APPROXIMATION SCHEMES FOR MONOTONE SYSTEMS OF NONLINEAR SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS: CONVERGENCE RESULT AND ERROR ESTIMATE

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Abstract. We consider approximation schemes for monotone systems of fully nonlinear second order partial differential equations. We first prove a general convergence result for monotone, consistent and regular schemes. This result is a generalization to the well known framework of Barles-Souganidis, in the case of scalar nonlinear equation. Our second main result provides the convergence rate of approximation schemes for weakly coupled systems of Hamilton-Jacobi-Bellman equations. Examples including finite difference schemes and Semi-Lagrangian schemes are discussed.

1. Introduction

In this paper we study approximation schemes for a system of nonlinear second order partial differential equations

$$F_i\left(x, u, Du_i, D^2u_i\right) = 0, \qquad x \in \mathbb{R}^N, \quad i = 1, \dots, M, \tag{1.1}$$

where $u = (u_1, \dots, u_M)$ denotes the unknown function, and $F = (F_1, \dots, F_M)$ is a given function.

The theory of viscosity solution, initially developed for the scalar equation has been extended to systems in [13, 17, 19, 26]. In this framework the monotonicity of F with respect to the variable u (see (2.2c)) is essential to guarantee the validity of a maximum principle. Note that this property involves not only the single component F_i, but all the system at the same time. Given this property and standard regularity assumptions on F, it is possible to prove a strong comparison principle, hence the uniqueness of the viscosity solution to (1.1).

For a large class of monotone systems, the existence of the solution can be obtained either via the Perron's method ([17, 19]) or via control-theoretic representation formulas, ([13, 22]). We refer to [15, 16] for various applications of systems of PDEs in many areas, in particular we mention [5] for a Black–Scholes pricing model with jumping volatility.

Here, we consider approximation schemes of the type

$$S_i(h, x, u^h(x), u_i^h) = 0 \qquad x \in \mathbb{R}^N,\ i = 1, \dots, M \tag{1.2}$$

where S_i are consistent, monotone and uniformly continuous approximation of F_i in (1.1) and the coupling among the equations is only in the variable $u^h(x)$ (where $u^h = (u_1^h, \dots, u_M^h)$ represents the solution of the approximate system (1.2), and is expected to be an approximation to the solution u of system (1.1)). Typical approximation of this type, like finite difference methods and semi-Lagrangian schemes, will be discussed in details.

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For the case of a scalar fully nonlinear equation, the work of Barles-Souganidis [4] provides a general setting for convergence of approximation schemes. This setting says that any monotone, consistent, continuous scheme converges to the unique viscosity solution of the limit problem, provided that it satisfies a comparison principle. To extend the result of [4] to the case of systems of PDEs, the crucial point is to introduce an appropriate monotonicity condition for (1.2) (the monotonicity of the single scheme S_i being not sufficient). Under this appropriate monotonicity, (assumption **(C1)** in Section 3), and the standard consistency and regularity conditions, by applying the general idea of [4] based on the use of the half-relaxed limits, we get the convergence result for the monotone systems.

While results on convergence rates for viscosity solutions of 1^{st} order equation were obtained from the beginning of the viscosity theory, only quite recently Krylov [22, 21, 23] and Barles-Jakobsen [1, 2] success in proving similar rates for second order Hamilton-Jacobi-Bellman equations. Convergence rate of approximation schemes for particular Isaac equations have been obtained in [6, 20]. We refer also to [10] for convergence rate in the case of elliptic fully nonlinear equations, and to [14] for convergence rate of probabilistic approximation schemes. Our aim, in the second part of the paper, is to extend the previous convergence rates to the case of convex systems. In order to simplify the presentation we choose as a model problem a weakly coupled systems of Hamilton-Jacobi-Bellman equations, but the approach is sufficiently general to hold for other classes of problems, f.e. switching control problems.

To obtain the rate of convergence, we will use the same arguments developed in [1, 2, 7] and adapt them to the case of monotone system of PDEs. As usual, the upper bound for $u-u^h$ is easier and it is obtained via a *Krylov regularization* and *shaking coefficient* techniques. These techniques allow to define a smooth subsolution to the system. So by using the consistency property, we obtain the upper bound. The proof of the lower bound is more involved and requires an approximation of the weakly coupled system with a switching system with a bigger number of components. By this procedure it is possible to build regular *"local"* supersolutions of the continuous problems. Then, we derive the lower error estimate by using the consistency and monotonicity conditions. Our result gives an upper bound of $h^{1/2}$ and a lower bound of $h^{1/5}$ for the finite differences scheme [9]. For a Semi-Lagrangian scheme these estimates become $h^{1/4}$ for the upper bound, and $h^{1/10}$ for the lower bound.

The paper is organized as follows. In Section 2 we recall definitions and basic results for the continuous problem. In Section 3 we state the main assumptions for the scheme and we prove the convergence theorem. Section 4 is devoted to the proof of the error estimates, while in section 5.1-5.2 and 5.3 we discuss respectively finite difference schemes and semi-Lagrangian schemes. The Appendix 6 is devoted to the proofs of some technical results.

Notation: We will use the following norms

$$|f|_0 = \operatorname{ess\,sup}_{x\in\mathbb{R}^N}|f(x)|, \quad [f]_1 = |Df|_0, \quad \text{and} \quad |f|_1 = |f|_0 + [f]_1.$$

The space of real symmetric $N\times N$ matrices is denoted by S^N, and $X \geq Y$ in S^N will mean that $X-Y$ is positive semi-definite.

For a function $u:\mathbb{R}^N \to \mathbb{R}^M$, we say that $u=(u_1,\dots,u_M)$ is upper semicontinuous (u.s.c. for short), respectively lower semicontinuous (l.s.c. for short), if all the components u_i, $i=1,\dots,M$, are u.s.c., respectively l.s.c.. If $u=(u_1,\dots,u_M)$, $v=(v_1,\dots,v_M)$, are two functions defined in a set E we write $u\leq v$ in E if $u_i\leq v_i$ in E, $i=1,\dots,M$.

2. The continuous problem: definitions and assumptions

Consider the system of nonlinear second order equations

$$F_i\left(x, u, Du_i, D^2 u_i\right) = 0, \qquad x \in \mathbb{R}^N, \quad i = 1, \dots, M \tag{2.1}$$

where $u := (u_1, \dots, u_M)$ and u_i is a real valued function defined in $\mathbb{R}^N$ and $F := (F_1, \dots, F_M) : \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^N \times S^N \to \mathbb{R}^M$ is a continuous given function.

Let us first recall the definition of viscosity solution for system (2.1) (see [19]).

Definition 2.1.
i) An u.s.c. function $u : \mathbb{R}^N \to \mathbb{R}^M$ is said a viscosity subsolution of (2.1) if whenever $\phi \in C^2(\mathbb{R}^N)$, $i = 1, \dots, M$ and $u_i - \phi$ attains a local maximum at $x \in \mathbb{R}^N$, then

$$F_i(x, u(x), D\phi(x), D^2\phi(x)) \le 0.$$

ii) A l.s.c. function $v : \mathbb{R}^N \to \mathbb{R}^M$ is said a viscosity supersolution of (2.1) if whenever $\phi \in C^2(\mathbb{R}^N)$, $i = 1, \dots, M$ and $v_i - \phi$ attains a local minimum at $x \in \mathbb{R}^N$, then

$$F_i(x, v(x), D\phi(x), D^2\phi(x)) \ge 0.$$

iii) A continuous function u is said a viscosity solution of (2.1) if it is both viscosity sub- and supersolution of (2.1).

The existence of a solution to (2.1) can be obtained, for a large class of monotone systems, either via Perron's method ([17], [19]) or via the control-theoretic interpretation of the problem [13, 26]. To get a comparison principle for system (2.1), we shall assume the following conditions on function F:

$$F_i \in C(\mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^N \times S^N) \quad i = 1, \dots, M; \tag{2.2a}$$

$$\begin{gathered} \text{There exists a modulus of continuity } \omega \text{ s.t. if } X,\ Y \in S^N,\ \beta > 1 \text{ and} \\ -3\beta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le -3\beta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \\ \text{then } F_i(y, r, \beta(x-y), -Y) - F_i(x, r, \beta(x-y), X) \le \omega(\beta|x-y|^2 + \beta^{-1}); \end{gathered} \tag{2.2b}$$

$$\begin{gathered} \text{There exists } c_0 > 0 \text{ s.t. for } r,\ s \in \mathbb{R}^M \text{ satisfying} \\ r_i - s_i = \max_{j=1,\dots,M} \{r_j - s_j\} \ge 0, \text{ then for all } x,\ y,\ p \in \mathbb{R}^N,\ X \in S^N \\ F_i(x, r, p, X) - F_i(x, s, p, X) \ge c_0(r_i - s_i). \end{gathered} \tag{2.2c}$$

Theorem 2.2 (see [19])**.** *Assume $F_i : \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^N \times S^N \to \mathbb{R}$, $i = 1, \dots, M$, to be continuous and satisfy* (2.2)*. Let u and v be respectively a bounded subsolution and a bounded supersolution of* (2.1)*. Then $u \le v$ in $\mathbb{R}^N$.*

Remark 2.3. Assumption (2.2c) is the condition giving the monotonicity of the system (2.1), while (2.2a), (2.2b) are standard regularity assumptions in viscosity solution theory.

