

CONVERGENCE RATES FOR THE VANISHING VISCOSITY APPROXIMATION OF HAMILTON-JACOBI EQUATIONS: THE CONVEX CASE

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ABSTRACT. We study the speed of convergence in L^∞ norm of the vanishing viscosity process for Hamilton-Jacobi equations with uniformly or strictly convex Hamiltonian terms with superquadratic behavior. Our analysis boosts previous findings on the rate of convergence for this procedure in L^p norms, showing rates in sup-norm of order $\mathcal{O}(\varepsilon^\beta)$, $\beta \in (1/2, 1)$, or $\mathcal{O}(\varepsilon |\log \varepsilon|)$ with respect to the vanishing viscosity parameter ε , depending on the regularity of the initial datum of the problem and convexity properties of the Hamiltonian. Our proofs are based on integral methods and avoid the use of techniques based on stochastic control or the maximum principle.

1. INTRODUCTION

This paper studies quantitative estimates of the rate of convergence of the vanishing viscosity approximation of periodic solutions to the viscous Hamilton-Jacobi equation

$$(1) \quad \begin{cases} -\partial_t u_\varepsilon - \varepsilon \Delta u_\varepsilon + H(Du_\varepsilon) = f(x, t) & \text{in } Q_T := \mathbb{T}^n \times (0, T), \\ u_\varepsilon(x, T) = u_T(x) & \text{in } \mathbb{T}^n. \end{cases}$$

towards its first-order counterpart

$$(2) \quad \begin{cases} -\partial_t u + H(Du) = f(x, t) & \text{in } Q_T := \mathbb{T}^n \times (0, T), \\ u(x, T) = u_T(x) & \text{in } \mathbb{T}^n. \end{cases}$$

Here, $H : \mathbb{R}^n \rightarrow \mathbb{R}$ is the so-called Hamiltonian, while $f : Q_T \rightarrow \mathbb{R}$ and $u_T : \mathbb{T}^n \rightarrow \mathbb{R}$ are the periodic source and the terminal condition of the problem.

The rate of convergence of this regularization approach has been widely investigated in the context of viscosity solutions and it is sensitive of the regularity and the geometric properties of the Hamiltonian and the data of the problem. In the case of Lipschitz solutions and *locally Lipschitz* H the $\mathcal{O}(\sqrt{\varepsilon})$ rate of convergence for $\|u_\varepsilon - u\|_\infty$ was established in [Fle64, Lio82, CL84, Eva10] using probabilistic techniques, maximum principle approaches and integral duality methods respectively, see also [Cal18] for a proof via the regularization by sup/inf-convolution. In the absence of further assumptions, this rate is sharp in view of the recent examples discussed in [QSTY24, Section 4], and it becomes even slower if one weakens the regularity conditions on H and u or considers homogenization problems, see for example [BCD97, CCM11].

One may then wonder whether the $\mathcal{O}(\sqrt{\varepsilon})$ rate can be improved under further assumptions. First, following the common principle in PDE theory that smallness conditions imply smoothness, one can deduce a faster $\mathcal{O}(\varepsilon)$ rate of convergence in sup-norms when $\|D^2 u_T\|_\infty$ is *small* or for *short time horizons* [Lio82, Chapter 12]. This rate is a consequence of $C^{1,1}$ estimates that are independent of the viscosity, and that hold typically in these smallness regimes. In this case, applying the usual maximum principle to the (linearized) equation solved by $u_\varepsilon - u$, one obtains

$$\|u_\varepsilon - u\|_{L^\infty(Q_T)} \leq C\varepsilon.$$

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This agrees with L^1 rates of convergence for the vanishing viscosity process of the one-dimensional Burger's equation [Wan99, Theorem 3.1].

If H is assumed to be *uniformly convex*, well-known results provide *one-side* $\mathcal{O}(\varepsilon)$ rate in sup-norm [Lio82] (this rate is also well-known for conservation laws under the Oleinik one-side Lipschitz condition [NT92]), and $\mathcal{O}(\varepsilon)$ rate in the weaker L^1 norm [LT01, CGM23]. These rates can be interpolated, giving new convergence results in L^p spaces [CGM23]. A special role in the uniformly convex framework is played by the purely *quadratic* Hamiltonian: by means of the Hopf-Cole reduction and the explicit representation formula for the heat flow [Bre20], the $\mathcal{O}(\varepsilon|\log \varepsilon|)$ rate in sup-norm has been observed in the literature. See [Wan99] for conservation laws, and [QSTY24, Proposition 4.4] for the one-dimensional Hamilton-Jacobi equation. This result has been very recently generalized to the multidimensional case [CD25, Section 3], for equations posed on the whole space \mathbb{R}^n and Lipschitz data.

The first goal of this paper is to prove that the $\mathcal{O}(\varepsilon|\log \varepsilon|)$ rate holds beyond the purely quadratic case, that is, for uniformly convex Hamiltonians. We assume, in particular, that for some $0 < \theta \leq \Theta$,

$$(H) \quad H \in C^2 \quad \text{and} \quad \theta \mathbb{I}_n \leq D_{pp}^2 H(p) \leq \Theta \mathbb{I}_n \quad \text{for all } p.$$

Throughout the paper, u_T will be implicitly assumed to be of class $C^{2,\alpha}$, and f of class $C^{2+\alpha, 1+\alpha/2}$. This will guarantee that the unique solution to (1) is classical for each $\varepsilon > 0$. Nevertheless, all the estimates obtained below will depend on Lipschitz or semiconcavity properties of u_T and f , that will be specified below.

The first result, stated in Theorem 3.1, shows that the general $\mathcal{O}(\sqrt{\varepsilon})$ rate can be boosted to

$$(3) \quad -C_{1,\beta} \varepsilon^\beta \leq u_\varepsilon - u \leq C_2 \varepsilon, \quad \text{on } Q_T, \quad \text{for any } \beta \in (1/2, 1).$$

Here, $C_{1,\beta}, C_2$ depend on some semiconcavity properties of u_T and f . This estimate implies also a new $\mathcal{O}(\varepsilon^{\beta/2})$ rate of convergence of gradients in $L_t^\infty(L_x^2)$ norms and, consequently, the convergence rate for solutions of certain hyperbolic systems of conservation laws.

Secondly we prove, in Theorem 4.1, the analytic bound

$$(4) \quad \|(u_\varepsilon - u)(\tau)\|_{L^\infty(\mathbb{T}^n)} \leq C_\tau \varepsilon |\log \varepsilon|, \quad \text{for any } \tau \in [0, T].$$

Note that C_τ deteriorates as $\tau \rightarrow T$, as we are using only the information $u_T \in W^{1,\infty}(\mathbb{T}^n)$ on the final condition. The bounds (3)-(4) complete the picture of the vanishing viscosity approximation for *uniformly convex* Hamiltonians, answering a question posed in [QSTY24, p.17], which provided numerical evidence in favor of the validity of these results. Moreover, we observe in Remark 3.3 that the order ε can be reached in special situations, such as the one of convex data.

In the second part of the paper, we continue our analysis and consider Hamiltonians that are merely *strictly convex*. More precisely, we relax (H) to

$$(H_\gamma) \quad H \in C^2 \quad \text{satisfies} \quad \theta |\xi|^2 |p|^{\gamma-2} \leq D_{pp}^2 H(p) \xi \cdot \xi \leq \Theta |\xi|^2, \quad \xi \in \mathbb{R}^n, \quad \xi \neq 0, \quad \gamma > 2, \quad |p| \leq C,$$

for some $0 < \theta < \Theta$, a model being the superquadratic Hamiltonian

$$H(p) = |p|^\gamma, \quad \gamma > 2.$$

We prove in Theorem 5.1 the following estimate: there exists $\beta_\gamma \in (\frac{1}{2}, 1)$ such that

$$(5) \quad -C_{3,\beta} \varepsilon^\beta \leq u_\varepsilon - u \leq C_2 \varepsilon \quad \text{on } Q_T, \quad \text{for each } \beta < \beta_\gamma.$$

The upper bound β_γ is explicit. One can check that the leftmost rate $\mathcal{O}(\varepsilon^{\beta_\gamma})$ deteriorates to $\mathcal{O}(\sqrt{\varepsilon})$ as $\gamma \rightarrow +\infty$ and agrees with (3) in the limit $\gamma \rightarrow 2$. To the best of our knowledge, these intermediate rates β_γ are new within the framework of strictly convex Hamiltonians. Numerical experiments have been carried out in [QSTY24], suggesting that one should indeed expect some intermediate situation $\beta_\gamma \in (1/2, 1)$. We do not know whether the upper bound β_γ obtained here is sharp or not, and we believe that further investigation of strictly but not uniformly convex

problems can be an interesting research direction. See Remark 5.2 for further comments.

The proofs presented here exploit a duality approach [Eva10], being based on the analysis of the adjoint of the linearization of (1). A more detailed description of the arguments is given in the next section.

We conclude by mentioning that one of our main motivations to address this vanishing viscosity problem was the more complicated study of the rate of the convergence problem in Mean Field Control, where, roughly speaking, one fixes $\varepsilon > 0$ in (1) and let the dimension $n \rightarrow \infty$. In some specific situations (see for example [DDJ24]), this problem can be reduced in fact to a question of vanishing viscosity for Hamilton-Jacobi equations posed on the Euclidean space (with fixed dimension). We report the implications of our results in the convergence problem for Mean Field Control in Remark 5.3.

