

CONSISTENCY FOR THE SURFACE DIFFUSION FLAT FLOW IN THREE DIMENSIONS

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ABSTRACT. We investigate the flat flow solution for the surface diffusion equation via the discrete minimizing movements scheme proposed by Cahn and Taylor [9]. We prove that in dimension three the scheme converges to the unique smooth solution of the equation, provided that the initial set is sufficiently regular.

1. INTRODUCTION

In this paper we study the evolution of smooth sets governed by the surface diffusion equation

$$(1.1) \quad V_t = \Delta_{\partial E_t} H_{E_t} \quad \text{on } \partial E_t,$$

where V_t is the normal velocity, H_{E_t} the mean curvature and $\Delta_{\partial E_t}$ the Laplace-Beltrami operator. We adopt the convention where the unit normal vector is outward and H_{E_t} is nonnegative for convex sets. It is immediate that this geometric flow preserves the volume of every component of the evolving sets, while decreasing the perimeter.

The equation was proposed first by Mullins [33] as a model to describe the development of surface grooves at the grain boundaries of a heated polycrystal. In the physically relevant three dimensional case it describes the evolution of interfaces in a variety of physical settings, including phase transition, epitaxial deposition and grain growth. The derivation of (1.1) was later revisited by Davì and Gurtin [14] and by Cahn and Taylor [9], who extended it to a more general mathematical context (see also [22]). In fact, Cahn and Taylor derived a more general equation of type

$$V_t = (\delta - \varepsilon \Delta_{\partial E_t})^{-1} \Delta_{\partial E_t} H_{E_t} \quad \text{on } \partial E_t.$$

Letting $\varepsilon \rightarrow 0$ one obtains the surface-diffusion (with a constant δ), while $\delta \rightarrow 0$ yields the volume preserving mean curvature flow

$$(1.2) \quad V_t = -\varepsilon(H_{E_t} - \bar{H}_{E_t}) \quad \text{on } \partial E_t.$$

In [9] Cahn and Taylor also pointed out that the surface diffusion equation could be obtained as the sharp-interface limit ($\varepsilon \rightarrow 0$) of the Cahn-Hilliard equation

$$(1.3) \quad \partial_t u = \operatorname{div} \left(m(u) \nabla (-\varepsilon^2 \Delta u + W'(u)) \right),$$

where W is a double well potential and $m(\cdot)$ is the mobility. This fact was formally proven in low dimensions \mathbb{R}^n , $n \leq 3$, by Cahn, Elliot and Novik-Cohen in [8] with the mobility $m(u) = 1 - u^2$, the potential

$$W(u) = \varepsilon^\alpha ((1+u) \log(1+u) + (1-u) \log(1-u)) + (1-u^2),$$

and by rescaling the time $t \mapsto \varepsilon^2 t$. To the best of our knowledge the rigorous argument for this convergence is still missing, nor it is known that the limit yields a weak solution to (1.1). We remark that by choosing the constant mobility $m(u) = 1$, the standard double well potential $W(u) = \frac{1}{4}(1-u^2)^2$ and rescaling the time $t \mapsto \varepsilon t$ yields the Mullins-Sekerka flow as a limit as $\varepsilon \rightarrow 0$, which was first formally derived by Pego [34], and then rigorously proven by Alikakos, Bates and Chen [3]. We also mention the recent work [23] where the authors consider a more general type Cahn-Hilliard equation and its sharp interface limit.

For regular initial sets, the short time existence of a classical solution of (1.1) was first proven in the planar case in [6, 17, 21] and later in every dimension by Escher, Mayer and Simonett [18] for $C^{2,\alpha}$ -regular initial sets. The result in [18] holds even for immersed surfaces and the authors prove the global existence and exponential convergence for initial sets close to the ball. In the flat torus \mathbb{T}^n similar long time existence and convergence results near any stable critical set have been obtained in [1], for $n = 3$, and in [16] for $n \geq 4$.

One expects that for generic initial sets the equation (1.1) forms singularities in finite time. Moreover, unlike the curve shortening flow, the flow under surface diffusion for embedded curves may exhibit self-intersections in finite time as shown in [21]. For immersed curves, singularities are also expected, as supported by the numerical simulation in [18]. We stress that in this paper, we require that the evolution in (1.1) is given by smooth and bounded sets and not immersed surfaces.

The construction of a weak solution to (1.1), which exists globally in time, presents a significant challenge due to the fourth-order nature of the equation and the absence of a maximum principle. A possible candidate for a weak solution, referred to as the flat curvature flow, was introduced by Cahn and Taylor in [9]. The starting point is the fact that (1.1) can be seen as the H^{-1} -gradient flow of the area functional, see [38]. One may use the gradient flow structure to construct the solution using a minimizing movement scheme similar to the one proposed by Almgren, Taylor and Wang [4] and Luckhaus and Stürzenhecker [27] for the mean curvature flow. More precisely, given any initial bounded set of finite perimeter $E_0 \subset \mathbb{R}^n$ and a small time step $h > 0$, one inductively defines $E_0^h = E_0$ and E_k^h for $k = 1, 2, \dots$ as a minimizer of the functional

$$(1.4) \quad P(F) + \frac{d_{H^{-1}}(F; E_{k-1}^h)^2}{2h},$$

where

$$(1.5) \quad d_{H^{-1}}(F; E) = \sup_{\|\nabla_{\partial E} f\|_{L^2(\partial E)} \leq 1} \int_{\mathbb{R}^n} f(\pi_{\partial E}(x)) (\chi_F(x) - \chi_E(x)) dx.$$

Above, $\nabla_{\partial E}$ denotes the tangential gradient, χ_E the characteristic function of E and $\pi_{\partial E}$ the projection on the boundary ∂E . Note that if $d_{H^{-1}}(F; E)$ is finite, then necessarily $|F| = |E|$, ensuring the scheme to be volume preserving. However, (1.5) does not define a real distance, nor does $d_{H^{-1}}(F; E) = 0$ in general imply $F = E$. Then Cahn and Taylor set $E^h(t) = E_k^h$ for $t \in [kh, (k+1)h)$ and define a flat flow as any cluster point of $E^h(t)$ as $h \rightarrow 0$. To the best of our knowledge, even the existence of a minimizer of the functional (1.4) is not known, let alone the convergence of the scheme as $h \rightarrow 0$ to a short time smooth solution of (1.1) when the initial set E_0 is smooth. In order to highlight the difficulty of the problem, let us consider the functional (1.4) in the flat torus \mathbb{T}^n and the associated minimum problem

$$\inf \left\{ P_{\mathbb{T}^n}(F) + \frac{1}{2h} \left(\sup_{\|\nabla_{\partial E} f\|_{L^2(\partial E)} \leq 1} \int_{\mathbb{T}^n} f(\pi_{\partial E}(x)) (\chi_F(x) - \chi_E(x)) dx \right)^2 \right\}.$$

In the case of the strip $E = (0, 1)^{n-1} \times (0, \frac{1}{2}) \subset \mathbb{T}^n$, minimizers of the above problem are E and $\mathbb{T}^n \setminus E$ for every h . This shows that at least in the case of flat torus, the scheme may fail to converge as $h \rightarrow 0$ unless the minimizer is chosen appropriately.

The only result related to the existence of the flat flow for the surface diffusion is contained in the thesis of Kin Yan Chung [12], where the author replaces the distance (1.5) with

$$(1.6) \quad \tilde{d}_{H^{-1}}(F; E) = \sup \left\{ \int_{\mathbb{R}^n} f(x) (\chi_F(x) - \chi_E(x)) dx : f : \mathbb{R}^n \rightarrow \mathbb{R}, \text{Lip}(f) \leq \Lambda, \|\nabla_{\partial E} f\|_{L^2(\partial E)} \leq 1 \right\}.$$

Using (1.6) in place of (1.5) Chung shows the existence and continuity in time of the limiting flow of the minimizing movement scheme as $h \rightarrow 0$ and $\Lambda \rightarrow \infty$ (or by choosing $\Lambda = h^{-\alpha}$ for some $\alpha > 0$). If F and E are smooth sets, whose Hausdorff distance is of order h , then it holds

$$|\tilde{d}_{H^{-1}}(F; E) - d_{H^{-1}}(F; E)| \leq Ch^2,$$

for a constant that depends on E and Λ , which means that (1.6) is formally a small perturbation of the original distance (1.5). However, due to the double constraint it is not clear if the definition (1.6) in the minimizing movement scheme provides even formally the surface diffusion (1.1) as a limit as $h \rightarrow 0$. Therefore the existence of a flat flow solution which is defined globally in time and agrees with the classical solution for its existence time was still an open problem.

The main result of this paper concerns the consistency of the scheme proposed by Cahn and Taylor in the physically relevant three dimensional case. More precisely, given a C^5 -regular bounded initial set $E_0 \subset \mathbb{R}^3$, we fix a small $\delta > 0$, set $E_0^{h,\delta} = E_0$ and define $E_k^{h,\delta}$ as a minimizer of the following incremental minimum problem

$$(1.7) \quad \inf \left\{ P(F) + \frac{d_{H^{-1}}(F; E_{k-1}^{h,\delta})^2}{2h} : F \Delta E_{k-1}^{h,\delta} \subset n_\delta(\partial E_{k-1}^{h,\delta}) \right\},$$

where $n_\delta(\Gamma)$ denotes the tubular neighborhood of a set $\Gamma \subset \mathbb{R}^3$ (see (2.1) for the definition). Due to the constraint condition $F \Delta E_{k-1}^{h,\delta} \subset n_\delta(\partial E_{k-1}^{h,\delta})$ the existence of a minimizer of the

above problem now easily follows from the direct methods of the Calculus of Variations. However, one of the key point of our proof is to show by means of quantitative geometric estimates, that any minimizer $E_k^{h,\delta}$ of (1.7) satisfies

$$(1.8) \quad E_k^{h,\delta} \Delta E_{k-1}^{h,\delta} \subset \mathcal{N}_{\frac{\delta}{2}}(\partial E_{k-1}^{h,\delta}),$$

when h is sufficiently small, provided $E_{k-1}^{h,\delta}$ is sufficiently regular. Therefore the additional constraint in (1.7) is merely a technical tool needed to overcome the lack of coercivity in the original functional (1.4). We note that our main theorem holds also in the flat torus \mathbb{T}^3 , in which case the constraint in (1.7) ensures that we choose a proper minimizer of the problem. We define the approximate flat flow $E^{h,\delta}(t)$ for all $t \geq 0$ setting $E^{h,\delta}(t) = E_k^{h,\delta}$ for $t \in [kh, (k+1)h)$ and define the constrained flat flow $(E^\delta(t))_{t \geq 0}$ as any cluster point of $E^{h,\delta}(t)$ as $h \rightarrow 0$ (see the precise definition in Section 4).

