|\nabla^2 d_E - \nabla \nu_{E_t}\|_{L^\infty(\partial E_t)} \leqslant 2\varepsilon\,. \tag{2.37}$$

We estimate $X_{\tau_t} = X - \langle X|\nu_{E_t}\rangle\nu_{E_t}$ (recall that $X = \langle X|\nabla d_E\rangle\nabla d_E$),

$$\begin{aligned}
|X_{\tau_t}| &= |X - \langle X|\nu_{E_t}\rangle\nu_{E_t}| \\
&= |\langle X|\nabla d_E\rangle\nabla d_E - \langle X|\nu_{E_t}\rangle\nu_{E_t}| \\
&= |\langle X|\nabla d_E\rangle\nabla d_E - \langle X|\nu_{E_t}\rangle\nabla d_E + \langle X|\nu_{E_t}\rangle\nabla d_E - \langle X|\nu_{E_t}\rangle\nu_{E_t}| \\
&\leqslant |\langle X|(\nabla d_E - \nu_{E_t})\rangle\nabla d_E| + |\langle X|\nu_{E_t}\rangle(\nabla d_E - \nu_{E_t})| \\
&\leqslant 2|X|\,|\nabla d_E - \nu_{E_t}| \\
&\leqslant 4\varepsilon|X|\,,
\end{aligned}$$

then

$$|X_{\tau_t}| \leqslant 4\varepsilon |X_{\tau_t} + \langle X|\nu_{E_t}\rangle \nu_{E_t}| \leqslant 4\varepsilon |X_{\tau_t}| + |\langle X|\nu_{E_t}\rangle| \,,$$

hence,

$$|X_{\tau_t}| \leqslant C|\langle X|\nu_{E_t}\rangle| \,. \tag{2.38}$$

We now estimate the covariant derivative of X_{τ_t} along ∂E_t, that is,

$$\begin{aligned}
|\nabla X_{\tau_t}| &= |\nabla X - \nabla(\langle X|\nu_{E_t}\rangle \nu_{E_t})| \\
&= |\nabla(\langle X|\nabla d_E\rangle \nabla d_E) - \nabla(\langle X|\nu_{E_t}\rangle \nu_{E_t})| \\
&= |\nabla(\langle X|\nabla d_E\rangle \nabla d_E) - \nabla(\langle X|\nu_{E_t}\rangle \nabla d_E) + \nabla(\langle X|\nu_{E_t}\rangle \nabla d_E) - \nabla(\langle X|\nu_{E_t}\rangle \nu_{E_t})| \\
&\leqslant |\nabla(\langle X|(\nabla d_E - \nu_{E_t})\rangle \nabla d_E)| + |\nabla(\langle X|\nu_{E_t}\rangle (\nabla d_E - \nu_{E_t}))| \\
&\leqslant C\varepsilon\big[|\nabla X| + |\nabla\langle X|\nu_{E_t}\rangle|\big] + C|X|\big[|\nabla(\nabla d_E)| + |\nabla \nu_{E_t}|\big] \\
&\leqslant C\varepsilon\big[|\nabla(\langle X|\nu_{E_t}\rangle \nu_{E_t} + X_{\tau_t})| + |\nabla\langle X|\nu_{E_t}\rangle|\big] + C\big(|\langle X|\nu_{E_t}\rangle| + |X_{\tau_t}|\big)\,\big[|\nabla^2 d_E| + |\nabla \nu_{E_t}|\big]
\end{aligned}$$

hence, using inequality (2.38) and arguing as above, there holds

$$|\nabla X_{\tau_t}| \leqslant C|\nabla\langle X|\nu_{E_t}\rangle| + C|\langle X|\nu_{E_t}\rangle|\big[|\nabla^2 d_E| + |\nabla \nu_{E_t}|\big] \,.$$

Then, we get

$$\begin{aligned}
\|\nabla X_{\tau_t}\|^2_{L^2(\partial E_t)} &\leqslant C\|\nabla\langle X|\nu_{E_t}\rangle\|^2_{L^2(\partial E_t)} + C\int_{\partial E_t} |\langle X|\nu_{E_t}\rangle|^2 \big[|\nabla^2 d_E| + |\nabla \nu_{E_t}|\big]^2 \, d\mu \\
&\leqslant C\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)} + C\|\langle X|\nu_{E_t}\rangle\|^2_{L^{\frac{2p}{p-2}}(\partial E_t)} \big\||\nabla^2 d_E| + |\nabla \nu_{E_t}|\big\|^2_{L^p(\partial E_t)} \\
&\leqslant C\,\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}
\end{aligned}$$

where in the last inequality we used as usual Sobolev embeddings, as $p > \max\{2, n-1\}$ and the fact that $\|\nabla \nu_{E_t}\|_{L^p(\partial E_t)}$ is bounded by the inequality (2.37) (as $\|\nabla^2 d_E\|_{L^p(\partial E_t)}$).
Considering the covariant derivative of $X = X_{\tau_t} + \langle X|\nu_{E_t}\rangle \nu_{E_t}$, by means of this estimate, the trivial one

$$\|\nabla\langle X|\nu_{E_t}\rangle\|_{L^2(\partial E_t)} \leqslant \|\langle X|\nu_{E_t}\rangle\|_{H^1(\partial E_t)}$$

and inequality (2.38), we obtain estimate (2.36). □

We now show that any smooth set E sufficiently $W^{2,p}$–close to another smooth set F, can be "translated" by a vector $\eta \in \mathbb{R}^n$ such that $\partial E - \eta = \{y + \psi_\eta(y)\nu_F(y) \,:\, y \in \partial F\}$, for a function $\psi_\eta \in C^\infty(\partial F)$ having a suitable small "projection" on $T(\partial F)$ (see the definitions and the discussion after Remark 2.2.4).

Lemma 2.2.15. *Let $p > n-1$ and $F \subseteq \mathbb{T}^n$ a smooth set with a tubular neighborhood N_ε. For any $\tau > 0$ there exist constants $\delta, C > 0$ such that if another smooth set $E \subseteq \mathbb{T}^n$ satisfies $\mathrm{Vol}(E \triangle F) < \delta$ and $\partial E = \{y + \psi(y)\nu_F(y) \,:\, y \in \partial F\} \subseteq N_\varepsilon$ for a function $\psi \in C^\infty(\mathbb{R})$ with $\|\psi\|_{W^{2,p}(\partial F)} < \delta$, then there exist $\eta \in \mathbb{R}^n$ and $\psi_\eta \in C^\infty(\partial F)$ with the following properties:*

$$\partial E - \eta = \{y + \psi_\eta(y)\nu_F(y) \,:\, y \in \partial F\} \subseteq N_\varepsilon \,,$$

$$|\eta| \leqslant C\|\psi\|_{W^{2,p}(\partial F)}, \qquad \|\psi_\eta\|_{W^{2,p}(\partial F)} \leqslant C\|\psi\|_{W^{2,p}(\partial F)}$$

and

$$\left|\int_{\partial F} \psi_\eta \nu_F \, d\mu\right| \leqslant \tau \|\psi_\eta\|_{L^2(\partial F)} \,.$$

Proof. We let d_F to be the signed distance function from ∂F, as in formula (1.14). We underline that, throughout the proof, the various constants will be all independent of $\psi : \partial F \to \mathbb{R}$.
As observed in Remark 2.2.7, there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ such that the functions $\langle \nu_F | e_i \rangle$ are orthogonal in $L^2(\partial F)$, that is,

$$\int_{\partial F} \langle \nu_F | e_i \rangle \langle \nu_F | e_j \rangle \, d\mu = 0, \tag{2.39}$$

for all $i \neq j$. Given a smooth function $\psi : \partial F \to \mathbb{R}$, we set

$$\eta = \sum_{i=1}^{n} \eta_i e_i \,,$$

where

$$\eta_i = \begin{cases} \frac{1}{\|\langle \nu_F | e_i \rangle\|^2_{L^2(\partial F)}} \int_{\partial F} \psi(x) \langle \nu_F(x) | e_i \rangle \, d\mu & \text{if } i \in \mathrm{I}_F, \\ \eta_i = 0 & \text{otherwise} \end{cases} \tag{2.40}$$

and I_F is the set of the indices $i \in \{1, \ldots, n\}$ such that $\|\langle \nu_F | e_i \rangle\|_{L^2(\partial F)} > 0$. Note that, from Hölder inequality, it follows

$$|\eta| \leqslant C_1 \|\psi\|_{L^2(\partial F)} \,. \tag{2.41}$$

Step 1. Let $T_\psi : \partial F \to \partial F$ be the map

$$T_\psi(y) = \pi_F(y + \psi(y)\nu_F(y) - \eta) \,. \tag{2.42}$$

It is easily checked that there exists $\varepsilon_0 > 0$ such that if

$$\|\psi\|_{W^{2,p}(\partial F)} + |\eta| \leqslant \varepsilon_0 \leqslant 1 \,, \tag{2.43}$$

then T_ψ is a smooth diffeomorphism, moreover,

$$\|JT_\psi - 1\|_{L^\infty(\partial F)} \leqslant C\|\psi\|_{C^1(\partial F)} \tag{2.44}$$

(here JT_ψ is the Jacobian relative to ∂F) and

$$\|T_\psi - \mathrm{Id}\|_{W^{2,p}(\partial F)} + \|T_\psi^{-1} - \mathrm{Id}\|_{W^{2,p}(\partial F)} \leqslant C(\|\psi\|_{W^{2,p}(\partial F)} + |\eta|) \,. \tag{2.45}$$

Therefore, setting $\widehat{E} = E - \eta$, we have

$$\partial \widehat{E} = \{z + \psi_\eta(z)\nu_F(z) \,:\, z \in \partial F\} \,,$$

for some function ψ_η which is linked to ψ by the following relation: for all $y \in \partial F$, we let $z = z(y) \in \partial F$ such that

$$y + \psi(y)\nu_F(y) - \eta = z + \psi_\eta(z)\nu_F(z) \,,$$

then,

$$T_\psi(y) = \pi_F(y + \psi(y)\nu_F(y) - \eta) = \pi_F(z + \psi_\eta(z)\nu_F(z)) = z,$$

that is, $y = T_\psi^{-1}(z)$ and

$$\begin{aligned} \psi_\eta(z) &= \psi_\eta(T_\psi(y)) \\ &= d_F(z + \psi_\eta(z)\nu_F(z)) \\ &= d_F(y + \psi(y)\nu_F(y) - \eta) \\ &= d_F(T_\psi^{-1}(z) + \psi(T_\psi^{-1}(z))\nu_F(T_\psi(y)) - \eta). \end{aligned}$$

Thus, using inequality (2.45), we have

$$\|\psi_\eta\|_{W^{2,p}(\partial F)} \leqslant C_2\big(\|\psi\|_{W^{2,p}(\partial F)} + |\eta|\big), \tag{2.46}$$

for some constant $C_2 > 1$. We now estimate

$$\begin{aligned}\int_{\partial F} \psi_\eta(z)\nu_F(z)\,d\mu(z) &= \int_{\partial F} \psi_\eta(T_\psi(y))\nu_F(T_\psi(y))JT_\psi(y)\,d\mu(y)\\ &= \int_{\partial F} \psi_\eta(T_\psi(y))\nu_F(T_\psi(y))\,d\mu(y) + R_1\,,\end{aligned} \tag{2.47}$$

where

$$|R_1| = \left|\int_{\partial F} \psi_\eta(T_\psi(y))\nu_F(T_\psi(y))\,[JT_\psi(y)-1]\,d\mu(y)\right| \leqslant C_3\|\psi\|_{C^1(\partial F)}\|\psi_\eta\|_{L^2(\partial F)}\,, \tag{2.48}$$

by inequality (2.44).
On the other hand,

$$\begin{aligned}&\int_{\partial F} \psi_\eta(T_\psi(y))\nu_F(T_\psi(y))\,d\mu(y)\\ &\quad= \int_{\partial F} \big[y + \psi(y)\nu_F(y) - \eta - T_\psi(y)\big]\,d\mu(y)\\ &\quad= \int_{\partial F} \big[y + \psi(y)\nu_F(y) - \eta - \pi_F(y + \psi(y)\nu_F(y) - \eta)\big]\,d\mu(y)\\ &\quad= \int_{\partial F} \big\{\psi(y)\nu_F(y) - \eta + \big[\pi_F(y) - \pi_F(y + \psi(y)\nu_F(y) - \eta)\big]\big\}\,d\mu(y)\\ &\quad= \int_{\partial F} (\psi(y)\nu_F(y) - \eta)\,d\mu(y) + R_2\,,\end{aligned} \tag{2.49}$$

where

$$\begin{aligned}R_2 &= \int_{\partial F} \big[\pi_F(y) - \pi_F(y + \psi(y)\nu_F(y) - \eta)\big]\,d\mu(y)\\ &= -\int_{\partial F} d\mu(y)\int_0^1 \nabla\pi_F(y + t(\psi(y)\nu(y) - \eta))(\psi(y)\nu_F(y) - \eta)\,dt\\ &= -\int_{\partial F} \nabla\pi_F(y)(\psi(y)\nu_F(y) - \eta)\,d\mu(y) + R_3\,.\end{aligned} \tag{2.50}$$

In turn, recalling inequality (2.41), we get

$$|R_3| \leqslant \int_{\partial F} d\mu(y)\int_0^1 |\nabla\pi_F(y + t(\psi(y)\nu_F(y) - \eta)) - \nabla\pi_F(y)|\,|\psi(y)\nu_F(y) - \eta|\,dt \leqslant C_4\|\psi\|^2_{L^2(\partial F)}\,. \tag{2.51}$$

Since $\pi_F(x) = x - d_F(x)\nabla d_F(x)$ for $x \in N_\varepsilon$ (by equation (1.15)), it follows

$$\frac{\partial\pi_F^i}{\partial x_j}(x) = \delta_{ij} - \frac{\partial d_F}{\partial x_i}(x)\frac{\partial d_F}{\partial x_j}(x) - d_F(x)\frac{\partial^2 d_F}{\partial x_i\partial x_j}(x),$$

thus, for all $y \in \partial F$, there holds

$$\frac{\partial\pi_F^i}{\partial x_j}(y) = \delta_{ij} - \frac{\partial d_F}{\partial x_i}(y)\frac{\partial d_F}{\partial x_j}(y)\,.$$

From this identity and equalities (2.47), (2.49) and (2.50), we conclude

$$\int_{\partial F} \psi_\eta(z)\nu_F(z)\,d\mu(z) = \int_{\partial F} \big[\psi(x)\nu_F(x) - \langle\eta\,|\,\nu_F(x)\rangle\nu_F(x)\big]\,d\mu(x) + R_1 + R_3\,.$$

As the integral at the right–hand side vanishes by relations (2.39) and (2.40), estimates (2.48) and (2.51) imply

$$\begin{aligned}\Big|\int_{\partial F}\psi_\eta(y)\nu_F(y)\,d\mu(y)\Big| &\leqslant C_3\|\psi\|_{C^1(\partial F)}\|\psi_\eta\|_{L^2(\partial F)}+C_4\|\psi\|^2_{L^2(\partial F)}\\ &\leqslant C\|\psi\|_{C^1(\partial F)}\big(\|\psi_\eta\|_{L^2(\partial F)}+\|\psi\|_{L^2(\partial F)}\big)\\ &\leqslant C_5\|\psi\|^{1-\vartheta}_{W^{2,p}(\partial F)}\|\psi\|^{\vartheta}_{L^2(\partial F)}\big(\|\psi_\eta\|_{L^2(\partial F)}+\|\psi\|_{L^2(\partial F)}\big)\,, \end{aligned}\tag{2.52}$$

where in the last passage we used a well known interpolation inequality, with $\vartheta\in(0,1)$ depending only on $p>n-1$ (see [7, Theorem 3.70] and Proposition 3.3.4 below).

Step 2. The previous estimate does not allow to conclude directly, but we have to rely on the following iteration procedure. Fix any number $K>1$ and assume that $\delta\in(0,1)$ is such that (possibly considering a smaller τ)

$$\tau+\delta<\varepsilon_0/2,\qquad C_2\delta(1+2C_1)\leqslant\tau,\qquad 2C_5\delta^\vartheta K\leqslant\tau\,.\tag{2.53}$$

Given ψ, we set $\psi_{\eta,0}=\psi$ and we denote by η^1 the vector defined as in (2.40). We set $E_1=E-\eta^1$ and denote by $\psi_{\eta,1}$ the function such that $\partial E_1=\{x+\psi_{\eta,1}(x)\nu_F(x)\,:\,x\in\partial F\}$. As before, $\psi_{\eta,1}$ satisfies

$$y+\psi_{\eta,0}(y)\nu_F(y)-\eta^1=z+\psi_{\eta,1}(z)\nu_F(z)\,.$$

Since $\|\psi\|_{W^{2,p}(\partial F)}\leqslant\delta$ and $|\eta|\leqslant C_1\|\psi\|_{L^2(\partial F)}$, by inequalities (2.41), (2.46) and (2.53) we have

$$\|\psi_{\eta,1}\|_{W^{2,p}(\partial F)}\leqslant C_2\delta(1+C_1)\leqslant\tau\,.\tag{2.54}$$

Using again that $\|\psi\|_{W^{2,p}(\partial F)}<\delta<1$, by estimate (2.52) we obtain

$$\Big|\int_{\partial F}\psi_{\eta,1}(y)\nu_F(y)\,d\mu(y))\Big|\leqslant C_5\|\psi_{\eta,0}\|^{\vartheta}_{L^2(\partial F)}\big(\|\psi_{\eta,1}\|_{L^2(\partial F)}+\|\psi_{\eta,0}\|_{L^2(\partial F)}\big)\,,$$

where we have $\|\psi_{\eta,0}\|_{L^2(\partial F)}\leqslant\delta$.

We now distinguish two cases.

If $\|\psi_{\eta,0}\|_{L^2(\partial F)}\leqslant K\|\psi_{\eta,1}\|_{L^2(\partial F)}$, from the previous inequality and (2.53), we get

$$\begin{aligned}\Big|\int_{\partial F}\psi_{\eta,1}(y)\nu_F(y)\,d\mu(y)\Big| &\leqslant C_5\delta^\vartheta\big(\|\psi_{\eta,1}\|_{L^2(\partial F)}+\|\psi_{\eta,0}\|_{L^2(\partial F)}\big)\\ &\leqslant 2C_5\delta^\vartheta K\|\psi_{\eta,1}\|_{L^2(\partial F)}\\ &\leqslant \delta\|\psi_{\eta,1}\|_{L^2(\partial F)}\,,\end{aligned}$$

thus, the conclusion follows with $\eta=\eta^1$.

In the other case,

$$\|\psi_{\eta,1}\|_{L^2(\partial F)}\leqslant\frac{\|\psi_{\eta,0}\|_{L^2(\partial F)}}{K}\leqslant\frac{\delta}{K}\leqslant\delta\,.\tag{2.55}$$

We then repeat the whole procedure: we denote by η^2 the vector defined as in formula (2.40) with ψ replaced by $\psi_{\eta,1}$, we set $E_2=E_1-\eta^2=E-\eta^1-\eta^2$ and we consider the corresponding $\psi_{\eta,2}$ which satisfies

$$w+\psi_{\eta,2}(w)\nu_F(w)=z+\psi_{\eta,1}(z)\nu_F(z)-\eta^2=y+\psi_{\eta,0}(y)\nu_F(y)-\eta^1-\eta^2\,.$$

Since

$$\begin{aligned}\|\psi_{\eta,0}\|_{W^{2,p}(\partial F)}+|\eta^1+\eta^2| &\leqslant\delta+C_1\delta+C_1\|\psi_{\eta,1}\|_{L^2(\partial F)}\\ &\leqslant\delta+C_1\delta\Big(1+\frac{1}{K}\Big)\\ &\leqslant C_2\delta(1+2C_1)\\ &\leqslant\tau\,,\end{aligned}$$

the map $T_{\psi_{\eta,0}}(y) = \pi_F(y + \psi_{\eta,0}(y)\nu_F(y) - (\eta^1 + \eta^2))$ is actually a diffeomorphism, thanks to formula (2.43) (having chosen τ and δ small enough).
Thus, by applying inequalities (2.46) (with $\eta = \eta^1 + \eta^2$), (2.41), (2.53) and (2.55), we get

$$\|\psi_{\eta,2}\|_{W^{2,p}(\partial F)} \leqslant C_2\big(\|\psi_{\eta,0}\|_{W^{2,p}(\partial F)} + |\eta^1 + \eta^2|\big) \leqslant C_2\delta\Big(1 + C_1 + \frac{C_1}{K}\Big) \leqslant \tau\,,$$

as $K > 1$, analogously to conclusion (2.54). On the other hand, by estimates (2.41), (2.54) and (2.55),

$$\|\psi_{\eta,1}\|_{W^{2,p}(\partial F)} + \eta^2 \leqslant C_2\delta(1 + C_1) + C_1\frac{\delta}{K} \leqslant C_2\delta(1 + 2C_1) \leqslant \tau\,,$$

hence, also the map $T_{\psi_{\eta,1}}(x) = \pi_F(x + \psi_{\eta,1}(x)\nu_F(x) - \eta^2)$ is a diffeomorphism satisfying inequalities (2.43) and (2.44). Therefore, arguing as before, we obtain

$$\Big|\int_{\partial F} \psi_{\eta,2}(y)\nu_F(y)\,d\mu(y)\Big| \leqslant C_5\|\psi_{\eta,1}\|^{\vartheta}_{L^2(\partial F)}\big(\|\psi_{\eta,2}\|_{L^2(\partial F)} + \|\psi_{\eta,1}\|_{L^2(\partial F)}\big)\,.$$

Since $\|\psi_{\eta,1}\|_{L^2(\partial F)} \leqslant \delta$ by inequality (2.55), if $\|\psi_{\eta,1}\|_{L^2(\partial F)} \leqslant K\|\psi_{\eta,2}\|_{L^2(\partial F)}$ the conclusion follows with $\eta = \eta^1 + \eta^2$. Otherwise, we iterate the procedure observing that

$$\|\psi_{\eta,2}\|_{L^2(\partial F)} \leqslant \frac{\|\psi_{\eta,1}\|_{L^2(\partial F)}}{K} \leqslant \frac{\|\psi_{\eta,0}\|_{L^2(\partial F)}}{K^2} \leqslant \frac{\delta}{K^2}\,.$$

This construction leads to three (possibly finite) sequences η^n, E_n and $\psi_{\eta,n}$ such that

$$\begin{cases} E_n = E - \eta^1 - \cdots - \eta^n, \qquad |\eta^n| \leqslant \frac{C_1\delta}{K^{n-1}} \\ \|\psi_{\eta,n}\|_{W^{2,p}(\partial F)} \leqslant C_2\big(\|\psi_{\eta,0}\|_{W^{2,p}(\partial F)} + |\eta^1 + \cdots + \eta^n|\big) \leqslant C_2\delta(1 + 2C_1) \\ \|\psi_{\eta,n}\|_{L^2(\partial F)} \leqslant \frac{\delta}{K^n} \\ \partial E_n = \{x + \psi_{\eta,n}(x)\nu_F(x)\,:\, x \in \partial F\} \end{cases}$$

If for some $n \in \mathbb{N}$ we have $\|\psi_{\eta,n-1}\|_{L^2(\partial F)} \leqslant K\|\psi_{\eta,n}\|_{L^2(\partial F)}$, the construction stops, since, arguing as before,

$$\Big|\int_{\partial F} \psi_{\eta,n}(y)\nu_F(y)\,d\mu(y)\Big| \leqslant \delta\|\psi_{\eta,n}\|_{L^2(\partial F)}$$

and the conclusion follows with $\eta = \eta^1 + \cdots + \eta^n$ and $\psi_\eta = \psi_{\eta,n}$. Otherwise, the iteration continues indefinitely and we get the thesis with

$$\eta = \sum_{n=1}^{\infty} \eta^n, \qquad \psi_\eta = 0\,,$$

(notice that the series is converging), which actually means that $E = \eta + F$. □

We finally show Theorem 2.2.10.

Proof of Theorem 2.2.10.
Step 1. We first want to see that

$$m_0 = \inf\Big\{\Pi_E(\psi)\,:\, \psi \in T^{\perp}(\partial E), \|\psi\|_{H^1(\partial E)} = 1\Big\} > 0.$$

To this aim, we consider a minimizing sequence ψ_i for the above infimum and we assume that $\psi_i \rightharpoonup \psi_0$ weakly in $H^1(\partial E)$, then $\psi_0 \in T^{\perp}(\partial E)$ (since it is a closed subspace of $H^1(\partial E)$) and if $\psi_0 \neq 0$, there holds

$$m_0 = \lim_{i\to+\infty} \Pi_E(\psi_i) \geqslant \Pi_E(\psi_0) > 0$$

due to the strict stability of E and the lower semicontinuity of Π_E (recall formula (2.21) and the fact that the weak convergence in $H^1(\partial E)$ implies strong convergence in $L^2(\partial E)$ by Sobolev embeddings). On the other hand, if instead $\psi_0 = 0$, again by the strong convergence of $\psi_i \to \psi_0$ in $L^2(\partial E)$, by looking at formula (2.21), we have

$$m_0 = \lim_{i\to\infty} \Pi_E(\psi_i) = \lim_{i\to\infty} \int_{\partial E} |\nabla \psi_i|^2 \, d\mu = \lim_{i\to\infty} \|\psi_i\|^2_{H^1(\partial E)} = 1$$

since $\|\psi_i\|_{L^2(\partial E)} \to 0$.

Step 2. Now we show that there exists a constant $\delta_1 > 0$ such that if E is like in the statement and $\partial F = \{y + \psi_F(y)\nu_E(y) \, : \, y \in \partial E\}$, with $\|\psi_F\|_{W^{2,p}(\partial E)} \leqslant \delta_1$ and $\mathrm{Vol}(F) = \mathrm{Vol}(E)$, then

$$\inf \left\{ \Pi_F(\psi) \, : \, \psi \in \widetilde{H}^1(\partial F), \|\psi\|_{H^1(\partial F)} = 1, \left| \int_{\partial F} \psi \nu_F \, d\mu \right| \leqslant \delta_1 \right\} \geqslant \frac{m_0}{2}.$$

We argue by contradiction assuming that there exists a sequence of sets F_i with $\partial F_i = \{y + \psi_{F_i}(y)\nu_E(y) \, : \, y \in \partial E\}$ with $\|\psi_{F_i}\|_{W^{2,p}(\partial E)} \to 0$ and $\mathrm{Vol}(F_i) = \mathrm{Vol}(E)$ and a sequence of functions $\psi_i \in \widetilde{H}^1(\partial F_i)$ with $\|\psi_i\|_{H^1(\partial F_i)} = 1$ and $\int_{\partial F_i} \psi_i \nu_{F_i} \, d\mu_i \to 0$, such that

$$\Pi_{F_i}(\psi_i) < \frac{m_0}{2}.$$

We then define the following sequence of smooth functions

$$\widetilde{\psi}_i(y) = \psi_i(y + \psi_{F_i}(y)\nu_E(y)) - \fint_{\partial E} \psi_i(y + \psi_{F_i}(y)\nu_E(y)) \, d\mu(y) \tag{2.56}$$

which clearly belong to $\widetilde{H}^1(\partial E)$. Setting $\theta_i(y) = y + \psi_{F_i}(y)\nu_E(y)$, as $p > \max\{2, n-1\}$, by the Sobolev embeddings, $\theta_i \to \mathrm{Id}$ in $C^{1,\alpha}$ and $\nu_{F_i} \circ \theta_i \to \nu_E$ in $C^{0,\alpha}(\partial E)$, hence, the sequence $\widetilde{\psi}_i$ is bounded in $H^1(\partial E)$ and if $\{e_k\}$ is the special orthonormal basis found in Remark 2.2.7, we have $\langle \nu_{F_i} \circ \theta_i | e_k \rangle \to \langle \nu_E | e_k \rangle$ uniformly for all $k \in \{1, \ldots, n\}$. Thus,

$$\int_{\partial E} \widetilde{\psi}_i \langle \nu_E | \varepsilon_i \rangle \, d\mu \to 0,$$

as $i \to \infty$, indeed,

$$\int_{\partial E} \widetilde{\psi}_i \langle \nu_E | e_k \rangle \, d\mu - \int_{\partial E} \widetilde{\psi}_i \langle \nu_{F_i} \circ \theta_i | e_k \rangle \, d\mu \to 0$$

and

$$\int_{\partial E} \widetilde{\psi}_i \langle \nu_{F_i} \circ \theta_i | e_k \rangle \, d\mu = \int_{\partial F_i} \psi_i \langle \nu_{F_i} | e_k \rangle \, J\theta_i^{-1} d\mu_i \to 0,$$

as the Jacobians (notice that $J\theta_i$ are Jacobians "relative" to the hypersurface ∂E) $J\theta_i^{-1} \to 1$ uniformly and we assumed

$$\int_{\partial F_i} \psi_i \nu_{F_i} \, d\mu_i \to 0\,.$$

Hence, using expression (2.26), for the projection map π on $T^{\perp}(\partial E)$, it follows

$$\|\pi(\widetilde{\psi}_i) - \widetilde{\psi}_i\|_{H^1(\partial E)} \to 0$$

as $i \to \infty$ and

$$\lim_{i\to\infty} \|\pi(\widetilde{\psi}_i)\|_{H^1(\partial E)} = \lim_{i\to\infty} \|\widetilde{\psi}_i\|_{H^1(\partial E)} = \lim_{i\to\infty} \|\psi_i\|_{H^1(\partial F_i)} = 1, \tag{2.57}$$

since $\|\psi_{F_i}\|_{W^{2,p}(\partial E)} \to 0$, thus $\|\psi_i\|_{C^{1,\alpha}(\partial E)} \to 0$, by looking at the definition of the functions $\widetilde{\psi}_i$ in formula (2.56).

Note now that the $W^{2,p}$– convergence of F_i to E (the second fundamental form $B_{\partial F_i}$ of ∂F_i is "morally" the Hessian of ψ_{F_i}) implies

$$\mathrm{B}_{\partial F_i} \circ \theta_i \to \mathrm{B}_{\partial E} \qquad \text{in } L^p(\partial E)\,,$$

as $i \to \infty$, then, by Sobolev embeddings again (in particular $H^1(\partial E) \hookrightarrow L^q(\partial E)$ for any $q \in [1, 2^*)$, with $2^* = 2(n-1)/(n-3)$ which is larger than 2) and the $W^{2,p}$–convergence of F_i to E, we get

$$\int_{\partial F_i} |B_{\partial F_i}|^2 \psi_i^2 \, d\mu_i - \int_{\partial E} |B_{\partial E}|^2 \widetilde{\psi}_i^2 \, d\mu \to 0\,.$$

Finally, recalling expression (2.21), we conclude

$$\Pi_{F_i}(\psi_i) - \Pi_E(\widetilde{\psi}_i) \to 0\,,$$

since we have

$$\|\psi_i\|_{L^2(\partial F_i)} - \|\widetilde{\psi}_i\|_{L^2(\partial E)} \to 0\,,$$

which easily follows again by looking at the definition of the functions $\widetilde{\psi}_i$ in formula (2.56) and taking into account that $\|\psi_{F_i}\|_{C^{1,\alpha}(\partial E)} \to 0$, hence limits (2.57) imply

$$\|\nabla \psi_i\|_{L^2(\partial F_i)} - \|\nabla \widetilde{\psi}_i\|_{L^2(\partial E)} \to 0\,.$$

By the previous conclusion $\|\pi(\widetilde{\psi}_i) - \widetilde{\psi}_i\|_{H^1(\partial E)} \to 0$ and Sobolev embeddings, it this then straightforward, arguing as above, to get also

$$\Pi_E(\widetilde{\psi}_i) - \Pi_E(\pi(\widetilde{\psi}_i)) \to 0,$$

hence,

$$\Pi_{F_i}(\psi_i) - \Pi_E(\pi(\widetilde{\psi}_i)) \to 0.$$

Since we assumed that $\Pi_{F_i}(\psi_i) < m_0/2$, we conclude that for $i \in \mathbb{N}$, large enough there holds

$$\Pi_E(\pi(\widetilde{\psi}_i)) \leqslant \frac{m_0}{2} < m_0,$$

which is a contradiction to Step 1, as $\pi(\widetilde{\psi}_i) \in T^{\perp}(\partial E)$.

