Università di Trento

DOTTORATO DI RICERCA IN MATEMATICA

PhD Thesis

Boundary Properties for Almost-Minimizers of the Relative Perimeter

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Contents

Introduction

Through the present dissertation, we aim to illustrate some outcomes related to the study of the boundary behavior of almost-minimizers of the relative perimeter in an Euclidean open subset. The main results obtained are on the one hand a boundary Monotonicity Formula, for which we refer to [23], and on the other hand two results regarding the behavior of an almost-minimizer of the relative perimeter in an open set Ω near a *vertex-type singularity* of $\partial\Omega$ [24, 25]. The latter results are discussed in Chapter 4 and Chapter 5. As it will be carefully explained later, we remark that the definition of vertex appearing in Chapter 5 is different from the classical notion of vertex for a cone, since it is quite more general.

Given an open set $\Omega \subset \mathbb{R}^n$ and a measurable set $E \subset \Omega$, we define the relative perimeter of E in Ω restricted to an open set $A \subset \mathbb{R}^n$ as

$$P_\Omega(E; A) := P(E; \Omega \cap A)$$

with the short form $P_\Omega(E)$ when $A \supset \Omega$, and with $P(E; B)$ denoting the standard perimeter of E in an open set B (see Definition 1.2.2). We say that E is an almost-minimizer of the relative perimeter in Ω if, roughly speaking, E minimizes P_Ω among those measurable sets $F \subset \Omega$ that are obtained from E via a compact variation into $\overline{\Omega}$ (Definition 2.1.1). In particular, the competitor F could differ from E up to the boundary of Ω. Accordingly, the present definition of almost-minimality wishes for someway generalize the notion of free-boundary area-minimizing surface.

Boundary Monotonicity Formula

In the third chapter, we focus on the proof of the boundary Monotonicity Formula, and then we apply it to the proof of a perimeter-minimizing cone property for the limit of a blow-up sequence of an almost-minimizer. The Monotonicity Formula is a tool that for instance permits proving that if E is an almost-minimizer of the relative perimeter in an open set $\Omega \subset \mathbb{R}^n$ and $x_0 \in \overline{\Omega} \cap \partial E$, the density ratios between the perimeter of E in the relative ball $B_r(x_0) \cap \Omega$ and r^{n-1} are monotonically non-decreasing in r up to an error term that goes to 0 as $r \to 0^+$ in a quantified way. The Monotonicity Formula is known and classical when x_0 is internal to Ω, and a proof can be found in [13]. It is known also for $x_0 \in \partial\Omega$, but typically under smoothness assumptions on $\partial\Omega$ ([2], [14]) In our main result Theorem 3.3.2, we prove a boundary Monotonicity Formula at a boundary point $x_0 \in \partial\Omega$ provided the domain Ω satisfies a *visibility condition* at x_0. Roughly speaking, this condition requires the existence of $R > 0$ and a point V_r, for $0 < r < R$, slightly displaced with respect to x_0 such that each point $x \in \partial\Omega \cap B_r(x_0)$ is visible from V_r, i.e. the segment connecting V_r with x does not intersect Ω. This condition leads to the construction of a quasi-conical competitor for E, allowing to prove the boundary Monotonicity Formula through a suitable adaptation of the classical argument developed by Giusti in [13], where indeed the construction of a conical competitor represents a key passage. The aforementioned visibility condition does not constitute a smoothness assumption on Ω, since it is satisfied, for instance, by a generic convex set, possibly singular at 0, provided it is suitably approximated by its tangent cone at 0 (see Example 3.2.12), and even by other open sets with quite rough boundary (Example 3.2.13). Then, in the final part of Chapter 3, we show how the boundary Monotonicity Formula previously obtained can be employed to prove that any blow-up sequence of an almost-minimizer E of the relative perimeter admits a subsequence that L^1_{loc}-converges to a minimal cone in the tangent cone to Ω at x_0, denoted by Ω_{x_0}.

The Monotonicity Formula is an important tool in the study of the regularity of an almost-minimizer of the relative perimeter ([7], [13], [26]), and also of more general objects, like varifolds. In his seminal work [1], Allard proves an interior Monotonicity Formula for integral, k-dimensional varifolds with L^p-mean curvature, $p > k$, that is subsequently used in the proof of an internal regularity result for the same varifolds. In this case, the proof of the Monotonicity Formula is made using a first variation of the varifold along a suitable vector field. We also mention [2], where the same author proves a boundary Monotonicity Formula for varifolds with $C^{1,1}$ boundary. In [14], the authors focus on the study of the boundary regularity for free-boundary integral k-varifolds. In particular, they prove that if the varifold V intersects $\partial\Omega$ orthogonally, V has L^p mean curvature with $p > k$ and the mass of V inside small balls centered at points of $\partial\Omega$ is close to the volume of a half ball, then V is a $C^{1,\alpha}$ submanifold of $\mathbb{R}^n$ with boundary, that is diffeomorphic to a half ball. Also in this case, the Monotonicity Formula represents a key tool in the proof of the main regularity result, while its proof is based in particular on a reflection through $\partial\Omega$ procedure, that requires Ω at least of class C^2. The common aspect among all the aforementioned works is the presence of a smoothness hypothesis on Ω. Indeed, a key passage in the proof of the boundary Monotonicity Formula is to vary the varifold along a suitable vector-field tangent to $\partial\Omega$. The technical difficulties arising in implementing variation arguments in the presence of singularities of $\partial\Omega$ motivate why the big majority of the works available in the literature concerning the boundary Monotonicity Formula, and consequently the study of the boundary regularity, require some smoothness assumptions on Ω. As we remarked above, the proof of the boundary Monotonicity Formula developed in Chapter 3 is not based on a variation argument, but exploits the construction of a quasi-conical competitor, allowing to skip the strong smoothness assumptions on $\partial\Omega$ that are typically required. It is worth mentioning that such a boundary Monotonicity Formula could be applied to the study of the boundary regularity for almost-minimizers of the relative perimeter. A regularity result in this sense is known only if $\partial\Omega \in C^2$ [14]. An interesting development could be to exploit the boundary Monotonicity Formula proved in the present thesis in order to extend the regularity result in [14] to more general domains Ω, for instance for those with $\partial\Omega \in C^{1,\beta}$.

Behavior of the almost-minimizers near vertices

A fascinating question regarding the theory of the free-boundary minimal surfaces is that about their behavior near the boundary of their container Ω. In the broader context of capillarity, it is known that if Ω is of class $C^{1,1}$, a capillary surface interface in Ω satisfies Young's law (see [30]), i.e., forms a contact angle $\beta = \arccos\gamma$ with $\partial\Omega$, where $\gamma \in [-1, 1]$

is the wetting coefficient appearing in the capillary energy (without bulk terms)

$$P_{\Omega}(E) + \gamma P(E; \partial\Omega)\,,$$

this resulting in the relative perimeter when $\gamma = 0$. The proof of Young's law relies on a boundary regularity result for local minimizers of the capillary energy, which requires $\partial\Omega$ to be sufficiently smooth [8]. However, despite of its relevance in applications [12], much fewer is known, in general, about the behavior of a capillary surface near singular points of $\partial\Omega$. We mention here some works by Concus-Finn [5], Lancaster [21], Chen-Finn-Miersemann [4], Lancaster-Siegel [20], Tamanini [29], Leonardi-Saracco [22], where the authors focus their attention on the behavior of a capillary surface close to a wedge- or corner-type singularity. About the case of free-boundary minimal surfaces ($\gamma = 0$), we also recall the study on free-boundary minimal surfaces in 3-dimensional wedge domains, mainly due to Hildebrandt and Sauvigny [16–19], and the recent contribution by Edelen and Li [9], where the authors demonstrate an ε-regularity result for free-boundary minimal surfaces in domains that are close to a polyhedral-cone, i.e. are the image of a polyhedral cone through a transformation that is C^2-close to the identity. The present dissertation aspires to take place among the literature mentioned above, providing a survey about the behavior of an almost-minimizer of the relative perimeter (or a free-boundary minimal surface) near a vertex-type singularity of the container. This analysis is carried out in two different but complementary directions. In Chapter 4, we focus on a specific setting. Given an axially-symmetric, $(n+1)$-dimensional convex cone, that up to translations is assumed to coincide with

$$\Omega_\lambda := \left\{ (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \,:\, x_n > \lambda\sqrt{x_1^2 + ... + x_{n-1}^2 + x_{n+1}^2} \right\}\,,$$

for some $\lambda > 0$, we study the stability of the free-boundary minimal surface Σ obtained by the intersection of Ω_λ with a n-plane containing the axis of Ω_λ. For our stability analysis, we introduce a Lipschitz flow $\Sigma_t[f]$ of deformations of Σ associated with a compactly-supported, scalar deformation field f, which satisfies the key property $\partial\Sigma_t[f] \subset \partial\Omega_\lambda$, for all $t \in \mathbb{R}$. The construction is actually performed in more general convex domains, in particular those that can be represented as epigraphs of a convex map. By computing the lower-right second variation of the area of Σ along this flow, we discover that the stability of Σ into Ω_λ depends on $\dim\Sigma = n$. When $n = 2$, in Theorem 4.3.3, we prove that Σ is not stable; consequently, Σ fails to be area-minimizing in a 3-dimensional circular cone. While, when $n \geq 3$, in Theorem 4.3.5 we show something rather surprising, namely that, as Ω_λ has a sufficiently large aperture, i.e. λ is small enough, then Σ is strictly stable. This behavior sinks its roots in the feature assumed by the non-negativity condition for the second variation of the area of Σ. Indeed, we proved that the latter is equivalent to the validity of the following functional relation, for every $f : \Sigma \to \mathbb{R}$ Lipschitz continuous and compactly supported into $\overline{\Sigma}$:

$$\int_\Sigma |\nabla f|^2 \geq \lambda \int_{\partial\Sigma} \frac{f(y)^2}{|y|} \, d\mathcal{H}^{n-1}(y) \, .$$

This functional inequality, in a slightly different shape, is known in literature, in particular in the context of reaction-diffusion problems [6], and is called *Kato's Inequality*. This inequality fails when $n = 2$ (because, if f is compactly supported and $f(0) \neq 0$, then the right hand side explodes), while it is proved to hold when $\dim\Sigma \geq 3$, with a suitable dimensional constant in place of λ. Thus, as $n \geq 3$ and λ is smaller than a suitable threshold $\lambda^* \equiv \lambda^*(n)$, the stability of Σ follows. The different (and, at first sight, quite unexpected) stability properties of Σ have a correspondence in some literature on minimal surfaces within cones. In particular, a result of Morgan [27] implies that the free-boundary $\partial\Sigma$ is an area-minimizing $(n-1)$-surface in $\partial\Omega_\lambda$ as soon as $n \geq 4$ and λ is small enough. The investigation initiated in Chapter 4 opens the doors to other interesting questions. For instance, one could ask whether, for $\lambda \leq \lambda^*$, Σ is only stable or even area-minimizing into Ω_λ. A possible approach for the proof of the minimality of Σ into Ω could be the construction of an appropriate (sub-)calibration of Σ into Ω. The calibration method is the same employed by Morgan in [27] to demonstrate his minimality result for $\partial\Sigma$ into $\partial\Omega_\lambda$ in dimension $n \geq 4$. We then expect that, via a suitable adaptation of this construction, we should be able to prove the minimality of Σ into Ω_λ at least for $n \geq 4$. However, the same kind of analysis in the case $n = 3$ seems harder, as it is not fully clear whether $\partial\Sigma$ is unstable, or at least not area-minimizing, in $\partial\Omega_\lambda$. We limit to notice that our result seems to support the conjecture that Σ is locally area-minimizing also when $n = 3$.

In the last Chapter of the thesis, we concentrate on the proof of a so-called *Vertex-skipping Theorem*. The question underlying this result generalizes that which moved us in the study of the minimality of Σ in an axially symmetric cone: our purpose is to understand whether the boundary of an almost-minimizer of the relative perimeter in an open, convex set $\Omega \subset \mathbb{R}^n$ could contain vertex-type singularities of $\partial\Omega$. We say that a point $x_0 \in \partial\Omega$ is a vertex for Ω if the tangent cone Ω_{x_0} to Ω at x_0 does not contain lines, i.e., up to isometries, Ω_{x_0} cannot be written as $\mathbb{R} \times C$, for some convex cone $C \subset \mathbb{R}^{n-1}$. We notice that here a vertex x_0 of Ω is not the vertex of a cone, i.e. it is not required that $x_0 + t(x - x_0) \in \Omega$, for every $x \in \Omega$. Nonetheless, we remark that, when Ω is a cone having a vertex at x_0, the two definitions trivially coincide. With a little abuse of terminology, saying vertex, we will refer both to the vertex of a cone and to the vertex of an open, convex set Ω in the sense of the definition just provided. The context shall clarify which is the correct interpretation of the terminology. In our main result Theorem 5.3.1, we prove what follows.

Let $\Omega \subset \mathbb{R}^3$ be an open, convex set, and let $x_0 \in \partial\Omega$ be a vertex for Ω. If $E \subset \Omega$ is a local almost-minimizer of the relative perimeter in Ω, then

$$x_0 \notin \overline{\partial E \cap \Omega}\,,$$

i.e. the closure of the boundary of E does not contain vertex-type singularities of the boundary of Ω.

This result generalizes Theorem 4.3.4 proved in Chapter 4. On the other hand, it is worth to mention that the validity of such a result in dimension $n \geq 4$ seems improbable, owing to the stability result Theorem 4.3.5. The proof of the statement above is based on a contradiction argument, that is composed of three main steps. The first one is a double blow-up procedure, that allows to reduce the problem to the case of a conical minimizer of the relative perimeter in the tangent cone to Ω at x_0, and containing x_0. We notice that, since more classical Monotonicity Formulas (such as that proved in [13]) are enough for the proof of Theorem 5.3.1, the boundary Monotonicity Formula proved in Chapter 3 does not play a direct role here. Nevertheless, its application permits to show that, under some further assumptions on Ω (mainly those allowing to apply Theorem 3.4.1), a single blow-up of Ω at x_0 is enough in the context of our proof (Remark 5.1.5). The second step consists of the characterization of the boundary of the conical minimizer obtained through the aforementioned blow-up operation. We in particular show that the boundary of this minimizer is a plane containing the vertex. The proof of this fact is very delicate: in particular, it is based on the properties of geodesic triangles on the sphere. The final step is devoted to construct a competitor with less area than the plane, achieving the desired contradiction. To do so, the idea is to *pack* the tangent cone into a cone with rectangular section, that we called *pyramid-cone*, performing then the construction of the competitor in this special cone. This construction is non-trivial: to build a competitor with less area than the plane in the pyramid-cone, the idea is to displace the plane itself along its normal up to a suitable height, and then find an astute connection with the original plane to produce the competitor. We conclude by highlighting an important consequence of the Vertex-skipping Theorem 5.3.1 in combination with Theorem 1.1 proved in the already cited paper by Edelen-Li [9]: the singular set of a 2-current in a 3-dimensional plyhedral domain is empty. This improves Theorem 1.2 demonstrated in [9].

Chapter 1

Bounded Variation functions

In this Chapter, we introduce the notions of variation for the gradient of a measurable function and of perimeter for a measurable set, namely the total variation of its characteristic map. Then we illustrate the main properties satisfied by the functions with bounded variation.

1.1 Differentiation of Radon measures

For a definition of Radon measure, real Radon measure, or vector Radon measure we refer to [3, Definition 1.40]. With a little abuse of terminology, saying Radon measure we will equivalently refer to a Radon measure, a real Radon measure, or a vector Radon measure. If μ is a Radon measure on $\mathbb{R}^n$ taking values in $\mathbb{R}^p$, we denote by $|\mu|$ its total variation (see [3, Definition 1.4]). Let $u = (u_1, ..., u_p) : \mathbb{R}^n \to \mathbb{R}^p$ be summable with respect to $|\mu|$, then we denote by $u \cdot \mu$ the Radon measure defined by

$$u \cdot \mu(E) := \int_E u \cdot d\mu = \sum_{q=1}^{p} \int_E u_q d\mu_q \, .$$

It can be proved (see [3, Proposition 1.23]) that

$$|u \cdot \mu| = |u| \cdot \mu \, . \tag{1.1.1}$$

We have the following result, whose proof can be found in [26, Corollary 5.11]. From now on, by $B_r(x)$ we will denote a ball of center x and radius r, and denoting by B_r the ball $B_r(0)$.

Lemma 1.1.1. *Let μ, ν be Radon measures taking values in $\mathbb{R}$, $\mathbb{R}^p$ respectively. Then, for μ-a.e. $x \in \mathbb{R}^n$, the following limit exists*

$$D_\mu \nu(x) := \lim_{r \to 0^+} \frac{\nu(B_r(x))}{\mu(B_r(x))},$$

and the corresponding function f is such that $f \in L^1_{\text{loc}}(\mathbb{R}^n, \mu; \mathbb{R}^p)$ and

$$\nu = D_\mu \nu \cdot \mu + \nu^s_\mu,$$

where ν^s_μ is singular with respect to μ.

The function $D_\mu \nu$ is called the *μ-density of ν*.

1.2 Variation and perimeter

We start with the following

Definition 1.2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open subset and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. The* variation of f in Ω *is defined by*

$$|Df|(\Omega) = \sup \left\{ \int_\Omega f \, \operatorname{div} \phi \, dx \ : \ \phi \in C^1_c(\Omega; \mathbb{R}^n), \ ||\phi||_{L^\infty(\Omega)} \le 1 \right\}.$$

We say that f has bounded variation *in Ω, and we write $f \in BV(\Omega)$, provided $f \in L^1(\Omega)$ and*

$$|Df|(\Omega) < +\infty.$$

When f coincides with the characteristic of a measurable set, we use the following terminology.

Definition 1.2.2. *Let $\Omega \subset \mathbb{R}^n$ be an open subset and $E \subset \mathbb{R}^n$ be a measurable set. We call* perimeter *of E in Ω the following quantity*

$$P(E; \Omega) = |D\mathbf{1}_E|(\Omega),$$

We say that E has finite perimeter in Ω *if* $P(E;\Omega) < +\infty$.

In the next Chapters, we will often use the following notation: given two open sets Ω, $A \subset \mathbb{R}^n$ and a measurable set $E \subset \Omega$, we set

$$P_\Omega(E;A) := P(E;A\cap\Omega)\,,$$

with the short form $P_\Omega(E;A) := P(E)$ when A contains Ω. We call P_Ω the relative perimeter in Ω, and $P_\Omega(E;A)$ the relative perimeter of E in Ω restricted to A.
Sometimes we will also use the following terminology.

Definition 1.2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set.*

(i) We say that $f \in L^1_{\text{loc}}(\Omega)$ has locally bounded variation *in Ω if*

$$|Df|(A) < +\infty\,, \qquad \textit{for any } A \subset\subset \Omega,\ A \textit{ open.}$$

(ii) We say that a measurable subset $E \subset \Omega$ is a Caccioppoli Set *(or that has* locally finite perimeter*) in Ω, if*

$$P(E;A) < +\infty\,, \qquad \textit{for any } A \subset\subset \Omega,\ A \textit{ open.}$$

We will denote by $BV_{\text{loc}}(\Omega)$ the space of the functions $f \in L^1_{\text{loc}}(\Omega)$ such that

$$f \in BV(\Omega\cap A)\,, \qquad \text{for all open, bounded sets } A \subset \Omega\,.$$

We remark that $f \in BV_{\text{loc}}(\Omega)$ implies that f has locally bounded variation in Ω, but the viceversa does not hold.

A bounded variation function can be characterized by its distributional gradient. The proof of the result below corresponds to the proof of Theorem 5.1 in [10].

Theorem 1.2.4. *Let $\Omega \subset \mathbb{R}^n$ be open. A function $f \in L^1(\Omega)$ has bounded variation in Ω if and only if its distributional gradient is a Radon measure, i.e. there exists a vector-valued Radon measure $\mu_f : \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}^n$ with the following property:*

$$-\int_\Omega f\ \mathrm{div}\,\phi\,dx = (\phi\cdot\mu_f)\,\Omega\,, \qquad \textit{for all } \phi \in C^1_c(\Omega;\mathbb{R}^n).$$

Moreover, the total variation of μ_f and the variation of f coincide, i.e.

$$|\mu_f|(\Omega) = |Df|(\Omega)\,.$$

From now on, with a little abuse of notation, we will identify Df and μ_f. Let now $f \in BV(\Omega)$. As a consequence of Lemma 1.1.1 we can consider the $|Df|$-density of Df, that we call ν_f, which is defined by

$$\nu_f(x) = \lim_{r\to 0^+} \frac{Df(B_r(x))}{|Df|(B_r(x))}\,, \qquad \text{for } |Df|\text{-a.e. } x \in \mathbb{R}^n\,.$$

We have $\nu_f \in L^1(\mathbb{R}^n, |Df|; \mathbb{R}^n)$. Since trivially Df is absolutely continuous with respect to $|Df|$, ν_f realizes

$$Df = \nu_f \cdot |Df|\,. \tag{1.2.1}$$

Moreover, from (1.1.1), (1.2.1), it follows that

$$|\nu_f| \cdot |Df| = |\nu_f \cdot |Df|| = |Df|\,,$$

and so that $\nu_f(x) = 1$ at $|Df|$-a.e. $x \in \mathbb{R}^n$. When E has finite perimeter in Ω, we set

$$\nu_E = \nu_{\mathbf{1}_E}\,. \tag{1.2.2}$$

The variation $|Df(\Omega)|$ is lower-semicontinuous with respect to the L^1_{loc}-convergence. The proof of the following result is given in [10].

Lemma 1.2.5. *Let $\Omega \subset \mathbb{R}^n$ be open, and $\{f_j\}_{j\geq 1} \subset L^1_{\text{loc}}(\Omega)$ be a sequence of functions having locally bounded variation in Ω that L^1_{loc} converges to a function f. Then the following inequality holds:*

$$|Df|(\Omega) \leq \liminf_j |Df_j|(\Omega)\,.$$

1.3 Properties of BV functions

1.3.1 Approximation and compactness

BV functions can be approximated by smooth functions in a suitable sense.

Theorem 1.3.1. *Let $\Omega \subset \mathbb{R}^n$ be open and $f \in BV(\Omega)$. Then there exists a sequence $\{f_j\}_{j\geq 1} \in C^\infty(\Omega) \cap BV(\Omega)$ such that*

$$||f_j - f||_{L^1(\Omega)} \to 0\,, \qquad |Df_j|(\Omega) \to |Df|(\Omega)\,. \tag{1.3.1}$$

Moreover, provided (1.3.1) *holds, we also have $Df_j \rightharpoonup^* Df$, i.e., for all $\phi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \phi \cdot dDf_j \to \int_{\mathbb{R}^n} \phi \cdot dDf\,, \qquad \text{as } j \to \infty\,.$$

For the proof of the Theorem above, see Theorems 5.3 and 5.4 in [10].

Remark 1.3.2. *If condition* (1.3.1) *holds, we say that the sequence f_j* strictly converges *to f (see [3, Definition 3.14]). Thus Theorem 1.3.1 tells us in particular that $C^\infty(\Omega) \cap BV(\Omega)$ is dense in $BV(\Omega)$ with respect to the strict convergence.*

Remark 1.3.3. *The sequence $\{f_j\}_{j\geq 1}$ built for the proof Theorem 1.3.1 turns out to have the following, further property (see [13, Remark 1.18])*

$$\lim_{\rho \to 0^+} \rho^{-N} \int_{\Omega \cap B_\rho(x_0)} |f_j - f| dx = 0\,, \qquad \text{for all } N > 0,\ x_0 \in \partial\Omega,\ j \geq 1\,. \tag{1.3.2}$$

Combining Theorem 1.3.1 with the compactness result holding for Sobolev functions, one can prove the following compactness result for BV functions, provided the domain is regular enough (see [10, Theorem 5.5]).

Theorem 1.3.4. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with Lipschitz boundary, and $\{f_j\}_{j\geq 1} \in BV(\Omega)$ be such that, for some constant $C > 0$,*

$$||f_j||_{BV(\Omega)} := ||f_j||_{L^1(\Omega)} + |Df_j|(\Omega) \leq C\,, \qquad \text{for every } j\,. \tag{1.3.3}$$

Then there exists a function $f \in BV(\Omega)$ and a subsequence f_{j_k} of f_j such that

$$||f_{j_k} - f||_{L^1(\Omega)} \to 0\,.$$

1.3.2 Coarea

Theorem 1.3.5 (Coarea for BV functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and $f \in L^1(\Omega)$. For $t \in \mathbb{R}$, let*

$$E_t = \{x \in \Omega \, : \, f(x) > t\} \, .$$

Then the following statements hold:

(i) if $f \in BV(\Omega)$, then E_t has finite perimeter for a.e. $t \in \mathbb{R}$, $t \mapsto P(E_t;\Omega)$ is measurable, and

$$|Df|(\Omega) = \int_{\mathbb{R}} P(E_t;\Omega)dt \, ;$$

(ii) conversely, if $f \in L^1(\Omega)$, $t \mapsto P(E_t;\Omega)$ is measurable, and

$$\int_{\mathbb{R}} P(E_t;\Omega)dt < +\infty \, ,$$

then $f \in BV(\Omega)$.

Proof. The proof of the measurability of $t \mapsto P(E_t;\Omega)$ is given in [10, Lemma 5.1]. The other statements are proved in Theorem 5.9 of the same book. □

We also give the following Coarea result for Lipschitz continuous maps.

Theorem 1.3.6 (Coarea for Lipschitz functions)**.** *If $\phi \in \mathrm{Lip}(\mathbb{R}^n)$, $E \subset \mathbb{R}^n$ is a Borel set, and $f : \mathbb{R}^n \to [0,\infty)$ is a Borel function, then*

$$\int_E f|\nabla \phi| = \int_{\mathbb{R}} \int_{E \cap \{\phi = t\}} f \, d\mathcal{H}^{n-1} \, dt \, .$$

1.3.3 Traces

Theorem 1.3.7. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with Lipschitz boundary and denote by ν_Ω the outer, unit, normal vector defined $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$. Then there exists a unique linear and bounded operator*

$$\mathrm{Tr}_(f, \partial\Omega) : BV(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1})$$

called trace operator *such that, for all* $f \in BV(\Omega)$,

$$\lim_{r\to 0^+} \fint_{B_r(x)\cap\Omega} |\mathrm{Tr}(f,\partial\Omega)(x) - f(y)|\, dy = 0\,, \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Omega\,. \tag{1.3.4}$$

Moreover, for all $f \in BV(\Omega)$, *the following identity holds:*

$$\int_\Omega f\, \operatorname{div}\phi\, dx = -(\phi\cdot Df)\Omega + \int_{\partial\Omega} \langle \phi, \nu_\Omega\rangle\, \mathrm{Tr}(f,\partial\Omega)\, d\mathcal{H}^{n-1}\,, \qquad \text{for any } \phi \in C^1_c(\mathbb{R}^n,\mathbb{R}^n)\,. \tag{1.3.5}$$

Proof. The linearity and the uniqueness of $\mathrm{Tr}(f,\partial\Omega)$ immediately follow by condition (1.3.4). The existence is proven in [13, Theorem 2.10]. □

The function $\mathrm{Tr}(f,\partial\Omega)$ is called the *trace* of f in Ω. Identity (1.3.4) ensures in particular that

$$\mathrm{Tr}(f,\partial\Omega)(x) = \lim_{r\to 0^+} \fint_{B_r(x)\cap\Omega} f(y)\, dy\,, \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial\Omega\,. \tag{1.3.6}$$

Hence, the trace can be interpreted as the boundary value on $\partial\Omega$ assumed by the function f. In particular, if $f \in BV(\Omega)\cap C^0(\overline{\Omega})$, then (1.3.6) yields

$$\mathrm{Tr}(f,\partial\Omega) = f|_{\partial\Omega}\,, \qquad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega\,.$$

Remark 1.3.8. *Under the assumptions of Theorem 1.3.7, the operator* $\mathrm{Tr}(\cdot,\partial\Omega)$ *is also continuous with respect to the strict convergence (see [13, Theorem 2.11] or [3, Theorem 3.88]). We also observe that, selecting a suitable sequence* $\{f_j\}_{j\ge 1} \subset C^\infty(\Omega)\cap BV(\Omega)$ *realizing the thesis of Theorem 1.3.1, by applying* (1.3.2) *and* (1.3.6) *we have*

$$\begin{aligned} |\mathrm{Tr}(f_j,\partial\Omega)(x) - \mathrm{Tr}(f,\partial\Omega)(x)| &= \left| \lim_{r\to 0^+} \fint_{B_r(x)\cap\Omega} f_j(y)\, dy - \lim_{r\to 0^+} \fint_{B_r(x)\cap\Omega} f(y)\, dy \right| \\ &\le \lim_{r\to 0^+} \fint_{B_r(x)\cap\Omega} |f_j(y) - f(y)|\, dy \\ &= 0\,, \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial\Omega\,, \end{aligned} \tag{1.3.7}$$

and thus $\mathrm{Tr}(f_j,\partial\Omega) = \mathrm{Tr}(f,\partial\Omega)$, $\mathcal{H}^{n-1}$*-a.e. on* $\partial\Omega$.