It is worth mentioning that other approximations of Hamilton-Jacobi equations have been considered in the literature, for example regularizations via nonlocal operators [DI06, Gof24]. Finally, it would be interesting to apply this approach to the study of the convergence rate of the vanishing viscosity approximation of Mean Field Games systems [TZ23], numerical methods, cf. [BCD97, Cal18, CGT13], stationary problems and homogenization.

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2. PRELIMINARIES AND METHOD OF PROOF

Following a classical approach (see for instance [FS86, Theorem 5.1]) we first observe that the estimate of the quantity $\|u_\varepsilon - u\|_{L^\infty}$ can be obtained, via an asymptotic expansion, from an estimate of $\|\partial_\varepsilon u_\varepsilon\|_{L^\infty}$, see in particular Remark 2.2 below. Note also that, by differentiating (1) with respect to ε , $v^\varepsilon = \partial_\varepsilon u_\varepsilon$ solves the linear drift-diffusion PDE

$$-\partial_t v^\varepsilon - \varepsilon \Delta v^\varepsilon + D_p H(Du_\varepsilon) \cdot Dv^\varepsilon = \Delta u_\varepsilon$$

equipped with homogenous final condition $v_\varepsilon(T) = 0$. Now, for any $(\bar{x}, \tau) \in Q_T$, by means of the dual problem solved by $\rho = \rho_{\varepsilon, \tau, \bar{x}}$

$$(6) \quad \begin{cases} \partial_t \rho - \varepsilon \Delta \rho - \operatorname{div}(D_p H(Du_\varepsilon) \rho) = 0 & \text{in } Q_{\tau, T} := \mathbb{T}^n \times (\tau, T), \\ \rho(\tau) = \delta_{\bar{x}} & \text{in } \mathbb{T}^n \end{cases}$$

we get a representation formula, for arbitrary $s \in [\tau, T]$, as follows

$$v_\varepsilon(\bar{x}, \tau) = \int_{\mathbb{T}^n} v_\varepsilon(s) \rho(s) \, dx + \iint_{\mathbb{T}^n \times (\tau, s)} \Delta u_\varepsilon \rho \, dx \, dt = \text{(I)} + \text{(II)}.$$

One can take for simplicity $s = T$, so that the integral (I) is zero. Therefore, the estimate on v^ε boils down to a control on (II), that involves second-order derivatives $D^2 u_\varepsilon$ weighted by ρ . The crucial step of the approach presented here is in fact to exploit the weight ρ , which, from perspective of optimal control, describes the flow of optimal trajectories originating from \bar{x} at time τ .

There are two main ingredients that allow to control the quantity $\iint \Delta u_\varepsilon \rho$. These are the presence of the diffusion and the convexity of the Hamiltonian. The diffusion gives bounds of the form

$$(7) \quad \int_\tau^T \int_{\mathbb{T}^n} |D^2 u_\varepsilon|^2 \rho \, dx \, dt \lesssim \frac{1}{\varepsilon},$$

see Lemma 2.3 below. Note that this implies immediately, by Hölder's inequality and conservation of mass for ρ , that

$$\left| \iint_{\mathbb{T}^n \times (\tau, T)} \Delta u_\varepsilon \rho \right| \leq \sqrt{\frac{T - \tau}{\varepsilon}}.$$

This crude bound gives the $\mathcal{O}(\sqrt{\varepsilon})$ control on $\|u_\varepsilon - u\|_{L^\infty}$. To improve it, one needs to get further information out of the convex structure of H . To this aim, we follow a general principle of optimal control, which roughly states that optimal trajectories should be “stable”, as well as the optimal drift, when restricted to time intervals of the form $[t, T] \subset (\tau, T]$, that is, away from the originating point (\bar{x}, τ) . We make this observation quantitative by proving estimates of the form

$$(8) \quad \int_{\tau+\delta}^T \int |D^2 u_\varepsilon|^2 \rho dx dt \lesssim \frac{1}{\delta^\alpha},$$

where $\delta > 0$ and $\alpha > 1$ (see (10)). This, together with (7), allows to improve the previous bounds, by splitting the time interval into $[\tau, \tau + \delta] \cup [\tau + \delta, T]$ as follows:

$$\left| \iint_{\mathbb{T}^n \times (\tau, T)} \Delta u_\varepsilon \rho \right| \lesssim \frac{1}{\delta^\alpha} + \sqrt{\frac{\delta}{\varepsilon}}.$$

A suitable choice of $\delta = \delta(\varepsilon)$ gives then the desired rate $\mathcal{O}(\varepsilon^\beta)$, for $\beta < 1$.

To reach the optimal rate $\mathcal{O}(\varepsilon |\log \varepsilon|)$, a more delicate analysis is needed, as it corresponds to the “critical” case $\alpha = 1$ in the previous bound. We exploit here some additional entropy and first-order bounds for (6) when the initial datum does not belong to the Orlicz space $L \log L$ (Lemma 4.2). We refer to [NFSS22] for a related analysis under different assumptions on the velocity field and the initial datum.

The analysis of the strictly convex case is even more delicate, as (8) is meaningful only when $|Du_\varepsilon|$ is bounded away from zero. Under our strict convexity assumptions, (8) becomes

$$\int_{\tau+\delta}^T \int_{|Du|^2 \geq m} |D^2 u_\varepsilon|^2 \rho dx dt \lesssim \frac{1}{m^{\frac{\gamma-2}{2}} \delta^\alpha},$$

To compensate the lack of information when $|Du|$ is close to zero, we need to sharpen (7) in regions where the gradient is small. The proof of the crucial estimate (20) is obtained via a Bochner-type identity for $\tilde{w} = \varphi(|Du|^2)$,

$$\Delta \tilde{w} = 4\varphi'' \sum_{j=1}^n (Du_{x_j} \cdot Du)^2 + 2\varphi' D\Delta u \cdot Du + 2\varphi' |D^2 u|^2,$$

where φ is a carefully chosen truncation function. We proceed with a hole-filling technique in the spirit of K.-O. Widman [Wid71]. This technique has been historically applied to obtain interior Morrey estimates, and local Hölder continuity as a byproduct, for elliptic equations and systems. We apply here a similar method to get integral bounds of (weighted) second order derivatives over sublevel sets of the gradient (instead of concentric balls as in the classical literature).

We conclude by stressing again that it is crucial to look at integral estimates on second derivatives, as classical pointwise estimates (coming from example from the theory of viscosity solutions) merely give

$$\|\partial_t u_\varepsilon\|_{L_{x,t}^\infty}, \|Du_\varepsilon\|_{L_{x,t}^\infty} \leq C \implies \|\Delta u_\varepsilon\|_{L_{x,t}^\infty} \lesssim \frac{1}{\varepsilon}.$$

These bounds have no (straightforward) use in the control of the rate of convergence.

The next lemmata make some of the steps that have been previously described rigorous.

Lemma 2.1. *Assume that H satisfies (H) or (H_γ) . For every $(x, t) \in \mathbb{T}^n \times [0, T]$ and $\varepsilon > 0$, the derivative $v^\varepsilon(x, t) = \partial_\varepsilon u_\varepsilon(x, t)$ is well-defined. The function v^ε is of class $C^{2+\alpha, 1+\alpha/2}(\mathbb{T}^n \times [0, T])$, and it solves*

$$(9) \quad \begin{cases} -\partial_t v^\varepsilon - \varepsilon \Delta v^\varepsilon + D_p H(Du_\varepsilon) \cdot Dv^\varepsilon = \Delta u_\varepsilon & \text{in } \mathbb{T}^n \times (0, T), \\ v^\varepsilon(x, T) = 0 & \text{in } \mathbb{T}^n \end{cases}$$

in the classical sense.

Proof. We denote by $C^{2k+\alpha, k+\alpha/2}$ the classical Hölder space with respect to the parabolic distance. Note first that by Schauder estimates (for quasilinear parabolic equations), u_ε is of class $C^{2+\alpha, 1+\alpha/2}(\mathbb{T}^n \times [0, T])$, hence $\Delta u_\varepsilon \in C^{\alpha, \alpha/2}(\mathbb{T}^n \times [0, T])$ and the (linear) Cauchy problem (9) admits a unique classical solution $v^\varepsilon \in C^{2+\alpha, 1+\alpha/2}(\mathbb{T}^n \times [0, T])$. Consider now, for $\eta > 0$, the finite difference

$$v_\eta^\varepsilon = \frac{u_{\varepsilon+\eta} - u_\varepsilon}{\eta}$$

and note that it satisfies $v_\eta^\varepsilon(T) = 0$ and

$$-\partial_t v_\eta^\varepsilon - (\varepsilon + \eta) \Delta v_\eta^\varepsilon + \frac{H(Du_{\varepsilon+\eta}) - H(Du_\varepsilon)}{\eta} = \Delta u_\varepsilon.$$

Here we use the regularity of H to show that, by the second-order Taylor expansion,

$$H(Du_{\varepsilon+\eta}) = H(Du_\varepsilon) + D_p H(Du_\varepsilon) \cdot D(u_{\varepsilon+\eta} - u_\varepsilon) + \Psi D(u_{\varepsilon+\eta} - u_\varepsilon) \cdot D(u_{\varepsilon+\eta} - u_\varepsilon),$$

where the matrix-valued function Ψ is defined as

$$\Psi_{\varepsilon, \eta}(x, t) = \frac{1}{2} \int_0^1 (1 - \zeta) D_{pp}^2 H(Du_\varepsilon(x, t) + \zeta(Du_{\varepsilon+\eta}(x, t) - Du_\varepsilon(x, t))) d\zeta.$$