Our main result is the short time regularity and the consistency of the minimizing movement scheme defined above.

Theorem 1.1. *Assume $E_0 \subset \mathbb{R}^3$ is a bounded and C^5 -regular set. There exist δ_0, σ_0 and C_0 with the following property: for every $\delta < \delta_0$ there exists h_0 such that the sets $E^{h,\delta}(t)$ are uniformly bounded in H^4 , i.e.,*

$$\partial E^{h,\delta}(t) = \{x + u^{h,\delta}(x, t) \nu_{E_0}(x) : x \in \partial E_0\}, \quad \|u^{h,\delta}(\cdot, t)\|_{H^4(\partial E_0)} \leq C_0$$

for all $t \in [0, \sigma_0]$ and $h \leq h_0$. Moreover, the functions $u^{h,\delta}$ converge in $L^\infty(0, \sigma_0; C^{2,\alpha}(\partial E_0))$ for all $0 < \alpha < 1$ to a function u^δ such that the family $(E^\delta(t))_{t \in [0, \sigma_0]}$ with

$$\partial E^\delta(t) = \{x + u^\delta(x, t) \nu_{E_0}(x) : x \in \partial E_0\}$$

is the unique smooth solution of (1.1) starting from E_0 .

Note that, in particular, the above result shows that the limit flow $(E^\delta(t))$ is the same for all $\delta < \delta_0$, which again indicates that the constraint in (1.7) is irrelevant when $t \in [0, \sigma_0]$. We note that the novelty of Theorem 1.1 is the uniform higher regularity of the discrete flow, from where the consistency follows rather easily due to the uniqueness of the smooth solution. In case the classical solution of (1.7) is defined in a larger time interval $[0, T_0)$, $T_0 > \sigma_0$, we can prove the full consistency.

Theorem 1.2. *Assume $E_0 \subset \mathbb{R}^3$ is as in Theorem 1.1 and assume the classical solution (E_t) of (1.1) starting from E_0 exists in the time interval $[0, T_0)$. Then for every $T < T_0$ there exists $\delta(T)$ such that the constrained flat flow $(E^\delta(t))_{t \geq 0}$ agrees with E_t in $[0, T]$ for all $\delta < \delta(T)$, i.e., $E_t = E^\delta(t)$ for all $t \in [0, T]$. In addition, the sets $E^{h,\delta}(t)$ converge to E_t in $C^{2,\alpha}$ as $h \rightarrow 0$ uniformly in $[0, T]$, for all $0 < \alpha < 1$.*

Note that our theorems are stated in \mathbb{R}^3 , but the results hold also in \mathbb{R}^2 following the same proof, with much less technical difficulties.

As already mentioned, we are not aware of previous results on consistency of the flat flow solution or, in fact, any weak solution for the surface diffusion. Construction of a varifold solution via minimizing movement scheme to the Willmore flow, which is somewhat similar to the surface diffusion flow, is proven in [7], but even in this case the consistency is not known. The situation is better understood in the case of Mullins-Sekerka, where the flat flow solution constructed by Luckhaus and Stürzenhecker [27] is known to be a weak solution of the associated equation in low dimension $n \leq 3$, due to results by Röger and Schätzle [35, 36] and the consistency, or the weak-strong uniqueness, is known to hold in the planar case due to the recent result proven in [19]. For the volume preserving mean curvature flow (1.2), the third author with Niinikoski [25] proved the consistency of the flat flow solution constructed in [32]. Finally we recall that the consistency of the flat flow for the mean curvature flow follows from the comparison principle for the level set solutions [10, 11].

1.1. Overview of the proof of Theorem 1.1. The crucial step in the proof of Theorem 1.1 is to show that the speed of the discrete flow $E^{h,\delta}(t)$ is bounded in a suitable distance, i.e., the distance of the subsequent sets is linear in h . This is not too difficult in the case when the dissipation is given by an L^2 -type distance as in the case of the mean curvature flow and volume preserving mean curvature flow [25, Proposition 3.1], where it follows from a direct comparison argument. This argument cannot be applied to the case when the dissipation is given by the H^{-1} -distance, which contains less geometric information than the L^2 -distance. In order to solve this issue we recall first that if a set E satisfies the uniform ball condition, i.e., interior and exterior ball condition, with radius r , then it is a Λ -minimizer of the perimeter in the sense that for all sets of finite perimeter F it holds (see [2, Lemma 4.1])

$$P(E) \leq P(F) + \Lambda |E \Delta F|.$$

The above estimate is almost sharp in the sense that any Λ -minimizer of the perimeter is $C^{1,\alpha}$ -regular for any $\alpha \in (0, 1)$ in \mathbb{R}^n for $n \leq 7$. In Proposition 3.1 we prove that if $E \subset \mathbb{R}^3$ satisfies the uniform ball condition with radius r and in addition $\|\Delta_{\partial E} H_E\|_{L^2(\partial E)} \leq C$, then for all sets of finite perimeter F it holds

$$P(E) \leq P(F) + \Lambda d_{H^{-1}}(F; E),$$

where $d_{H^{-1}}$ is defined in (1.5). This result can be seen as a higher order version of the Λ -minimality. Applying it to the minimization problem (1.7) we immediately obtain that the $d_{H^{-1}}$ -distance between the sets $E_{k-1}^{h,\delta}$ and $E_k^{h,\delta}$ is linear in h as long as they satisfy the regularity assumptions in Proposition 3.1.

The second step is to show that we are able to keep the regularity assumed in Proposition 3.1 for a short time. To this aim we first prove in Proposition 4.1 that the sets $E_k^{h,\delta}$ are (Λ, α) -minimizers of the perimeter (see Definition 2.6) and use the well known results of [37] to deduce that when the constraint parameter δ is small enough, the sets $E_k^{h,\delta}$ are uniformly C^1 -regular with mean curvature bounded in H^1 , and can be written as a normal deformation over E_0 as

described in Theorem 1.1. This, together with the linear estimate $d_{H^{-1}}(E_k^{h,\delta}; E_{k-1}^{h,\delta}) \leq Ch$, implies the estimate (1.8), which is important since it means that the constraint in (1.7) is not active and that we may write the Euler-Lagrange equation. We point out that the argument for Proposition 3.1 can be generalized to higher dimension, but the proof of Proposition 4.1 is constrained to low dimensions. The difficulties are similar to those of the Mullins-Sekerka flow [35, 36], where the results for the flat flow hold in \mathbb{R}^3 .

In order to get a uniform bound for $\|\Delta_{\partial E_k^{h,\delta}} H_{E_k^{h,\delta}}\|_{L^2}$, which we need for Proposition 3.1, we use the fact that the Euler-Lagrange equation (4.11) is at each time step a fourth order elliptic equation. This allows us to upgrade the H^3 -regularity to H^4 (see Lemmas 4.2 and 4.3), with estimates which however are not uniform in time. The final step is to prove an iteration lemma (Lemma 5.1), where we use the parabolic nature of the flow and the estimates from the previous lemmas, to obtain H^4 -estimates uniform in time. With these high order estimates at hand, it is not surprising for the experts that Theorems 1.1 and 1.2 follow, but the proofs still require delicate technical arguments, which we chose to give in detail.

2. PRELIMINARIES

2.1. Notation. We denote the Lebesgue measure of a set $E \subset \mathbb{R}^n$ by $|E|$. We denote the distance function by $\text{dist}_E(x) := \inf_{y \in E} |x - y|$ and the signed distance function by $d_E : \mathbb{R}^n \rightarrow \mathbb{R}$, which is defined as

$$d_E(x) := \begin{cases} \text{dist}_E(x), & \text{for } x \in \mathbb{R}^n \setminus E \\ -\text{dist}_{\mathbb{R}^n \setminus E}(x), & \text{for } x \in E. \end{cases}$$

Clearly it holds $\text{dist}_{\partial E} = |d_E|$. If for a given point $x \in \mathbb{R}^n$ there is a unique distance minimizer y on ∂E (that is $\text{dist}_{\partial E}(x) = |x - y|$), we call y the projection of x onto ∂E and set $y = \pi_{\partial E}(x)$. Given $\sigma > 0$ we define the tubular neighborhood of ∂E as

$$(2.1) \quad n_\sigma(\partial E) := \{x \in \mathbb{R}^n : \text{dist}_{\partial E}(x) < \sigma\}.$$

We denote the ball centred at x with radius r by $B_r(x)$ and by B_r if it is centred at the origin. We say that a set $E \subset \mathbb{R}^n$ satisfies the uniform ball condition (UBC) with a given radius $r > 0$, if it simultaneously satisfies the exterior and interior ball condition with radius r at every boundary point. To be more precise, for every $x \in \partial E$ there are balls $B_r(x_+)$ and $B_r(x_-)$ such that

$$B_r(x_+) \subset \mathbb{R}^n \setminus E, \quad B_r(x_-) \subset E \quad \text{and} \quad x \in \partial B_r(x_+) \cap \partial B_r(x_-).$$