Step 3. In order to simplify the notation, in the rest of the proof we denote $\psi_F = \psi$.
Let us now consider F such that $\mathrm{Vol}(F) = \mathrm{Vol}(E)$, $\mathrm{Vol}(F \triangle E) < \delta$ and

$$\partial F = \{y + \psi(y)\nu_E(y) \,:\, y \in \partial E\} \subseteq N_\varepsilon,$$

with $\|\psi\|_{W^{2,p}(\partial E)} \leqslant \delta$ where $\delta > 0$ is smaller than δ_1 given by Step 2.
Taking a possibly smaller $\delta > 0$, we consider the field X and the associated flow Φ found in Lemma 2.2.13. Hence, $\operatorname{div} X = 0$ in N_ε and $\Phi(1, y) = y + \psi(y)\nu_E(y)$, for all $y \in \partial E$, that is, $\Phi(1, \partial E) = \partial F \subseteq N_\varepsilon$ which implies $E_1 = \Phi_1(E) = F$ and $\mathrm{Vol}(E_1) = \mathrm{Vol}(F) = \mathrm{Vol}(E)$. Then the special variation $E_t = \Phi_t(E)$ is volume–preserving, for $t \in [-1, 1]$ and the vector field X is admissible, by the last part of such lemma.
By Lemma 2.2.15, choosing an even smaller $\delta > 0$ if necessary, possibly replacing F with a translate $F - \sigma$ for some $\eta \in \mathbb{R}^n$ if needed, we can assume that

$$\left| \int_{\partial E} \psi \, \nu_E \, d\mu \right| \leqslant \frac{\delta_1}{2} \|\psi\|_{L^2(\partial E)}. \tag{2.58}$$

We now claim that

$$\left| \int_{\partial E} \langle X | \nu_{E_t} \rangle \nu_{E_t} \, d\mu_t \right| \leqslant \delta_1 \|\langle X | \nu_{E_t} \rangle\|_{L^2(\partial E_t)}\,, \tag{2.59}$$

for every $t \in [0,1]$. To this aim, we write

$$\begin{aligned}\int_{\partial E} \langle X|\nu_{E_t}\rangle \nu_{E_t}\, d\mu_t &= \int_{\partial E} \langle X\circ\Phi_t|\nu_{E_t}\circ\Phi_t\rangle(\nu_{E_t}\circ\Phi_t)\, J\Phi_t\, d\mu\\ &= \int_{\partial E} \langle X\circ\Phi_t|\nu_E\rangle\nu_E\, d\mu + R_1\\ &= \int_{\partial E} \langle X(x)|\nu_E\rangle\nu_E\, d\mu + R_1 + R_2\\ &= \int_{\partial E} \psi\nu_E\, d\mu + R_1 + R_2 + R_3\end{aligned}$$

with appropriate R_1, R_2 and R_3 (see below).
By the definition of X in formula (2.30) (in the proof of Lemma 2.2.13), the bounds $0 < C_1 \leqslant \xi \leqslant C_2$ and $\|J(\pi_E\circ\Phi_t)^{-1}\|_{L^\infty(\partial E)} \leqslant C_3$ (by inequality (2.28) and Sobolev embeddings, as $p > \max\{2, n-1\}$, we have $\|\Phi(t,\cdot) - \mathrm{Id}\|_{C^{1,\alpha}(\partial E)} \leqslant C\|\psi\|_{W^{2,p}(\partial E)} \leqslant C\delta$), the following inequality holds

$$\begin{aligned}\int_{\partial E} |X(\Phi(t,x))|\, d\mu &= \int_{\partial E}\left|\int_0^{\psi(\pi_E(\Phi(t,x)))} \frac{\xi(\Phi(t,x))\nabla d_E(\Phi(t,x))}{\xi(\Phi(t,x)+s\nu(\pi_E(\Phi(t,x))))}\, ds\right| d\mu\\ &\leqslant C\int_{\partial E} |\psi(\pi_E(\Phi(t,x)))|\, d\mu\\ &= \int_{\partial E} |\psi(z)| J(\pi_E\circ\Phi_t)^{-1}(z)\, d\mu(z)\\ &\leqslant C\|\psi\|_{L^2(\partial E)}.\end{aligned} \tag{2.60}$$

for every $t \in [0,1]$.
We want now to prove that for every $\overline{\varepsilon} > 0$, choosing a suitably small $\delta > 0$ we have the estimate

$$|R_1| + |R_2| + |R_3| \leqslant \varepsilon\|\psi\|_{L^2(\partial E)}. \tag{2.61}$$

First,

$$\begin{aligned}R_1 &= \int_{\partial E} \langle X\circ\Phi_t|\nu_{E_t}\circ\Phi\rangle\nu_{E_t}\circ\Phi_t[J\Phi_t - 1]\, d\mu\\ &\quad + \int_{\partial E} \langle X\circ\Phi_t|\nu_{E_t}\circ\Phi_t\rangle\nu_{E_t}\circ\Phi_t\, d\mu - \int_{\partial E} \langle X\circ\Phi_t, \nu_E\rangle\nu_E\, d\mu\\ &= \int_{\partial E} \langle X\circ\Phi_t|\nu_{E_t}\circ\Phi_t\rangle\nu_{E_t}\circ\Phi_t\,[J\Phi_t - 1]\, d\mu + \int_{\partial E} \langle X\circ\Phi_t|\nu_{E_t}\circ\Phi_t - \nu_E\rangle\nu_E\, d\mu\\ &\quad + \int_{\partial E} \langle X\circ\Phi_t|\nu_{E_t}\circ\Phi_t\rangle(\nu_{E_t}\circ\Phi_t - \nu_E)\, d\mu\\ &\leqslant \int_{\partial E} |X\circ\Phi_t|\, \|J\Phi_t - 1\|_{L^\infty(\partial E)}\, d\mu + \int_{\partial E} |X\circ\Phi_t|\, \|\nu_E - \nu_{E_t}\circ\Phi_t\|_{L^\infty(\partial E)}\, d\mu\,,\end{aligned}$$

then, since by equality (2.27), it follow that for every $t \in [0,1]$ the two terms

$$\|\nu_E - \nu_{E_t}\circ\Phi(t,x)\|_{L^\infty(\partial E)} \qquad \text{and} \qquad \|J\Phi_t - 1\|_{L^\infty(\partial E)}$$

can be made (uniformly in $t \in [0,1]$) small as we want, if $\delta > 0$ is small enough, by using inequality (2.60), we obtain

$$|R_1| \leqslant \overline{\varepsilon}\|\psi\|_{L^2(\partial E)}/3.$$

Then we estimate, by means of inequality (2.27) and where $s = s(t, y) \in [t, 1]$,

$$\begin{aligned}
|R_2| &\leqslant \int_{\partial E} |X(\Phi(t,x)) - X(\Phi(1,x))| + |X(\Phi(1,x)) - X(x)|\, d\mu \\
&\leqslant \int_{\partial E} |X(\Phi(t,x)) - X(\Phi(1,x))| + \|\nabla X\|_{L^2(N_\varepsilon)}\|\psi\|_{L^2(\partial E)} \\
&= \int_{\partial E} (1-t)|\nabla X(\Phi_s(y))| \left|\frac{\partial \Phi_s}{\partial t}(y)\right| d\mu(y) + \|\nabla X\|_{L^2(N_\varepsilon)}\|\psi\|_{L^2(\partial E)} \\
&\leqslant \int_{\partial E} |\nabla X(\Phi(s,x))||\Phi(t,x) - \Phi(1,x)| + \|\nabla X\|_{L^2(N_\varepsilon)}\|\psi\|_{L^2(\partial E)} \\
&\leqslant C\|\nabla X\|_{L^\infty(N_\varepsilon)} C\|\psi\|_{L^2(\partial E)} + \|\nabla X\|_{L^2(N_\varepsilon)}\|\psi\|_{L^2(\partial E)},
\end{aligned}$$

where in the last inequality we use equation (2.60). Hence, using equality (2.31) and Sobolev embeddings, as $p > \max\{2, n-1\}$, we get

$$|R_2| \leqslant C\|\psi\|_{W^{2,p}(\partial E)}\|\psi\|_{L^2(\partial E)},$$

then, since $\|\psi\|_{W^{2,p}(\partial E)} < \delta$, we obtain

$$|R_2| < \overline{\varepsilon}\|\psi\|_{L^2(\partial E)}/3,$$

if δ_2 is small enough.
Arguing similarly, recalling the definition of X given by formula (2.30), we also obtain $|R_3| \leqslant \overline{\varepsilon}\|\psi\|_{L^2(\partial E)}$, hence estimate (2.61) follows. We can then conclude that, for $\delta > 0$ small enough, we have

$$\left|\int_{\partial E} \langle X|\nu_{E_t}\rangle \nu_{E_t}\, d\mu_t\right| \leqslant \left|\int_{\partial E} \psi \nu_E\, d\mu\right| + \overline{\varepsilon}\|\psi\|_{L^2(\partial E)} \leqslant \Big(\frac{\delta_1}{2} + \overline{\varepsilon}\Big)\|\psi\|_{L^2(\partial E)}$$

for any $t \in [0, 1]$, where in the last inequality we used the assumption (2.58), thus choosing $\overline{\varepsilon} = \delta_1/4$ we get

$$\left|\int_{\partial E} \langle X|\nu_{E_t}\rangle \nu_{E_t}\, d\mu_t\right| \leqslant \frac{3\delta_1}{4}\|\psi\|_{L^2(\partial E)}.$$

Along the same line, it is then easy to prove that

$$\|\langle X|\nu_{E_t}\rangle\|_{L^2(\partial E_t)} \geqslant (1-\varepsilon)\|\psi\|_{L^2(\partial E)},$$

for any $t \in [0, 1]$, hence claim (2.59) follows.
As a consequence, since $\langle X|\nu_{E_t}\rangle \in \widetilde{H}^1(\partial E_t)$, being X admissible for E_t (recalling computation 2.6) and ∂E_t can be described as a graph over ∂E with a function with small norm in $W^{2,p}(\partial E)$ (by estimate (2.28) of Lemma 2.2.13), we can apply Step 2 with $F = E_t$ to the function $\langle X|\nu_{E_t}\rangle / \|\langle X|\nu_{E_t}\rangle\|_{H^1(\partial E_t)}$, concluding

$$\Pi_{E_t}(\langle X|\nu_{E_t}\rangle) \geqslant \frac{m_0}{2}\|\langle X|\nu_{E_t}\rangle\|_{H^1(\partial E_t)}. \tag{2.62}$$

By means of Lemma 2.2.14, for $\delta > 0$ small enough, we now show the following inequality on ∂E_t (here div is the divergence operator and $X_{\tau_t} = X - \langle X|\nu_{E_t}\rangle\nu_{E_t}$ is a tangent vector field on ∂E_t), for any $t \in [0,1]$,

$$\begin{aligned}
\|\operatorname{div}(X_{\tau_t}\langle X|\nu_{E_t}\rangle)\|_{L^{\frac{p}{p-1}}(\partial E_t)} &= \|\operatorname{div} X_{\tau_t}\langle X|\nu_{E_t}\rangle + \langle X_{\tau_t}|\nabla\langle X|\nu_{E_t}\rangle\rangle\|_{L^{\frac{p}{p-1}}(\partial E_t)} \\
&\leq C\|\nabla X_{\tau_t}\|_{L^2(\partial E_t)}\|\langle X|\nu_{E_t}\rangle\|_{L^{\frac{2p}{p-2}}(\partial E_t)} \\
&\quad + C\|X_{\tau_t}\|_{L^{\frac{2p}{p-2}}(\partial E_t)}\|\nabla\langle X|\nu_{E_t}\rangle\|_{L^2(\partial E_t)} \\
&\leq C\|X\|_{H^1(\partial E_t)}\|X\|_{L^{\frac{2p}{p-2}}(\partial E_t)} \\
&\leq C\|X\|^2_{H^1(\partial E_t)} \\
&\leq C\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)},
\end{aligned} \tag{2.63}$$

where we used the Sobolev embedding $H^1(\partial E_t) \hookrightarrow L^{\frac{2p}{p-2}}(\partial E_t)$, as $p > \max\{2, n-1\}$.

Then, we compute (here X_{τ_t} is the tangent component of X and H_t is the mean curvature)

$$\begin{aligned}
\mathcal{A}(\partial F) - \mathcal{A}(\partial E) &= \mathcal{A}(\partial E_1) - \mathcal{A}(\partial E) \\
&= \int_0^1 (1-t)\frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\,dt \\
&= \int_0^1 (1-t)\big(\Pi_{E_t}(\langle X|\nu_{E_t}\rangle)\big)\,dt \\
&= \int_0^1 (1-t)\Pi_{E_t}(\langle X|\nu_{E_t}\rangle)\,dt \\
&\quad - \int_0^1 (1-t)\int_{\partial E} \mathrm{H}_t \operatorname{div}(X_{\tau_t}\langle X|\nu_{E_t}\rangle)\,d\mu_t\,dt\,,
\end{aligned}$$

by Theorem 2.1.12, the definition of Π_{E_t} in formula (2.21) and taking into account that $\operatorname{div} X = 0$ in N_ε and that $X_t = X$, as the variation is special.
Hence, by estimate (2.62), we have (recall that $\mathrm{H} = \mathrm{H}_0 = \lambda$ constant, as E is a critical set)

$$\begin{aligned}
\mathcal{A}(\partial F) - \mathcal{A}(\partial E) \geq{}& \frac{m_0}{2}\int_0^1 (1-t)\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}\,dt \\
&- \int_0^1 (1-t)\int_{\partial E_t} \mathrm{H}_t\ \operatorname{div}(X_{\tau_t}\langle X|\nu_{E_t}\rangle)\,d\mu_t\,dt \\
={}& \frac{m_0}{2}\int_0^1 (1-t)\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}\,dt \\
&- \int_0^1 (1-t)\int_{\partial E_t} (\mathrm{H}_t - \lambda)\operatorname{div}(X_{\tau_t}\langle X|\nu_{E_t}\rangle)\,d\mu_t\,dt \\
\geqslant{}& \frac{m_0}{2}\int_0^1 (1-t)\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}\,dt \\
&- \int_0^1 (1-t)\|\mathrm{H}_t - \lambda\|_{L^p(\partial E_t)}\|\operatorname{div}(X_{\tau_t}\langle X|\nu_{E_t}\rangle)\|_{L^{\frac{p}{p-1}}(\partial E_t)}\,dt \\
\geqslant{}& \frac{m_0}{2}\int_0^1 (1-t)\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}\,dt \\
&- C\int_0^1 (1-t)\|\mathrm{H}_t - \lambda\|_{L^p(\partial E_t)}\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}\,dt,
\end{aligned}$$

by estimate (2.63).
If $\delta > 0$ is sufficiently small, as E_t is $W^{2,p}$–close to E, we have

$$\|\mathrm{H}_t - \lambda\|_{L^p(\partial E_t)} < m_0/4C\,,$$

hence,

$$\mathcal{A}(\partial F) - \mathcal{A}(\partial E) \geqslant \frac{m_0}{4}\int_0^1 (1-t)\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}\,dt.$$

Then, we can conclude the proof of the theorem with the following series of inequalities, holding for a suitably small $\delta > 0$ as in the statement,

$$\begin{aligned}\mathcal{A}(\partial F) &\geqslant \mathcal{A}(\partial E) + \frac{m_0}{2}\int_0^1 (1-t)\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}\,dt\\ &\geqslant \mathcal{A}(\partial E) + C\|\langle X|\nu_E\rangle\|^2_{L^2(\partial E)}\\ &\geqslant \mathcal{A}(\partial E) + C\|\psi\|^2_{L^2(\partial E)}\\ &\geqslant \mathcal{A}(\partial E) + C[\mathrm{Vol}(E\triangle F)]^2\\ &\geqslant \mathcal{A}(\partial E) + C[\alpha(E,F)]^2,\end{aligned}$$

where the first inequality is due to the $W^{2,p}$–closedness of E_t to E, the second one by the very expression (2.30) of the vector field X on ∂E,

$$|\langle X(y)|\nu_E(y)\rangle| = \left|\int_0^{\psi(y)} \frac{ds}{\xi(y+s\nu_E(y))}\right| \leqslant C|\psi(y)|,$$

the third follows by a straightforward computation (involving the map L defined by formula (1.49) and its Jacobian), as ∂E is a "normal graph" over ∂F with ψ as "height function", finally the last one simply by the definition of the "distance" α, recalling that we possibly translated the "original" set F by a vector $\eta \in \mathbb{R}^n$, at the beginning of this step. □

We conclude this section by proving two propositions that will be used later. The first one says that when a set is sufficiently $W^{2,p}$–close to a strictly stable critical set of the Area functional $\mathcal{A}$, then the quadratic form (2.21) remains uniformly positive definite (on the orthogonal complement of its degenerate subspace, see the discussion at the end of the previous subsection).

Proposition 2.2.16. *Let $p > \max\{2, n-1\}$ and $E \subseteq \mathbb{T}^n$ be a smooth strictly stable critical set with N_ε a tubular neighborhood of ∂E. Then, for every $\theta \in (0,1]$ there exist $\sigma_\theta, \delta > 0$ such that if a smooth set $F \subseteq \mathbb{T}^n$ is $W^{2,p}$–close to E, that is, $\mathrm{Vol}(F\triangle E) < \delta$ and $\partial F \subseteq N_\varepsilon$ with*

$$\partial F = \{y + \psi_F(y)\nu_E(y)\,:\, y \in \partial E\}$$

for a smooth ψ_F with $\|\psi_F\|_{W^{2,p}(\partial E)} < \delta$, there holds

$$\Pi_F(\psi) \geqslant \sigma_\theta \|\psi\|^2_{H^1(\partial F)},$$

for all $\psi \in \widetilde{H}^1(\partial F)$ satisfying

$$\min_{\eta\in \mathrm{O}_E} \|\psi - \langle \eta|\nu_F\rangle\|_{L^2(\partial F)} \geqslant \theta\|\psi\|_{L^2(\partial F)},$$

where O_E is defined by formula (2.25).

Proof.
Step 1. We first show that for every $\theta \in (0,1]$ there holds

$$m_\theta = \inf\Big\{\Pi_E(\psi) : \psi \in \widetilde{H}^1(\partial E)\,, \|\psi\|_{H^1(\partial E)} = 1 \text{ and } \min_{\eta\in\mathrm{O}_E}\|\psi - \langle\eta|\nu_E\rangle\|_{L^2(\partial E)} \geqslant \theta\|\psi\|_{L^2(\partial E)}\Big\} > 0\,. \tag{2.64}$$

Indeed, let ψ_i be a minimizing sequence for this infimum and assume that $\psi_i \rightharpoonup \psi_0 \in \widetilde{H}^1(\partial E)$ weakly in $H^1(\partial E)$.
If $\psi_0 \neq 0$, as the weak convergence in $H^1(\partial E)$ implies strong convergence in $L^2(\partial E)$ by Sobolev embeddings, for every $\eta \in \mathrm{O}_E$ we have

$$\|\psi_0 - \langle \eta | \nu_E \rangle\|_{L^2(\partial E)} = \lim_{i\to\infty} \|\psi_i - \langle \eta | \nu_E \rangle\|_{L^2(\partial E)} \geqslant \lim_{i\to\infty} \theta \|\psi_i\|_{L^2(\partial E)} = \theta \|\psi_0\|_{L^2(\partial E)},$$

hence,

$$\min_{\eta \in \mathrm{O}_E} \|\psi_0 - \langle \eta | \nu_E \rangle\|_{L^2(\partial E)} \geqslant \theta \|\psi_0\|_{L^2(\partial E)} > 0,$$

thus, we conclude $\psi_0 \in \widetilde{H}^1(\partial E) \setminus T(\partial E)$ and

$$m_\theta = \lim_{i\to\infty} \Pi_E(\psi_i) \geqslant \Pi_E(\psi_0) > 0\,,$$

where the last inequality follows from estimate (2.22) in Remark 2.2.6.
If $\psi_0 = 0$, then again by the strong convergence of $\psi_i \to \psi_0$ in $L^2(\partial E)$, by looking at formula (2.21), we have

$$m_\theta = \lim_{i\to\infty} \Pi_E(\psi_i) = \lim_{i\to\infty} \int_{\partial E} |\nabla \psi_i|^2 \, d\mu = \lim_{i\to\infty} \|\psi_i\|^2_{H^1(\partial E)} = 1$$

since $\|\psi_i\|_{L^2(\partial E)} \to 0$.
Step 2. In order to finish the proof it is enough to show the existence of some $\delta > 0$ such that if $\mathrm{Vol}(F \triangle E) < \delta$ and $\partial F = \big\{ y + \psi_F(y)\nu_E(y) \,:\, y \in \partial E \big\}$ with $\|\psi_F\|_{W^{2,p}(\partial E)} < \delta$, then

$$\begin{aligned} \inf\Big\{ \Pi_F(\psi) \,:\, \psi \in \widetilde{H}^1(\partial F)\,, \|\psi\|_{H^1(\partial F)} = 1 \;\text{ and }\; \min_{\eta \in \mathrm{O}_E} \|\psi - \langle \eta | \nu_F \rangle\|_{L^2(\partial F)} \geqslant \theta \|\psi\|_{L^2(\partial F)} \Big\} \\ \geqslant \sigma_\theta = \frac{1}{2} \min\{m_{\theta/2}, 1\}\,, \end{aligned} \tag{2.65}$$

where $m_{\theta/2}$ is defined by formula (2.64), with $\theta/2$ in place of θ.

Assume by contradiction that there exist a sequence of smooth sets $F_i \subseteq \mathbb{T}^n$, with $\partial F_i = \{y + \psi_{F_i}(y)\nu_E(y) \,:\, y \in \partial E\}$ and $\|\psi_{F_i}\|_{W^{2,p}(\partial E)} \to 0$ and a sequence $\psi_i \in \widetilde{H}^1(\partial F_i)$, with $\|\psi_i\|_{H^1(\partial F_i)} = 1$ and $\min_{\eta \in \mathrm{O}_E} \|\psi_i - \langle \eta | \nu_{F_i} \rangle\|_{L^2(\partial F_i)} \geqslant \theta \|\psi_i\|_{L^2(\partial F_i)}$, such that

$$\Pi_{F_i}(\psi_i) < \sigma_\theta \leqslant m_{\theta/2}/2\,. \tag{2.66}$$

Let us suppose first that $\lim_{i\to\infty} \|\psi_i\|_{L^2(\partial F_i)} = 0$ and observe that by the Sobolev embeddings $\|\psi_i\|_{L^q(\partial F_i)} \to 0$ for some $q > 2$, thus, since the functions ψ_{F_i} are uniformly bounded in $W^{2,p}(\partial E)$ for $p > \max\{2, n-1\}$, recalling formula (2.21), it is easy to see that

$$\lim_{i\to\infty} \Pi_{F_i}(\psi_i) = \lim_{i\to\infty} \int_{\partial F_i} |\nabla \psi_i|^2 \, d\mu_i = \lim_{i\to\infty} \|\psi_i\|^2_{H^1(\partial F_i)} = 1\,,$$

which is a contradiction with assumption (2.66).
Hence, we may assume that

$$\lim_{i\to\infty} \|\psi_i\|_{L^2(\partial F_i)} > 0. \tag{2.67}$$

The idea now is to write every ψ_i as a function on ∂E. We define the functions $\widetilde{\psi}_i(\partial E) \to \mathbb{R}$, given by

$$\widetilde{\psi}_i(y) = \psi_i\big(y + \psi_{F_i}(y)\nu_E(y)\big) - \fint_{\partial E} \psi_i(y + \psi_{F_i}(y)\nu_E(y))\, d\mu(y)\,,$$

for every $y \in \partial E$.
As $\psi_{F_i} \to 0$ in $W^{2,p}(\partial E)$, we have in particular that

$$\widetilde{\psi}_i \in \widetilde{H}^1(\partial E)\,, \qquad \|\widetilde{\psi}_i\|_{H^1(\partial E)} \to 1 \qquad \text{and} \qquad \frac{\|\widetilde{\psi}_i\|_{L^2(\partial E)}}{\|\psi_i\|_{L^2(\partial F_i)}} \to 1\,,$$

moreover, note also that $\nu_{F_i}(\cdot + \psi_{F_i}(\cdot)\nu_E(\cdot)) \to \nu_E$ in $W^{1,p}(\partial E)$ and thus in $C^{0,\alpha}(\partial E)$ for a suitable $\alpha \in (0,1)$, depending on p, by Sobolev embeddings. Using this fact and taking into account the third limit above and inequality (2.67), one can easily show that

$$\liminf_{i\to\infty} \frac{\min_{\eta\in \mathrm{O}_E} \|\widetilde{\psi}_i - \langle \eta|\nu_E\rangle\|_{L^2(\partial E)}}{\|\widetilde{\psi}_i\|_{L^2(\partial E)}} \geqslant \liminf_{i\to\infty} \frac{\min_{\eta\in \mathrm{O}_E} \|\psi_i - \langle \eta|\nu_{F_i}\rangle\|_{L^2(\partial F_i)}}{\|\psi_i\|_{L^2(\partial E_i)}} \geqslant \theta\,.$$

Hence, for $i \in \mathbb{N}$ large enough, we have

$$\|\widetilde{\psi}_i\|_{H^1(\partial E)} \geqslant 3/4 \qquad \text{and} \qquad \min_{\eta\in \mathrm{O}_E} \|\widetilde{\psi}_i - \langle \eta|\nu_E\rangle\|_{L^2(\partial E)} \geqslant \frac{\theta}{2}\|\widetilde{\psi}_i\|_{L^2(\partial E)}\,,$$

then, in turn, by Step 1, we infer

$$\Pi_E(\widetilde{\psi}_i) \geqslant \frac{9}{16} m_{\theta/2}\,. \tag{2.68}$$

Arguing now exactly like in the final part of Step 2 in the proof of Theorem 2.2.10, we have that all the terms of $\Pi_{F_i}(\psi_i)$ are asymptotically close to the corresponding terms of $\Pi_E(\widetilde{\psi}_i)$, thus

$$\Pi_{F_i}(\psi_i) - \Pi_E(\widetilde{\psi}_i) \to 0\,,$$

which is a contradiction, by inequalities (2.66) and (2.68). This establishes inequality (2.65) and concludes the proof. □

The following final result of this section states that close to a strictly stable critical set there are no other smooth critical sets (up to translations).

Proposition 2.2.17. *Let p and $E \subseteq \mathbb{T}^n$ be as in Proposition 2.2.16. Then, there exists $\delta > 0$ such that if $E' \subseteq \mathbb{T}^n$ is a smooth critical set with $\mathrm{Vol}(E') = \mathrm{Vol}(E)$, $\mathrm{Vol}(E\triangle E') < \delta$, $\partial E' \subseteq N_\varepsilon$ and*

$$\partial E' = \{y + \psi(y)\nu_E(y) \,:\, y \in \partial E\}$$

for a smooth ψ with $\|\psi\|_{W^{2,p}(\partial E)} < \delta$, then E' is a translate of E.

Proof. In Step 3 of the proof of Theorem 2.2.10, it is shown that under these hypotheses on E and E', if $\delta > 0$ is small enough, we may find a small vector $\eta \in \mathbb{R}^n$ and a volume–preserving variation E_t such that$E_0 = E$, $E_1 = E' - \eta$ and

$$\frac{d^2}{dt^2}\mathcal{A}(\partial E_t) \geqslant C[\mathrm{Vol}(E\triangle(E'-\eta))]^2\,,$$

for all $t \in [0,1]$, where C is a positive constant independent of E'.
Assume that E' is a smooth critical set as in the statement, which is not a translate of E, then $\frac{d}{dt}\mathcal{A}(\partial E_t)\big|_{t=0} = 0$, but from the above formula it follows $\frac{d}{dt}\mathcal{A}(\partial E_t)\big|_{t=1} > 0$, which implies that $E' - \eta$ cannot be critical, hence neither E', which is a contradiction. Indeed, $s \mapsto E_{1-s}$ is a volume–preserving variation for $E' - \eta$ and

$$\frac{d}{ds}\mathcal{A}(\partial E_{1-s})\Big|_{s=0} = -\frac{d}{dt}\mathcal{A}(\partial E_t)\Big|_{t=1} < 0\,,$$

showing that $E' - \eta$ is not critical. □

3 THE SURFACE DIFFUSION FLOW

3.1 GEOMETRIC FEATURES

We start this section with the general notion of smooth flow of sets.

Definition 3.1.1. Let $E_t \subseteq \mathbb{T}^n$ for $t \in [0,T)$ be a one-parameter family of sets, then we say that it is a *smooth flow* if there exists a smooth *reference set* $F \subseteq \mathbb{T}^n$ and a map $\Psi \in C^\infty([0,T) \times \mathbb{T}^n; \mathbb{T}^n)$ such that $\Psi_t = \Psi(t,\cdot)$ is a smooth diffeomorphism from $\mathbb{T}^n$ to $\mathbb{T}^n$ and $E_t = \Psi_t(F)$, for all $t \in [0,T)$.

The *velocity* of the motion of any point $x = \Psi_t(y)$ of the set E_t, with $y \in F$, is then given by

$$X_t(x) = X_t(\Psi_t(y)) = \frac{\partial \Psi_t}{\partial t}(y).$$

Remark 3.1.2. Notice that, in general, the smooth vector field X_t, defined in the whole $\mathbb{T}^n$ by $X_t(\Psi_t(z)) = \frac{\partial \Psi_t}{\partial t}(z)$ for every $z \in \mathbb{T}^n$, is not independent of t.

When $x \in \partial E_t$, we define the *outer normal velocity* of the flow of the boundaries ∂E_t, which are smooth hypersurfaces of $\mathbb{T}^n$, as

$$V_t(x) = \langle X_t(x) | \nu_t(x) \rangle,$$

for every $t \in [0,T)$, where ν_t is the outer normal vector to E_t.

However, we only use the following definition which is obtained by representing the smooth hypersurfaces ∂E_t in $\mathbb{T}^n$ with a family of smooth embeddings. This is actually the more standard way to define the surface diffusion flow, in the more general situation of smooth and possibly *immersed–only* hypersurfaces (usually in $\mathbb{R}^n$), without being the boundary of any set.