Example 2.4. *Weakly coupled system:* Consider the weakly coupled system of M equations:
(2.3)

$$F_i(x, u(x), Du_i(x), D_i^2 u(x)) := \sup_{\alpha \in \mathcal{A}^i} \left\{ \mathcal{L}_i^\alpha(x, u(x), Du_i(x), D^2 u_i(x)) + \sum_{j=1}^M d_{ij}(x, \alpha) u_j(x) \right\}$$

where the operator $\mathcal{L}^\alpha$ is defined by:

$$\mathcal{L}_i^\alpha(x,r,p,X) = -\frac{1}{2}Tr[a_i(x,\alpha)X] - b_i(x,\alpha)\cdot p - f_i(x,\alpha) + \lambda_i r_i. \tag{2.4}$$

and with $a_i(x,\alpha) = \sigma_i(x,\alpha)\sigma_i^T(x,\alpha)$. We assume that the following assumptions hold:

$$\begin{aligned} &\mathcal{A}^i \text{ are compact metric spaces, } i = 1,\dots,M\\ &b_i : \mathbb{R}^N \times \mathcal{A}^i \longrightarrow \mathbb{R}^N,\ \sigma_i : \mathbb{R}^N \times \mathcal{A}^i \to \mathbb{R}^{N\times D}, f_i : \mathbb{R}^N \times \mathcal{A}^i \to \mathbb{R}\\ &|f_i(\cdot,\alpha)|_1 + |\sigma_i(\cdot,\alpha)|_1 + |b_i(\cdot,\alpha)|_1 \le L, \text{for any}\, \alpha \in \mathcal{A}^i,\ i = 1,\dots,M \end{aligned} \tag{2.5a}$$

and

$$\begin{aligned} &d_{ij} : \mathbb{R}^N \times \mathcal{A}^i \to \mathbb{R},\\ &|d_{ij}(\cdot,\alpha)|_1 \le L, \text{ for any}\, \alpha \in \mathcal{A}^i,\ i,j = 1,\dots,M,\\ &d_{ii}(x,\alpha) \ge 0,\ d_{ij}(x,\alpha) \le 0 \text{ for } i \ne j,\ \lambda_i \ge 0,\\ &\lambda_i + \sum_{j=1}^M d_{ij}(x,\alpha) \ge \lambda_0 > 0 \qquad (x,\alpha) \in \mathbb{R}^N \times \mathcal{A}^i,\ i = 1,\dots,M. \end{aligned} \tag{2.5b}$$

It is easy to check that under assumptions (2.5a)-(2.5b), system (2.4) satisfies conditions (2.2a)-(2.2c) (in particular the last condition in (2.5b) implies the monotonicity of the system with $c_0 = \lambda_0$). Moreover, we have the following result whose proof is given in the appendix.

Proposition 2.5. *Under assumptions* (2.5a)-(2.5b)*, system* (2.3) *admits a unique bounded continuous solution u. Moreover, if we have:*

$$\lambda_0 > \max_{i=1,\dots,M} \sup_{\alpha\in\mathcal{A}^i} |\sigma_i(\cdot,\alpha)|_1^2 + \max_{i=1,\dots,M} \sup_{\alpha\in\mathcal{A}^i} |b_i(\cdot,\alpha)|_1, \tag{2.6}$$

then u is bounded and Lipschitz continuous in $\mathbb{R}^N$.

Finally, let us also mention that system (2.3) comes from infinite horizon optimal control problems of hybrid systems and random evolution processes.

3. The approximation scheme: definitions and assumptions

For a fixed $h > 0$, we consider the following approximation scheme:

$$S_i(h,x,u^h(x),u_i^h) = 0 \qquad x \in \mathbb{R}^N,\ i = 1,\dots,M \tag{3.1}$$

where the function $u^h : \mathbb{R}^N \to \mathbb{R}^M$, $u^h = (u_1^h,\dots,u_M^h)$, represents the solution of (3.1). In the sequel, we will state a set of assumptions on the scheme $S(h,.,.,.)$:

- **(C1)** *(Monotonicity)* There exists $c_0 > 0$ such that for any $h > 0$, $x \in \mathbb{R}^N$, bounded functions u, v such that $u \le v$ in $\mathbb{R}^N$ and r, $s \in \mathbb{R}^M$ such that $\theta := r_i - s_i = \max_{j=1,\dots,M}\{r_j - s_j\} \ge 0$, then
$$S_i(h,x,r,u_i+\theta) - S_i(h,x,s,v_i) \ge c_0\theta.$$
- **(C2)** *(Regularity)* For any $h > 0$ and any continuous, bounded function $\phi : \mathbb{R}^N \to \mathbb{R}$, the functions
$$x \mapsto S_i(h,x,r,\phi) \qquad i = 1,\dots,M$$
are bounded and continuous on $\mathbb{R}^N$ and the functions
$$r \mapsto S_i(h,x,r,\phi)$$

are uniformly continuous for bounded r, uniformly with respect to $x \in \mathbb{R}^N$.

(C3) *(Consistency)* Fix $i = 1, \dots, M$. For all $h > 0$, $x \in \mathbb{R}^N$ and for any continuous function $\Phi \in C^0(\mathbb{R}^N, \mathbb{R}^M)$ with a smooth i-th component Φ_i, we have:

$$|S_i(h, x, \Phi(x), \Phi_i) - F_i(x, \Phi, D\Phi_i, D^2\Phi_i)| \le \omega(h)$$

with $\omega(h) \to 0$ for $h \to 0$.

(C4) *(Stronger consistency)* There exist $n, k_j > 0$, $j \in J \subset \{1, \dots, n\}$ and a constant $K_c > 0$ such that: for all $h > 0$, $i = 1, \dots, M$, $x \in \mathbb{R}^N$, and for any continuous function $\Phi \in C^0(\mathbb{R}^N; \mathbb{R}^M)$ such that its i-th component Φ_i is smooth with $|D^j\Phi_i(x)|$ bounded in $\mathbb{R}^N$, for every $j \in J$, we have:

$$|S_i(h, x, \Phi(x), \Phi_i) - F_i(x, \Phi, D\Phi_i, D^2\Phi_i)| \le K_C Q(\Phi_i) \qquad \text{in } \mathbb{R}^N,$$

where $Q(\phi) := \sum_{j \in J} |D^j\phi| h^{k_j}$ for every smooth function ϕ.

Definition 3.1. We say that a function $u : \mathbb{R}^N \to \mathbb{R}^M$ is a subsolution (respectively, a supersolution) of (3.1) if it satisfies

$$S_i(h, x, u(x), u_i) \ge 0 \quad (\text{respectively } S_i(h, x, u(x), u_i) \le 0) \qquad x \in \mathbb{R}^N,\ i = 1, \dots, M.$$

The next result is a comparison principle for the scheme (3.1).