We will mostly deal with regular and bounded sets $E \subset \mathbb{R}^n$. We denote the outer unit normal by ν_E . As usual, a bounded set $E \subset \mathbb{R}^n$ is said to be $C^{k,\alpha}$ -regular, for some integer $k \geq 1$ and an exponent $0 < \alpha \leq 1$, if its boundary is a $C^{k,\alpha}$ -regular hypersurface, i.e., for every $x \in \partial E$ we may find a radius $r > 0$ and a function $f \in C^{k,\alpha}(\mathbb{R}^{n-1})$ such that, up to a

rotation,

$$E \cap B_r(x) = \{(y', y_n) \in \mathbb{R}^n : y_n < f(y')\} \cap B_r(x).$$

We say E is smooth if $k = \infty$. If E satisfies the UBC with radius r_0 , then by [25, Proposition 2.7] (see also [13, Theorem 2.6] and [31, Proposition 2.1]) the above holds for $r = r_0/2$, i.e., up to a rotation

$$(2.2) \quad E \cap B_{r_0/2}(x) = \{(y', y_n) \in \mathbb{R}^n : y_n < f(y')\} \cap B_{r_0/2}(x),$$

and the function f satisfies $\|f\|_{C^{1,1}(B_{r_0/2}^{n-1}(x'))} \leq \frac{10}{r_0}$, assuming $r_0 \leq 1$. In particular, the radius is independent of the point x . Here by $B_r^{n-1}(x')$ we denote a ball in \mathbb{R}^{n-1} . We say that E is uniformly $C^{k,\alpha}$ -regular for $k \geq 2$ with constants r_0, C_0 , if it satisfies the UBC with radius r_0 and the function in (2.2) satisfies $\|f\|_{C^{k,\alpha}(B_{r_0/2}^{n-1}(x'))} \leq C_0$. The notions of C^k -regular and uniformly C^k -regular function are defined similarly.

We define the matrix-valued function $P_{\partial E} : \partial E \rightarrow \mathbb{R}^n \otimes \mathbb{R}^n$ by setting $P_{\partial E} = I - \nu_E \otimes \nu_E$. For a given point $x \in \partial E$ the map $P_{\partial E}(x)$ is the orthogonal projection onto the tangent hyperplane $T_x \partial E := \langle \nu_E(x) \rangle^\perp$.

For a given map $X \in C^1(\mathbb{R}^n; \mathbb{R}^m)$ we define its tangential differential on ∂E as a matrix-valued function $\nabla_{\partial E} X : \partial E \rightarrow \mathbb{R}^m \otimes \mathbb{R}^n$ by setting

$$\nabla_{\partial E} X := \nabla X P_{\partial E} = \nabla X - (\nabla X \nu_E) \otimes \nu_E$$

and, by slight abuse of notation, the tangential gradient of a real valued function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\nabla_{\partial E} u = P_{\partial E} \nabla u$. In the case $m = n$, the tangential divergence of X is defined as $\text{div}_{\partial E} X := \text{Tr}(\nabla_{\partial E} X)$. In order to define the tangential Jacobian we denote the matrix associated with the natural inclusion by $I_x : T_x \partial E \rightarrow \mathbb{R}^n$, $I_x \tau = \tau$ for $\tau \in T_x \partial E$, and define the tangential Jacobian as $J_{\partial E} X(x) := \sqrt{\det((\nabla_{\partial E} X(x) I_x)^T (\nabla_{\partial E} X(x) I_x))}$. If E is C^2 -regular, we may define the tangential differential for maps $X : \partial E \rightarrow \mathbb{R}^m$ by extending them in a tubular neighborhood of ∂E as $\tilde{X} := X \circ \pi_{\partial E}$ and defining $\nabla_{\partial E} X := \nabla_{\partial E} \tilde{X}$ (see the discussion on the regularity of the projection below).

If E is C^2 -regular, we define the symmetric matrix associated with the second fundamental form as $B_E := \nabla_{\partial E} \nu_E$. Note that this is not the standard definition for the second fundamental form as B_E is an $n \times n$ -matrix with one eigenvalue always zero, but for our purposes this is the most convenient choice of definition. The mean curvature can be then written as $H_E = \text{Tr}(B_E) = \text{div}_{\partial E} \nu_E$ and the Gaussian curvature $K_E(x) = \det(I_x^T B_E(x) I_x)$, where $I_x : T_x \partial E \rightarrow \mathbb{R}^n$ is the inclusion defined above. We remark that for $E \subset \mathbb{R}^3$ we may write $B_E(x)$ for every $x \in \partial E$ as

$$B_E(x) = \kappa_1 \tau_1 \otimes \tau_1 + \kappa_2 \tau_2 \otimes \tau_2,$$

where κ_1, κ_2 are the principal curvatures and τ_1, τ_2 are the associated principal directions on the tangent hyperplane $T_x \partial E$. Therefore it is clear that our definition of B_E is equivalent to the standard notion of the second fundamental form and, in particular, it holds $H_E = \kappa_1 + \kappa_2$

and $K_E = \kappa_1 \kappa_2$. We use throughout the paper the fact that if E satisfies the UBC with radius r , then $|B_E(x)| \leq \frac{n}{r}$. We have the divergence theorem for all $X \in C^1(\partial E; \mathbb{R}^n)$

$$\int_{\partial E} \operatorname{div}_{\partial E} X \, d\mathcal{H}^{n-1} = \int_{\partial E} H_E(X \cdot \nu_E) \, d\mathcal{H}^{n-1}.$$

Finally, we define the Laplace-Beltrami operator for a given $u \in C^2(\partial E)$ as

$$\Delta_{\partial E} u := \operatorname{div}_{\partial E}(\nabla_{\partial E} u).$$

If a bounded set E satisfies the UBC with radius r , then the projection $\pi_{\partial E}$ is well defined in $\mathcal{N}_r(\partial E)$ and can be written as

$$\pi_{\partial E}(x) = x - d_E(x) \nabla d_E(x).$$

If in addition E is C^2 -regular, then $d_E \in C^2(\mathcal{N}_r(\partial E))$ and for any $x \in \mathcal{N}_r(\partial E)$ its Hessian at x is given by

$$(2.3) \quad \nabla^2 d_E(x) = B_E(\pi_{\partial E}(x))(I + d_E(x) B_E(\pi_{\partial E}(x)))^{-1},$$

see [25, (2.33)]. We note that it is necessary to define the second fundamental form as the $n \times n$ -matrix $\nabla_{\partial E} \nu_E$ for the formula (2.3) to hold. Note also that the projection $\pi_{\partial E}$ is C^1 -regular and we may write its differential using (2.3) and the fact that $\nabla d_E(x) = \nu_E(\pi_{\partial E}(x))$

$$(2.4) \quad \begin{aligned} \nabla \pi_{\partial E}(x) &= I - \nabla d_E \otimes \nabla d_E - d_E \nabla^2 d_E \\ &= I - (\nu_E \circ \pi_{\partial E}) \otimes (\nu_E \circ \pi_{\partial E}) - d_E (B_E \circ \pi_{\partial E})(I + d_E(B_E \circ \pi_{\partial E}))^{-1}. \end{aligned}$$

We need also some notation from Riemannian geometry and as an introduction to the topic we refer to [26]. Let us assume that $E \subset \mathbb{R}^n$ is a smooth and bounded set and set $\Sigma = \partial E$. Since Σ is embedded in \mathbb{R}^n it has a natural metric g induced by the Euclidean metric. Then (Σ, g) is a Riemannian manifold and we denote the inner product of any two vectors X, Y in the tangent space $T_x \Sigma$ by $\langle X, Y \rangle$. Then, in local coordinates we have

$$\langle X, Y \rangle = g(X, Y) = g_{ij} X^i Y^j.$$

We extend the inner product in a natural way for tensors, while we denote by $x \cdot y$ the inner product of two vectors in \mathbb{R}^n . We denote the space of smooth vector fields on Σ by $\mathcal{T}(\Sigma)$ and by a slight abuse of notation we denote the space of smooth k -th order tensor fields on Σ by $\mathcal{T}^k(\Sigma)$. We denote the Riemannian connection on Σ by $\bar{\nabla}$ and recall that for a function $u \in C^\infty(\Sigma)$ the covariant derivative $\bar{\nabla} u$ is a covector field defined for $X \in \mathcal{T}(\Sigma)$ as

$$\bar{\nabla} u(X) := \bar{\nabla}_X u = Xu,$$

i.e., the derivative of u in the direction of X . We denote the covariant derivative of a smooth k -tensor field $F \in \mathcal{T}^k(\Sigma)$ by $\bar{\nabla} F$, which is a $(k+1)$ -tensor field, for the definition see [26], and the k -th order covariant derivative of a function u on Σ by $\bar{\nabla}^k u \in \mathcal{T}^k(\Sigma)$.

We will always identify the tangent hyperplane $T_x \partial E := \langle \nu_E(x) \rangle^\perp$ and the tangent space $T_x \Sigma$ as defined in Differential Geometry. With this convention the tangential gradient of

$u : \Sigma = \partial E \rightarrow \mathbb{R}$ is equivalent to its covariant derivative in the sense that for every tangent vector $\tau \in T_x \partial E$ it holds

$$\bar{\nabla} u(\tau) = \nabla_{\partial E} u \cdot \tau.$$

Since Σ is embedded in \mathbb{R}^n we may define the second fundamental form and denote it by B_Σ . Note that in this setting $B_\Sigma(x)$ for $x \in \Sigma$ is understood as a bilinear form in $T_x \Sigma \times T_x \Sigma$.