The function Ψ is continuous, and bounded in $\mathbb{T}^n \times [0, T]$ uniformly with respect to η by the uniform bound $D_{pp}^2 H \leq \Theta$. Hence,

$$-\partial_t v_\eta^\varepsilon - (\varepsilon + \eta) \Delta v_\eta^\varepsilon + D_p H(Du_\varepsilon) \cdot Dv_\eta^\varepsilon + \eta \Psi Dv_\eta^\varepsilon \cdot Dv_\eta^\varepsilon = \Delta u_\varepsilon.$$

To pass to the limit $\eta \rightarrow 0$, we use the Hölder regularity estimates on v_η^ε from [LSU68] (uniform in η), that go through gradient bounds first, then $W_p^{2,1}$ (the parabolic Sobolev space of second order) regularity and Sobolev embeddings finally. We can then extract a subsequence, that we keep denoting by v_η^ε , that converges uniformly to some continuous function v^ε by Ascoli-Arzelà Theorem. To show that the limit v^ε (along the specific subsequence) solves (9), we use a standard viscosity solution argument: $\eta \Psi$ vanishes uniformly on $\mathbb{T}^n \times [0, T]$ and $\varepsilon + \eta \rightarrow \varepsilon$. Therefore, by the stability of viscosity solutions, v^ε is a viscosity solution of (9). Since viscosity solutions to the linear problem (9) are unique, v^ε enjoys in fact the regularity claimed at the beginning of the proof. Moreover, this uniqueness property also shows that $v_\eta^\varepsilon \rightarrow v^\varepsilon$ uniformly on $\mathbb{T}^n \times [0, T]$ along the full limit $\eta \rightarrow 0$, providing the desired assertion. \square

Remark 2.2. The proof of the rate of convergence will be reduced to an L^∞ estimate on v^ε . Having this at our disposal, we then get that, for every $\varepsilon'' > \varepsilon' > 0$, $(x, t) \in \mathbb{T}^n \times [0, T]$,

$$|u_{\varepsilon''}(x, t) - u_{\varepsilon'}(x, t)| \leq \int_{\varepsilon'}^{\varepsilon''} |v^\varepsilon(x, t)| d\varepsilon \leq \int_{\varepsilon'}^{\varepsilon''} \|v^\varepsilon\|_\infty d\varepsilon.$$

Hence, whenever $\varepsilon \mapsto \|v^\varepsilon\|_\infty$ is integrable on $(0, 1)$, we have that u_ε is a Cauchy sequence in the space of continuous functions on $\mathbb{T}^n \times [0, T]$, and its uniform limit u satisfies

$$\|u_{\varepsilon''} - u\|_\infty \leq \int_0^{\varepsilon''} \|v^\varepsilon\|_\infty d\varepsilon.$$

Moreover, by uniform convergence and stability of the notion of viscosity solution, u turns out to be a (the) viscosity solution of (2). Similarly, an upper bound on $(v^\varepsilon)^+$ yields an upper bound on $(u_\varepsilon - u)^+$.

Lemma 2.3. *Assume that H satisfies (H) or (H_γ) (in fact, H can be assumed to be only locally Lipschitz). Then, for each $(x, \tau) \in \mathbb{T}^n \times [0, T]$ and ρ solving (6) we have*

$$|Du_\varepsilon(x, \tau)| + \varepsilon \int_\tau^T \int_{\mathbb{T}^n} |D^2 u_\varepsilon|^2 \rho dx dt \leq C_L,$$

where C_L depends on $\|Du_T\|_{L_x^\infty}$, $\|Df\|_{L_{x,t}^\infty}$ and T .

Proof. The estimate is well-known but we briefly sketch its proof here for completeness. It is enough to look at the equation solved by $w = |Du_\varepsilon|^2$, namely

$$-\partial_t w - \varepsilon \Delta w + 2\varepsilon |D^2 u_\varepsilon|^2 + DH(Du_\varepsilon) \cdot Dw = Df \cdot Du_\varepsilon.$$

By duality with (6), we have

$$\begin{aligned} |Du_\varepsilon(x, \tau)|^2 + 2\varepsilon \int_\tau^T \int_{\mathbb{T}^n} |D^2 u_\varepsilon|^2 \rho dx dt &\leq \int_{\mathbb{T}^n} |Du_T|^2 \rho(x, 0) dx + \int_\tau^T \int_{\mathbb{T}^n} Df \cdot Du_\varepsilon \rho dx dt \\ &\leq \|Du_T\|_\infty^2 + T \|Df\|_{\infty; Q_T} \|Du\|_{\infty; Q_T} \end{aligned}$$

Using now Young's inequality and taking the supremum over $(x, \tau) \in \mathbb{T}^n \times [0, T]$ one concludes. \square

3. RATE OF CONVERGENCE FOR UNIFORMLY CONVEX HAMILTONIANS AND SEMICONCAVE DATA

In this section we prove that the rate of convergence of the quantity $\|(u_\varepsilon - u)^-\|_{L^\infty}$ for semi-concave solutions of Hamilton-Jacobi equation can be boosted from $\mathcal{O}(\sqrt{\varepsilon})$ to $\mathcal{O}(\varepsilon^\beta)$, $\beta \in (1/2, 1)$ in L^∞ norm. This is usually known to be $\mathcal{O}(\sqrt{\varepsilon})$ in L^∞ norm or $\mathcal{O}(\varepsilon)$ in the weaker L^1 norm or in a certain average sense [Tra21], see also [Gof24] for different rates of order $\mathcal{O}(\varepsilon)$ with nonlocal regularization of the PDE. The proof uses the nonlinear adjoint method and exploits only the conservation of mass property of the dual equation (6).

Theorem 3.1. *Let H be such that (H) holds. Then, if u_ε solves (1) and u solves the first-order equation (2) we have that*

$$-C\varepsilon^\beta \leq u_\varepsilon - u \leq (T\|(\Delta u_T)^+\|_{L_x^\infty} + \|(\Delta f)^+\|_{L_t^1(L_x^\infty)})\varepsilon, \quad \beta \in (1/2, 1).$$

where C depends on $n, \theta, \|(\Delta u_T)^+\|_{L_x^\infty}, \|(\Delta f)^+\|_{L_{x,t}^\infty}, T, \|Du_T\|_{L_x^\infty}, \|Df\|_{L_{x,t}^\infty}$.

Proof. Step 1 ($\mathcal{O}(\varepsilon)$ bound from above). The linear upper bound on $(u_\varepsilon - u)^+$ is well-known, even under weaker assumptions on H , see [CGM23] and [Lio82, Proposition 11.2]. We briefly recall the proof for completeness. First, note that by Lemma 2.1 the function $v = \partial_\varepsilon u_\varepsilon$ solves (9). Testing (9) against the solution of the adjoint problem (6), we find by the conservation of mass for (6) that

$$\begin{aligned} \rho(\bar{x}, \tau) &= \int_\tau^T \int_{\mathbb{T}^n} \Delta u_\varepsilon \rho dx dt \leq \|(\Delta u_\varepsilon)^+\|_{L_t^1(L_x^\infty)} \\ &\leq T\|(\Delta u_T)^+\|_{L^\infty(\mathbb{T}^n)} + \|(\Delta f)^+\|_{L^1(0, T; L^\infty(\mathbb{T}^n))}, \end{aligned}$$

where we used the semi-superharmonic bound in [CGM23, Remarks 3.6 and 4.9], see also [Lio82]. By Remark 2.2 we conclude the estimate

$$\|(u_\varepsilon - u)^+\|_{L^\infty(Q_T)} \leq \varepsilon(T\|(\Delta u_T)^+\|_{L^\infty(\mathbb{T}^n)} + \|(\Delta f)^+\|_{L^1(0, T; L^\infty(\mathbb{T}^n))}).$$

Step 2 (Second order estimate). We prove that for $\alpha \in (1, 2)$ we have the bound

$$(10) \quad \int_\tau^T \int_{\mathbb{T}^n} (t - \tau)^\alpha |D^2 u_\varepsilon|^2 \rho dx dt \leq K,$$

where ρ solves (6) and K depends on $n, \alpha, T, \theta, u_T, f$. We first find, by differentiating twice the equation for u_ε and uniform convexity of H , the following inequality solved by the function $z(x, t) = (t - \tau)^\alpha \Delta u_\varepsilon(x, t)$:

$$\begin{aligned} -\partial_t z - \varepsilon \Delta z + \theta(t - \tau)^\alpha |D^2 u_\varepsilon|^2 + D_p H(Du_\varepsilon) \cdot Dz \\ \leq -\alpha(t - \tau)^{\alpha-1} \Delta u_\varepsilon(x, t) + (t - \tau)^\alpha \Delta f(x, t) \quad \text{in } Q_T. \end{aligned}$$

By duality and integrating in $\mathbb{T}^n \times (\tau, T)$ we have

$$\begin{aligned} (11) \quad \underbrace{\int_{\mathbb{T}^n} z(\tau) \rho(\tau) dx}_{=0} + \theta \int_\tau^T \int_{\mathbb{T}^n} (t - \tau)^\alpha |D^2 u_\varepsilon|^2 \rho dx dt &= \underbrace{\int_{\mathbb{T}^n} z(T) \rho(T) dx}_{\leq (T - \tau)^\alpha \|(\Delta u_T)^+\|_{L^\infty(\mathbb{T}^n)}} \\ &\quad - \alpha \int_\tau^T \int_{\mathbb{T}^n} (t - \tau)^{\alpha-1} \Delta u_\varepsilon \rho dx dt + \int_\tau^T \int_{\mathbb{T}^n} (t - \tau)^\alpha \Delta f \rho dx dt. \end{aligned}$$