Definition 3.1.3. Let $E \subseteq \mathbb{T}^n$ be a smooth set. We say that the family $E_t \subseteq \mathbb{T}^n$, for $t \in [0,T)$ with $E_0 = E$, is a *surface diffusion flow* starting from E if the map $t \mapsto \chi_{E_t}$ is continuous from $[0,T)$ to $L^1(\mathbb{T}^n)$ and the hypersurfaces ∂E_t move by surface diffusion, that is, there exists a smooth family of embeddings $\varphi_t : \partial E \to \mathbb{T}^n$, for $t \in [0,T)$, with $\varphi_0 = \mathrm{Id}$ and $\varphi_t(\partial E) = \partial E_t$, such that

$$\frac{\partial \varphi_t}{\partial t} = (\Delta \mathrm{H})\nu\,, \tag{3.1}$$

where, at every point and time, H and Δ are respectively the mean curvature and the Laplacian (with the Riemannian metric induced by $\mathbb{T}^n$, that is, by $\mathbb{R}^n$) of the moving hypersurface ∂E_t, while ν is the "outer" normal to the smooth set E_t.

Remark 3.1.4. An alternative way to describe the flow is to speak of the sets "enclosed" by the boundary hypersurfaces moving by surface diffusion. This anyway would introduce an ambiguity, since every hypersurface ∂E_t clearly "separate" $\mathbb{T}^n$ in components and one should indicate which ones are actually the sets E_t at every time t. The use of the continuity of the map $t \mapsto \chi_{E_t}$ is a way to avoid such ambiguity. Moreover, it follows easily that being the solution of the PDE system (3.1) unique, by Theorem 3.2.1 below, the sets E_t are uniquely determined (being a "geometric flow", actually the same "geometric" uniqueness also holds for the hypersurfaces ∂E_t, like for the mean curvature flow, see [48, Section 1.3]).

By means of equation (1.8), the system (3.1) can be rewritten as

$$\frac{\partial \varphi_t}{\partial t} = -\Delta_t \Delta_t \varphi_t + \text{ lower order terms} \tag{3.2}$$

and it can be seen that it is a fourth order, *quasilinear* and *degenerate*, parabolic system of PDEs. Indeed, it is quasilinear, as the coefficients (as second order partial differential operator) of the Laplacian associated to the induced metrics g_t on the evolving hypersurfaces, that is,

$$\Delta_t \varphi_t(p) = \Delta_{g_t(p)} \varphi_t(p) = g_t^{ij}(p) \nabla_i^{g_t(p)} \nabla_j^{g_t(p)} \varphi_t(p)$$

depend on the first order derivatives of φ_t, as g_t (and the coefficient of $\Delta_t \Delta_t$ on the third order derivatives). Moreover, the operator at the right hand side of system (0.3) is degenerate, as its symbol (the symbol of the linearized operator) admits zero eigenvalues due to the invariance of the Laplacian by diffeomorphisms.

Like the Area functional, the flow is obviously invariant by isometries of $\mathbb{T}^n$ (or of $\mathbb{R}^n$) and reparametrizations. The volume–preserving property follows immediately arguing as in computation (2.6), indeed, if $E_t = \Psi_t(F)$ is a surface diffusion flow, described by $\Psi \in C^\infty([0,T) \times \mathbb{T}^n, \mathbb{T}^n)$ (as in Definition 3.1.1), with associated smooth vector field X_t satisfying

$$\frac{\partial \Psi_t}{\partial t}(y) = X_t(\Psi_t(y)),$$

we have

$$\begin{aligned}
\frac{d}{dt}\mathrm{Vol}(E_t) &= \int_F \frac{\partial}{\partial t} J\Psi_t(y)\, dy \\
&= \int_F \mathrm{div}\, X_t(\Psi(t,y)) J\Psi(t,y)\, dy \\
&= \int_{E_t} \mathrm{div}\, X_t(x)\, dx \\
&= \int_{\partial E_t} \langle X, \nu_t \rangle \, d\mu_t \\
&= \int_{\partial E_t} V_t \, d\mu_t \\
&= \int_{\partial E_t} \Delta_t H_t \, d\mu_t \\
&= 0,
\end{aligned}$$

where μ_t is in the canonical measure induced on ∂E_t by the flat metric of $\mathbb{T}^n$ and the last equality follows from the divergence theorem (1.3).

Moreover, the surface diffusion flow can be regarded as the $\widetilde{H}^{-1}$–gradient flow of the volume–constrained Area functional, in the following sense (see [32], for instance).
For a smooth set $E \subseteq \mathbb{T}^n$, we let the space $\widetilde{H}^{-1}(\partial E) \subseteq L^2(\partial E)$ to be the dual of $\widetilde{H}^1(\partial E)$ with the norm $\|u\|_{\widetilde{H}^1(\partial E)} = \int_{\partial E} |\nabla u|^2 \, d\mu$ (the functions in $H^1(\partial E)$ with zero integral) and the pairing between $\widetilde{H}^1(\partial E)$ and $\widetilde{H}^{-1}(\partial E)$ simply being the integral of the product of the functions on ∂E. Then, it follows easily that the norm of a smooth function $v \in \widetilde{H}^{-1}(\partial E)$ is given by

$$\|v\|^2_{\widetilde{H}^{-1}(\partial E)} = \int_{\partial E} v(-\Delta)^{-1} v \, d\mu = \int_{\partial E} \langle \nabla(-\Delta)^{-1} v, \nabla(-\Delta)^{-1} v \rangle \, d\mu$$

and, by polarization, we have the $\widetilde{H}^{-1}(\partial E)$–scalar product between a pair of smooth functions $u, v : \partial E \to \mathbb{R}$ with zero integral,

$$\langle u, v \rangle_{\widetilde{H}^{-1}(\partial E)} = \int_{\partial E} \langle \nabla(-\Delta)^{-1} u, \nabla(-\Delta)^{-1} v \rangle \, d\mu = \int_{\partial E} u(-\Delta)^{-1} v \, d\mu,$$

integrating by parts.
This scalar product, extended to the whole space $\widetilde{H}^{-1}(\partial E)$, make it a Hilbert space, hence, by the

Riesz representation theorem, there exists a function $\nabla^{\widetilde{H}^{-1}}_{\partial E}\mathcal{A} \in \widetilde{H}^{-1}(\partial E)$ such that, for every smooth function $v \in \widetilde{H}^{-1}(\partial E)$, there holds

$$\int_{\partial E} v\mathrm{H}\, d\mu = \delta\mathcal{A}_{\partial E}(v) = \langle v, \nabla^{\widetilde{H}^{-1}}_{\partial E}\mathcal{A}\rangle_{\widetilde{H}^{-1}(\partial E)} = \int_{\partial E} v(-\Delta)^{-1}\nabla^{\widetilde{H}^{-1}}_{\partial E}\mathcal{A}\, d\mu\,,$$

by Theorem 2.1.5.
Then, by the *fundamental lemma of calculus of variations*, we conclude

$$(-\Delta)^{-1}\nabla^{\widetilde{H}^{-1}}_{\partial E}\mathcal{A} = \mathrm{H} + c\,,$$

for a constant $c \in \mathbb{R}$, that is,

$$\nabla^{\widetilde{H}^{-1}}_{\partial E}\mathcal{A} = -\Delta\mathrm{H}\,.$$

It clearly follows that the outer normal velocity of the moving boundaries of a surface diffusion flow $V_t = \Delta_t\mathrm{H}_t$ is minus the $\widetilde{H}^{-1}$–gradient of the volume–constrained functional $\mathcal{A}$.

3.2 SHORT–TIME EXISTENCE AND UNIQUENESS OF THE FLOW

The following existence/uniqueness theorem of classical solutions for the surface diffusion flow was proved by Escher, Mayer and Simonett in [26]. It should be expected, by the explicit parabolic nature of system (3.1), as shown by the formula (3.2).
As we mentioned in the introduction, it deals with the evolution in the whole space $\mathbb{R}^n$ of a generic hypersurface, even only immersed, hence possibly with self–intersections. It is then straightforward to adapt the same arguments to our case, when the ambient is the flat torus $\mathbb{T}^n$ and the hypersurfaces are the boundaries of the sets E_t, as in Definition 3.1.3, getting a (unique) surface diffusion flow in a positive time interval $[0, T)$, for every initial smooth set $E_0 \subseteq \mathbb{T}^n$.

Theorem 3.2.1. *Let $\varphi_0 : M \to \mathbb{R}^n$ be a smooth and compact, immersed hypersurface. Then, there exists a unique smooth surface diffusion flow $\varphi : [0, T) \times M \to \mathbb{R}^n$, starting from $M_0 = \varphi_0(M)$ and solving system (3.1), for some maximal time of existence $T > 0$. Moreover, such flow and the maximal time of existence depend continuously on the $C^{2,\alpha}$–norm of the initial hypersurface.*

As an easy consequence, we have the following theorem, well suited for our setting.

Theorem 3.2.2. *Let $E \subseteq \mathbb{T}^n$ be a smooth set, N_ε a tubular neighborhood of ∂E and $M_E < \varepsilon/2$. For every $E_0 \subseteq \mathbb{T}^n$ smooth set in $\mathfrak{C}^1_{M_E}(E)$ with*

$$\partial E_0 = \{y + \psi_0(y)\nu_E(y) \,:\, y \in \partial E\}$$

for a smooth function $\psi_0 : \partial E \to \mathbb{R}$, there exists a unique surface diffusion flow E_t, starting from E_0, such that

$$\partial E_t = \{y + \psi_t(y)\nu_E(y) \,:\, y \in \partial E\}$$

for smooth functions $\psi_t : \partial E \to \mathbb{R}$, for $t \in [0, T(E_0))$, with $T(E_0)$ depending on the $C^{2,\alpha}$–norm of ψ_0.

Instead of proving Theorem 3.2.1 (hence, Theorem 3.2.2), which is well known, we show the following alternative short–time and existence result. Moreover, we provide higher order regularity estimates depending on the $C^{1,1}$–bound on the initial datum only.

Theorem 3.2.3. *Let $E \subseteq \mathbb{T}^n$ be a smooth set and $\varepsilon > 0$. Then, there exist $\delta = \delta(E, \varepsilon)$ and $T = T(E, \varepsilon) > 0$ with the following property: if E_0 is the normal deformation of E induced by $\psi_0 \in C^{1,1}(\partial E)$ (as in Definition 1.3.1), $\|\psi_0\|_{C^{1,1}(\partial E)} \leqslant \delta$ and $\mathrm{Vol}(E_0) = \mathrm{Vol}(E)$, then the surface diffusion flow E_t starting from E_0 exists in $[0, T)$, the sets E_t are normal deformations of E induced by $\psi(t, \cdot) \in C^\infty(\partial E)$ for all $t \in (0, T)$ and*

$$\sup_{t\in(0,T)} \|\psi\|_{C^2(\partial E)} \leqslant \varepsilon. \tag{3.3}$$

Moreover, for every $k \in \mathbb{N} \setminus \{0\}$, there exist constants $C_k = C_k(E, \varepsilon) > 0$ such that

$$\sup_{t\in[T/2,T)} \|\nabla^{k+2}\psi\|_{C^0(\partial E)} \leqslant C_k(\|\psi_0\|_{C^{1,1}(\partial E)} + 1). \tag{3.4}$$

To prove this theorem we use the classical linearization and fixed point approach in order to solve the nonlinear evolution problem. Then, following closely what was done in [30] (combined with the results of [41]), we employ some Schauder–type estimates to show the higher order regularity of the flow. Before doing that, we recall some useful facts and lemmas.

Let $E \subseteq \mathbb{T}^n$ be a smooth set and N_ε a tubular neighborhood of ∂E. It is well known that any small deformation of ∂E can be represented as the graph of a "height" function ψ and conversely, to any smooth function $\psi : \partial E \to \mathbb{R}$ we can associate a set E_ψ such that the hypersurface ∂E_ψ is given by $\varphi(x) = x + \psi(x)\nu_E(x)$ (see [48] for more details). We aim to compute the equation for a smooth (time dependent) function $\psi(t, x)$, so that $\varphi_t = x + \psi(t, x)\nu_E(x)$ satisfies system (3.1). Obviously, we set $\psi(0, x) = 0$, for every $x \in \partial E$.
Arguing as in [48, Section 1.5], we deduce that ψ must satisfy the following evolution equation:

$$\begin{aligned}\frac{\partial\psi}{\partial t} &= -\Delta_t^2\psi - \frac{1}{\langle\nu_E\,|\,\nu_t\rangle}\Delta_t\langle\nu_E\,|\,\nu_t\rangle\Delta_t\psi + \frac{1}{\langle\nu_E\,|\,\nu_t\rangle}\Delta_t P(x,\psi,\nabla\psi)\\ &= -\Delta_t^2\psi + \widetilde{J}(x,\psi,\nabla\psi,\nabla^2\psi,\nabla^3\psi),\end{aligned} \tag{3.5}$$

where P and $\widetilde{J}$ are smooth functions (assuming that ψ and $\nabla\psi$ are small). So, denoting by $\otimes$ the usual tensor product, it follows that the function $\widetilde{J}$ can be written as

$$\widetilde{J}(x,\psi,\nabla\psi,\nabla^2\psi,\nabla^3\psi) = \langle\widetilde{Q}_1\,|\,\nabla^2\psi\rangle + \langle\widetilde{Q}_2\,|\,\nabla^2\psi\otimes\nabla^2\psi\rangle + \langle\widetilde{Q}_3\,|\,\nabla^3\psi\rangle + \widetilde{q}_4$$

where $\widetilde{Q}_1, \widetilde{Q}_2, \widetilde{Q}_3$ and $\widetilde{q}_4$ are, respectively, tensor–valued and scalar–valued functions depending on $(x, \psi, \nabla\psi)$. Moreover, they are smooth if their arguments are small enough.
Hence, linearizing the Laplace–Beltrami operator yields the following evolution equation (compare with [30, Section 3.1])

$$\frac{\partial\psi}{\partial t} = -\Delta_E^2\psi + \langle A(x,\psi,\nabla\psi)\,|\,\nabla^4\psi\rangle + J(x,\psi,\nabla\psi,\nabla^2\psi,\nabla^3\psi), \tag{3.6}$$

where A is a smooth 4th–order tensor, vanishing when both ψ and $\nabla\psi$ vanish and J is given by

$$\begin{aligned}J = &\langle Q_1\,|\,\nabla^3\psi\otimes\nabla^2\psi\rangle + \langle Q_2\,|\,\nabla^3\psi\rangle + \langle Q_3\,|\,\nabla^2\psi\otimes\nabla^2\psi\otimes\nabla^2\psi\rangle\\ &+ \langle Q_4\,|\,\nabla^2\psi\otimes\nabla^2\psi\rangle + \langle Q_5\,|\,\nabla^2\psi\rangle + q_6,\end{aligned} \tag{3.7}$$

where Q_i, for $i = 1, \dots, 5$ and q_6 are, respectively, smooth tensor–valued and scalar–valued functions depending on $(x, \psi, \nabla\psi)$.

3.2.1 *The biharmonic heat equation on a Riemannian manifold*

We collect some classical results concerning the biharmonic heat equation on a smooth Riemannian manifold (Σ, g), that is, the following problem:

$$\begin{cases}\partial_t u = -\Delta_\Sigma^2 u + f & \text{in } [0,+\infty)\times\Sigma\\ u(\cdot,0) = u_0 & \text{on } \Sigma\end{cases}$$

for some given functions $f : [0, +\infty) \times \Sigma \to \mathbb{R}$ and $u_0 : \Sigma \to \mathbb{R}$.
For the detailed proofs, see [29, 41] and the references therein.

Theorem 3.2.4 ([29, Theorem 2]). *Given a smooth Riemannian manifold (Σ, g), there exists a unique biharmonic heat kernel with respect to g, denoted by $b_g \in C^\infty\big((0,+\infty)\times\Sigma\times\Sigma\big)$. Moreover, let $T>0$, for any integers $k,p,q \geqslant 0$ and for any $(t,x,y)\in(0,T)\times\Sigma\times\Sigma$ we have*

$$|\partial_t^k \nabla_x^p \nabla_y^q b_g(t,x,y)| \leqslant C t^{-\frac{n+4k+p+q}{4}} \exp\{-\delta\big(t^{-\frac{1}{4}} d(x,y)\big)^{\frac{4}{3}}\},$$

where ∇_x, ∇_y are covariant derivatives with respect to g and the constants $C,\delta>0$ depend on T, g and $p+q+4k$.

Given the biharmonic heat kernel $b_g \in C^\infty\big((0,+\infty)\times\Sigma\times\Sigma\big)$ and a function $u_0\in C^0(\Sigma)$, for $(t,x)\in(0,+\infty)\times\Sigma$ we define

$$Su_0(t,x) = \int_\Sigma b_g(t,x,y)u_0(y)\,d\mu(y)\,. \tag{3.8}$$

Hence, Su_0 is the solution of the homogeneous problem

$$\begin{cases}\partial_t v + \Delta_\Sigma^2 v = 0 & \text{in } (0,+\infty)\times\Sigma\\ v(\cdot,0)=u_0 & \text{on } \Sigma\end{cases} \tag{3.9}$$

and it is smooth, since the biharmonic heat kernel is smooth for every $t>0$.

We now collect some results from [41]. We start with the following Schauder–type estimates on the solution of the homogeneous problem (3.9), which are a slight reformulation of [41, Theorem 3.8] that better fit our purposes.

Theorem 3.2.5 ([41, Theorem 3.8]). *Let $T>0$ and $u_0\in C^{1,1}(\Sigma)$. Then, there exists $C(\Sigma,T)>0$ such that*

$$\sup_{t\in(0,T)} \|Su_0\|_{C^{1,1}(\Sigma)} \leqslant C\|u_0\|_{C^{1,1}(\Sigma)}\,. \tag{3.10}$$

Furthermore, for any $l,k\in\mathbb{N}$, we have

$$\sup_{t\in(0,T)} t^{l+\frac{k}{4}} \Big\|\partial_t^l \nabla_g^{k+2} Su_0(t)\Big\|_{C^0(\Sigma)} \leqslant C_{l,k}\|u_0\|_{C^{1,1}(\Sigma)}, \tag{3.11}$$

for some constants $C_{l,k}>0$ depending on l, k as well as Σ, T.

Definition 3.2.6. Fix $0<T<+\infty$ and $0<\beta<1$. We define

$$Y_T = \big\{u\in C^0\big((0,T)\times\Sigma\big) : \|u\|_{Y_T} < +\infty\big\}$$

with the norm

$$\|u\|_{Y_T} = \sup_{t\in(0,T)} \Big(t^{\frac{1}{2}}\|u(t,\cdot)\|_{C^0(\Sigma)} + t^{\frac{1}{2}+\frac{\beta}{4}}[u(t,\cdot)]_{C^\beta(\Sigma)}\Big) + \sup_{(t,x)\in(0,T)\times\Sigma}\ \sup_{0<h<T-t} t^{\frac{1}{2}+\frac{\beta}{4}} \frac{|u(t+h,x)-u(t,x)|}{|h|^{\frac{\beta}{4}}}\,,$$

where $[\,\cdot\,]_{C^\beta}$ is the Hölder seminorm.

Similarly, we define

$$X_T = \big\{u\in C^0\big((0,T)\times\Sigma\big) : u(t,\cdot)\in C^4(\Sigma),\ \|u\|_{X_T} < +\infty\big\}$$

with the norm

$$\begin{aligned}\|u\|_{X_T} = \sup_{t\in(0,T)} \Bigg(&\sum_{k=0}^{4} t^{-\frac{1}{2}+\frac{k}{4}} \|\nabla^k u(t,\cdot)\|_{C^0(\Sigma)} + t^{\frac{1}{2}+\frac{\beta}{4}} [\nabla^4 u(t,\cdot)]_{C^\beta(\Sigma)} \\ &+ t^{\frac{1}{2}} \|\partial_t u(t,\cdot)\|_{C^0(\Sigma)} + t^{\frac{1}{2}+\frac{\beta}{4}} [\partial_t u(t,\cdot)]_{C^\beta(\Sigma)} \Bigg) \\ &+ \sup_{(t,x)\in(0,T)\times\Sigma} \ \sup_{0<h<T-t} t^{\frac{1}{2}+\frac{\beta}{4}} \frac{|\nabla^4 u(t+h,x) - \nabla^4 u(t,x)|}{|h|^{\frac{\beta}{4}}} \\ &+ \sup_{(t,x)\in(0,T)\times\Sigma} \ \sup_{0<h<T-t} t^{\frac{1}{2}+\frac{\beta}{4}} \frac{|\partial_t u(t+h,x) - \partial_t u(t,x)|}{|h|^{\frac{\beta}{4}}} \,. \end{aligned} \tag{3.12}$$

Proposition 3.2.7. *The spaces $(Y_T, \|\cdot\|_{Y_T})$ and $(X_T, \|\cdot\|_{X_T})$ are Banach spaces.*

The proof of the completeness of the spaces Y_T and X_T is standard. Indeed, one can prove directly that all Cauchy sequences converge to a function in the space and the candidate limit is obtained by means of a diagonal argument.

Remark 3.2.8. Since the norm $\sum_{k=0}^4 \|\nabla^k u\|_{C^0}$ is equivalent to the norm $\|u\|_{C^0} + \|\nabla^4 u\|_{C^0}$ in the function space $C^4(\Sigma)$, we have that the norm $\|\cdot\|_{X_T}$ defined in formula (3.12) is equivalent to the following norm:

$$\|u\|'_{X_T} = \|u\|_{X_T} + \sum_{k=0}^{3} \sup_{(t,x)\in(0,T)\times\Sigma} \ \sup_{0<h<T-t} t^{-\frac{1}{2}+\frac{k}{4}+\frac{\beta}{4}} \frac{|\nabla^k u(t+h,x) - \nabla^k u(t,x)|}{|h|^{\frac{\beta}{4}}} \,.$$

Given the biharmonic heat kernel $b_g \in C^\infty\big((0,T)\times\Sigma\times\Sigma\big)$, the solution (if it exists) to the nonhomogeneous problem

$$\begin{cases} \partial_t u + \Delta^2_\Sigma u = f & \text{in } (0,T)\times\Sigma \\ u(\cdot,0) = 0 & \text{on } \Sigma \end{cases} \tag{3.13}$$

where f is a fixed function on $(0,T)\times\Sigma$, is given (by Duhamel's principle) by

$$Vf(t,x) = \int_0^t \int_\Sigma b_g(t-s,x,y) f(s,y)\, d\mu(y)\, ds \tag{3.14}$$

and $Vf \in C^\infty((\lambda/2,\lambda)\times\Sigma)$, for every $\lambda > 0$.

We conclude this section by recalling the following fundamental Schauder–type estimates for solutions of problem (3.13), proved in [41] (see [41, Remark 3.12] for the final comments on the constant C).

Theorem 3.2.9 ([41, Theorem 3.10])**.** *Let $0 < T < +\infty$, if $f \in Y_T$, then $Vf \in X_T$ and there exists a constant $C > 0$ depending on Σ, T such that*

$$\|Vf\|_{X_T} \leqslant C\|f\|_{Y_T}\,.$$

Moreover, the equation $(\partial_t + \Delta^2_\Sigma)Vf = f$ holds in classical sense on $(0,T)\times\Sigma$ and $Vf \in C^\infty((0,T)\times\Sigma)$.

3.2.2 *A new proof of the short–time existence and uniqueness result*

In order to prove Theorem 3.2.3, we need some fundamental estimates which follows from the results above (with $\Sigma = \partial E$). We consider the map $\psi \mapsto f[\psi]$ with represents the nonlinear error term generated in linearizing equation (3.5)

$$f[\psi](x) = \langle A(x,\psi,\nabla\psi), \nabla^4\psi\rangle + J(x,\psi,\nabla\psi,\nabla^2\psi,\nabla^3\psi), \tag{3.15}$$

where A and J are the operators defined in formula (3.6). The following lemma provides such estimates on $f[\psi]$.

Lemma 3.2.10. *For any $\varepsilon, m > 0$ there exist $T, \delta > 0$ depending on E and ε, with the following properties:*

(i) *for every $\psi_0 \in C^{1,1}(\Sigma)$ and $\zeta \in X_T$ satisfying $\|\zeta\|_{X_T} \leqslant m$, we have*

$$f[\zeta + S\psi_0] \in Y_T\,, \tag{3.16}$$

(ii) *if, moreover, $\|\psi_0\|_{C^{1,1}(\Sigma)} \leqslant \delta$, there holds*

$$\|f[S\psi_0]\|_{Y_T} \leqslant \varepsilon(\|\psi_0\|_{C^{1,1}(\Sigma)} + 1)\,, \tag{3.17}$$

(iii) *for every $\zeta_1, \zeta_2 \in X_T$ satisfying $\|\zeta_i\|_{X_T} \leqslant m$, there holds*

$$\|f[\zeta_1 + S\psi_0] - f[\zeta_2 + S\psi_0]\|_{Y_T} \leqslant \varepsilon\|\zeta_1 - \zeta_2\|_{X_T}\,. \tag{3.18}$$

Proof. Let $T < 1$ to be chosen later and $\varepsilon, m > 0$.
We only prove equation (3.17) and we give a sketch of the proof for estimates (3.16) and (3.18), as they are similar.

We will drop the dependence on the set E in the norms and we will write $A(t, x)$ assuming implicitly the dependence on ψ and $\nabla\psi$. Moreover, for clarity of exposition, we prove the results for the simplified error term

$$\widetilde{f}[\psi](x,t) = \langle A(x, \psi(x,t), \nabla\psi(x,t)), \nabla^4\psi(x,t)\rangle + \langle Q, \nabla^3\psi(x,t) \otimes \nabla^2\psi(x,t)\rangle\,, \tag{3.19}$$

where Q is a (constant) tensor and $\|Q\| < 1$. Then, we briefly analyze other terms of J.

From the very definition of $\widetilde{f}$, denoting $\psi = S\psi_0$, we have

$$\|\widetilde{f}[\psi]\|_{C^0} \leqslant \|A\|_{C^0}\|\nabla^4\psi\|_{C^0} + \|\nabla^3\psi\|_{C^0}\|\nabla^2\psi\|_{C^0} \tag{3.20}$$

and

$$\begin{aligned}[\widetilde{f}[\psi]]_{C^\beta} \leqslant{} & \|\nabla^4\psi\|_{C^0} \sup_{\tau\in\mathbb{T}^N} \big(|\tau|^{-\beta}|A(t, x+\tau) - A(t,x)|\big) \\ & + \|A\|_{C^0}[\nabla^4\psi]_{C^\beta} + [\nabla^3\psi]_{C^\beta}\|\nabla^2\psi\|_{C^0} + \|\nabla^3\psi\|_{C^0}[\nabla^2\psi]_{C^\beta}\,.\end{aligned} \tag{3.21}$$

We multiply by $t^{1/2}$ both sides of inequality (3.20) to get

$$t^{1/2}\|\widetilde{f}[\psi]\|_{C^0} \leqslant \|A\|_{C^0} t^{1/2}\|\nabla^4\psi\|_{C^0} + t^{1/2}\|\nabla^3\psi\|_{C^0}\|\nabla^2\psi\|_{C^0}\,. \tag{3.22}$$

By means of inequalities (3.11) with $l = 0$, $k = 0, 1, 2$, we have that the right–hand term in estimate (3.22) is bounded by $\|\psi\|_{C^{1,1}}$ (up to a constant that depends on E).
We now fix $\delta > 0$ sufficiently small, depending on ε and E, so that $\|A\|_{C^0}$ is bounded by ε, which can be done since A is a smooth tensor and $A(\cdot, 0, 0) = 0$. Finally, taking T small enough, depending on ε and E, we conclude

$$\sup_{t\in(0,T)} t^{1/2}\|\widetilde{f}[\psi]\|_{C^0} \leqslant \varepsilon\|\psi_0\|_{C^{1,1}}.$$

Taking into account the full expression for the error term $f[\psi]$ in (3.15) (see the very definition of J in formula (3.7)), arguing as above, we show that

$$\sup_{t\in(0,T)} t^{1/2}\|f[\psi]\|_{C^0} \leqslant C\varepsilon\left(\|\psi_0\|_{C^{1,1}} + 1\right),$$

where the extra constant term comes from the fact that $\|q_6\|_{C^0} \leqslant C$, hence

$$\sup_{t\in(0,T)} t^{1/2}\|q_6\|_{C^0} \leqslant \varepsilon\,.$$

Concerning the Hölder seminorm in space, we first remark that

$$\sup_{\tau\in\mathbb{T}^N} \frac{|A(t,x+\tau)-A(t,x)|}{|\tau|^\beta} \leqslant [A(\cdot,\psi,\nabla\psi)]_{C^\beta} + \|\partial_2 A\|_{C^0}[\psi]_{C^\beta} + \|\partial_3 A\|_{C^0}[\nabla\psi]_{C^\beta}\,,$$

where $\partial_2 A$ and $\partial_3 A$ denote the derivative of $A(x,y,z)$ with respect to the second and third components, respectively. Therefore, employing again the bounds (3.10) and (3.11), we have

$$t^{1/2}\|\nabla^4\psi\|_{C^0}\sup_\tau \frac{|A(t,x+\tau)-A(t,x)|}{|\tau|^\beta} \leqslant \varepsilon\|\psi_0\|_{C^{1,1}}\,,$$

where we choose $\delta>0$ sufficiently small, depending on ε, E, such that

$$[A(\cdot,\psi,\nabla\psi)]_{C^\beta} + \|\partial_2 A\|_{C^0}[\psi]_{C^\beta} + \|\partial_3 A\|_{C^0}[\nabla\psi]_{C^\beta} \leqslant \varepsilon,$$

which is possible since A is smooth and $A(\cdot,0,0)=0$. Thus, multiplying by $t^{\frac12+\frac\beta4}$ both sides of inequality (3.21), we obtain

$$\begin{aligned}t^{\frac12+\frac\beta4}[\widetilde f[\psi]]_{C^\beta} \leqslant{}& t^{\frac\beta4}\varepsilon\|\psi_0\|_{C^{1,1}} + \|A\|_{C^0}t^{\frac12+\frac\beta4}[\nabla^4\psi]_{C^\beta}\\ &+ t^{\frac14}t^{\frac14+\frac\beta4}\|\nabla^3\psi\|_{C^\beta}\|\nabla^2\psi\|_{C^0} + t^{\frac14}t^{\frac14}\|\nabla^3\psi\|_{C^0}t^{\frac\beta4}\|\nabla^2\psi\|_{C^\beta}\,.\end{aligned}$$