Proposition 3.2. *Assume* ***(C1)*** *and* ***(C2)****. If u and v are bounded sub- and supersolutions of* (3.1)*, respectively, then $u \le v$ in $\mathbb{R}^N$.*

Proof. We assume $\delta := \sup_{i, \mathbb{R}^N}(u_i - v_i) > 0$ and we derive a contradiction. Let $\{x_n\}$ in $\mathbb{R}^N$ and $\{i_n\} \in \{1, \dots, M\}$ be such that $\delta_n = u_{i_n}(x_n) - v_{i_n}(x_n) = \max_j\{u_j(x_n) - v_j(x_n)\} \to \delta$ for $n \to \infty$. Since u and v are respectively sub- and supersolutions, we get:

$$0 \ge S_{i_n}(h, x_n, u(x_n), u_{i_n}) - S_{i_n}(h, x_n, v(x_n), v_{i_n}).$$

Moreover, we know that $u(x_n) \le v(x_n) + \delta_n$, $0 = \max_j(u_j(x_n) - (v_j(x_n) + \delta_n))$ and $u_{i_n} \le v_{i_n} + \delta$ in $\mathbb{R}^N$. Then, the monotonicity yields to:

$$S_{i_n}(h, x_n, u(x_n), u_{i_n}) \ge S_{i_n}(h, x_n, v(x_n) + \delta_n, v_{i_n} + \delta).$$

Therefore, by assumption **(C2)**, we have:

$$0 \ge S_{i_n}(h, x_n, v(x_n) + \delta, v_{i_n} + \delta) - S_{i_n}(h, x_n, v(x_n), v_{i_n}) + \omega(\delta_n - \delta),$$

where $\omega(t) \to 0$ when $t \to 0^+$. Finally, by using (**C1**) again, we obtain:

$$0 \ge c_0\delta + \omega(\delta_n - \delta),$$

which leads to a contradiction when $n \to \infty$. □

In all the sequel, we assume that:

(C5) *(Existence of discrete solution)* For every $h > 0$, system (3.1) admits a solution u^h.

We give a convergence result for the scheme based on the classical argument by Barles-Souganidis, [4].

Proposition 3.3. *Assume* ***(C1)****–****(C3)*** *and* ***(C5)****. Let $u^h = (u_1^h, \dots, u_M^h)$ be a locally uniformly bounded family of solutions of* (3.1) *and let u be the solution of* (2.1)*. Then $u^h \to u$ for $h \to 0$ locally uniformly in $\mathbb{R}^N$.*

Proof. Define the relaxed half-limits by:

$$\underline{u}_i(x) = \liminf_{h\to 0, z\to x} u_i^h(z), \qquad \overline{u}_i(x) = \limsup_{h\to 0,\ z\to x} u_i^h(z), \qquad i = 1, \dots, M.$$

Following the arguments introduced in [4], it is sufficient to prove that $\underline{u}$ is a supersolution and $\overline{u}$ is a subsolution of (2.1). Then by the comparison principle (Theorem 2.2), it follows that $\overline{u} \le \underline{u}$. Since the other inequality is obvious we get $\overline{u} = \underline{u} = \lim_{h\to 0} u^h$ and the local uniform convergence in $\mathbb{R}^N$.

We will only prove that $\overline{u}$ is a subsolution of (2.1) (the same arguments can be used to prove that $\underline{u}$ is a supersolution). Let ϕ be a smooth function, and $i \in \{1, \dots, M\}$, such that $\overline{u}_i - \phi$ has a maximum point at x_0, $\overline{u}_i(x_0) = \phi(x_0)$. By using standard arguments in the viscosity theory, there exists a sequence $(h_n)_n$ of positive numbers satisfying:

$$\begin{aligned} &h_n \to 0 \text{ and } x_n := x_{h_n} \to x_0 \quad \text{ when } n \to \infty,\\ &u_i^{h_n} - \phi \text{ has a maximum point at } x_n, \quad \text{ and } u_i^{h_n}(x_n) \to \overline{u}_i(x_0). \end{aligned}$$

Set $\delta_n := (u_i^{h_n} - \phi)(x_n)$, and define the function:

$$\Phi^n(x) = (u_1^{h_n}(x), \dots, u_{i-1}^{h_n}(x), \phi(x), u_{i+1}^{h_n}(x), \dots, u_M^{h_n}(x)) \qquad \text{in } \mathbb{R}^N.$$

Up to a subsequence, $\Phi^n(x_n)$ converges to a vector $r \in \mathbb{R}^M$, where $r_i = \overline{u}_i(x_0)$, and $r_j \le \overline{u}_j(x_0)$ for $j \ne i$. Then, using the fact that: $u_i^{h_n} - \delta_n \le \phi$ in $\mathbb{R}^N$ and $\delta_n = u_i^{h_n}(x_n) - \phi(x_n) = \max_j(u_j^{h_n}(x_n) - \Phi_j^n(x_n))$, and taking into account the monotonicity assumption **(C1)**, we get:

$$\begin{aligned} 0 = S_i(h_n, x_n, u^{h_n}(x_n), u_i^{h_n}) = S_i(h_n, x_n, u^{h_n}(x_n), (u_i^{h_n} - \delta_n) + \delta_n) \ge \\ S_i(h_n, x_n, \Phi^n(x_n), \phi) + c_0\delta_n \ge F_i(x_n, \Phi^n(x_n), D\phi(x_n), D^2\phi(x_n)) + c_0\delta_n - \omega(h) \end{aligned}$$

where the last inequality is due to the consistency assumption **(C3)**. Passing to the limit when $n \to \infty$, we get

$$0 \ge F_i(x_0, \overline{u}(x_0), D\phi(x_0), D^2\phi(x_0)),$$

which proves that $\overline{u}$ is a subsolution of (2.1).

□

4. The error estimate for the weakly coupled systems

In this section, we consider again system (2.3) considered in Example 2.4, with $\mathcal{L}_i^\alpha$ as in (2.4), i.e.

$$F_i(x, r, p, X) := \sup_{\alpha\in\mathcal{A}^i} \left\{ -\frac{1}{2} Tr[a_i(x,\alpha)X] - b_i(x,\alpha)\cdot p - f_i(x,\alpha) + \lambda_i r_i + \sum_{j=1}^M d_{ij}(x,\alpha) r_j \right\}.$$

We will always assume that (2.5) and (2.6) are satisfied and therefore we will denote by u the unique solution given by Proposition 2.5. For this particular system, we will derive an error estimate for $|u - u^h|$, where u^h is the solution of a scheme satisfying **(C1)**,**(C2)**,**(C4)** and **(C5)**.

4.1. **The upper bound.**

Proposition 4.1. *There exists $C > 0$ such that:*

$$u_i(x) - u_i^h(x) \le C h^{\overline{\gamma}} \qquad \forall i = 1, \dots, M \tag{4.1}$$

where $\overline{\gamma} = \min_{j \in J} \left\{ \frac{k_j}{j} \right\}$, with J and k_j being defined in ***(C4)****.*

We need a preliminary lemma which will be proved in the Appendix.

Lemma 4.2.

i) *There is a unique solution u^ε of the system*

$$\max_{|e| \le \varepsilon} F_i(x + e, u(x), Du_i, D^2 u_i) = 0 \quad x \in \mathbb{R}^N,\ i = 1, \dots, M \tag{4.2}$$

and two constants C_0, C_1 independent of h, ε such that

$$\max_{i=1,\dots,M} |u_i^\varepsilon|_1 \le C_0 \qquad \textit{and} \qquad \max_{i=1,\dots,M} |u_i - u_i^\varepsilon|_0 \le C_1 \varepsilon. \tag{4.3}$$

where u is a solution of (2.1)*.*

ii) *Let $(\rho_\varepsilon)_\varepsilon$ be a family of standard mollifiers and define $u_\varepsilon = (\rho_\varepsilon * u_1^\varepsilon, \dots, \rho_\varepsilon * u_M^\varepsilon) := (u_{1,\varepsilon}, \dots, u_{M,\varepsilon})$. Then u_ε is a classical subsolution of* (2.1)*.*

Proof of Proposition 4.1. Thanks to Lemma 4.2 *ii)*, u_ε is classical subsolution of (2.1). Therefore, by **(C4)** we have, for each $i = 1, \dots, M$

$$S_i(h, x, u_\varepsilon(x), u_{i,\varepsilon}) \le F_i(x, u_\varepsilon, Du_{i,\varepsilon}, D^2 u_{i,\varepsilon}) + Q(u_{i,\varepsilon}) \le Q(u_{i,\varepsilon}). \tag{4.4}$$

Set $\varepsilon = h^{\overline{\gamma}}$ with $\overline{\gamma}$ as in the statement. By [7, Lemma 4.2], we can estimate

$$Q(u_{i,\varepsilon}) \le K|J|C_0 h^{\overline{\gamma}} := \overline{C} h^{\overline{\gamma}},$$

where C_0 as in (4.3). By **(C1)** and (4.4), we have that $u_\varepsilon - \frac{C}{c_0} h^{\overline{\gamma}} = (u_{1,\varepsilon} - \frac{C}{c_0} h^{\overline{\gamma}}, \dots, u_{M,\varepsilon} - \frac{C}{c_0} h^{\overline{\gamma}})$ is a subsolution of the scheme. Hence, by the comparison principle of the scheme (Proposition 3.2), we get:

$$u_{i,\varepsilon} - u_i^h \le \frac{\overline{C}}{c_0} h^{\overline{\gamma}} \qquad \forall i = 1, \dots, M. \tag{4.5}$$