We now use Young's inequality as follows

$$\begin{aligned}
& -\alpha \int_{\tau}^T \int_{\mathbb{T}^n} (t-\tau)^{\alpha-1} \Delta u_{\varepsilon} \rho dx dt \\
& \leq \frac{\theta}{2n} \int_{\tau}^T \int_{\mathbb{T}^n} (t-\tau)^{\alpha} |\Delta u_{\varepsilon}|^2 \rho dx dt + \frac{n\alpha^2}{2\theta} \int_{\tau}^T \int_{\mathbb{T}^n} (t-\tau)^{\alpha-2} \rho dx dt \\
& \leq \frac{\theta}{2} \int_{\tau}^T \int_{\mathbb{T}^n} (t-\tau)^{\alpha} |D^2 u_{\varepsilon}|^2 \rho dx dt + \frac{n\alpha^2}{2\theta} \int_{\tau}^T (t-\tau)^{\alpha-2} dt \\
& \leq \frac{\theta}{2} \int_{\tau}^T \int_{\mathbb{T}^n} (t-\tau)^{\alpha} |D^2 u_{\varepsilon}|^2 \rho dx dt + \frac{n\alpha^2}{2\theta(\alpha-1)} T^{\alpha-1}.
\end{aligned}$$

We then obtain

$$\begin{aligned}
(12) \quad & \int_{\tau}^T \int_{\mathbb{T}^n} (t-\tau)^{\alpha} |D^2 u_{\varepsilon}|^2 \rho dx dt \\
& \leq \frac{n\alpha^2}{\theta^2(\alpha-1)} T^{\alpha-1} + \frac{2}{\theta} T^{\alpha} \|(\Delta u_T)^+\|_{L^{\infty}(\mathbb{T}^n)} + \frac{T^{\alpha+1}}{\alpha+1} \|(\Delta f)^+\|_{L^{\infty}(Q_T)} =: K.
\end{aligned}$$

Step 3 ($\mathcal{O}(\varepsilon^{\beta})$ bound from below). We now proceed to estimate the rate of convergence: define again $v = v^{\varepsilon}(x, t) = \partial_{\varepsilon} u_{\varepsilon}(x, t)$ and find by Lemma 2.1 the PDE (9). By duality we find

$$(13) \quad v(\bar{x}, \tau) = \int_{\mathbb{T}^n} v(T) \rho(T) dx + \int_{\tau}^T \int_{\mathbb{T}^n} \Delta u_{\varepsilon} \rho dx dt.$$

We only estimate the last term on the right-hand side since $v^{\varepsilon}(T) = 0$. Assume first that $\tau + \varepsilon < T$. Then,

$$\begin{aligned}
\left| \int_{\tau}^T \int_{\mathbb{T}^n} \Delta u_{\varepsilon} \rho dx dt \right| & \leq \sqrt{n} \left(\int_{\tau+\varepsilon}^T \int_{\mathbb{T}^n} (t-\tau)^{\alpha/2} |D^2 u_{\varepsilon}| \rho (t-\tau)^{-\alpha/2} dx dt + \int_{\tau}^{\tau+\varepsilon} \int_{\mathbb{T}^n} |D^2 u_{\varepsilon}| \rho dx dt \right) \\
& \leq \sqrt{n} \left(\int_{\tau+\varepsilon}^T \int_{\mathbb{T}^n} (t-\tau)^{\alpha} |D^2 u_{\varepsilon}|^2 \rho dx dt \right)^{\frac{1}{2}} \left(\int_{\tau+\varepsilon}^T \int_{\mathbb{T}^n} (t-\tau)^{-\alpha} \rho dx dt \right)^{\frac{1}{2}} \\
& \quad + \sqrt{n} \left(\int_{\tau}^{\tau+\varepsilon} \int_{\mathbb{T}^n} |D^2 u_{\varepsilon}|^2 \rho dx dt \right)^{\frac{1}{2}} \left(\int_{\tau}^{\tau+\varepsilon} \int_{\mathbb{T}^n} \rho dx dt \right)^{\frac{1}{2}} \\
& \leq \sqrt{\frac{nK\varepsilon^{1-\alpha}}{\alpha-1}} + \sqrt{C_L},
\end{aligned}$$

where we used the (12) and Lemma 2.3. Note that if $\tau \geq T - \varepsilon$, then the same estimate holds, without the term $\sqrt{\frac{nK\varepsilon^{1-\alpha}}{\alpha-1}}$, as in this case there is no need to split the first integral.

Back to (13), this implies that

$$v^{\varepsilon}(\bar{x}, \tau) \geq -\tilde{C}_{\varepsilon}^{\frac{1-\alpha}{2}}.$$

Since the previous estimate holds for all \bar{x}, τ , by Remark 2.2 we obtain

$$u_{\varepsilon} - u \geq -\tilde{C}_{\varepsilon}^{\frac{3-\alpha}{2}},$$

and we conclude by setting $\frac{3-\alpha}{2} = \beta \in (1/2, 1)$. \square

Remark 3.2. Since K in the previous proof grows linearly in the dimension n , one can easily check that the rate (from below) of the previous result grows linearly in n .

The previous proof extends to more general Hamiltonians $H = H(x, t, p) \in C^2(\mathbb{R}^n \times (0, T) \times \mathbb{R}^n)$ under the conditions

$$\sup_{(x,t) \in Q_T} |D_x H(x, t, p)|, \sup_{(x,t) \in Q_T} |D_{xp}^2 H(x, t, p)|, \sup_{(x,t) \in Q_T} |D_{xx}^2 H(x, t, p)| \leq C_H(1 + |p|),$$

together with

$$D_{pp}^2 H(x, t, p) \xi \cdot \xi \geq a(x, t) |\xi|^2, \quad a > 0,$$

and the constant of the estimates will depend also on C_H, a . More precisely, the bound (10) under these assumptions was proved in [Tra11], while the second order estimate can be found starting from equation (36) in [CG19] and following the proof of Proposition 3.6 therein.

Remark 3.3 (Reaching $\mathcal{O}(\varepsilon)$). As we mentioned in the introduction, the linear order $\mathcal{O}(\varepsilon)$ can be reached in exceptional cases only, for example under smallness conditions. We mention here two other possible situations where the presence of a uniformly convex Hamiltonian can be exploited. If one takes $\alpha = 0$ in the previous proof, then (11) reads

$$\theta \int_{\tau}^T \int_{\mathbb{T}^n} |D^2 u_{\varepsilon}|^2 \rho dx dt \leq \|(\Delta u_T)^+\|_{L^\infty(\mathbb{T}^n)} + \int_{\tau}^T \int_{\mathbb{T}^n} \Delta f \rho dx dt - \Delta u_{\varepsilon}(\bar{x}, \tau).$$

The last term $-\Delta u_{\varepsilon}(\bar{x}, \tau)$ is in general unbounded as $\varepsilon \rightarrow 0$, and that is exactly why we put the weight $(T - \tau)^\alpha$ in the previous proof. Nevertheless, if one assumes that u_T and $f(\cdot, t)$ are *convex*, then it is known that $u_{\varepsilon}(\cdot, t)$ is convex as well for all t and ε (for example by employing coupling methods, see for example [Con24] or [CM24]). Therefore, $-\Delta u_{\varepsilon}(\bar{x}, \tau) \leq 0$ and the previous inequality gives a uniform bound on $\iint |D^2 u_{\varepsilon}|^2 \rho$ that is independent of $\tau, \bar{x}, \varepsilon$. Arguing as in the previous proof (with an even simpler argument, since there is no need to split the time interval), one reaches the order $\mathcal{O}(\varepsilon)$ in sup-norm. Clearly the hypothesis on convexity of the data is extremely restrictive if periodicity is also assumed, but the arguments presented here can be easily adapted to the nonperiodic setting, paying some attention on the usual technicalities arising from the presence of unbounded domains.

In fact, even if the data are not assumed to be convex, every point (\bar{x}, τ) such that $-\Delta u_{\varepsilon}(\bar{x}, \tau)$ remains bounded above uniformly with respect to ε is a point where convergence of $u_{\varepsilon}(\bar{x}, \tau)$ to $u(\bar{x}, \tau)$ is linear in ε . The points where convergence is linear are known to form a set of full Lebesgue measure when the Hamiltonian is purely quadratic, see [QSTY24, Proposition 4-(ii)]. We expect also this to be true when H is uniformly convex.

The first consequence of the quantitative bound in Theorem 3.1 concerns the speed of convergence of gradients in the vanishing viscosity approximation.