We define the Sobolev space $W^{l,p}(\Sigma)$ in a standard way for $p \in [1, \infty]$, see e.g. [5], denote the Hilbert space $H^l(\Sigma) = W^{l,2}(\Sigma)$ and define the associated norm for $f \in W^{l,p}(\Sigma)$ as

$$\|f\|_{W^{l,p}(\Sigma)}^p = \sum_{k=0}^l \int_{\Sigma} |\bar{\nabla}^k f|^p d\mathcal{H}^{n-1}$$

and for $p = \infty$

$$\|f\|_{W^{l,\infty}(\Sigma)} = \sum_{k=0}^l \operatorname{ess\,sup}_{x \in \Sigma} |\bar{\nabla}^k f|.$$

The above definition extends naturally to tensor fields. We adopt the convention that $\|u\|_{H^0(\Sigma)} = \|u\|_{L^2(\Sigma)}$ and denote $\|u\|_{C^m(\Sigma)} = \|u\|_{W^{m,\infty}(\Sigma)}$ for $m \geq 0$. We remark that we may define the space $W^{k,p}(\Sigma)$ for $k \geq 2$ as above assuming that Σ (i.e., the set E for which $\Sigma = \partial E$) is $C^{k-1,1}$ -regular. We remark that we may define the first order Sobolev space for real valued functions by using the tangential derivatives and we will do this throughout the paper. However, when we need higher order Sobolev spaces or Sobolev space for tensor fields we need to rely on covariant derivatives.

Finally, we define the Hölder norm of a continuous function $u : \Sigma \rightarrow \mathbb{R}$ by

$$\|u\|_{C^\alpha(\Sigma)} = \sup_{\substack{x \neq y \\ x, y \in \Sigma}} \frac{|u(y) - u(x)|}{d(y, x)^\alpha} + \|u\|_{L^\infty(\Sigma)},$$

where $d(y, x)$ denotes the geodesic distance between x and y on Σ . We define the Hölder norm of a tensor field $T \in \mathcal{T}^k(\Sigma)$ as in [24]

$$\|T\|_{C^\alpha(\Sigma)} = \sup\{\|T(X_1, \dots, X_k)\|_{C^\alpha(\Sigma)} : X_i \in \mathcal{T}(\Sigma) \text{ with } \|X_i\|_{C^1(\Sigma)} \leq 1\}.$$

2.2. Preliminary Results. We begin by recalling the interpolation inequalities with Sobolev-norms on embedded surfaces. We use the result from [29, Proposition 6.5] (see also [16, Proposition 4.3]) which states that under a curvature bound the standard interpolation inequalities hold with a uniform constant.

Proposition 2.1. *Assume $\|B_\Sigma\|_{L^\infty}, \mathcal{H}^{n-1}(\Sigma) \leq C_0$ and Σ is C^m -regular for $m \geq 2$. Then for integers $0 \leq k < l \leq m$ and numbers $p \in [1, \infty)$, $q, r \in [1, \infty]$ there is $\theta \in [k/l, 1]$ such that for every C^l -regular tensor field T on Σ it holds*

$$\|\bar{\nabla}^k T\|_{L^p(\Sigma)} \leq C \|T\|_{W^{l,q}(\Sigma)}^\theta \|T\|_{L^r(\Sigma)}^{1-\theta}$$

for a constant $C = C(k, l, n, p, q, r, \theta, C_0)$, provided that the following condition is satisfied

$$\frac{1}{p} = \frac{k}{n-1} + \theta \left(\frac{1}{q} - \frac{l}{n-1} \right) + \frac{1}{r}(1-\theta).$$

Moreover, if $f : \Sigma \rightarrow \mathbb{R}$ is a smooth function with $\int_{\Sigma_i} f d\mathcal{H}^{n-1} = 0$ on every component Σ_i of Σ , the above inequality can be written as

$$\|\bar{\nabla}^k f\|_{L^p(\Sigma)} \leq C \|\bar{\nabla}^l f\|_{L^q(\Sigma)}^\theta \|f\|_{L^r(\Sigma)}^{1-\theta}.$$

With the above assumptions on Σ , we also have the standard Sobolev embedding $\|f\|_{C^\alpha(\Sigma)} \leq C_p \|f\|_{W^{1,p}(\Sigma)}$ for $\alpha = 1 - \frac{n-1}{p}$ and $p > n-1$. If $E \subset B_{R_0} \subset \mathbb{R}^n$ is bounded and uniformly $C^{m,\beta}$ -regular, for $m \geq 2$ with constants r_0, C_0 , then we have also the classical interpolation in Hölder norms, i.e., for $0 < \alpha < \beta \leq 1$ and $0 \leq k \leq l \leq m$ it holds

$$(2.5) \quad \|f\|_{C^{k,\alpha}(\Sigma)} \leq C \|f\|_{C^{l,\beta}(\Sigma)}^\theta \|f\|_{C^0(\Sigma)}^{1-\theta}, \quad \text{for } \theta = \frac{k+\alpha}{l+\beta},$$

where the constant depends on $r_0, R_0, C_0, k, l, \alpha, \beta$. The above formula follows from the Euclidean result, see for instance [28, Example 1.9].

The interpolation inequality in Proposition 2.1 implies the following useful estimate. The proof is standard and we refer to [25, Proposition 2.3], and note that the proof is similar as in the Euclidean case [39, Proposition 3.7]. We denote the norm of an index vector $\alpha \in \mathbb{N}^l$ by

$$|\alpha| := \alpha_1 + \dots + \alpha_l$$

Lemma 2.2. *Assume $\|B_\Sigma\|_{L^\infty}, \mathcal{H}^{n-1}(\Sigma) \leq C_0$ and Σ is C^m -regular for $m \geq 2$, and assume T_1, \dots, T_l are C^m -regular tensor fields. Then for an index vector $\alpha \in \mathbb{N}^l$ with norm $|\alpha| \leq k \leq m$ it holds*

$$\|\bar{\nabla}^{\alpha_1} T_1 \cdots \bar{\nabla}^{\alpha_l} T_l\|_{L^2(\Sigma)} \leq C \sum_{\sigma \in S_l} \|T_{\sigma(1)}\|_{L^\infty(\Sigma)} \cdots \|T_{\sigma(l-1)}\|_{L^\infty(\Sigma)} \|T_{\sigma(l)}\|_{H^k(\Sigma)},$$

where the sum is over the permutation group S_l . In particular,

$$\|\bar{\nabla}^k \langle T_1, T_2 \rangle\|_{L^2(\Sigma)} \leq C \|T_1\|_{L^\infty(\Sigma)} \|T_2\|_{H^k(\Sigma)} + C \|T_2\|_{L^\infty(\Sigma)} \|T_1\|_{H^k(\Sigma)}.$$

We also need to bound the whole $\|u\|_{H^{2k}}$ -norm with the norm $\|\Delta^k u\|_{L^2}$. Similarly we need to bound $\|B_\Sigma\|_{H^{2k}}$ -norm with the norm $\|\Delta^k H_{\partial E}\|_{L^2}$. This is given by the following lemma whose proof can be found e.g. in [25, Lemma 2.5 & Proposition 2.6].

Lemma 2.3. *Assume $\Sigma = \partial E \subset \mathbb{R}^n$ is C^{2k+1} -regular with $\mathcal{H}^{n-1}(\Sigma), \|B_\Sigma\|_{L^\infty} \leq C$. Then for all $f \in C^{2k}(\Sigma)$ it holds*

$$\begin{aligned} \|f\|_{H^{2k}(\Sigma)} &\leq C_k (\|\Delta^k f\|_{L^2(\partial E)} + (1 + \|\nabla_{\partial E} \Delta^{k-1} H_{\partial E}\|_{L^2(\partial E)}) \|f\|_{L^\infty(\partial E)}), \\ \|f\|_{H^{2k-1}(\Sigma)} &\leq C_k (\|\nabla_{\partial E} \Delta^{k-1} f\|_{L^2(\partial E)} + (1 + \|\Delta^{k-1} H_{\partial E}\|_{L^2(\partial E)}) \|f\|_{L^\infty(\partial E)}). \end{aligned}$$

Moreover, it holds

$$\begin{aligned} \|B_\Sigma\|_{H^{2k-2}(\Sigma)} &\leq C_k(1 + \|\Delta^{k-1}H_{\partial E}\|_{L^2(\partial E)}), \\ \|B_\Sigma\|_{H^{2k-1}(\Sigma)} &\leq C_k(1 + \|\nabla_{\partial E}\Delta^{k-1}H_{\partial E}\|_{L^2(\partial E)}). \end{aligned}$$

Here the constant C_k depends only on n , k and C .

The estimates for $\|f\|_{H^{2k}(\Sigma)}$ in Lemma 2.3 are sharp with respect to the norm on the curvature. However, since we assume that the second fundamental form is point-wise uniformly bounded the following holds under the assumptions of Lemma 2.3

$$(2.6) \quad \|f\|_{H^2(\Sigma)} \leq C(\|\Delta f\|_{L^2(\partial E)} + \|f\|_{L^2(\partial E)}).$$

This follows from a similar argument as the Lichnerowicz's theorem [5, Theorem 4.19] (see e.g. [20, Remark 2.4 & 2.5]).

The above results hold for standard Sobolev-norms. In the definition of the minimizing movement scheme it appears a geometric H^{-1} -distance, which is defined for sets of finite perimeter E, F as

$$(2.7) \quad d_{H^{-1}}(F; E) = \sup_{\|\nabla_{\partial E} f\|_{L^2(\partial E)} \leq 1} \int_{\mathbb{R}^n} f(\pi_{\partial E}(x))(\chi_F(x) - \chi_E(x)) dx.$$

Note that the projection $\pi_{\partial E}$ is well-defined almost everywhere in \mathbb{R}^n . Indeed, this follows from the fact that the projection is unique at points where the signed distance function d_E is differentiable. Note that (2.7) is not symmetric as it is defined using the projection on E .