Then, all the terms at the right–hand side of this inequality can be bounded employing inequalities (3.10) and (3.11), thus we can make such right–hand side above as small as needed taking T,δ small enough. Analogous computations show a similar inequality for the complete error term $f[\psi]$, once we notice that, since the terms Q_i for $i=1,\dots,5$ are not constant, some (bounded) derivatives appear.
Finally, we show how to bound the Hölder seminorm in time appearing in $\|\widetilde f[\psi]\|_{Y_T}$. We fix $t\in(0,T)$ and $h\in(0,T-t)$. So, by the very definition of $\widetilde f[\psi](t)$, we have,

$$\begin{aligned}|\widetilde f[\psi](t+h)-\widetilde f[\psi](t)| \leqslant{}& |\langle A(\psi(t+h),\nabla\psi(t+h)),\nabla^4\psi(t+h)\rangle - \langle A(\psi(t),\nabla\psi(t)),\nabla^4\psi(t)\rangle|\\ &+ |\langle Q,\big(\nabla^3\psi(t+h)\otimes\nabla^2\psi(t+h)\big)\rangle - \langle Q,\big(\nabla^3\psi(t)\otimes\nabla^2\psi(t)\big)\rangle|\,,\end{aligned}$$

where we omitted the dependence on x, in order to simplify the notation.
By the triangular inequality, we obtain

$$\begin{aligned}&|\langle A(\psi(t+h),\nabla\psi(t+h)),\nabla^4\psi(t+h)\rangle - \langle A(\psi(t),\nabla\psi(t)),\nabla^4\psi(t)\rangle|\\ &\quad\leqslant \|A\|_{C^0}|\nabla^4\psi(t+h)-\nabla^4\psi(t)| + \|\partial_3 A\|_{C^0}|\nabla\psi(t+h)-\nabla\psi(t)|\|\nabla^4\psi(t)\|_{C^0}\\ &\quad\quad+ \|\partial_2 A\|_{C^0}|\psi(t+h)-\psi(t)|\|\nabla^4\psi\|_{C^0}\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}&|\langle Q,(\nabla^3\psi(t+h)\otimes\nabla^2\psi(t+h))\rangle - \langle Q,\big(\nabla^3\psi(x,t)\otimes\nabla^2\psi(x,t)\big)\rangle|\\ &\quad\leqslant |\nabla^3\psi(t+h)-\nabla^3\psi(t)|\|\nabla^2\psi\|_{C^0} + \|\nabla^3\psi\|_{C^0}|\nabla^2\psi(t+h)-\nabla^2\psi(t)|\,.\end{aligned} \tag{3.24}$$

Therefore, from formulas (3.23) and (3.24), we get

$$\begin{aligned}&|\widetilde f[\psi](t+h)-\widetilde f[\psi](t)|\\ &\quad\leqslant \big(\|\partial_2 A\|_{C^0}|\psi(t+h)-\psi(t)| + \|\partial_3 A\|_{C^0}|\nabla\psi(t+h)-\nabla\psi(t)|\big)\|\nabla^4\psi(t)\|_{C^0}\\ &\quad\quad+ \|A\|_{C^0}|\nabla^4\psi(t+h)-\nabla^4\psi(t)| + |\nabla^3\psi(t+h)-\nabla^3\psi(t)|\|\nabla^2\psi\|_{C^0}\\ &\quad\quad+ \|\nabla^3\psi\|_{C^0}|\nabla^2\psi(t+h)-\nabla^2\psi(t)|\,.\end{aligned}$$

Applying again estimates (3.10), (3.11) and using the smallness of $\|A\|_{C^0}$, we obtain inequality (3.17) by taking T and δ small enough. Then, as above, the same conclusion hold for $f[\psi]$, once we notice that the derivatives of Q_i for $i = 1, \dots, 5$ and q_6 are bounded.
Hence, given $\psi_0 \in C^{1,1}(\Sigma)$ and $\zeta \in X_T$ such that $\|\zeta\|_{X_T} \leqslant m$, recalling the definition of $\|\cdot\|_{X_T}$ in formula (3.12) and arguing as above, we can show that

$$f[\zeta + S\psi_0] \in Y_T\,.$$

The proof of inequality (3.18) is quite similar. We show the computations only for the term $\sup_{t\in(0,T)} t^{1/2}\|\cdot\|_{C^0}$ appearing in the norm of Y_T and for the simplified error term (3.19). Setting $\psi_i = \zeta_i + S\psi_0$, we have

$$\begin{aligned}
|\widetilde{f}[\psi_1] - &\widetilde{f}[\psi_2]| \\
&= \big|\langle A(x,\psi_1,\nabla\psi_1),\nabla^4\psi_1\rangle - \langle A(x,\psi_2,\nabla\psi_2),\nabla^4\psi_2\rangle + \langle Q, (\nabla^3\psi_1\otimes\nabla^2\psi_1 - \nabla^3\psi_2\otimes\nabla^2\psi_2)\rangle\big| \\
&\leqslant \|\nabla^4\psi_1\|_{C^0}\left(\|\partial_1 A\|_{C^0}|\zeta_1-\zeta_2| + \|\partial_2 A\|_{C^0}|\nabla\zeta_1 - \nabla\zeta_2|\right) + \|A\|_{C^0}|\nabla^2\zeta_1 - \nabla^2\zeta_2| \\
&\quad + \|\nabla^3\psi_1\|_{C^0}|\nabla^2\zeta_1 - \nabla^2\zeta_2| + \|\nabla^2\psi_2\|_{C^0}|\nabla^3\zeta_1 - \nabla^3\zeta_2|\,.
\end{aligned}$$

Multiplying both sides of this inequality by $t^{1/2}$, we get

$$\begin{aligned}
t^{1/2}|\widetilde{f}[\psi_1] &- \widetilde{f}[\psi_2]| \\
&\leqslant \Big(\|\nabla^4\psi_1\|_{C^0}\left(t\|\partial_1 A\|_{C^0} + t^{\frac{3}{4}}\|\partial_2 A\|_{C^0}\right) + t^{1/2}\left(\|A\|_{C^0} + \|\nabla^3\psi_1\|_{C^0}\right) \\
&\qquad + t^{1/4}\|\nabla^2\psi_2\|_{C^0}\Big)\|\zeta_1-\zeta_2\|_{X_T} \\
&\leqslant t^{1/4}\Big(t^{1/2}\|\nabla^4\psi_1\|_{C^0}\|A\|_{C^1} + \|A\|_{C^0} + t^{1/4}\|\nabla^3\psi_1\|_{C^0} + |\nabla^2\psi_2\|_{C^0}\Big)\|\zeta_1-\zeta_2\|_{X_T}\,.
\end{aligned}$$

By definition of $\|\cdot\|_{X_T}$ and by the estimates (3.10) and (3.11), we conclude by taking T and δ small enough. Using the observations above, the same conclusion holds for the full $f[\psi]$. □

We will denote with $B_r(x)$ the ball in $\mathbb{R}^n$ of center x and radius r, while B_r and B will be a short–hand notations for $B_r(0)$ and $B_1(0)$ (that is, the unit ball), respectively. Moreover, given $x \in \mathbb{R}^n$, we will write $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Similarly, we will denote with $B'_r(x') \subseteq \mathbb{R}^{n-1}$ the ball in $\mathbb{R}^{n-1}$ with radius $r > 0$ and center $x' \in \mathbb{R}^{n-1}$.

We are now ready to prove our short–time existence result for the surface diffusion evolution.

Proof of Theorem 3.2.3. Let us fix $\varepsilon > 0$. We underline that in the whole proof the constant C depends on n, ε and E.
Step 1. We show the existence of a solution of equation (3.6) via a fixed point argument.
Let $T < 1$ and $\delta < 1$ to be chosen later and let $\psi_1 \in C^\infty((0,T); C^\infty(\partial E))$ be the unique solution of the problem

$$\begin{cases} \partial_t\psi_1 = -\Delta^2\psi_1 & \text{in } [0,T)\times\partial E \\ \psi_1(\cdot,0) = \psi_0 & \text{on } \partial E \end{cases}$$

where $\psi_0 \in C^{1,1}(\partial E)$ is such that $\|\psi_0\|_{C^{1,1}(\partial E)} \leqslant \delta$. Thus, recalling the definition of $S\psi_0$ in formula (3.8), we have $\psi_1 = S\psi_0$. From Theorem 3.2.5 and for δ small enough depending on ε, the solution ψ_1 satisfies the estimates (3.3) and (3.4).
Let now ψ_2 be the unique solution of the problem

$$\begin{cases} \partial_t\psi_2 = -\Delta^2\psi_2 + f[\psi_1] & \text{in } [0,T)\times\partial E \\ \psi_2(\cdot,0) = \psi_0 & \text{on } \partial E \end{cases}$$

where $f[\psi]$ is defined in equation (3.15). By formulas (3.8) and (3.14), such solution is given by

$$\psi_2 = Vf[\psi_1] + S\psi_0 = Vf[S\psi_0] + S\psi_0 \,.$$

We then define an iterative scheme. For $i \geqslant 3$, we let ψ_i be the unique solution of the problem

$$\begin{cases} \partial_t \psi_i = -\Delta^2 \psi_i + f[\psi_{i-1}] & \text{in } [0,T) \times \partial E \\ \psi_i(\cdot, 0) = \psi_0 & \text{on } \partial E \end{cases} \tag{3.25}$$

that is, $\psi_i = Vf[\psi_{i-1}] + S\psi_0$. Let us denote by $\zeta_i = \psi_i - S\psi_0$, that is $\zeta_i = Vf[\psi_{i-1}]$. We aim to show that the sequence ζ_i converges in X_T. To do so, assume that $\zeta_j \in X_T$ for $j = 1, \dots, i-1$ with $\|\zeta_j\|_{X_T} \leqslant m$, then, by Theorem 3.2.9 and Lemma 3.2.10, we get $\zeta_i \in X_T$ and

$$\begin{aligned} \|\zeta_i\|_{X_T} &= \|Vf[\psi_{i-1}]\|_{X_T} \leqslant C\|f[\psi_{i-1}]\|_{Y_T} = C\|f[\zeta_{i-1} + S\psi_0]\|_{Y_T} \\ &\leqslant C \sum_{j=2}^{i-1} \|f[\zeta_j + S\psi_0] - f[\zeta_{j-1} + S\psi_0]\|_{Y_T} + C\|f[S\psi_0]\|_{Y_T} \\ &\leqslant C\Big(\sum_{j=1}^{i-1} \varepsilon^j\Big)(\|\psi_0\|_{C^{1,1}(\partial E)} + 1) \leqslant C\varepsilon\Big(1 + \sum_{j=1}^{+\infty} \varepsilon^j\Big)(\|\psi_0\|_{C^{1,1}(\partial E)} + 1) \\ &\leqslant C\varepsilon(\|\psi_0\|_{C^{1,1}(\partial E)} + 1) \leqslant m \,. \end{aligned}$$

Moreover, Lemma 3.2.10 implies that, for $\delta(\varepsilon, E)$ and $T(\varepsilon, E)$ small enough, there holds

$$\|\zeta_{i+1} - \zeta_i\|_{X_T} \leqslant \varepsilon \|\zeta_i - \zeta_{i-1}\|_{X_T} \,,$$

for all $i \geqslant 3$. Therefore, ζ_i is a Cauchy sequence in X_T, hence it admits a limit function ζ satisfying

$$\|\zeta\|_{X_T} \leqslant C\varepsilon(\|\psi_0\|_{C^{1,1}(\partial E)} + 1) \tag{3.26}$$

and, passing to the limit in problem (3.25), we get

$$\begin{cases} \partial_t \psi = -\Delta^2 \psi + f[\psi] & \text{in } [0,T) \times \partial E \\ \psi_i(\cdot, 0) = \psi_0 & \text{on } \partial E \end{cases}$$

with $\psi = \zeta + S\psi_0$.
Moreover, by estimates (3.10) and (3.26), there holds

$$\|\psi\|_{C^2(\partial E)} = \|\zeta + S\psi_0\|_{C^2(\partial E)} \leqslant \|\zeta\|_{X_T} + \|S\psi_0\|_{C^2(\partial E)} \leqslant C\varepsilon(\|\psi_0\|_{C^{1,1}(\partial E)} + 1) \,. \tag{3.27}$$

Step 2. By inequality (3.27) we get immediately that estimate (3.4) holds for $k = 0, 1, 2$. In order to prove such estimate for $k \geqslant 3$, we fix a point $x \in \partial E$ and we use normal coordinate around x. In particular, we fix $B'_r = U \subseteq \partial E$ such that the inverse g_E^{ij} of the metric g_E of ∂E (induced by the flat metric of $\mathbb{T}^n$) satisfies $\frac{1}{2}\delta_{ij} \leqslant g_E^{ij} \leqslant 2\delta_{ij}$.
Then, we observe that by the previous step, the function ψ restricted to $[T/2, T) \times B'_r$ is of class C^∞. Moreover, recalling that $\psi = \zeta + S\psi_0$, we have that the function ζ satisfies

$$\partial_t \zeta = -\Delta_t^2 \zeta + \widetilde{f} \,, \tag{3.28}$$

where we denoted by $\widetilde{f} = (\partial_t + \Delta_t^2)(S\psi_0) + f'$. Taking the covariant derivative ∇ (with respect to the metric g_E) in this equality, we get that the function $\nabla\zeta$ satisfies the equation

$$\begin{aligned} \partial_t \nabla\zeta &= -\Delta_t^2 \nabla\zeta - (\nabla g_t^{ij}) g_t^{kl} (\zeta)_{ijkl} - g_t^{ij} (\nabla g_t^{kl})(\zeta)_{ijkl} + \nabla\widetilde{f} \\ &= -\Delta_t^2 \nabla\zeta + F \,, \end{aligned}$$

where the error term F contains the derivatives of ζ up to order four and we denoted by g_t the metric on $\partial E_{\psi(t,\cdot)}$.
In order to estimate $\|F\|_{C^{\beta/4}([T/2,T];C^\beta(B'_r))}$ we first observe that by inequalities (3.11), it follows

$$\|\nabla\left((\partial_t+\Delta_t^2)(S\psi_0)\right)\|_{C^{\beta/4}([T/2,T);C^\beta(B'_1))} \leqslant C\varepsilon(\|\psi_0\|_{C^{1,1}(\partial E)}+1)\,,$$

then we remark that the other terms in F can be bounded analogously, recalling that they contain derivatives of ζ up to order four. So, by means of the bound (3.26), we obtain that

$$\|F\|_{C^{\beta/4}([T/2,T);C^\beta(B'_r))} \leqslant C\varepsilon(\|\psi_0\|_{C^{1,1}(\partial E)}+1)\,. \tag{3.29}$$

Since the coefficients of Δ_t^2 are close to the ones of Δ^2, depending on $\|\psi(t,\cdot)\|_{C^{1,1}(\partial E)}$ as $g_t^{ij}-g_E^{ij}=Q(x,\psi,\nabla\psi)$, where Q is a smooth function with $Q(x,0,0)=0$ (see [48], for instance), we have that $\partial_t+\Delta_t^2$ is a uniformly parabolic operator. Then, by standard interior Schauder estimates and the bound (3.29), there exists a constant $C>0$, which depends on T, ε and E, such that

$$\begin{aligned}\|\nabla\zeta\|_{C^{1,\beta/4}([T/2,T);C^{4,\beta}(B'_{r/2}))} &\leqslant C\left(\|F\|_{C^{\beta/4}([T/4,T);C^\beta(B'_r))}+\|\nabla\zeta\|_{C^0([T/4,T)\times B'_r)}\right)\\ &\leqslant C\varepsilon(\|\psi_0\|_{C^{1,1}(\partial E)}+1)\,,\end{aligned}$$

where we used the estimate $\|\zeta\|_{C^1(([T/4,T)\times B'_r))} \leqslant \|\zeta\|_{X_T}$ and the bound (3.29).
Hence, we finally conclude

$$\sup_{t\in[T/2,T)}\|\nabla^5\psi\|_{C^0(\partial E)} \leqslant C(\|\psi_0\|_{C^{1,1}(\partial E)}+1)\,.$$

Then, estimate (3.4) follows by induction, for every $k\in\mathbb{N}$. Indeed, let us suppone that inequality (3.4) holds for $k\in\mathbb{N}$, we want to show that it holds for $k+1$. Taking $k-1$ covariant derivatives (with respect to the metric g) in formula (3.28), we get the following equation

$$\partial_t\nabla^{k-1}\zeta = -\Delta_t^2\nabla^{k-1}\zeta+\widetilde{F}\,,$$

where the error term $\widetilde{F}$ contains the derivatives of ζ up to the order $k+2$. Then, we estimate $\|\widetilde{F}\|_{C^{\beta/4}([T/2,T];C^\beta(B'_r))}$ by means of inequality (3.4) and we conclude by means of the same argument above. □

3.3 LONG–TIME BEHAVIOR – I

From now on we drop the t–subscript on H_t, B_t, Δ_t, μ_t and we simply write H, B, Δ, μ for the mean curvature, second fundamental form, Laplacian and canonical measure, respectively, when it is clear that they refer to the set E_t and its boundary.

3.3.1 *Evolution of geometric quantities*

Along any surface diffusion flow $\varphi_t : M \to \mathbb{T}^n$ (or when the ambient is a general flat space) we have the following evolution equations (computed in detail in [47, Proposition 3.4] for a general geometric flow of hypersurfaces),

$$\frac{\partial}{\partial t}g_{ij} = 2\Delta\mathrm{H}h_{ij}\,,\qquad \frac{\partial}{\partial t}g^{ij} = -2\Delta\mathrm{H}h^{ij}\,,\qquad \frac{\partial}{\partial t}\mu = \mathrm{H}\Delta\mathrm{H}\mu \tag{3.30}$$

and

$$\frac{\partial}{\partial t}\Gamma^i_{jk} = \nabla\mathrm{B}*\Delta\mathrm{H}+\mathrm{B}*\nabla\Delta\mathrm{H} \tag{3.31}$$

where the symbol $*$ was introduced in Section 1.2.3.
Then, arguing as in [48, Proposition 2.3.1], we get the following evolution equation for the mean curvature

$$\frac{\partial}{\partial t}\mathrm{H} = -\Delta\Delta\mathrm{H} - \Delta\mathrm{H}|\mathrm{B}|^2\,. \tag{3.32}$$

We now introduce some notation which will be useful for the computations that follow (see [47]). If $T_1, \dots, T_l$ is a finite family of tensors (here l is not an index of the tensor T), with the symbol

$$\mathop{\circledast}_{i=1}^{l} T_i$$

we will mean $T_1 * T_2 * \dots * T_l$.
With the symbol $\mathfrak{p}_s(\nabla^\alpha T, \nabla^\beta S, \dots, \nabla^\gamma R)$ we will denote a "polynomial" in the tensors $T, S, \dots, R$ and their iterated covariant derivatives with the $*$ product as

$$\mathfrak{p}_s(\nabla^\alpha T, \nabla^\beta S, \dots, \nabla^\gamma R) \;=\; \sum_{i+j+\dots+k=s} c_{ij\dots k}\, \nabla^i T * \nabla^j S * \dots * \nabla^k R$$

where the $c_{i_1\dots i_l}$ are some real constants and $i \leqslant \alpha$, $j \leqslant \beta$, ... , $k \leqslant \gamma$. Moreover, we set $\mathfrak{p}_0(\,\cdot\,) = 0$. Notice that every tensor must be present in every additive term of $\mathfrak{p}_s(\nabla^\alpha T, \nabla^\beta S, \dots, \nabla^\gamma R)$ and there are no repetitions.
We will use instead the symbol $\mathfrak{q}^s(\nabla^\alpha \mathrm{B}, \nabla^\beta \mathrm{H})$ for a completely contracted "polynomial" (hence a function) of the iterated covariant derivatives of B and H, respectively up to α and β (repetitions are allowed), where in every additive term both B and H must be present and H without derivatives is considered as a contracted B–factor. That is,

$$\mathfrak{q}^s(\nabla^\alpha \mathrm{B}, \nabla^\beta \mathrm{H}) = \sum \mathop{\circledast}_{k=1}^{p} \nabla^{i_k}\mathrm{B} \mathop{\circledast}_{l=1}^{q} \nabla^{j_l}\mathrm{H}$$

with $p, q \geqslant 1$, $i_1, \dots, i_p \leqslant \alpha$ and $1 \leqslant j_1, \dots, j_q \leqslant \beta$, then the coefficient s denotes the sum

$$s = \sum_{k=1}^{p}(i_k + 1) \,+\, \sum_{l=1}^{q}(j_l + 1)\,. \tag{3.33}$$

We advise the reader that in the following the "polynomials" $\mathfrak{p}_s$ and $\mathfrak{q}^s$ could vary from a line to another in a computation, by addition of "similar" terms.

With this notation, we have the following "computation" lemmas.

Lemma 3.3.1. *For every tensor T and function f on M, we have*

$$\frac{\partial}{\partial t}\nabla^s T = \nabla^s \frac{\partial T}{\partial t} + \mathfrak{p}_s(\nabla^{s-1}T, \nabla^s \mathrm{B}, \nabla^s \Delta\mathrm{H}) \quad \textit{for every } s \geqslant 1 \tag{3.34}$$

$$\frac{\partial}{\partial t} df = d\frac{\partial f}{\partial t} \qquad \textit{and} \qquad \frac{\partial}{\partial t}\nabla^s f = \nabla^s \frac{\partial f}{\partial t} + \mathfrak{p}_{s-1}(\nabla^{s-2}(\nabla f), \nabla^{s-1}\mathrm{B}, \nabla^{s-1}\Delta\mathrm{H}) \tag{3.35}$$

for every $s \geqslant 2$.

Proof. We show the first equation by induction on $s \in \mathbb{N}$. If $s = 1$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla T &= \frac{\partial}{\partial t}(\partial T + T\Gamma) = \frac{\partial}{\partial t}\partial T + \frac{\partial}{\partial t}(T\Gamma) = \partial\frac{\partial T}{\partial t} + \frac{\partial T}{\partial t}\Gamma + T\frac{\partial \Gamma}{\partial t}\\ &= \nabla\frac{\partial T}{\partial t} + T * \nabla\mathrm{B} * \Delta\mathrm{H} + T * \mathrm{B} * \nabla\Delta\mathrm{H} = \nabla\frac{\partial T}{\partial t} + \mathfrak{p}_1(T, \nabla\mathrm{B}, \nabla\Delta\mathrm{H})\,,\end{aligned}$$

where we computed "schematically", denoting with ∂ the standard derivative in coordinates (with commute with $\frac{\partial}{\partial t}$) and with Γ the Christoffel symbols, moreover, we used formula (3.31).
Now, assuming that formula (3.34) holds up to $s-1$, we apply it to the tensor $S = \nabla T$

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^s T = \frac{\partial}{\partial t}\nabla^{s-1}S &= \nabla^{s-1}\frac{\partial S}{\partial t} + \mathfrak{p}_{s-1}(\nabla^{s-2}S, \nabla^{s-1}\mathrm{B}, \nabla^{s-1}\Delta\mathrm{H})\\ &= \nabla^{s-1}\frac{\partial}{\partial t}\nabla T + \mathfrak{p}_s(\nabla^{s-1}T, \nabla^{s-1}\mathrm{B}, \nabla^{s-1}\Delta\mathrm{H})\\ &= \nabla^{s-1}\Big(\nabla\frac{\partial T}{\partial t} + \mathfrak{p}_1(T, \nabla\mathrm{B}, \nabla\Delta\mathrm{H})\Big) + \mathfrak{p}_s(\nabla^{s-1}T, \nabla^{s-1}\mathrm{B}, \nabla^{s-1}\Delta\mathrm{H})\\ &= \nabla^s\frac{\partial T}{\partial t} + \mathfrak{p}_s(\nabla^{s-1}T, \nabla^s\mathrm{B}, \nabla^s\Delta\mathrm{H})\end{aligned}$$

by the properties of the $*$–product. Hence, formula (3.34) is proved.
To get equation (3.35), we apply the previous formula to $T = \nabla f$ as follows

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^s f = \frac{\partial}{\partial t}\nabla^{s-1}\nabla f &= \nabla^{s-1}\frac{\partial}{\partial t}\nabla f + \mathfrak{p}_{s-1}(\nabla^{s-2}(\nabla f), \nabla^{s-1}\mathrm{B}, \nabla^{s-1}\Delta\mathrm{H})\\ &= \nabla^s\frac{\partial f}{\partial t} + \mathfrak{p}_{s-1}(\nabla^{s-2}(\nabla f), \nabla^{s-1}\mathrm{B}, \nabla^{s-1}\Delta\mathrm{H})\end{aligned}$$

and we are done. □

Proposition 3.3.2. *Let $E_t \subseteq \mathbb{T}^n$ be a surface diffusion flow. Then, the following equations hold*

$$\frac{d}{dt}\int_{\partial E_t}|\nabla\mathrm{H}|^2\,d\mu_t = -2\Pi_{E_t}(\Delta\mathrm{H}) + \int_{\partial E_t}\mathrm{H}|\nabla\mathrm{H}|^2\Delta\mathrm{H}\,d\mu_t - \int_{\partial E_t}2\mathrm{B}(\nabla\mathrm{H},\nabla\mathrm{H})\,\Delta\mathrm{H}\,d\mu_t \tag{3.36}$$

$$\begin{aligned}\frac{d}{dt}\int_{\partial E_t}|\nabla^{n-2}\mathrm{H}|^2\,d\mu_t = &-2\int_{\partial E_t}|\nabla^n\mathrm{H}|^2\,d\mu_t + \int_{\partial E_t}\mathfrak{q}^{2n+2}(\nabla^{n-4}\mathrm{B}, \nabla^{n-1}\mathrm{H})\,d\mu_t\\ &+\int_{\partial E_t}\mathfrak{q}^{2n+2}(\nabla^{n-3}(\mathrm{B}^2), \nabla^n\mathrm{H})\,d\mu_t\end{aligned} \tag{3.37}$$

where:

- *Every "monomial" of $\mathfrak{q}^{2n+2}(\nabla^{n-4}\mathrm{B}, \nabla^{n-1}\mathrm{H})$ has 4 factors in B, $\nabla\mathrm{H}$ and their covariant derivatives. The factor B (or H without derivatives) or its covariant derivative up to $\nabla^{n-4}\mathrm{B}$ is present exactly one time and the other three factors are derivatives of $\nabla\mathrm{H}$ up to $\nabla^{n-1}\mathrm{H}$, with $\nabla^{n-1}\mathrm{H}$ or $\nabla^{n-2}\mathrm{H}$ present at least one time. Moreover, if the factor $\nabla^{n-1}\mathrm{H}$ is not present, B cannot appear without derivatives.*
- *Every "monomial" of $\mathfrak{q}^{2n+2}(\nabla^{n-3}(\mathrm{B}^2), \nabla^n\mathrm{H})$ has 3 factors in B^2, $\nabla\mathrm{H}$ and their covariant derivatives. The factor B^2 or its covariant derivative up to $\nabla^{n-3}(\mathrm{B}^2)$ is present exactly one time, the other two factors are derivatives of $\nabla\mathrm{H}$ up to $\nabla^n\mathrm{H}$. The factor $\nabla^n\mathrm{H}$ is present exactly one time, with the exception of "monomials" of kind $\nabla^{n-1}\mathrm{H} * \mathrm{B}^2 * \nabla^{n-1}\mathrm{H}$.*

Finally, the coefficients of these "polynomials" are algebraic, that is, they are the result of formal manipulations, in particular, they are independent of E_t.