Corollary 3.4. *Under the assumptions of Theorem 3.1 we have*

$$\|Du_{\varepsilon} - Du\|_{L^\infty(0, T; L^2(\mathbb{T}^n))} \leq C\varepsilon^{\frac{\beta}{2}}.$$

Proof. We first note that the semiconcavity estimate implies

$$\|\Delta u_{\varepsilon}\|_{L^\infty(0, T; L^1(\mathbb{T}^n))} \leq C_1$$

for a constant C_1 independent of ε , see [Kru66] or [CGM23, Theorem 4.14]. Integrating by parts we get for a.e. $t \in (0, T)$

$$\begin{aligned} \int_{\mathbb{T}^n} |Du_{\varepsilon}(t) - Du(t)|^2 dx &= - \int_{\mathbb{T}^n} \Delta(u_{\varepsilon} - u)(u_{\varepsilon} - u)(t) dx \\ &\leq \|(u_{\varepsilon} - u)(t)\|_{L^\infty(\mathbb{T}^n)} \|\Delta(u_{\varepsilon} - u)(t)\|_{L^1(\mathbb{T}^n)} \leq C\varepsilon^\beta, \end{aligned}$$

where C is independent of ε . \square

We conclude this section with an application to the convergence rate of the solution to the quasilinear parabolic system

$$\begin{cases} \partial_t p_i^\varepsilon + \partial_{x_i} H(p_1^\varepsilon, \dots, p_n^\varepsilon) = \varepsilon \Delta p_i^\varepsilon & \text{in } Q_T, \ i = 1, \dots, n \\ p_i^\varepsilon(x, 0) = p_i^0(x) & \text{in } \mathbb{T}^n \end{cases}$$

to the solution of the hyperbolic system

$$(14) \quad \begin{cases} \partial_t p_i + \partial_{x_i} H(p_1, \dots, p_n) = 0 & \text{in } Q_T, \ i = 1, \dots, n \\ p_i(x, 0) = p_i^0(x) & \text{in } \mathbb{T}^n. \end{cases}$$

The convergence of the parabolic equation to the system of conservation laws (14) was proved by S.N. Kruzhkov in Theorem 8 of [Kru67], see also [Lio82, Section 16.1]. The connection among (2) and (14) is the following: if u is a solution of (2), then $p(x, t) = Du(x, t)$ with $p_i(x, t) = \partial_{x_i} u(x, t)$ and $p(x, 0) = Du(x, 0)$ lead to (14). Corollary 3.4 implies the following:

Corollary 3.5. *If H is uniformly convex and $p_0 = Du_0$ with $p_0 \in C^1(\mathbb{T}^n)$, we have*

$$\|p_i^\varepsilon - p_i\|_{L^2(Q_T)} \leq C\varepsilon^{\frac{\beta}{2}}, \quad \beta \in (1/2, 1).$$

4. AN IMPROVED RATE FOR UNIFORMLY CONVEX HAMILTONIANS AND LIPSCHITZ DATA

We now improve the left-side rate of convergence of Section 3 to $\mathcal{O}(\varepsilon|\log \varepsilon|)$. While in the previous result we rely on the semiconcavity of u_T , the following argument will depend merely on its Lipschitz continuity, at the price of an estimate that deteriorates close to T . This provides an analytic proof to general uniformly convex Hamilton-Jacobi equations of the one-dimensional result in Proposition 4.4-(i) of [QSTY24].

Theorem 4.1. *Let u_ε be a solution of (1) with H satisfying (H). For all $\varepsilon \in (0, 1)$ and $\tau \in [0, T]$ we have*

$$\|(u_\varepsilon - u)(\tau)\|_{L^\infty(\mathbb{T}^n)} \leq \frac{C}{(T - \tau)^{1/2}} \varepsilon |\log \varepsilon|$$

where C depends on $n, \|Du_T\|_{L_x^\infty}, T, \|(\Delta f)^+\|_{L_{x,t}^\infty}, \theta, \Theta$.

The proof of this result is divided in several steps and requires a careful analysis of entropy-type bounds for solutions of the adjoint problem (6). The next lemma shows a certain control with respect to ε of the $W^{1,1}$ norm for solutions of Fokker-Planck equations with bounded velocity fields and Dirac initial condition, in the regime of small viscosity. These estimates are known when the drift of the Fokker-Planck equation belongs to some Sobolev space and the initial data belongs to an Orlicz class, cf. [LBL19]. In particular, this complements the analysis of [NFSS22], see Remark 1 therein, where a decay result of the Sobolev norm is studied when the velocity field $b = b(x, t)$ of the advection equation is weakly compressible, i.e.

$$[\operatorname{div}(b)]^- \in L_t^1(L_x^\infty),$$

and $\rho(\tau) \in \operatorname{LLogL}(\mathbb{T}^n)$. These available results cannot be applied in our setting since here $\rho(\tau)$ is just a probability measure. Quantitative convergence rates for Fokker-Planck equations within the Diperna-Lions framework have been the matter of [BCC22], see also the references therein.

Lemma 4.2. *Assume that $0 < \varepsilon \leq 1$ and $\tau \leq T - 4\varepsilon$. Let $\rho := \rho_{\varepsilon, \tau, x}$ be a solution of*

$$\begin{cases} \partial_t \rho - \varepsilon \Delta \rho + \operatorname{div}(b\rho) = 0 & \text{in } Q_{\tau, T}, \\ \rho(\tau) = \delta_x \end{cases}$$

and assume that $\|b\|_{L^\infty(\mathbb{T}^n \times (\tau, T))} \leq K$. Then, there exists $C > 0$ depending on n, K and

$$t_1 \in [\tau + \varepsilon, \tau + 2\varepsilon], \quad t_2 \in [T - 2\varepsilon, T - \varepsilon]$$

such that

$$\int_{\mathbb{T}^n} |D\rho(y, t_i)| dy \leq \frac{C}{\varepsilon} (1 + |\log \varepsilon|)^{1/2}, \quad i = 1, 2.$$

Proof. After the time change $t \mapsto t - \tau$ the problem reads

$$\begin{cases} \partial_t \hat{\rho} - \varepsilon \Delta \hat{\rho} + \operatorname{div}(b\hat{\rho}) = 0 & \text{in } \mathbb{T}^n \times (0, T - \tau), \\ \hat{\rho}(0) = \delta_x \end{cases}$$

We can apply [BKRS15, Corollary 7.2.3] with the choice $A = \varepsilon \mathbb{I}_n$, $B = \hat{b}$, $c = 0$, $\lambda_0 = \varepsilon$, $\Theta = \frac{1}{2}$, and any $\nu > (n + 2)/2$ to get

$$\begin{aligned} \hat{\rho}(x, t) &\leq C_{\nu, n} \left(1 + \frac{1}{\varepsilon}\right)^\nu t^{-\frac{n+2}{2}} \int_{t/2}^t \int_{\mathbb{T}^n} \left[1 + \varepsilon^\nu + \frac{t^{2\nu}}{\varepsilon^\nu} |\hat{b}|^{2\nu}\right] \hat{\rho} dy ds \\ &\leq C_{\nu, n, K} \left(1 + \frac{1}{\varepsilon}\right)^\nu t^{-\frac{n+2}{2}} \left[1 + \frac{t^{2\nu}}{\varepsilon^\nu}\right] \int_{t/2}^t \int_{\mathbb{T}^n} \hat{\rho} dy ds = C_{\nu, n, K} \left(1 + \frac{1}{\varepsilon}\right)^\nu t^{-\frac{n}{2}} \left[1 + \frac{t^{2\nu}}{\varepsilon^\nu}\right] \end{aligned}$$

for all $(x, t) \in \mathbb{T}^n \times (0, \tau)$. Therefore,

$$|\log \hat{\rho}(x, t)| \leq C_1 \left(1 + |\log \varepsilon| + |\log t| + \left|\log \frac{t^2}{\varepsilon}\right|\right),$$

yielding by the conservation of mass

$$(15) \quad \int_{\mathbb{T}^n} \hat{\rho}(t) |\log \hat{\rho}(y, t)| dy \leq C_1 \left(1 + |\log \varepsilon| + |\log t| + \left|\log \frac{t^2}{\varepsilon}\right|\right).$$

Let us now test the equation for $\hat{\rho}$ by $\log \hat{\rho}$ and integrate to obtain, for $t \in (0, \tau)$,

$$\frac{d}{dt} \int_{\mathbb{T}^n} \hat{\rho}(t) \log \hat{\rho}(t) dx + \varepsilon \int_{\mathbb{T}^n} \frac{|D\hat{\rho}(t)|^2}{\hat{\rho}(t)} dx = \int_{\mathbb{T}^n} (\hat{b} D\hat{\rho})(t) dx,$$

hence by Young's inequality and the conservation of mass

$$\frac{d}{dt} \int_{\mathbb{T}^n} \hat{\rho}(t) \log \hat{\rho}(t) dx + \frac{\varepsilon}{2} \int_{\mathbb{T}^n} \frac{|D\hat{\rho}(t)|^2}{\hat{\rho}(t)} dx \leq \frac{1}{2\varepsilon} \int_{\mathbb{T}^n} |\hat{b}|^2 \hat{\rho}(t) dx \leq \frac{K^2}{2\varepsilon}.$$

Integrating on intervals $(t_1, t_2) \subset (0, \tau)$ and plugging in (15) gives then

$$(16) \quad \varepsilon \int_{t_1}^{t_2} \int_{\mathbb{T}^n} \frac{|D\hat{\rho}(t)|^2}{\hat{\rho}(t)} dt \leq C_2 \left(1 + |\log \varepsilon| + |\log t_1| + |\log t_2| + \left| \log \frac{t_1^2}{\varepsilon} \right| + \left| \log \frac{t_2^2}{\varepsilon} \right| + \frac{t_2 - t_1}{\varepsilon} \right).$$