By estimate (4.3) and (4.5), we conclude

$$u_i - u_i^h \le \frac{\overline{C}}{c_0} h^{\overline{\gamma}} + C h^{\overline{\gamma}} \qquad \forall i = 1, \dots, M$$

and therefore (4.1) is satisfied. □

4.2. **The lower bound.** For the lower bound we need an additional assumption

$$\begin{aligned} &\text{For each control set } \mathcal{A}^i \text{ and } \delta > 0 \text{, there is a finite family of controls } \{\alpha_{ij}\}_{j=1}^{L_i} \\ &\text{such that for any } \alpha \in \mathcal{A}^i,\ \inf_{j=1,\dots,L_i} \big\{ |\sigma_i(\cdot,\alpha) - \sigma_i(\cdot,\alpha_{ij})|_0 + |b_i(\cdot,\alpha) - b_i(\cdot,\alpha_{ij})|_0 + \\ &|f_i(\cdot,\alpha) - f_i(\cdot,\alpha_{ij})|_0 + \sum_{l=1}^{M} |d_{il}(\cdot,\alpha) - d_{il}(\cdot,\alpha_{ij})|_0 \big\} \le \delta. \end{aligned} \tag{4.6}$$

Remark 4.3. The previous assumption is satisfied for instance when the sets $\mathcal{A}^i$ are finite or if the functions $\sigma_i, b_i, f_i, d_{ij}$ are uniformly continuous in α, uniformly in x. A slight more general assumption is considered in [3, 21, 22]. To simplify the notation, we assume in the following w.l.o.g. that all the families $\{\alpha_{ij}\}_{j=1}^{L_i}$ have the same cardinality L.

Proposition 4.4. *Assume that* (2.5), (2.6) *and* (4.6) *hold. Then there exists* $C > 0$ *such that:*

$$-Ch^{\underline{\gamma}} \leq u_i(x) - u_i^h(x) \qquad \forall i = 1, \dots, M \tag{4.7}$$

where $\underline{\gamma} = \min_{j\in J}\left\{\frac{k_j}{3j-2}\right\}$ *with* J*,* k_j *being defined in **(C4)**.*

For every $\ell > 0$, we introduce the following switching system

$$F_{ij}^{\varepsilon,\ell}(x, V^{\varepsilon,\ell}, DV_{ij}^{\varepsilon,\ell}, D^2V_{ij}^{\varepsilon,\ell}) = 0 \qquad x \in \mathbb{R}^N,\ i = 1, \dots, M,\ j = 1, \dots, L \tag{4.8}$$

where

$$F_{ij}^{\varepsilon,\ell}(x, R, p, X) = \max\Big\{ \min_{|e|\leq\varepsilon} \big\{\mathcal{L}_i^{\alpha_{ij}}(x+e, R_{ij}, p, X) + \sum_{l=1}^{M} d_{il}(x+e, \alpha_{ij})R_{lj}\big\}, \mathcal{M}_{ij}(R)\Big\}, \tag{4.9}$$

$$\mathcal{M}_{ij}(R) := R_{ij} - \min_{\substack{l=1,\dots,L\\ l\neq j}} \{R_{il} + \ell\}, \tag{4.10}$$

where $\{\alpha_{ij}\}_{j=1}^{L}$, $i = 1, \dots, M$ are defined in (4.6). The solution of the system (4.8) is a function from $\mathbb{R}^N$ to $\mathbb{R}^{M\times L}$. The (i, j) component of the solution of (4.8) is coupled with other L components by means of the switching term $\mathcal{M}_{ij}(R)$ and with other $(M - 1)$ components by means of the term $\sum_{l=1}^{M} d_{il}(x, \alpha)R_{lj}$.

Lemma 4.5. *Let* $\delta, \varepsilon, \ell > 0$ *be fixed. Let* u *be the solution of* (2.1) *and* $V^{\varepsilon,\ell}$ *be the solution of* (4.8)*.*

i) There exists $C > 0$ *such that:*

$$\max_{j=1,\dots,L} |V_{ij}^{\varepsilon,\ell} - u_i|_0 \leq C(\varepsilon + \ell^{\frac{1}{3}} + \delta) \qquad i = 1, \dots, M. \tag{4.11}$$

ii) Set $V_{\varepsilon,ij}^{\ell} := \rho_\varepsilon * V_{ij}^{\varepsilon,\ell}$, $i = 1, \dots, M$, $j = 1, \dots, L$ *where* ρ_ε *is a standard mollifier. Then*

$$\max_{j,l=1,\dots,L} |V_{ij}^{\varepsilon,\ell} - V_{\varepsilon,il}^{\ell}|_0 \leq C(\varepsilon + \ell) \tag{4.12}$$

iii) Moreover, for $\varepsilon \leq (4 \sup_{ij}[V_{ij}^{\varepsilon,\ell}]_1)^{-1}\ell$ *and for for any* $x \in \mathbb{R}^N$*, if we set* $j = j(\varepsilon, i, x) := \arg\min_{l=1,\dots,J} V_{\varepsilon,il}^{\ell}(x)$*, then*

$$\mathcal{L}_i^{\alpha_{ij}}(x, V_{\varepsilon,ij}^{\ell}, DV_{\varepsilon,ij}^{\ell}, D^2V_{\varepsilon,ij}^{\ell}) + \sum_{l=1}^{M} d_{il}(x, \alpha_{ij})V_{\varepsilon,lj}^{\ell} \geq 0.$$

The proof of the previous lemma is postponed to the Appendix.

Proof of Proposition 4.4. The proof is based on the same arguments as in [2, Theorem 3.5]. We fix $\delta > 0$ in such a way that (4.6) is satisfied, we consider the solution $V^{\varepsilon,\ell}$ of (4.8) and its mollification (component by component) V_ε. We define a function $w : \mathbb{R}^N \to \mathbb{R}^M$ by

$w_i := \min_{j=1,\dots,L} V^\ell_{\varepsilon,ij}$. By Lemma 4.5.(iii), it follows that w is a supersolution of (2.1). We define

$$m = \sup_{y\in\mathbb{R}^N, i=1,\dots,M} \{u^h_i(y) - w_i(y)\}$$

and

$$m_k = \sup_{y\in\mathbb{R}^N, i=1,\dots,M} \{u^h_i(y) - w_i(y) - k\phi(y)\} \tag{4.13}$$

where $\phi(y) = (1+|y|^2)^{1/2}$. Since w and u^h are bounded continuous functions, the supremum in (4.13) is achieved at a point x and at an index i, which is also a maximum point for $y \longmapsto u^h_i(y) - V^\ell_{\varepsilon,ij}(y) - k\phi(y)$, where $j = \arg\min_{l=1,\dots,L} V^\ell_{\varepsilon,il}(x)$. By Lemma 4.5.(iii), the definition of ϕ , (2.5a) and (2.5b), we get

$$\begin{aligned} &\sup_{\alpha\in\mathcal{A}^i} \mathcal{L}^\alpha_i(x, (V^\ell_{\varepsilon,ij} + k\phi)(x), D(V^\ell_{\varepsilon,ij} + k\phi)(x), D^2(V^\ell_{\varepsilon,ij} + k\phi)(x)) \\ &\qquad\qquad\qquad + \sum_{l=1}^M d_{il}(x,\alpha)(V^\ell_{\varepsilon,lj} + k\phi)(x) \ge -Ck. \end{aligned} \tag{4.14}$$

With the consistency assumption **(C4)**, we obtain

$$-Ck \le S_i(h, x, (\{V^\ell_{\varepsilon,lj}\}_{l=1}^M + k\Phi)(x), V^\ell_{\varepsilon,ij} + k\phi) + Q(V^\ell_{\varepsilon,ij} + k\phi) \tag{4.15}$$

where $\Phi = (\phi,\dots,\phi)$. Taking into account the definition of $Q(\cdot)$, $V^\ell_{\varepsilon,ij}$ and Φ, we obtain

$$-C\sum_{l\in J} \varepsilon^{1-l} h^{k_l} + O(k) \le S_i(h, x, (\{V^\ell_{\varepsilon,lj}\}_{l=1}^M + k\Phi)(x), V^\ell_{\varepsilon,ij} + k\phi). \tag{4.16}$$