Proof. Taking into account the evolution equations (3.30) and (3.32), integrating by parts, we compute

$$\begin{aligned}\frac{d}{dt}\int_{\partial E_t}|\nabla \mathrm{H}|^2\,d\mu_t &= \int_{\partial E_t}\mathrm{H}|\nabla\mathrm{H}|^2\,\Delta\mathrm{H}\,d\mu_t - \int_{\partial E_t}2h^{ij}\nabla_i\mathrm{H}\nabla_j\mathrm{H}\,\Delta\mathrm{H}\,d\mu_t\\ &\quad - \int_{\partial E_t}2g^{ij}\nabla_i\mathrm{H}\nabla_j\big(|\mathrm{B}|^2\Delta\mathrm{H}+\Delta\Delta\mathrm{H}\big)\,d\mu_t\\ &= \int_{\partial E_t}\mathrm{H}|\nabla\mathrm{H}|^2\,\Delta\mathrm{H}\,d\mu_t - \int_{\partial E_t}2\mathrm{B}(\nabla\mathrm{H},\nabla\mathrm{H})\,\Delta\mathrm{H}\,d\mu_t\\ &\quad + \int_{\partial E_t}2|\mathrm{B}|^2(\Delta\mathrm{H})^2\,d\mu_t + \int_{\partial E_t}2\Delta\mathrm{H}\,\Delta\Delta\mathrm{H}\,d\mu_t\\ &= \int_{\partial E_t}\mathrm{H}|\nabla\mathrm{H}|^2\,\Delta\mathrm{H}\,d\mu_t - \int_{\partial E_t}2\mathrm{B}(\nabla\mathrm{H},\nabla\mathrm{H})\,\Delta\mathrm{H}\,d\mu_t\\ &\quad + \int_{\partial E_t}2|\mathrm{B}|^2(\Delta\mathrm{H})^2\,d\mu_t - \int_{\partial E_t}2|\nabla\Delta\mathrm{H}|^2\,d\mu_t\,,\end{aligned}$$

where the first term on the right hand side comes from the area measure variation and the second one from the evolution equation of the inverse of the metric. Then, we have formula (3.36), recalling Definition 2.2.1 of the form Π_{E_t}.
To get equation (3.37), we compute analogously

$$\begin{aligned}\frac{d}{dt}\int_{\partial E_t}|\nabla^{n-2}\mathrm{H}|^2\,d\mu_t &= \int_{\partial E_t}|\nabla^{n-2}\mathrm{H}|^2\mathrm{H}\Delta\mathrm{H}\,d\mu_t + 2\int_{\partial E_t}g\Big(\nabla^{n-2}\mathrm{H},\frac{\partial}{\partial t}\nabla^{n-2}\mathrm{H}\Big)\,d\mu_t\\ &\quad - 2\sum_{k=1}^{n-2}\int_{\partial E_t}\Delta\mathrm{H}h^{i_kj_k}\prod_{l\neq k,l=1}^{n-2}g^{i_lj_l}\nabla^{n-2}_{i_1\dots i_{n-2}}\mathrm{H}\,\nabla^{n-2}_{j_1\dots j_{n-2}}\mathrm{H}\,d\mu_t\,.\end{aligned} \tag{3.38}$$

We focus on the second integral, noticing that we can collect the terms inside the other integrals in a "polynomial" of kind $\mathfrak{q}^{2n+2}(\mathrm{B},\nabla^{n-3}(\nabla\mathrm{H}))$ such that every "monomial" has 4 factors in B, ∇H and its covariant derivatives up to ∇^{n-2}H (remember that we consider H as a contracted B–factor, in the first term – we will always do the same also in the following). Moreover, the factor ∇^{n-2}H appears at least one time.
By equation (3.32) and formula (3.35) in Lemma 3.3.1 with $f = \mathrm{H}$ and $s = n-2$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^{n-2}\mathrm{H} &= \nabla^{n-2}\frac{\partial}{\partial t}\mathrm{H} + \mathfrak{p}_{n-3}(\nabla^{n-4}(\nabla\mathrm{H}),\nabla^{n-3}\mathrm{B},\nabla^{n-3}\Delta\mathrm{H})\\ &= \nabla^{n-2}\big(-\Delta\Delta\mathrm{H}-\Delta\mathrm{H}|\mathrm{B}|^2\big) + \mathfrak{p}_{n-3}(\nabla^{n-4}(\nabla\mathrm{H}),\nabla^{n-3}\mathrm{B},\nabla^{n-3}\Delta\mathrm{H})\end{aligned}$$

hence, the second integral in formula (3.38) is equal to

$$\int_{\partial E_t}g\big(\nabla^{n-2}\mathrm{H},\nabla^{n-2}\big(-\Delta\Delta\mathrm{H}-\Delta\mathrm{H}|\mathrm{B}|^2\big)\big) + g\big(\nabla^{n-2}\mathrm{H},\mathfrak{p}_{n-3}(\nabla^{n-4}(\nabla\mathrm{H}),\nabla^{n-3}\mathrm{B},\nabla^{n-3}\Delta\mathrm{H})\big)\,d\mu_t\,.$$

Then, recalling the properties of $\mathfrak{p}_{n-3}(\nabla^{n-4}(\nabla\mathrm{H}),\nabla^{n-3}\mathrm{B},\nabla^{n-3}\Delta\mathrm{H})$, integrating by parts in the second term inside the integral, we can "take away" one derivative from B (in the "monomials" containing it) and "move" it on the other three factors, which are derivatives of H. Hence, the integral of such term becomes of kind $\int_{\partial E_t}\mathfrak{q}^{2n+2}(\nabla^{n-4}\mathrm{B},\nabla^{n-1}\mathrm{H})\,d\mu_t$, noticing that ∇^nH cannot appear, as by the properties of $\mathfrak{p}_{n-3}(\nabla^{n-4}(\nabla\mathrm{H}),\nabla^{n-3}\mathrm{B},\nabla^{n-3}\Delta\mathrm{H})$ either it contains ∇^{n-3}B or $\nabla^{n-3}\Delta$H, but not both together in any of its "monomials". Summarizing, we have a sum of integrals each one like

$$\int_{\partial E_t}\nabla^j\mathrm{B}*\nabla^{i_1}\mathrm{H}*\nabla^{i_2}\mathrm{H}*\nabla^{i_3}\mathrm{H}\,d\mu_t\,,$$

with $0 \leqslant j \leqslant n-4$, $1 \leqslant i_1 \leqslant i_2 \leqslant i_3 \leqslant n-1$, with i_3 equal either to $n-1$ or $n-2$ and

$$2n+2 = (j+1) + \sum_{l=1}^{3}(i_l+1)\,,$$

by formula (3.33). Then, if $i_3 = n-2$, that is the factor $\nabla^{n-1}\mathrm{H}$ is not present, we can integrate repeatedly by parts, "carrying away" derivatives from $\nabla^{i_1}\mathrm{H}$ and distributing them on the other three factors. It is then easy to see that at some point either the term $\nabla^{n-1}\mathrm{H}$ appears or some derivative must go on B.
Hence, from equation (3.38) and since the above "polynomial" of kind $\mathfrak{q}^{2n+2}(\mathrm{B}, \nabla^{n-3}(\nabla\mathrm{H}))$ is *a fortiori* of kind $\mathfrak{q}^{2n+2}(\nabla^{n-4}\mathrm{B}, \nabla^{n-1}\mathrm{H})$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\partial E_t}|\nabla^{n-2}\mathrm{H}|^2\,d\mu_t = &\int_{\partial E_t}|\nabla^{n-2}\mathrm{H}|^2\mathrm{H}\Delta\mathrm{H}\,d\mu_t\\ &-2\int_{\partial E_t} g\big(\nabla^{n-2}\mathrm{H}, \nabla^{n-2}(\Delta\Delta\mathrm{H})\big)\,d\mu_t\\ &-2\int_{\partial E_t} g\big(\nabla^{n-2}\mathrm{H}, \nabla^{n-2}(\Delta\mathrm{H}|\mathrm{B}|^2)\big)\,d\mu_t\\ &+\int_{\partial E_t}\mathfrak{q}^{2n+2}(\nabla^{n-4}\mathrm{B}, \nabla^{n-1}\mathrm{H})\,d\mu_t \end{aligned} \tag{3.39}$$

where every "monomial" of $\mathfrak{q}^{2n+2}(\nabla^{n-4}\mathrm{B}, \nabla^{n-1}\mathrm{H})$ has 4 factors in B, $\nabla\mathrm{H}$ and their covariant derivatives, moreover

- the factor B (or H, without derivatives) or its derivatives up to order $n-4$ is present exactly one time,
- the other three factors are derivatives of $\nabla\mathrm{H}$ up to $\nabla^{n-1}\mathrm{H}$,
- the higher order factor $\nabla^{n-1}\mathrm{H}$ or $\nabla^{n-2}\mathrm{H}$ is present at least one time,
- if the factor $\nabla^{n-1}\mathrm{H}$ is not present, B cannot appear without derivatives.

Now we deal with the second integral in the right hand side of equation (3.39) which can be written as

$$-2\int_{\partial E_t} g^{i_1 j_1}\cdots g^{i_{n-2}j_{n-2}}g^{ms}g^{pq}\,\nabla^{n-2}_{j_1\cdots j_{n-2}}\mathrm{H}\,\nabla^{n-2}_{i_1\dots i_{n-2}}\nabla^4_{mspq}\mathrm{H}\,d\mu_t\,.$$

We interchange repeatedly the covariant derivatives in the last factor inside the integral in order to have

$$-2\int_{\partial E_t} g^{i_1 j_1}\cdots g^{i_{n-2}j_{n-2}}g^{ms}g^{pq}\,\nabla^{n-2}_{j_1\cdots j_{n-2}}\mathrm{H}\,\nabla^2_{pm}\nabla^2_{sq}\nabla^{n-2}_{i_1\dots i_{n-2}}\mathrm{H}\,d\mu_t + \text{Error Terms}\,,$$

where any "error term" introduced at every interchange has the form $\nabla^l(\mathrm{Riem} * \nabla^{n-l}\mathrm{H}) = \nabla^l(\mathrm{B}^2 * \nabla^{n-l}\mathrm{H})$, for $l = 0, \dots, n-1$, by the Gauss equations (1.12).
Integrating by parts twice, "moving" the double derivative ∇^2_{pm} on the other factor, we get

$$-2\int_{\partial E_t} g^{i_1 j_1}\cdots g^{i_{n-2}j_{n-2}}g^{ms}g^{pq}\,\nabla^2_{mp}\nabla^{n-2}_{j_1\cdots j_{n-2}}\mathrm{H}\,\nabla^2_{sq}\nabla^{n-2}_{i_1\dots i_{n-2}}\mathrm{H}\,d\mu_t + \text{Error Terms}\,,$$

which is equal to

$$-2\int_{\partial E_t}|\nabla^n\mathrm{H}|^2\,d\mu_t + \sum_{l=0}^{n-1}\int_{\partial E_t}\nabla^{n-2}\mathrm{H} * \nabla^l(\mathrm{B}^2 * \nabla^{n-l}\mathrm{H})\,d\mu_t\,,$$

where we made explicit the error terms, by what we observed above. Then, we notice that, integrating twice by parts in every integral in the sum above with $l \geqslant 2$ and only one time when $l = 1$, we have

$$-2\int_{\partial E_t} |\nabla^n \mathrm{H}|^2 \, d\mu_t + \int_{\partial E_t} \nabla^{n-2}\mathrm{H} * \mathrm{B}^2 * \nabla^n \mathrm{H} \, d\mu_t + \sum_{l=2}^{n-1} \int_{\partial E_t} \nabla^n \mathrm{H} * \nabla^{l-2}(\mathrm{B}^2 * \nabla^{n-l}\mathrm{H}) \, d\mu_t$$
$$+ \int_{\partial E_t} \nabla^{n-1}\mathrm{H} * \mathrm{B}^2 * \nabla^{n-1}\mathrm{H} \, d\mu_t \,,$$

hence, the last two integrals on the first line contain the factor $\nabla^n \mathrm{H}$ exactly one time and we can finally write the second integral in the right hand side of equation (3.39) as

$$-2\int_{\partial E_t} |\nabla^n \mathrm{H}|^2 \, d\mu_t + \int_{\partial E_t} \mathfrak{q}^{2n+2}(\nabla^{n-3}(\mathrm{B}^2), \nabla^n \mathrm{H}) \, d\mu_t + \int_{\partial E_t} \nabla^{n-1}\mathrm{H} * \mathrm{B}^2 * \nabla^{n-1}\mathrm{H} \, d\mu_t \,,$$

where every "monomial" of $\mathfrak{q}^{2n+2}(\nabla^{n-3}(\mathrm{B}^2), \nabla^n \mathrm{H})$ has 3 factors in B^2, $\nabla \mathrm{H}$ and their covariant derivatives, moreover

- the factor B^2 or its derivatives up to order $n-3$ is present exactly one time,
- the other two factors are a derivatives of $\nabla \mathrm{H}$ up to $\nabla^n \mathrm{H}$,
- the factor $\nabla^n \mathrm{H}$ appears exactly one time.

Finally, integrating by parts two times the third integral in formula (3.39), the integrand becomes a contraction of $\nabla^n \mathrm{H}$ with $\nabla^{n-4}(\Delta \mathrm{H}|\mathrm{B}|^2)$, which is clearly also a "polynomial" of the form $\mathfrak{q}^{2n+2}(\nabla^{n-3}(\mathrm{B}^2), \nabla^n \mathrm{H})$ satisfying these same properties and we are done. □

Remark 3.3.3. We notice that if $n = 3$, the expressions (3.36) and (3.37) coincide. Hence, we will actually never use Lemmas 3.3.9 and 3.3.10 for the estimates of the next section, in the special case $n = 3$. In other words, whenever we will work on quantities involving $(n-2)$–derivatives of the mean curvature, we will assume that $n \geqslant 4$ without specifying further.

3.3.2 *Estimate of the energy variation and other basic estimates*

In all the following, we will be interested in having uniform estimates for the families of sets in $\mathfrak{C}^1_{M_E}(E)$, given a smooth set $E \subseteq \mathbb{T}^n$ and a tubular neighborhood N_ε of ∂E, for $M_E < \varepsilon$. To this aim, we need that the constants in the Sobolev, Poincaré, Gagliardo–Nirenberg interpolation and Calderón–Zygmund inequalities relative to all the hypersurfaces ∂F boundaries of the sets $F \in \mathfrak{C}^1_{M_E}(E)$, are uniform (for the Calderón–Zygmund inequalities, we actually need that $F \in \mathfrak{C}^1_{M_E}(E)$, with $M_E > 0$ small enough). This is the content of Section 1.2 , where such uniformity is proved in detail. Hence, from now on we will use the adjective "uniform" in order to underline such fact. We also highlight that in all the following we will denote with C a constant which may vary from a line to another and depends only on E and M_E.

Proposition 3.3.4 (Gagliardo–Nirenberg interpolation inequalities)**.** *Let $E \subseteq \mathbb{T}^n$ be a smooth set, $j, m \in \mathbb{N}$ with $0 \leqslant j < m$ and $0 < r, q \leqslant +\infty$. Then, for every $F \in \mathfrak{C}^1_{M_E}(E)$ and every covariant tensor $T = T_{i_1 \ldots i_l}$ the following uniform interpolation inequalities hold:*

$$\|\nabla^j T\|_{L^p(\partial F)} \leqslant C\left(\|\nabla^m T\|_{L^r(\partial F)} + \|T\|_{L^r(\partial F)}\right)^\theta \|T\|^{1-\theta}_{L^q(\partial F)} \,, \qquad (3.40)$$

with the compatibility condition

$$\frac{1}{p} = \frac{j}{n-1} + \theta\Big(\frac{1}{r} - \frac{m}{n-1}\Big) + \frac{1-\theta}{q} \,,$$

for all $\theta \in [j/m, 1]$ *for which* $p \in [1, +\infty)$ *is nonnegative, with the exception of the case* $r = \frac{n-1}{m-j} \neq 1$ *for which the inequality is not valid for* $\theta = 1$*. The constant* C *depends only on* n, j, m, p, q, r, E *and* M_E*. Moreover, if* $f : \partial F \to \mathbb{R}$ *is a smooth function, inequality* (3.40) *becomes*

$$\|\nabla^j f\|_{L^p(\partial F)} \leqslant C \,\|\nabla^m f\|^{\theta}_{L^r(\partial F)} \|f\|^{1-\theta}_{L^q(\partial F)}\,, \tag{3.41}$$

if $j \geqslant 1$ *or* $j = 0$ *and* $\fint_{\partial F} f \, d\mu = 0$.
By density, all these inequalities clearly extend to functions and tensors in the appropriate Sobolev spaces.

Proof – Sketch. For a single fixed regular hypersurface ∂F, inequality (3.41) is given by Theorem 3.70 in [7], while inequality (3.40) for T equal to a function $f : \partial F \to \mathbb{R}$ can be obtained by repeating step by step the proof of such theorem, once established the following Sobolev–type inequality for hypersurfaces without boundary,

$$\|f\|_{L^{p^*}(\partial F)} \leqslant C \left(\|\nabla f\|_{L^p(\partial F)} + \|f\|_{L^p(\partial F)} \right),$$

for every $p \in [1, n-1)$ (an example of such argument can be found in [47, Section 6]).
The extension of inequality (3.40) to tensors can be obtained as in [47, Sections 5 and 6], by means of the estimate (see [7], Proposition 2.11 and also [12, 13]),

$$\left| \nabla \sqrt{|T|^2 + \varepsilon^2} \right| = \left| \frac{\langle \nabla T, T \rangle}{\sqrt{|T|^2 + \varepsilon^2}} \right| \leqslant \frac{|T|}{\sqrt{|T|^2 + \varepsilon^2}} \, |\nabla T| \leqslant |\nabla T|$$

clearly leading to the previous Sobolev inequality for tensors, as $\sqrt{|T|^2 + \varepsilon^2}$ converges to $|T|$ when $\varepsilon \to 0$ (this argument is necessary as $|T|$ is not necessarily smooth).
Finally, the "uniformity" in the constants of the inequalities, independently of $F \in \mathfrak{C}^1_{M_E}(E)$, follows by the same independence in the Sobolev inequalities, as it is shown and discussed in detail in Section 1.2 (Theorem 1.2.1 – point (vi)). □

Remark 3.3.5. Notice that in the same hypotheses of this proposition, by means of the uniform Sobolev–Poincarè inequality

$$\|f - \overline{f}\|_{L^{q^*}(\partial F)} \leqslant C \,\|\nabla f\|_{L^q(\partial F)}\,,$$

for every $q \in [1, n-1)$ which can be easily deduced by estimate (3.41), we have the following uniform Poincaré inequalities

$$\|f - \overline{f}\|_{L^p(\partial F)} \leqslant C \,\|\nabla f\|_{L^p(\partial F)}\,, \tag{3.42}$$

for every $p \in [1, +\infty]$.

Remark 3.3.6. Very similar uniform interpolation inequalities are worked out in [47], for any family of n–dimensional, regular hypersurfaces $N \subseteq \mathbb{R}^{n+1}$ satisfying $\mathrm{Vol}(N) + \|\mathrm{H}\|_{L^{n+\delta}(N)} \leqslant C$, for some $\delta > 0$, instead of being boundaries of sets belonging to $\mathfrak{C}^1_{M_E}(E)$.

As a direct consequence of Proposition 3.3.4, we have the following lemma that will be used very often in the sequel.

Lemma 3.3.7. *Let* $E \subseteq \mathbb{T}^n$ *be a smooth set and* $j, m \in \mathbb{N}$ *with* $1 \leqslant j < m$*. Then, for every* $F \in \mathfrak{C}^1_{M_E}(E)$ *and every covariant tensor* T*, the following uniform inequalities hold, for every* $\varepsilon > 0$,

$$\|\nabla^j T\|^2_{L^p(\partial F)} \leqslant C \|\nabla^m T\|^{2\theta}_{L^2(\partial F)} \|\nabla T\|^{2(1-\theta)}_{L^2(\partial F)} + C \|\nabla T\|^2_{L^2(\partial F)} \leqslant \varepsilon \|\nabla^m T\|^2_{L^2(\partial F)} + C \|\nabla T\|^2_{L^2(\partial F)}\,, \tag{3.43}$$

with the compatibility condition

$$\frac{1}{p} = \frac{j-1}{n-1} - \theta \Big(\frac{m-1}{n-1} \Big) + \frac{1}{2}\,,$$

for all $\theta \in \left[\frac{j-1}{m-1}, 1\right]$ *for which* $p \in [1, +\infty)$ *is nonnegative, with the exception of the case* $\frac{n-1}{m-j} = 2$ *for which the inequality is not valid for* $\theta = 1$ *and*

$$\|\nabla^j T\|_{L^p(\partial F)} \leqslant C\|\nabla^m T\|_{L^p(\partial F)}^{\frac{j-1}{m-1}}\|\nabla T\|_{L^p(\partial F)}^{\frac{m-j}{m-1}} + C\|\nabla T\|_{L^p(\partial F)} \leqslant \varepsilon\|\nabla^m T\|_{L^p(\partial F)} + C\|\nabla T\|_{L^p(\partial F)}\,, \tag{3.44}$$

for every $p \in (1, +\infty)$.
The constants C *depends only on* n, j, m, p, E, M_E *and* ε.

Proof. The first inequality in formula (3.43) comes from inequality (3.40), by substituting ∇T in place of T, while the second one follows by Young inequality. Analogously, one gets formula (3.44). □

Lemma 3.3.8. *Let* $E \subseteq \mathbb{T}^n$ *be a smooth set,* $F \in \mathfrak{C}^1_{M_E}(E)$ *and* $f_1, \dots, f_l$ *smooth functions such that* $\|f_i\|_{L^\infty(\partial F)} \leqslant C$. *Then, for every* $\alpha_1, \dots, \alpha_l \in \mathbb{N}$ *with* $\alpha_1 + \dots + \alpha_l \leqslant k$ *and* $p \in (1, +\infty)$, *there holds*

$$\||\nabla^{\alpha_1} f_1| \cdots |\nabla^{\alpha_l} f_l|\|_{L^p(\partial F)} \leqslant C_k \sum_{i=1}^{l} \left(\|\nabla^k f_i\|_{L^p(\partial F)} + \|\nabla f_i\|_{L^p(\partial F)}\right), \tag{3.45}$$

for some uniform constant C_k.

Proof. Without loss of generality, we may assume that $\alpha_1 + \dots + \alpha_l = k$, otherwise we argue with $k' = \alpha_1 + \dots + \alpha_l$ in place of k and then we apply the previous lemma (inequality (3.44)). Moreover, we can also assume that $\alpha_i \geqslant 1$, for every $i \in \{1, \dots, l\}$, as we can simply estimate any $|f_i|$ with C, if it appears in the left hand side of inequality (3.45).
We first use Hölder inequality,

$$\||\nabla^{\alpha_1} f_1| \cdots |\nabla^{\alpha_l} f_l|\|_{L^p(\partial F)} \leqslant \prod_{i=1}^{l} \|\nabla^{\alpha_i} f_i\|_{L^{\frac{pk}{\alpha_i}}(\partial F)}\,.$$

Then, by the uniform interpolation inequalities (3.41) (being every $\alpha_i \geqslant 1$), we have

$$\|\nabla^{\alpha_i} f_i\|_{L^{\frac{pk}{\alpha_i}}(\partial F)} \leqslant \|\nabla^k f_i\|_{L^p(\partial F)}^{\frac{\alpha_i}{k}} \|f_i\|_{L^\infty(\partial F)}^{1-\frac{\alpha_i}{k}} \leqslant \|\nabla^k f_i\|_{L^p(\partial F)}\,,$$

hence, the thesis follows by Young inequality, as $\alpha_1 + \dots + \alpha_l = k$. □

Lemma 3.3.9. *Let* $E_t \subseteq \mathbb{T}^n$ *be a surface diffusion flow such that* $E_t \in \mathfrak{C}^1_{M_E}(E)$, *for some smooth set* E *and* $\mathfrak{q}^{2n+2}(\nabla^{n-3}(\mathrm{B}^2), \nabla^n \mathrm{H})$ *is a "polynomial" as in Proposition 3.3.2. Then,*

$$\int_{\partial E_t} \mathfrak{q}^{2n+2}(\nabla^{n-3}(\mathrm{B}^2), \nabla^n \mathrm{H})\, d\mu_t \leqslant -\|\nabla^n \mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)},$$

for some constant C *which depends on* E *and* M_E *and for any* $j \leqslant n-3$, *also on*

- $\|\nabla^j \mathrm{B}\|_{L^{\frac{2n-2}{2j-n+3}}(\partial E_t)}$ *if* $j > (n-3)/2$,
- $\|\nabla^j \mathrm{B}\|_{L^p(\partial E_t)}$, *for every* $1 < p < +\infty$ *if* $j = (n-3)/2$,
- $\|\nabla^j \mathrm{B}\|_{L^\infty(\partial E_t)}$ *if* $j < (n-3)/2$.

Proof. In Proposition 3.3.2, we found out that every "monomial" of $\mathfrak{q}^{2n+2}(\nabla^{n-3}(\mathrm{B}^2), \nabla^n \mathrm{H})$ has 3 factors in B^2, $\nabla \mathrm{H}$ and their covariant derivatives. The factor B^2 or its covariant derivative up to $\nabla^{n-3}(\mathrm{B}^2)$ is present exactly one time, the other two factors are derivatives of $\nabla \mathrm{H}$ up to $\nabla^n \mathrm{H}$. The factor $\nabla^n \mathrm{H}$ is present exactly one time, with the exception of "monomials" of kind

$\nabla^{n-1}\mathrm{H} * \mathrm{B}^2 * \nabla^{n-1}\mathrm{H}$.
Hence, after expanding the iterated derivatives of B^2, the integrals of the *non–exceptional* "monomials" have the form

$$\int_{\partial E_t} \nabla^k \mathrm{B} * \nabla^j \mathrm{B} * \nabla^i \mathrm{H} * \nabla^n \mathrm{H}\, d\mu_t\,,$$

with $j + k \leqslant n - 3$ and $i + j + k = n - 2$, by formula (3.33).
We now estimate the modulus of these integrals (after "carrying" the modulus inside the integrals and using the properties of the $*$–product). Actually different cases may happen:

1. If $k, j < \frac{n-3}{2}$, by Peter–Paul inequality, for every $\varepsilon > 0$, we get

$$\int_{\partial E_t} |\nabla^k \mathrm{B}||\nabla^j \mathrm{B}||\nabla^i \mathrm{H}||\nabla^n \mathrm{H}|\, d\mu_t \leqslant \varepsilon \int_{\partial E_t} |\nabla^n \mathrm{H}|^2\, d\mu_t + C \int_{\partial E_t} |\nabla^i \mathrm{H}|^2\, d\mu_t\,,$$

for any $\varepsilon > 0$, with $C = C\big(\varepsilon, \|\nabla^k \mathrm{B}\|_{L^\infty(\partial E_t)}, \|\nabla^j \mathrm{B}\|_{L^\infty(\partial E_t)}\big)$. Then, estimating the last integral by means of Lemma 3.3.7 (inequality (3.43) with $\theta = \frac{i-1}{n-1}$), we conclude

$$\int_{\partial E_t} |\nabla^k \mathrm{B}||\nabla^j \mathrm{B}||\nabla^i \mathrm{H}||\nabla^n \mathrm{H}|\, d\mu_t \leqslant 2\varepsilon \int_{\partial E_t} |\nabla^n \mathrm{H}|^2\, d\mu_t + C \int_{\partial E_t} |\nabla \mathrm{H}|^2\, d\mu_t\,.$$

2. If $k < \frac{n-3}{2}$ and $j > \frac{n-3}{2}$, as above we get

$$\int_{\partial E_t} |\nabla^k \mathrm{B}||\nabla^j \mathrm{B}||\nabla^i \mathrm{H}||\nabla^n \mathrm{H}|\, d\mu_t \leqslant \varepsilon \int_{\partial E_t} |\nabla^n \mathrm{H}|^2\, d\mu_t + C \int_{\partial E_t} |\nabla^j \mathrm{B}|^2 |\nabla^i \mathrm{H}|^2\, d\mu_t\,, \tag{3.46}$$

for any $\varepsilon > 0$, with $C = C\big(\varepsilon, \|\nabla^k \mathrm{B}\|_{L^\infty(\partial E_t)}\big)$. Hence, using the Hölder inequality on the last integral, we have

$$\int_{\partial E_t} |\nabla^j \mathrm{B}|^2 |\nabla^i \mathrm{H}|^2\, d\mu_t \leqslant C \|\nabla^j \mathrm{B}\|^2_{L^{\frac{2(n-1)}{2j-n+3}}(\partial E_t)} \|\nabla^i \mathrm{H}\|^2_{L^{\frac{n-1}{n-j-2}}(\partial E_t)} \leqslant C \|\nabla^i \mathrm{H}\|^2_{L^{\frac{n-1}{n-j-2}}(\partial E_t)}\,,$$

with $C = C\Big(\|\nabla^j \mathrm{B}\|_{L^{\frac{2n-2}{2j-n+3}}(\partial E_t)}\Big)$.
Then, we estimate the last term by means of inequality (3.43) with $\theta = \frac{1}{2} - \frac{i+j-n+1}{n-1} \in \big[\frac{i-1}{n-1}, 1\big)$, that is,

$$C \|\nabla^i \mathrm{H}\|^2_{L^{\frac{n-1}{n-j-2}}(\partial E_t)} \leqslant \varepsilon \|\nabla^n \mathrm{H}\|^2_{L^2(\partial E_t)} + C \|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)}\,.$$

Hence, getting back to inequality (3.46), we conclude

$$\int_{\partial E_t} |\nabla^k \mathrm{B}||\nabla^j \mathrm{B}||\nabla^i \mathrm{H}||\nabla^n \mathrm{H}|\, d\mu_t \leqslant 2\varepsilon \|\nabla^n \mathrm{H}\|^2_{L^2(\partial E_t)} + C \|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)}\,,$$

for any $\varepsilon > 0$, with $C = C\Big(\varepsilon, \|\nabla^k \mathrm{B}\|_{L^\infty(\partial E_t)}, \|\nabla^j \mathrm{B}\|_{L^{\frac{2n-2}{2j-n+3}}(\partial E_t)}\Big)$.

3. If $k = j = \frac{n-3}{2}$ (hence $i = 1$), by means of Young and Hölder inequalities, we have

$$\begin{aligned}\int_{\partial E_t} \big|\nabla^{\frac{n-3}{2}} \mathrm{B}\big|^2 |\nabla \mathrm{H}||\nabla^n \mathrm{H}|\, d\mu_t &\leqslant \varepsilon \|\nabla^n \mathrm{H}\|^2_{L^2(\partial E_t)} + C \|\nabla^{\frac{n-3}{2}} \mathrm{B}\|^4_{L^{\frac{4n-4}{n-3}}(\partial E_t)} \|\nabla \mathrm{H}\|^2_{L^{n-1}(\partial E_t)} \\ &\leqslant \varepsilon \|\nabla^n \mathrm{H}\|^2_{L^2(\partial E_t)} + C \|\nabla \mathrm{H}\|^2_{L^{n-1}(\partial E_t)}\,,\end{aligned} \tag{3.47}$$

with $C = C\Big(\varepsilon, \big\|\nabla^{\frac{n-3}{2}} \mathrm{B}\big\|_{L^{\frac{4n-4}{n-3}}(\partial E_t)}\Big)$. Then, by the uniform inequality (3.43) with $\theta = \frac{n-3}{2n-2} \in [0, 1)$, we get

$$\|\nabla \mathrm{H}\|^2_{L^{n-1}(\partial E_t)} \leqslant \varepsilon \|\nabla^n \mathrm{H}\|^2_{L^2(\partial E_t)} + C \|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)}\,,$$

hence, estimating the last term in inequality (3.47) with this one, we conclude

$$\int_{\partial E_t} \big|\nabla^{\frac{n-3}{2}}\mathrm{B}\big|^2 |\nabla\mathrm{H}||\nabla^n\mathrm{H}|\,d\mu_t \leqslant 2\varepsilon\|\nabla^n\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)}\,,$$

with $C = C\Big(\varepsilon, \big\|\nabla^{\frac{n-3}{2}}\mathrm{B}\big\|_{L^{\frac{4n-4}{n-3}}(\partial E_t)}\Big)$.

4. If $k = \frac{n-3}{2}$ and $j < \frac{n-3}{2}$, we argue as in the previous case. Indeed, we have

$$\int_{\partial E_t} |\nabla^{\frac{n-3}{2}}\mathrm{B}||\nabla^j\mathrm{B}||\nabla^i\mathrm{H}||\nabla^n\mathrm{H}|\,d\mu_t \leqslant \varepsilon\|\nabla^n\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla^{\frac{n-3}{2}}\mathrm{B}\|^2_{L^{\frac{4n-4}{n-3}}(\partial E_t)}\|\nabla^i\mathrm{H}\|^2_{L^{\frac{4n-4}{n+1}}(\partial E_t)}$$

$$\leqslant \varepsilon\|\nabla^n\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla^i\mathrm{H}\|^2_{L^{\frac{4n-4}{n+1}}(\partial E_t)}$$

where $C = C\Big(\varepsilon, \big\|\nabla^{\frac{n-3}{2}}\mathrm{B}\big\|_{L^{\frac{4n-4}{n-3}}(\partial E_t)}, \|\nabla^j\mathrm{B}\|_{L^\infty(\partial E_t)}\Big)$. Then, by means of the uniform inequality (3.43) with $\theta = \frac{i-1}{n-1} + \frac{1}{4} - \frac{1}{2(n-1)} \in \big[\frac{i-1}{n-1}, 1\big)$, we get

$$\|\nabla^i\mathrm{H}\|^2_{L^{\frac{4n-4}{n+1}}(\partial E_t)} \leqslant \varepsilon\|\nabla^n\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)}$$

and we conclude, as above,

$$\int_{\partial E_t} |\nabla^{\frac{n-3}{2}}\mathrm{B}||\nabla^j\mathrm{B}||\nabla^i\mathrm{H}||\nabla^n\mathrm{H}|\,d\mu_t \leqslant 2\varepsilon\|\nabla^n\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla^i\mathrm{H}\|^2_{L^2(\partial E_t)}\,,$$

where $C = C\Big(\varepsilon, \big\|\nabla^{\frac{n-3}{2}}\mathrm{B}\big\|_{L^p(\partial E_t)}, \|\nabla^j\mathrm{B}\|_{L^\infty(\partial E_t)}\Big)$.