Choose now $t_1 = \varepsilon$ and $t_2 = 2\varepsilon$. By the Mean Value Theorem there exists $\hat{t} \in [\varepsilon, 2\varepsilon]$ such that

$$\int_{\mathbb{T}^n} \frac{|D\hat{\rho}(\hat{t})|^2}{\hat{\rho}(\hat{t})} dx = \frac{1}{\varepsilon} \int_{\varepsilon}^{2\varepsilon} \int_{\mathbb{T}^n} \frac{|D\hat{\rho}(t)|^2}{\hat{\rho}(t)} dx dt \leq \frac{C_3}{\varepsilon^2} (1 + |\log \varepsilon|).$$

Thus, by Hölder's inequality we get

$$\int_{\mathbb{T}^n} |D\hat{\rho}(\hat{t})| dx \leq \left(\int_{\mathbb{T}^n} \frac{|D\hat{\rho}(\hat{t})|^2}{\hat{\rho}(\hat{t})} dx \right)^{1/2} \left(\int_{\mathbb{T}^n} \hat{\rho}(\hat{t}) dx \right)^{1/2} \leq \frac{C_3^{1/2}}{\varepsilon} (1 + |\log \varepsilon|)^{1/2}.$$

Similarly, by choosing $t_1 = T - \tau - 2\varepsilon$ and $t_2 = T - \tau - \varepsilon$ we have the existence of $\tilde{t} \in [T - \tau - 2\varepsilon, T - \tau - \varepsilon]$ such that

$$\int_{\mathbb{T}^n} |D\hat{\rho}(\tilde{t})| dx \leq \frac{C_4^{1/2}}{\varepsilon} (1 + |\log \varepsilon|)^{1/2}.$$

Going back to the original time variable we obtain the assertion. \square

In what follows, we will use again the adjoint problem for $\rho = \rho_{\varepsilon, \bar{x}, \tau}$, where $\bar{x} \in \mathbb{T}^n$, $\tau \in [0, T]$

$$\begin{cases} \partial_t \rho - \varepsilon \Delta \rho - \operatorname{div}(D_p H(Du_\varepsilon) \rho) = 0 & \text{in } Q_{\tau, T}, \\ \rho(\tau) = \delta_{\bar{x}} \end{cases}$$

Lemma 4.3. *Assume that $0 < \varepsilon \leq 1$ and $\tau \leq T - 4\varepsilon$. Then, there exists C depending on $n, \|Du_\varepsilon(T)\|_{L_x^\infty}, T$, and t_1, t_2 satisfying*

$$t_1 \in [\tau + \varepsilon, \tau + 2\varepsilon], \quad t_2 \in [T - 2\varepsilon, T - \varepsilon]$$

such that

$$\int_{t_1}^{t_2} \int_{\mathbb{T}^n} (t - \tau)(T - t) |D^2 u_\varepsilon|^2 \rho dx dt \leq C(1 + |\log \varepsilon|).$$

Proof. We find, by uniform convexity of H , the following inequality solved by the function $z(x, t) = \chi(t) \Delta u_\varepsilon(x, t)$, where $\chi(t) = (t - \tau)(T - t)$:

$$-\partial_t z - \varepsilon \Delta z + \theta \chi |D^2 u_\varepsilon|^2 + DH(Du_\varepsilon) \cdot Dz \leq -\chi' \Delta u_\varepsilon + \chi \Delta f \text{ in } Q_{\tau, T}.$$

Let now t_1, t_2 be as in Lemma 4.2, applied with $b = -D_p H(Du_\varepsilon)$, that gives

$$(17) \quad \int_{\mathbb{T}^n} |D\rho(t_i)| dx \leq \frac{C}{\varepsilon} (1 + |\log \varepsilon|)^{1/2}, \quad i = 1, 2.$$

for some

$$(18) \quad t_1 \in [\tau + \varepsilon, \tau + 2\varepsilon], \quad t_2 \in [T - 2\varepsilon, T - \varepsilon].$$

By duality between ρ and z and integrating in $\mathbb{T}^n \times (t_1, t_2)$ we have

$$(19) \quad \theta \int_{t_1}^{t_2} \int_{\mathbb{T}^n} \chi |D^2 u_\varepsilon|^2 \rho dx dt = \int_{\mathbb{T}^n} z(t_2) \rho(t_2) dx - \int_{\mathbb{T}^n} z(t_1) \rho(t_1) dx - \int_{t_1}^{t_2} \int_{\mathbb{T}^n} \chi' \Delta u_\varepsilon \rho dx dt + \int_{t_1}^{t_2} \int_{\mathbb{T}^n} \chi \Delta f \rho dx dt.$$

Note that $(\Delta u_\varepsilon)^2 \leq n|D^2 u_\varepsilon|^2$, hence by Young's inequality we get

$$\begin{aligned} - \int_{t_1}^{t_2} \int_{\mathbb{T}^n} \chi' \Delta u_\varepsilon \rho dx dt &\leq \frac{\theta}{2} \int_{t_1}^{t_2} \int_{\mathbb{T}^n} \chi |D^2 u_\varepsilon|^2 \rho dx dt + \frac{n}{2\theta} \int_{t_1}^{t_2} \frac{(\chi')^2}{\chi} \left(\int_{\mathbb{T}^n} \rho dx \right) dt \leq \\ &\quad \frac{\theta}{2} \int_{t_1}^{t_2} \int_{\mathbb{T}^n} \chi |D^2 u_\varepsilon|^2 \rho dx dt + \\ &\quad + \frac{1}{2\theta} (4(t_2 - t_1) + (T - \tau)(|\log(t_2 - \tau)| + |\log(t_1 - \tau)| + |\log(T - t_2)| + |\log(T - t_1)|)). \end{aligned}$$

On the other hand, for $i = 1, 2$, integrating by parts yields, together with (17) and Lipschitz estimates of Lemma 2.3

$$\left| \int_{\mathbb{T}^n} z(t_i) \rho(t_i) dx \right| \leq \chi(t_i) \int_{\mathbb{T}^n} |Du_\varepsilon(t_i)| |D\rho(t_i)| dx \leq (t_i - \tau)(T - t_i) \frac{C(1 + |\log \varepsilon|)^{1/2}}{\varepsilon}.$$

Plugging the previous inequality into (19), and using also (18), we conclude that

$$\frac{\theta}{2} \int_{t_1}^{t_2} \int_{\mathbb{T}^n} \chi |D^2 u_\varepsilon|^2 \rho dx dt \leq C_5 \left((1 + |\log \varepsilon|)^{1/2} + 1 + |\log \varepsilon| \right)$$

for some C_5 depending on $n, \|Du_T\|_{L_x^\infty}, T, \|(\Delta f)^+\|_{L_{x,t}^\infty}$. \square

Proof of Theorem 4.1. As in Theorem 3.1 we have to estimate $\int_\tau^T \int_{\mathbb{T}^n} \Delta u_\varepsilon \rho dx dt$. Let t_1 and t_2 be as in Lemma 4.2 and $\tau \leq T - 4\varepsilon$. Then,

$$\begin{aligned} \left| \int_\tau^T \int_{\mathbb{T}^n} \Delta u_\varepsilon \rho dx dt \right| &\leq \sqrt{n} \int_\tau^T \int_{\mathbb{T}^n} |D^2 u_\varepsilon| \rho dx dt \\ &= \sqrt{n} \int_\tau^{t_1} \int_{\mathbb{T}^n} |D^2 u_\varepsilon| \rho dx dt + \sqrt{n} \int_{t_2}^T \int_{\mathbb{T}^n} |D^2 u_\varepsilon| \rho dx dt + \sqrt{n} \int_{t_1}^{t_2} \int_{\mathbb{T}^n} |D^2 u_\varepsilon| \rho dx dt. \end{aligned}$$

First, since $t_1 - \tau \leq 2\varepsilon$ and $T - t_2 \leq 2\varepsilon$, by Lemma 2.3

$$\int_\tau^{t_1} \int_{\mathbb{T}^n} |D^2 u_\varepsilon| \rho dx dt \leq \left(\int_\tau^{t_1} \int_{\mathbb{T}^n} |D^2 u_\varepsilon|^2 \rho dx dt \right)^{1/2} (t_1 - \tau)^{1/2} \leq \left(\frac{C}{\varepsilon} \right)^{1/2} (2\varepsilon)^{1/2},$$

and similarly

$$\int_{t_2}^T \int_{\mathbb{T}^n} |D^2 u_\varepsilon| \rho dx dt \leq 2C^{1/2}.$$

On the other hand, by Lemma 4.3,

$$\begin{aligned} \left(\int_{t_1}^{t_2} \int_{\mathbb{T}^n} |D^2 u_\varepsilon| \rho dx dt \right)^2 &= \left(\int_{t_1}^{t_2} \int_{\mathbb{T}^n} \frac{(t - \tau)^{1/2} (T - t)^{1/2}}{(t - \tau)^{1/2} (T - t)^{1/2}} |D^2 u_\varepsilon| \rho dx dt \right)^2 \\ &\leq \left(\int_{t_1}^{t_2} \int_{\mathbb{T}^n} (t - \tau)(T - t) |D^2 u_\varepsilon|^2 \rho dx dt \right) \left(\int_{t_1}^{t_2} \frac{1}{(t - \tau)(T - t)} dt \right) \\ &\leq C_6 (1 + |\log \varepsilon|) \frac{\log |t_1 - T| - \log |t_1 - \tau| - \log |t_2 - T| + \log |t_2 - \tau|}{T - \tau} \\ &\leq C_7 (1 + |\log \varepsilon|) \frac{|\log \varepsilon|}{T - \tau}. \end{aligned}$$

Therefore,

$$\left| \int_\tau^T \int_{\mathbb{T}^n} \Delta u_\varepsilon \rho dx dt \right| \leq C_8 \frac{1 + |\log \varepsilon|}{(T - \tau)^{1/2}}.$$

The rate of convergence

$$\|u_\varepsilon - u\|_{L^\infty(Q_T)} \leq \frac{C_9}{(T - \tau)^{1/2}} \varepsilon |\log \varepsilon|$$

is straightforward from Remark 2.2. \square

5. RATE OF CONVERGENCE FOR HAMILTONIANS WITH SUPERQUADRATIC GROWTH AND SEMICONCAVE DATA

In this section we consider strictly convex Hamiltonians with superquadratic growth. We will need to prove a refined version of (10) under (H_γ) : this will be performed through a Bernstein-type argument.