Let Ω, $\Omega' \subset \mathbb{R}^n$ open, bounded sets with Lipschitz boundary such that $\Omega \subset\subset \Omega'$.

We observe that $\partial\Omega$ can be oriented by ν_Ω, the unit, normal vector that points out with respect to Ω, or by $-\nu_\Omega$, the unit, normal vector that points out with respect to $\Omega' \setminus \overline{\Omega}$. For any $f \in BV(\Omega')$, we can then consider two traces for f on $\partial\Omega$, the first one with respect to Ω, the second one with respect to $\Omega' \setminus \overline{\Omega}$:

$$\mathrm{Tr}^+(f, \partial\Omega) = \mathrm{Tr}(f, \partial\Omega)\,, \qquad \mathrm{Tr}^-(f, \partial\Omega) = \mathrm{Tr}(f, \partial(\Omega' \setminus \overline{\Omega}))\,.$$

From (1.3.6), it follows that the definition of $\mathrm{Tr}^-(f, \partial\Omega)$ does not depend on Ω'. The functions $\mathrm{Tr}^\pm(f, \partial\Omega) \in L^1(\partial\Omega; \mathcal{H}^{n-1})$ are called the *inner trace* and the *outer trace* of f in Ω respectively. The following extension of BV functions result holds.

Lemma 1.3.9. *Let Ω, $\Omega' \subset \mathbb{R}^n$ be open subsets. In addition, let Ω be bounded and have Lipschitz boundary, and assume that $\Omega \subset\subset \Omega'$. For all $f \in BV(\Omega')$, we have*

$$|Df|(\Omega') = |Df|(\Omega) + |Df|(\Omega' \setminus \overline{\Omega}) + ||\mathrm{Tr}^+(f, \partial\Omega) - \mathrm{Tr}^-(f, \partial\Omega)||_{L^1(\partial\Omega;\mathcal{H}^{n-1})}\,. \tag{1.3.8}$$

Conversely, for any $f_1 \in BV(\Omega)$, $f_2 \in BV(\Omega' \setminus \overline{\Omega})$, if we set $f = f_1\,\mathbf{1}_\Omega + f_2\,\mathbf{1}_{\Omega' \setminus \overline{\Omega}}$, then

$$f \in BV(\Omega')\,.$$

Proof. The argument is very similar to that exploited for the proof of Theorem 5.8 in [10]. □

Remark 1.3.10. *We observe in particular that, by* (1.3.8)*,*

$$|Df|(\partial\Omega) = ||\mathrm{Tr}^+(f_1, \partial\Omega) - \mathrm{Tr}^-(f_2, \partial\Omega)||_{L^1(\partial\Omega;\mathcal{H}^{n-1})}. \tag{1.3.9}$$

Lemma 1.3.11. *Fix an open set $A \subset \mathbb{R}^n$ and a Lipschitz function $\phi : A \to \mathbb{R}$ of class C^1, such that $|\nabla\phi(x)| > 0$ for all $x \in A$. Set $A_r = \phi^{-1}(-\infty, r)$. Then for all $f \in BV(A)$, for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$, and $\mathcal{H}^{n-1}$-a.e. $x \in \partial A_r \cap A$, we have*

$$f(x) = \mathrm{Tr}^+(f, \partial A_r)(x) = \mathrm{Tr}^-(f, \partial A_r)(x)\,. \tag{1.3.10}$$

Proof. We observe that, for a.e. $r \in \mathbb{R}$,

$$\mathrm{Tr}^+(f, \partial A_r) = \mathrm{Tr}^-(f, \partial A_r)\,, \qquad \mathcal{H}^{n-1}\text{-a.e. on } \partial A_r \cap A. \tag{1.3.11}$$

Indeed, for the proof of (1.3.11) we can combine (1.3.8) with $|Df|(\partial A_r \cap A) = 0$ for a.e. $r \in \mathbb{R}$, which in turn comes from the fact that $|Df|$ is a finite measure and $\partial A_r \cap \partial A_s \cap A = \emptyset$ whenever $r \neq s$. Let $x \in A$ be a Lebesgue point for f, then $x \in \partial A_r$ if and only if $r = \phi(x)$. Thanks to the smoothness of $\partial A_r \cap A$ (a consequence of the Implicit Function Theorem) we have

$$|A_r \cap B_\rho(x)| = \frac{1}{2}\,\rho^n + o(\rho^n)\,, \qquad \text{as } \rho \to 0^+\,,$$

and consequently

$$\begin{aligned} f(x) &= \lim_{\rho\to 0^+} \frac{1}{\omega_n\,\rho^n} \int_{B_\rho(x)} f(y)\,dy \\ &= \lim_{\rho\to 0^+} \frac{1}{\omega_n\,\rho^n} \left(\int_{A_r \cap B_\rho(x)} f(y)\,dy + \int_{B_\rho(x)\setminus A_r} f(y)\,dy \right) \\ &= \lim_{\rho\to 0^+} \frac{1}{2}\,\frac{1}{|A_r \cap B_\rho(x)|} \int_{A_r\cap B_\rho(x)} f(y)\,dy + \frac{1}{2}\,\frac{1}{|B_\rho(x)\setminus A_r|} \int_{B_\rho(x)\setminus A_r} f(y)\,dy \\ &= \frac{1}{2}\,\mathrm{Tr}^+(f,\partial A_r)(x) + \frac{1}{2}\,\mathrm{Tr}^-(f,\partial A_r)(x) \\ &= \mathrm{Tr}^\pm(f,\partial A_r)(x)\,. \end{aligned}$$

Since the set of Lebesgue points for f coincides with A up to a $\mathcal{L}^n$-negligible set, by Theorem 1.3.6 we obtain that the first equality in (1.3.10) is verified for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$ and $\mathcal{H}^{n-1}$-a.e. $x \in \partial A_r \cap A$, which together with (1.3.11) concludes the proof. □

Proposition 1.3.12. *Under the assumptions of Lemma 1.3.11, we take $f, f_j \in BV(A)$ for $j \in \mathbb{N}$, such that*

$$||f_j - f||_{L^1(A)} \longrightarrow 0\,, \qquad |Df_j|(A) \longrightarrow |Df|(A)\,.$$

Then, for a.e. $0 < r < 1$, we have

$$\mathrm{Tr}(f,\partial A_r) := \mathrm{Tr}^+(f,\partial A_r) = \mathrm{Tr}^-(f,\partial A_r) \tag{1.3.12}$$

$$\mathrm{Tr}(f_j,\partial A_r) := \mathrm{Tr}^+(f_j,\partial A_r) = \mathrm{Tr}^-(f_j,\partial A_r)\,, \qquad \textit{for all } j \geq 1\,, \tag{1.3.13}$$

and

$$|Df_j|(A_r) \longrightarrow |Df|(A_r)\,, \qquad ||\mathrm{Tr}(f_j, \partial A_r) - \mathrm{Tr}(f, \partial A_r)||_{L^1(\partial A_r)} \longrightarrow 0\,, \tag{1.3.14}$$

hence in particular f_j strictly converges to f on A_r.

Proof. Thanks to Lemma 1.3.11, the two identities (1.3.12) and (1.3.13) hold for a.e. $0 < r < 1$. In particular, for such r, we deduce that $|Df|(\partial A_r) = 0$, hence $|Df|(\overline{A_r}) = |Df|(A_r)$. Moreover, by Lemma 1.2.5, we have

$$\begin{aligned}
\liminf_{j\to\infty} |Df_j|(A_r) &\geq |Df|(A_r) \\
&= |Df|(\overline{A_r}) = |Df|(A) - |Df|(A \setminus \overline{A_r}) \\
&= \lim_{j\to\infty} |Df_j|(A) - |Df|(A \setminus \overline{A_r}) \\
&\geq \limsup_{j\to\infty} |Df_j|(A) - |Df_j|(A \setminus \overline{A_r}) \\
&= \limsup_{j\to\infty} |Df_j|(\overline{A_r}) \\
&\geq \limsup_{j\to\infty} |Df_j|(A_r)\,,
\end{aligned}$$

which proves that

$$|Df|(A_r) = \lim_{j\to\infty} |Df_j|(A_r)\,.$$

Since $||f_j - f||_{L^1(A)} \to 0$, we have in particular $||f_j - f||_{L^1(A_r)} \to 0$, and thus $\{f_j\}_{j\geq 1}$ strictly converges to f in A_r. Finally, (1.3.14) holds because, as observed in Remark 1.3.8, the inner trace operator is continuous with respect to the strict convergence. □

1.4 Structure properties of Caccioppoli Sets

The perimeter of a set can be roughly described as a generalization of the measure of the boundary of the set itself. In this section, we give a very important structure Theorem for sets having (locally) finite perimeter. The proof of this capital result is due to Ennio De Giorgi. We start stating the following Lemma, whose proof is exactly that of Proposition 12.19 contained in [26]. The other parts of the present section are mostly inspired by Chapter 5 of [26].

Lemma 1.4.1. *Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter. Then*

$$\operatorname{spt} D\mathbf{1}_E = \{x \in \mathbb{R}^n \,:\, 0 < |E \cap B_r(x)| < \omega_n r^n \,, \text{ for all } r > 0\} \subset \partial E\,.$$

Moreover, there exists a measurable set $E' \subset \mathbb{R}^n$ such that

$$|E' \Delta E| = 0\,, \qquad \operatorname{spt} |D\mathbf{1}_{E'}| = \partial E'\,. \tag{1.4.1}$$

Remark 1.4.2. *In other words, the support of $D\mathbf{1}_E$ is contained in the topological boundary of E. In addition, up to choosing suitably a representative for E, the support of $D\mathbf{1}_E$ coincides with ∂E.*

Definition 1.4.3. *Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter. We define the* reduced boundary *of E, and we denote it by $\partial^* E$, as the set of those points $x \in \mathbb{R}^n$ such that $|D\mathbf{1}_E(B_r(x))| > 0$, for all $r > 0$, and the following limit exists*

$$\nu_E(x) = \lim_{r \to 0^+} \frac{D\mathbf{1}_E(B_r(x))}{|D\mathbf{1}_E(B_r(x))|}\,,$$

with $|\nu_E(x)| = 1$.

Remark 1.4.4. *We observe that $\nu_E(x)$ is precisely the $|D\mathbf{1}_E|$-density of $D\mathbf{1}_E$ that we introduced in* (1.2.2)*. Then*

$$\begin{aligned} &|\nu_E(x)| = 1\,, \qquad \text{for } |D\mathbf{1}_E|\text{-almost all points } x \in \mathbb{R}^n\,, \\ &D\mathbf{1}_E = \nu_E \cdot |D\mathbf{1}_E| \llcorner \partial^* E\,. \end{aligned} \tag{1.4.2}$$

Remark 1.4.5. *By definition, $\partial^* E \subset \operatorname{spt} D\mathbf{1}_E$. Moreover, we observe that* (1.4.2) *implies*

$$\operatorname{spt} D\mathbf{1}_E \subset \overline{\partial^* E}.$$

Hence $\partial^* E \subset \operatorname{spt} D\mathbf{1}_E \subset \overline{\partial^* E}$. By Lemma 1.4.1, we infer that up to a suitable choice of a representative of E realizing (1.4.1)*, we also have*

$$\partial^* E \subset \partial E = \overline{\partial^* E}\,.$$

We finally state the following structure Theorem due to De Giorgi. The proof of this result can be found in [26].

Theorem 1.4.6. *Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter. Then*

$$|D\mathbf{1}_E| = \mathcal{H}^{n-1} \llcorner \partial^* E\,, \qquad D\mathbf{1}_E = \nu_E \cdot \mathcal{H}^{n-1} \llcorner \partial^* E\,,$$

and the following generalization of the Gauss-Green Formula holds:

$$\int_E \nabla\phi\, dx = \int_{\partial^* E} \phi \nu_E\, d\mathcal{H}^{n-1}\,.$$

Moreover $\partial^* E$ is countably $(n-1)$-rectifiable, i.e. there exist countably many C^1 hypersurfaces $M_j \subset \mathbb{R}^n$, compact sets $K_j \subset M_j$ and a Borel set F with $\mathcal{H}^{n-1}(F) = 0$ realizing

$$\partial^* E = F \cup \bigcup_{j\geq 1} K_j\,.$$

Finally, for every $x \in K_j$, we have $\nu_E(x)^\perp = T_x M_j$.

1.4.1 A local extension result

For this part, we assume that $\Omega \subset \mathbb{R}^n$ is an open set with Lipschitz boundary. We fix $\rho > 0$ such that, up to an isometry, there exist a Lipschitz function $\omega : B'_\rho \to \mathbb{R}$ and a constant $m > 0$, with $\omega(0) = 0$ and $m > \|\omega\|_{L^\infty(B'_\rho)}$, satisfying the following property: if we set $\mathcal{C}_{\rho,m} = B'_\rho \times (-m, m)$, we have

$$\Omega \cap \mathcal{C}_{\rho,3m} = \{x = (x', x_n) \in \mathbb{R}^n \,:\, x' \in B'_\rho\,,\, \omega(x') < x_n < 3m\}\,. \tag{1.4.3}$$

We aim to prove that, under this assumption, any measurable set $E \subset \Omega$ with $\mathbf{1}_E \in BV_{\text{loc}}(\Omega)$[1] can be extended to a locally finite perimeter set $\tilde{E}$ in $\Omega \cup \mathcal{C}_{\rho,m}$, in such a way that $\tilde{E} \cap \Omega = E \cap \Omega$, $P(\tilde{E}; \partial\Omega \cap \mathcal{C}_{\rho,m}) = 0$, and $P(\tilde{E}; S(B)) \leq C\, P(E; B)$, for all Borel sets $B \subset \mathcal{C}_{\rho,m} \setminus \Omega$ and for some constant $C > 0$ depending on the dimension n and the function ω. In what follows, we will denote by $T_x E$ the approximate tangent space to $\partial^* E$ at x[2].

[1] We recall that in this paper $f \in BV_{\text{loc}}(\Omega)$ means $f \in BV(A)$ for all $A \subset \Omega$ open and bounded.

[2] The approximate tangent space $T_x E$ is given by the orthogonal complement of $\nu_E(x)$.

Set $\mathcal{C}_\rho = B'_\rho \times \mathbb{R}$ and define the map $S : \mathcal{C}_\rho \to \mathcal{C}_\rho$ as

$$S(x) := (x', 2\omega(x') - x_n)\,. \tag{1.4.4}$$

Note that S satisfies $S^2(x) = x$ for all x. Moreover, elementary computations show that

$$\mathrm{Lip}(S) \leq \sqrt{3 + 6\mathrm{Lip}(\omega)^2}\,. \tag{1.4.5}$$

Given $E \subset \Omega$ measurable with $\mathbf{1}_E \in BV_{\mathrm{loc}}(\Omega)$, we define $\tilde{E} \subset \Omega \cup \mathcal{C}_\rho$ as

$$\tilde{E} = E \cup (S_\rho(E) \setminus \Omega)\,, \tag{1.4.6}$$

where $S_\rho(E) = S(E \cap \mathcal{C}_\rho)$. Clearly, we have $\tilde{E} \cap \Omega = E \cap \Omega$. Further properties of $\tilde{E}$ are stated in the next lemma.

Lemma 1.4.7. *Let $E \subset \Omega$ be a measurable set with $\mathbf{1}_E \in BV_{\mathrm{loc}}(\Omega)$. Then, for almost all $x \in \partial^* E$, S restricted to $x + T_x E$ is differentiable at x, and we denote by $d^E S_x : T_x E \to \mathbb{R}^n$ its differential. Moreover, if $\tilde{E}$ is the set defined in* (1.4.6)*, we have*

$$P(\tilde{E}; \partial\Omega \cap \mathcal{C}_{\rho,m}) = 0 \tag{1.4.7}$$

and, for all Borel sets $B \subset S(\mathcal{C}_{\rho,m} \cap \Omega)$,

$$P(\tilde{E}; B) = P(S_\rho(E); B) = \int_{\partial^* E \cap S(B)} J^E S(x)\, d\mathcal{H}^{n-1}(x) \leq C\, P(E; S(B))\,, \tag{1.4.8}$$

where $J^E S(x) := \sqrt{\det(d^E S_x^* \circ d^E S_x)}$, $d^E S_x^*$ is the adjoint of $d^E S_x$, and $C = \mathrm{Lip}(S)^{n-1}$.

Proof. Owing to Theorem 1.4.6, we know that $\partial^* E$ is countably $(n-1)$-rectifiable. Then the fact that $S|_{T_x E}$ is differentiable at $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$ follows immediately from [26, Theorem 11.4].

Now, we prove (1.4.7) in the following way. Thanks to Lemma 1.3.9 we only need to check that, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega \cap \mathcal{C}_{\rho,m}$, we have

$$\mathrm{Tr}^+(\mathbf{1}_{\tilde{E}}, \partial\Omega)(x) = \mathrm{Tr}^-(\mathbf{1}_{\tilde{E}}, \partial\Omega)(x)\,,$$

that is,

$$\mathrm{Tr}^+(\mathbf{1}_E, \partial\Omega)(x) = \mathrm{Tr}^+(\mathbf{1}_{S_\rho(E)}, \partial(\mathbb{R}^n \setminus \Omega))(x)\,. \tag{1.4.9}$$

The proof of (1.4.9) goes as follows. We employ the characterization of the trace as a limit of averages: for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$ we have

$$\mathrm{Tr}^+(\mathbf{1}_E, \partial\Omega)(x) = \lim_{r\to 0^+} \frac{|E \cap B_r(x) \cap \Omega|}{|B_r(x) \cap \Omega|}$$

and

$$\mathrm{Tr}^+(\mathbf{1}_{S_\rho(E)}, \partial(\mathbb{R}^n \setminus \Omega))(x) = \lim_{r\to 0^+} \frac{|S_\rho(E) \cap B_r(x) \setminus \Omega|}{|B_r(x) \setminus \Omega|}\,.$$

Then we combine this characterization with a consequence of (1.3.4), i.e. that the trace of a BV characteristic function coincides with a characteristic function $\mathcal{H}^{n-1}$-almost everywhere on $\partial\Omega$, to infer that we only need to show the equivalence

$$\mathrm{Tr}^+(\mathbf{1}_E, \partial\Omega)(x) = 0 \quad \Leftrightarrow \quad \mathrm{Tr}^+(\mathbf{1}_{S_\rho(E)}, \partial(\mathbb{R}^n \setminus \Omega))(x) = 0\,.$$

One of the two required implications (the other can be discussed similarly) is

$$\mathrm{Tr}^+(\mathbf{1}_E, \partial\Omega)(x) = 0 \quad \Rightarrow \quad \mathrm{Tr}^+(\mathbf{1}_{S_\rho(E)}, \partial(\mathbb{R}^n \setminus \Omega))(x) = 0\,.$$

This implication can be restated as

$$|E \cap B_r(x) \cap \Omega| = o(r^n) \quad \Rightarrow \quad |S_\rho(E) \cap B_r(x) \setminus \Omega| = o(r^n) \qquad \text{as } r \to 0^+. \tag{1.4.10}$$

Up to taking $r > 0$ small enough, we have $B_r(x) \subset \mathcal{C}_{\rho,m}$, hence setting $L = \mathrm{Lip}(S)$ we get

$$\begin{aligned} |S_\rho(E) \cap B_r(x) \setminus \Omega| &\le |S\big(E \cap S(B_r(x)) \cap \Omega\big)| \\ &\le L^n |E \cap B_{Lr}(x) \cap \Omega| \\ &= L^n o(L^n r^n) = o(r^n) \qquad \text{as } r \to 0^+\,, \end{aligned}$$

which proves the implication (1.4.10) and concludes the proof of (1.4.7).

Finally, for the proof of (1.4.8), it is enough to show that

$$\mathcal{H}^{n-1}(\partial^* S(E) \Delta S(\partial^* E)) = 0 \,. \tag{1.4.11}$$

Indeed, if (1.4.11) holds, Theorem 1.4.6 ensures that

$$P(S(E); B) = \mathcal{H}^{n-1}(\partial^* S(E) \cap B) = \mathcal{H}^{n-1}(S(\partial^* E) \cap B) = \mathcal{H}^{n-1}(S(\partial^* E \cap S(B))) \,,$$

thus (1.4.8) is an immediate consequence of the Area Formula for rectifiable sets (see [26, Theorem 11.6]). Let us demonstrate (1.4.11). Again by Theorem 1.4.6, it suffices to prove that

$$S(E)^{(0)} = S(E^{(0)}) \,, \qquad S(E)^{(1)} = S(E^{(1)}) \,. \tag{1.4.12}$$

Let us prove the first of the previous identities, as the proof of the other one is obtained by observing that $(\mathbb{R}^n \setminus E)^{(0)} = E^{(1)}$. Let $0 < \delta < \rho$, and $x \in \Omega \cap (B'_{\rho-\delta} \times \mathbb{R})$. Set $L = \mathrm{Lip}(S)$ as before, then by construction, for any $0 < r < \delta$, $S(B_r(S(x))) \subset B_{Lr}(x)$, and thus the Area Formula yields

$$|S(E) \cap B_r(S(x))| = \int_{E \cap B_r(S(x))} JS \, dx \le L^n |B_{Lr}(x)| \,. \tag{1.4.13}$$

Since r is arbitrary and $S^{-1} = S$, it is easy to check that, thanks to (1.4.13), if $x \in E^{(0)}$ then $S(x) \in S(E)^{(0)}$, which proves the inclusion $S(E^{(0)}) \subset S(E)^{(0)}$. The reverse inclusion is proved in a completely analogous way. The proof of the lemma is then achieved thanks to (1.4.5). □

Chapter 2

Almost minimality

In this Chapter, we introduce the definition of (local) almost-minimizer of the relative perimeter in an open subset. This notion will play a key role in the thesis, since the most part of our results will apply to almost minimizers.

2.1 Almost minimizers

Definition 2.1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $E \subset \mathbb{R}^n$ be measurable. We say that E is a* local almost-minimizer *of P_Ω if, for any $x \in \overline{\Omega}$ there exists $r_x > 0$ such that, for any $0 < r < r_x$ and any measurable subset F of Ω with $F \Delta E \subset\subset B_r(x)$, one has*

$$P_\Omega(E; B_r(x)) \leq P_\Omega(F; B_r(x)) + |F \Delta E|^{\frac{n-1}{n}} \psi_\Omega(E; x, r)\,, \tag{2.1.1}$$

for a suitable function $\psi_\Omega(E; x, r)$ such that $\lim_{r \to 0^+} \psi_\Omega(E; x, r) = 0$.
If $\psi_\Omega(E; x, r) = 0$, for all $x \in \overline{\Omega}$, we say that E is a minimizer *of P_Ω.*
If there exists a function $\psi_\Omega(E; r) : (0, r_0) \longrightarrow \mathbb{R}$ such that $\lim_{r \to 0^+} \psi_\Omega(E; r) = 0$, and

$$\psi_\Omega(E; x, r) \leq \psi_\Omega(E; r)\,, \qquad \text{for all } x \in \overline{\Omega}\,,$$

we say that E is a almost-minimizer *of P_Ω.*

Various notions of almost minimality can be found in literature. The notion chosen in

Definition 2.1.1 turns out to be rather weak, in the sense that the other, most common definitions of almost-minimality require conditions that are stronger than (2.1.1). This is the case of the so-called Λ-minimizers. We say that E is a Λ-*minimizer* of P_Ω if, for any $x \in \overline{\Omega}$ there exists $r_x > 0$ such that, for any $0 < r < r_x$ and any measurable subset F of Ω with $F\Delta E \subset\subset B_r(x)$, one has

$$P_\Omega(E; B_r(x)) \le P_\Omega(F; B_r(x)) + \Lambda |F\Delta E| \,. \tag{2.1.2}$$

It is then clear that condition (2.1.2) is stronger than (2.1.1): in particular, Λ-minimizers satisfy (2.1.1) with

$$\psi_\Omega(E; x, r) = \Lambda\, r\, \omega_n^{1/n} \,.$$

2.1.1 Minimality gap

Definition 2.1.2. *Let Ω, $A \subset \mathbb{R}^n$ be open sets and $f \in BV_{\mathrm{loc}}(\Omega)$. The* minimality gap *of f in A relative to Ω is*

$$\Psi_\Omega(f; A) = |Df|(\Omega\cap A) - \inf\{|Dg|(\Omega\cap A) \,:\, g \in BV_{\mathrm{loc}}(\Omega)\,,\ g - f \text{ has compact support in } A\} \,.$$

If $f = \mathbf{1}_E$ for some measurable set E with $\mathbf{1}_E \in BV_{\mathrm{loc}}(\Omega)$, the minimality gap measures how far is a measurable subset E from minimizing P_Ω. When $f = \mathbf{1}_E$, for some measurable subset $E \subset \mathbb{R}^n$, we denote $\Psi_\Omega(\mathbf{1}_E; A)$ by $\Psi_\Omega(E; A)$.