Moreover by the definition of m_k, it follows that

$$(V^\ell_{\varepsilon,ij} + k\phi)(y) \ge u^h_i(y) - m_k \qquad \forall y \in \mathbb{R}^N,$$

and

$$u^h_i(x) - V^\ell_{\varepsilon,ij}(x) - k\phi(x) \ge u^h_l(x) - w_l(x) - k\phi(x) \ge u^h_l(x) - V^\ell_{\varepsilon,lj}(x) - k\phi(x)$$

and therefore

$$(u^h_i(x) - m_k) - (V^\ell_{\varepsilon,ij}(x) + k\phi(x)) = \max_l\{(u^h_l(x) - m_k) - (V^\ell_{\varepsilon,lj}(x) + k\phi(x))\}.$$

This with assumption **(C1)** yields:

$$\begin{aligned} S_i(h, x, (\{V^\ell_{\varepsilon,lj}\}_{l=1}^M + k\Phi)(x), V^\ell_{\varepsilon,ij} + k\phi) &\le S_i(h, x, (u^h - m_k)(x), u^h_i - m_k) \\ &\le -c_0 m_k + S_i(h, x, u^h(x), u^h_i) = -c_0 m_k. \end{aligned}$$

From the previous inequality and (4.16), sending $k \to 0$ we get the estimate

$$c_0 m \le C\sum_{l\in J} \varepsilon^{1-l} h^{k_l}. \tag{4.17}$$

Fix $y \in \mathbb{R}^N$ and $l \in \{1,\dots,M\}$, then for any $j \in \{1,\dots,L\}$

$$\begin{aligned} u^h_l(y) - u_l(y) \le u^h_l(y) - V^\ell_{\varepsilon,lj}(y) + V^\ell_{\varepsilon,lj}(y) - u_l(y) &\le u^h_l(y) - w_l(y) + V^\ell_{\varepsilon,lj}(y) - u_l(y) \\ &\le m + V^\ell_{\varepsilon,lj}(y) - u_l(y) \end{aligned}$$

By (4.17), Lemma 4.5(i) and (ii), we get

$$u_l^h(y) - u_l(y) \le C(\sum_{l\in J} \varepsilon^{1-l}h^{k_l} + \varepsilon + \ell + \ell^{1/3} + \delta),$$

and the statement of the theorem follows by taking $\varepsilon = \max_{l\in J} h^{\frac{3k_l}{3l-2}}$ and $\ell = 4\sup_{ij}[V_{ij}^{\varepsilon,\ell}]_1\varepsilon$ and sending δ to 0. ☐

5. Examples of approximation schemes

5.1. **Finite differences, one dimensional problem.** Let x be in $\mathbb{R}$, ϕ in $C^4(\mathbb{R})$, h in $\mathbb{R}_+^*$ and define

$$\delta_\pm\phi(x) = \frac{\phi(x\pm h) - \phi(x)}{\pm h}, \quad \Delta\phi(x) = \frac{\phi(x+h) - 2\phi(x) + \phi(x-h)}{h^2}.$$

In particular, by a Taylor expansion, we obtain

$$|\delta_\pm\phi(x) - D\phi(x)| \le \frac{1}{2}h|D^2\phi|, \quad |\Delta_h\phi(x) - D^2\phi(x)| \le \frac{1}{12}h^2|D^4\phi|.$$

Now, an approximation u^h to the solution of the coupled system (2.4) in dimension $N=1$, can be obtained by the finite difference scheme in $\mathbb{R}$:

$$\sup_{\alpha\in\mathcal{A}^i}\Big\{-\frac{1}{2}a_i(x,\alpha)\Delta u_i^h(x) - b_i^+(x,\alpha)\delta_+u_i^h(x) - b_i^-(x,\alpha)\delta_-u_i^h(x) + \lambda_i u_i^h(x) - f_i(x,\alpha)$$
$$+\sum_{j=1}^M d_{ij}(x,\alpha)u_j^h(x)\Big\} = 0,$$

for $i=1,\dots,M$, where $b_i^+(x,\alpha) = \max(b_i(x,\alpha),0)$, and $b_i^-(x,\alpha) = \min(b_i(x,\alpha),0)$. This scheme can be rewritten as:

$$S_i(h,x,u^h(x),u_i^h) = 0 \quad \text{in } \mathbb{R}, \text{ for } i=1,\dots,M$$

where the operator S_i is defined on $\mathbb{R}_+^*\times\mathbb{R}\times\mathbb{R}^M\times C_0(\mathbb{R})$ by:

$$S_i(h,x,r,w) := \sup_{\alpha\in\mathcal{A}^i}\Big\{-\frac{1}{2}a_i(x,\alpha)\frac{w(x+h)-2r_i+w(x-h)}{h^2} - b_i^+(x,\alpha)\frac{w(x+h)-r_i}{h}$$
$$-b_i^-(x,\alpha)\frac{r_i - w(x-h)}{h} + \lambda_i r_i - f_i(x,\alpha) + \sum_{j=1}^M d_{ij}(x,\alpha)r_j\Big\}, \tag{5.1}$$

From the Taylor expansion, one can easily check that the monotonicity **(C1)** holds (with $c_0 = \lambda_0$), and the consistency hypothesis **(C4)** is satisfied with $Q(\phi) = |D^2\phi|h + |D^4\phi|h^2$, i.e. $k_2 = 1$ and $k_4 = 2$. Then, by Propositions 4.1 and 4.4, we have

$$-Ch^{1/5} \le u - u_h \le Ch^{1/2}.$$

5.2. **The generalized finite differences scheme.** We consider the generalized finite differences scheme defined in [8]. Let ϕ be a real valued function. Let $h > 0$, $\xi \in \mathbb{Z}^N$ and consider the finite difference operator along direction ξ:

$$\Delta_\xi \phi(x) := \frac{\phi(x+\xi h) + \phi(x-\xi h) - 2\phi(x)}{h^2}.$$

On the other hand, we consider

$$(D^{\pm}\phi(x))_j = \begin{cases} \dfrac{\phi(x+he_j)-\phi(x)}{h} & \text{if } [b_i(x,\alpha)]_j \geq 0, \\ \dfrac{\phi(x)-\phi(x-he_j)}{h} & \text{if } [b_i(x,\alpha)]_j \leq 0. \end{cases}$$

Let $\mathcal{S}$ be a finite set of $\mathbb{Z} \setminus \{0\}$ containing $\{e_1, \dots, e_N\}$. We consider the following scheme:

$$\sup_{\alpha\in\mathcal{A}^i} \left\{ -\frac{1}{2}\sum_{\xi\in\mathcal{S}} \bar{\gamma}^i_\xi(x,\alpha)\Delta_\xi \Phi_i(x) - b_i(x,\alpha)D^{\pm}\Phi_i(x) + \lambda_i\Phi_i(x) - f_i(x,\alpha) \right.$$
$$\left. + \sum_{j=1}^{M} d_{ij}(x,\alpha)\Phi_j(x) \right\} = 0, \tag{5.2}$$

where the coefficients $\bar{\gamma}^i_\xi$ are given by:

$$\|a_i(x,\alpha) - \sum_{\xi\in\mathcal{S}} \bar{\gamma}^i_\xi \xi\xi^T\| = \min_{(\gamma_\xi)\in(\mathbb{R}_+)^{|\mathcal{S}|}} \|a_i(x,\alpha) - \sum \gamma_\xi \xi\xi^T\|.$$

For fast computations of the coefficients γ^i_ξ we refer to [9]. In the sequel, we make the strong consistency hypothesis (see also [23]):

$$a_i(x,\alpha) = \sum_{\xi\in\mathcal{S}} \bar{\gamma}^i_\xi(x,\alpha)\xi\xi^T, \quad \forall\, x \in \mathbb{R}^N, \forall\, \alpha \in \mathcal{A}^i. \tag{5.3}$$

The scheme defined in (5.2), can be rewritten as $S_i(h, x, u^h(x), u^h_i) = 0$, where for $i = 1, \dots, M$, the operator S_i is defined in $\mathbb{R}^*_+ \times \mathbb{R}^N \times \mathbb{R}^M \times C_0(\mathbb{R}^N)$ by:

$$\begin{aligned} S_i(h,x,r,w) \quad := \quad & \sup_{\alpha\in\mathcal{A}^i} \left\{ -\frac{1}{2}\sum_{\xi\in\mathcal{S}} \bar{\gamma}^i_\xi(x,\alpha)\frac{w(x+h\xi) - 2r_i + w(x-h\xi)}{h^2} \right. \\ & -\sum_{j=1}^{N}\left[\max\left(0, [b_i(x,\alpha)]_j\right)\frac{w(x+he_j)-r_i}{h} - \min\left(0,[b_i(x,\alpha)]_j\right)\frac{r_i - w(x-he_j)}{h}\right] \\ & \left. +\lambda_i r_i - f_i(x,\alpha) + \sum_{j=1}^{M} d_{ij}(x,\alpha) r_j \right\}. \end{aligned}$$