The integrals of the *exceptional* "monomials" can be estimated by means of inequality (3.43), with $\theta = \frac{n-2}{n-1}$, as follows

$$\int_{\partial E_t} |\mathrm{B}|^2|\nabla^{n-1}\mathrm{H}|^2\,d\mu_t \leqslant C\|\nabla^{n-1}\mathrm{H}\|^2_{L^2(\partial E_t)} \leqslant \varepsilon\|\nabla^n\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)}\,,$$

where the constant C depends on $\|\mathrm{B}\|_{L^\infty}$ and we used the Young inequality.
Hence, adding together the estimates on the integrals of all the terms in $\mathfrak{q}^{2n+2}(\nabla^{n-3}(\mathrm{B}^2), \nabla^n\mathrm{H})$ (belonging to all the above cases) and choosing suitable $\varepsilon > 0$, we obtain the thesis of the lemma. □

Lemma 3.3.10. *Let $E_t \subseteq \mathbb{T}^n$ be a surface diffusion flow such that $E_t \in \mathfrak{C}^1_{M_E}(E)$, for some smooth set E and $\mathfrak{q}^{2n+2}(\nabla^{n-4}\mathrm{B}, \nabla^{n-1}\mathrm{H})$ be a family of "polynomials" as in Proposition 3.3.2. Then,*

$$\int_{\partial E_t} \mathfrak{q}^{2n+2}(\nabla^{n-4}\mathrm{B}, \nabla^{n-1}\mathrm{H})\,d\mu_t \leqslant -\|\nabla^n\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)}$$

for some constant C which depends on E and M_E and, for any $j \leqslant n-4$, also on

- $\|\mathrm{B}\|_{L^\infty}$ *and* $\|\nabla^j\mathrm{B}\|_{L^{\frac{2n-2}{2j-n+3}}(\partial E_t)}$ *if* $j > (n-3)/2$,
- $\|\mathrm{B}\|_{L^\infty}$ *and* $\|\nabla^j\mathrm{B}\|_{L^p(\partial E_t)}$, *for every* $1 < p < +\infty$ *if* $j = (n-3)/2$,
- $\|\mathrm{B}\|_{L^\infty}$ *and* $\|\nabla^j\mathrm{B}\|_{L^\infty(\partial E_t)}$ *if* $j < (n-3)/2$.

Proof. From Proposition 3.3.2, we have that every "monomial" of $\mathfrak{q}^{2n+2}(\nabla^{n-4}\mathrm{B}, \nabla^{n-1}\mathrm{H})$ has 4 factors in B, ∇H and their covariant derivatives. The factor B (or H without derivatives) or its covariant derivative up to $\nabla^{n-4}\mathrm{B}$ is present exactly one time and the other three factors are derivatives of ∇H up to $\nabla^{n-1}\mathrm{H}$, with $\nabla^{n-1}\mathrm{H}$ or $\nabla^{n-2}\mathrm{H}$ present at least one time. Moreover, if the factor $\nabla^{n-1}\mathrm{H}$ is not present, B cannot appear without derivatives.
Hence, we have a sum of integrals each one like

$$\int_{\partial E_t} \nabla^j \mathrm{B} * \nabla^{i_1}\mathrm{H} * \nabla^{i_2}\mathrm{H} * \nabla^{i_3}\mathrm{H}\, d\mu_t\,,$$

with $0 \leqslant j \leqslant n-4$, $1 \leqslant i_1 \leqslant i_2 \leqslant i_3 \leqslant n-1$, $i_3 \in \{n-1, n-2\}$ and $j + i_1 + i_2 + i_3 = 2n-2$, by formula (3.33).
We now estimate the modulus of these integrals (after "carrying" the modulus inside the integrals and using the properties of the $*$–product). Arguing as in Lemma 3.3.9, we have different cases:

1. If $j > \frac{n-3}{2}$, by Peter–Paul inequality, we get

$$\int_{\partial E_t} |\nabla^j \mathrm{B}||\nabla^{i_1}\mathrm{H}||\nabla^{i_2}\mathrm{H}||\nabla^{i_3}\mathrm{H}|\, d\mu_t \leqslant C\int_{\partial E_t} |\nabla^j \mathrm{B}|^2|\nabla^{i_1}\mathrm{H}|^2|\nabla^{i_2}\mathrm{H}|^2\, d\mu_t + C\int_{\partial E_t} |\nabla^{i_3}\mathrm{H}|^2\, d\mu_t\,.$$

Then, we estimate the last integral by means of Lemma 3.3.7 (inequality (3.43) with $\theta = \frac{i_3-1}{n-1}$), while for the first one we use the Hölder inequality with $p = \frac{n-1}{2j-n+3}$, as follows,

$$C\int_{\partial E_t} |\nabla^j \mathrm{B}|^2|\nabla^{i_1}\mathrm{H}|^2|\nabla^{i_2}\mathrm{H}|^2\, d\mu_t \leqslant C\|\nabla^j\mathrm{B}\|^2_{L^{\frac{2(n-1)}{2j-n+3}}(\partial E_t)}\,\||\nabla^{i_1}\mathrm{H}||\nabla^{i_2}\mathrm{H}|\|^2_{L^{\frac{n-1}{n-j-2}}(\partial E_t)}\,.$$

We now bound the last norm in this inequality by means of Lemma 3.3.8, with $f_1 = f_2 = \mathrm{H}$, $p = \frac{n-1}{n-j-2}$, $\alpha = (i_1, i_2)$ and $k = \left[\frac{n+3}{2}\right]$ (the *integer part* of (n+3)/2), noticing that $k > i_1 + i_2$ as $i_3 \in \{n-2, n-1\}$. In doing this, we underline that in our case, the constants in inequality (3.45) depend on $\|\mathrm{H}\|_{L^\infty}$ which is clearly bounded by $\|\mathrm{B}\|_{L^\infty}$. Hence, we get

$$\||\nabla^{i_1}\mathrm{H}||\nabla^{i_2}\mathrm{H}|\|^2_{L^{\frac{n-1}{n-j-2}}(\partial E_t)} \leqslant C\|\nabla^k\mathrm{H}\|^2_{L^{\frac{n-1}{n-j-2}}(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^{\frac{n-1}{n-j-2}}(\partial E_t)}$$

and, since by inequality (3.43), we have

$$\begin{aligned} C\|\nabla^k\mathrm{H}\|^2_{L^{\frac{n-1}{n-j-2}}(\partial E_t)} &\leqslant C\|\nabla^n\mathrm{H}\|^{2\theta}_{L^2(\partial E_t)}\|\nabla\mathrm{H}\|^{2(1-\theta)}_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)}\\ &\leqslant \varepsilon\|\nabla^n\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)}\,,\end{aligned}$$

with $\theta = \frac{k-n+j+1}{n-1} + \frac{1}{2} \in \left[\frac{k-1}{n-1}, 1\right)$ and

$$\begin{aligned} C\|\nabla\mathrm{H}\|^2_{L^{\frac{n-1}{n-j-2}}(\partial E_t)} &\leqslant C\|\nabla^n\mathrm{H}\|^{2\theta}_{L^2(\partial E_t)}\|\nabla\mathrm{H}\|^{2(1-\theta)}_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)}\\ &\leqslant \varepsilon\|\nabla^n\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)}\,,\end{aligned}$$

with $\theta = \frac{j-n+2}{n-1} + \frac{1}{2} \in [0,1)$, we conclude

$$\||\nabla^{i_1}\mathrm{H}||\nabla^{i_2}\mathrm{H}|\|^2_{L^{\frac{n-1}{n-j-2}}(\partial E_t)} \leqslant 2\varepsilon\|\nabla^n\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)}\,.$$

Thus, we easily get

$$\int_{\partial E_t} |\nabla^j \mathrm{B}|^2|\nabla^{i_1}\mathrm{H}|^2|\nabla^{i_2}\mathrm{H}|^2\, d\mu_t \leqslant 2\varepsilon\|\nabla^n\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)}\,,$$

for any $\varepsilon > 0$, with $C = C\Big(\varepsilon, \|\mathrm{B}\|_{L^\infty}, \|\nabla^j\mathrm{B}\|_{L^{\frac{2n-2}{2j-n+3}}(\partial E_t)}\Big)$.

2. If $j = \frac{n-3}{2}$, again by Peter–Paul inequality, we have

$$\int_{\partial E_t} |\nabla^{\frac{n-3}{2}}\mathrm{B}||\nabla^{i_1}\mathrm{H}||\nabla^{i_2}\mathrm{H}||\nabla^{i_3}\mathrm{H}|\,d\mu_t \leqslant C\int_{\partial E_t} |\nabla^{i_1}\mathrm{H}|^2|\nabla^{i_2}\mathrm{H}|^2\,d\mu_t + C\int_{\partial E_t} |\nabla^{\frac{n-3}{2}}\mathrm{B}|^2|\nabla^{i_3}\mathrm{H}|^2\,d\mu_t\,. \tag{3.48}$$

We now use Hölder inequality for the last integral, that is

$$\begin{aligned} C\int_{\partial E_t} |\nabla^{\frac{n-3}{2}}\mathrm{B}|^2|\nabla^{i_3}\mathrm{H}|^2\,d\mu_t &\leqslant C\|\nabla^{\frac{n-3}{2}}\mathrm{B}\|^2_{L^{\frac{n-1}{n-i_3}}(\partial E_t)}\|\nabla^{i_3}\mathrm{H}\|^2_{L^{\frac{2n-2}{2i_3-n-1}}(\partial E_t)} \\ &\leqslant C\|\nabla^{i_3}\mathrm{H}\|^2_{L^{\frac{2n-2}{2i_3-n-1}}(\partial E_t)}\,, \end{aligned} \tag{3.49}$$

where $C = C\Big(\varepsilon, \|\nabla^{\frac{n-3}{2}}\mathrm{B}\|_{L^{\frac{n-1}{n-i_3}}(\partial E_t)}\Big)$. Then, we estimate the last term in (3.49) by means of Lemma 3.3.7 (inequality (3.43) with $\theta = 1$), so we get

$$\|\nabla^{i_3}\mathrm{H}\|^2_{L^{\frac{2n-2}{2i_3-n-1}}(\partial E_t)} \leqslant \varepsilon\|\nabla^n\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)}\,, \tag{3.50}$$

where $C = C\Big(\varepsilon, \|\nabla^{\frac{n-3}{2}}\mathrm{B}\|_{L^{\frac{n-1}{n-i_3}}(\partial E_t)}\Big)$.

As in the previous step, in the first integral in (3.48) we use Lemma 3.3.8 with $f_1 = f_2 = \mathrm{H}$, $p = 2$, $\alpha = (i_1, i_2)$ and $k = \big[\frac{n+3}{2}\big]$ (the *integer part* of (n+3)/2), noticing that $k \geqslant i_1 + i_2$. Hence, we have

$$\||\nabla^{i_1}\mathrm{H}||\nabla^{i_2}\mathrm{H}|\|^2_{L^2(\partial E_t)} \leqslant C\|\nabla^k\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)}\,. \tag{3.51}$$

Holding, by inequality (3.43) with $\theta = \frac{k-1}{n-1}$,

$$\begin{aligned} C\|\nabla^k\mathrm{H}\|^2_{L^2(\partial E_t)} &\leqslant C\|\nabla^n\mathrm{H}\|^{2\theta}_{L^2(\partial E_t)}\|\nabla\mathrm{H}\|^{2(1-\theta)}_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)} \\ &\leqslant \varepsilon\|\nabla^n\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)}\,, \end{aligned} \tag{3.52}$$

we conclude

$$\||\nabla^{i_1}\mathrm{H}||\nabla^{i_2}\mathrm{H}|\|^2_{L^2(\partial E_t)} \leqslant 2\varepsilon\|\nabla^n\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)}\,. \tag{3.53}$$

Hence,

$$\int_{\partial E_t} |\nabla^{\frac{n-3}{2}}\mathrm{B}||\nabla^{i_1}\mathrm{H}||\nabla^{i_2}\mathrm{H}||\nabla^{i_3}\mathrm{H}|\,d\mu_t \leqslant 2\varepsilon\|\nabla^n\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla\mathrm{H}\|^2_{L^2(\partial E_t)}\,,$$

for any $\varepsilon > 0$, with $C = C\Big(\varepsilon, \|\mathrm{B}\|_{L^\infty}, \|\nabla^{\frac{n-3}{2}}\mathrm{B}\|_{L^{\frac{n-1}{n-i_3}}(\partial E_t)}\Big)$.

3. If $j < \frac{n-3}{2}$, we have

$$\int_{\partial E_t} |\nabla^j\mathrm{B}||\nabla^{i_1}\mathrm{H}||\nabla^{i_2}\mathrm{H}||\nabla^{i_3}\mathrm{H}|\,d\mu_t \leqslant C\int_{\partial E_t} |\nabla^{i_1}\mathrm{H}|^2|\nabla^{i_2}\mathrm{H}|^2\,d\mu_t + C\int_{\partial E_t} |\nabla^{i_3}\mathrm{H}|^2\,d\mu_t\,,$$

where $C = C\big(\|\nabla^j\mathrm{B}\|_{L^\infty(\partial E_t)}\big)$.

Then, the last integral can be estimated by means of Lemma 3.3.7 (inequality (3.43), getting a bound as in inequality (3.50), with $C = C\big(\varepsilon, \|\nabla^j\mathrm{B}\|_{L^\infty(\partial E_t)}\big)$.

The first integral in the right hand side can be estimated by means of Lemma 3.3.8 with $f_1 = f_2 = \mathrm{H}$, $p = 2$, $\alpha = (i_1, i_2)$ and $k \geqslant i_1 + i_2 = 2n - 2 - i_3 - j$. We have two cases:

- if $i_3 = n - 1$, since $k \geqslant i_1 + i_2 = n - 1 - j$, by means of inequalities (3.51) and (3.52) we get the same conclusion as in inequality (3.53), for any $0 \leqslant j < \frac{n-3}{2}$, for a constant $C = C(\varepsilon, \|\mathrm{B}\|_{L^\infty}, \|\nabla^j\mathrm{B}\|_{L^\infty})$;

- if $i_3 = n-2$, since $k \geqslant i_1 + i_2 = n - j$ and estimate (3.52) holds for $k \leqslant n-1$, using again inequalities (3.51) and (3.52), we get the the same conclusion (3.53), for any $1 \leqslant j < \frac{n-3}{2}$, for a constant $C = C(\varepsilon, \|\mathrm{B}\|_{L^\infty}, \|\nabla^j \mathrm{B}\|_{L^\infty})$. This is sufficient to conclude the proof, as we remind that in the "polynomial" $\mathfrak{q}^{2n+2}(\nabla^{n-4}\mathrm{B}, \nabla^{n-1}\mathrm{H})$, the "monomials" $\mathrm{B} * \nabla^{i_1}\mathrm{H} * \nabla^{i_2}\mathrm{H} * \nabla^{n-2}\mathrm{H}$ are not present, that is, we do not need to deal with the case $i_3 = n-2$ and $j = 0$.

□

Proposition 3.3.11. *Let $E \subseteq \mathbb{T}^n$ be a smooth set and $E_t \in \mathfrak{C}^1_{M_E}(E)$ a surface diffusion flow. Then,*

$$\frac{d}{dt}\int_{\partial E_t} |\nabla \mathrm{H}|^2 \, d\mu_t \leqslant -2\Pi_{E_t}(\Delta \mathrm{H}) + \varepsilon \|\nabla^n \mathrm{H}\|^2_{L^2(\partial E_t)} + C_1\big(1 + \|\nabla \mathrm{H}\|^\tau_{L^2(\partial E_t)}\big)\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)}$$

$$\frac{d}{dt}\int_{\partial E_t} |\nabla^{n-2} \mathrm{H}|^2 \, d\mu_t \leqslant -\|\nabla^n \mathrm{H}\|^2_{L^2(\partial E_t)} + C_2 \|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)}$$

for any $\varepsilon > 0$, with some $\tau > 0$ and constants C_1, C_2 depending on E, M_E, ε, $\|\nabla^{n-3}\mathrm{B}\|_{L^{\frac{2n-2}{n-3}}(\partial E)}$ and $\|\mathrm{B}\|_{L^\infty(\partial E)}$.

Proof. To get the first inequality, we start estimating the second and third terms in formula (3.36) as follows,

$$C\int_{\partial E_t} |\mathrm{B}||\nabla \mathrm{H}|^2 |\nabla^2 \mathrm{H}| \, d\mu_t \leqslant C\int_{\partial E_t} |\mathrm{B}| \prod_{l=1}^{3} |\nabla^{j_l}\mathrm{H}| \, d\mu_t \leqslant C\|\mathrm{B}\|_{L^\infty(\partial E_t)} \prod_{l=1}^{3} \|\nabla^{j_l}\mathrm{H}\|_{L^{\beta_l}(\partial E_t)},$$

where we used Hölder inequality, with exponents $\beta_l = \frac{7}{j_l+1} > 2$, noticing that since $\sum_{l=1}^3 j_l = 4$, we have

$$\sum_{l=1}^{3} \frac{1}{\beta_l} = \sum_{l=1}^{3} \frac{j_l+1}{7} = 1\,.$$

Then, by the uniform interpolation inequalities (3.43), we get

$$\|\nabla^{j_l}\mathrm{H}\|_{L^{\beta_l}(\partial E_t)} \leqslant C\|\nabla^n \mathrm{H}\|^{\theta_l}_{L^2(\partial E_t)}\|\nabla \mathrm{H}\|^{1-\theta_l}_{L^2(\partial E_t)} + \|\nabla \mathrm{H}\|_{L^2(\partial E_t)}$$

with

$$\theta_l = \frac{j_l - 1}{n-1} + \frac{1}{2} - \frac{1}{\beta_l} \in \Big(\frac{j_l-1}{n-1}, 1\Big),$$

for some uniform constants C, Hence,

$$\begin{aligned} C\int_{\partial E_t} |\mathrm{B}||\nabla \mathrm{H}|^2|\nabla^2 \mathrm{H}| \, d\mu_t \leqslant & C(\|\mathrm{B}\|_{L^\infty(\partial E_t)})\Big[\|\nabla^n \mathrm{H}\|^\Theta_{L^2(\partial E_t)}\|\nabla \mathrm{H}\|^{3-\Theta}_{L^2(\partial E_t)} \\ & + \sum_{l=1}^{3} \|\nabla^4 \mathrm{H}\|^{\Theta-\theta_l}_{L^2(\partial E_t)}\|\nabla \mathrm{H}\|^{3-\Theta+\theta_l}_{L^2(\partial E_t)} \\ & + \sum_{l=1}^{3} \|\nabla^4 \mathrm{H}\|^{\theta_l}_{L^2(\partial E_t)}\|\nabla \mathrm{H}\|^{3-\theta_l}_{L^2(\partial E_t)} + \|\nabla \mathrm{H}\|^3_{L^2(\partial E_t)}\Big] \end{aligned}$$

where

$$\Theta = \sum_{l=1}^{3} \theta_l = \sum_{l=1}^{3} \frac{j_l - 1}{n-1} + \frac{3}{2} - 1 = \frac{1}{n-1} + 1/2 \leqslant 2\,,$$

as $\sum_{l=1}^{3} j_l = 4$.
Finally, by the Young inequality, we conclude

$$C\int_{\partial E_t} |\mathrm{B}||\nabla \mathrm{H}|^2|\nabla^2 \mathrm{H}|\, d\mu_t \leqslant \varepsilon\|\nabla^n \mathrm{H}\|^2_{L^2(\partial E_t)} + C\big(1+\|\nabla \mathrm{H}\|^\tau_{L^2(\partial E_t)}\big)\|\nabla \mathrm{H}\|^2_{L^2}$$

for any $\varepsilon > 0$, with $C = C(\varepsilon, \|\mathrm{B}\|_{L^\infty(\partial E_t)})$ and $\tau > 0$.
About the second inequality, recalling formula (3.37), by Lemmas 3.3.9 and 3.3.10, we have the thesis once we uniformly control with $\|\nabla^{n-3}\mathrm{B}\|_{L^{\frac{2n-2}{n-3}}(\partial F)}$ and $\|\mathrm{B}\|_{L^\infty(\partial F)}$ the following norms:

- $\|\nabla^j \mathrm{B}\|_{L^{\frac{2n-2}{2j-n+3}}(\partial E_t)}$ if $j > (n-3)/2$,
- $\|\nabla^j \mathrm{B}\|_{L^p(\partial E_t)}$, for every $1 < p < +\infty$ if $j = (n-3)/2$,
- $\|\nabla^j \mathrm{B}\|_{L^\infty(\partial E_t)}$ if $j < (n-3)/2$,

for any $j \leqslant n-3$.

According to inequality (3.40), we have

$$\|\nabla^j \mathrm{B}\|_{L^p(\partial F)} \leqslant C\Big(\|\nabla^{n-3}\mathrm{B}\|_{L^{\frac{2n-2}{n-3}}(\partial F)} + \|\mathrm{B}\|_{L^{\frac{2n-2}{n-3}}(\partial F)}\Big)^\theta \|\mathrm{B}\|^{1-\theta}_{L^\infty(\partial F)}$$

with

$$\theta = \frac{2n-2}{n-3}\Big(\frac{j}{n-1} - \frac{1}{p}\Big) \in \Big[\frac{j}{n-3}, 1\Big]$$

and a uniform constant C.
If $j > (n-3)/2$ we have the admissible case (see the conditions on θ in Proposition 3.3.4)

$$\theta = \frac{2n-2}{n-3}\Big(\frac{j}{n-1} - \frac{2j-n+3}{2n-2}\Big) = 1\,,$$

with $p = \frac{2n-2}{2j-n+3}$.
If $j = (n-3)/2$, we have

$$\frac{1}{2} \leqslant \theta = \frac{2n-2}{n-3}\Big(\frac{n-3}{2n-2} - \frac{1}{p}\Big) < 1\,,$$

holding for $p \in \big[\frac{4(n-1)}{n-3}, +\infty\big)$, then clearly also for all the smaller $p \geqslant 1$.
If $j < (n-3)/2$, taking into account Remark 1.2.4 and the discussion that precedes it, we can adapt the uniform Sobolev embeddings (Theorem 1.2.1–(iii)) to covariant tensors, as we did for instance in Proposition 3.3.4. Hence, we have

$$\|\nabla^j \mathrm{B}\|_{L^\infty(\partial E_t)} \leqslant C\Big(\|\nabla^{n-3}\mathrm{B}\|_{L^{\frac{2n-2}{n-3}}(\partial F)} + \|\mathrm{B}\|_{L^\infty}\Big)\,,$$

with a uniform constant C. □

Remark 3.3.12. Recalling Remark 3.3.6, in the proof of this proposition we could alternatively uniformly control the constants in the interpolation inequalities by a function of the quantity $\mathrm{Vol}(\partial E_t) + \|\mathrm{H}\|_{L^n(\partial E_t)}$, instead of using Proposition 3.3.4, as it is done in [47], for instance. It follows that this proposition holds also for only immersed (not boundaries of sets) smooth hypersurfaces moving by the surface diffusion flow.

3.3.3 *Compactness*

Lemma 3.3.13. *Let $E \subseteq \mathbb{T}^n$ be a smooth set and N_ε be a tubular neighborhood of ∂E. For M_E small enough and $\delta > 0$, there exists a constant $C = C(E, M_E, \delta)$ such that if $F \in \mathfrak{C}^1_{M_E}(E)$ with*

$$\partial F = \{y + \psi_F(y)\nu_E(y) \,:\, y \in \partial E\}$$

for a smooth function $\psi_F : \partial E \to \mathbb{R}$ and

$$\int_{\partial F} |\nabla^{n-2}\mathrm{H}|^2 \, d\mu + \int_{\partial F} |\nabla \mathrm{H}|^2 \, d\mu \leqslant \delta \,,$$

there hold

$$\|\mathrm{B}\|_{L^\infty(\partial F)} + \|\nabla^{n-3}\mathrm{B}\|_{L^{\frac{2n-2}{n-3}}(\partial F)} \leqslant C \qquad \text{and} \qquad \|\psi_F\|_{W^{n,2}(\partial E)} \leqslant C \,.$$

Moreover, for every $1 \leqslant p < \frac{2n-2}{n-3}$, there exists a monotone non–decreasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, depending only on E and M_E, with $\lim_{\delta\to 0^+} \omega(\delta) = 0$ and such that if F satisfies the further condition

$$\mathrm{Vol}(F \triangle E) \leqslant \delta \,,$$

then $\|\psi_F\|_{W^{n-1,p}(\partial E)} \leqslant \omega(\delta)$.
As a consequence, if $E_i \subseteq \mathfrak{C}^1_{M_E}(E)$ is a sequence of smooth sets such that

$$\sup_{i\in\mathbb{N}} \int_{\partial E_i} |\nabla^2\mathrm{H}|^{n-2} \, d\mu_i + \int_{\partial E_i} |\nabla \mathrm{H}|^2 \, d\mu_i < +\infty \,,$$

then there exists a (non necessarily smooth) set $E' \in \mathfrak{C}^1_{M_E}(E)$ such that, up to a (non relabeled) subsequence, $E_i \to E'$ in $W^{n-1,p}$ as $i \to \infty$, for all $1 \leqslant p < \frac{2n-2}{n-3}$. Moreover, if

$$\int_{\partial E_i} |\nabla^{n-2}\mathrm{H}|^2 \, d\mu_i + \int_{\partial E_i} |\nabla \mathrm{H}|^2 \, d\mu_i \to 0 \,,$$

as $i \to \infty$, the set E' is critical for the volume–constrained Area functional, that is, its mean curvature is constant.

Proof. Let $F \in \mathfrak{C}^1_{M_E}(E)$ with an associate function $\psi_F : \partial E \to \mathbb{R}$ as in the statement. We start by observing that, by the first inequality (3.43), we have

$$\|\nabla \mathrm{H}\|_{L^{\frac{2n-2}{n-3}}(\partial F)} \leqslant C\big(\|\nabla^{n-2}\mathrm{H}\|^{\theta}_{L^2(\partial F)}\|\nabla \mathrm{H}\|^{(1-\theta)}_{L^2(\partial F)} + \|\nabla \mathrm{H}\|_{L^2(\partial F)}\big) \leqslant C\sqrt{\delta} \,,$$

with $\theta = \frac{1}{n-3}$ and

$$\|\nabla \mathrm{H}\|_{L^n(\partial F)} \leqslant C\big(\|\nabla^{n-2}\mathrm{H}\|^{\theta'}_{L^2(\partial F)}\|\nabla \mathrm{H}\|^{(1-\theta')}_{L^2(\partial F)} + \|\nabla \mathrm{H}\|_{L^2(\partial F)}\big) \leqslant C\sqrt{\delta} \,,$$

with $\theta' = \frac{(n-1)(n-2)}{2n(n-3)}$.
Then, by means of the uniform Sobolev embeddings (Theorem 1.2.1–(iii)), we get

$$\|\mathrm{H} - \overline{\mathrm{H}}\|_{L^\infty(\partial F)} \leqslant C \,\|\nabla \mathrm{H}\|_{L^n(\partial F)} \leqslant C\sqrt{\delta} \tag{3.54}$$

where $\overline{\mathrm{H}} = \fint_{\partial F} \mathrm{H} \, d\mu$ and all the constants depends only on E and M_E.
By the uniform C^1–bounds on ∂F, we may find a finite family (only depending on E and M_E) of "solid" cylinders of the form $\mathcal{C}_k = D_k + \nu_E(x_k)\mathbb{R}$, with $D_k \subseteq T_{x_k}E$ a closed disk of fixed radius

$R > 0$ centered at the origin, for a finite family of points $x_k \in E$, such that $\partial F \cap \mathcal{C}_k$ is the graph on D_k of a smooth function $f_k : D_k \to \mathbb{R}$, with

$$\|f_k\|_{C^1(D_k)} \leqslant M_E \tag{3.55}$$

for every k and $\partial F = \bigcup \partial F \cap \mathcal{C}_k$.
Since we want to estimate $\int_{\partial F \cap \mathcal{C}_k} \mathrm{H}\, d\mu$, which is a "geometric" quantity, we can assume (by means of an isometry) that $T_{x_k}E = \langle e_1, \dots, e_{n-1}\rangle$, hence $\nu_E(x_k) = e_n$, in the canonical orthonormal basis of $\mathbb{R}^n$ and

$$\partial F \cap \mathcal{C}_k = \{(x, f_k(x)) \,:\, x \in D_k\}\,.$$

Then, by formulas in Remark 1.1.1 we have

$$\mathrm{H} = -\operatorname{div}\left(\frac{\nabla f_k}{\sqrt{1+|\nabla f_k|^2}}\right),$$

hence,

$$\begin{aligned}\int_{D_k} \mathrm{H}\, dx &= -\int_{D_k} \operatorname{div}\left(\frac{\nabla f_k}{\sqrt{1+|\nabla f_k|^2}}\right) dx = -\int_{\partial D_k} \left\langle \frac{\nabla f_k}{\sqrt{1+|\nabla f_k|^2}} \,\middle|\, \frac{x}{|x|} \right\rangle d\sigma \\ &= \int_{\partial D_k} \left\langle \nu_F \,\middle|\, \frac{x}{|x|} \right\rangle d\sigma\end{aligned}$$

where σ is the canonical (standard) $(n-2)$–dimensional measure on the sphere ∂D_k. Thus, being the last term at most equal to the area of the sphere ∂D_k, we get

$$\overline{\mathrm{H}}\operatorname{Vol}(D_k) = \int_{D_k} (\overline{\mathrm{H}} - \mathrm{H})\, dx + \int_{D_k} \mathrm{H}\, dx \leqslant \int_{D_k} |\mathrm{H} - \overline{\mathrm{H}}|\, dx + C \leqslant C \int_{\partial F \cap \mathcal{C}_k} |\mathrm{H} - \overline{\mathrm{H}}|\, dx + C$$

where in the last inequality we kept into account estimate (3.55) in changing the domain (and variables) of integration. Hence, controlling the last term of this inequality by estimate (3.54), it follows that $\overline{\mathrm{H}}$ is bounded by a constant depending on E, M_E, δ and the same then holds also for H. In particular, recalling that the volume of ∂F is uniformly bounded (as $F \in \mathfrak{C}^1_{M_E}(E)$), we have that $\mathrm{H} \in L^q(\partial F)$ for every $q \in [1, +\infty)$. Then, choosing M_E small enough, Theorem 1.2.3, says that we have an analogous uniform estimate on B in $L^q(\partial F)$, for every $q \in [1, +\infty)$.
Once we have a control on $\|\mathrm{B}\|_{L^q(\partial F)}$, for some exponent q larger than the dimension of the hypersurfaces, again if M_E is small enough, we have the following uniform higher order Calderón–Zygmund–type inequalities (inequalities (1.42))

$$\|\nabla^k \mathrm{B}\|_{L^2(\partial F)} \leqslant C_k\big(1 + \|\nabla^k \mathrm{H}\|_{L^2(\partial F)}\big)$$

for every $k \in \mathbb{N}$, where the constants C_k depend on E, M_E and $\|\mathrm{B}\|_{L^q(\partial F)}$ and the dimension.
It then follows

$$\|\nabla^{n-2}\mathrm{B}\|_{L^2(\partial F)} \leqslant C(E, M_E, \delta) \tag{3.56}$$

and, by inequality (3.40), we have

$$\|\nabla^{n-3}\mathrm{B}\|_{L^{\frac{2n-2}{n-3}}(\partial F)} \leqslant C\Big(\|\nabla^{n-2}\mathrm{B}\|_{L^2(\partial F)} + \|\mathrm{B}\|_{L^2(\partial F)}\Big)^{\theta} \|\mathrm{B}\|^{1-\theta}_{L^2(\partial F)}$$

with $\theta = \frac{n-1}{n-2}\big(\frac{n-3}{2n-1} - \frac{1}{2}\big) = 1$. Hence, we conclude

$$\|\mathrm{B}\|_{L^q(\partial F)} + \|\nabla^{n-3}\mathrm{B}\|_{L^{\frac{2n-2}{n-3}}(\partial F)} \leqslant C(E, M_E, \delta)\,, \tag{3.57}$$

for every $q \in [1, +\infty)$.
These geometric estimates on B and their derivatives, can be "transferred" to estimates on the

function $\psi_F : \partial E \to \mathbb{R}$, by means of the technique of localization/representation for any "graphical" hypersurface on ∂E introduced by Langer in [45] for surfaces, generalized to any dimension by Delladio [20] and fully developed in details by Breuning in the papers [8, 9, 10] (such technique is similar to the one we used to estimate $\overline{\mathrm{H}}$ above). In particular, by the results in [10], under a uniform control on $\|\mathrm{B}\|_{L^q(\partial F)}$ with q larger than the dimension of the hypersurface, we have that an estimate on $\|\mathrm{B}\|_{W^{k,p}(\partial F)}$ implies a uniform estimate on $\|\psi_F\|_{W^{k+2,p}(\partial F)}$ and viceversa, for every set $F \in \mathfrak{C}^1_{M_E}(E)$. Hence, by the previous estimates (3.56) and (3.57) on B and its derivatives, we conclude

$$\|\psi_F\|_{W^{n,2}(\partial E)} \leqslant C(E, M_E, \delta)\,.$$

Then, we notice that, by the uniform Sobolev embeddings, we have

$$\|\nabla^2\psi_F\|_{L^\infty(\partial E)} \leqslant C(E, M_E, \delta)$$

which in turn implies $\|\mathrm{B}\|_{L^\infty(\partial F)} \leqslant C(E, M_E, \delta)$, by what we said above.
Now, in the hypotheses of the lemma on a sequence of sets E_i, writing

$$\partial E_i = \{y + \psi_i(y)\nu_E(y) \,:\, y \in \partial E\}\,,$$

by the previous estimates and the uniform Sobolev compact embeddings

$$W^{n,2}(\partial E) \hookrightarrow W^{n-1,p}(\partial E) \hookrightarrow C^1(\partial E)$$

for all $1 \leqslant p < \frac{2n-2}{n-3}$, up to a (not relabeled) subsequence there exists a set $E' \in \mathfrak{C}^1_{M_E}(E)$ such that $\psi_i \to \psi_{E'}$ in $W^{n-1,p}(\partial E)$ (and in $C^1(\partial E)$) where

$$\partial E' = \{y + \psi_{E'}(y)\nu_E(y) \,:\, y \in \partial E\}\,,$$

for all $1 \leqslant p < \frac{2n-2}{n-3}$.
If actually

$$\int_{\partial E_i} |\nabla^{n-2}\mathrm{H}|^2\,d\mu_i + \int_{\partial E_i} |\nabla \mathrm{H}|^2\,d\mu_i \to 0\,,$$

clearly for the limit set E' the mean curvature must be constant.