Lemma 2.1.3. *Let Ω, $A \subset \mathbb{R}^n$ be open sets and $E \subset \Omega$ be such that $\mathbf{1}_E \in BV_{\mathrm{loc}}(\Omega)$. Then*

$$\Psi_\Omega(E; A) = P_\Omega(E; A) - \inf\{P_\Omega(F; A) \,:\, \mathbf{1}_F \in BV_{\mathrm{loc}}(\Omega)\,,\ F\Delta E \subset\subset A \cap \overline{\Omega}\} \,.$$

Proof. Let us define

$$\mathcal{I}_1 = \inf\{|Dg|(\Omega \cap A) \,:\, g \in BV_{\mathrm{loc}}(\Omega)\,,\ \mathrm{spt}(g - \mathbf{1}_E) \subset\subset A \cap \overline{\Omega}\}$$
$$\mathcal{I}_2 = \inf\{P_\Omega(F; A) \,:\, \mathbf{1}_F \in BV_{\mathrm{loc}}(\Omega)\,,\ F\Delta E \subset\subset A \cap \overline{\Omega}\} \,.$$

It suffices to show that $\mathcal{I}_1 = \mathcal{I}_2$. For sure, $\mathcal{I}_1 \le \mathcal{I}_2$ because we can take $g = \mathbf{1}_F$ in the definition of $\mathcal{I}_1$. Fix now $\varepsilon > 0$, and let $g \in BV_{\mathrm{loc}}(\Omega)$ be such that $\mathrm{spt}(g - \mathbf{1}_E) \subset\subset A \cap \overline{A}$

and

$$|Dg|(\Omega \cap A) \leq \mathcal{I}_1 + \varepsilon \,. \tag{2.1.3}$$

For $t \in \mathbb{R}$, let us set

$$G_t = \{x \in \Omega \,:\, g(x) > t\} \,.$$

We observe that, for each $0 < t < 1$, $G_t \setminus \operatorname{spt}(g - \mathbf{1}_E) = E \setminus \operatorname{spt}(g - \mathbf{1}_E)$, and so

$$G_t \Delta E \subset \operatorname{spt}(g - \mathbf{1}_E) \subset\subset A \cap \overline{\Omega} \,. \tag{2.1.4}$$

We can now exploit Theorem 1.3.5 and (2.1.4) to infer that

$$|Dg|(\Omega \cap A) = \int_{\mathbb{R}} P_\Omega(G_t; A) dt \geq \int_0^1 P_\Omega(G_t; A) dt \geq \mathcal{I}_2 \,. \tag{2.1.5}$$

By (2.1.3) and (2.1.5), we deduce that

$$\mathcal{I}_1 \leq \mathcal{I}_2 \leq \mathcal{I}_1 + \varepsilon \,.$$

Owing to the arbitrariness of ε, we finally conclude that $\mathcal{I}_1 = \mathcal{I}_2$. □

Lemma 2.1.4. *Under the assumptions of Definition 2.1.2, if $A \subset\subset A'$, then*

$$\Psi_\Omega(f; A) \leq \Psi_\Omega(f; A') \,.$$

Proof. Let us fix $\varepsilon > 0$, and let g be such that

$$|Df|(\Omega \cap A) - |Dg|(\Omega \cap A) \geq \Psi_\Omega(f; A) - \varepsilon \,.$$

We can take

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in \Omega \cap A \\ f(x) & \text{if } x \in \Omega \setminus \overline{A}. \end{cases}$$

Since $A \subset\subset A'$ and the fact that f and $\tilde{g}$ coincide out of A, by (1.3.8), we get

$$\begin{aligned}\Psi_\Omega(f;A') &\geq |Df|(\Omega\cap A') - |D\tilde{g}|(\Omega\cap A')\\ &= |Df|(\Omega\cap A) + |Df|(\Omega(A'\setminus\overline{A})) + |Df|(\Omega\cap\partial A)-\\ &\quad - |D\tilde{g}|(\Omega\cap A) - |D\tilde{g}|(\Omega\cap(A'\setminus\overline{A})) - |D\tilde{g}|(\Omega\cap\partial A)\\ &= |Df|(\Omega\cap A) - |Dg|(\Omega\cap A)\\ &\geq \Psi_\Omega(f;A) - \varepsilon\,.\end{aligned}$$

Since ε is arbitrary, we conclude. □

Remark 2.1.5. *The minimality gap of a local almost-minimizer satisfies a suitable decay on balls. Indeed, under the assumptions of Definition 2.1.2, if E is a local almost-minimizer (resp. a minimizer) of P_Ω, then, owing to Lemma 2.1.3 and the trivial estimate $|F\Delta E| \leq \omega_n r^n$, we have*

$$\Psi_\Omega(E;B_r(x)) \leq \omega_n^{\frac{n-1}{n}} r^{n-1}\psi_\Omega(E;x,r) \qquad \textit{(resp. } \Psi_\Omega(E;B_r(x)) = 0\textit{)}, \tag{2.1.6}$$

for all $x \in \overline{\Omega}$, $0 < r < r_x$, where $\psi_\Omega(E;x,r)$ is the function appearing in (2.1.1)*. Conversely, if for any $x \in \overline{\Omega}$ there exists $r_x > 0$ such that for all $0 < r < r_x$,*

$$\Psi_\Omega(E;B_r(x)) = 0\,,$$

then E is a minimizer of P_Ω.

2.1.2 Density estimates

In this subsection we establish perimeter and volume density estimates for almost-minimizers at a point either in Ω or on $\partial\Omega$.

Lemma 2.1.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set with a Lipschitz boundary. Let E be an almost-minimizer in Ω, and let $x \in \overline{\Omega}$. Assume that $P_\Omega(E;B_r(x)) > 0$ for all $r > 0$.*

Then, there exist constants $C \geq 1$, $\overline{r} > 0$, depending on Ω, E and x, such that

$$C^{-1} r^{n-1} \leq P_\Omega(E; B_r(x)) \leq C r^{n-1} \tag{2.1.7}$$

$$\min\Big(|E \cap B_r(x) \cap \Omega|,\ |(B_r(x) \cap \Omega) \setminus E|\Big) \geq C^{-1} r^n\,, \tag{2.1.8}$$

for all $0 < r < \overline{r}$.

Proof. Up to a translation, we assume that $x = 0$. We start proving (2.1.8). Given $r > 0$ we set

$$m(r) := |B_r \cap \Omega \cap E|\,, \quad \mu(r) := |B_r \cap \Omega \setminus E|\,.$$

Both m and μ are non-decreasing, thus differentiable for almost all $r > 0$. By [26, Example 13.3], for almost all $r > 0$, we have

$$m'(r) = \mathcal{H}^{n-1}(E \cap \partial B_r \cap \Omega)\,, \quad \mu'(r) = \mathcal{H}^{n-1}(\partial B_r \cap \Omega \setminus E)\,.$$

Since the one-parameter family of rescaled domains $D_r := r^{-1}(\Omega \cap B_r)$, $0 < r \leq 1$ is precompact with respect to the L^1-convergence in the class of Lipschitz and connected domains, there exists a constant $\overline{C} > 0$ such that the following, relative isoperimetric inequality holds

$$P_\Omega(E; B_r) \geq \overline{C} \min\{m(r), \mu(r)\}^{\frac{n-1}{n}}\,, \tag{2.1.9}$$

for all $0 < r < 1$. Set $0 < t < r$ and define the competitor

$$F_t = \begin{cases} E \cup B_t \cap \Omega & \text{if } m(t) > \mu(t), \\ E \setminus B_t \cap \Omega & \text{otherwise.} \end{cases}$$

We note that in the first case $F_t \Delta E = B_t \cap \Omega \setminus E$, while in the second case $F_t \Delta E = B_t \cap \Omega \cap E$. In any case, we have $F_t \Delta E \subset\subset B_r \cap \Omega$. Thus, by the almost-minimality of E in Ω, and for almost all $0 < t < r$, we infer that either

$$\begin{aligned} P_\Omega(E; B_r) &\leq P_\Omega(F_t; B_r) + \mu(t)^{\frac{n-1}{n}} \psi(r) \\ &\leq P_\Omega(E; B_r \setminus \overline{B_t}) + \mathcal{H}^{n-1}(\partial B_t \cap \Omega \setminus E) + \mu(r)^{\frac{n-1}{n}} \psi(r)\,, \end{aligned} \tag{2.1.10}$$

or

$$\begin{aligned} P_\Omega(E;B_r) &\le P_\Omega(F_t;B_r) + m(t)^{\frac{n-1}{n}}\psi(r) \\ &\le P_\Omega(E;B_r\setminus \overline{B_t}) + \mathcal{H}^{n-1}(\partial B_t\cap\Omega\cap E) + m(r)^{\frac{n-1}{n}}\psi(r)\,. \end{aligned} \tag{2.1.11}$$

where $\psi(r) := \psi_\Omega(E;0,r)$. Taking the limit as $t \nearrow r$ in (2.1.10) and (2.1.11), and using (2.1.9), we deduce that, if $m(r) > \mu(r)$, then for almost all $0 < r < r_0$ we have

$$\mu'(r) + \mu(r)^{\frac{n-1}{n}}\psi(r) \ge \overline{C}\mu(r)^{\frac{n-1}{n}}\,,$$

while otherwise, we have

$$m'(r) + m(r)^{\frac{n-1}{n}}\psi(r) \ge \overline{C}m(r)^{\frac{n-1}{n}}\,.$$

Therefore, calling $s(r) := \min\{m(r),\mu(r)\}$ and owing to the infinitesimality of $\psi(r)$ as $r \to 0$, we obtain

$$\frac{s'(r)}{s(r)^{\frac{n-1}{n}}} \ge C\,,$$

for $0 < r < \overline{r}$, for some $C, \overline{r} > 0$. Integrating this inequality on the interval $(0, r)$ we obtain (2.1.8). Then, the first inequality in (2.1.7) follows from (2.1.8) and (2.1.9). Finally, the second inequality in (2.1.7) follows from the observation that, taking the limit as $t \nearrow r$ in (2.1.10) and possibly redefining $\overline{r}$ and C, we have

$$P_\Omega(E;B_r) \le \mathcal{H}(\partial B_r\cap\Omega\setminus E) + \mu(r)^{\frac{n-1}{n}}\psi(r) \le Cr^{n-1}\,,$$

for every $0 < r < \overline{r}$. □

2.1.3 Some estimates for the minimality gap

Here we prove two key properties of the minimality gap. The first one is the lower semicontinuity property of the minimality gap for uniform sequences of local almost-minimizers. The second is an upper bound for the difference between the minimality gaps of two BV_{loc} functions.

In what follows, we will say that $\Omega_j \to \Omega$ locally in Hausdorff distance if there exist

$r_0, m, L > 0$, with $r_0 < m$, and L-Lipschitz functions $\omega_j, \omega : B'_{r_0} \to (-m, m)$ providing local graphical representations of $\partial\Omega_j,\ \partial\Omega$, respectively, as in (1.4.3), such that $\omega_j \to \omega$ uniformly in B'_{r_0}.

Lemma 2.1.7. *For $j \in \mathbb{N}$ we let $\Omega_j, \Omega \subset \mathbb{R}^n$ be open sets with uniformly Lipschitz boundary, such that $0 \in \partial\Omega$ and $\Omega_j \to \Omega$ locally in Hausdorff distance. Let E_j, E be sets of locally finite perimeter, such that $E_j \subset \Omega_j,\ E \subset \Omega$, and $E_j \to E$ in $L^1(B_{r_0})$. Finally, we assume that E_j satisfies* (2.1.1) *for all $0 < r < r_0$ and $x = 0$, and that moreover we have*

$$\lim_{r \to 0^+} \sup_j \psi_{\Omega_j}(E_j; 0, r) = 0\,.$$

Then, E satisfies (2.1.1) *for all $0 < r < r_0$, with $\psi_\Omega(E; 0, r) = \sup_j \psi_{\Omega_j}(E_j; 0, r)$, and*

$$\liminf_j \Psi_{\Omega_j}(E_j; B_{r_0}) \geq \Psi_\Omega(E; B_{r_0})\,. \tag{2.1.12}$$

Moreover, if $\Psi_{\Omega_j}(E_j; B_{r_0}) \to 0$ as $j \to \infty$, then for almost all $0 < r < r_0$ we have

$$\lim_{j \to \infty} P_{\Omega_j}(E_j; B_r) = P_\Omega(E; B_r)\,, \qquad \textit{for almost all } r > 0\,. \tag{2.1.13}$$

Proof. Let us fix $\varepsilon > 0$. Let $F \subset \Omega$ be such that $F \Delta E \subset\subset B_{r_0}$ and

$$\Psi_\Omega(E; B_{r_0}) \leq P_\Omega(E; B_{r_0}) - P_\Omega(F; B_{r_0}) + \varepsilon\,. \tag{2.1.14}$$

By Lemma 1.3.11, for all $j \geq 1$, for a.e. $0 < r < r_0$ and $\mathcal{H}^{n-1}$-a.e. $x \in \partial B_r$, we have

$$\mathbf{1}_{E_j}(x) = \mathrm{Tr}^\pm(E_j, \partial B_r)(x) \quad \text{and} \quad \mathbf{1}_E(x) = \mathrm{Tr}^\pm(E, \partial B_r)(x)\,, \tag{2.1.15}$$

where $\mathrm{Tr}^\pm(A, \partial B_r) := \mathrm{Tr}^\pm(\mathbf{1}_A, \partial B_r)$. By the L^1_{loc}-convergence of E_j to E, we can choose $r < r_0$ with the above property and the additional

$$P_\Omega(E; B_r) \geq P_\Omega(E; B_{r_0}) - \varepsilon\,, \tag{2.1.16}$$

then take j_ε large enough, such that

$$F\Delta E \subset\subset B_r\,, \tag{2.1.17}$$

$$\int_{\partial B_r} |\mathbf{1}_{E_j} - \mathbf{1}_E| d\mathcal{H}^{n-1} < \varepsilon \qquad \text{for } j \geq j_\varepsilon\,. \tag{2.1.18}$$

Let us fix $\delta > 0$, define

$$U_\delta := \Big\{x = (x', x_n) \in \mathcal{C}_{r,m} :\ \omega(x') - \delta < x_n \leq \omega(x')\Big\}\,,$$

and assume δ so small that

$$\mathcal{H}^{n-1}(\partial B_r \cap U_\delta) < \varepsilon \tag{2.1.19}$$

and

$$P_\Omega(E; B_r) \leq P(E; B_r \cap (\Omega + \delta e_n)) + \varepsilon\,. \tag{2.1.20}$$

Owing to the uniform convergence of ω_j to ω, we can select $j_\delta \geq 1$ such that $\|\omega_j - \omega\|_\infty < \delta$ for all $j \geq j_\delta$, then for those j we define

$$A_j := \Omega_j \cap \Omega \cap B_r\,, \qquad B_j := (\Omega_j \setminus \Omega) \cap B_r\,, \tag{2.1.21}$$

and observe that $B_j \subset U_\delta$. Now we set

$$F_j := (\tilde{F} \cap \Omega_j \cap B_r) \cup \Big(E_j \cap (B_{r_0} \setminus B_r)\Big)\,, \tag{2.1.22}$$

where $\tilde{F} = F \cup (S(F) \setminus \Omega)$ and S is the symmetry through $\partial\Omega$ defined in (1.4.4) (with $\rho = r_0$). Thanks to (2.1.17), we have $F_j \subset \Omega_j$ and $F_j \Delta E_j \subset\subset B_{r_0}$, which means that F_j is a competitor for E_j in the definition of $\Psi_{\Omega_j}(E_j, B_{r_0})$. Moreover, by (2.1.22), (2.1.18), and (2.1.19), we have

$$\begin{aligned} P_{\Omega_j}(F_j; B_{r_0}) &\leq P_{\Omega_j}(F_j; B_r) + P_{\Omega_j}(E_j; B_{r_0} \setminus \overline{B_r}) \\ &\qquad + \int_{\partial B_r \cap \Omega_j \cap \Omega} |\mathbf{1}_{E_j} - \mathbf{1}_E| d\mathcal{H}^{n-1} + \mathcal{H}^{n-1}(\partial B_r \cap U_\delta) \\ &\leq P_{\Omega_j}(F_j; B_r) + P_{\Omega_j}(E_j; B_{r_0} \setminus \overline{B_r}) + 2\varepsilon\,. \end{aligned} \tag{2.1.23}$$

Let us compute $P(F_j; B_r \cap \Omega_j)$. By Lemma 1.4.7, $P(\tilde{F}; \partial\Omega \cap \mathcal{C}_{r_0,m}) = 0$, hence

$$P_{\Omega_j}(F_j; B_r) = P(F; A_j) + P(S(F); B_j)\,. \tag{2.1.24}$$

Again by Lemma 1.4.7, up to possibly taking a smaller δ, we have

$$P(S(F); B_j) = P(\tilde{F}; B_j) \leq P(\tilde{F}; U_\delta) \leq \varepsilon\,. \tag{2.1.25}$$

Putting together (2.1.23), (2.1.24), and (2.1.25), and taking into account (2.1.15), we obtain

$$\begin{aligned} \Psi_{\Omega_j}(E_j; B_{r_0}) &\geq P_{\Omega_j}(E_j; B_{r_0}) - P_{\Omega_j}(F_j; B_{r_0}) \\ &\geq P_{\Omega_j}(E_j; B_r) - P(F; A_j) - 3\varepsilon \\ &\geq P_{\Omega_j}(E_j; B_r) - P_\Omega(F; B_{r_0}) - 3\varepsilon\,. \end{aligned} \tag{2.1.26}$$

Now, since for all $j \geq j_\varepsilon$ we have $\|\omega_j - \omega\|_\infty < \delta$, we infer that

$$B_r \cap (\Omega + \delta e_n) \subset B_r \cap \Omega_j\,.$$

This inclusion combined with (2.1.26) and (2.1.20) gives

$$\Psi_{\Omega_j}(E_j; B_{r_0}) \geq P(E_j; B_r \cap (\Omega + \delta e_n)) - P(F; B_{r_0} \cap \Omega) - 3\varepsilon\,, \tag{2.1.27}$$

so that taking the liminf as $j \to \infty$ in (2.1.27), and using the lower semicontinuity of the perimeter, (2.1.20), (2.1.16), and (2.1.14), we find

$$\begin{aligned} \liminf_j \Psi_{\Omega_j}(E_j; B_{r_0}) &\geq \liminf_j P(E_j; B_r \cap (\Omega + \delta e_n)) - P_\Omega(F; B_{r_0}) - 3\varepsilon \\ &\geq P(E; B_r \cap (\Omega + \delta e_n)) - P_\Omega(F; B_{r_0}) - 3\varepsilon \\ &\geq P_\Omega(E; B_r) - P_\Omega(F; B_{r_0}) - 4\varepsilon \\ &\geq P_\Omega(E; B_{r_0}) - P_\Omega(F; B_{r_0}) - 5\varepsilon \\ &\geq \Psi_\Omega(E; B_{r_0}) - 6\varepsilon\,. \end{aligned}$$

Then, the arbitrary choice of ε implies (2.1.12). Now, the fact that E satisfies (2.1.1) with $\psi_\Omega(E;0,r)$ as in the statement can be proved with the same argument used to show (2.1.12), also taking into account that $|E_j \Delta F_j| \to |E\Delta F|$ as $j \to \infty$. Finally, to prove (2.1.13) we consider the sequence F_j defined as before, but now with $F = E$. Choosing $\varepsilon > 0$ arbitrarily, for j large enough we obtain as before

$$\begin{aligned}\Psi_{\Omega_j}(E_j;B_{r_0}) &\geq P_{\Omega_j}(E_j;B_{r_0}) - P_{\Omega_j}(F_j;B_{r_0})\\ &\geq P_{\Omega_j}(E_j;B_r) - P_\Omega(E;B_r) - 3\varepsilon\,,\end{aligned}$$

which gives the desired conclusion. □

Lemma 2.1.8. *Let $\Omega, A \subset \mathbb{R}^n$ be open sets, with $\partial\Omega$ Lipschitz and A bounded, of class C^2, and such that $\mathcal{H}^{n-1}(\partial A \cap \partial\Omega) = 0$. Let $f, g \in BV_{\mathrm{loc}}(\Omega)$, then*

$$\begin{aligned}|\Psi_\Omega(f,A) - \Psi_\Omega(g,A)| \leq &\Big||Df|(\Omega\cap A) - |Dg|(\Omega\cap A)\Big| \\ &+ \|\mathrm{Tr}^+(f_0,\partial A) - \mathrm{Tr}^+(g_0,\partial A)\|_{L^1(\partial A)}\,,\end{aligned} \tag{2.1.28}$$

where f_0, g_0 denote the zero-extensions of, respectively, f and g on $A \setminus \Omega$.

Proof. Given $\varepsilon > 0$, there exists $h \in BV_{\mathrm{loc}}(\Omega)$ with $\mathrm{spt}(h - f) \subset\subset A$, such that

$$\begin{aligned}\Psi_\Omega(f,A) &\leq |Df|(A\cap\Omega) - |Dh|(A\cap\Omega) + \varepsilon \\ &\leq \Big||Df|(A\cap\Omega) - |Dg|(A\cap\Omega)\Big| + \Psi_\Omega(g,A) + |D\tilde{h}|(A\cap\Omega) - |Dh|(A\cap\Omega) + \varepsilon\,,\end{aligned} \tag{2.1.29}$$

where $\tilde{h} \in BV_{\mathrm{loc}}(\Omega)$ will be suitably chosen, so that in particular $\mathrm{spt}(\tilde{h} - g) \subset\subset A$. For the definition of $\tilde{h}$, we claim that it is possible to construct a sequence $A^{(k)}$ of inner parallel sets of A that converge to A, for which $|Df_0|(\partial A^{(k)}) = |Dg_0|(\partial A^{(k)}) = 0$ and, moreover,

$$\lim_k \int_{\partial A^{(k)}} |\mathrm{Tr}^+(f_0 - g_0, \partial A^{(k)})|\, d\mathcal{H}^{n-1} = \int_{\partial A} |\mathrm{Tr}^+(f_0 - g_0, \partial A)|\, d\mathcal{H}^{n-1}\,. \tag{2.1.30}$$

For the proof of (2.1.30) we argue as follows. Since A is of class C^2, there exists $\delta > 0$ such that, for all $0 < t < \delta$, the map $\zeta_t(x) = x + t\nu_A(x)$ is a diffeomorphism of class C^1 between ∂A and the boundary ∂A_t of the inner parallel set $A_t = \{x \in A :\ \mathrm{dist}(x, \partial A) > t\}$. Now, we consider two sequences $f_{0,j}, g_{0,j}$ of smooth approximations of f_0, g_0 on A, with traces $\mathrm{Tr}^+(f_{0,j}, \partial A) = \mathrm{Tr}^+(f_0, \partial A)$ and $\mathrm{Tr}^+(g_{0,j}, \partial A) = \mathrm{Tr}^+(g_0, \partial A)$, respectively (see Remark

1.3.8). By inspecting the proof of Anzellotti-Giaquinta's approximation theorem, it is not restrictive to ask that the sequences $f_{0,j}, g_{0,j}$ also satisfy

$$\int_{A\setminus A_{1/k}} (|f_{0,j} - f_0| + |g_{0,j} - g_0|)\, dx \leq \frac{1}{k^2}\,, \tag{2.1.31}$$

for all j and for $k > \delta^{-1}$. We note that the tangential Jacobian of ζ_t satisfies $J\zeta_t(x) = 1 + O(t)$, hence the area formula gives

$$\int_{\partial A_t} |f_{0,j}(y) - g_{0,j}(y)|\, d\mathcal{H}^{n-1}(y) = (1 + O(t)) \int_{\partial A} |f_{0,j}(\zeta_t(x)) - g_{0,j}(\zeta_t(x))|\, d\mathcal{H}^{n-1}(x)\,. \tag{2.1.32}$$

As $t \to 0^+$ we have $\zeta_t(x) \to x$ uniformly. Therefore, by following the same Cauchy-sequence argument as in the classical construction of the trace (see, e.g., [10]), the compositions $f_{0,j}(\zeta_t(x))$ and $g_{0,j}(\zeta_t(x))$ can be shown to converge in $L^1(\partial A)$ to some limits $\hat{f}_{0,j}$ and $\hat{g}_{0,j}$, respectively. Hence (2.1.32) implies

$$\lim_{t\to 0^+} \int_{\partial A_t} |f_{0,j}(y) - g_{0,j}(y)|\, d\mathcal{H}^{n-1}(y) = \int_{\partial A} |\hat{f}_{0,j}(x) - \hat{g}_{0,j}(x)|\, d\mathcal{H}^{n-1}(x)\,. \tag{2.1.33}$$

At the same time, if we choose a vector field $\xi \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ and set either $u_j = f_{0,j}$ or $u_j = g_{0,j}$, by Gauss-Green Theorem we obtain

$$\begin{aligned}\int_{A_t} (u_j \div \xi + \nabla u_j \cdot \xi)\, dx &= -\int_{\partial A_t} u_j\, \xi \cdot \nu_{A_t}\, d\mathcal{H}^{n-1} \\ &= -(1 + O(t)) \int_{\partial A} u_j(\zeta_t(x))\, \xi(\zeta_t(x)) \cdot \nu_A(x)\, d\mathcal{H}^{n-1}(x)\,,\end{aligned}$$

hence taking the limit as $t \to 0^+$ gives

$$\int_A (u_j \div \xi + \nabla u_j \cdot \xi)\, dx = -\int_{\partial A} \hat{u}_j\, \xi \cdot \nu_A\, d\mathcal{H}^{n-1}\,,$$

which means that $\hat{f}_{0,j}$ and $\hat{g}_{0,j}$ coincide, respectively, with $\mathrm{Tr}^+(f_0, \partial A)$ and $\mathrm{Tr}^+(g_0, \partial A)$ up to $\mathcal{H}^{n-1}$-null sets and for all j, by the uniqueness of the trace. We can thus rewrite (2.1.33) as

$$\lim_{t\to 0^+} \int_{\partial A_t} |f_{0,j}(y) - g_{0,j}(y)|\, d\mathcal{H}^{n-1}(y) = \int_{\partial A} |\mathrm{Tr}^+(f_0 - g_0, \partial A)(x)|\, d\mathcal{H}^{n-1}(x)\,. \tag{2.1.34}$$

To get (2.1.30) from (2.1.34), we must choose $A^{(k)}$ appropriately. To this aim, we apply the coarea formula to the integral in (2.1.31) and average the resulting inequality, deducing the existence of $0 < t_k < 1/k$ such that for all j

$$\int_{\partial A^{(k)}} (|f_{0,j} - \mathrm{Tr}(f_0, \partial A^{(k)})| + |g_{0,j} - \mathrm{Tr}(g_0, \partial A^{(k)})|) \, d\mathcal{H}^{n-1} \leq \frac{1}{k}\,, \tag{2.1.35}$$

where we have set $A^{(k)} = A_{t_k}$. By the triangle inequality and (2.1.35) we obtain

$$\int_{\partial A^{(k)}} |\mathrm{Tr}^+(f_0 - g_0, \partial A^{(k)})| \, d\mathcal{H}^{n-1} \leq \int_{\partial A^{(k)}} |f_{0,j} - g_{0,j}| \, d\mathcal{H}^{n-1} + \frac{1}{k}\,,$$

which gives (2.1.30) at once from (2.1.34).

Now we observe that $|Df_0|(\partial A^{(k)}) = |Dg_0|(\partial A^{(k)}) = 0$ because the inner and outer traces of f_0 and g_0 on $\partial A^{(k)}$ coincide, hence we can define

$$\tilde{h} = h\mathbf{1}_{A^{(k)}} + g\mathbf{1}_{A \setminus A^{(k)}}\,.$$

Note that $\mathrm{spt}(\tilde{h} - g) \subset\subset A$ and, if k is large enough, $\mathrm{spt}(h - f) \subset\subset A^{(k)}$, so that

$$|D\tilde{h}|(A \cap \Omega) \leq |Dh|(A^{(k)} \cap \Omega) + |Dg_0|(A \setminus \overline{A^{(k)}}) + \int_{\partial A^{(k)}} |\mathrm{Tr}(f_0 - g_0, \partial A^{(k)})| \, d\mathcal{H}^{n-1}\,.$$

By choosing k large enough we obtain $|Dg_0|(A \setminus A^{(k)}) < \varepsilon$ and thus

$$|D\tilde{h}|(A \cap \Omega) - |Dh|(A \cap \Omega) \leq \int_{\partial A} |\mathrm{Tr}^+(f_0 - g_0, \partial A)| \, d\mathcal{H}^{n-1} + \varepsilon. \tag{2.1.36}$$

By combining (2.1.29) and (2.1.36), we get

$$\Psi_\Omega(f, A) \leq \Big||Df|(A \cap \Omega) - |Dg|(A \cap \Omega)\Big| + \Psi_\Omega(g, A) + \int_{\partial A} |\mathrm{Tr}^+(f_0 - g_0, \partial A)| \, d\mathcal{H}^{n-1} + 2\varepsilon\,.$$

Since ε is arbitrary, we conclude

$$\Psi_\Omega(f, A) - \Psi_\Omega(g, A) \leq \Big||Df|(A \cap \Omega) - |Dg|(A \cap \Omega)\Big| + \int_{\partial A} |\mathrm{Tr}^+(f_0 - g_0, \partial A)| \, d\mathcal{H}^{n-1}$$

and, by exchanging the role of f and g in the argument above, we obtain (2.1.28). □

Chapter 3

Boundary Monotonicity Formula

The aim of the present Chapter is to prove a boundary Monotonicity Formula holding for local almost-minimizers of the relative perimeter at a boundary point x_0 of an open set $\Omega \subset \mathbb{R}^n$ with Lipschitz continuous boundary, under the assumption that Ω satisfies a *visibility condition* at x_0. The main results proved in this chapter are contained in [23]. From now on, given $x \in \mathbb{R}^n$, we write $x = (x', x_n)$, where $x' = (x_1, ..., x_{n-1}) \in \mathbb{R}^n$. We denote by $B'_r(x')$ the $(n-1)$-dimensional ball of radius r and center x'. Moreover, given a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^p$, we define $\mathrm{Lip}(f, A)$ as the Lipschitz constant of $f|_A$, for all sets $A \subset \mathbb{R}^n$, and we set $\mathrm{Lip}(f) := \mathrm{Lip}(f, \mathbb{R}^n)$.