With straightforward calculations, one can check that the above scheme satisfies **(C1)** and **(C2)**. Moreover, under condition (5.3), if we consider a function $\Phi \in C^0(\mathbb{R}^N, \mathbb{R}^M)$ with $\Phi_i \in C^4(\mathbb{R}^N)$, then by applying a Taylor expansion, we obtain

$$|F_i(x,\Phi(x),D\Phi_i,D^2\Phi_i) - S_i(x,h,\Phi(x),\Phi_i)| \leq \sup_{\alpha\in\mathcal{A}^i}|b_i(.,\alpha)|_0|D^2\Phi_i|_0 h + \sup_{\alpha\in\mathcal{A}^i}|\sigma_i(\cdot,\alpha)|_0^2|D^4\Phi_i|_0 h^2.$$

Then the scheme satisfies the strong consistency **(C4)** with $k_2 = 1$ and $k_4 = 2$. We conclude that when the stencil $\mathcal{S}$ is chosen in such way condition (5.3) is satisfied, for h sufficiently small, the upper bound of the error estimate for the generalized finite difference scheme is of order $h^{1/2}$ and the lower bound is of order $h^{1/5}$.

Remark 5.1. Equation (5.2) can be rewritten as a fixed-point equation with a contraction operator. Indeed, let us introduce a *fictitious* time step $\tau > 0$ and introduce the operator $\mathcal{T} : C^0(\mathbb{R}^N, \mathbb{R}^M) \to C^0(\mathbb{R}^N, \mathbb{R}^M)$ defined by

$$[\mathcal{T}w]_i(x) := (1 - \tau\lambda_i w_i(x)) - \tau \sup_{\alpha\in\mathcal{A}^i} \Bigg\{ -\frac{1}{2}\sum_{\xi\in\mathcal{S}} \bar{\gamma}_\xi(x,\alpha)\Delta_\xi w_i(x) - b_i(x,\alpha)D^{\pm}w_i(x) \\ -f_i(x,\alpha) + \sum_{j=1}^{M} d_{ij}(x,\alpha)w_j(x) \Bigg\}.$$

Then, we can easily see that (5.2) is equivalent to

$$u(x) = [\mathcal{T}u](x), \qquad \text{on } \mathbb{R}^N. \tag{5.4}$$

Moreover, for τ small enough, the operator $\mathcal{T}$ is a monotone contraction, then the fixed-point equation $u = \mathcal{T}u$ admits a unique solution, and this solution is limit to any sequence defined by $u^{n+1} = \mathcal{T}u^n$, with $u^0 \in C(\mathbb{R}^N, \mathbb{R}^M)$. This iterative process, called *value iterations method* can be used to compute a numerical solution u^h on a grid $\mathcal{G}$.

5.3. **Semi-Lagrangian schemes.** Semi-Lagrangian schemes for second order Hamilton-Jacobi equations have been already studied in several papers, we refer to [1, 11, 12, 24, 25] for more details. Here, we use the Semi-Lagrangian scheme to approximate the weakly coupled system given in Example 2.4. We recall that the system (2.1) with F_i as in (2.3) is the dynamic programming equation of an infinite horizon optimal control problem with dynamics given by the stochastic differential equation

$$dX_t = b_{\nu_t}(X_t, \alpha_t)dt + \sigma_{\nu_t}(X_t, \alpha_t)dW_t \tag{5.5}$$

where $X_0 = x$, W_t is a standard Brownian motion, α_t is the control process and ν_t is a continuous-time random process with state space $\{1, \dots, M\}$ for which

$$\mathbb{P}\{\nu_{t+\Delta t} = j \,|\, \nu_t = i,\, X_t = x, \alpha_t = \alpha\} = c_{ij}(x,\alpha)\Delta t + O(\Delta t) \quad \text{for } \Delta t \to 0.\ j \neq i, \tag{5.6}$$

for any $i, j = 1, \dots, M$, $i \neq j$. We consider an approximation of (5.5) via a discrete-time control process $(X_n, \nu_n) \in \mathbb{R}^N \times \{1, \dots, M\}$ which evolves according to the following rule

$$\begin{cases} X_0 = x, \\ X_{n+1} = X_n + \left[hb_{\nu_n}(X_n,\alpha_n) + \sqrt{h}\sum_{m=1}^{d} \sigma_{\nu_n,m}(X_n,\alpha_n)\xi_n^m\right]\delta_{\nu_n,\nu_{n+1}}, \\ \mathbb{P}\{\nu_{n+1} = j \,|\, \nu_n = i,\, X_n = x,\, \alpha_n = \alpha\} = hc_{ij}(x,\alpha) \qquad j \neq i \end{cases} \tag{5.7}$$

for $n \in \mathbb{N}$, where $\sigma_{i,m}$ denote the m-th column of σ_i and ξ_n^m, $m = 1, \dots, D$, are random variables taking values in $\{-1, 0, 1\}$ such that

$$\mathbb{P}[\{\xi_n^i = \pm 1\}] = \frac{1}{2D} \qquad \text{and} \qquad \mathbb{P}[\{\xi_n^i \neq 0\} \cap \{\xi_n^j \neq 0\}] = 0, \quad i \neq j$$

and $\delta_{n,n} = 1$ and $\delta_{n,m} = 0$ for $n \neq m$. The discrete control $\{\alpha_n\}$ is a random variable which is measurable with respect to the σ-algebra generated by $X_1, \dots, X_n$ and such that $\alpha_n \in \mathcal{A}^{\nu_n}$.

Set $d_{ii} = \sum_{j=1}^{M} c_{ij}$, and $d_{ij} = -c_{ij}$ for every $j \neq i$, then the generator of the discrete process is

$$\mathcal{L}^h_{i,\alpha}(x, \Phi(x), \Phi_i) = \frac{\mathbb{E}_{x,i}[\Phi_{\nu_1}(X_1)] - \Phi_i(x)}{h} = L^h_{i,\alpha}(x, \Phi(x), \Phi_i) - \sum_{j=1}^{M} d_{ij}(x, \alpha)\Phi_j(x)$$

for $\Phi \in C^0(\mathbb{R}^N, \mathbb{R}^M)$, where

$$\begin{aligned} L^h_{i,\alpha}(x, r, \phi) =& \frac{1}{2Dh} \sum_{m=1}^{D} \Big[\phi(x + hb_i(x, \alpha) + \sqrt{h}\sigma_{i,m}(x, \alpha)) + \phi(x + hb_i(x, \alpha) \\ & - \sqrt{h}\sigma_{i,m}(x, \alpha)) - 2r_i\Big], \end{aligned} \tag{5.8}$$

Let $u^h = (u^h_1, \dots, u^h_M)$ be a solution of the system

$$S_i(h, x, u^h(x), u^h) = 0 \quad \text{in} \quad \mathbb{R}^N,\ i = 1, \dots, M \tag{5.9}$$

where

$$S_i(h, x, r, \phi) := \sup_{\alpha \in \mathcal{A}^i} \left\{ -(1 - \lambda_i h) L^h_{i,\alpha}(x, r, \phi) - f_i(x, \alpha) - (1 - \lambda_i h) \sum_{j=1}^{M} d_{ij}(x, a) r_j \right\}$$

and $L^h_{i,a}$ is as in (5.8). It is easy to see that the scheme (5.9) satisfies assumption **(C1)**–**(C3)**. Moreover, for $0 < h < 1$, $i = 1, \dots, N$ and for any $\Phi \in C^4(\mathbb{R}^N, \mathbb{R}^M)$ satisfying $|\Phi_i|_0 + \dots + |D^4\Phi_i|_0 < \infty$, we have

$$|F_i(x, \Phi(x), D\Phi_i(x), D^2\Phi_i(x)) - S_i(h, x, \Phi(x), \Phi_i)| \leq C_1 h(|D^2\Phi_i|_0 + |D^3\Phi_i|_0 + |D^4\Phi_i|_0) \tag{5.10}$$

also assumption **(C4)** is satisfied, giving a rate of convergence of order $h^{1/4}$ as in the case of the single equation (see [2]).

6. Appendix

We start by proving the regularity result given in Proposition 2.5.