It is immediate that if $d_{H^{-1}}(F; E)$ is finite then necessarily $|F| = |E|$. In general however, it is not clear what is the sufficient condition for the set F to have finite distance in (2.7). The situation becomes more clear if we assume that E satisfies the UBC with radius r_0 . Indeed, then the signed distance function d_E is differentiable in $\mathcal{N}_{r_0}(\partial E)$, the projection is well defined there and we may define a map $\Phi_\tau : \partial E \rightarrow \{x \in \mathbb{R}^n : d_E(x) = \tau\}$, with $\tau \in (-r_0, r_0)$, as

$$(2.8) \quad \Phi_\tau(x) = x + \tau \nu_E(x)$$

which is a bijection. Moreover, the boundary ∂E has finite number of components $\Sigma_1, \dots, \Sigma_N$. In this case we may characterize the sets F with finite $d_{H^{-1}}$ -distance from E when we constraint them to satisfy $F \Delta E \subset \mathcal{N}_{r_0}(\partial E)$. We state this as a lemma. We state the lemma in three dimensions in order to simplify the technicalities, but the result easily generalizes to any dimension.

Lemma 2.4. *Assume that $E \subset \mathbb{R}^3$ is C^2 -regular, $E \subset B_{R_0}$, satisfies the UBC with radius r_0 and denote the components of ∂E by Σ_i . Let F be a measurable set such that*

$$F \Delta E \subset \mathcal{N}_{r_0}(\partial E)$$

and define the function $\xi_{F,E} : \partial E \rightarrow \mathbb{R}$ as

$$(2.9) \quad \xi_{F,E}(x) := \int_{-r_0}^{r_0} (\chi_F(x + \tau \nu_E(x)) - \chi_E(x + \tau \nu_E(x))) (1 + H_E(x)\tau + K_E(x)\tau^2) d\tau.$$

Then it holds $d_{H^{-1}}(F; E) < \infty$ if and only if

$$(2.10) \quad \int_{\Sigma_i} \xi_{F,E} d\mathcal{H}^2 = 0$$

for every component Σ_i of ∂E .

Proof. The lemma follows from the co-area formula and the fact that the tangential Jacobian of Φ_τ defined in (2.8) is

$$J_{\partial E} \Phi_\tau(x) = 1 + H_E(x)\tau + K_E(x)\tau^2.$$

In fact, the assumption $F \Delta E \subset n_{r_0}(\partial E)$, combined with the co-area formula and a change of variables, implies that we may write for $f \in H^1(\partial E)$

$$\begin{aligned} \int_{\mathbb{R}^3} f(\pi_{\partial E}(x))(\chi_F(x) - \chi_E(x)) dx &= \int_{n_{r_0}(\partial E)} f(\pi_{\partial E}(x))(\chi_F(x) - \chi_E(x)) dx \\ &= \int_{-r_0}^{r_0} \int_{\{x: d_E(x)=\tau\}} f(\pi_{\partial E}(x))(\chi_F(x) - \chi_E(x)) d\mathcal{H}^2(x) d\tau \\ &= \int_{-r_0}^{r_0} \int_{\partial E} f(x)(\chi_F(\Phi_\tau(x)) - \chi_E(\Phi_\tau(x))) J_{\partial E} \Phi_\tau(x) d\mathcal{H}^2(x) d\tau \\ &= \int_{\partial E} f(x) \xi_{F,E}(x) d\mathcal{H}^2(x). \end{aligned}$$

If (2.10) holds, we may assume that also f satisfies $\int_{\Sigma_i} f d\mathcal{H}^2 = 0$ in every component Σ_i . By Poincaré inequality it holds

$$\int_{\partial E} f(x) \xi_{F,E}(x) d\mathcal{H}^2(x) \leq C \|f\|_{L^2(\partial E)} \leq C \|\nabla_{\partial E} f\|_{L^2(\partial E)}$$

and therefore $d_{H^{-1}}(F; E)$ is finite. It is also immediate that (2.10) is a necessary condition for $d_{H^{-1}}(F; E) < \infty$. \square

If in addition to the assumptions of Lemma 2.4, the set F is C^1 -regular and ∂F is a normal deformation of ∂E , i.e.,

$$\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\},$$

then we may write the function (2.9) explicitly as

$$(2.11) \quad \xi_{F,E} = \psi + \frac{H_E}{2}\psi^2 + \frac{K_E}{3}\psi^3.$$

We may also characterize the function $f \in H^1(\partial E)$ attaining the supremum in (2.7) as a solution of the equation

$$(2.12) \quad -\Delta_{\partial E} f = \frac{\xi_{F,E}}{d_{H^{-1}}(F; E)} \quad \text{on } \partial E.$$

Finally note that if E is in addition $C^{2,\alpha}$ -regular, then f in (2.12) is also $C^{2,\alpha}$ -regular. This follows from standard elliptic regularity estimates.

Let us define the standard H^{-1} -norm for functions on ∂E as

$$(2.13) \quad \|f\|_{H^{-1}(\partial E)} = \sup_{\|\nabla_{\partial E} g\|_{L^2(\partial E)} \leq 1} \int_{\partial E} f g d\mathcal{H}^2.$$

Then we have trivially the following interpolation inequality

$$(2.14) \quad \|f\|_{L^2(\partial E)} \leq \|\nabla_{\partial E} f\|_{L^2(\partial E)}^{\frac{1}{2}} \|f\|_{H^{-1}(\partial E)}^{\frac{1}{2}}$$

for all $f \in H^1(\partial E)$. If f has zero average on every components of Σ_i , then Poincaré inequality implies $\|f\|_{H^{-1}(\partial E)} \leq C\|f\|_{L^2(\partial E)}$. Moreover, by the proof of Lemma 2.4 it holds

$$(2.15) \quad d_{H^{-1}}(F; E) = \|\xi_{F,E}\|_{H^{-1}(\partial E)},$$

where $\xi_{F,E}$ is defined in (2.9).

2.3. Useful formulas. In case ∂F is a normal deformation of ∂E and is parametrized by the heightfunction ψ , we may write the mean curvature H_F in terms of ψ . These calculations are standard and can be found e.g. in [30] (see also [20]). However, we need this formula with an explicit dependence on the reference set ∂E , which we did not find in the literature. The key point is that the expansion of the mean curvature of F depends on the derivative of B_E , which presents a significant challenge in the forthcoming analysis. This dependence leads to a loss of regularity when expanding the mean curvature using the height function. Specifically, it requires more regularity from E than what the formula provides for F . We address this issue by using the fact that the term involving the derivatives of B_E is quadratically small.

We assume that E is C^3 -regular, satisfies the UBC with radius r , and F is C^2 -regular such that $F\Delta E \subset \mathcal{N}_r(\partial E)$ and we may write its boundary as

$$(2.16) \quad \partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}$$

for $\psi \in C^2(\partial E)$. We define a diffeomorphism $\Psi : \partial E \rightarrow \partial F$ as

$$(2.17) \quad \Psi(x) := x + \psi(x)\nu_E(x).$$

Let us first make some easy observations. Since $F\Delta E \subset \mathcal{N}_r(\partial E)$, the projection $\pi_{\partial E}$ is well defined on ∂F and it holds for $y \in \partial F$, setting $x = \pi_{\partial E}(y)$,

$$(2.18) \quad d_E(y) = \psi(x), \quad \nabla \pi_{\partial E}(y) = I - \nu_E(x) \otimes \nu_E(x) - \psi(x) B_E(x) (I + \psi(x) B_E(x))^{-1},$$

where the latter follows from (2.4). We may extend a given function $f \in C^1(\partial E)$ as $f \circ \pi_{\partial E} : \mathcal{N}_r(\partial E) \rightarrow \mathbb{R}$. In particular, this function is defined on ∂F and we may calculate the tangential gradient of $f \circ \pi_{\partial E} : \partial F \rightarrow \mathbb{R}$ at $y \in \partial F$ as

$$(2.19) \quad \nabla_{\partial F}(f \circ \pi_{\partial E})(y) = (\nabla_{\partial F} \pi_{\partial E}(y))^T \nabla_{\partial E} f(\pi_{\partial E}(y)).$$

Indeed, we obtain (2.19) by using the fact that in $\mathcal{N}_r(\partial E)$ it holds $f \circ \pi_{\partial E} = f \circ \pi_{\partial E} \circ \pi_{\partial E}$. We differentiate this and have for all $y \in \mathcal{N}_r(\partial E)$

$$\nabla(f \circ \pi_{\partial E})(y) = (\nabla \pi_{\partial E}(y))^T \nabla(f \circ \pi_{\partial E})(\pi_{\partial E}(y)).$$

The equality (2.19) then follows from above, by observing that since $\pi_{\partial E}(y) \in \partial E$ it holds by the definition of tangential gradient $\nabla(f \circ \pi_{\partial E})(\pi_{\partial E}(y)) = \nabla_{\partial E} f(\pi_{\partial E}(y))$.

We may now derive our formula for the expansion of the mean curvature.

Lemma 2.5. *Assume $E \subset \mathbb{R}^n$ is C^3 -regular and satisfies the UBC with radius r , and $F \subset \mathbb{R}^n$ is C^2 -regular such that $F \Delta E \subset \mathcal{N}_r(\partial E)$ and we may write its boundary as (2.16) for $\psi \in C^2(\partial E)$. Then it holds for $x \in \partial E = \Sigma$*

$$H_F(\Psi(x)) = -\Delta_{\partial E} \psi(x) + H_E(x) + R_0(x),$$

where Ψ is defined in (2.17). The error term is of the form

$$(2.20) \quad R_0 = \langle A_1(\psi B_\Sigma, \bar{\nabla} \psi), \bar{\nabla}^2 \psi \rangle + \langle A_2(\psi B_\Sigma, \bar{\nabla} \psi), \bar{\nabla}(\psi B_\Sigma) \rangle + a_0(\psi, \bar{\nabla} \psi, B_\Sigma)$$

where A_1, A_2 are smooth tensor fields such that $A_i(0, 0) = 0$ and a_0 is a smooth function with $a_0(0, 0, \cdot) = 0$.