3.1 A result on the uniform convergence.

In the next section, we will use the following technical Lemma.

Lemma 3.1.1. *Let $f_j \in \mathrm{Lip}(\mathbb{R}^n)$ be a sequence of Lipschitz continuous functions with the following properties:*

$$\sup_{j \geq 1} |f_j(0)| < \infty\,, \qquad L := \sup_{j \geq 1} \mathrm{Lip}(f_j) < \infty\,. \tag{3.1.1}$$

Let us assume that there exists a function f such that

$$f_j(x) \to f(x)\,, \qquad \textit{for all } x \in \mathbb{R}^n\,.$$

Then f_j converges to f locally uniformly in $\mathbb{R}^n$, i.e.

$$||f_j - f||_{L^\infty(B_R)} \to 0\,, \qquad \text{for all } R > 0\,.$$

Proof. We argue by contradiction. Let us assume that there exists $R > 0$ such that

$$||f_j - f||_{L^\infty(B_R)} \nrightarrow 0\,.$$

Then we need to have $\ell = \limsup_{j\to\infty} ||f_j - f||_{L^\infty(B_R)} > 0$. Hence, in particular, for a suitable subsequence $g_k = f_{j_k}$, we should have

$$||g_k - f||_{L^\infty(B_R)} \to \ell\,. \tag{3.1.2}$$

Owing to (3.1.1), we are in position to apply Ascoli-Arzelà Theorem to g_k deducing that it admits a subsequence uniformly convergent to f in B_R. Thus we conclude, because this condition contradicts (3.1.2). □

3.2 The visibility property

In this section we introduce the visibility property and its main consequences. In what follows, $\Omega \subset \mathbb{R}^n$ denotes an open set with Lipschitz boundary such that $0 \in \partial\Omega$ and $\partial\Omega$ is a graph in a neighborhood of 0, as in (1.4.3). For notational convenience, we will only consider the visibility property at 0, but of course, we could equally define the property at a generic point of $\partial\Omega$.

Definition 3.2.1. *We say that Ω satisfies the visibility property provided there exists $T > 0$ and a function $u \in C^1([0,T))$[1] such that:*

(V1) $u(0) = u'(0) = 0$ and $0 \le u' \le 2^{-1}$;

(V2) The function

$$\gamma_u(t) := t^{-1} \sup_{0<s\le t} \sqrt{\frac{u(s)}{s} + u'(s)}$$

[1]By $u \in C^1([0,T))$ we mean that $u \in C^1(0,T)$ and there exist finite the limits of $u(t)$ and $u'(t)$ as $t \to 0$.

is summable on $(0, T)$*;*

(V3) for all $0 < t < T$*, the segment joining the point* $U_t = -u(t)\, e_n$ *with a point* x *belonging to* $\partial\Omega \cap B_t$ *does not intersect* Ω*.*

Remark 3.2.2. *We note that (V1) and (V2) imply that* $u(t) = o(t)$ *and*

$$t\gamma_u(t) \to 0\,, \tag{3.2.1}$$

as $t \to 0$*. Moreover, the summability of* $\gamma_u(t)$ *implies that of* $t^{-2}u(t)$*, indeed* $0 \le u(t) \le t/2$ *by (V1) and thus*

$$0 \le \frac{u(t)}{t^2} = t^{-1}\frac{u(t)}{t} \le t^{-1}\sqrt{\frac{u(t)}{t}} \le \gamma_u(t)\,. \tag{3.2.2}$$

In the following proposition, we rewrite the assumption (V3) in the form of a property involving the functions $\omega(x')$ and $u(t)$. This will be particularly useful when checking the visibility property for relevant classes of domains (see the examples at the end of the section).

Proposition 3.2.3. Ω *satisfies the property (V3) in Definition 3.2.1 if and only if, for any* $\nu \in \partial B'_1$ *and for all* $0 < t < T$*, the slope* $m_t(s)$ *of the line connecting* U_t *with* $(s, \omega(s\nu))$*, that is given by*

$$m_t(s) = \frac{\omega(s\nu) + u(t)}{s}\,,$$

is non-increasing as a function of s*, for* $s > 0$ *such that* $s^2 + \omega(s\nu)^2 < t^2$*.*

Proof. Let us assume that (V3) holds, and set $\omega_\nu(s) = \omega(s\nu)$ for more simplicity. By contradiction, let $s_1 < s_2$ be such that $s_i^2 + \omega_\nu(s_i)^2 < t^2$, for $i = 1, 2$, and $m_t(s_1) < m_t(s_2)$. By definition of m_t, we have

$$\frac{\omega_\nu(s_1) + u(t)}{s_1} < \frac{\omega_\nu(s_2) + u(t)}{s_2}\,,$$

and so equivalently

$$\omega_\nu(s_1) < \frac{s_1}{s_2}\Big(\omega_\nu(s_2) + u(t)\Big) - u(t)\,.$$

This implies that the point

$$x = \left(s_1\nu, \frac{s_1}{s_2}\Big(\omega_\nu(s_2) + u(t)\Big) - u(t)\right)$$

is internal to Ω and lies on the segment connecting $(s_2\nu, \omega_\nu(s_2))$. This contradicts (V3).

Conversely, let us suppose that $m_t(s)$ is non-increasing in s, for $s > 0$ such that $s^2 + \omega_\nu(s)^2 < t^2$. Set $P(s) = (s\nu, \omega_\nu(s))$, then, arguing by contradiction, assume that $s_2 > 0$ is such that $s_2^2 + \omega_\nu(s_2)^2 < t^2$ and there exists $\lambda \in (0,1)$ with the property

$$(1-\lambda)U_t + \lambda P(s_2) = (\lambda s_2, \lambda(u(t) + \omega_\nu(s_2)) - u(t)) \in \Omega\,.$$

Then $\omega_\nu(\lambda s_2) < \lambda(u(t)+\omega_\nu(s_2)) - u(t)$. By continuity, for all $\delta > 0$ there exists $\lambda_\delta \in (\lambda, 1)$ such that

$$\lambda_\delta(u(t) + \omega_\nu(s_2)) - u(t) - \delta < \omega_\nu(\lambda_\delta\, s_2) < \lambda_\delta(u(t) + \omega_\nu(s_2)) - u(t)\,. \tag{3.2.3}$$

Since the segment $[U_t, P(s_2)]$ is compactly contained in B_t, by (3.2.3), we can pick $\delta > 0$ small enough and a correspondent λ_δ such that

$$P(\lambda_\delta s_2) \in B_t\,. \tag{3.2.4}$$

Let $s_1 = \lambda_\delta s_2$. We observe that (3.2.4) and (3.2.3) imply

$$m_t(s_1) < m_t(s_2)\,, \qquad s_1^2 + \omega_\nu(s_1)^2 < t^2\,,$$

and this contradicts our hypothesis. This completes the proof of the proposition. □

Corollary 3.2.4. *Assume that ω satisfies*

$$\langle x', \nabla\omega(x')\rangle \le \omega(x') + u(|x'|) \qquad \text{for a.e. } x' \in B'_\rho\,, \tag{3.2.5}$$

where $u : (0,T) \to \mathbb{R}$ is a non-decreasing function satisfying properties (V1) and (V2). Then Ω satisfies the visibility property.

Proof. Since ω is Lipschitz, the function m_t defined in the statement of Proposition 3.2.3 is a.e. differentiable, thus m_t is non-increasing if and only if $m_t'(s) \le 0$ at almost every s. We observe that

$$m_t'(s) = \frac{\omega_\nu'(s)}{s} - \frac{\omega_\nu(s) + u(t)}{s^2}\,,$$

thus $m_t' \leq 0$ if and only if

$$s\,\omega_\nu'(s) \leq \omega_\nu(s) + u(t)\,. \tag{3.2.6}$$

The hypothesis (3.2.5) implies that

$$s\,\omega_\nu'(s) \leq \omega_\nu(s) + u(s) \qquad \text{for almost all } 0 < s < T\,. \tag{3.2.7}$$

Hence if $s > 0$ is such that $s^2 + \omega_\nu(s)^2 < t^2$, then $s < t$, and by (3.2.7), since u is non-decreasing, we obtain

$$s\,\omega_\nu'(s) \leq \omega_\nu(s) + u(t)\,,$$

that is precisely (3.2.6). Consequently, (V3) is verified thanks to Proposition 3.2.3. □

3.2.1 Existence of the tangent cone

An important consequence of the visibility property is the existence of the tangent cone to Ω at 0.

Proposition 3.2.5. *Let $\Omega \subset \mathbb{R}^n$ satisfy the visibility property. Then there exists the tangent cone to Ω at 0, denoted by Ω_0. More precisely, if we set $\Omega_s := s^{-1}\Omega$ for $s > 0$, we obtain*

$$\lim_{s\to 0^+} \operatorname{dist}_{\mathcal{H}}(\Omega_s \cap B_R, \Omega_0 \cap B_R) = 0\,, \qquad \textit{for all } R > 0. \tag{3.2.8}$$

Proof. Let us fix $\nu \in \partial B_1'$, and let

$$\omega_\nu(s) = \omega(s\nu)\,, \qquad s \geq 0\,,$$

where ω is the function realizing (1.4.3). Let $s > 0$ be a point where ω_ν is differentiable and let $t = \sqrt{s^2 + \omega_\nu(s)^2}$. By (V3) in Definition 3.2.1, we deduce that the slope of the line connecting $(s, \omega_\nu(s))$ with U_t needs to be bounded below by $\omega_\nu'(s)$, that is,

$$\omega_\nu'(s) \leq \frac{\omega_\nu(s) + u(t)}{s}\,.$$

We set $L = \sqrt{1 + \operatorname{Lip}(\omega)^2}$ and observe that $t \leq Ls$. Since u is non-decreasing by (V1), we infer

$$\omega_\nu'(s) \leq \frac{\omega_\nu(s) + u(Ls)}{s}\,,$$

hence we get

$$\left(\frac{\omega_\nu(s)}{s}\right)' = \frac{\omega_\nu'(s)}{s} - \frac{\omega_\nu(s)}{s^2} \le \frac{u(Ls)}{s^2} \le L^2 \frac{u(Ls)}{(Ls)^2}\,. \tag{3.2.9}$$

We integrate (3.2.9) between $s_1 < s_2$ thanks to (V2) (see Remark 3.2.2), and obtain

$$\frac{\omega_\nu(s_2)}{s_2} - \frac{\omega_\nu(s_1)}{s_1} \le L^2 \int_{s_1}^{s_2} \frac{u(Ls)}{(Ls)^2}\,ds = L \int_{Ls_1}^{Ls_2} \frac{u(t)}{t^2}\,dt\,.$$

Thus we conclude that the function

$$s \mapsto \frac{\omega_\nu(s)}{s} - L\int_0^{Ls} \frac{u(t)}{t^2}\,dt$$

is monotonically non-increasing in s and bounded by $\mathrm{Lip}(\omega) + L\int_0^{Ls} t^{-2}u(t)\,dt$, for $0 < s < L^{-1}T$. Therefore there exists

$$D_\nu^+\omega(0) := \lim_{s\to 0^+} \frac{\omega_\nu(s)}{s} = \lim_{s\to 0^+} \frac{\omega_\nu(s)}{s} - L\int_0^{Ls} \frac{u(t)}{t^2}\,dt \in \mathbb{R}\,.$$

Let us define

$$\omega_0(x') = \begin{cases} |x'| D^+_{\frac{x'}{|x'|}}\omega(0) & \text{if } x' \neq 0, \\ 0 & \text{if } x' = 0\,. \end{cases}$$

The function ω_0 is 1-homogeneous, therefore the set

$$\Omega_0 = \{x \in \mathbb{R}^n \,:\, x_n > \omega_0(x')\}$$

is a cone with vertex at 0. Now, for all $s > 0$, we set

$$\omega_s(x') = \begin{cases} \dfrac{\omega(sx')}{s} & \text{if } x' \neq 0, \\ 0 & \text{if } x' = 0\,. \end{cases} \tag{3.2.10}$$

It is immediate to observe that $\omega_s(0) = 0$ and, setting $t = s|x'|$,

$$\omega_s(x') = \frac{\omega\left(t(x'/|x'|)\right)}{t}|x'| \to |x'|\, D^+_{\frac{x'}{|x'|}}\omega(0) = \omega_0(x')\,, \qquad \text{as } s \to 0^+\,.$$

Since $\{\omega_s\}_{s>0}$ is a one-parameter family of locally equi-bounded and equi-Lipschitz functions that pointwisely converge to ω_0 as $s \to 0^+$, we can apply Lemma 3.1.1 to conclude that this convergence is locally uniform. This easily implies the Hausdorff convergence stated in (3.2.8). $\square$

Remark 3.2.6. *We note that for the proof of Proposition 3.2.5, the hypothesis (V2) of Definition 3.2.1 can be replaced by the weaker hypothesis of summability of $t^{-2}u(t)$ on $(0,T)$. We also observe that, if Ω is convex, then the existence of the tangent cone Ω_0 is always granted, even though (V2) may not be satisfied.*

3.2.2 An off-centric visibility property

The next lemma shows that the assumption (V3) in Definition 3.2.1 can be replaced by an equivalent assumption, where off-centric balls are taken instead of balls centered at 0. This off-centric visibility property will be useful later on.

Lemma 3.2.7. *The following properties are equivalent:*

(i) Ω satisfies the visibility property;

(ii) There exist $R > 0$ and a function $v \in C^1(0,R)$ satisfying properties (V1), (V2) of Definition 3.2.1, and

> *(V3') for all $0 < r < R$, any segment joining the point $V_r = -v(r)\, e_n$ with a point x belonging to $\partial\Omega \cap B_r(V_r)$ does not intersect Ω.*

Proof. We prove that (i) implies (ii). Let $z(t) = t - u(t)$ where u is as in Definition 3.2.1. We can find $0 < T' < T$ such that $z(t)$ is an increasing C^1 diffeomorphism of the interval $(0,T')$ with the property

$$\frac{1}{2}\, t \le z(t) \le t\,.$$

Let $R = z(T')$. Then we can consider the inverse z^{-1} of z in $(0,R)$, that is an increasing diffeomorphism such that

$$r \le z^{-1}(r) \le 2\,r\,. \tag{3.2.11}$$

Setting $U_t = -u(t)e_n$, it follows that $B_{t-u(t)}(U_t) \subset B_t$, for all $0 < t < T$, thus (V3) holds for all points $x \in \partial\Omega \cap B_{t-u(t)}(U_t)$. We then have that

$$B_r(U_{z^{-1}(r)}) \subset B_{z^{-1}(r)}\,, \qquad \text{for any } 0 < r < R.$$

Any line segment joining $x \in \partial\Omega \cap B_{z^{-1}(r)}$ with $U_{z^{-1}(r)}$ does not intersect Ω, hence the same property holds for any $x \in \partial\Omega \cap B_r(U_{z^{-1}(r)})$. Let

$$v(r) = u(z^{-1}(r)), \qquad 0 < r < R.$$

It is clear that (V3') holds. By (3.2.11), up to possibly reducing the value of R, (V2) and $v' \leq 2^{-1}$ follow. Since both u and z^{-1} are non-decreasing, v is non-decreasing, thus also (V1) is satisfied. A completely analogous argument shows that (ii) implies (i), and the proof is concluded. □

For all $0 < r < R$, we define

$$\mathcal{C}_r = \{V_r + t(z - V_r) \,:\, z \in \partial B_r(V_r) \cap \Omega\,,\, t > 0\}\,. \tag{3.2.12}$$

The set $\mathcal{C}_r$ is an open cone with vertex at V_r.

Lemma 3.2.8. *Assume that Ω satisfies the (off-centric) visibility property. Then, for all $0 < r < R$, the cone $\mathcal{C}_r$ contains $\Omega \cap B_r(V_r)$.*

Proof. By contradiction, let $x \in (\Omega \cap B_r(V_r)) \setminus \mathcal{C}_r$. For all $0 < t \leq r$, let

$$x_t = V_r + t\,\frac{x - V_r}{|x - V_r|}\,,$$

and note that $x = x_{t_0}$ for a suitable $0 < t_0 < r$. It's clear that $x_r \notin \Omega$, otherwise x_{t_0} would belong to $\mathcal{C}_r$, for all $t > 0$, which is against our assumption. We can then select a value $s \in (t_0, r]$ such that $x_s \in \partial\Omega$. This leads to a contradiction with the visibility because the segment joining V_r and $x_s \in B_r(V_r) \cap \partial\Omega$ contains $x_{t_0} = x \in \Omega$. □

3.2.3 Foliation by off-centric spheres

Let us consider the family of off-centric balls $B_r(V_r)$, with $V_r = -v(r)e_n$, for $0 < r < R$. By the Implicit Function Theorem we can easily show the existence of a smooth function ϕ, such that $\partial B_r(V_r)$ is the r-level set of ϕ. This means that the punctured ball $B_R(V_R) \setminus \{0\}$ is foliated by the spheres $\partial B_s(V_s) = \phi^{-1}(s)$ for $0 < s < R$.

Lemma 3.2.9. *There exists a function $\phi \in C^1(B_R(V_R) \setminus \{0\})$ such that $0 \le \phi < R$ and*

$$\partial B_r(V_r) = \phi^{-1}(r)\,, \qquad \text{for any } 0 < r < R.$$

In particular, for any $x \in B_R(V_R) \setminus \{0\}$ we have

$$\frac{\nabla \phi(x)}{|\nabla \phi(x)|} = \frac{x - V_{\phi(x)}}{|x - V_{\phi(x)}|} \tag{3.2.13}$$

and

$$\left| \nabla \phi(x) - \frac{x}{|x|} \right| \le 7 \sqrt{\frac{v(\phi(x))}{\phi(x)} + v'(\phi(x))}\,. \tag{3.2.14}$$

Proof. If $v(r)$ is identically 0, then there is nothing to prove because $\phi(x) = |x|$ in this case. We then suppose $v \neq 0$. Let us start by proving the existence of the function ϕ. We observe that, for any $x \in B_R(V_R)$, there exists a unique $r = r_x \in [0, R)$ such that $x \in \partial B_r(V_r)$. Indeed, if $x = 0$, we can take $r = 0$. Otherwise, let $F : (B_R(V_R) \setminus \{0\}) \times (0, R) \to \mathbb{R}$ be the function defined by

$$F(x, r) = |x - V_r|^2 - r^2\,. \tag{3.2.15}$$

It is immediate to observe that F is continuous. Moreover,

$$F(x, 0) = |x|^2 > 0 \quad \text{and} \quad F(x, R) < 0\,, \qquad \text{for all } x \in B_R(V_R) \setminus \{0\}\,.$$

Hence we can find $r \in (0, R)$ such that $F(x, r) = 0$, i.e. such that $x \in \partial B_r(V_r)$. Let us show the uniqueness. Indeed, if r, $r' \in (0, R)$ have the property that

$$x \in \partial B_r(V_r) \cap \partial B_{r'}(V_{r'})\,,$$

then we get

$$|r - r'| = ||x - V_r| - |x - V_{r'}|| \le |V_r - V_{r'}| = |v(r) - v(r')| \le \frac{1}{2}\,|r - r'|\,,$$

thus we must have $r = r'$. Now we can define $\phi(x) = r_x$. Let us show that $\phi \in$

$\mathcal{C}^1(B_R(R) \setminus \{0\})$. To do so, we note that ϕ is implicitly defined by

$$F(x, \phi(x)) = 0\,, \tag{3.2.16}$$

where F is the function defined in (3.2.15). Easy computations give

$$\partial_r F(x,r) = -2r + 2v'(r)(x_n + v(r))\,.$$

Therefore, if we assume $F(x,r) = 0$ (that is, $x \in B_r(V_r)$) we obtain

$$\partial_r F(x,r) = -2r + 2v'(r)(x_n + v(r)) \leq 2r + |x_n + v(r)| \leq -r\,,$$

where the first inequality follows from the assumption $0 \leq v' \leq 2^{-1}$, while the second inequality from

$$|x_n + v(r)| = |\langle x - V_r, e_n\rangle| \leq |x - V_r| = r\,.$$

By the Implicit Function Theorem we deduce that $\phi \in \mathcal{C}^1(B_R(V_R) \setminus \{0\})$. The identity (3.2.13) is a consequence of the fact that, if $\phi(x) = r$, then the vector $\nabla\phi(x)$ is orthogonal to the level set $\partial B_r(V_r) = \phi^{-1}(r)$ at x.

Let us now prove (3.2.14). We first observe that, if $\phi(x) = r > 0$, then

$$\nabla\phi(x) = -\frac{\partial_x F(x,r)}{\partial_r F(x,r)} = \frac{x + v(r)e_n}{r - v'(r)(x_n + v(r))} \tag{3.2.17}$$

Then (3.2.17) yields

$$\begin{aligned}
\left|\nabla\phi(x)-\frac{x}{|x|}\right|^2 &= \frac{|x+v(r)e_n|^2}{(r-v'(r)(x_n+v(r)))^2}+1-2\left\langle\frac{x+v(r)e_n}{r-v'(r)(x_n+v(r))},\frac{x}{|x|}\right\rangle\\
&= \frac{|x+v(r)e_n|^2}{(r-v'(r)(x_n+v(r)))^2}+1-2\,\frac{|x+v(r)e_n|^2-v(r)(x_n+v(r))}{(r-v'(r)(x_n+v(r)))|x|}\\
&= \frac{|x|\,r^2+|x|(r-v'(r)(x_n+v(r)))^2-2(r-v'(r)(x_n+v(r)))[r^2-v(r)(x_n+v(r))]}{|x|(r-v'(r)(x_n+v(r)))^2}\\
&= \frac{\frac{|x|}{r}+\frac{|x|}{r}\left(1-\frac{v'(r)(x_n+v(r))}{r}\right)^2-2\left(1-\frac{v'(r)(x_n+v(r))}{r}\right)\left(1-\frac{v(r)}{r}\,\frac{x_n+v(r)}{r}\right)}{\frac{|x|}{r}\left(1-\frac{v'(r)(x_n+v(r))}{r}\right)^2}\\
&= \frac{1+\left(1-\frac{v'(r)(x_n+v(r))}{r}\right)^2-2\,\frac{r}{|x|}\left(1-\frac{v'(r)(x_n+v(r))}{r}\right)\left(1-\frac{v(r)}{r}\,\frac{x_n+v(r)}{r}\right)}{\left(1-\frac{v'(r)(x_n+v(r))}{r}\right)^2}\,.
\end{aligned}\tag{3.2.18}$$

Next we observe that $\Big||x|-r\Big|=\Big|x-|x-V_r|\Big|$, hence

$$r-v(r)\le|x|\le r+v(r)\,.\tag{3.2.19}$$

Exploiting (3.2.19), then summing and subtracting 2 in the numerator of the last term of (3.2.18), we get

$$\left|\nabla\phi(x)-\frac{x}{|x|}\right|^2\le\frac{2-2\left(1-\frac{v(r)}{r-v(r)}\right)\left(1-\frac{v'(r)(x_n+v(r))}{r}\right)\left(1-\frac{v(r)}{r}\,\frac{x_n+v(r)}{r}\right)}{\left(1-\frac{v'(r)(x_n+v(r))}{r}\right)^2}\,.$$

We now recall that $v'\le2^{-1}$, thus in particular $v(r)\le2^{-1}r$, and we observe that $x_n+v(r)\le r$, which leads to the final estimate

$$\begin{aligned}
\left|\nabla\phi(x)-\frac{x}{|x|}\right|^2 &\le 8\left\{\frac{v(r)}{r-v(r)}+\frac{v'(r)(x_n+v(r))}{r}+\frac{v(r)(x_n+v(r))}{r^2}+\frac{v(r)^2v'(r)(x_n+v(r))^2}{(r-v(r))r^2}\right\}\\
&\le 8\left\{\frac{v(r)}{r-v(r)}+v'(r)+\frac{v(r)}{r}+\frac{v(r)}{2(r-v(r))}\right\}\\
&\le 40\left\{\frac{v(r)}{r}+v'(r)\right\}\,,
\end{aligned}$$

and this concludes the proof. □

3.2.4 Some examples

We exhibit some relevant classes of domains for which the visibility holds. We recall that

$$\omega_\nu(s) = \omega(s\nu)\,, \qquad \text{for } s \geq 0.$$

Example 3.2.10 (Lipschitz cones and outer star-shaped sets)**.** *Let Ω be either a cone with respect to 0, or such that its complement $\mathbb{R}^n \setminus \Omega$ is locally star-shaped with respect to 0. It is immediate to check that Ω satisfies the visibility property with visibility function $u(t) \equiv 0$.*

Example 3.2.11 ($C^{1,\beta}$-sets)**.** *Let Ω have $C^{1,\beta}$ boundary and assume $0 \in \partial\Omega$. We show that Ω satisfies the visibility property. Up to a rotation we can assume that Ω admits a representation as in* (1.4.3) *with $\omega \in C^{1,\beta}(B'_\rho)$. By Corollary 3.2.4, it is enough to show that ω satisfies* (3.2.5)*. Since $\nabla\omega$ is β-Hölder, we have*

$$\langle x', \nabla\omega(x')\rangle \leq \langle x', \nabla\omega(0)\rangle + u(|x'|)\,. \tag{3.2.20}$$

where $u(t) = C\,t^{1+\beta}$ for some $C > 0$. Set $\overline{\omega}(x') := \omega(x') - \langle \nabla\omega(0), x'\rangle$ and note that $\overline{\omega}$ is $C^{1,\beta}$, $\overline{\omega}(0) = 0$, $\nabla\overline{\omega}(0) = 0$, and

$$\begin{aligned} |\omega(x') - \langle x', \nabla\omega(0)\rangle| &\leq \max_{|y'|\leq|x'|} |\nabla\overline{\omega}(y')||x'| \\ &\leq (|\nabla\overline{\omega}(0)| + C|x|^\beta)|x'| \\ &= C\,|x'|^{1+\beta}\,. \end{aligned} \tag{3.2.21}$$

Putting together (3.2.20) *and* (3.2.21)*, we finally get*

$$\langle x', \nabla\omega(x')\rangle \leq \omega(x') + u(|x'|), \qquad \textit{for a.e. } x' \in B'_\rho\,,$$

that is precisely (3.2.5)*. Since trivially u is non-decreasing and satisfies (V2), by Corollary 3.2.4 we infer that Ω satisfies the visibility property.*