The fact that $\|\psi_F\|_{W^{n-1,p}(\partial E)}$ goes uniformly to zero as $\delta \to 0$, hence we have a function ω as in the statement, follows by the fact that, assuming $F_i \in \mathfrak{C}^1_{M_E}(E)$ and

$$\mathrm{Vol}(F_i \triangle E) \leqslant \delta_i\,, \qquad\qquad \int_{\partial F_i} |\nabla^{n-1}\mathrm{H}|^2\,d\mu_i + \int_{\partial F_i} |\nabla \mathrm{H}|^2\,d\mu_i \leqslant \delta_i$$

with $\delta_i \to 0$, as $i \to \infty$, by the previous argument we have that $\psi_{F_i} : \partial E \to \mathbb{R}$ converges to some $\psi : \partial E \to \mathbb{R}$ in $W^{n-1,p}(\partial E)$, hence in $L^1(\partial E)$, while the limit $\mathrm{Vol}(F_i \triangle E) \to 0$ implies that $\|\psi_{F_i}\|_{L^1(\partial E)} \to 0$, then we conclude that ψ must be zero and we have the thesis. □

3.3.4 *Global existence and stability – I*

Theorem 3.3.14. *Let $E \subseteq \mathbb{T}^n$, for $n \geqslant 3$, be a strictly stable critical set for the Area functional under a volume constraint and let N_ε be a tubular neighborhood of ∂E. For $M_E < \varepsilon/2$ small enough, there exists $\delta > 0$ such that, if E_0 is a smooth set in $\mathfrak{C}^1_{M_E}(E)$ satisfying $\mathrm{Vol}(E_0) = \mathrm{Vol}(E)$ and*

$$\mathrm{Vol}(E_0 \triangle E) \leqslant \delta \qquad \textit{and} \qquad \int_{\partial E_0} |\nabla^{n-2}\mathrm{H}|^2\,d\mu_0 + \int_{\partial E_0} |\nabla \mathrm{H}|^2\,d\mu_0 \leqslant \delta\,, \tag{3.58}$$

then, the unique smooth surface diffusion flow E_t starting from E_0, given by Proposition 3.2.2, is defined for all $t \geqslant 0$. Moreover, E_t converges smoothly to $E' = E + \eta$ exponentially fast as $t \to +\infty$, for some

$\eta \in \mathbb{R}^n$, with the meaning that the sequence of smooth functions $\psi_t : \partial E \to \mathbb{R}$ representing ∂E_t as "normal graphs" on ∂E, that is,

$$\partial E_t = \{y + \psi_t(y)\nu_E(y) \,:\, y \in \partial E\},$$

satisfy, for every $k \in \mathbb{N}$,

$$\|\psi_t - \psi\|_{C^k(\partial E)} \leqslant C_k e^{-\beta_k t}\,,$$

for every $t \in [0, +\infty)$, for some positive constants C_k and β_k, where $\psi : \partial E \to \mathbb{R}$ represents $\partial E' = \partial E + \eta$ as a "normal graph" on ∂E.

Remark 3.3.15. The request that E_0 belongs to $\mathfrak{C}^1_{M_E}(E)$ with M_E small enough, is necessary only in order to be able to represent its boundary as a graph of a function with bounded gradient on ∂E and to have uniform Sobolev, interpolation and Calderón–Zygmund inequalities, as proved in Section 1.2, while the first condition (3.58) is a "closedness" assumption in L^1 for E_0 and E (that is, on ψ_0). The second "small energy" condition (3.58) in the theorem implies (see the last part of Lemma 3.3.13 and its proof) that the mean curvature of ∂E_0 is "close" to be constant, as it is for the strictly stable set E (actually for any critical set). Notice that this latter is a condition "of order n" for the boundary of E_0 and that all these assumptions are clearly implied by an appropriate $W^{n,2}$–closedness of ∂E to ∂E, arguing as in Lemma 3.3.13.

Before showing the proof of Theorem 3.3.14, we recall the following lemma, which is Proposition 2.2.16 under stronger assumptions.

Lemma 3.3.16. *Let $E \subseteq \mathbb{T}^n$ be a strictly stable critical set for the Area functional under a volume constraint. For every $\theta \in (0, 1]$ there exist a constant $\sigma_\theta > 0$ such that if $F \in \mathfrak{C}^1_{2M_E}(E)$ satisfies*

$$\mathrm{Vol}(F \triangle E) \leqslant \delta_0 \qquad \text{and} \qquad \int_{\partial F} |\nabla \mathrm{H}|^2 \, d\mu \leqslant \delta_0\,, \tag{3.59}$$

for $\delta_0 > 0$ small enough, there holds

$$\Pi_F(\psi) \geqslant \sigma_\theta \|\psi\|^2_{L^2(\partial F)},$$

for all $\psi \in \widetilde{H}^1(\partial F)$ satisfying

$$\min_{\eta \in \mathrm{O}_E} \|\psi - \langle \eta \,|\, \nu_F \rangle\|_{L^2(\partial F)} \geqslant \theta \|\psi\|_{L^2(\partial F)}$$

where the vector subspace $\mathrm{O}_E \subseteq \mathbb{R}^4$ was defined in formula (2.24).

Proof. Representing the boundary of $F \in \mathfrak{C}^1_{2M_E}(E)$ as $\partial F = \{y + \psi_F(y)\nu_E(y) \,:\, y \in \partial E\}$ for a smooth function $\psi_F : \partial E \to \mathbb{R}$, according to Proposition 2.2.16, fixed some $p > n - 1$, there exists a positive constant $C = C(\theta, p)$ such that the conclusion follows if $\|\psi_F\|_{W^{2,p}(\partial E)} \leqslant C$. This inequality follows if conditions (3.59) hold with δ_0 small enough, by the properties of the function ω stated in Lemma 3.3.13 (and Sobolev embeddings). □

Proof of Theorem 3.3.14. By choosing M_E small enough, we assume that for every set $F \in \mathfrak{C}^1_{2M_E}(E)$, all the constants in the inequalities we are going to consider for functions on ∂F are uniform, depending on E and M_E, as it is shown in Section 1.2.
After choosing some small $\delta_0 > 0$, we consider the surface diffusion flow E_t starting from $E_0 \in \mathfrak{C}^1_{M_E}(E)$ satisfying

$$\mathrm{Vol}(E_0 \triangle E) \leqslant \delta \qquad \text{and} \qquad \int_{\partial E_0} |\nabla^{n-2}\mathrm{H}|^2 \, d\mu_0 + \int_{\partial E_0} |\nabla \mathrm{H}|^2 \, d\mu_0 \leqslant \delta\,,$$

for $\delta < \delta_0/2$ and we let $T(E_0) \in (0, +\infty]$ be the maximal time such that the flow is defined for t in the interval $[0, T(E_0))$, $E_t \in \mathfrak{C}^1_{2M_E}(E)$,

$$\mathrm{Vol}(E_t \triangle E) \leqslant \delta_0 \qquad \text{and} \qquad \mathcal{F}(t) = \int_{\partial E_t} |\nabla^{n-2}\mathrm{H}|^2 \, d\mu_t + \int_{\partial E_t} |\nabla \mathrm{H}|^2 \, d\mu_t \leqslant \delta_0 \, .$$

All the moving boundaries ∂E_t can be represented as normal graphs on ∂E as

$$\partial E_t = \{y + \psi_t(y)\nu_E(y) \, : \, y \in \partial E\}$$

for some smooth functions $\psi_t : \partial E \to \mathbb{R}$. Moreover, if $T(E_0) < +\infty$, then at least one of the three following conditions must hold:

- $\limsup_{t \to T(E_0)} \|\psi_t\|_{C^1(\partial E)} = 2M_E$
- $\limsup_{t \to T(E_0)} \mathcal{F}(t) = \delta_0$
- $\limsup_{t \to T(E_0)} \mathrm{Vol}(E_t \triangle E) = \delta_0$

otherwise, restarting the flow from a time $\bar{t}$ close enough to $T(E_0)$ by means of Proposition 3.2.2, we have the contradiction that $T(E_0)$ cannot be the maximal time defined above. Indeed, the time interval of smooth existence of the flow given by such proposition is bounded below by a constant depending on the $C^{2,\alpha}$–norm of $\psi_{\bar{t}}$ and this latter by a constant depending on δ_0, by the first point of Lemma 3.3.13 and Sobolev (uniform) embeddings.
We are going to show that if δ_0 was chosen small enough, there exists $\delta > 0$ such that none of these conditions can occur, hence $T(E_0) = +\infty$, that is, the surface diffusion flow of E_0 exists for all time.

Let us define, for $K > 2$, the following "energy" function

$$\mathcal{E}(t) = \int_{\partial E_t} |\nabla^{n-2}\mathrm{H}|^2 \, d\mu_t + K \int_{\partial E_t} |\nabla \mathrm{H}|^2 \, d\mu_t \geqslant \mathcal{F}(t)$$

(notice that also holds $\mathcal{E}(t) \leqslant K\mathcal{F}(t)$). From Lemma 3.3.13 we easily have

$$\|\mathrm{B}\|_{L^\infty(\partial E_t)} + \|\nabla^{n-3}\mathrm{B}\|_{L^{\frac{2n-2}{n-3}}(\partial E_t)} \leqslant S_0(\mathcal{F}(t)) \leqslant S_0(\mathcal{E}(t)) \, , \tag{3.60}$$

for $t \in [0, T(E_0))$, where the function $S_0 : [0, +\infty) \to \mathbb{R}^+$ is continuous and monotone non–decreasing and it is determined by E and M_E.

We now split the rest of the proof into steps. Our first goal will be to show that the function $\mathcal{E}$ decreases in time if δ is small enough, for an appropriate constant K.

Step 1 (*Monotonicity of $\mathcal{E}$*).
By Proposition 3.3.11, for any $t \in [0, T(E_0))$, we have

$$\frac{d}{dt}\mathcal{E}(t) \leqslant -2K\Pi_{E_t}(\Delta \mathrm{H}) - (1 - \varepsilon K)\|\nabla^n \mathrm{H}\|^2_{L^2(\partial E_t)} + (C_2 + KC_1 + KC_1\|\nabla \mathrm{H}\|^\tau_{L^2(\partial E_t)})\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)} \, ,$$

for any $\varepsilon > 0$, $\tau > 0$ and some constants C_1, C_2 depending on E, M_E, ε, $\|\nabla^{n-3}\mathrm{B}\|_{L^{\frac{2n-2}{n-3}}(\partial E)}$ and $\|\mathrm{B}\|_{L^\infty(\partial E)}$. Then, choosing $\varepsilon = 1/2K$, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) \leqslant & -2K\Pi_{E_t}(\Delta \mathrm{H}) - \frac{1}{2}\|\nabla^n \mathrm{H}\|^2_{L^2(\partial E_t)} \\ & + \big(S_2(\mathcal{E}(t)) + KS_1(\mathcal{E}(t)) + KS_1(\mathcal{E}(t))\|\nabla \mathrm{H}\|^\tau_{L^2(\partial E_t)}\big)\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)} \, ,\end{aligned} \tag{3.61}$$

where $S_1, S_2 : [0, +\infty) \to \mathbb{R}^+$ are two continuous, monotone non–decreasing functions depending on E, M_E, by inequality (3.60).
By inequality (3.43) in Lemma 3.3.7 (with $\varepsilon = 1/2$), we have

$$\|\nabla^{n-2}\mathrm{H}\|^2_{L^2(\partial E_t)} \leqslant \frac{1}{2}\|\nabla^n \mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)}\,,$$

that is,

$$-\frac{1}{2}\|\nabla^n \mathrm{H}\|^2_{L^2(\partial E_t)} \leqslant -\|\nabla^{n-2}\mathrm{H}\|^2_{L^2(\partial E_t)} + C\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)}$$

and substituting into inequality (3.61), we get (recalling that $K > 2$)

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) \leqslant & -2K\Pi_{E_t}(\Delta \mathrm{H}) - \|\nabla^{n-2}\mathrm{H}\|^2_{L^2(\partial E_t)} \\
& + \big(S_2(\mathcal{E}(t)) + KS_1(\mathcal{E}(t)) + KS_1(\mathcal{E}(t))\|\nabla \mathrm{H}\|^\tau_{L^2(\partial E_t)}\big)\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)} \\
\leqslant & -2K\Pi_{E_t}(\Delta \mathrm{H}) - \|\nabla^{n-2}\mathrm{H}\|^2_{L^2(\partial E_t)}/K - \|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)} \\
& + \big(1 + S_2(\mathcal{E}(t)) + KS_1(\mathcal{E}(t)) + KS_1(\mathcal{E}(t))\|\nabla \mathrm{H}\|^\tau_{L^2(\partial E_t)}\big)\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)} \\
= & -2K\Pi_{E_t}(\Delta \mathrm{H}) - \mathcal{E}(t)/K \\
& + \big(1 + S_2(\mathcal{E}(t)) + KS_1(\mathcal{E}(t)) + KS_1(\mathcal{E}(t))\|\nabla \mathrm{H}\|^\tau_{L^2(\partial E_t)}\big)\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)}\,.
\end{aligned}$$

If we assume that, for every $t \in [0, T(E_0))$, there holds

$$\Pi_{E_t}(\Delta \mathrm{H}) \geqslant \sigma\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)}\,, \tag{3.62}$$

for some constant $\sigma > 0$, then

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) \leqslant & -\big[2K\sigma - 1 - S_2(\mathcal{E}(t)) - KS_1(\mathcal{E}(t)) - KS_1(\mathcal{E}(t))\|\nabla \mathrm{H}\|^\tau_{L^2(\partial E_t)}\big]\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)} \\
& - \mathcal{E}(t)/K \\
\leqslant & -\big[2K\sigma - S(\mathcal{E}(t))(1 + K + K^{1-\tau/2}\mathcal{E}(t)^{\tau/2})\big]\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)} - \mathcal{E}(t)/K\,,
\end{aligned}$$

with $S = \max\{S_1, S_2 + 1\} : [0, +\infty) \to \mathbb{R}^+$ continuous, monotone non–decreasing and depending on E and M_E.
Hence,

$$\frac{d}{dt}\mathcal{E}(t) \leqslant -P(\mathcal{E}(t)\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)} - \mathcal{E}(t)/K\,,$$

with $P(s) = 2K\sigma - (1+K)S(s) - S(s)K^{1-\tau/2}s^{\tau/2}$, which is a continuous and monotone non–increasing function, determined by E and M_E.
It is then an exercise of qualitative analysis of ordinary differential inequalities, to conclude that if $P(0)$ is positive, the first term starts and stays negative and the "energy" $\mathcal{E}$ satisfies

$$\frac{d}{dt}\mathcal{E}(t) \leqslant -\mathcal{E}(t)/K \tag{3.63}$$

for every $t \in [0, T(E_0))$, that is, the function $\mathcal{E}$ is never increasing, so it remains bounded by $\mathcal{E}(0)$ (moreover, it decreases exponentially and converges to zero, as $t \to +\infty$, if the flow is "eternal"). Thus, choosing an appropriate constant K, by the definition of the function S, it is easy to see that we can make $P(0) > 0$, hence if $\delta > 0$ is small, since $\mathcal{E}(0) \leq K\mathcal{F}(0) \leq \delta K$, it follows that $\mathcal{E}(0)$ is small enough and we have the above conclusion.

Step 2 (*Proof of estimate* (3.62)).
We now want to apply Lemma 3.3.16 with $F = E_t$ and $\varphi = \Delta \mathrm{H}$, for all $t \in [0, T(E_0))$, hence, we need to show that there exists a small constant $\theta > 0$ such that

$$\min_{\eta \in \mathrm{O}_E} \|\Delta \mathrm{H} - \langle \eta \,|\, \nu_t \rangle\|_{L^2(\partial E_t)} \geqslant \theta\|\Delta \mathrm{H}\|_{L^2(\partial E_t)} \qquad \text{for all } t \in [0, T(E_0))\,. \tag{3.64}$$

Considering the special basis $\{e_i\}$ of $\mathbb{R}^n$ and the associated set $i \in \mathrm{I}_E$ in the discussion just after Definition 2.2.5, by the properties of the function ω stated in Lemma 3.3.13, if δ_0 is small enough we have that for every $t \in [0, T(E_0))$ the norm $\|\psi_F\|_{W^{n,2}(\partial E)}$ is small, hence the same holds for $\|\psi_F\|_{C^1(\partial E)}$. Then, it follows that there exists a constant $C_0 = C_0(E, M_E) > 0$ such that, for every $i \in \mathrm{I}_E$, we have $\|\langle e_i \,|\, \nu_t\rangle\|_{L^2(\partial E_t)} \geqslant C_0 > 0$, holding $\|\langle e_i \,|\, \nu_E\rangle\|_{L^2(\partial E)} > 0$ (notice that this argument also shows that, with an appropriate choice of small δ_0 and δ, the condition $\limsup_{t\to T(E_0)} \|\psi_t\|_{C^1(\partial E)} = 2M_E$ cannot occur). It is then easy to see that the vector $\eta_t \in \mathrm{O}_E$ realizing the above minimum for E_t is unique and satisfies

$$\Delta \mathrm{H} = \langle \eta_t \,|\, \nu_t\rangle + g, \tag{3.65}$$

where $g \in L^2(\partial E_t)$ is a function L^2–orthogonal (with respect to the measure μ_t on ∂E_t) to the vector subspace of $L^2(\partial E_t)$ spanned by the functions $\langle e_i \,|\, \nu_t\rangle$. Moreover, letting $\eta_t = \eta_t^i e_i$, from relation (3.64) we have

$$\|\Delta \mathrm{H}\|^2_{L^2(\partial E_t)} \geqslant \|\langle \eta_t \,|\, \nu_t\rangle\|^2_{L^2(\partial E_t)} = \int_{\partial E_t} |\eta_t^i \langle e_i \,|\, \nu_t\rangle|^2 \, d\mu_t \geqslant C_0^2 |\eta_t^i|^2 = C|\eta_t|^2\,, \tag{3.66}$$

where C is a constant depending only on E and M_E.
We now argue by contradiction, assuming $\|g\|_{L^2(\partial E_t)} < \theta \|\Delta \mathrm{H}\|_{L^2(\partial E_t)}$.
We recall that, thanks to the uniform Poincaré inequality (3.42), we have

$$\int_{\partial E_t} |\mathrm{H} - \overline{\mathrm{H}}|^2 \, d\mu_t \leqslant C \int_{\partial E_t} |\nabla \mathrm{H}|^2 \, d\mu_t \leqslant C \|\Delta \mathrm{H}\|^2_{L^2(\partial E_t)} \tag{3.67}$$

where the second estimate can be obtained integrating by parts and using the Cauchy–Schwarz inequality.
Hence, by multiplying relation (3.65) by $\mathrm{H} - \overline{\mathrm{H}}$ and integrating over ∂E_t, we get

$$\begin{aligned}\left| \int_{\partial E_t} (\mathrm{H} - \overline{\mathrm{H}}) \Delta \mathrm{H} \, d\mu_t \right| &= \left| \int_{\partial E_t} (\mathrm{H} - \overline{\mathrm{H}}) g \, d\mu_t \right| \\ &< \theta \|\mathrm{H} - \overline{\mathrm{H}}\|_{L^2(\partial E_t)} \|\Delta \mathrm{H}\|_{L^2(\partial E_t)} \\ &\leqslant C\theta \|\Delta \mathrm{H}\|^2_{L^2(\partial E_t)}\,,\end{aligned} \tag{3.68}$$

where the equality follows from the identities

$$\int_{\partial E_t} \mathrm{H}\, \nu_t \, d\mu_t = 0 \qquad \text{and} \qquad \int_{\partial E_t} \nu_t \, d\mu_t = 0$$

holding for every embedded hypersurface. Then, recalling estimate (3.66) and the fact that g is L^2–orthogonal to $\langle \eta_t \,|\, \nu_t\rangle$, we have

$$\begin{aligned}\|\langle \eta_t \,|\, \nu_t\rangle\|^2_{L^2(\partial E_t)} &= \int_{\partial E_t} \Delta \mathrm{H} \langle \eta_t \,|\, \nu_t\rangle \, d\mu_t \\ &= -\int_{\partial E_t} \langle \nabla \mathrm{H} \,|\, \nabla \langle \eta_t \,|\, \nu_t\rangle\rangle \, d\mu_t \\ &\leqslant |\eta_t| \|\nabla \nu_t\|_{L^2(\partial E_t)} \|\nabla \mathrm{H}\|_{L^2(\partial E_t)} \\ &\leqslant C \|\Delta \mathrm{H}\|_{L^2(\partial E_t)} \|\nabla \nu_t\|_{L^2(\partial E_t)} \left| \int_{\partial E_t} (\mathrm{H} - \overline{\mathrm{H}}) \Delta \mathrm{H} \, d\mu_t \right|^{1/2} \\ &\leqslant C\sqrt{\theta} \|\Delta \mathrm{H}\|^2_{L^2(\partial E_t)}\,,\end{aligned}$$

where in the last inequality we used relation (3.68) and we estimated $\|\nabla \nu_t\|_{L^2(\partial E_t)}$ by inequality (3.60) and the fact that $\mathcal{F}(t) \leqslant \delta_0$, as $\nabla \nu_t = \mathrm{B}$ by the Gauss–Weingarten relations (1.7).

If then $\theta > 0$ is chosen so small that $C\sqrt{\theta} < 1 - \theta^2$ in the last inequality, we have a contradiction since equality (3.65) and the fact that $\|g\|_{L^2(\partial E_t)} < \theta \|\Delta \mathrm{H}\|_{L^2(\partial E_t)}$ imply (by L^2–orthogonality) that

$$\|\langle \eta_t \,|\, \nu_t \rangle\|^2_{L^2(\partial E_t)} > (1-\theta^2)\|\Delta \mathrm{H}\|^2_{L^2(\partial E_t)}\,.$$

All this argument shows that with such a suitable choice of θ, condition (3.64) holds, hence by Lemma 3.3.16, we conclude

$$\Pi_{E_t}(\Delta \mathrm{H}) \geqslant \sigma_\theta \|\Delta \mathrm{H}\|^2_{L^2(\partial E_t)} \qquad \text{for all } t \in [0, T(E_0)).$$

Then, the second estimate (3.67) clearly proves assumption (3.62) and the proof of monotonicity of $\mathcal{E}$ in Step 1 is concluded. Hence, if δ is small enough, $\mathcal{E}(t)$ remains bounded by δ during the flow, up to the time $t = T(E_0)$, thus the same clearly holds for $\mathcal{F}(t)$.

Step 3 (*Global existence of the flow*).
We have seen at Step 1 that choosing an appropriate constant K, if δ is small enough, then the "energy" $\mathcal{E}(t)$ is uniformly bounded and decreasing. More precisely, integrating the differential inequality (3.63), there holds

$$\mathcal{E}(t) \leqslant \mathcal{E}(0)e^{-t/K} \leqslant \delta e^{-t/K} \leqslant \delta \tag{3.69}$$

hence, we also have $\mathcal{F}(t) \leqslant \delta e^{-t/K} \leqslant \delta$, for every $t \in [0, T(E_0))$.
Moreover, at Step 2 we already saw that if δ_0 is chosen small enough,

$$\limsup_{t \to T(E_0)} \|\psi_t\|_{C^1(\partial E)} = 2M_E$$

is not possible. Hence, in order to obtain the global existence of the flow, we only have to show that also

$$\limsup_{t \to T(E_0)} \mathrm{Vol}(E_t \triangle E) = \delta_0 \tag{3.70}$$

cannot occur.

We define the following quantity

$$D(t) = \int_{E_t \triangle E} d(x, \partial E)\, dx = \int_{E_t} d_E(x)\, dx - \int_E d_E(x)\, dx, \tag{3.71}$$

where $d_E : N_\varepsilon \to \mathbb{R}$ is the signed distance function defined in formula (1.14). We observe that,

$$\mathrm{Vol}(E_t \triangle E) \leqslant C\|\psi_t\|_{L^1(\partial E)} \leqslant C\|\psi_t\|_{L^2(\partial E)}$$

and

$$\begin{aligned}
\|\psi_t\|^2_{L^2(\partial E)} &= 2\int_{\partial E}\int_0^{|\psi_t(y)|} t\, dt\, d\mu(y) \\
&= 2\int_{\partial E}\int_0^{|\psi_t(y)|} d(L(y,t), \partial E)\, dt\, d\mu(y) \\
&= 2\int_{E_t \triangle E} d(x, \partial E)\, JL^{-1}(x)\, dx \\
&\leqslant CD(t)\,,
\end{aligned}$$

where the constants depend on E and M_E, $L : \partial E \times (-\varepsilon, \varepsilon) \to N_\varepsilon$ is the smooth diffeomorphism defined in formula (1.49) and JL is its Jacobian. It clearly follows

$$\mathrm{Vol}(E_t \triangle E) \leqslant C\|\psi_t\|_{L^1(\partial E)} \leqslant C\|\psi_t\|_{L^2(\partial E)} \leqslant C\sqrt{D(t)}\,, \tag{3.72}$$

and

$$D(t) \leqslant \int_{E_t \triangle E} 2M_E \, dx = 2M_E \mathrm{Vol}(E_t \triangle E)\,. \tag{3.73}$$

Then, recalling formula (3.71), we compute

$$\frac{d}{dt} D(t) = \frac{d}{dt} \int_{E_t \triangle E} d(x, \partial E)\, dx = \int_{\partial E_t} d_E \, \Delta \mathrm{H} \, d\mu_t \leqslant C \|\Delta \mathrm{H}\|_{L^2(\partial E_t)} \leqslant C\sqrt{\delta}\, e^{-t/2K},$$

for all $t \leqslant T(E_0)$, where the last inequality clearly follows from the above estimate (3.69) for $\mathcal{E}(t)$. By integrating this differential inequality on $[0, \bar{t})$ with $\bar{t} \in [0, T(E_0))$ and taking into account estimate (3.72), we get

$$\mathrm{Vol}(E_{\bar{t}} \triangle E) \leqslant C \|\psi_{\bar{t}}\|_{L^2(\partial E)} \leqslant C \sqrt{D(0) + 2KC\sqrt{\delta}} \leqslant C \sqrt[4]{\delta}\,,$$

as $D(0) \leqslant C\mathrm{Vol}(E_0 \triangle E) \leqslant C\delta$, by inequality (3.73) with $t = 0$. Hence, if $\delta > 0$ is small enough such that $C\sqrt[4]{\delta} < \delta_0$, we have that also condition (3.70) cannot happen.

We conclude that the surface diffusion flow of E_0 exists smooth for every time, moreover $E_t \in \mathfrak{C}^1_{2M_E}(E)$ and

$$\mathrm{Vol}(E_t \triangle E) \leqslant C\sqrt[4]{\delta}\,, \qquad \int_{\partial E_t} |\nabla^{n-2}\mathrm{H}|^2 \, d\mu_t + \int_{\partial E_t} |\nabla \mathrm{H}|^2 \, d\mu_t \leqslant \delta e^{-t/K}\,, \tag{3.74}$$

for every $t \in [0, +\infty)$.

Step 4 (*Convergence, up to a subsequence, to a translate of E*).
Let $t_i \to +\infty$, then by estimates (3.74), the sets E_{t_i} satisfy the hypotheses of the last point of Lemma 3.3.13, hence, up to a (not relabeled) subsequence, we have that there exists a critical set $E' \in \mathfrak{C}^1_{2M_E}(E)$ such that $E_{t_i} \to E'$ in $W^{n-1,p}$ for $p < \frac{2n-2}{n-3}$, that is $\|\psi_{t_i} - \psi\|_{W^{n-1,p}(\partial E)} \to 0$ for some $\psi : \partial E \to \mathbb{R}$ representing $\partial E'$ as a "normal graph" on ∂E. As $\partial E'$ has constant mean curvature and it is a graph over ∂E of a C^1 function (by Sobolev embeddings), it follows by standard regularity theory for quasilinear equations that it is smooth (see [34] for instance), then by Proposition 2.2.17, we have that $E' = E + \eta$ for some (small) $\eta \in \mathbb{R}^n$. Such proposition actually states that E is a strict local minimum for the volume–constrained Area functional, up to translations and that a smooth set "close enough" to E (as E' in our situation) can be a critical set if and only if it is a translate of E.