Proof of Proposition 2.5. The bound on the solution u follows by the comparison principle after checking that $\frac{C}{\lambda_0}$ and $\frac{-C}{L}$ are, respectively, a super and a subsolution of (2.1).
To get the bound on the gradient of u consider

$$m := \sup_{i=1,\dots,M,\ x,y \in \mathbb{R}^N} \{u_i(x) - u_i(y) - L|x - y|\}.$$

If, by choosing a

$$L > \frac{\max_{i=1,\dots,M} \sup_{\alpha \in \mathcal{A}^i} |f_i(\cdot, a)|_1 + \max_{i,j=1,\dots,M} \sup_{\alpha \in \mathcal{A}^i} |d_{i,j}(\cdot, a)|_1 \max_{i=1,\dots,M} |u_i|_0}{\lambda_0 - \max_{i=1,\dots,M} \sup_{\alpha \in \mathcal{A}^i} |\sigma_i(\cdot, a)|_1^2 - \max_{i=1,\dots,M} \sup_{\alpha \in \mathcal{A}^i} |b_i(\cdot, a)|_1}$$

we con conclude that $m \leq 0$, the proof is achieved. Assume, for simplicity that the maximum is attained at $(\hat{x}, \hat{y})$ (if it is not one can modify the test function in a standard way). If $\hat{x} = \hat{y}$ then $m = 0$ and we are done. If not, at $(\hat{x}, \hat{y})$ the function $L|x - y|$ is smooth and the proof just follows a classical doubling argument and the same computation as in the proof of Proposition 6.1 below. For more detail see also the proof of [18, Theorem 5].

$\square$

In order to prove Lemma 4.2, we need the the following continuous dependence estimate.

Proposition 6.1. *Let u, $v \in C^{0,1}(\mathbb{R}^N)$ be a solution of* (2.3) *with $\mathcal{L}_i^\alpha$ as in* (2.4) *with coefficients $\{\sigma_i, b_i, \lambda_i, f_i, d_{ij}\}_{i=1}^M$ and, respectively, $\{\overline{\sigma}_i, \overline{b}_i, \overline{\lambda}_i, \overline{f}_i, \overline{d}_{ij}\}_{i=1}^M$ satisfying* (2.5a), (2.5b) *(with the same constant λ_0). Then there is a constant C such that*

$$\lambda_0 \max_{i=1,\dots,M} |u_i - v_i|_0 \le C \sup_{i=1,\dots,M} \sup_{\alpha\in\mathcal{A}^i} \{|\lambda_i - \overline{\lambda}_i|(|u_i|_0 \wedge |v_i|_0) + |\sigma_i(\cdot,a) - \overline{\sigma}_i(\cdot,a)|_0$$
$$+ |b_i(\cdot,a) - \overline{b}_i(\cdot,a)|_0 + |f_i(\cdot,a) - \overline{f}_i(\cdot,a)|_0 + \sum_{j=1}^M |d_{ij}(\cdot,a) - \overline{d}_{ij}(\cdot,a)|_0\}$$

Proof. The proof is a modification of [1, Theorem A.1], therefore we only details the difference with that. Define $m = \sup_{x\in\mathbb{R}^N, i=1,\dots,M}(u_i - v_i)$, $\phi(x,y) := \alpha|x-y|^2 + \varepsilon(|x|^2 + |y|^2)$ and $\psi_i(x,y) = u_i(x) - v_i(y) - \phi(x,y)$. Then set $m_{\alpha,\varepsilon} := \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N, i=1,\dots,M} \psi_i(x,y)$. A standard computation gives that there exists $(x_0,y_0) \in \mathbb{R}^N \times \mathbb{R}^N$ and $i_0 \in \{1,\dots,M\}$ such that $m_{\alpha,\varepsilon} = \psi_{i_0}(x_0,y_0)$. In the following we drop any dependence on α and ε. Arguing as in [1, Lemma A.2], we get

$$\begin{aligned}
0 \le \sup_{\alpha\in\mathcal{A}^{i_0}} \big\{ &- \frac{1}{2} Tr[\overline{a}_{i_0}(y_0,a)Y - a_{i_0}(x_0,a)X] - \overline{b}_{i_0}(y_0,a)(2\alpha(x_0-y_0) - 2\varepsilon y_0) \\
&+ b_{i_0}(x_0,a)(2\alpha(x_0-y_0) + 2\varepsilon x_0) - \overline{f}_{i_0}(y_0,a) + f_{i_0}(x_0,a) \\
&+ \sum_{j=1}^M (\overline{d}_{i_0 j}(y_0,a)v_j(y_0) - d_{i_0 j}(x_0,a)u_j(x_0)) + \overline{\lambda}_{i_0} v_{i_0}(y_0) - \lambda_{i_0} u_{i_0}(x_0)\big\}.
\end{aligned}$$

for some matrices X, $Y \in S^N$. The only additional term to estimate with respect to Lemma [1, Theorem A.1] is the last line of the previous inequality. By

$$m_{\alpha,\varepsilon} = u_{i_0}(x_0) - v_{i_0}(y_0) - \phi(x_0,y_0) \ge u_j(x_0) - v_j(y_0) - \phi(x_0,y_0),$$

we get

$$\begin{aligned}
v_{i_0}(y_0) - u_{i_0}(x_0) &\le -m_{\alpha,\varepsilon} \\
u_j(x_0) - v_j(y_0) &\le m_{\alpha,\varepsilon} + \phi(x_0,y_0).
\end{aligned}$$

Therefore, recalling that $d_{ii} \ge 0$, $d_{ij} \le 0$ we get

$$\sum_{j=1}^M (\overline{d}_{i_0 j}(y_0,a)v_j(y_0) - d_{i_0 j}(x_0,a)u_j(x_0)) + \overline{\lambda}_{i_0} v_{i_0}(y_0) - \lambda_{i_0} u_{i_0}(x_0)$$
$$\le \sum_{j=1}^M |\overline{d}_{i_0 j}(y_0,a) - d_{i_0 j}(x_0,a)||v_j(y_0)| - m_{\alpha,\varepsilon}(\sum_{j=1}^M \overline{d}_{i_0 j}(x_0,a) + \lambda_{i_0}) + |\lambda_{i_0} - \overline{\lambda}_{i_0}||v_{i_0}(y_0)|$$
$$\le \sum_{j=1}^M |\overline{d}_{i_0 j}(\cdot,a) - d_{i_0 j}(\cdot,a)|_0 \max_i |v_i|_0 + |\lambda_{i_0} - \overline{\lambda}_{i_0}| \max_i |v_i|_0 - \lambda_0 m_{\alpha,\varepsilon}$$

and we conclude as in [1, Lemma A.2]. □

Proof of Lemma 4.2. For existence and uniqueness of the solution to (4.2), we refer to [17, 19] (see also Theorem 2.2). The regularity of the solution, see (4.3), follows by Proposition 2.5, while the second inequality in (4.3) is consequence of Proposition 6.1.

For the statement *ii)* in (4.3), we refer to [1], Lemma 2.7. Note that, although here we are dealing with a system, the *weakly coupling* term $\sum_{j=1}^{M} d_{ij}(x,a)u_j$ is linear in u_j, $j = 1,\dots,N$ therefore the proof of [1, Lemma A3] can be straightforward adapted. □

We now give the proof of Lemma 4.5. We need a preliminary result.

Lemma 6.2. *Let $V : \mathbb{R}^N \to \mathbb{R}^{M\times L}$ be the solution of*

$$F_{ij}(x,V,DV_{ij},D^2V_{ij}) = 0 \qquad x\in\mathbb{R}^N,\ i=1,\dots,M,\ j=1,\dots,L \tag{6.1}$$

where

$$F_{ij}(x,R,p,X) = \max\left\{\sup_{\alpha\in\mathcal{A}^{ij}}\left(\mathcal{L}_i^\alpha(x,R_{ij},p,X)+\sum_{l=1}^{M} d_{il}(x,\alpha)R_{lj}\right),\mathcal{M}_{ij}(R)\right\},$$

where $\mathcal{A}^{ij}\subset\mathcal{A}^i$, $i=1,\dots,M$ and $\mathcal{M}_{ij}(R)$ as in (4.10)*. Let $u:\mathbb{R}^N\to\mathbb{R}^M$ be the solution of the system*

$$\sup_{\alpha\in\cup_{j=1}^M\mathcal{A}^{ij}}\left\{\mathcal{L}_i^\alpha(x,u_i,Du_i,D^2u_i)+\sum_{l=1}^{M} d_{il}(x,\alpha)u_l\right\} = 0 \quad i=1,\dots,M. \tag{6.2}$$

Then

$$0\le V_{ij}-u_i\le C\ell^{\frac{1}{3}} \qquad i=1,\dots,M,\ j=1,\dots,L \tag{6.3}$$

where C depends only on the bounds in the assumptions (2.5a) *and* (2.5b).