Example 3.2.12 (Convex sets satisfying (V2))**.** *Let* Ω *be a convex set with* $0 \in \partial\Omega$*. For* $s > 0$*, let*

$$\Omega_s := \frac{1}{s}\,\Omega\,, \qquad \Omega_0 := \bigcup_{s>0} \frac{1}{s}\,\Omega\,.$$

The set Ω_0 *is the tangent cone to* Ω *at* 0*. Let*

$$u(r) := \mathrm{dist}_{\mathcal{H}}(\Omega \cap B_r, \Omega_0 \cap B_r)\,, \tag{3.2.22}$$

and assume that $u(r)$ *satisfies (V2).*

We observe that u *is monotonically non-decreasing in* r*. Let us prove that* Ω *satisfies the visibility property. As before, we assume that* Ω *admits a graphical representation as in* (1.4.3) *with the further property that* $\omega : B'_\rho \to \mathbb{R}$ *is convex. Using the notations introduced in the proof of Proposition 3.2.5, by the convexity of* ω*, we can define*

$$\omega_0(x') = \begin{cases} |x'| D^+_{\frac{x'}{|x'|}}\omega(0) & \text{if } x' \neq 0, \\ 0 & \text{if } x' = 0\,, \end{cases}$$

and deduce that

$$\Omega_0 = \{x \in \mathbb{R}^n \,:\, x_n > \omega_0(x')\}\,.$$

By the definition of u *given in* (3.2.22)*, we have*

$$||\omega - \omega_0||_{L^\infty(B'_r)} \leq Cu(r)\,, \qquad \text{for some } C > 0\,. \tag{3.2.23}$$

Owing to Corollary 3.2.4, the visibility property can be proved by showing that (3.2.5) *holds. Thanks to the convexity of* ω*, for all* $\nu \in \partial B'_1$*, we have*

$$D^+_\nu \omega(0) \leq \omega'_\nu\left(\frac{s}{2}^+\right) := \lim_{\sigma \to 0^+} \frac{\omega_\nu(s/2+\sigma) - \omega_\nu(s/2)}{\sigma}\,, \qquad \text{for all } 0 \leq s < \rho\,. \tag{3.2.24}$$

Moreover, for all $0 < \sigma < s/2$, *by* (3.2.23) *and the convexity of* ω_ν, *we have*

$$\begin{aligned}
\frac{\omega_\nu(s/2+\sigma) - \omega_\nu(s/2)}{\sigma} &\leq \frac{\omega_\nu(s) - \omega_\nu(s/2)}{s/2} \\
&= 2\,\frac{\omega_\nu(s)}{s} - \frac{\omega_\nu(s/2)}{s/2} \\
&= 2\,\frac{\omega_\nu(s)}{s} - 2D^+_\nu\omega(0) + D^+_\nu\omega(0) - \frac{\omega_\nu(s/2)}{s/2} + D^+_\nu\omega(0) \qquad (3.2.25) \\
&= \frac{2}{s}\,(\omega_\nu(s) - \omega_0(s\nu) + \omega_0(s\nu/2) - \omega_\nu(s/2)) + D^+_\nu\omega(0) \\
&\leq D^+_\nu\omega(0) + \tilde{C}\,\frac{u(s)}{s}\,, \qquad \text{for some } \tilde{C} > 0\,.
\end{aligned}$$

Putting together (3.2.24) *and* (3.2.25), *we obtain*

$$\left| D^+_\nu\omega(0) - \omega'_\nu\left(\frac{s}{2}^+\right)\right| \leq \tilde{C}\,\frac{u(s)}{s}\,.$$

This suffices to conclude. In fact, if $x' \in B'_\rho \setminus \{0\}$ *is such that* ω *is differentiable at* x', *setting* $s := |x'|$, $\nu := x'/|x'|$, *for some* $C > 0$, *we achieve*

$$\begin{aligned}
\langle x', \nabla\omega(x')\rangle = \langle s\nu, \nabla\omega(s\nu)\rangle &= s\,\omega'_\nu(s^+) \\
&\leq s\,D^+_\nu\omega(0) + C\,u(s) \\
&\leq \omega(s\nu) + C\,u(s) = \omega(x') + C\,u(|x'|)\,.
\end{aligned}$$

Since ω *is convex, it is differentiable a.e. in* B'_ρ, *and so* (3.2.5) *is verified. Moreover,* u *is non-decreasing and satisfies (V2) by our assumption, thus we conclude.*

Example 3.2.13 (A piece-wise affine profile)**.** *For* $i \in \mathbb{N}$, *let*

$$y_i = 2^{-i}\,, \qquad \tilde{y}_i = 2^{-i} + 2^{-2i}\,.$$

We observe that $y_{i+1} < y_{i+1} < y_i$. *For* $i \in \mathbb{N}$, *we define*

$$P_i = (y_i, \tilde{y}_i)\,, \qquad Q_i = (\tilde{y}_i, \tilde{y}_i)\,.$$

We can consider the polygonal curve formed by the segments $\overline{P_iQ_{i+1}}$, $\overline{Q_{i+1}P_{i+1}}$, $i \geq 0$. *It is*

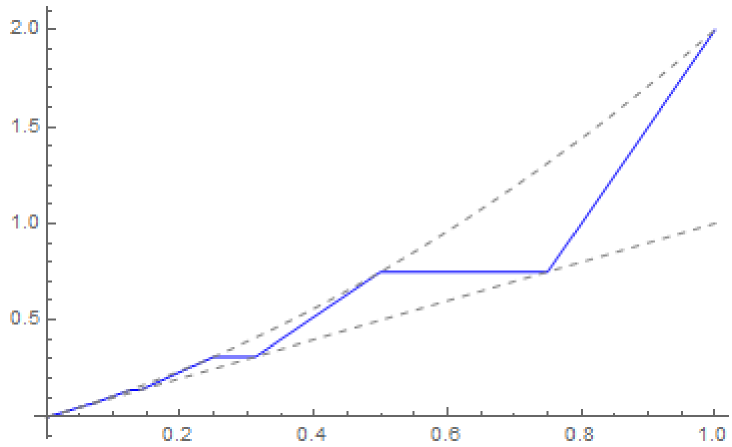

Figure 3.1: The graph of $\omega(y)$ "bounces" between the graphs of y and $y + y^2$.

immediate to observe that this curve coincides with the graph of the function $\omega : [0, 1] \to \mathbb{R}$ *uniquely defined by*

$$\omega(y) = \begin{cases} 2^{-(i+1)} + 2^{-2(i+1)} & \text{if } y \in (y_{i+1}, \tilde{y}_{i+1}], \\ a_i\, y - b_i & \text{if } y \in (\tilde{y}_{i+1}, y_i]\,, \end{cases} \tag{3.2.26}$$

for $i \geq 0$, *where*

$$a_i = \frac{1 + 3 \cdot 2^{-(i+1)}}{1 - 2^{-(i+1)}}\,, \qquad b_i = \frac{2^{-2i} + 2^{-(3i+1)}}{1 - 2^{-(i+1)}}\,.$$

Let $\Omega \subset \mathbb{R}^2$ *be an open set such that*

$$\Omega \cap ((-1, 1) \times \mathbb{R}) = \{x = (y, z) \in \mathbb{R}^2 \,:\, z > \omega(|y|)\}\,.$$

Let us show that Ω *satisfies the visibility condition at* 0*. Owing to Corollary 3.2.4, it suffices to show the existence of a non-decreasing function* $u : (0, 1) \to \mathbb{R}$ *satisfying (V1), (V2) and such that*

$$y\, \omega'(y) \leq \omega(y) + u(y)\,, \qquad \text{for a.e. } y \in (-1, 1)\,. \tag{3.2.27}$$

By (3.2.26)*, for every* $i \in \mathbb{N}$*, we have*

$$y\,\omega'(y) = \begin{cases} 0 & \text{if } y \in (y_{i+1}, \tilde{y}_{i+1}], \\ a_i\, y & \text{if } y \in (\tilde{y}_{i+1}, y_i]\,. \end{cases}$$

Thus, in order to realize (3.2.27)*, it suffices to choose a function* $u = u(y)$ *greater or equal than the function*

$$\overline{u}(y) = \begin{cases} 0 & \text{if } y \in (y_{i+1}, \tilde{y}_{i+1}], \\ b_i & \text{if } y \in (\tilde{y}_{i+1}, y_i]\,. \end{cases}$$

We look for $\alpha > 0$ *such that*

$$u_\alpha(y) = \alpha y^2 \geq \overline{u}(y)\,. \tag{3.2.28}$$

In order to obtain the validity of (3.2.28)*, it suffices to impose that*

$$u_\alpha(\tilde{y}_{i+1}) = \alpha\,\tilde{y}_{i+1}^2 \geq b_i\,.$$

It is immediate to observe that

$$\alpha\,\tilde{y}_{n+1}^2 \geq \frac{\alpha}{4}\,2^{-2i}\,, \qquad b_i \leq 4 \cdot 2^{-2i}\,,$$

hence, if for instance we take $\alpha = 16$*,* (3.2.28) *holds. Since the function* $u(y) = 16\,y^2$ *trivially fulfills (V1) and (V2), we conclude that* Ω *satisfies the visibility condition at* 0*.*

3.3 Boundary Monotonicity Formula

The present section aims to prove a boundary Monotonicity Formula for local almost-minimizers of P_Ω at a point $x_0 \in \partial\Omega$ satisfying the visibility property up to an isometry (hence, from now on, we will directly assume $x_0 = 0$). In the notation of the previous section, given $0 < r_1 < r_2 < R$, we recall $V_r = -v(r)e_n$ and define

$$\mathcal{A}_{r_1,r_2} := B_{r_2}(V_{r_2}) \setminus \overline{B_{r_1}(V_{r_1})} = \phi^{-1}(r_1, r_2)\,,$$

where $\phi(x)$ is the function defined in Lemma 3.2.9. We also conveniently introduce some further notation. Given $f \in BV_{\mathrm{loc}}(\Omega)$ and $0 < r_1 < r_2$, we set

$$\mu_f(r) = \frac{|Df|(\Omega \cap B_r(V_r))}{r^{n-1}}$$

and

$$\begin{aligned} G(f; r_1, r_2) = &\int_{r_1}^{r_2} \frac{n-1}{\rho^n} \int_{\Omega \cap B_\rho(V_\rho)} (|\nabla \phi| - 1)\, d|Df|\, d\rho \\ &+ \frac{1}{r_2^{n-1}} \int_{\Omega \cap B_{r_2}(V_{r_2})} (|\nabla \phi| - 1)\, d|Df| - \frac{1}{r_1^{n-1}} \int_{\Omega \cap B_{r_1}(V_{r_1})} (|\nabla \phi| - 1)\, d|Df| \,. \end{aligned} \tag{3.3.1}$$

In the next proposition, we combine the visibility property with an upper bound on $\mu_f(r)$ and obtain the finiteness of $\lim_{\rho \to 0} G(f; \rho, r)$.

Proposition 3.3.1. *Let $\Omega \subset \mathbb{R}^n$ satisfy the visibility property as in Definition 3.2.1, and let $f \in BV_{\mathrm{loc}}(\Omega)$. Assume that $\mu_f(r) \le C$ for some constant $C > 0$ and for all $r \in (0, R)$. Then for $r \in (0, R)$ the limit*

$$\begin{aligned} G(f; r) &:= \lim_{\rho \to 0} G(f; \rho, r) \\ &= \int_0^r \frac{n-1}{\rho^n} \int_{\Omega \cap B_\rho(V_\rho)} (|\nabla \phi| - 1)\, d|Df|\, d\rho + \frac{1}{r^{n-1}} \int_{\Omega \cap B_r(V_r)} (|\nabla \phi| - 1)\, d|Df| \end{aligned} \tag{3.3.2}$$

exists and is finite.

Proof. By (3.2.14), for all $x \in B_\rho(V_\rho)$ we have

$$||\nabla \phi(x)| - 1| \le \left| \nabla \phi(x) - \frac{x}{|x|} \right| \le 7 \sqrt{\frac{v(\phi(x))}{\phi(x)} + v'(\phi(x))} \le 7 \sup_{0 < r \le \rho} \sqrt{\frac{v(r)}{r} + v'(r)} = 7\rho\gamma_v(\rho)\,,$$

where γ_v is the function defined in the visibility property (V2). Then, using the upper bound on μ_f, for $0 < \rho < r < R$ we obtain

$$\left| \rho^{1-n} \int_{\Omega \cap B_\rho(V_\rho)} (|\nabla \phi| - 1)\, d|Df| \right| \le 7C\rho\gamma_v(\rho)\,.$$

Thanks to the summability of γ_v (see property (V2) of the visibility property) and to

(3.2.1), from the last inequality we easily get the proof of the proposition. □

Our monotonicity formula will then follow from the general inequality proved in the next theorem.

Theorem 3.3.2 (Monotonicity inequality). *Let $\Omega \subset \mathbb{R}^n$ satisfy the visibility property, and let $f \in BV_{\mathrm{loc}}(\Omega)$. Then, for $R > 0$ small enough, for almost every $0 < r_1 < r_2 < R$, we have*

$$\left(\int_{\Omega\cap\mathcal{A}_{r_1,r_2}} \phi^{1-n}\, |\langle \nu_f, \nabla\phi\rangle|\; d|Df|\right)^2 \leq 2\left(\int_{\Omega\cap\mathcal{A}_{r_1,r_2}} \frac{|\nabla\phi(x)|}{\phi(x)^{n-1}}\, d|Df|\right) \cdot \left[\mu_f(r_2) - \mu_f(r_1) + \int_{r_1}^{r_2} \frac{n-1}{\rho^n}\, \Psi_\Omega(f; B_\rho(V_\rho))d\rho + G(f; r_1, r_2)\right], \tag{3.3.3}$$

where ν_f is such that $Df = \nu_f|Df|$.

Proof. We start by assuming $f \in BV(\Omega)\cap C^1(\Omega)$. For all $0 < r < R$ and $x \in \mathcal{C}_r \cap B_r(V_r)$, where $\mathcal{C}_r$ is defined in (3.2.12), we let

$$Y_r(x) = V_r + r\,\frac{x - V_r}{|x - V_r|}\,.$$

Standard computations yield

$$\begin{aligned} DY_r(x) &= r\left[\frac{1}{|x-V_r|}D(x-V_r) + (x-V_r)\otimes\nabla|x-V_r|^{-1}\right] \\ &= \frac{r}{|x-V_r|}\left[\mathrm{Id} - \frac{x-V_r}{|x-V_r|}\otimes\frac{x-V_r}{|x-V_r|}\right]. \end{aligned} \tag{3.3.4}$$

We define

$$g_r(x) = f(Y_r(x)) \qquad \text{for all } x \in \mathcal{C}_r\,,$$

then the "off-centric conical competitor" is

$$f_r(x) = \begin{cases} g_r(x) & \text{if } x \in \mathcal{C}_r \cap B_r(V_r) \\ f(x) & \text{if } x \in \Omega\setminus B_r(V_r)\,. \end{cases} \tag{3.3.5}$$

By definition, f_r coincides with f in $\Omega \setminus B_r(V_r)$, hence we infer

$$\Psi_\Omega(f; B_r(V_r)) \geq |Df|(\Omega \cap B_r(V_r)) - |Df_r|(\Omega \cap B_r(V_r)) \,. \tag{3.3.6}$$

Then, by (3.3.6), we deduce that

$$\begin{aligned} |Df|(\Omega \cap B_r(V_r)) - \Psi_\Omega(f; B_r(V_r)) &\leq |Df_r|(\Omega \cap B_r(V_r)) \\ &\leq |Df_r|(\mathcal{C}_r \cap B_r(V_r)) \\ &= \int_{\mathcal{C}_r \cap B_r(V_r)} |\nabla g_r(x)| dx \,. \end{aligned} \tag{3.3.7}$$

Let us now compute the gradient of g_r. By (3.3.4), setting

$$\nu_r(x) = \frac{x - V_r}{|x - V_r|} \,, \qquad Y_r = Y_r(x) \,,$$

we obtain

$$\nabla g_r(x) = DY_r \cdot \nabla f(Y_r) = \frac{r}{|x - V_r|} \nabla f(Y_r)^{\nu_r(x)^\perp} \,,$$

where $\nabla f(Y_r)^{\nu_r(x)^\perp}$ denotes the projection of $\nabla f(Y_r)$ onto the hyperplane

$$\nu_r(x)^\perp := \{y \in \mathbb{R}^n \,:\, \langle \nu_r(x), y \rangle = 0\} \,.$$

Going on with the computations, we obtain

$$\begin{aligned} |\nabla g_r(x)| &= \frac{r}{|x - V_r|} \sqrt{|\nabla f(Y_r)|^2 - \langle \nabla f(Y_r), \nu_r(x) \rangle^2} \\ &= \frac{r}{|x - V_r|} \, |\nabla f(Y_r)| \sqrt{1 - \frac{\langle \nabla f(Y_r), \nu_r(x) \rangle^2}{|\nabla f(Y_r)|^2}} \,. \end{aligned}$$

Consequently, we get

$$\begin{aligned}\int_{\mathcal{C}_r\cap B_r(V_r)} |\nabla g_r| dx &= \int_0^r \int_{\mathcal{C}_r\cap\partial B_\rho(V_r)} \frac{r}{\rho}\, |\nabla f(Y_r)| \sqrt{1-\frac{\langle \nabla f(Y_r),\nu_r\rangle^2}{|\nabla f(Y_r)|^2}}\, d\mathcal{H}^{n-1}\, d\rho \\ &= \int_0^r \left(\frac{\rho}{r}\right)^{n-2} d\rho \int_{\mathcal{C}_r\cap\partial B_r(V_r)} |\nabla f| \sqrt{1-\frac{\langle \nabla f,\nu_r\rangle^2}{|\nabla f|^2}}\, d\mathcal{H}^{n-1} \\ &\le \frac{r}{n-1}\left\{ \int_{\Omega\cap\partial B_r(V_r)} |\nabla f| d\mathcal{H}^{n-1} - \frac{1}{2}\int_{\Omega\cap\partial B_r(V_r)} \frac{\langle \nabla f,\nu_r\rangle^2}{|\nabla f|}\, d\mathcal{H}^{n-1}\right\}.\end{aligned} \tag{3.3.8}$$

Combining (3.3.7) and (3.3.8), we get

$$\begin{aligned}&\frac{r}{2(n-1)} \int_{\Omega\cap\partial B_r(V_r)} \frac{\langle \nabla f,\nu_r\rangle^2}{|\nabla f|} d\mathcal{H}^{n-1} \\ &\quad \le \frac{r}{n-1}\int_{\Omega\cap\partial B_r(V_r)} |\nabla f| d\mathcal{H}^{n-1} - \int_{\Omega\cap B_r(V_r)} |\nabla f| dx + \Psi_\Omega(f; B_r(V_r))\,.\end{aligned} \tag{3.3.9}$$

Multiplying both sides of (3.3.9) by $(n-1)r^{-n}$ and observing that $r = \phi(y)$ for any $y \in \partial B_r(V_r)$, we get

$$\begin{aligned}&\frac{1}{2}\int_{\Omega\cap\partial B_r(V_r)} \left\langle \nabla f, \frac{\nabla\phi}{|\nabla\phi|}\right\rangle^2 \frac{1}{|\nabla f|\,\phi^{n-1}}\, d\mathcal{H}^{n-1} \\ &\quad \le \frac{1}{r^{n-1}}\int_{\Omega\cap\partial B_r(V_r)} |\nabla f|\, d\mathcal{H}^{n-1} + \frac{n-1}{r^n}\int_{\Omega\cap B_r(V_r)} |\nabla f| dx + \frac{n-1}{r^n}\,\Psi_\Omega(f; B_r(V_r)) \\ &\quad = \frac{d}{dr}\left(\frac{1}{r^{n-1}}\int_{\Omega\cap B_r(V_r)} |\nabla f||\nabla\phi| dx\right) \\ &\qquad + \frac{n-1}{r^n}\int_{\Omega\cap B_r(V_r)} |\nabla f|(|\nabla\phi|-1) dx + \frac{n-1}{r^n}\,\Psi_\Omega(f; B_r(V_r))\,.\end{aligned} \tag{3.3.10}$$

Let us now integrate (3.3.10) between $0 < r_1 < r_2 < R$. We then achieve

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega\cap\mathcal{A}_{r_1,r_2}}\left\langle\nabla f(x),\frac{\nabla\phi(x)}{|\nabla\phi(x)|}\right\rangle^2\frac{|\nabla\phi(x)|}{|\nabla f(x)|\,\phi(x)^{n-1}}\,dx\\
&\quad\leq\frac{1}{r_2^{n-1}}\int_{\Omega\cap B_{r_2}(V_{r_2})}|\nabla f(x)||\nabla\phi(x)|dx-\frac{1}{r_1^{n-1}}\int_{\Omega\cap B_{r_1}(V_{r_1})}|\nabla f(x)||\nabla\phi(x)|dx\\
&\qquad+\int_{r_1}^{r_2}\frac{n-1}{r^n}\int_{\Omega\cap B_r(V_r)}|\nabla f(x)|(|\nabla\phi(x)|-1)dx\,dr+\int_{r_1}^{r_2}\frac{n-1}{r^n}\,\Psi_\Omega(f;B_r(V_r))dr\,.
\end{aligned}\tag{3.3.11}$$

By Hölder's Inequality, we get

$$\begin{aligned}
&\left(\int_{\Omega\cap\mathcal{A}_{r_1,r_2}}|\langle\nabla f(x),\nabla\phi(x)\rangle|\,\frac{dx}{\phi(x)^{n-1}}\right)^2\\
&\quad\leq\left(\int_{\Omega\cap\mathcal{A}_{r_1,r_2}}\frac{|\nabla\phi(x)|}{\phi(x)^{n-1}}|\nabla f(x)|dx\right)\cdot\left(\int_{\Omega\cap\mathcal{A}_{r_1,r_2}}\left\langle\nabla f(x),\frac{\nabla\phi(x)}{|\nabla\phi(x)|}\right\rangle^2\frac{|\nabla\phi(x)|\,dx}{|\nabla f(x)|\phi(x)^{n-1}}\right).
\end{aligned}\tag{3.3.12}$$

Putting together (3.3.11) and (3.3.12), we obtain

$$\begin{aligned}
&\left(\int_{\Omega\cap\mathcal{A}_{r_1,r_2}}|\langle\nabla f(x),\nabla\phi(x)\rangle|\,\frac{dx}{\phi(x)^{n-1}}\right)^2\leq2\left(\int_{\Omega\cap\mathcal{A}_{r_1,r_2}}\frac{|\nabla\phi(x)|}{\phi(x)^{n-1}}|\nabla f(x)|dx\right)\\
&\quad\cdot\left(\frac{1}{r_2^{n-1}}\int_{\Omega\cap B_{r_2}(V_{r_2})}|\nabla f(x)|dx-\frac{1}{r_1^{n-1}}\int_{\Omega\cap B_{r_1}(V_{r_1})}|\nabla f(x)|dx\right.\\
&\qquad\left.+\int_{r_1}^{r_2}\frac{n-1}{\rho^n}\,\Psi_\Omega(f;B_\rho(V_\rho))d\rho+G(f;r_1,r_2)\right),
\end{aligned}\tag{3.3.13}$$

where $G(f;r_1,r_2)$ is as in (3.3.1). This proves (3.3.3) for all $f\in BV(\Omega)\cap C^1(\Omega)$.

Let now $f\in BV(\Omega)$. We can select a sequence $f_j\in BV(\Omega)\cap C^1(\Omega)$ such that

$$||f_j-f||_{L^1(\Omega)}\to0\,,\qquad|Df_j|(\Omega)\to|Df|(\Omega)\,,\qquad Df_j\overset{*}{\rightharpoonup}Df\quad\text{in }\Omega\,.\tag{3.3.14}$$

In particular, by the continuity of the trace with respect to the strict convergence, we have

$$||\mathrm{Tr}^+(f_j,\partial\Omega)-\mathrm{Tr}^+(f,\partial\Omega)||_{L^1(\partial\Omega,\mathcal{H}^{n-1})}\to0\,.\tag{3.3.15}$$

Let us consider the extensions

$$f_{0,j}(x) = \begin{cases} f_j(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega\,, \end{cases} \qquad f_0(x) = \begin{cases} f(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega\,. \end{cases}$$

We observe that, by (3.3.14), (3.3.15),

$$||f_{0,j} - f_0||_{L^1(\mathbb{R}^n)} \to 0\,, \qquad |Df_{0,j}|(\mathbb{R}^n) \to |Df_0|(\mathbb{R}^n)\,.$$

By Proposition 1.3.12, for almost all $0 < r < R$,

$$|Df_{0,j}|(B_r(V_r)) \to |Df_0|(B_r(V_r))\,, \qquad ||\mathrm{Tr}^+(f_{0,j}, \partial B_r(V_r)) - \mathrm{Tr}^+(f_0, \partial B_r(V_r))||_{L^1(\partial B_r(V_r))} \to 0\,, \tag{3.3.16}$$

and in particular, owing to (3.3.15),

$$|Df_j|(\Omega \cap B_r(V_r)) = |Df_{0,j}|(\Omega \cap B_r(V_r)) \to |Df_0|(\Omega \cap B_r(V_r)) = |Df|(\Omega \cap B_r(V_r))\,. \tag{3.3.17}$$

Now (3.3.16), (3.3.17) allow to apply Lemma 2.1.8, deducing that

$$|\Psi_\Omega(f_j; B_r(V_r)) - \Psi_\Omega(f; B_r(V_r))| \to 0\,, \qquad \text{as } j \to \infty\,,$$

for almost all $0 < r < R$. This implies that

$$\int_{r_1}^{r_2} \frac{n-1}{\rho^n}\, \Psi_\Omega(f_j; B_\rho(V_\rho))d\rho \to \int_{r_1}^{r_2} \frac{n-1}{\rho^n}\, \Psi_\Omega(f; B_\rho(V_\rho))d\rho\,.$$

Finally, to conclude that the RHS of (3.3.13) for $f = f_j$, passes to the limit as $j \to \infty$, giving precisely the RHS of (3.3.3), it suffices to show that the terms

$$\int_{\Omega\cap\mathcal{A}_{r_1,r_2}} \phi^{1-n}\, d|Df_j|\,, \quad \int_{\Omega\cap B_{r_1}(V_{r_1})} (|\nabla\phi| - 1)\, d|Df_j|\,, \quad \int_{\Omega\cap B_{r_2}(V_{r_2})} (|\nabla\phi| - 1)\, d|Df_j|\,,$$

converge as $j \to \infty$ respectively to

$$\int_{\Omega\cap\mathcal{A}_{r_1,r_2}} \phi^{1-n}\, d|Df|\,, \quad \int_{\Omega\cap B_{r_1}(V_{r_1})} (|\nabla\phi| - 1)\, d|Df|\,, \quad \int_{\Omega\cap B_{r_2}(V_{r_2})} (|\nabla\phi| - 1)\, d|Df|\,.$$

To see this, it suffices to construct a suitable partition of each domain, for instance using portions of circular annuli whose boundaries are negligible for $|Df|$ and $|Df_j|$ for all $j \geq 1$, to uniformly approximate each integrand by simple functions (up to removing a small neighborhood of 0 in the case of the last two integrals). About the LHS of (3.3.13), we observe that (3.3.14) implies

$$Df_j \overset{*}{\rightharpoonup} Df \qquad \text{in } \Omega \cap \mathcal{A}_{r_1,r_2} .$$

Now, for f smooth, the LHS of (3.3.3) and the LHS of (3.3.13) coincide. Moreover, we have

$$\int_{\Omega \cap \mathcal{A}_{r_1,r_2}} \phi(x)^{1-n} \left|\langle \nu_f(x), \nabla\phi(x)\rangle\right| \, d|Df|(x) = \left|\phi^{1-n}\nabla\phi \cdot Df_j\right| (\Omega \cap \mathcal{A}_{r_1,r_2}) .$$

In particular, (3.3.14) implies that

$$\phi^{1-n}\nabla\phi \cdot Df_j \overset{*}{\rightharpoonup} \phi^{1-n}\nabla\phi \cdot Df ,$$

and well-known properties of the weak-star convergence of Radon measures (see [26]) ensure that

$$\left|\phi^{1-n}\nabla\phi \cdot Df\right| (\Omega \cap \mathcal{A}_{r_1,r_2}) \leq \liminf_{j\to\infty} \left|\phi^{1-n}\nabla\phi \cdot Df_j\right| (\Omega \cap \mathcal{A}_{r_1,r_2}) .$$

This implies (3.3.3) and concludes the proof of the theorem. □

The next corollary, a first important consequence of Theorem 3.3.2, states the monotonicity of a suitable function of the radius r, which is defined by three terms: the renormalized perimeter $\mu_E(r)$, the integral of a renormalized minimality gap $\Psi_\Omega(E; B_r(V_r))$, and the visibility error $G(E, r)$. In particular, when E is an almost-minimizer, the infinitesimality of the second and third terms implies that $\mu_E(r)$ is "almost-increasing", hence that it admits a finite limit as $r \to 0$. This limit represents the perimeter density of E at 0, see Remark 3.3.4 below.