Proof. We observe that for $\Psi : \partial E \rightarrow \partial F$ defined as in (2.17) it holds $\Psi^{-1} = \pi_{\partial E} : \partial F \rightarrow \partial E$. For every tangent vector $\tau \in T_x \partial E$ it holds $\nabla_{\partial E} \Psi(x) \tau \in T_{\Psi(x)} \partial F$ and

$$(2.21) \quad \nabla_{\partial E} \Psi(x) \tau = (I + \psi(x) B_E(x)) \tau + (\nabla_{\partial E} \psi \cdot \tau) \nu_E(x).$$

We define $N : \partial E \rightarrow \mathbb{R}^n$ as

$$(2.22) \quad N(x) = -(I + \psi(x) B_E(x))^{-1} \nabla_{\partial E} \psi(x) + \nu_E(x).$$

Then it holds for every $\tau \in T_x \partial E$ that

$$(\nabla_{\partial E} \Psi(x) \tau) \cdot N(x) = 0.$$

It follows from (2.16) that $\nu_F(\Psi(x)) \cdot \nu_E(x) \geq 0$. Therefore at every $y \in \partial F$ the vector $N(\pi_{\partial E}(y))$ is in the direction of the normal $\nu_F(y)$, i.e.,

$$(2.23) \quad \nu_F(y) = \frac{N(\pi_{\partial E}(y))}{|N(\pi_{\partial E}(y))|}.$$

Therefore it holds

$$(2.24) \quad \begin{aligned} H_F(y) &= (\operatorname{div}_{\partial F} \nu_F)(y) \\ &= \frac{1}{|N(\pi_{\partial E}(y))|} (\operatorname{div}_{\partial F} (N \circ \pi_{\partial E}))(y) + \nabla_{\partial F} (|N(\pi_{\partial E}(y))|^{-1}) \cdot N(\pi_{\partial E}(y)) \\ &= \frac{1}{|N(\pi_{\partial E}(y))|} (\operatorname{div}_{\partial F} (N \circ \pi_{\partial E}))(y), \end{aligned}$$

where the last equality follows from the fact that $N(\pi_{\partial E}(y))$ is in the direction of $\nu_F(y)$.

We split $N = N_1 + N_2$ as

$$N_1 = -(I + \psi B_E)^{-1} \nabla_{\partial E} \psi \quad \text{and} \quad N_2 = \nu_E.$$

Note that N_1 and N_2 are orthogonal and N_1 is a tangent vector field, by which we mean that $N_1(x) \in T_x \partial E$ for every $x \in \partial E$. By (2.24) we need to calculate $\operatorname{div}_{\partial F}(N_1 \circ \pi_{\partial E})$ and $\operatorname{div}_{\partial F}(N_2 \circ \pi_{\partial E})$. We write

$$(2.25) \quad |N(\pi_{\partial E}(y))| = 1 + a_1(\psi B_\Sigma, \bar{\nabla} \psi),$$

where the function satisfies $a_1(0, 0) = 0$.

Let us first calculate $\operatorname{div}_{\partial F}(N_1 \circ \pi_{\partial E})$. Fix $y \in \partial F$ and denote $x = \pi_{\partial E}(y)$. We use (2.19) component-wise to infer

$$(2.26) \quad \operatorname{div}_{\partial F}(N_1 \circ \pi_{\partial E})(y) = \operatorname{Tr}(\nabla_{\partial E} N_1(x) \nabla \pi_{\partial E}(y) P_{\partial F}(y)),$$

where $P_{\partial F} = I - \nu_F \otimes \nu_F$. By (2.23), (2.25) and by the definition of N in (2.22) we find that there are smooth tangent vector fields $\hat{A}_i = \hat{A}_i(\psi B_E, \nabla_{\partial E} \psi)$, with $i = 1, 2$, and a smooth function $a_2 = a_2(\psi(x) B_E(x), \nabla_{\partial E} \psi(x))$ such that

$$(2.27) \quad P_{\partial F}(y) - P_{\partial E}(x) = -\hat{A}_1 \otimes \hat{A}_1 + \hat{A}_2 \otimes \nu_E(x) + \nu_E(x) \otimes \hat{A}_2 + a_2 \nu_E(x) \otimes \nu_E(x),$$

and it holds $\hat{A}_i(0, 0) = 0$, $a_2(0, 0) = 0$. We may thus write (2.26) using (2.18), (2.27) and recalling that $N_1 = -(I + \psi B_E)^{-1} \nabla_{\partial E} \psi$ is a tangent vector field

$$(2.28) \quad \operatorname{div}_{\partial F}(N_1 \circ \pi_{\partial E}) = -\Delta_{\partial E} \psi + \langle \tilde{A}_1(\psi B_\Sigma, \bar{\nabla} \psi), \bar{\nabla}^2 \psi \rangle + \langle \tilde{A}_2(\psi B_\Sigma, \bar{\nabla} \psi), \bar{\nabla}(\psi B_\Sigma) \rangle,$$

where \tilde{A}_1, \tilde{A}_2 are smooth tensor fields such that $\tilde{A}_i(0, 0) = 0$.

Let us then calculate $\operatorname{div}_{\partial F}(N_2 \circ \pi_{\partial E})$. We have $N_2 \circ \pi_{\partial E} = \nu_E \circ \pi_{\partial E} = \nabla d_E$. By (2.3) it holds

$$\nabla^2 d_E(y) = B_E(x)(I + \psi(x) B_E(x))^{-1}.$$

Therefore we have by (2.27) (denote the vector field by $\hat{A}_1 = \hat{A}_1(\psi B_E, \nabla_{\partial E} \psi)$) for $y \in \partial F$ that

$$(2.29) \quad \begin{aligned} \operatorname{div}_{\partial F}(N_2 \circ \pi_{\partial E}) &= \operatorname{Tr}(\nabla_{\partial F} \nabla d_E) = \operatorname{Tr}(P_{\partial F}(y) \nabla^2 d_E(y)) \\ &= \operatorname{Tr}(\nabla^2 d_E(y)) - (\nabla^2 d_E(y) \hat{A}_1) \cdot \hat{A}_1 \\ &= \operatorname{Tr}(B_E(x)(I + \psi(x) B_E(x))^{-1}) - (B_E(x)(I + \psi(x) B_E(x))^{-1} \hat{A}_1) \cdot \hat{A}_1 \\ &= H_E(x) + a_3(\psi, \nabla_{\partial E} \psi, B_E), \end{aligned}$$

where a_3 is a smooth function such that $a_3(0, 0, \cdot) = 0$. The claim then follows from (2.24), (2.25), from $N = N_1 + N_2$, (2.28) and from (2.29). \square

We remark that if E and F are as in Lemma 2.5, then we may use (2.27) to write the tangential derivative of a function $f : \partial E \rightarrow \mathbb{R}$ in (2.19) on $y \in \partial F$, setting $x = \pi_{\partial E}(y)$, as

$$(2.30) \quad \nabla_{\partial F}(f \circ \pi_{\partial E})(y) = (I + A(\psi(x), \nabla_{\partial E} \psi(x), \nu_E(x), B_E(x))) \nabla_{\partial E} f(x)$$

for a smooth matrix-valued function with $A(0, 0, \cdot, \cdot) = 0$.

2.4. Almost minimizers of the perimeter. In this section we discuss properties of almost minimizers of the perimeter. We will use the following definition.

Definition 2.6. *A set of finite perimeter $F \subset \mathbb{R}^n$ is a (Λ, α) -minimizer of the perimeter for $\Lambda \geq 0$ and $\alpha \in [0, \frac{1}{n})$ if for every set of finite perimeter $G \subset \mathbb{R}^n$ it holds*

$$P(F) \leq P(G) + \Lambda |F \Delta G|^{1-\alpha}.$$

Moreover, we say that F is a Λ -minimizer of the perimeter if it is $(\Lambda, 0)$ -minimizer of the perimeter.

We point out that the definition of (Λ, α) -minimizer is closely related to the one of ω -minimizer of the perimeter (see [37]), where a set of finite perimeter $F \subset \mathbb{R}^n$ is an (ω_0, r_0, β) -minimizer of the perimeter if for all x and $r \leq r_0$ and every set of finite perimeter $G \subset \mathbb{R}^n$ with $G \Delta F \subset B_r(x)$ it holds

$$P(F) \leq P(G) + \omega_0 r^{n-1+\beta}.$$

It is clear that if a set is a (Λ, α) -minimizer of the perimeter according to Definition 2.6, then it is (ω_0, r_0, β) -minimizer for $\omega_0 = \Lambda |B_1|^{1-\alpha}$, say for $r_0 = 1$, and $\beta = 1 - n\alpha$. Therefore Definition 2.6 is stronger than the ω -minimality defined above as it does not require the perturbation to be local.

It is well-known that if $E \subset \mathbb{R}^n$ satisfies the UBC with radius r_0 , then it is a Λ -minimizer of the perimeter, where $\Lambda = \Lambda(n, r_0)$. We state this result below as a lemma whose proof can be found in [2, Lemma 4.1].

Lemma 2.7. *Assume that $E \subset \mathbb{R}^n$ is bounded and satisfies the UBC with radius r_0 , then it is a Λ -minimizer of the perimeter, i.e., every set of finite perimeter $G \subset \mathbb{R}^n$ it holds*

$$P(E) \leq P(G) + \Lambda |E \Delta G|,$$

for $\Lambda = \frac{C}{r_0}$, for a dimensional constant C . In particular, $P(E) \leq \Lambda |E|$.

We remark that if $E \subset \mathbb{R}^n$ satisfies the UBC with radius r_0 and $E \subset B_{R_0}$, then we may bound its perimeter by Lemma 2.7 as $P(E) \leq \Lambda |E| \leq \Lambda |B_{R_0}|$.