Step 5 (*Smooth exponential convergence of the full sequence*).
Arguing similarly as above, we consider the function

$$\overline{D}(t) = \int_{E_t \triangle E'} d(x, \partial E)\, dx$$

with derivative

$$\frac{d}{dt}\overline{D}(t) = \frac{d}{dt}\int_{E_t \triangle E'} d(x, \partial E)\, dx = \int_{\partial E_t} \mathrm{sgn}(\psi_t - \psi)\, d_{\partial E}\, \Delta \mathrm{H}\, d\mu_t\,, \tag{3.75}$$

where sgn is the "sign function". By the exponential second estimate (3.74) and the fact that $E_t \in \mathfrak{C}^1_{2M_E}(E)$, we have

$$\left|\frac{d}{dt}\overline{D}(t)\right| \leqslant C\|\Delta \mathrm{H}\|_{L^2(\partial E_t)} \leqslant C\sqrt{\delta}\, e^{-t/2K}$$

for all $t \geqslant 0$, moreover,

$$\overline{D}(t) \leqslant \int_{E_t \triangle E'} 2M_E\, dx = 2M_E \mathrm{Vol}(E_t \triangle E') \leqslant C\|\psi_t - \psi\|_{L^1(\partial E)} \leqslant C\|\psi_t - \psi\|_{L^2(\partial E)}$$

which implies $\overline{D}(t_i) \to 0$, as $i \to \infty$, by the previous step.
Integrating the differential inequality (3.75), we get

$$\begin{aligned}\overline{D}(t) - \overline{D}(t_i) &= -\int_t^{t_i} \frac{d}{ds}\overline{D}(s)\,ds \leqslant \int_t^{+\infty} \left|\frac{d}{ds}\overline{D}(s)\,ds\right| \leqslant \int_t^{+\infty} C\sqrt{\delta}e^{-s/2K}\,ds \\ &\leqslant 2CK\sqrt{\delta}e^{-t/2K}\,,\end{aligned}$$

hence, passing to the limit as $i \to \infty$, we conclude

$$\overline{D}(t) \leqslant Ce^{-t/2K}$$

for every $t \geqslant 0$, thus $\lim_{t\to+\infty} \overline{D}(t) = 0$. Then, we have

$$\begin{aligned}\|\psi_t - \psi\|^2_{L^2(\partial E)} &= 2\int_{\partial E}\left|\int_{\psi(y)}^{\psi_t(y)} s\,ds\right| d\mu(y) \\ &= 2\int_{\partial E}\left|\int_{\psi(y)}^{\psi_t(y)} d(L(y,s),\partial E)\,ds\right| d\mu(y) \\ &= 2\int_{E_t \triangle E'} d(x,\partial E)\,JL^{-1}(x)\,dx \\ &\leqslant C\overline{D}(t) \\ &\leqslant Ce^{-t/2K}\,,\end{aligned}$$

where $L : \partial E \times (-\varepsilon, \varepsilon) \to N_\varepsilon$ is, as before, the smooth diffeomorphism defined in formula (1.49) with Jacobian JL. By this exponential decay and the uniform bound on $\|\psi_t - \psi\|_{W^{n,2}(\partial E)}$ following from estimates (3.74) by means of Lemma 3.3.13, we obtain the convergence of the full sequence E_t to E' in $W^{n-1,p}$.
Finally, we have that the convergence of $E_t \to E + \eta$ is actually exponentially smooth, by arguing as in the proof of Theorem 5.1 in [30] (see also [16]), that is, via standard parabolic estimates and the uniform interpolation inequalities (and Sobolev embeddings), holding the exponential convergence in $W^{n-1,p}$.

□

3.4 LONG–TIME BEHAVIOR – II

As in the previous section, we aim to study the evolution by surface diffusion of normal deformations of a strictly stable set E. In this second line, the main tool will be a generalization of a quantitative version of Alexandrov theorem.

3.4.1 *A quantitative generalized Alexandrov theorem*

The following is a famous and "classical" theorem due to Alexandrov (see the original paper [4] for a complete and detailed proof, for instance).

Theorem 3.4.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set of class C^2. Then, $\mathrm{H}_{\partial\Omega}$ is constant if and only if Ω is a ball.*

A quantitative version of this result was first proven in [44, Theorem 1.10] and then rephrased in [51, Theorem 1.3] as follows.

Theorem 3.4.2 ([51], Theorem 1.3)**.** *There exist $\delta \in (0, 1/2)$ and $C > 0$ with the following property: for any $\psi \in C^1(\partial B) \cap H^2(\partial B)$ such that $\|\psi\|_{C^1(\partial B)} \leqslant \delta$, $\mathrm{Vol}(E_\psi) = \omega_n$ and $\mathrm{Bar}(E_\psi) = 0$, we have*

$$\|\psi\|_{H^1(\partial B)} \leqslant C\|\mathrm{H} - \overline{\mathrm{H}}\|_{L^2(\partial B)}\,,$$

where $\mathrm{Bar}(E_\psi)$ *and* H *are, respectively, the barycenter of* E_ψ *and the mean curvature of* ∂E_ψ *and* $\overline{\mathrm{H}} = \fint_{\partial B} \mathrm{H}\, d\mathcal{H}^{n-1}$.

Later on, in [17] the authors showed that in the periodic setting the above quantitative estimate holds with B replaced by any strictly stable critical set. We call the conclusion in the following theorem *quantitive generalized Alexandrov inequality*.

Theorem 3.4.3 ([17], Theorem 1.3)**.** *Let* $E \subseteq \mathbb{T}^n$ *be a strictly stable critical set. Then, there exist* $\delta^* \in (0, 1/2)$ *and* $C > 0$ *with the following property: for any* $\psi \in C^1(\partial E) \cap H^2(\partial E)$ *such that* $\|\psi\|_{C^1(\partial E)} \leqslant \delta^*$ *and satisfying*

$$\left| \int_{\partial E} \psi \, d\mu \right| \leqslant \delta^* \|\psi\|_{L^2(\partial E)} , \qquad \left| \int_{\partial E} \psi \nu_E \, d\mu \right| \leqslant \delta^* \|\psi\|_{L^2(\partial E)} \tag{3.76}$$

we have

$$\|\psi\|_{H^1(\partial E)} \leqslant C \|\mathrm{H}_{E_\psi} - \overline{\mathrm{H}}_{E_\psi}\|_{L^2(\partial E)} .$$

Moreover, as it is shown in [17, Section 3], the first condition (3.76) can be replaced with the equality $\mathrm{Vol}(E_\psi) = \mathrm{Vol}(E)$.

Theorem 3.4.4. *Let* $E \subseteq \mathbb{T}^n$ *be a strictly stable critical set. Then, there exist* $\delta^* \in (0, 1/2)$ *and* $C > 0$ *with the following property: for any* $\psi \in C^1(\partial E) \cap H^2(\partial E)$ *such that* $\|\psi\|_{C^1(\partial E)} \leqslant \delta^*$ *and satisfying*

$$\mathrm{Vol}(E_\psi) = \mathrm{Vol}(E) , \qquad \left| \int_{\partial E} \psi \nu_E \, d\mu \right| \leqslant \delta^* \|\psi\|_{L^2(\partial E)} ,$$

we have

$$\|\psi\|_{H^1(\partial E)} \leqslant C \|\mathrm{H}_{E_\psi} - \overline{\mathrm{H}}_{E_\psi}\|_{L^2(\partial E)} . \tag{3.77}$$

Finally, we notice that inequality (3.77) implies

$$\|\psi\|_{H^1(\partial E)} \leqslant C \|\mathrm{H}_{E_\psi} - \lambda\|_{L^2(\partial E)} ,$$

for any $\lambda \in \mathbb{R}$.

3.4.2 *Global existence and stability – II*

In Theorem 3.2.3 we showed that the surface diffusion flow starting from $E_0 = E_{\psi_0}$ exists in a short–time interval and the evolving sets E_t can be parametrized as normal deformations of a fixed set smooth E, induced by functions $\psi(t, \cdot)$ satisfying

$$\begin{cases} \partial_t \psi(t,x) \nu_t(p) \cdot \nu_E(x) = \Delta_t \mathrm{H}_t(p) \\ \psi(0,x) = \psi_0(x) \end{cases}$$

for every $x \in \partial E$, with $p = x + \psi(t,x)\nu_E(x)$. Moreover,

$$\nu_t(p) \cdot \nu_E(x) = \left(1 + \sum_{j=1}^{n-1} \frac{(\partial_{\tau_j} \psi(t,x))^2}{(1 + \kappa_j(x)\psi(t,x))^2} \right)^{-1/2} ,$$

where $\kappa_j(x)$ and $\tau_j(x)$ for $j = 1, \dots, n-1$ are, respectively, the principal curvatures and the principal directions of ∂E at x (see for instance [17, eq. (3.4)]). In particular, we remark that

$$\nu_t(p) \cdot \nu_E(x) = 1 + O(\|\psi(t, \cdot)\|_{H^1}) .$$

Definition 3.4.5. We say that an open set $E \subseteq \mathbb{T}^n$ satisfies a *uniform inner (respectively outer) ball condition* of radius r if there exists $r > 0$ such that for every $x \in \partial E$ there exists a ball $B_r(y) \subseteq E$ (resp. $B_r(y) \subseteq \mathbb{T}^n \setminus E$) with $x \in \partial B_r(y)$.

Notice that every smooth set satisfies a uniform inner and outer ball condition.

Remark 3.4.6. Let $E \subseteq \mathbb{T}^n$ be a set satisfying a uniform ball condition of radius r_E. Then, every small $C^{1,1}$–normal deformations of E satisfy a uniform ball condition of radius $r \approx r_E$. Indeed, it is easy to see that if E_ψ is the normal deformation of E induced by $\psi \in C^{1,1}(\partial E)$, then the Hausdorff distance between E and E_ψ is bounded by $\|\psi\|_{C^0(\partial E)}$. Furthermore, since $\nabla d_{E_\psi} = \nu_{E_\psi}$ can be written as

$$\nu_{E_\psi} = \left(\nu_E - \sum_{i=1}^{n-1} \frac{\nabla\psi \cdot v_i}{1+\kappa_i\psi} v_i\right)\left(1 + \sum_{i=1}^{n-1} \frac{(\nabla\psi \cdot v_i)^2}{(1+\kappa_i\psi)^2}\right)^{-1/2},$$

where the family v_i denotes an orthonormal frame of the tangent space of ∂E. By differentiating this formula, one can see that

$$\|d_{E_\psi} - d_E\|_{C^{1,1}(\partial E)} \leqslant C_E \|\psi\|_{C^{1,1}(\partial E)},$$

which then implies that $E_\psi \to E$ in $C^{1,1}$, if $\|\psi\|_{C^{1,1}} \to 0$. Therefore, by [15, Theorem 2.6] and [15, Remark 2.7] one infers that the radius r of the uniform ball condition of the set E_ψ depends continuously on $\|\psi\|_{C^{1,1}}$ when this latter is small enough. In particular, for every $\varepsilon > 0$ there exists $\delta(r_E, \varepsilon) > 0$ such that, if $\|\psi\|_{C^{1,1}} \leqslant \delta$ then

$$|r_E - r| \leqslant \varepsilon.$$

Lemma 3.4.7. *Let $E \subseteq \mathbb{T}^n$ be a smooth set and $m > 0$. There exists $\eta = \eta(m, E) > 0$ such that, for every $k \in \mathbb{N}$, $\psi \in C^k(\partial E)$ with $\|\psi\|_{C^k(\partial E)} \leqslant m$, $\|\psi\|_{C^0(\partial E)} \leqslant \eta$ and for every $\sigma \in \mathbb{T}^n$ with $|\sigma| \leqslant \eta$, then the normal deformation of E induced by ψ (as in Definition 1.3.1) and translated by σ, that is $E_\psi + \sigma$, can be written as a normal deformation of E induced by a function $\widetilde{\psi} : \partial E \to \partial E$ such that*

$$\|\widetilde{\psi}\|_{C^0(\partial E)} \leqslant 2\eta, \quad \|\widetilde{\psi}\|_{C^k(\partial E)} \leqslant C(\|\psi\|_{C^k(\partial E)} + |\sigma|).$$

where $C = C(E) > 0$.

Proof. Being the set E smooth, it satisfies a uniform inner and outer ball condition, hence there exists a positive radius $r > 0$ such that the signed distance d_E from the set E, defined in formula (1.14), is a smooth function in the tubular neighborhood N_r (see Definition (1.13)). Since, for some $k \geqslant 2$, ψ has C^k–norm bounded by m, we also have $\|\psi\|_{C^{1,1}(\partial E)} \leqslant m$. Then, there exists a radius $\rho = \rho(m, E)$ such that the normal deformation E_ψ of E induced by ψ satisfies a uniform inner and outer ball condition of radius ρ and we can clearly assume without loss of generality that $\rho < r$.
We now let $\eta \leqslant \rho/2$ to be chosen later, take any $|\sigma| < \eta$ and set $F = E_\psi + \sigma$. Clearly, F still satisfies a uniform inner and outer ball condition of radius ρ. Then, for every $y \in \partial F$ there exists $x \in \partial E_\psi$ such that $y = x + \sigma$, hence we have

$$d(y, \partial E) \leqslant |\sigma| + d(x, \partial E) < \eta + \|\psi\|_{C^0(\partial E)} \leqslant 2\eta$$

and in particular $\partial F \subseteq N_{2\eta} \subseteq N_r$. We now define the map $T_\psi : \partial E \to \partial E$ as in formula (2.42). By choosing η small enough and using standard interpolation inequalities, there holds $\|\psi\|_{C^1(\partial E)} + |\sigma| < 1/2$, which implies that the function $x \mapsto x + \psi(x)\nu_E(x) + \sigma$ is a diffeomorphism (since it is a small perturbation of the identity). Since E is smooth (and possibly considering a smaller η), we have that $\pi_E\big|_{\partial F} : \partial F \to \partial E$ is a smooth diffeomorphism, C^k–close to the identity. Hence, by inequality (2.45), we conclude

$$\|T_\psi - I\|_{C^k(\partial E)} \leqslant C(\|\psi\|_{C^k(\partial E)} + |\sigma|), \tag{3.78}$$

moreover, by the invertibility of the map $x \mapsto x + \psi(x)\nu_E(x) + \sigma$, we also obtain

$$\|T_\psi^{-1} - I\|_{C^k(\partial E)} \leqslant C(\|\psi\|_{C^k(\partial E)} + |\sigma|)\,. \tag{3.79}$$

Then, we can find a function $\widetilde{\psi} : \partial E \to \mathbb{R}$ such that F is the normal deformation of E induced by $\widetilde{\psi}$, more precisely for every $x \in \partial E$, there holds

$$x + \psi(x)\nu_E(x) + \sigma = T_\psi(x) + \widetilde{\psi}(T_\psi(x))\nu_E(T_\psi(x))\,.$$

Finally, using the above expression and the estimates (3.78) and (3.79), we conclude that

$$\|\widetilde{\psi}\|_{C^k(\partial E)} \leqslant \|T_\psi^{-1}\|_{C^k(\partial E)}(\|\psi\|_{C^k(\partial E)} + |\sigma| + \|T_\psi - I\|_{C^k(\partial E)}) \leqslant C(\|\psi\|_{C^k(\partial E)} + |\sigma|)\,,$$

for some constant $C = C(E) > 0$. □

We are ready to state and prove the second version of our stability result.

Theorem 3.4.8. *Let $E \subseteq \mathbb{T}^n$ be a strictly stable set and let $E_0 = E_{\psi_0}$ be the normal deformation of E induced by $\psi_0 \in C^{1,1}(\partial E)$ (as in Definition 1.3.1) with $\mathrm{Vol}(E_0) = \mathrm{Vol}(E)$. There exists $\delta = \delta(E) > 0$ such that if $\|\psi_0\|_{C^{1,1}(\partial E)} \leqslant \delta$, then the surface diffusion flow E_t starting from E_0 exists smooth for all times $t \geqslant 0$ and E_t converges smoothly to $E + \tau$ exponentially fast as $t \to +\infty$, for some $\tau \in \mathbb{T}^n$, with the same meaning of Theorem 3.3.14.*

Proof. Let $\varepsilon > 0$ and $\delta(\varepsilon) \in (0, 1)$ to be chosen later (smaller than the constant given by Theorem 3.2.3). We split the proof into steps.

Step 1. Our first goal is to show that the function $\mathscr{G}(t) = \mathcal{A}(\partial E_t) - \mathcal{A}(\partial E)$ is non–increasing in time and in particular, $\mathscr{G}(t) \leqslant Ce^{-ct}$ as long as the flow exists.
Let $\psi_0 \in C^{1,1}(\partial E)$ with $\|\psi_0\|_{C^{1,1}} \leqslant \delta < 1$. By Theorem 3.2.3 there exists a time $T > 0$, which depends on ε, E and a smooth flow E_t starting from E_0, for $t \in [0, T)$. Moreover, $E_t = E_\psi$ and $\psi(t, \cdot)$ satisfies estimates (3.3) and (3.4). Without loss of generality, we can assume $T < +\infty$.
We recall that

$$\frac{d}{dt}\mathcal{A}(\partial E_t) = \int_{\partial E_t} \mathrm{H}\Delta\mathrm{H}\, d\mu = -\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)} \leqslant -C\|\mathrm{H} - \overline{\mathrm{H}}\|^2_{L^2(\partial E_t)}\,,$$

where the constant C coming from the Poincaré inequality is uniform since $\|\psi(t, \cdot)\|_{C^{1,1}(\partial E)}$ remains bounded and small, for every $t \in (0, T)$ (see Section 1.2). So, the function $\mathscr{G}$ is non–increasing.
Let δ^* be the constant given by Theorem 3.4.4, $p > n - 1$ and $\xi = \xi(\delta^*, p)$ given by Lemma 2.2.15. By estimates (3.3), (3.4) and by interpolation, we have that $\|\psi(t, \cdot)\|_{W^{2,p}(\partial E)} \leqslant \xi$, for every $t \in [T/2, T)$, up to taking ε and δ small enough. Thus, for any $t \in [T/2, T)$, by Lemma 2.2.15, there exists η_t and a function $\widetilde{\psi}(t, \cdot)$ such that $E_t + \eta_t = E_{\widetilde{\psi}}$

$$|\eta_t| \leqslant C\|\psi(t, \cdot)\|_{W^{2,p}(\partial E)}\,, \qquad \|\widetilde{\psi}(t, \cdot)\|_{W^{2,p}(\partial E)} \leqslant C\|\psi(t, \cdot)\|_{W^{2,p}(\partial E)}$$

and

$$\left|\int_{\partial E_t} \widetilde{\psi}(t, \cdot)\nu\, d\mu_t\right| \leqslant \delta^* \|\widetilde{\psi}(t, \cdot)\|_{L^2(\partial E)}\,.$$

Furthermore, Lemma 3.4.7 (taking δ smaller, if needed) implies that $\|\widetilde{\psi}(t, \cdot)\|_{C^1(\partial E)} \leqslant \delta^*$.
We then apply Theorem 3.4.4 to the set $E_t + \eta_t = E_{\widetilde{\psi}}$ and we obtain

$$\|\widetilde{\psi}(t, \cdot)\|_{H^1(\partial E)} \leqslant C\|\mathrm{H}_{E_{\widetilde{\psi}}} - \lambda\|_{L^2(\partial E)}\,,$$

for any $\lambda \in \mathbb{R}$. By means of the bounds on $\widetilde{\psi}$ and by the translation invariance, we thus get

$$\|\widetilde{\psi}(t,\cdot)\|_{H^1(\partial E)} \leqslant C\|\mathrm{H}_{E_{\widetilde{\psi}}} - \lambda\|_{L^2(\partial E_{\widetilde{\psi}})} = C\|\mathrm{H}_{E_t} - \lambda\|_{L^2(\partial E_t)}. \tag{3.80}$$

We claim that

$$\mathcal{A}(\partial E_{\widetilde{\psi}(t,\cdot)}) - \mathcal{A}(\partial E) \leqslant C\|\widetilde{\psi}(t,\cdot)\|^2_{H^1(\partial E)}. \tag{3.81}$$

Indeed, by defining for every $x \in \partial E$ the function

$$\mathcal{Q}(x) = \left(1 + \sum_{j=1}^{n-1} \frac{(\partial_{\tau_j}\widetilde{\psi}(t,x))^2}{(1+\kappa_j(x)\widetilde{\psi}(t,x))^2}\right)^{1/2},$$

we have

$$\begin{aligned}
\mathcal{A}(\partial E_{\widetilde{\psi}}) &= \int_{\partial E} \mathcal{Q}(x) \prod_{i=1}^{n-1}\Big(1+\kappa_i(x)\widetilde{\psi}(t,x)\Big)\, d\mu \\
&= \mathcal{A}(\partial E) + \int_{\partial E} \big(\mathrm{H}_E\widetilde{\psi}(t,\cdot) + O(\widetilde{\psi}(t,\cdot)^2) + O(|\nabla\widetilde{\psi}(t,\cdot)|^2)\big)\, d\mu \\
&\leqslant \mathcal{A}(\partial E) + \mathrm{H}_E \int_{\partial E} \widetilde{\psi}(t,\cdot)\, d\mu + C\int_{\partial E} \big((\widetilde{\psi}(t,\cdot))^2 + |\nabla\widetilde{\psi}(t,\cdot)|^2\big)\, d\mu \\
&\leqslant \mathcal{A}(\partial E) + C\|\widetilde{\psi}(t,\cdot)\|^2_{H^1(\partial E)},
\end{aligned}$$

where we used [17, Lemma 3.1], relation $\mathrm{H}_E = \sum_{i=1}^{n-1}\kappa_i$ and inequality

$$\left|\int_{\partial E} \widetilde{\psi}(t,\cdot)\, d\mu\right| \leqslant C\int_{\partial E} \widetilde{\psi}(t,\cdot)^2\, d\mu,$$

which follows from the fact that $\mathrm{Vol}(E_t) = \mathrm{Vol}(E_0)$ (see [17, Remark 3.2] for more details).
We now notice that, by the translation invariance and inequalities (3.80) and (3.81), for any $\lambda \in \mathbb{R}$, we have

$$\mathcal{A}(\partial E_t) - \mathcal{A}(\partial E) = \mathcal{A}(\partial E_{\widetilde{\psi}}) - \mathcal{A}(\partial E) \leqslant C\|\mathrm{H}_{E_t} - \lambda\|^2_{L^2(\partial E_t)}. \tag{3.82}$$

Since for any $t \in (0,T)$, equation (3.82) for the particular choice of $\lambda = \overline{\mathrm{H}}$ implies

$$\mathscr{G}'(t) = -\|\mathrm{H}_{E_t} - \overline{\mathrm{H}}_{E_t}\|^2_{L^2(\partial E_t)} \leqslant -C\mathscr{G}(t),$$

by Gronwall's inequality we conclude (recalling that $\mathscr{G}(0) \geqslant \mathscr{G}(T/2)$)

$$\mathscr{G}(t) \leqslant \mathscr{G}(0)e^{-C(t-T/2)} \tag{3.83}$$

for every $t \in [T/2, T)$.

Step 2. We now show that the flow exists for every $t \geqslant 0$ and it converges exponentially fast to E up to translations.
Possibly taking a smaller $\delta > 0$, by means of the quantitative isoperimetric inequality in Theorem 2.2.10, we get a family of translations τ_t such that

$$C\mathrm{Vol}(E \triangle (E_t + \tau_t))^2 \leqslant \mathcal{A}(\partial E_t) - \mathcal{A}(\partial E) \leqslant \mathcal{A}(\partial E_0) - \mathcal{A}(\partial E).$$

Furthermore, since all the evolving sets E_t, for $t \in [T/2, T)$, satisfy a uniform inner and outer ball condition by Remark 3.4.6, by classical convergence results (see [15, Theorem 3.2], for instance) we have that $E_t + \tau_t$ is C^1-close to E. In particular, by the implicit map theorem, there exist smooth functions $v(t,\cdot) : \partial E \to \mathbb{R}$ such that $E_t + \tau_t = E_{v(t,\cdot)}$ and

$$|\tau_t| \leqslant \max_{x \in \partial(E_t+\tau_t)} d(x, \partial E_t) \leqslant \|\psi(t,\cdot)\|_{C^0(\partial E)} + \|v(t,\cdot)\|_{C^0(\partial E)} \leqslant 2\varepsilon,$$

up to taking δ small. Therefore, recalling inequality (3.83), we have

$$\|v(t,\cdot)\|^2_{L^1(\partial E)} \leqslant C(\mathcal{A}(\partial E_0) - \mathcal{A}(\partial E))e^{-C(t-T/2)}\,. \tag{3.84}$$

By Lemma 3.4.7, we also have

$$\|v(t,\cdot)\|_{C^k(\partial E)} \leqslant C(\|\psi(t,\cdot)\|_{C^k(\partial E)} + |\tau_t|)\,,$$

for every $k \geqslant 2$.
Thus, for every $t \in [T/2, T)$, combining this estimate with (3.4), (3.84) and using the interpolation inequalities, for any $l \in \mathbb{N}$ there exist $k(l) \in \mathbb{N}, \theta(l) \in (0,1)$ and $C = C(E,l) > 0$ such that

$$\|\nabla^l v(t,\cdot)\|_{C^0} \leqslant C\|v(t,\cdot)\|^{\theta}_{L^1}\|v(t,\cdot)\|^{1-\theta}_{C^k} \leqslant C(\mathcal{A}(\partial E_0) - \mathcal{A}(\partial E))^{\frac{\theta}{2}} e^{-C(t-T/2)}\,. \tag{3.85}$$

Choosing $\mathscr{G}(0) = \mathcal{A}(\partial E_0) - \mathcal{A}(\partial E)$ small (hence, choosing δ small) we can then apply again Theorem 3.2.3 with $E_{v(T/2,\cdot)} = E_{T/2} + \tau_{T/2}$ as initial set to get existence of the translated flow up to the time $3T/2$. We remark that, by uniqueness, the flow above is well defined since it coincides in $[T/2, T)$ with the flow E_t translated by τ_t and estimate (3.83) holds for all $t \in [T/2, 3T/2)$. Since the bound (3.85) is uniform along the flow, choosing at every step the times $t = kT/2$, for $k \in \mathbb{N}$, we can iterate the procedure above to prove that the flow exists for all times $t \in [0, +\infty)$. Moreover, for every $t \in (0, +\infty)$ there exists a translation τ_t such that $E_t + \tau_t = E_{v(t,\cdot)}$, with v satisfying estimate (3.85). In particular, we have that $v \to 0$ exponentially in C^k for any $k \in \mathbb{N}$, as $t \to +\infty$, hence $E_t + \tau_t \to E$ in C^k for every k. This also implies (arguing as in the previous section, that is, "transferring" estimates on the function to geometric estimates on H – see for instance Lemma 3.3.13) that $\|\Delta \mathrm{H}\|_{L^2(\partial E_t)} \to 0$ exponentially fast.

Step 3. We conclude the proof by showing the convergence of the whole flow to a translate of E. Let us prove the convergence of the translations τ_t. By compactness, we can find a sequence $t_n \to +\infty$ such that $\tau_{t_n} \to \tau$. Defining, $D(t)$ as in formula (3.71), we get

$$\begin{aligned}
\left|\frac{d}{dt}D(t)\right| &= \left|\frac{d}{dt}\int_{E_t \triangle (E-\tau)} d(x, \partial E - \tau)\, dx\right| \\
&= \left|\int_{E_t} \operatorname{div}(d_{E-\tau}(x)V_t(x)\nu(x))\, dx\right| \\
&= \left|\int_{\partial E_t} d_{E-\tau}(x)\Delta \mathrm{H}\, d\mu\right| \\
&\leqslant \mathcal{A}(\partial E_0)\|\Delta \mathrm{H}\|_{L^2(\partial E_t)}\left(\sup_{x\in\partial E_t} d(x, \partial E - \tau)\right) \\
&\leqslant Ce^{-Ct}\,,
\end{aligned} \tag{3.86}$$

where we recall that V_t is the velocity of the flow in the normal direction.
Clearly, the estimate (3.86) implies that $D(t)$ admits a limit as $t \to +\infty$. By the previous step and by the fact that $\tau_{t_n} \to \tau$, we deduce

$$D(t) \to 0 \quad \text{as } t \to +\infty\,.$$

Assume now that σ is the limit of τ_n up to a (non relabeled) subsequence. Thus, $E_{\tau_n} \to E - \sigma$ and

$$0 = \lim_{n\to+\infty} D(\tau_n) = \int_{E-\sigma\triangle E-\tau} d(x, \partial E\tau)\, dx\,,$$

which implies $\sigma = \tau$. This concludes the proof as the exponential convergence follows from the second step. $\square$

3.5 THE CLASSIFICATION OF THE STABLE CRITICAL SETS

In this final section, we discuss the classes of smooth sets to which Theorem 3.3.14 and Theorem 3.4.8 can be applied, hence, "dynamically exponentially stable" for the surface diffusion flow. We observe that it is easy to see that (by a dilation/contraction argument) any strictly stable smooth critical set must be connected, but actually, being the normal velocity of the surface diffusion flow at every point defined by the *local* quantity ΔH, it follows that Theorem 3.3.14 can be applied also to finite unions of boundaries of strictly stable critical sets (see [30] and Figure 1 below). Moreover, by Definition 2.2.5, if ∂E in $\mathbb{T}^n$ is composed by flat pieces, hence its second fundamental form B is identically zero, the set E is critical and stable and with a little effort, actually strictly stable. It is a little more difficult to show that any ball in any dimension $n \in \mathbb{N}$ is strictly stable (it is obviously a critical set), which is connected to the study of the eigenvalues of the Laplacian on the sphere $\mathbb{S}^{n-1}$, see [36, Theorem 5.4.1], for instance. The same then holds for all the "cylinders" $\mathbb{R}^k \times \mathbb{S}^{n-k-1} \subseteq \mathbb{R}^n$, bounding $E \subseteq \mathbb{T}^n$ after taking their quotient by the same equivalence relation defining $\mathbb{T}^n$, determined by the standard integer lattice of $\mathbb{R}^n$.

Notice that if $n = 2$, it follows that the only bounded strictly stable critical sets of the (in this case) *Length* functional in the plane are the disks and in $\mathbb{T}^2$ they are the disks and the "strips" with straight borders.

In the three–dimensional case, a first classification of the smooth stable "periodic" critical sets for the volume–constrained Area functional, was given by Ros in [55], where it is shown that in the flat torus $\mathbb{T}^3$, they are *balls*, 2–*tori*, *gyroids* or *lamellae*.

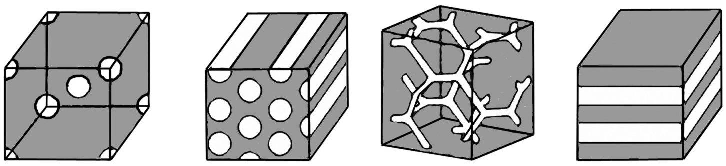


Figure 1: From left to right: balls, 2–tori, gyroids and lamellae.

Notice that, despite their name, the *lamellae* are (after taking the quotient) parallel planar 2–tori and the 2–*tori* are quotients of circular cylinders in $\mathbb{R}^3$. As we said, with the balls, these surfaces are actually strictly stable, while in [37, 38, 56] the authors established the strict stability of gyroids only in some cases. To give an example, we refer to [38] where Grosse–Brauckmann and Wohlgemuth showed the strict stability of the gyroids that are fixed with respect to translations. We remind that the gyroids, that were discovered by the crystallographer Schoen in the 1970 (see [58]), are the unique non–trivial embedded members of the family of the Schwarz' P and D surfaces, namely, the simplest and most well known triply–periodic minimal surfaces (see [56]).

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