Corollary 3.3.3 (Boundary monotonicity for almost-minimizers)**.** *Let Ω be an open set satisfying the visibility property. Let $E \subset \Omega$ be a local almost-minimizer, such that*

$P_\Omega(E, B_r) > 0$ for all $r > 0$ and

$$\int_0^R \rho^{-n}\Psi_\Omega(E; B_\rho(V_\rho))\, d\rho < +\infty\,.$$

Then there exists R' such that the function

$$r \mapsto \mu_E(r) + (n-1)\int_0^r \rho^{-n}\Psi_\Omega(E; B_\rho(V_\rho))\, d\rho + G(E; r)$$

is non-decreasing on $(0, R')$. Moreover, the two terms $\int_0^r \rho^{-n}\Psi_\Omega(E; B_\rho(V_\rho))\, d\rho$ and $G(E; r)$ are infinitesimal as $r \to 0$, hence in particular $\mu_E(r)$ is "almost-monotone" and the limit

$$\theta_E(0) := \lim_{r\to 0^+} \mu_E(r)$$

exists and is finite.

Proof of Corollary 3.3.3. By Lemma 2.1.6 and the fact that $B_r(V_r) \subset B_{r+v(r)}$, we can find constants $C, R > 0$ such that

$$\frac{|Df|(\Omega \cap B_r(V_r))}{r^{n-1}} \le \frac{|Df|(\Omega \cap B_{r+v(r)})}{(r+v(r))^{n-1}}\left(1 + \frac{v(r)}{r}\right)^{n-1} \le C\,, \qquad \text{for all } 0 < r < R.$$

By combining Proposition 3.3.1 with (3.2.14) and the previous bound, up to redefining the constants $C, R > 0$, we obtain

$$|G(E; r)| \le C\left(\int_0^r \gamma_v(\rho)\, d\rho + r\gamma_v(r)\right)\,, \qquad \text{for all } 0 < r < R. \tag{3.3.18}$$

Finally, the proof of the corollary follows directly from Theorem 3.3.2 and from the observation that the RHS of (3.3.18) is infinitesimal as $r \to 0^+$. □

Remark 3.3.4. *It is easy to check that, under the assumptions of Corollary 3.3.3, one has*

$$\exists \lim_{r\to 0^+} \frac{P_\Omega(E; B_r)}{r^{n-1}} = \theta_E(0)\,. \tag{3.3.19}$$

Indeed, this is an immediate consequence of the inclusions

$$B_{r-v(r)}(V_r) \subset B_r \subset B_{r+v(r)}(V_r)$$

combined with $v(r) = o(r)$ as $r \to 0$.

3.4 Blow-up limits of almost-minimizers are cones

We now apply Theorem 3.3.2 and prove that any blow-up limit of a local almost-minimizer E of P_Ω is a perimeter-minimizing cone.

Theorem 3.4.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set satisfying the visibility property. Let $E \subset \Omega$ be a local almost-minimizer in Ω such that*

$$\int_0^R \frac{\Psi_\Omega(E; B_r)}{r^n} dr < \infty \,. \tag{3.4.1}$$

Fix a decreasing sequence $t_j \to 0$ and set $E_{t_j} = t_j^{-1}E$. Then, up to subsequences, E_{t_j} converges to $E_0 \subset \Omega_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, E_0 is a nontrivial cone minimizing the relative perimeter in Ω_0.

Proof. Set $E_j = E_{t_j}$ and $\Omega_j = \Omega_{t_j}$ for more simplicity. Then by the upper density estimate on the relative perimeter of E (Lemma 2.1.6) coupled with analogous estimates satisfied by Ω Lipschitz, we can find a constant $C > 0$ such that, for every fixed $R > 0$,

$$\begin{aligned} P(E_j; B_R) &\le P(\Omega_j; B_R) + P_{\Omega_j}(E_j; B_R) \\ &= t_j^{1-n}\Big(P(\Omega; B_{Rt_j}) + P_\Omega(E; B_{Rt_j})\Big) \\ &\le CR^{n-1} \,. \end{aligned}$$

By the compactness property of sequences of sets with uniformly bounded relative perimeter the ball B_R, we conclude that there exists a not relabeled subsequence E_j and a set E_0 of finite perimeter in B_R, such that $E_j \to E_0$ in $L^1(B_R)$ as $j \to \infty$. The fact that $E_0 \subset \Omega_0$ up to null sets is immediate, since $E_j \subset \Omega_j$, for all j, and the sequence Ω_j converges to the tangent cone Ω_0 locally in Hausdorff distance (hence, in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$) thanks to Proposition 3.2.5. Up to a standard diagonal argument we can assume that the subsequence E_j converges to E_0 in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover by the lower-density estimates on the volume of E we also deduce that E_0 can be neither the empty set, nor the whole Ω_0 up to null sets (that is, E_0 is nontrivial).

By the scaling properties of the perimeter, for any fixed $R > 0$ we have

$$\Psi_{\Omega_{t_j}}(E_{t_j}; B_R) = \frac{1}{t_j^{n-1}} \Psi_\Omega(E; B_{t_j R}) \le \omega_n^{1-\frac{1}{n}} R^{n-1} \psi_\Omega(E; 0, t_j R) \longrightarrow 0\,,$$

therefore we can apply Lemma 2.1.7 and deduce that

$$\Psi_{\Omega_0}(E_0; B_R) \le \liminf_j \Psi_{\Omega_{t_j}}(E_{t_j}; B_R) = 0$$

for all $R > 0$ and, also owing to Corollary 3.3.3,

$$P_{\Omega_0}(E_0; B_R) = \lim_j P_{\Omega_{t_j}}(E_{t_j}; B_R) = \lim_j t_j^{1-n} P_\Omega(E; B_{t_j R} \cap \Omega) = R^{n-1} \theta_E(0)\,.$$

Thus, E_0 is a minimizer for the relative perimeter in the cone Ω_0, such that

$$\frac{P(E_0; B_R \cap \Omega_0)}{R^{n-1}} = \theta_E(0) \qquad \text{for all } R > 0.$$

Now, the monotonicity inequality (3.3.3) written for $f = \mathbf{1}_{E_0}$ and $\Omega = \Omega_0$ takes the form

$$\begin{aligned}\left(\int_{\Omega_0 \cap (B_{r_2} \setminus B_{r_1})} \frac{|\langle \nu_{E_0}(x), x \rangle|}{|x|^n}\, d|D\mathbf{1}_E|(x) \right)^2 \\ \le \left(\int_{\Omega_0 \cap (B_{r_2} \setminus B_{r_1})} |x|^{1-n}\, d|D\mathbf{1}_{E_0}(x)| \right) \cdot \left(\frac{P_{\Omega_0}(E_0; B_{r_2})}{r_2^{n-1}} - \frac{P_{\Omega_0}(E_0; B_{r_1})}{r_1^{n-1}} \right) = 0\,,\end{aligned}$$

for almost all $0 < r_1 < r_2$. The only possibility is then that $\langle \nu_{E_0}(x), x \rangle = 0$ at $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E_0$. By [26, Proposition 28.8] we infer that E_0 is a cone with vertex at the origin, up to negligible sets, and the proof is concluded. □

Chapter 4

Free-boundary variations in non-smooth domains

In this Chapter, we develop a technique for constructing "free-boundary variations" in domains with non-smooth boundary. We then apply this technique to the study of the stability of a n-plane containing the vertex of a circular cone in $\mathbb{R}^{n+1}$.

4.1 Some preliminaries

4.1.1 Main notations

Along the present Chapter, will adopt the following notations: for $n \geq 2$, given $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, we define $\tilde{x} = (x, t) \in \mathbb{R}^{n+1}$ and $x' \in \mathbb{R}^{n-1}$ such that $x = (x', x_n)$, with a slight abuse of notation. We identify x with $(x, 0) \in \mathbb{R}^{n+1}$ and x' with $(x', 0, 0) \in \mathbb{R}^{n+1}$, whenever this does not create confusion. We denote by $\tilde{B}_r(\tilde{x})$ the open ball of radius $r > 0$ centered at $\tilde{x}$, and we set $\tilde{B}_r := \tilde{B}_r(0)$. For any $1 \leq i \leq n+1$, we denote by p_i the hyperplane of $\mathbb{R}^{n+1}$ of equation $x_i = 0$, and use the same notation for the orthogonal projection of $\mathbb{R}^{n+1}$ onto this hyperplane. Given a differentiable function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, we denote the partial derivative of f with respect to x_i as $\partial_i f$.

4.1.2 A technical result about wedge-products

Let $v_1, ..., v_k \in \mathbb{R}^{n+1}$ and denote by $S[v_1, ..., v_k]$ the $k \times k$ symmetric matrix which elements are the scalar products $\langle v_i, v_j \rangle$, for $1 \le i, j \le k$. By applying the definition of determinant and the Cauchy-Binet Formula (see, e.g., [28, Chapter 2]) it is immediate to show that

$$\det(S[v_1, ..., v_k]) = |v_1 \wedge ... \wedge v_k|^2 \,, \tag{4.1.1}$$

where $\wedge$ denotes the wedge-product of $v_1, ..., v_k$ (we refer to [11] for a formal definition of $\wedge$ and its main properties). We limit to mention that $\wedge$ is alternating, i.e. $v_1 \wedge ... \wedge v_k$ vanishes whenever $v_i = v_j$, for some $i \neq j$. Moreover, the wedge products $v_1 \wedge ... \wedge v_k$ form a linear vector space, usually denoted by $\bigwedge^k \mathbb{R}^{n+1}$. If $e_1, ..., e_{n+1}$ denote the canonical basis of $\mathbb{R}^{n+1}$, then the wedge products $e_{i_1} \wedge ... \wedge e_{i_k}$, for all $1 \le i_1 < ... < i_k \le n+1$, form a basis of $\bigwedge^k \mathbb{R}^{n+1}$. We assume that $\bigwedge^k \mathbb{R}^{n+1}$ is endowed with the unique scalar product such that $e_{i_1} \wedge ... \wedge e_{i_k}$, for $1 \le i_1 < ... < i_k \le n+1$, are orthonormal. The norm associated with this scalar product is denoted by $|\cdot|$.
Let us introduce some convenient notation. We let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a multi-index with $\alpha_\ell \in \{1, \dots, n+1\}$. Then, we set $e_\alpha = e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k}$ and, given $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n+1\}$, we define

$$e^j_{\alpha,i} = e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{i-1}} \wedge e_j \wedge e_{\alpha_{i+1}} \wedge \cdots \wedge e_{\alpha_k}$$

Lemma 4.1.1. *For $i = 1, ..., n$, let $v_i = e_i + A_i\, e_n + B_i\, e_{n+1}$, where A_i, B_i are real numbers. Then*

$$\det(S[v_1, ..., v_n]) = (1 + A_n)^2 \left(1 + \sum_{i=1}^{n-1} B_i^2\right) + B_n^2 \left(1 + \sum_{i=1}^{n-1} A_i^2\right) - 2\,(1 + A_n) B_n \sum_{i=1}^{n-1} A_i B_i \,. \tag{4.1.2}$$

Proof. Let us set $\alpha = (1, 2, \dots, n-1)$. By definition of v_n, using the alternating property of the wedge product and the fact that $e^n_{\alpha,i} \wedge e_{n+1} = -e^{n+1}_{\alpha,i} \wedge e_n$, we get

$$\begin{aligned} v_1 \wedge \ldots \wedge v_n = (1+A_n)\, e_\alpha \wedge e_n + B_n\, e_\alpha \wedge e_{n+1} + \\ + B_n \sum_{i=1}^{n-1} A_i\, e^n_{\alpha,i} \wedge e_{n+1} + (1+A_n) \sum_{i=1}^{n-1} B_i\, e^{n+1}_{\alpha,i} \wedge e_n \\ = (1+A_n)\, e_\alpha \wedge e_n + B_n\, e_\alpha \wedge e_{n+1} + \sum_{i=1}^{n-1} \left[B_n A_i - (1+A_n)\, B_i\right]\, e^n_{\alpha,i} \wedge e_{n+1}\,. \end{aligned} \tag{4.1.3}$$

By the orthonormality of the wedge products $e_\alpha \wedge e_n, e_\alpha \wedge e_{n+1}$, and $e^n_{\alpha,i} \wedge e_{n+1}$ for $i = 1, \ldots, n-1$, we obtain

$$|v_1 \wedge \ldots \wedge v_n|^2 = (1+A_n)^2 \left(1 + \sum_{i=1}^{n-1} B_i^2\right) + B_n^2 \left(1 + \sum_{i=1}^{n-1} A_i^2\right) - 2\,(1+A_n) B_n \sum_{i=1}^{n-1} A_i B_i\,.$$

Then (4.1.2) immediately follows from (4.1.1). □

4.2 Construction of the flow

We now construct one-parameter families of compact deformations of a fixed, n-dimensional hyperplane Σ restricted to a Lipschitz epigraph Ω. The deformations are defined on $\overline{\Omega}$ and are "tangential", in the sense that they send $\partial\Omega$ into itself.

Specifically, given a Lipschitz function $\omega = \omega(x', t) : \mathbb{R}^n \to \mathbb{R}$, we let

$$\Omega = \{\tilde{x} = (x', x_n, t) \in \mathbb{R}^{n+1} \,:\, x_n > \omega(x', t)\}\,. \tag{4.2.1}$$

We also let $\Sigma := \Omega \cap p_{n+1}$ and set $\partial\Sigma = \overline{\Sigma} \cap \partial\Omega$. Given $x = (x', x_n) \in \overline{\Sigma}$, we consider the parametric curve $\Gamma_x : \mathbb{R} \to \mathbb{R}^{n+1}$ defined as

$$\Gamma_x(t) := (x', x_n + \omega(x', t) - \omega(x', 0), t) \tag{4.2.2}$$

and, with a slight abuse of notation, we identify Γ_x with $\Gamma_x(\mathbb{R})$.

The next lemma collects some key properties of the family $\{\Gamma_x :\; x \in \overline{\Sigma}\}$.

Lemma 4.2.1. *The family of parametric curves $\{\Gamma_x :\; x \in \overline{\Sigma}\}$ defines a foliation of $\overline{\Omega}$. More precisely, the following properties hold:*

(i) $\Gamma_x(0) = x$ *and* $\Gamma_x \subset \overline{\Omega}$*, for all* $x \in \overline{\Sigma}$*;*

(ii) if $x \neq y \in \overline{\Sigma}$ *then* $\Gamma_x \cap \Gamma_y = \emptyset$*;*

(iii) $\Gamma_x(t) \in \partial\Omega$ *for some* $t \in \mathbb{R}$ *if and only if* $\Gamma_x \subset \partial\Omega$*.*

(iv) the map $(x, t) \mapsto \Gamma_x(t)$ *is Lipschitz.*

Proof. Property (i) directly follows from the definition of Γ_x and by observing that $x \in \overline{\Sigma}$ implies $x_n \geq \omega(x', 0)$, hence

$$x_n + \omega(x', t) - \omega(x', 0) \geq \omega(x', t)\,.$$

Then, concerning property (ii), if $\Gamma_x(t) = \Gamma_y(u)$ for some $x, y \in \overline{\Sigma}$ and some $t, u \in \mathbb{R}$, by definition we must have $t = u$, $x' = y'$, and

$$x_n + (\omega(x', t) - \omega(x', 0)) = y_n + (\omega(y', u) - \omega(y', 0))\,,$$

which implies $x_n = y_n$ and hence $x = y$, which proves (ii). The less obvious implication of (iii) is the only if part: if $\Gamma_x(t) \in \partial\Omega$ for some $t \in \mathbb{R}$, then

$$x_n + \omega(x', t)) - \omega(x', 0)) = \omega(x', t)\,,$$

i.e., $x_n = \omega(x', 0)$. Consequently,

$$\Gamma_x(u) = (x', \omega(x', 0) + \omega(x', u) - \omega(x', 0), u) = (x', \omega(x', u), u) \in \partial\Omega$$

for all $u \in \mathbb{R}$. Finally, the proof of (iv) easily follows from the estimate

$$\begin{aligned} |\Gamma_x(t) - \Gamma_y(u)| &\leq |x' - y'| + |x_n - y_n| + |\omega(x', t) - \omega(y', u)| + |\omega(x', 0) + \omega(y', 0)| + |t - u| \\ &\leq (1 + 2\mathrm{Lip}(\omega))(|x' - y'| + |x_n - y_n| + |t - u|). \end{aligned}$$

□

Let now $f : \overline{\Sigma} \to \mathbb{R}$ be a Lipschitz function with compact support in $\overline{\Sigma}$. We define the *flow associated to* f as

$$\Phi[f](x, t) := \Gamma_x(tf(x)) \tag{4.2.3}$$

for all $(x,t) \in \overline{\Sigma} \times \mathbb{R}$. The map $\Phi[f]$ satisfies $\Phi[f](x,t) = x$ and, thanks to Lemma 4.2.1, it is Lipschitz and one-to-one. When $\Phi[f](\cdot,t)$ is differentiable at x (which is true for almost all $x \in \Sigma$), we denote by

$$D\Phi[f](x,t) = (\partial_1 \Phi[f](x,t), \dots, \partial_n \Phi[f](x,t))$$

the Jacobian matrix of $\Phi[f](\cdot,t)$ at x, and by $D\Phi[f](x,t)^T$ its transpose. We also define

$$S[f](x,t) := D\Phi[f](x,t)^T \cdot D\Phi[f](x,t)$$

and

$$J[f](x,t) := \sqrt{\det(S[f](x,t))}\,,$$

and note for future reference that

$$J[f](x,0) = 1 \quad \forall\, x \in \Sigma\,. \tag{4.2.4}$$

Then, we define

$$\Sigma_t[f] = \Phi[f](\Sigma,t)$$

and

$$A[f](t) := \mathcal{H}^n(\Phi[f](\Sigma \cap \operatorname{spt} f, t)) = \int_{\operatorname{spt} f} J[f](x,t)\,dx\,,$$

where the second identity follows from the Area Formula (see, e.g., [3, Theorem 2.91]). The function $A[f](t)$ represents the area of the compact portion of the hyperplane Σ deformed via the flow map $\Phi[f](\cdot,t)$. The minimality/stability of Σ is related to the asymptotic properties of $A[f](t)$ when $t \to 0$. In general, we cannot expect that $A[f](t)$ is (twice) differentiable at $t = 0$, hence it will not be possible to compute the first and second variations of the area in a classical sense. However, we can test the local minimality of Σ by taking lower right variations, i.e.,

$$\underline{\partial_t} A[f](0^+) := \liminf_{t \to 0^+} \frac{A[f](t) - A[f](0)}{t}$$

and, assuming $\underline{\partial_t} A[f](0^+) = 0$,

$$\underline{\partial_t^2} A[f](0^+) := 2 \liminf_{t \to 0^+} \frac{A[f](t) - A[f](0)}{t^2} .$$

Therefore, a first-order necessary condition for local minimality is

$$\underline{\partial_t} A[f](0^+) \geq 0 , \qquad \forall \, f \in C_c^{0,1}(\overline{\Sigma}) , \tag{4.2.5}$$

with strict inequality only if the local minimality is strict. Then, a second-order necessary condition is

$$\underline{\partial_t} A[f](0^+) = 0 \implies \underline{\partial_t^2} A[f](0^+) \geq 0 , \tag{4.2.6}$$

for all $f \in C_c^{0,1}(\overline{\Sigma})$.

4.3 The case of the $(n+1)$-dimensional circular cone

From now on, Ω_λ will denote the $(n+1)$-dimensional circular cone defined as the epigraph of the function

$$\omega_\lambda(x', x_{n+1}) := \lambda \sqrt{|x'|^2 + x_{n+1}^2} ,$$

with $\lambda > 0$ a fixed parameter defining the aperture of the cone. In this case, for all $f \in C_c^{0,1}(\overline{\Sigma})$, the one-parameter flow associated with f is given by

$$\begin{aligned} \Phi[f](x,t) &:= (x' \, , \, \omega_\lambda(x', t f(x)) + x_n - \omega_\lambda(x', 0) \, , \, t \, f(x)) \\ &= \left(x' \, , \, \lambda \sqrt{|x'|^2 + t^2 f(x)^2} + x_n - \lambda |x'| \, , \, t \, f(x) \right) . \end{aligned}$$

It is immediate to check that $\Phi[f](x,t)$ is differentiable on the set

$$\Delta_f := \{ x \in \operatorname{spt} f \, : \, x \neq 0 \, , \, f \text{ is differentiable at } x \} \cup (\Sigma \setminus \operatorname{spt} f) \; .$$

In particular, since f is Lipschitz, we infer that

$$\mathcal{H}^n(\Sigma \setminus \Delta_f) = 0 \, .$$

For almost all $x \in \Delta_f$ and for $1 \le i \le n-1$, we have

$$\begin{aligned}\partial_i \Phi[f](x,t) &= e_i + [t\,\partial_i f(x)\,\partial_{n+1}\omega_\lambda(x', tf(x)) + \partial_i \omega_\lambda(x', tf(x)) - \partial_i\omega_\lambda(x',0)]\,e_n + t\,\partial_i f(x) e_{n+1} \\ &= e_i + \lambda\left(\frac{t^2 f(x)\,\partial_i f(x)}{\sqrt{|x'|^2 + t^2 f(x)^2}} + \frac{x_i}{\sqrt{|x'|^2+t^2 f(x)^2}} - \frac{x_i}{|x|}\right) e_n + t\,\partial_i f(x) e_{n+1}\end{aligned} \tag{4.3.1}$$

while, if $i = n$,

$$\begin{aligned}\partial_n \Phi[f](x,t) &= (t\,\partial_n f(x)\,\partial_{n+1}\omega_\lambda(x', tf(x)) + 1)\,e_n + t\,\partial_n f(x) e_{n+1} \\ &= \left(\lambda \frac{t^2 f(x)\,\partial_n f(x)}{\sqrt{|x'|^2+t^2 f(x)^2}} + 1\right) e_n + t\,\partial_n f(x) e_{n+1}\,.\end{aligned} \tag{4.3.2}$$

We introduce the following notations:

$$\alpha_i(x,t) := \begin{cases} \lambda\left(\dfrac{t^2 f(x)\,\partial_i f(x)}{\sqrt{|x'|^2+t^2 f(x)^2}} + \dfrac{x_i}{\sqrt{|x'|^2+t^2 f(x)^2}} - \dfrac{x_i}{|x|}\right) & \text{if } i = 1,...,n-1 \\ \lambda\, \dfrac{t^2 f(x) \partial_n f(x)}{\sqrt{|x'|^2+t^2 f(x)^2}} & \text{if } i = n\,, \end{cases} \tag{4.3.3}$$

$$\beta_i(x,t) := t\,\partial_i f(x)\,, \qquad \forall i = 1,...,n\,. \tag{4.3.4}$$

We thus have

$$\partial_i \Phi[f](x,t) = e_i + \alpha_i(x,t)\,e_n + \beta_i(x,t)\,e_{n+1}\,, \qquad \forall i = 1,...,n\,. \tag{4.3.5}$$

Lemma 4.3.1. *Given $f \in \mathcal{C}^{0,1}_c(\overline{\Sigma})$, for all $t \in \mathbb{R}$, we have*

$$J[f](x,t)^2 = 1 + t^2\left(|\nabla f(x)|^2 + \frac{2\,\lambda\, f(x)\,\partial_n f(x)}{\sqrt{|x'|^2+t^2 f(x)^2}}\right) + R[f](x,t)\,, \tag{4.3.6}$$

where $R[f](x,t)$ is such that

$$\operatorname{spt} R[f](t,\cdot) \subset \operatorname{spt} f\,, \qquad \lim_{t\to 0}\frac{R[f](x,t)}{t^2} = 0\,,\ \textit{a.e. in } \Sigma\,, \qquad \left|\frac{R[f](x,t)}{t^2}\right| \le C\,, \tag{4.3.7}$$

where $C \geq 0$ *is a constant depending on* Lip$f$ *only.*

Proof. The partial derivatives of $\Phi[f]$ satisfy the assumptions of Lemma 4.1.1, hence

$$J[f]^2 = (1+\alpha_n)^2 \left(1 + \sum_{i=1}^{n-1} \beta_i^2\right) + \beta_n^2 \left(1 + \sum_{i=1}^{n-1} \alpha_i^2\right) - 2\,(1+\alpha_n)\beta_n \sum_{i=1}^{n-1} \alpha_i \beta_i \,.$$

where α_i, β_i are defined in (4.3.3) and (4.3.4). We now observe that, for every $1 \leq i \leq n-1$,

$$\lim_{t \to 0^+} \alpha_i(t) = 0 \qquad \text{for } x \neq 0. \tag{4.3.8}$$

In addition,

$$\lim_{t \to 0} \frac{\alpha_n[f](t)}{t} = 0 \quad \text{for } x \neq 0, \qquad \lim_{t \to 0} \beta_i[f](t) = 0 \quad \text{for all } x, \tag{4.3.9}$$

and

$$\frac{\|\alpha_n[f](t)\|_{L^\infty(\Sigma)}}{t} \leq \lambda \|\partial_n f\|_{L^\infty(\Sigma)}\,, \qquad \frac{\|\beta_i[f](t)\|_{L^\infty(\Sigma)}}{t} \leq \|\partial_i f\|_{L^\infty(\Sigma)}\,. \tag{4.3.10}$$

Let us define

$$R[f] := \alpha_n^2 \left(1 + \sum_{i=1}^{n-1} \beta_i^2\right) + 2\alpha_n \sum_{i=1}^{n-1} \beta_i^2 + \beta_n^2 \sum_{i=1}^{n-1} \alpha_i^2 - 2\,(1+\alpha_n)\beta_n \sum_{i=1}^{n-1} \alpha_i \beta_i \,.$$

By the definition of α_i, β_i provided in (4.3.3), it is evident that α_i and β_i vanish as f vanishes, and this implies that $\operatorname{spt} R[f](t,\cdot) \subset \operatorname{spt} f$. Owing to (4.3.8), (4.3.9), we infer that

$$\lim_{t \to 0} \frac{R[f](x,t)}{t^2} = 0\,, \text{ a.e. in } \Sigma\,,$$

while (4.3.10) yields

$$\left| \frac{R[f](x,t)}{t^2} \right| \leq C\,, \qquad C \text{ constant depending on Lip} f.$$

This proves the validity of (4.3.7). On the other hand

$$J[f]^2 - R[f] \;=\; 1 + 2\alpha_n + \sum_{i=1}^{n} \beta_i^2 \;=\; 1 + t^2 \left(|\nabla f(x)|^2 + \frac{2\,\lambda\, f(x)\,\partial_n f(x)}{\sqrt{|x'|^2 + t^2 f(x)^2}} \right),$$

that is precisely what we claimed. □

We observe that (4.3.6) explicitly depends on t^2. This implies the vanishing of the first (lower) variation of the area, $\underline{\partial_t} A[f](0^+) = 0$, and the fact that the second variation can be computed as the first variation of the area with respect to the parameter $s = t^2$. To this end, given $s > 0$ we conveniently set $\mathcal{A}[f](s) = A[f](\sqrt{s})$ and $\mathcal{J}[f](x,s) = J[f](x,\sqrt{s})$. The following theorem holds.