The following lemma states some useful regularity properties that almost minimizers of the perimeter introduced in Definition 2.6 inherit from perimeter minimizers [37].

Lemma 2.8. *Assume that $E \subset B_{R_0} \subset \mathbb{R}^n$ satisfies the UBC with radius r_0 and is uniformly $C^{2,\beta}$ -regular with constants r_0, C_0 , and F is (Λ, α) -minimizer of the perimeter for $\alpha \in (0, \frac{1}{n})$. Then for $\gamma < \min\{\beta, \frac{1}{2} - \frac{n\alpha}{2}\}$ there is $\delta_0 \in (0, r_0/2)$, depending on $R_0, r_0, C_0, \Lambda, \alpha, \gamma$ and the dimension, such that if*

$$F \Delta E \subset n_{\delta_0}(\partial E)$$

then there is $\psi \in C^{1,\gamma}(\partial E)$ such that

$$\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}.$$

Moreover, for every $\varepsilon > 0$ there is $\delta_0 = \delta_0(\varepsilon)$ such that $\|\psi\|_{C^{1,\gamma}(\partial E)} \leq \varepsilon$.

Proof. We first note that by the assumption $F\Delta E \subset n_{\delta_0}(\partial E)$ for every $x \in \partial E$ the set $\partial F \cap B_{\delta_0}(x)$ is non-empty.

Let us fix $\varepsilon > 0$. We claim that there is $\delta_0 = \delta_0(\varepsilon) \in (0, \frac{r_0}{80}\varepsilon)$ such that if F and E are as in the assumptions, then for every $x \in \partial E$ and every $y \in \partial F \cap B_{\delta_0}(x)$ it holds

$$(2.31) \quad |\nu_E(x) - \nu_F(y)| < \varepsilon.$$

We argue by contradiction and assume that there exist E_k, F_k satisfying the assumptions with E and F replaced by E_k and F_k , respectively, such that $F_k\Delta E_k \subset n_{\frac{1}{k}}(\partial E_k)$ and points $x_k \in \partial E_k$, $y_k \in \partial F_k \cap B_{\frac{1}{k}}(x_k)$ such that

$$(2.32) \quad |\nu_{E_k}(x_k) - \nu_{F_k}(y_k)| \geq \varepsilon$$

for some $\varepsilon > 0$. By passing to a subsequence we may assume that $x_k \rightarrow x$, $y_k \rightarrow x$, $E_k \rightarrow E$ in $C^{2,\beta'}$ for all $0 < \beta' < \beta$ and $F_k \rightarrow E$ in the Hausdorff distance, E satisfies the UBC and is in particular a Λ -minimizer of the perimeter. Then $\nu_{E_k}(x_k) \rightarrow \nu_E(x)$, while from (Λ, α) -minimality of the sets F_k it follows by [37, Section 1.9] that $\nu_{F_k}(y_k) \rightarrow \nu_E(x)$, which contradicts (2.32).

The lemma follows from (2.31) by standard regularity argument. Indeed, let us fix $x_0 \in \partial E$. We may assume $x_0 = 0$ and $\nu_E(0) = e_n$. Since E satisfies the UBC with radius r_0 , we may write $E \cap B_{r_0/2}$ as a subgraph of a function $f : B_{r_0/2}^{n-1} \rightarrow \mathbb{R}$ with $\|f\|_{C^{1,1}(B_{r_0/2}^{n-1})} \leq \frac{10}{r_0}$, assuming $r_0 \leq 1$, see (2.2). Then it holds $|\nu_E(x) - e_n| < \varepsilon$ for all $x \in \partial E \cap B_{r_\varepsilon}$, where $r_\varepsilon = \frac{r_0}{20}\varepsilon$. We note that $\delta_0 < \frac{r_0}{80}\varepsilon$ implies $\delta_0 < \frac{r_\varepsilon}{4}$. Then by (2.31) we have $|\nu_F(y) - e_n| < 2\varepsilon$ for all $y \in \partial F \cap B_{\frac{3r_\varepsilon}{4}}$. Choose any point $y_0 \in \partial F \cap B_{\delta_0}$. By the previous inequality and the perimeter density estimates for (Λ, α) -minimizers we then deduce that the excess

$$\mathfrak{E}\left(y_0, \frac{r_\varepsilon}{2}\right) = \min_{|\omega|=1} \frac{1}{(r_\varepsilon/2)^{n-1}} \int_{\partial F \cap B_{\frac{r_\varepsilon}{2}}(y_0)} |\nu_F(y) - \omega|^2 d\mathcal{H}^{n-1}(y) \leq C\varepsilon^2,$$

provided $r_\varepsilon < r_1 = r_1(n, \Lambda, \alpha)$, for some constant $C = C(n, \Lambda, \alpha)$. Then, it follows from the so-called ε -regularity theorem, see for instance [37], and from the inclusion $B_{r_\varepsilon/4} \subset B_{r_\varepsilon/2}(y_0)$ that there is $\varphi : B_{r_\varepsilon/4}^{n-1} \rightarrow \mathbb{R}$ such that

$$F \cap B_{r_\varepsilon/4} = \{(y', y_n) \in \mathbb{R}^n : y_n < \varphi(y')\} \cap B_{r_\varepsilon/4}$$

with $\|\varphi\|_{C^{1,\gamma'}(B_{r_\varepsilon/4}^{n-1})} \leq C$ and $\gamma' = \frac{1}{2} - \frac{n\alpha}{2}$. The existence of the heightfunction $\psi \in C^{1,\gamma}(\partial E)$ then follows from the fact that ∂E is uniformly $C^{2,\beta}$ -regular, see [15, Section 1.2] for the precise argument. The smallness of the norm $\|\psi\|_{C^{1,\gamma}(\partial E)}$ for $\gamma < \min\{\beta, \gamma'\}$ when δ_0 is small, follows from interpolation inequality (2.5) observing that $\|\psi\|_{C^0} \leq \delta_0$. \square

As stated in Lemma 2.7, if $E \subset B_{R_0}$ satisfies the UBC with radius r_0 , then it is a Λ -minimizer of the perimeter. Moreover for every $s \in (-\frac{r_0}{2}, \frac{r_0}{2})$ also the set

$$(2.33) \quad E_s = \{x \in \mathbb{R}^n : d_E(x) < s\}$$

satisfies the UBC with radius $r_0/2$. Thus Lemma 2.7 implies that if $G \subset \mathbb{R}^n$ is a set of finite perimeter, then for every $s \in (-\frac{r_0}{2}, \frac{r_0}{2})$ it holds

$$(2.34) \quad P(G \cap E_s) \leq P(G) + \Lambda |G \setminus E_s| \quad \text{and} \quad P(G \cup E_s) \leq P(G) + \Lambda |E_s \setminus G|,$$

for $\Lambda = \Lambda(r_0, n)$, by possibly increasing Λ .

We will consider constrained minimization problems and to this aim we introduce the following definition.

Definition 2.9. *A set of finite perimeter $F \subset \mathbb{R}^n$ is a constrained (Λ, α) -minimizer of the perimeter for $\Lambda > 0$ and $\alpha \in [0, \frac{1}{n})$ if for every set of finite perimeter G such that*

$$G \Delta F \subset \overline{\mathcal{N}_\delta(\partial E)} \quad \text{and} \quad |G| = |F|$$

it holds

$$P(F) \leq P(G) + \Lambda |F \Delta G|^{1-\alpha}.$$

Lemma 2.10. *Assume that $E \subset \mathbb{R}^n$ satisfies the UBC with radius r_0 , $E \subset B_{R_0}$ and let F be a constrained (Λ, α) -minimizer such that $|F| = |E|$ and*

$$F \Delta E \subset \overline{\mathcal{N}_\delta(\partial E)}$$

for $\delta \in (0, r_0/4)$. Then F is a (Λ_1, α) -minimizer of the perimeter (without constraint) according to Definition 2.6, where $\Lambda_1 = \Lambda_1(\Lambda, r_0, R_0, n)$.

Proof. Let us fix a set of finite perimeter $G \subset \mathbb{R}^n$. We first notice that we may assume $|F \Delta G| \leq 1$. Indeed, if not then the inequality $P(F) \leq P(G) + \Lambda_1 |F \Delta G|^{1-\alpha}$ holds trivially by choosing $\Lambda_1 \geq P(F)$. Note also that we may bound the perimeter of F by choosing $G = E$ and using the assumption on F to have

$$P(F) \leq P(E) + \Lambda |F \Delta E|^{1-\alpha} \leq P(E) + \Lambda |B_{R_0}|.$$

The above estimate leads to the bound on the perimeter of F recalling that Lemma 2.7 yields $P(E) \leq \Lambda |B_{R_0}|$.

The rest of the proof is based on a calibration argument and to this aim we recall our notation (2.33) for $E_s = \{x \in \mathbb{R}^n : d_E(x) < s\}$ and that E_s satisfies the UBC for radius $r_0/2$ for all $s \in (-\frac{r_0}{2}, \frac{r_0}{2})$. The following argument will only rely on the estimate (2.34) and therefore all the constants below, denoted simply by Λ_0 , will depend only on r_0 and the dimension n .