Theorem 5.1. *Let H be such that (H_γ) holds. Then, if u_ε solves (1) and u solves the first-order equation (2), we have for some $\beta_\gamma \in (\frac{1}{2}, 1)$*

$$-C\varepsilon^\beta \leq u_\varepsilon - u \leq (T\|(\Delta u_T)^+\|_{L_x^\infty} + \|(\Delta f)^+\|_{L_t^1(L_x^\infty)})\varepsilon \quad \text{for all } \beta < \beta_\gamma.$$

where C depends on $n, \theta, \gamma, \beta, \|(\Delta u_T)^+\|_{L_x^\infty}, T, \|(\Delta f)^+\|_{L_{x,t}^\infty}, \|Du_T\|_{L^\infty}, \|Df\|_{L_{x,t}^\infty}$ and β_γ depends on γ .

The value of β_γ is explicit, see (23) below.

Proof. The proof of the bound from above is the same of Theorem 3.1, owing to the semiconcavity estimate in [CG19, Proposition 3.7 and Remark 3.8]. We prove the bound from below following similar steps.

Step 1. We prove that for m small enough and $\eta < 1/2$ it holds

$$(20) \quad \iint_{Q_\tau \cap \{|Du|^2 \leq m\}} |D^2 u_\varepsilon|^2 \rho dx dt \leq \frac{C_\eta m^\eta}{\varepsilon}.$$

Let us first observe that for $\tilde{w} = \varphi(|Du|^2)$ we have

$$\Delta \tilde{w} = 4 \sum_i \varphi''(Du_{x_i} \cdot Du)^2 + 2\varphi' D\Delta u \cdot Du + 2\varphi' |D^2 u|^2,$$

which leads to the evolution PDE solved by \tilde{w}

$$(21) \quad -\partial_t \tilde{w} - \varepsilon \Delta \tilde{w} + 2\varepsilon \varphi'(|Du|^2) |D^2 u|^2 + D_p H(Du) \cdot D\tilde{w} = -4\varepsilon \sum_i \varphi''(Du_{x_i} \cdot Du)^2 + 2\varphi' Df \cdot Du.$$

Notice that if φ is concave, we get

$$-\partial_t \tilde{w} - \varepsilon \Delta \tilde{w} + 2\varepsilon \varphi'(|Du|^2) |D^2 u|^2 + D_p H(Du) \cdot D\tilde{w} \leq -4\varepsilon \varphi''(|Du|^2) |Du|^2 |D^2 u|^2 + 2\varphi' Df \cdot Du.$$

Let now $0 < \delta < 1/2$, and choose

$$\varphi(z) = \varphi_m(z) = \begin{cases} z & z \in [0, m] \\ 2(\delta + 1)\sqrt{mz} - \delta z - \delta - 1 & z \in [m, \kappa m], \\ m(1 + 1/\delta) & z \in [\kappa m, +\infty). \end{cases} \quad \kappa = \frac{(1+\delta)^2}{\delta^2} > 1$$

One may verify with a direct computation that φ is nonnegative, increasing, concave and C^1 . Moreover,

$$\begin{aligned} \varphi' &= 1, \quad \varphi'' = 0 \quad \text{on } [0, m] \\ \varphi'(z) + 2\varphi''(z)z &= -\delta \quad \text{on } (m, \kappa m) \\ \varphi' &= \varphi'' = 0 \quad \text{elsewhere.} \end{aligned}$$

Using these properties, and testing the PDE (21) by ρ and integrating we conclude

$$\begin{aligned} 2\varepsilon \iint_{Q_\tau} \varphi' |D^2 u|^2 \rho dx dt &\leq \int_{\mathbb{T}^n} \tilde{w} \rho(x, t) dx \Big|_{t=\tau}^{t=T} - 4\varepsilon \iint_{Q_\tau} \varphi'' |Du|^2 |D^2 u|^2 \rho dx dt \\ &\quad + 2\|Df\|_{L^\infty(Q_\tau)} \iint_{Q_\tau} \varphi' |Du| dx dt, \end{aligned}$$

hence, using $\varphi'(z) + 2\varphi''(z)z = -\delta$ and $-\delta \leq \varphi' \leq 1$ on $(m, \kappa m)$, we have

$$\begin{aligned} \varepsilon \iint_{Q_\tau \cap \{|Du|^2 \leq m\}} |D^2 u|^2 \rho dx dt &\leq m \left(1 + \frac{1}{\delta}\right) + 2\delta \varepsilon \iint_{Q_\tau \cap \{m < |Du|^2 < \kappa m\}} |D^2 u|^2 \rho dx dt \\ &\quad + 2\|Df\|_{L^\infty(Q_\tau)} \iint_{Q_\tau \cap \{|Du|^2 < \kappa m\}} \varphi' |Du| dx dt. \end{aligned}$$

We now proceed by a hole-filling type technique, that is, we “fill the hole” by adding to both sides $2\delta \varepsilon \iint_{Q_\tau \cap \{|Du|^2 < m\}} |D^2 u|^2 \rho$ and obtain

$$\varepsilon \iint_{Q_\tau \cap \{|Du|^2 \leq m\}} |D^2 u|^2 \rho dx dt \leq mc_\delta + \frac{2\delta}{1+2\delta} \varepsilon \iint_{Q_\tau \cap \{|Du|^2 < \kappa m\}} |D^2 u|^2 \rho dx dt + \frac{c_f}{1+2\delta} \sqrt{\kappa m},$$

where $c_\delta > 0$ depends on δ only. If we now let, for $m \in (0, 1)$,

$$h(m) := \frac{\varepsilon \iint_{Q_\tau \cap \{|Du|^2 \leq m\}} |D^2 u|^2 \rho dx dt}{m^\eta},$$

the previous inequality reads, after dividing by m^η and using that $m < 1$ and $\eta < 1/2$, as

$$(22) \quad h(m) \leq c_\delta m^{1-\eta} + \frac{2\delta \kappa^\eta}{1+2\delta} h(\kappa m) + c_f \sqrt{\kappa m}^{\frac{1}{2}-\eta} \leq c_\delta + \frac{2\delta \kappa^\eta}{1+2\delta} h(\kappa m) + c_f \sqrt{\kappa}.$$

Recalling the definition of κ above, on one hand we have $\sqrt{\kappa} = \frac{1+\delta}{\delta}$, and, on the other hand, we can pick δ small enough so that

$$\frac{2\delta}{1+2\delta} \left(\frac{1+\delta}{\delta}\right)^{2\eta} \leq \frac{1}{2}.$$

Consider now the sequence $m_j = 1/\kappa^j$, $j = 1, 2, \dots$ and evaluate the previous inequality on m_j to get

$$h(m_j) \leq \tilde{c}_\delta + \frac{1}{2} h(m_{j-1}),$$

which yields by induction that $h(m_j) \leq 2\tilde{c}_\delta + \frac{1}{2} h(m_1)$ for all $j \geq 2$. Since h is increasing and $h(m_1)$ is bounded by Lemma 2.3, we get the claim.

Step 2. We now proceed by differentiating twice the equation and use (H_γ) to find the following inequality solved by the function $z(x, t) = (t - \tau)^\alpha \Delta u_\varepsilon$, ($\alpha > 0$ to be chosen later)

$$\begin{aligned} -\partial_t z - \varepsilon \Delta z + \theta(t - \tau)^\alpha |Du_\varepsilon|^{\gamma-2} |D^2 u_\varepsilon|^2 + D_p H(Du_\varepsilon) \cdot Dz \\ \leq -\alpha(t - \tau)^{\alpha-1} \Delta u_\varepsilon + \Delta f \text{ in } Q_T. \end{aligned}$$

By duality and integrating in $\mathbb{T}^n \times (\tau, T)$ we have

$$\begin{aligned} \underbrace{\int_{\mathbb{T}^n} z(\tau) \rho(\tau) dx}_{=0} + \theta \int_\tau^T \int_{\mathbb{T}^n} (t - \tau)^\alpha |Du_\varepsilon|^{\gamma-2} |D^2 u_\varepsilon|^2 \rho dx dt &= \underbrace{\int_{\mathbb{T}^n} z(T) \rho(T) dx}_{\leq (T-\tau)^\alpha \|(\Delta u_T)^+\|_{L^\infty(\mathbb{T}^n)}} \\ &\quad - \alpha \int_\tau^T \int_{\mathbb{T}^n} (t - \tau)^{\alpha-1} \Delta u_\varepsilon(x, t) \rho dx dt + \int_\tau^T \int_{\mathbb{T}^n} (t - \tau)^\alpha \Delta f(x, t) \rho dx dt. \end{aligned}$$