Theorem 4.3.2. *The following identity holds:*

$$\underline{\partial_s}\mathcal{A}[f](0^+) \;=\; \frac{1}{2}\left(\int_\Sigma |\nabla f(x)|^2\, dx - \lambda \int_{\mathbb{R}^{n-1}} \frac{f(x',\lambda|x'|)^2}{|x'|}\, dx' \right), \qquad \forall f \in C_c^{0,1}(\overline{\Sigma})\,. \tag{4.3.11}$$

Proof. Owing to (4.2.4), we have $\mathcal{J}[f](0) \equiv 1$, hence

$$\begin{aligned} \underline{\partial_s}\mathcal{A}[f](0^+) &= \liminf_{s\to 0^+} \int_{\operatorname{spt} f} s^{-1}\Big(\mathcal{J}[f](x,s) - \mathcal{J}[f](x,0)\Big) dx \\ &= \liminf_{s\to 0^+} \int_{\operatorname{spt} f} s^{-1}\Big(\mathcal{J}[f](x,s) - 1\Big)\, dx\,. \end{aligned} \tag{4.3.12}$$

Now, (4.3.6) guarantees that

$$\mathcal{J}[f](x,s)^2 = 1 + s\,G\,, \qquad \text{where} \quad G := \left(|\nabla f(x)|^2 + \frac{2\,\lambda\, f(x)\,\partial_n f(x)}{\sqrt{|x'|^2 + s f(x)^2}} + \frac{R[f](x,\sqrt{s})}{s} \right).$$

Thus by (4.3.12) we infer

$$\int_{\operatorname{spt} f} s^{-1}(\mathcal{J}[f](x,s) - 1)\, dx = \mathcal{I}_s^1 + \mathcal{I}_s^2\,,$$

where

$$\mathcal{I}_s^1 := \frac{1}{s}\int_{\operatorname{spt} f} \sqrt{1+sG} - \left(1 + s\,\frac{G}{2}\right) dx\,, \qquad \mathcal{I}_s^2 := \int_{\operatorname{spt} f} \frac{G}{2}\, dx\,.$$

An elementary computation yields

$$\sqrt{1+sG} - \left(1 + s\,\frac{G}{2}\right) = \frac{-s^2G^2/4}{\sqrt{1+sG} + \left(1+s\dfrac{G}{2}\right)}\,,$$

consequently

$$|\mathcal{I}_s^1| \leq \frac{1}{4}\int_{\operatorname{spt} f} \frac{sG^2}{\left|\sqrt{1+sG} + \left(1+s\dfrac{G}{2}\right)\right|} dx\,.$$

Now it is immediate to observe that $G = 0$ whenever $x \notin \operatorname{spt} f$, and, as $s \to 0^+$,

$$\sqrt{s}\,G \longrightarrow 0 \quad \text{a.e. in } \Sigma, \qquad |\sqrt{s}\,G| \leq C \quad \text{for some } C > 0 \text{ depending on } \operatorname{Lip} f \text{ only.}$$

By Dominated Convergence, we get

$$\lim_{s\to 0^+} \mathcal{I}_s^1 = 0\,.$$

Let us now study the limit as $s \to 0^+$ of

$$\mathcal{I}_s^2 = \int_\Sigma \left(\frac{1}{2}\,|\nabla f(x)|^2 + \frac{\lambda\, f(x)\,\partial_n f(x)}{\sqrt{|x'|^2 + s\,f(x)^2}} + \frac{1}{2}\,\frac{R[f](x,\sqrt{s})}{s}\right) dx$$

By (4.3.7) and Dominated Convergence, we have

$$\int_\Sigma \frac{R[f](x,\sqrt{s})}{s}\,dx \longrightarrow 0\,, \qquad \text{as } s \to 0^+\,.$$

On the other hand, denoting by $g(x') := f(x', \lambda|x'|)^2$, an integration by parts yields

$$\begin{aligned}\int_\Sigma \frac{\lambda\, f(x)\, \partial_n f(x)}{\sqrt{|x'|^2 + sf(x)^2}}\, dx &= \frac{\lambda}{s} \int_\Sigma \frac{s\, f(s)\, \partial_n f(x)}{\sqrt{|x'|^2 + sf(x)^2}}\, dx \\ &= \frac{\lambda}{s} \int_\Sigma \partial_n \sqrt{|x'|^2 + sf(x)^2}\, dx \\ &= \frac{\lambda}{s} \int_{\mathbb{R}^{n-1}} \left(\int_{\lambda|x'|}^{+\infty} \partial_n \sqrt{|x'|^2 + sf(x)^2} dx_n \right) dx' \\ &= \frac{\lambda}{s} \int_{\mathbb{R}^{n-1}} |x'| - \sqrt{|x'|^2 + sg(x')}\, dx' \\ &= -\lambda \int_{\mathbb{R}^{n-1}} \frac{g(x')}{|x'| + \sqrt{|x'|^2 + sg(x')}}\, dx'\end{aligned}$$

Now, the integrand

$$\frac{g(x')}{|x'| + \sqrt{|x'|^2 + sg(x')}}$$

is positive and monotonically non-increasing in s, for any $x' \neq 0$. Then, by Beppo Levi's Theorem, we obtain

$$\lim_{s \to 0^+} \int_{\mathbb{R}^{n-1}} \frac{g(x')}{|x'| + \sqrt{|x'|^2 + sg(x')}}\, dx' = \frac{1}{2} \int_{\mathbb{R}^{n-1}} \frac{g(x')}{|x'|}\, dx' \,.$$

This implies that

$$\lim_{s \to 0^+} \mathcal{I}_s^2 = \frac{1}{2} \int_\Sigma |\nabla f(x)|^2\, dx - \frac{\lambda}{2} \int_{\mathbb{R}^{n-1}} \frac{g(x')}{|x'|}\, dx' \,,$$

and the proof is concluded. □

Theorem 4.3.2 allows us to test in which cases Σ is stable. Indeed, by (4.3.11), the first variation of the area vanishes, hence the stability condition can be stated in terms of the lower-right second variation:

$$\underline{\partial_t^2} A[f](0^+) = 2\underline{\partial_s}\mathcal{A}[f](0^+) \geq 0 \,,$$

which is equivalent to the functional inequality

$$\int_\Sigma |\nabla f(x)|^2 \, dx \geq \lambda \int_{\mathbb{R}^{n-1}} \frac{f(x', \lambda|x'|)^2}{|x'|} \, dx' \,, \tag{4.3.13}$$

or equivalently to

$$\int_\Sigma |\nabla f(x)|^2 dx \geq \lambda \int_{\partial\Sigma} \frac{f(y)^2}{|y|} \, d\mathcal{H}^{n-1}(y) \,.$$

4.3.1 The case $\dim \Sigma = 2$

In this case, we see that Σ is always unstable, and in particular it is not area minimizing into Ω_λ. This result will be generalized in the next chapter with the Vertex-skipping Theorem 5.3.1.

Theorem 4.3.3. *Let $n = 2$. Then, for all $f \in C_c^{0,1}(\overline{\Sigma})$ with the property that $f(0) \neq 0$, we have*

$$\underline{\partial_s} \mathcal{A}[f](0^+) < 0 \,.$$

In particular, this shows that Σ is unstable.

Proof. When $n = 2$ the inequality (4.3.13) becomes

$$\int_\Sigma |\nabla f(x)|^2 \, dx \geq \lambda \int_{\mathbb{R}} \frac{f(x_1, \lambda|x_1|)^2}{|x_1|} \, dx_1 \,. \tag{4.3.14}$$

We now observe that, for all $f \in C_c^{0,1}(\overline{\Sigma})$, the left-hand side of (4.3.14) is always finite while the right-hand side is finite only if $f(0) = 0$. Thus (4.3.14) fails for all $f \in C_c^{0,1}(\overline{\Sigma})$ such that $f(0) \neq 0$. □

We can then prove the following

Theorem 4.3.4. *Let $n = 2$, and let us define*

$$E := \Omega \cap \{x \in \mathbb{R}^3 \,:\, x_1 < 0\} \,.$$

Then E is not a minimizer of the relative perimeter in Ω.

Proof. It suffices to demonstrate that, for all $r > 0$, there exists a measurable set $F_r \subset \Omega$ such that $F_r \Delta E \subset\subset B_r$, and

$$P_\Omega(F_r; B_r) < P_\Omega(E; B_r)\,. \tag{4.3.15}$$

Up to blowing up, it suffices to prove (4.3.15) for $r = 1$. Let $f \in C^{0,1}_c(\overline{\Sigma})$ be such that

$$f \geq 0\,, \qquad \operatorname{spt} f \subset\subset \overline{\Sigma} \cap B_1\,, \qquad f(0) > 0\,. \tag{4.3.16}$$

By Corollary 4.3.3, we know that

$$\underline{\partial_s} \mathcal{A}[f](0^+) < 0\,,$$

and thus we can select $\sigma > 0$ small enough to ensure

$$\frac{\mathcal{A}[f](\sigma) - \mathcal{A}[f](0)}{\sigma} \leq \max\left\{-1\,,\, \frac{1}{2}\underline{\partial_s}\mathcal{A}[f](0^+)\right\} < 0\,. \tag{4.3.17}$$

Since $\Phi[f](\cdot, 0)$ is the identity on $\overline{\Sigma}$, and $\operatorname{spt} f$ is compact in $\overline{\Sigma}$, up to possibly reducing σ, we can also suppose that

$$\Phi[f](x, \sqrt{s}) \in B_1\,, \qquad \text{for all } x \in \operatorname{spt} f\,.$$

Let us define

$$F := \{\Gamma_x(t) \,:\, x \in \overline{\Sigma}\,,\, t < \sqrt{\sigma} f(x)\}\,.$$

By construction, $F \Delta E \subset\subset B_1$. Owing to (iv) of Lemma 4.2.1 and (4.3.17), we can deduce that

$$P_\Omega(F; B_1) - P_\Omega(E; B_1) = \mathcal{A}[f](\sigma) - \mathcal{A}[f](0) < 0\,.$$

This concludes the proof. $\square$

4.3.2 The case $\dim\Sigma \geq 3$

In this case, the aperture of the cone Ω_λ plays a role. Indeed, we can prove that there exists a threshold aperture parameter $\lambda^* > 0$, such that for larger apertures (i.e., for $0 < \lambda \leq \lambda^*$), the hyperplane Σ becomes strictly stable.

Theorem 4.3.5. *Let $n \geq 3$. Then there exists $\lambda^* = \lambda^*(n) > 0$ such that, if $0 < \lambda \leq \lambda^*$, we have*

$$\underline{\partial_s}\mathcal{A}[f](0^+) \geq 0 \qquad \text{for all } f \in C_c^{0,1}(\overline{\Sigma}) \tag{4.3.18}$$

and the inequality is strict whenever f is not identically zero, which means that Σ is strictly stable.

To prove Theorem 4.3.5, we need the following result.

Lemma 4.3.6. *Let $n \geq 3$. Then*

$$\int_\Sigma |\nabla f(x)|^2\,dx \geq \frac{K_n}{(1+\lambda)^2}\int_{\mathbb{R}^{n-1}} \frac{f(x',\lambda|x'|)^2}{|x'|}\,dx', \qquad \text{for all } f \in H^1(\Sigma), \tag{4.3.19}$$

where $K_n = 2\Gamma^2(n/4)\Gamma^{-2}((n-2)/4)$ and $\Gamma(u)$ is Euler's Gamma function.

Inequality (4.3.19) is known as *Kato's Inequality.* A proof of a stronger version of (4.3.19), valid when Σ is a half-space (i.e. for $\lambda = 0$), is provided in [6, Theorem 1.4]. Although our domain Σ is not a half-space, our version directly follows from inequality (9) in the same work. We also mention that a former proof of Kato's Inequality was given by Herbst in [15].

Proof of Lemma 4.3.6. Let us denote by $\mathbb{R}^n_+$ the half-space of points $x \in \mathbb{R}^n$ with $x_n > 0$. Inequality (9) of [6] states that

$$\int_{\mathbb{R}^n_+} |\nabla g(x)|^2\,dx \geq K_n \int_{\mathbb{R}^{n-1}} \frac{g(x',0)^2}{|x'|}\,dx', \qquad \text{for all } g \in H^1(\mathbb{R}^n_+). \tag{4.3.20}$$

Given $f \in H^1(\Sigma)$, let us set

$$T(x) := (x', x_n + \lambda|x'|), \qquad g(x) := f(T(x))\,.$$

We have $g \in H^1(\mathbb{R}^n_+)$. Since $g(x', 0) = f(x', \lambda|x'|)$, we get

$$\int_{\mathbb{R}^{n-1}} \frac{g(x',0)^2}{|x'|}\, dx' = \int_{\mathbb{R}^{n-1}} \frac{f(x', \lambda|x'|)^2}{|x'|}\, dx' \,. \tag{4.3.21}$$

We observe that

$$\nabla g(x) = \nabla f(T(x)) + \lambda \partial_n f(T(x)) \left(\frac{x'}{|x'|}, 0 \right) \,,$$

and thus

$$|\nabla g(x)|^2 \le (1+\lambda)^2 |\nabla f(T(x))|^2 \,. \tag{4.3.22}$$

Combining (4.3.20), (4.3.21) and (4.3.22), and noting that $\det DT \equiv 1$, by the change of variable formula we get

$$\begin{aligned}(1+\lambda)^2 \int_\Sigma |\nabla f(x)|^2 dx &= (1+\lambda)^2 \int_{\mathbb{R}^n_+} |\nabla f(T(x))|^2 dx \\ &\ge \int_{\mathbb{R}^n_+} |\nabla g(x)|^2 dx \\ &\ge K_n \int_{\mathbb{R}^{n-1}} \frac{f(x', \lambda|x'|)^2}{|x'|}\, dx'\end{aligned}$$

and conclude the proof. □

Proof of Theorem 4.3.5. By Lemma 4.3.6, the validity of (4.3.13), for all $f \in C^{0,1}_c(\overline{\Sigma})$, is guaranteed by the condition

$$0 < \lambda \le \frac{K_n}{(1+\lambda)^2} \,. \tag{4.3.23}$$

Since the function $\lambda \mapsto \lambda(1+\lambda)^2$ is monotonically increasing from 0 to $+\infty$, there exists a unique $\lambda^* > 0$ such that

$$\lambda^*(1+\lambda^*)^2 = K_n$$

and, consequently, (4.3.23) is satisfied if and only if $0 < \lambda \le \lambda^*$, as wanted. □

Remark 4.3.7. *Theorem 4.3.5 leaves a lot of questions open. Indeed, we do not know whether the threshold $\alpha^* > 0$ is optimal, and it is not clear what happens if α is small. Moreover, Theorem 4.3.5 provides just a stability result, but it remains to understand whether Σ is really area-minimizing in Ω when $\alpha \ge \alpha^*$. We expect that the quantitative expansion of $J[f](t)$ provided in (4.3.6) could be employed to show that Σ is also mini-*

mizing. Finally, we did not treat the case of $n > 3$*, i.e. the case in which the ambient dimension is bigger than* 4*. The motivation of this is related to the difficulties arising in computing the determinant* $J[f](t)$*. On the other hand, we notice that Kato's Inequality holds also as* $n > 3$*, hence if we knew that* $J[f](t)$ *expands as in* (4.3.6) *in some dimension* $n > 3$*, we would deduce that* Σ *is stable in* Ω *for* $\alpha \geq \alpha^*$*, where* α^* *is expected to depend on* n*.*

Chapter 5

The Vertex-skipping Theorem

In this Chapter, we expose the argument for the proof of Theorem 5.3.1, as developed in [25]. The argument of the proof is by contradiction, and requires a characterization of the blow-up limit of a local almost-minimizer of the relative perimeter. This characterization will turn out to be a consequence of the boundary Monotonicity Formula. In what follows, Ω will always denote an open, convex subset of $\mathbb{R}^n$.

5.1 Preliminary results

5.1.1 Tangent cone and vertices

Definition 5.1.1. *Let $x_0 \in \overline{\Omega}$. We define the* tangent cone *to Ω at x_0 as*

$$\Omega_{x_0} := \lim_{t \to +\infty} t(\Omega - x_0) = \bigcup_{t>0} t(\Omega - x_0)\,.$$

We observe immediately that Ω_{x_0} is a cone with vertex at 0. Indeed, if $y \in \Omega_{x_0}$, by definition, there exists $t > 0$ such that

$$y \in t\,(\Omega - x_0)\,.$$

Then, for all $s > 0$, $sv \in st\,(\Omega - x_0) \subset \Omega_{x_0}$.
We also provide the definition of vertex for an Euclidean convex set.

Definition 5.1.2. *Let $x_0 \in \partial\Omega$. We say that x_0 is a* vertex *for Ω if Ω_{x_0} does not contain lines.*

5.1.2 Minimality in the tangent cone

We start showing that a sequence of dilations of an almost-minimizer E in Ω converges, up to subsequences, to a minimizer of the relative perimeter in the tangent cone Ω_0. This fact is an immediate consequence of Lemma 2.1.7.

Lemma 5.1.3. *Let E be an almost-minimizer in Ω, and assume $0 \in \partial\Omega$ and $P_\Omega(E; B_r) > 0$ for all $r > 0$. Let $t_j \searrow 0$ be a sequence and let*

$$\Omega_{t_j} := \frac{1}{t_j}\,\Omega\,, \qquad E_{t_j} := \frac{1}{t_j}\,E\,.$$

Then there exist a subsequence s_{j_k} and a measurable set $E_0 \subset \Omega_0$ such that $E_{s_{j_k}} \to E_0$ in L^1_{loc} and E_0 is a perimeter-minimizer in Ω_0, namely

$$\Psi_{\Omega_0}(E_0; B_R) = 0 \qquad \text{for any } R > 0.$$

Proof (of Lemma 5.1.3). First, we fix $R > 0$ and prove that there exist $t_0, C > 0$ such that

$$P(E_t; B_R) \le C R^{n-1} \qquad \forall\, 0 < t < t_0\,. \tag{5.1.1}$$

In what follows, for more simplicity, we will write C to denote a constant that might change from one line to another. To prove (5.1.1), we note that

$$P(\Omega_t; B_R) = t^{1-n} P(\Omega; B_{Rt}) \le C R^{n-1} \tag{5.1.2}$$

since $\partial\Omega$ is Lipschitz. Then owing to (2.1.7) and (5.1.2), and assuming $t < t_0 := \min(1, \bar{r}/R)$, we obtain

$$P(E_t; B_R) \le P_{\Omega_t}(E_t; B_R) + P(\Omega_t; B_R) = t^{1-n}\Big(P_\Omega(E; B_{tR}) + P(\Omega; B_{Rt})\Big) \le C\, R^{n-1}\,,$$

which proves (5.1.1). Consequently, we obtain the global perimeter bound

$$P(E_t) \leq P(E_t; B_R) + P(\Omega_t; B_R) \leq CR^{n-1},$$

hence any blow-up sequence E_{t_j} admits a (not relabeled) subsequence converging in $L^1(B_R)$ to a limit set E_0. By a standard diagonal argument, one can prove the existence of a subsequence and a limit set, still denoted respectively as E_{t_j} and E_0, such that $E_{t_j} \to E_0$ in $L^1_{loc}(\mathbb{R}^n)$. Since the corresponding sequence of rescaled domains Ω_{t_j} converges to the tangent cone Ω_0 in $L^1_{loc}(\mathbb{R}^n)$, we infer that $E_0 \subset \Omega_0$.

Now we have to show that $\Psi_{\Omega_0}(E_0; B_R) = 0$, for all $R > 0$. To do so, we can apply Lemma 2.1.7. Indeed, well-known properties of convex sets ensure that the sequence of rescaled domains Ω_{t_j} converges to the tangent cone Ω_0 locally in Hausdorff distance. Moreover, by the rescaling properties of the perimeter, since E is an almost-minimizer of the relative perimeter in Ω, we infer that E_{t_j} satisfies (2.1.1) for all $0 < r < r_0$ and $x = 0$ with the function

$$\psi_{\Omega_{t_j}}(E_{t_j}; 0, r) = \psi_\Omega(E; 0, t_j r), \qquad 0 < r < r_0/t_j.$$

Now $t_j \searrow 0$, hence there exists $M > 0$ such that $|t_j| \leq M$, for all $j \in \mathbb{N}$. Moreover, since $\psi_\Omega(E; 0, r)$ is infinitesimal as $r \to 0^+$, we infer that, for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that $0 \leq \psi_\Omega(E; 0, r) \leq \varepsilon$, for all $0 < r < r_\varepsilon$. If we take $r < r_\varepsilon/M$, we then have

$$\sup_j \psi_{\Omega_{t_j}}(E_{t_j}; 0, r) = \sup_j \psi_\Omega(E; 0, t_j r) \leq \varepsilon,$$

and this sufficees to show that

$$\lim_{r \to 0^+} \sup_j \psi_{\Omega_{t_j}}(E_{t_j}; 0, r) = 0.$$

We are then in position to apply Lemma 2.1.7 to deduce that

$$\liminf_j \Psi_{\Omega_{t_j}}(E_{t_j}; B_R) \geq \Psi_{\Omega_0}(E_0; B_R), \qquad \text{for all } R > 0. \tag{5.1.3}$$

Finally, by (2.1.6) we obtain

$$\Psi_{\Omega_{t_j}}(E_{t_j};B_R)=\frac{1}{t_j^{n-1}}\Psi_\Omega(E;B_{t_jR})\le\omega_n^{1-\frac{1}{n}}\,R^{n-1}\psi_\Omega(E;0,t_jR)\longrightarrow 0\,,$$

which concludes the proof. □

5.1.3 Flatness of the blow-up of an almost-minimizer

Thanks to Lemma 5.1.3, we know in particular that for a suitable choice of a sequence $t_j \searrow 0$ the rescalings E_{t_j} with respect to $0 \in \partial\Omega$ converge to a minimizer E_0 in the tangent cone Ω_0. Exploiting Theorem 3.4.1, we can prove the following result.

Proposition 5.1.4. *There exist a sequence $s_k \searrow 0$ and a subset $E_{00} \subset \Omega_0$ such that*

$$(E_0)_{s_k}\longrightarrow E_{00}\,,\qquad \text{in } L^1_{\mathrm{loc}}\,,$$

and E_{00} is a minimal cone in Ω_0, i.e. is a cone with vertex at 0 with the property that

$$\Psi_{\Omega_0}(E_{00};B_R)=0\qquad \text{for any } R>0. \tag{5.1.4}$$

Proof. We note that Ω_0 is a Lipschitz cone with vertex at 0, thus Ω_0 satisfies the visibility condition at 0 with the trivial choice $u(t)\equiv 0$ (see Example 3.2.10). Moreover, E_0 is a minimizer of the relative perimeter in Ω, i.e. $\Psi_{\Omega_0}(E_0;B_R)=0$, for all $R>0$, and this trivially implies (3.4.1). We can then apply Theorem 3.4.1 in this special setting deducing that, choosing a suitable sequence $s_k \searrow 0$, there exists $E_{00}\subset\Omega_0$ such that

$$(E_0)_{s_k}\longrightarrow E_0\,,\qquad \text{in } L^1_{\mathrm{loc}}\,,$$

and E_{00} is a minimal cone in Ω_0, i.e. is a cone with vertex at 0 such that (5.1.4) holds true, and this concludes the proof. □

Remark 5.1.5. *We observe that, if Ω satisfies the assumptions of Theorem 3.4.1, then the same result allows to prove that E_0 itself is a cone with vertex at 0. In this case, the argument performed in the proof of Proposition 5.1.4 becomes unuseful, since $E_0 = E_{00}$. In other words, a unique blow-up suffices.*

5.1.4 A Federer's Reduction Lemma

Next, we state a slight variant of the classical Federer's Reduction Lemma, which will be used in the proof of Theorem 5.2.1. In what follows, by "cone" we shall always mean a cone with respect to the origin.

Lemma 5.1.6. *Let $K \subset \mathbb{R}^n$ be a convex cone, $C \subset K$ be a minimizing cone for the relative perimeter, both with vertex at 0. Let $x_0 \in \partial C \cap \partial K \setminus \{0\}$. For $t > 0$, we set*

$$C_t := x_0 + \frac{C - x_0}{t}\,, \quad K_t := x_0 + \frac{K - x_0}{t}\,.$$

Then there exist a sequence $t_j \searrow 0$ and two sets C_{x_0}, K_{x_0} such that

$$C_j := C_{t_j} \longrightarrow C_{x_0}\,, \quad K_j := K_{t_j} \longrightarrow K_{x_0} \tag{5.1.5}$$

in $L^1_{\rm loc}$-topology, $C_{x_0} \subset K_{x_0}$ and C_{x_0}, K_{x_0} are cylinders with axis coinciding with the line joining 0 to x_0. Moreover, the sets $C'_{x_0} := C_{x_0} \cap x_0^\perp$ and $K'_{x_0} := K_{x_0} \cap x_0^\perp$ are cones with respect to 0 in the hyperplane $x_0^\perp$, and C'_{x_0} is perimeter-minimizing in K'_{x_0}.

To prove this result we need the following two technical Lemmas. The proof of Lemma 5.1.7 is part of the proof of Lemma 2.4 in [13], while the proof of Lemma 5.1.8 is a slight variation of that of Lemma 9.8 in [13].

Lemma 5.1.7. *Let f have locally bounded variation in $\mathbb{R}^n$. For $t \in \mathbb{R}$, let*

$$f_t : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$$

be the function defined by $f_t(y) := f(y,t)$. Then, for almost all $s < t$, $r > 0$, one has

$$\int_{B_r^{n-1}} |f_t - f_s| d\mathcal{H}^{n-1} \leq |D_n f|(B_r^{n-1} \times (s,t))\,, \tag{5.1.6}$$

where $B_r^{n-1} := B_r \cap \{x_n = 0\}$.

Lemma 5.1.8. *Let f have locally-bounded variation in $\mathbb{R}^n$, and Ω be an open, bounded set in $\mathbb{R}^{n-1}$. Then, for any $T > 0$,*

$$|Df|(\Omega \times (-T,T)) \geq \int_{-T}^{T} |Df_t|(\Omega) dt\,,$$

with equality holding if f is independent of the variable x_n.