Proof. The proof is an adaptation of Theorem 2.3 in [2] to the system (6.1) so we just sketch it. First observe that the function $W:\mathbb{R}^N\to\mathbb{R}^{M\times L}$ such that $W_{ij}=u_i$, for any $j=1,\dots,M$, where u is the solution of (6.2), is a subsolution of (6.1), hence

$$u_i\le V_{ij} \qquad j\in\{1,\dots,M\}. \tag{6.4}$$

Now consider the system

$$F^\varepsilon_{ij}(x,V^\varepsilon,DV^\varepsilon_{ij},D^2V^\varepsilon_{ij}) = 0 \qquad x\in\mathbb{R}^N,\ i=1,\dots,M,\ j=1,\dots,L \tag{6.5}$$

where

$$F^\varepsilon_{ij}(x,R,p,X) = \max\left\{\sup_{\alpha\in\mathcal{A}^{ij},|e|\le\varepsilon}\left(\mathcal{L}_i^\alpha(x+e,R_{ij},p,X)+\sum_{l=1}^{M} d_{il}(x+e,\alpha)R_{lj}\right),\mathcal{M}_{ij}(R)\right\}$$

and let V^ε the corresponding solution. For any e such $|e|\le\varepsilon$, we have

$$\sup_{\alpha\in\mathcal{A}^{ij}}\left(\mathcal{L}_i^\alpha(x+e,V^\varepsilon_{ij},DV^\varepsilon_{ij},D^2V^\varepsilon_{ij})+\sum_{l=1}^{M} d_{il}(x+e,\alpha)V^\varepsilon_{lj}\right)\le 0.$$

Therefore, by change of variable, for any $|e|\le\varepsilon$, $V^\varepsilon_{ij}(x-e)$ is a subsolution of the system

$$\sup_{\alpha\in\mathcal{A}^{ij}}\left(\mathcal{L}_i^\alpha(x,W_{ij},DW_{ij},D^2W_{ij})+\sum_{l=1}^{M} d_{il}(x,\alpha)W_{lj}\right) = 0. \tag{6.6}$$

Now define $V_\varepsilon : \mathbb{R}^N \to \mathbb{R}^{N\times M}$ by $V_{\varepsilon,ij} = V^\varepsilon_{ij} * \rho_\varepsilon$. Then, by Lemma 4.2 *ii)*, V_ε is a subsolution to (6.6). Moreover since V^ε is a subsolution of (6.5) we have

$$V^\varepsilon_{ij} \le \min_{l\neq j} V^\varepsilon_{il} + \ell, \qquad i = 1,\dots,M,\ j = 1,\dots,L.$$

It follows that

$$0 \le \max_j V^\varepsilon_{ij} - \min_l V^\varepsilon_{il} \le \ell \tag{6.7}$$

hence $|V^\varepsilon_{ij} - V^\varepsilon_{il}|_0 \le \ell$, $i = 1,\dots,M$, $l,j = 1,\dots,L$. By the definition of V_ε, we get

$$|D^n V_{\varepsilon,ij} - D^n V_{\varepsilon,il}|_0 \le C\frac{\ell}{\varepsilon^n} \qquad i = 1,\dots,M,\ j,l = 1,\dots,L,\ n \in \mathbb{N}$$

where C only depends on the mollifier ρ_ε. Therefore for any $j,l = 1\dots,L$

$$\Bigg|\sup_{\alpha\in\mathcal{A}^{ij}} \mathcal{L}^\alpha_i(x, V_{\varepsilon,ij}, DV_{\varepsilon,ij}, D^2V_{\varepsilon,ij}) + \sum_{r=1}^M d_{ir}(x,\alpha)V_{\varepsilon,rj}$$
$$- \sup_{\alpha\in\mathcal{A}^{ij}} \mathcal{L}^\alpha_i(x, V_{\varepsilon,il}, DV_{\varepsilon,il}, D^2V_{\varepsilon,il}) + \sum_{r=1}^M d_{ir}(x,\alpha)V_{\varepsilon,rl}\Bigg| \le C\frac{\ell}{\varepsilon^2}$$

where C now depends also on the bounds L in assumptions (2.5a) and (2.5b).
Since V_ε is a subsolution of (6.6), we get that $\forall i = 1,\dots,M$

$$\sup_{\alpha\in\cup_{j=1}^L\mathcal{A}^{ij}} \mathcal{L}^\alpha_i(x, V_{\varepsilon,ij}, DV_{\varepsilon,ij}, D^2V_{\varepsilon,ij}) + \sum_{l=1}^M d_{il}(x,\alpha)V_{\varepsilon,lj} \le C\frac{\ell}{\varepsilon^2}$$

It follows that $V_{\varepsilon,ij} - \frac{1}{\lambda_0}C\frac{\ell}{\varepsilon^2}$, where λ_0 as in (2.5b), is a subsolution of (6.2). By the comparison principle for (6.2), we get

$$V_{\varepsilon,ij} - u_i \le \frac{C}{\lambda_0}\frac{\ell}{\varepsilon^2}, \qquad i = 1,\dots,M,\ j = 1,\dots,L.$$

Hence we conclude

$$V_{ij} - u_i \le V_{ij} - V_{\varepsilon,ij} + V_{\varepsilon,ij} - u_i \le C\varepsilon + \frac{C}{\lambda_0}\frac{\ell}{\varepsilon^2},$$

recalling (6.4) and minimizing with respect to ε we get the result. □

Proof of Lemma 4.5. The statement *i)* follows by assumption (4.6) and Proposition 6.1 for the estimate in ε and δ and by Lemma 6.2 for the estimate in ℓ.

We now consider *ii)*. By the definition of V_ε, it follows that

$$|V_{\varepsilon,ij} - V^\varepsilon_{ij}|_0 \le C\varepsilon \qquad i = 1,\dots,M,\ j = 1,\dots,L \tag{6.8}$$

where $C = \max_{i,j}[V^\varepsilon_{ij}]_1$. Recalling (6.7) in the proof of Lemma 6.2, by (6.8) we get (4.12).

We conclude with the proof of *iii)*. Let $x \in \mathbb{R}^N$, $i \in \{1,\dots,M\}$ and j be such that

$$j = \arg\min_{l=1,\dots,L}\{V_{\varepsilon,il}\}.$$

Then

$$V_{\varepsilon,ij}(x) - \mathcal{M}_{ij}V_\varepsilon(x) = \max_{l\neq j}\{V_{\varepsilon,ij} - V_{\varepsilon,il} - \ell\} \le -\ell$$

since $V_{\varepsilon,ij}(x) \le V_{\varepsilon,il}$ for any $l = 1, \dots, L$. By (4.12), we have

$$V^\varepsilon_{ij}(x) - \mathcal{M}_{ij}V^\varepsilon(x) \le V_{\varepsilon,ij}(x) - \mathcal{M}_{ij}V_\varepsilon(x) + 2\max_l |V^\varepsilon_{il}(x) - V_{\varepsilon,il}(x)| \le -\ell + 2\max_{i,j}[V^\varepsilon_{ij}]_1\varepsilon.$$

Let $|x-y| \le \varepsilon$ and $\varepsilon \le (4\max_{i,j}[V^\varepsilon_{ij}]_1)^{-1}\ell$. Then by the Lipschitz continuity of V^ε, we get

$$V^\varepsilon_{ij}(x) - \mathcal{M}_{ij}V^\varepsilon(x) \le -\ell + 2\max_{i,j}[V^\varepsilon_{ij}]_1(\varepsilon + |x-y|) < 0.$$

Hence by (4.8), we get

$$\inf_{|e|\le\varepsilon} \big\{\mathcal{L}_i^{\alpha_{ij}}(y+e, V^\varepsilon_{ij}(y), DV^\varepsilon_{ij}(y), D^2V^\varepsilon_{ij}(y)) + \sum_{l=1}^M d_{il}(y+e,\alpha_{ij})V^\varepsilon_{lj}(y)\big\} = 0.$$

By this point the proof goes as in [2, Lemma 3.4(b)]. Note that the only difference linked with the fact that here we are dealing with a system can be proved by arguing as in the proof of Lemma 4.2 *ii)*. $\square$

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