The last integral can be bounded as in Theorem 3.1. We split the first integral in two regions (where the gradient is small and large respectively), apply the Hölder inequality and choose $\alpha > 1$

as follows

$$\begin{aligned}
& -\alpha \int_{\tau}^T \int_{\mathbb{T}^n} (t-\tau)^{\alpha-1} \Delta u_{\varepsilon} \rho dx dt = -\alpha \iint_{\{|Du|^2 \leq m\}} (t-\tau)^{\alpha-1} \Delta u_{\varepsilon} \rho dx dt \\
& \quad - \alpha \iint_{\{|Du|^2 > m\}} (t-\tau)^{\alpha-1} \Delta u_{\varepsilon} \rho dx dt \\
& \leq \alpha \sqrt{n} \left(\iint_{\{|Du|^2 \leq m\}} |D^2 u_{\varepsilon}|^2 \rho dx dt \right)^{\frac{1}{2}} \left(\iint_{Q_{\tau}} (t-\tau)^{2\alpha-2} \rho dx dt \right)^{\frac{1}{2}} \\
& \quad + \alpha \sqrt{n} \left(\iint_{\{|Du|^2 > m\}} (t-\tau)^{\alpha} |D^2 u_{\varepsilon}|^2 \rho dx dt \right)^{\frac{1}{2}} \left(\iint_{Q_{\tau}} (t-\tau)^{\alpha-2} \rho dx dt \right)^{\frac{1}{2}} \\
& \leq C_{\alpha, T} \sqrt{C_{\eta}} \sqrt{\frac{m^{\eta}}{\varepsilon}} + \frac{\theta}{2} \iint_{Q_{\tau}} (t-\tau)^{\alpha} |Du_{\varepsilon}|^{\gamma-2} |D^2 u_{\varepsilon}|^2 \rho dx dt + \frac{C_{\alpha, T}}{m^{\frac{\gamma-2}{2}}}.
\end{aligned}$$

This implies

$$\int_{\tau}^T \int_{\mathbb{T}^n} (t-\tau)^{\alpha} |Du_{\varepsilon}|^{\gamma-2} |D^2 u_{\varepsilon}|^2 \rho dx dt \leq \tilde{K}_1 \left(\sqrt{\frac{m^{\eta}}{\varepsilon}} + \frac{1}{m^{\frac{\gamma-2}{2}}} + \|(\Delta f)^+\|_{L^{\infty}(Q_T)} \right).$$

Choosing $m = \varepsilon^{\frac{1}{\eta+\gamma-2}}$ we get the following integral bound on superlevel sets $\{|Du|^2 > m\}$

$$\iint_{Q_{\tau} \cap \{|Du|^2 > m\}} (t-\tau)^{\alpha} |D^2 u_{\varepsilon}|^2 \rho dx dt \leq \tilde{K}_2 \left(\varepsilon^{\frac{2-\gamma}{2(\gamma+\eta-2)}} + 1 \right).$$

We can now conclude the proof. By the foregoing estimates and Lemma 2.3

$$\begin{aligned}
\int_{\tau}^T \int_{\mathbb{T}^n} \Delta u_{\varepsilon} \rho dx dt & \leq \sqrt{n} \int_{t_0}^T \int_{\mathbb{T}^n} (t-\tau)^{\alpha/2} |D^2 u_{\varepsilon}| (t-\tau)^{-\alpha/2} \rho dx dt + \sqrt{n} \int_{\tau}^{t_0} \int_{\mathbb{T}^n} |D^2 u_{\varepsilon}| \rho dx dt \\
& \leq \sqrt{n} \left(\int_{t_0}^T \int_{\mathbb{T}^n} (t-\tau)^{-\alpha} \rho dx dt \right)^{\frac{1}{2}} \left(\int_{t_0}^T \int_{\mathbb{T}^n} |D^2 u_{\varepsilon}|^2 (t-\tau)^{\alpha} \rho dx dt \right)^{\frac{1}{2}} \\
& \quad + \sqrt{n} \frac{C_L}{\sqrt{\varepsilon}} \sqrt{t_0 - \tau} \leq \frac{\sqrt{n}}{|\alpha-1|} ((t_0 - \tau)^{1-\alpha} + (T - \tau)^{1-\alpha})^{\frac{1}{2}} \times \\
& \quad \left(\iint_{\{|Du|^2 \leq m\}} |D^2 u_{\varepsilon}|^2 (t-\tau)^{\alpha} \rho dx dt + \iint_{Q_{\tau} \cap \{|Du|^2 > m\}} |D^2 u_{\varepsilon}|^2 (t-\tau)^{\alpha} \rho dx dt \right)^{\frac{1}{2}} \\
& \quad + \sqrt{n} \frac{C_L}{\sqrt{\varepsilon}} \sqrt{t_0 - \tau} \\
& \leq \tilde{K}_3 \sqrt{n} ((t_0 - \tau)^{1-\alpha} + (T - \tau)^{1-\alpha})^{\frac{1}{2}} \left(\frac{m^{\eta}}{\varepsilon} + \varepsilon^{\frac{2-\gamma}{2(\gamma+\eta-2)}} + 1 \right)^{\frac{1}{2}} \\
& \quad + \sqrt{n} \frac{C_L}{\sqrt{\varepsilon}} \sqrt{t_0 - \tau}.
\end{aligned}$$

With the choice of $m = \varepsilon^{\frac{1}{\eta+\gamma-2}}$ as before we get

$$\int_{\tau}^T \int_{\mathbb{T}^n} \Delta u_{\varepsilon} \rho dx dt \leq \tilde{K}_4 \left((t_0 - \tau)^{1-\alpha} \varepsilon^{\frac{2-\gamma}{2(\gamma+\eta-2)}} + 1 \right) + \sqrt{n} \frac{C_L}{\sqrt{\varepsilon}} \sqrt{t_0 - \tau}.$$

If $\tau + \varepsilon < T$ we choose $t_0 = \tau + \varepsilon$ and α arbitrarily close to 1 and get the following result integrating with respect to ε via Remark 2.2

$$\|u_{\varepsilon} - u\|_{L^{\infty}(Q_T)} \leq \tilde{K}_5 \varepsilon^{\beta}, \quad \beta < \frac{\gamma + 2\eta - 2}{2(\gamma + \eta - 2)}.$$

Then, the claim follows by choosing

$$(23) \quad \beta_{\gamma} = \sup_{\eta \in (0, 1/2)} \frac{\gamma + 2\eta - 2}{2(\gamma + \eta - 2)} = \frac{\gamma - 1}{2\gamma - 3}.$$

□

Remark 5.2 (On the optimal convergence rate for strictly convex Hamiltonians). Note first that

$$\begin{aligned}\beta_\gamma &= \frac{\gamma - 1}{2\gamma - 3} \rightarrow 1 && \text{as } \gamma \rightarrow 2, \\ \beta_\gamma &= \frac{\gamma - 1}{2\gamma - 3} \rightarrow \frac{1}{2} && \text{as } \gamma \rightarrow \infty.\end{aligned}$$

These limit rates are coherent with the rates obtained in the quadratic case and in the locally Lipschitz one. Our results are in the direction suggested by the numerical experiments in [QSTY24], that show a rate of order ε^β , $\beta \in (1/2, 1)$ in the case of strictly convex Hamiltonians. Example 5 therein for instance indicates the order $\mathcal{O}(\varepsilon^{\frac{2}{3}})$, when $H(p) = \frac{1}{4}|p|^4$ and $f = 0$. Our endpoint rate β_γ , once specialized to $\gamma = 4$ becomes $3/5$, that is slightly worse than $2/3$. We do not know what could be the optimal convergence rate for $H(p) = \frac{1}{4}|p|^4$, and we cannot exclude that it may depend on the regularity of f .

Note that, in our argument, the rate could be improved if we could allow η to vary in the wider range $(0, 1)$; for $\eta \rightarrow 1$, the exponent $2/3$ could be reached for quartic Hamiltonians: for general $\gamma > 2$, it would be $\frac{\gamma}{2(\gamma-1)}$. Is this the optimal one? Here, we can allow for $\eta < 1/2$ (see for example equation (22)), but that step of the proof “sees” only the Lipschitz property of f . We do not know now how to improve this step using the information that f is more regular.

Remark 5.3 (On the convergence rate for some Mean Field Control problems). The issue of the convergence problem in Mean Field Control has been in the last few years an active area of research, see for instance [DDJ24] and references therein. In general, it amounts to study the convergence of (symmetric) value functions of some control problems as the dimension n of the Euclidean space increases, towards a limit value function, which satisfies an equation in the space of probability measures. In some special cases, the problem can be recast into a vanishing viscosity approximation of (finite-dimensional) Hamilton-Jacobi equations, where in particular the viscosity satisfies $\varepsilon = \varepsilon_n = 1/n$ (see in particular [DDJ24, Proposition 2.10]). The convergence rate in Mean Field Control is known in general (under Lipschitz regularity assumptions) to be of order $\mathcal{O}(1/\sqrt{n})$; under convexity properties required by Theorems 3.1, 4.1 or 5.1, an improved rate of order $1/n^\beta$, $\beta \in (1/2, 1)$, or $(1/n)|\log(1/n)|$ shows up in the specific examples described in [DDJ24].

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