Proof of Lemma 5.1.6. Up to a rotation, we can suppose that $x_0 = (0, ..., 0, a)$, for some $a > 0$. We can argue as in the proof of Lemma 5.1.3 to deduce that there are a sequence $t_j \searrow 0$ and sets C_{x_0}, K_{x_0} such that (5.1.5) hold. We can also assume that $D\mathbf{1}_{C_{t_j}} \rightharpoonup^* D\mathbf{1}_{C_{x_0}}$. Furthermore, K_{x_0} is the tangent cone to K at x_0 and C_{x_0} is perimeter-minimizing in K_{x_0}. Let us show that C_{x_0} is a cylinder with axis coinciding with the x_n-axis. Let us denote by ν_C the inner, unit normal to $\partial^* C$, and let us set $\nu_C^n := \langle \nu_C, e_n \rangle$. As C is a cone with respect to 0, we have

$$\langle x, \nu_C(x) \rangle = 0 \,,$$

for $\mathcal{H}^{n-1}$-almost all $x \in \partial^* C$. Thus

$$-\langle x - x_0, \nu_C(x) \rangle = \langle x_0, \nu_C(x) \rangle = a\, \nu_C^n(x) \,,$$

and consequently we obtain ($a = |x_0|$)

$$|\nu_C^n(x)| \leq \frac{|x - x_0|}{|x_0|} \,.$$

So, for some constant $c > 0$, for all $\rho > 0$, we have

$$\begin{aligned} \int_{B_\rho(x_0) \cap \partial^* C_t} |\nu_{C_t}^n(x)| d\mathcal{H}^{n-1}(x) &= t^{1-n} \int_{B_{\rho t}(x_0) \cap \partial^* C_{x_0}} |\nu_C^n(x)| d\mathcal{H}^{n-1}(x) \\ &\leq \frac{t^{2-n}\rho}{|x_0|} P(C, B_{\rho t}(x_0)) \\ &\leq c \frac{\rho^n t}{|x_0|} \,. \end{aligned} \tag{5.1.7}$$

Now by Reshetnyak's lower semicontinuity Theorem (see Theorem 20.11 in [26], in particular we apply it for $\Phi(u) = |\langle u, e_n \rangle|$) we have

$$\int_{B_\rho(x_0) \cap \partial^* C_{x_0}} |\nu_{C_{x_0}}^n(x)| d\mathcal{H}^{n-1}(x) \leq \liminf_{j \to \infty} \int_{B_\rho(x_0) \cap \partial^* C_{t_j}} |\nu_{C_{t_j}}^n(x)| d\mathcal{H}^{n-1}(x) = 0 \,,$$

and consequently

$$\nu_{C_{x_0}}^n = 0 \,, \qquad \mathcal{H}^{n-1}\text{-a.e. on } \partial^* C_{x_0} \,.$$

Owing to Lemma 5.1.7, we deduce that $\mathbf{1}_{C_{x_0}}$ is independent of x_n, hence there exists $C'_{x_0} \subset \mathbb{R}^{n-1}$ such that

$$C_{x_0} = C'_{x_0} \times \mathbb{R}.$$

From the conicity of C_{x_0} with respect to 0, it follows that C'_{x_0} is a cone with respect to 0 in $\mathbb{R}^{n-1}$. The same argument developed up to now can be performed for K_{x_0}, deducing that there is a cone K'_{x_0} with respect to the origin in $\mathbb{R}^{n-1}$ with the property that

$$K_{x_0} = K'_{x_0} \times \mathbb{R}\,.$$

To conclude, it remains to show the minimality of C'_{x_0} in K'_{x_0}. By contradiction, let C'_{x_0} be not perimeter-minimizing in K'_{x_0}. Then we can pick a competitor E'_0 such that, for some $R, \epsilon > 0$, we have

$$E'_0 \Delta C'_{x_0} \subset\subset K'_{x_0} \cap B^{n-1}_R\,, \quad P_{K'_{x_0}}(E'_0, B^{n-1}_R) \leq P_{K'_{x_0}}(C'_{x_0}, B^{n-1}_R) - \epsilon\,.$$

Given $T > 0$, let

$$E_0 := E'_0 \times (-T, T) \cup (C_{x_0} \cap \{x_n \geq T\})\,.$$

By construction, $E_0 = C_{x_0}$ in $\mathbb{R}^n \setminus \Big(B^{n-1}_R \times (-T,T)\Big)$. In addition, by minimality of C_{x_0} in K_{x_0}, we observe that

$$P_{K_{x_0}}(C_{x_0}, B^{n-1}_R \times [-T,T]) \leq P_{K_{x_0}}(E_0, B^{n-1}_R \times [-T,T])\,. \tag{5.1.8}$$

On the other hand, we have

$$P_{K_{x_0}}(C_{x_0}, B^{n-1}_R \times [-T,T]) = 2T\, P_{K'_{x_0}}(C'_{x_0}, B^{n-1}_R)\,,$$

and also

$$\begin{aligned} P_{K_{x_0}}(E_0, B^{n-1}_R \times [-T,T]) &\leq 2T\, P_{K'_{x_0}}(E'_0, B^{n-1}_R) + 2\omega_{n-1}R^{n-1} \\ &\leq 2T\, P_{K'_{x_0}}(C'_{x_0}, B^{n-1}_R) - 2T\epsilon + 2\omega_{n-1}R^{n-1} \\ &= P_{K_{x_0}}(C_{x_0}, B^{n-1}_R \times [-T,T]) - 2T\epsilon + 2\omega_{n-1}R^{n-1}\,, \end{aligned}$$

and this estimate contradicts (5.1.8) for T large enough. So C'_{x_0} is a minimal cone in K'_{x_0}

and the proof is concluded. □

5.2 Characterization of the conical minimizer in $\mathbb{R}^3$.

Starting from an almost-minimizer E in Ω which satisfies volume density estimates at the origin, and applying Lemma 5.1.3 at most twice, we have obtained a conical minimizer E_{00} of the relative perimeter in the tangent cone Ω_0. The next theorem shows that, in dimension $n = 3$, $\partial E_{00} \cap \Omega_0$ coincides with a convex angle contained in a 2-plane through the origin, that meets $\partial\Omega_0$ orthogonally.

Theorem 5.2.1. *Let $n = 3$ and E_{00} be the conical minimizer obtained in the previous subsection. Then $\partial E_{00} \cap \Omega_0$ coincides with a 2-plane intersected with Ω_0, that meets $\partial\Omega_0$ orthogonally.*

Proof. Set $F = E_{00}$ for brevity, then the proof is accomplished by showing that there exists exactly one geodesic arc $\gamma \subset \partial B_1 \cap \Omega_0$ such that

$$\partial F \cap \partial B_1 \cap \Omega_0 = \gamma \,, \tag{5.2.1}$$

and γ meets $\partial\Omega_0 \cap \partial B_1$ orthogonally. We split the proof into some steps.

Step 1. We claim that $\partial F \cap \partial B_1 \cap \Omega_0$ is made of countably-many (open) geodesic arcs γ_i, $i \geq 1$ integer, such that

$$\gamma_i \cap \gamma_j = \emptyset, \text{ for } i \neq j, \quad \bigcup_i \gamma_i = \partial F \cap \partial B_1 \cap \Omega_0 \,.$$

By interior regularity, $\partial F \cap \Omega_0$ is smooth, and its outer normal vector ν_F is orthogonal to the radial directions (recall that F is a cone with vertex at the origin). Hence, ∂F intersects transversally $\partial B_1 \cap \Omega_0$ along smooth curves γ_i that cannot cross each other. Since F has locally-finite perimeter, the family of these curves is at most countable. Let us now show that γ_i is a geodesic arc, for all i. With a slight abuse of notation, we identify γ_i with its arc-length parametrization defined on the interval $(0, L_i)$, where L_i is the length of the curve. The connected component of ∂F that intersects ∂B_1 along γ_i can be then parametrized through

$$\sigma_i(s,t) := s\,\gamma_i(t)\,,\ \text{for } s>0,\ t\in(0,L_i). \tag{5.2.2}$$

We can choose the parametrization $\gamma_i(t)$ in such a way that $\nu_F(\sigma_i(s,t)) = \gamma_i(t)\times\gamma_i'(t)$, for all $s>0$. Exploiting (5.2.2), and using $\div_{\partial F}\nu_F = 0$ by the minimality of F, we infer

$$\begin{aligned} 0 &= \div_{\partial F}\nu_F(\sigma_i(s,t)) && (5.2.3)\\ &= \frac{d}{dt}(\gamma_i(t)\times\gamma_i'(t))\cdot\gamma_i'(t) \\ &= -\gamma_i(t)\times\gamma_i'(t)\cdot\gamma_i''(t), \qquad \text{for all } t\,. && (5.2.4)\end{aligned}$$

Since we also have $\gamma_i'(t)\cdot\gamma_i''(t)=0$ by the choice of the arc-length parametrization, and observing that $\{\gamma_i'(t),\gamma_i(t)\times\gamma_i'(t)\}$ is an orthonormal basis for the tangent space to ∂B_1 at $\gamma(t)$, we conclude that $\gamma_i''(t)$ is orthogonal to the tangent space to ∂B_1 at $\gamma(t)$, which is precisely the definition of geodesic arc.

Step 2. We prove that γ_i meets $\partial\Omega_0$ orthogonally at its endpoints. More precisely, if p is an endpoint of γ_i, then Ω_0 admits a unique supporting plane at p, hence the outer unit normal vector $\nu_0(p)$ to $\partial\Omega_0$ at p is well-defined, and moreover if we denote by ν_i the constant unit outer normal to the connected component of ∂F containing γ_i, we have

$$\nu_i\cdot\nu_0(p)=0\,. \tag{5.2.5}$$

Let us first prove that Ω_0 admits a unique supporting plane at p. Up to a rotation, we may assume that $p=(0,0,1)$, hence it follows that $\nu_i\cdot e_3=0$. Owing to Lemma 5.1.6, we can find a sequence $t_j\searrow 0$ such that $\Omega_0^{p,t_j} := t_j^{-1}(\Omega_0-p)$ locally converge to a cylinder of type $C\times\mathbb{R}$, and $F^{p,t_j} := t_j^{-1}(F-p)$ locally converge to a cylinder that can be written as $G\times\mathbb{R}$, where $G\subset C$. Moreover, both C, and G are cones with respect to 0 in the plane $z=0$, and G is perimeter-minimizing in C. By convexity of C, up to a further rotation, we can assume that

$$C=\{(x_1,x_2,0):\ x_2>\lambda|x_1|\}\,,$$

for some $\lambda\ge 0$. Clearly, $\lambda=0$ if and only if p admits a unique supporting plane for Ω_0. The only possibility is then that $\partial G\cap C$ is made of finitely many half-lines $L_1,\dots,L_k$ of

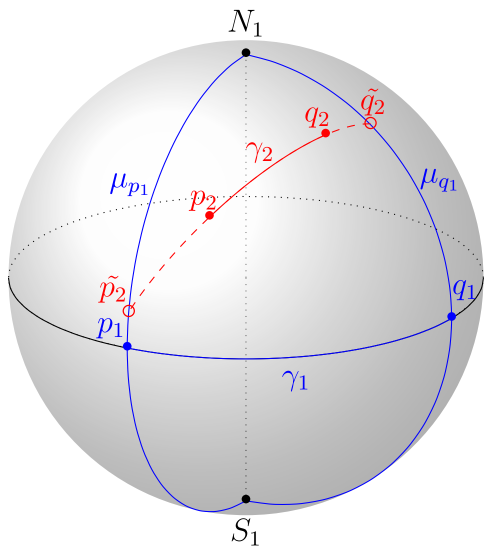

Figure 5.1

the form $L_j = \{t\, v_j : t \geq 0\}$, for some unit vectors v_j. Up to relabeling, we can assume

$$0 < v_1 \cdot e_2 \leq v_j \cdot e_2 \quad \text{for all } j\,.$$

If either $\lambda > 0$ or $k > 1$, we could replace an initial portion of L_1 with a projection segment onto the closest side of C, which strictly decreases the perimeter and thus contradicts the fact that G is perimeter-minimizing in C. Hence we necessarily have $\lambda = 0$ and $k = 1$, i.e., there exists a unique supporting plane to Ω_0 at p with $\nu_0(p) = -e_2$, and moreover $v_1 = e_2$. This proves the claim and, additionally, shows that two different geodesic arcs cannot share a common endpoint.

Step 3. Finally, we prove that $\partial F \cap \partial B_1 \cap \Omega_0$ is made of exactly one geodesic arc.

Suppose by contradiction that there exist two geodesic arcs $\gamma_1 \neq \gamma_2$ contained in $\partial F \cap \partial B_1 \cap \Omega_0$. From the previous step we know that

$$\overline{\gamma_1} \cap \overline{\gamma_2} = \emptyset\,.$$

For $i = 1, 2$ we denote by Π_i the plane through the origin that contains γ_i, and by p_i, q_i the boundary points of γ_i.

For $i = 1, 2$, we consider the point $N_i \in \partial B_1$ such that γ_i is contained in the equator with north pole N_i and south pole $S_i = -N_i$, and denote by μ_{p_i}, μ_{q_i} the corresponding meridians connecting N_i with S_i and passing through p_i and q_i, respectively. These meridians bound a region of ∂B_1, that we denote as Σ_i, which satisfies

$$\Omega_0 \cap \partial B_1 \subset \Sigma_i, \qquad i = 1, 2 .$$

Incidentally, thanks to the previous step, Σ_i is also obtained by intersecting the sphere ∂B_1 with the wedge W_i given by the intersection of the two supporting half-spaces to Ω_0 at p_i and q_i, respectively. Since in particular W_i is a convex cone, it is immediate to check that $\Sigma_i = W_i \cap \partial B_1$ is geodesically convex. Moreover, using the fact that the angle formed by the vectors p_i, q_i is strictly smaller than π (recall that the origin is an isolated vertex for Ω_0) we infer that the internal angle formed by the two geodesic sides of Σ_i, i.e. the meridians μ_{p_i} and μ_{q_i}, at N_i (or S_i) is strictly smaller than π.

Now, observe that the closure of Σ_i is the union of two closed geodesic triangles T_{N_i}, T_{S_i} with vertices p_i, q_i, N_i and p_i, q_i, S_i, respectively. Since in particular $\gamma_2 \subset \Omega_0 \cap \partial B_1 \subset \Sigma_1$, and γ_2 has a strictly positive distance from γ_1, we must have that either $\gamma_2 \subset T_{N_1}$ or $\gamma_2 \subset T_{S_1}$. Without loss of generality, we assume $\gamma_2 \subset T_{N_1}$.

Now, we set $\widetilde{p}_2 = \Pi_2 \cap \mu_{p_1}$ and $\widetilde{q}_2 = \Pi_2 \cap \mu_{q_1}$, and denote by $\widetilde{\gamma}_2$ the geodesic connecting $\widetilde{p}_2$ and $\widetilde{q}_2$, and by $\widetilde{\Sigma}_2$ the associated geodesically convex region bounded by the meridians $\mu_{\widetilde{p}_2}, \mu_{\widetilde{q}_2}$ meeting at poles $\widetilde{N}_2 = N_2$ and $\widetilde{S}_2 = S_2$. Clearly we have $\gamma_2 \subset \widetilde{\gamma}_2$ and thus $\Sigma_2 \subset \widetilde{\Sigma}_2$. Moreover, we have

$$\widetilde{p}_2, \widetilde{q}_2 \in T_{N_1} \setminus \{p_1, q_1, N_1\} .$$

Indeed, the geodesic $\widetilde{\gamma}_2$ cannot intersect γ_1, hence it is contained in T_{N_1} and its closure is disjoint from γ_1 because $\Pi_2 \cap \overline{\gamma_1} = \emptyset$; moreover, if we had $\widetilde{p}_2 = N_1$ (or $\widetilde{q}_2 = N_1$) we would conclude that $\mu_{q_1} \subset \Pi_2$ (respectively, $\mu_{p_1} \subset \Pi_2$), but this is impossible because Π_2 and $\overline{\gamma_1}$ are disjoint.

Now, consider the geodesic quadrilateral D determined by the four points $p_1, \widetilde{p}_2, \widetilde{q}_2, q_1$. By the previous argument, $D \subset T_{N_1}$. Denote by $\alpha_1, \widetilde{\alpha}_2, \widetilde{\beta}_2, \beta_1$ the angles formed by the pairs of geodesic sides meeting at the respective vertices. Then, consider the two geodesic triangles $R_1 = p_1\widetilde{p}_2 q_1$ and $R_2 = \widetilde{p}_2\widetilde{q}_2 q_1$. Call $\widetilde{\alpha}_{2,1}$ the internal angle to R_1 at $\widetilde{p}_2$, and $\widetilde{\alpha}_{2,2}$ the internal angle to R_2 at $\widetilde{p}_2$. Similarly, call $\beta_{1,1}$ the internal angle to R_1 at q_1, and

$\beta_{1,2}$ the internal angle to R_2 at q_1. We thus have $\widetilde{\alpha}_2 = \widetilde{\alpha}_{2,1} + \widetilde{\alpha}_{2,2}$, $\beta_1 = \beta_{1,1} + \beta_{1,2}$, and therefore we deduce

$$\alpha_1 = \beta_1 = \pi/2\,, \tag{5.2.6}$$

$$\max(\widetilde{\alpha}_2, \widetilde{\beta}_2) \leq \pi/2\,. \tag{5.2.7}$$

Indeed, (5.2.6) follows from the orthogonality of γ_1 with the meridians μ_{p_1} and μ_{q_1}, while (5.2.7) follows from the fact that the quadrilateral D is contained in one of the two geodesic triangles $\widetilde{T}_{N_2} = \widetilde{p}_2\widetilde{q}_2N_2$, $\widetilde{T}_{S_2} = \widetilde{p}_2\widetilde{q}_2S_2$ (indeed, by a symmetric argument, we have either $\gamma_1 \subset \widetilde{T}_{N_2}$ or $\gamma_1 \subset \widetilde{T}_{S_2}$) and we know by construction that the internal angles to $\widetilde{T}_{N_2}$ (or $\widetilde{T}_{S_2}$) at $\widetilde{p}_2$ and at $\widetilde{q}_2$ are both equal to $\pi/2$.

Now we notice that R_1 must be a non-degenerate geodesic triangle because it possesses an internal angle at p_1 measuring $\alpha_1 = \pi/2$, and the other two vertices do not coincide. Thus we have that the sum of the internal angles of R_1 satisfies

$$\alpha_1 + \widetilde{\alpha}_{2,1} + \beta_{1,1} > \pi\,. \tag{5.2.8}$$

At the same time, the sum of the internal angles of R_2 is not smaller than π:

$$\widetilde{\alpha}_{2,2} + \widetilde{\beta}_2 + \beta_{1,2} \geq \pi\,. \tag{5.2.9}$$

By combining (5.2.6), (5.2.7), (5.2.8) and (5.2.9), we reach the contradiction

$$2\pi < (\alpha_1 + \widetilde{\alpha}_{2,1} + \beta_{1,1}) + (\widetilde{\alpha}_{2,2} + \widetilde{\beta}_2 + \beta_{1,2}) = \alpha_1 + \widetilde{\alpha}_2 + \widetilde{\beta}_2 + \beta_1 \leq 2\pi$$

and this completes the proof of the theorem. □

5.3 Proof of the main result

We now dispose of all the necessary tools to prove the following

Theorem 5.3.1 (Vertex-skipping). *Let $\Omega \subset \mathbb{R}^3$ be an open, convex set, and $E \subset \Omega$ be a local almost-minimizer of P_Ω. Then $\overline{\partial E \cap \Omega}$ does not contain vertices of Ω.*

We argue by contradiction. By the results of the previous section, via a blow-up argument we can restrict the proof to the case of a domain Ω_0 being a convex cone with vertex at the origin, and of a minimizer given by the intersection with Ω_0 of a half-space whose boundary plane passes through the origin and meets $\partial\Omega_0$ orthogonally.

Before giving the proof of Theorem 5.3.1, we introduce a class of convex cones that will play a key role in the first part of the proof.

Definition 5.3.2. *We say that a convex cone $C \subset \mathbb{R}^3$ is a pyramid provided there exist two wedges W_1, W_2 with orthogonally incident spines such that $C = W_1 \cap W_2$.*

We note that a pyramid $C \subset \mathbb{R}^3$ is always a cone with vertex at the point V_0 in which the spines of the wedges intersect each other. Moreover, up to a rotation and a translation, there exist a, $b > 0$ such that

$$C = C_{a,b} := \{x \in \mathbb{R}^3 : \ x_3 \geq \max\{a|x_1|, b|x_2|\}\} .$$

Proof (of Theorem 5.3.1). The proof is split into two steps. In the first, we show the result under the assumption that the conical container is a pyramid, i.e., a cone over a rectangle. In the second, we employ a "packing-boy" technique that allows us to reduce the case of a general convex cone to that of a suitably associated pyramid.

Step 1. Consider a pyramid cone $C_{a,b}$. We want to show that the plane

$$\pi_0 = \{x \in \mathbb{R}^3 : \ x_1 = 0\}$$

is not locally area-minimizing in $C_{a,b}$. To do so, we build a family of competitors that improve the area of π_0 in $C_{a,b}$. For $\varepsilon \geq 0$, let $\pi_\varepsilon = \{x \in \mathbb{R}^3 : \ x_1 = \varepsilon\}$ and define

$$R_\varepsilon := C_{a,b} \cap \{x \in \pi_\varepsilon : \ x_3 \leq 1\} , \qquad A_\varepsilon := \mathcal{H}^2(R_\varepsilon) .$$

We note that R_0 is a triangle in the plane π_0, and that R_ε is a trapezium in the plane π_ε whenever $0 < \varepsilon < 1$. Moreover, up to translations, R_ε is obtained from R_0 by removing a triangle of area $\frac{a^2\varepsilon^2}{b}$, so that we have

$$A_\varepsilon = A_0 - \frac{a^2}{b}\varepsilon^2 . \tag{5.3.1}$$

The idea is now to connect the trapezium R_ε with $\pi_0 \cap C_{a,b}$ in order to obtain a local

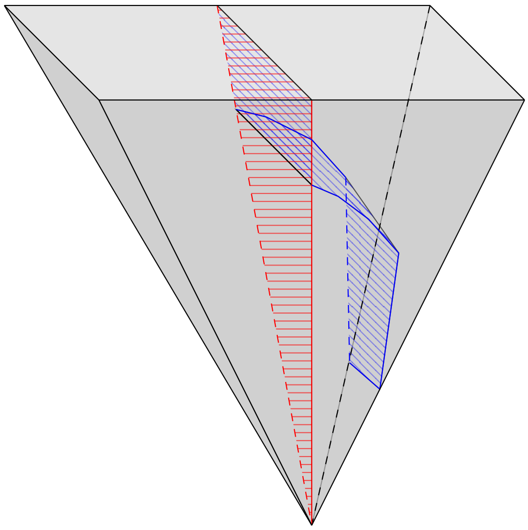

Figure 5.2

variation of $\pi_0 \cap C_{a,b}$. We formulate the problem in the following way. Let $h > 0$ and let T_h be the trapezium defined as

$$T_h = C_{a,b} \cap \{x \in \pi_0 : \ 1 \leq x_3 \leq 1 + h\} .$$

We immediately note that

$$\mathcal{H}^2(T_h) = \frac{h(2+h)}{b} . \tag{5.3.2}$$

We look for those smooth functions ϕ_h defined on the segment $\{1 \leq x_3 \leq 1+h\}$ satisfying the following conditions:

$$\phi_h(1) = 1 , \qquad \phi_h(1+h) = 0 . \tag{5.3.3}$$

We observe that, looking at ϕ_h as a function of both variables x_2 and x_3 defined in T_h, the ruled surface

$$G_\varepsilon(\phi_h) = \{(\varepsilon\phi_h(x_3), x_2, x_3) : \ (x_2, x_3) \in T_h\}$$

connects R_ε with $\pi_0 \cap C_{a,b}$ (see Figure 5.2, where the red region is $\pi_0 \cap C_{a,b}$, while the blue one coincides with $R_\varepsilon \cup G_\varepsilon(\phi_h)$). By suitably choosing h and the map ϕ_h, we claim that

$$A_\varepsilon + \mathcal{H}^2(G_\varepsilon(\phi_h)) < A_0 + \mathcal{H}^2(T_h) . \tag{5.3.4}$$

Using (5.3.1), (5.3.2), and the area formula, (5.3.4) turns out to be equivalent to

$$\iint_{T_h} \sqrt{1+\varepsilon^2|\phi_h'(x_3)|^2}dx_2dx_3 < \frac{a^2}{b}\varepsilon^2 + \frac{h(2+h)}{b}\,. \tag{5.3.5}$$

To guarantee (5.3.5), we only need to impose that the second-order derivative at 0 of the left-hand side is strictly smaller than the same derivative of the right-hand side. We first observe that, by Dominated Convergence, one has

$$\int_1^{1+h} t\,\phi_h'(t)^2dt < a^2\,. \tag{5.3.6}$$

Then, for $\alpha > 0$, we choose

$$\phi_h = \phi_{h,\alpha}(t) := \frac{h^\alpha t^{-\alpha} - 1}{h^\alpha - 1}\,.$$

We observe that $\phi_{h,\alpha}$ fulfills (5.3.3). Taking $\phi_h = \phi_{h,\alpha}$, condition (5.3.6) becomes

$$\frac{\alpha}{2}\frac{(1+h)^\alpha + 1}{(1+h)^\alpha - 1} = \int_1^{1+h} t\,\phi_{h,\alpha}'(t)^2dt < a^2\,. \tag{5.3.7}$$

As $h \to +\infty$, the term on the left-hand side of (5.3.7) tends to $\frac{\alpha}{2}$, hence it is enough to choose $\alpha < 2a^2$ and h large enough to enforce (5.3.7). This ultimately proves (5.3.4) and shows that π_0 cannot be area-minimizing in $C_{a,b}$.

Step 2. Let now Ω_0 be a generic convex cone with vertex at the origin. Thanks to Theorem 5.2.1, and up to rotations, we may suppose that the boundary of the minimizer E_{00} is the intersection of the plane π_0 with Ω_0, hence there exists $b > 0$ such that

$$\partial E_{00} \cap \Omega_0 = \{(0, x_2, x_3) :\ x_3 \geq b|x_2|\}\,.$$

Now, by Theorem 5.2.1 we have

$$\Omega_0 \subset W_1 := \{x \in \mathbb{R}^3 :\ x_3 \geq b|x_2|\}\,.$$

Since the origin is an isolated vertex for Ω_0, it is not possible that $\partial\Omega_0$ contains the whole line $\{(t,0,0): t\mathbb{R}\}$, hence there must exist $a > 0$ such that the pyramid $C_{a,b}$ verifies either

$$\Omega_0 \cap \{x_1 \geq 0\} \subset C_{a,b} \cap \{x_1 \geq 0\} \tag{5.3.8}$$

or

$$\Omega_0 \cap \{x_1 \leq 0\} \subset C_{a,b} \cap \{x_1 \leq 0\}\,. \tag{5.3.9}$$

We can assume for instance that (5.3.8) holds true, otherwise we simply flip the argument. We take $\varepsilon > 0$ and set

$$\hat{R}_\varepsilon := \Omega_0 \cap \{x \in \pi_\varepsilon : \ x_3 \leq 1\}\,, \qquad \hat{A}_\varepsilon := \mathcal{H}^2(\hat{R}_\varepsilon)\,.$$

With the choice of suitable values h and α, we already know that the connection map $\phi_{h,\alpha}$ constructed in the proof of Theorem 5.2.1 satisfies

$$A_\varepsilon + \mathcal{H}^2(G_\varepsilon(\phi_{h,\alpha})) < A_0 + \mathcal{H}^2(T_h)$$

whenever ε is small enough. Finally, we observe that

$$\hat{A}_\varepsilon + \mathcal{H}^2(G_\varepsilon(\phi_{h,\alpha}) \cap \Omega_0) \leq A_\varepsilon + \mathcal{H}^2(G_\varepsilon(\phi_{h,\alpha}))$$
$$\hat{A}(0) = A(0)\,,$$

which shows that π_0 cannot be a minimizer in Ω_0. This concludes the proof of Theorem 5.3.1.

□

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