Let $G \subset \mathbb{R}^n$ be such that $|F \Delta G| \leq 1$. We define

$$\tilde{G} = (G \cap E_\delta) \cup E_{-\delta}.$$

Then it holds $\tilde{G} \Delta E \subset \overline{\mathcal{N}_\delta(\partial E)}$ and $\tilde{G} \Delta G = (G \setminus E_\delta) \cup (E_{-\delta} \setminus G)$. Since $F \Delta E \subset \overline{\mathcal{N}_\delta(\partial E)}$ then $F \subset \overline{E}_\delta$ and $\text{int } E_{-\delta} \subset F$, which implies

$$(2.35) \quad |G \Delta \tilde{G}| \leq |G \Delta F|.$$

Using (2.34) and (2.35) we may estimate the perimeter as

$$(2.36) \quad P(\tilde{G}) \leq P(G) + \Lambda_0 |G \Delta \tilde{G}| \leq P(G) + \Lambda_0 |G \Delta F|.$$

It holds either $|\tilde{G}| \geq |F|$ or $|\tilde{G}| \leq |F|$. Let us assume the former, the other case being similar. Define a continuous function $\omega : [-\delta, \delta] \rightarrow \mathbb{R}$,

$$\omega(s) = |E_s \cap \tilde{G}|.$$

It holds by the construction of \tilde{G} and from $E_{-\delta} \subset F$ that

$$\omega(\delta) = |\tilde{G}| \geq |F| \quad \text{and} \quad \omega(-\delta) = |\tilde{G} \cap E_{-\delta}| = |E_{-\delta}| \leq |F|.$$

Therefore there exists $\tau \in [-\delta, \delta]$ such that $\omega(\tau) = |F|$, and thus the set $G_\tau = E_\tau \cap \tilde{G}$ satisfies $|G_\tau| = |F|$. Note also that it follows from $G_\tau \subset \tilde{G}$ and (2.35) that

$$(2.37) \quad \begin{aligned} |\tilde{G} \Delta G_\tau| &= |\tilde{G}| - |G_\tau| = |\tilde{G}| - |F| \leq |\tilde{G} \Delta F| \\ &\leq |G \Delta F| + |G \Delta \tilde{G}| \\ &\leq 2|G \Delta F|. \end{aligned}$$

Moreover, using (2.34) and (2.37) we have

$$(2.38) \quad P(G_\tau) \leq P(\tilde{G}) + \Lambda_0 |\tilde{G} \Delta G_\tau| \leq P(\tilde{G}) + 2\Lambda_0 |G \Delta F|.$$

Since $G_\tau \Delta E \subset \overline{n_\delta(\partial E)}$ and $|G_\tau| = |F|$, we have by the assumption on F that

$$P(F) \leq P(G_\tau) + \Lambda |G_\tau \Delta F|^{1-\alpha} \leq P(G_\tau) + \Lambda (|G \Delta F|^{1-\alpha} + |G \Delta \tilde{G}|^{1-\alpha} + |\tilde{G} \Delta G_\tau|^{1-\alpha}).$$

The claim then follows from (2.35), (2.36), (2.37) and (2.38). \square

3. GEOMETRIC ESTIMATE

We need a higher order version of the fact that if a bounded set E satisfies the UBC, then it is a Λ -minimizer of the perimeter (Lemma 2.7). While the result in Lemma 2.7 follows from standard calibration argument, its higher order version, which is formulated below, is considerably more challenging due to the degenerate nature of the H^{-1} -distance (2.7). Note that, since the UBC condition implies that E is of class $C^{1,1}$, this allows us to define the Sobolev space $H^2(\Sigma)$, $\Sigma = \partial E$ as discussed in the previous section.

Proposition 3.1. *Assume $E \subset \mathbb{R}^3$ satisfies the UBC with radius r_0 , $E \subset B_{R_0}$, and $\|H_E\|_{H^2(\Sigma)} \leq K_0$. Then there exist $\delta_1 \in (0, r_0)$ and $\Lambda_2 \geq 1$, depending on r_0, R_0 and K_0 , such that for every set of finite perimeter $F \Delta E \subset n_{\delta_1}(\partial E)$ it holds*

$$(3.1) \quad P(E) \leq P(F) + \Lambda_2 d_{H^{-1}}(F; E),$$

where $d_{H^{-1}}(F; E)$ is defined in (2.7). Moreover, if F is C^1 -regular and there is $\psi \in C^1(\partial E)$ such that

$$(3.2) \quad \partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\} \quad \text{with} \quad \|\psi\|_{C^1(\partial E)} < \delta_1$$

then

$$(3.3) \quad \frac{1}{4} \|\nabla_{\partial E} \psi\|_{L^2(\partial E)}^2 + P(E) \leq P(F) + \Lambda_2 d_{H^{-1}}(F; E).$$

Proof. We begin by proving the claim (3.3). Note first that the assumption $\|H_E\|_{H^2(\Sigma)} \leq K_0$ implies $\|B_\Sigma\|_{H^2(\Sigma)} \leq C$. Indeed, given $x \in \partial E$, up to a rotation we can write $E \cap B_{r_0/2}(x)$ as in (2.2), where the function $f \in C^{1,1}(B_{r_0/2}^2(x'))$ satisfies in $B_{r_0/2}^2(x')$ the mean-curvature equation with right-hand-side given by the function $\tilde{H}(x') = H_E((x', f(x'))) \in H^2(B_{r_0/2}^2(x'))$. As a result, by standard elliptic estimates we get that f is bounded in $H^4(B_{r_0/4}^2(x'))$, which in turn, implies that B_Σ is bounded in $H^2(\Sigma)$ and, by the Sobolev embedding theorem, that E is uniformly $C^{2,\alpha}$ -regular for any $\alpha \in (0, 1)$ and it holds

$$(3.4) \quad \|B_E\|_{L^\infty} + \|\bar{\nabla} B_\Sigma\|_{L^p} \leq C_p,$$

where $C_p = C_p(r_0, R_0, K_0, p)$ for every $p \in (1, \infty)$.

Since E is uniformly $C^{2,\alpha}$ -regular we may define a diffeomorphism $\Psi : \partial E \rightarrow \partial F$,

$$\Psi(x) = x + \psi(x) \nu_E(x).$$

Let us fix a point $x \in \partial E$ and choose an orthonormal basis τ_1, τ_2 of the tangent hyperplane $T_x \partial E$ such that $B_E(x) \tau_i = \kappa_i(x) \tau_i$, where $\kappa_i(x)$ are the principal curvatures of ∂E . By (2.21) it holds

$$\nabla_{\partial E} \Psi(x) \tau_i = (1 + \psi(x) \kappa_i(x)) \tau_i + \bar{\nabla}_i \psi(x) \nu_E(x),$$

where $\bar{\nabla}_i \psi(x)$ denotes the derivative in direction τ_i . Hence, the tangential Jacobian is

$$(3.5) \quad J_{\partial E} \Psi = ((1 + \psi \kappa_1)^2 (1 + \psi \kappa_2)^2 + (1 + \psi \kappa_1)^2 |\bar{\nabla}_2 \psi|^2 + (1 + \psi \kappa_2)^2 |\bar{\nabla}_1 \psi|^2)^{\frac{1}{2}}.$$

Since $\sqrt{1 + \tau} \geq 1 + \frac{\tau}{2} - \tau^2$ for $|\tau| \leq \frac{1}{2}$, we deduce that

$$J_{\partial E} \Psi \geq 1 + H_E \psi + \frac{1}{3} |\bar{\nabla} \psi|^2 - C \psi^2$$

when $\|\psi\|_{C^1} \leq \delta_1$ for δ_1 is small enough. We use the area formula to conclude

$$(3.6) \quad P(F) = \int_{\partial E} J_{\partial E} \Psi d\mathcal{H}^2 \geq P(E) + \int_{\partial E} H_E \psi d\mathcal{H}^2 + \frac{1}{3} \|\nabla_{\partial E} \psi\|_{L^2(\partial E)}^2 - C \|\psi\|_{L^2(\partial E)}^2.$$

Next we recall (2.15), i.e., $d_{H^{-1}}(F; E) = \|\xi_{F,E}\|_{H^{-1}(\partial E)}$, where $\xi_{F,E}$ is defined in (2.11). In the following we write simply $\xi = \xi_{F,E}$. We claim that when $\|\psi\|_{C^1} \leq \delta_1$, and when $\delta_1 = \delta(r_0, R_0, K_0)$ is small enough it holds

$$(3.7) \quad \begin{aligned} \frac{1}{2} |\psi(x)| &\leq |\xi(x)| \leq 2 |\psi(x)|, \\ \|\nabla_{\partial E} \xi\|_{L^2(\partial E)} &\leq 2 \|\nabla_{\partial E} \psi\|_{L^2(\partial E)} + \sqrt{\delta_1} \|\psi\|_{L^2(\partial E)}. \end{aligned}$$

The inequalities on the first line follow immediately from (3.4). For the inequality on the second line in (3.7) we have by straightforward calculation and by applying (3.4)

$$|\nabla_{\partial E} \xi(x)| \leq 2 |\nabla_{\partial E} \psi(x)| + C \psi^2(x) |\bar{\nabla} B_\Sigma(x)| \quad \text{for a.e. } x \in \partial E.$$

By Hölder's inequality and (3.4) we have

$$\begin{aligned}\|\nabla_{\partial E}\xi\|_{L^2(\partial E)} &\leq 2\|\nabla_{\partial E}\psi\|_{L^2(\partial E)} + C\|\psi\|_{L^8(\partial E)}^2\|\bar{\nabla}B_\Sigma\|_{L^4(\Sigma)} \\ &\leq 2\|\nabla_{\partial E}\psi\|_{L^2(\partial E)} + C\|\psi\|_{L^8(\partial E)}^2.\end{aligned}$$

The claim (3.7) then follows from interpolation inequality in Proposition 2.1, as

$$C\|\psi\|_{L^8(\partial E)}^2 \leq C\|\psi\|_{W^{1,4}(\partial E)}\|\psi\|_{L^2(\partial E)} \leq C\delta_1\|\psi\|_{L^2(\partial E)} \leq \sqrt{\delta_1}\|\psi\|_{L^2(\partial E)},$$

when δ_1 is small enough.

We proceed by using (3.7) to write (3.6) as

$$(3.8) \quad P(F) \geq P(E) + \int_{\partial E} H_E \xi d\mathcal{H}^2 + \frac{1}{3}\|\nabla_{\partial E}\psi\|_{L^2(\partial E)}^2 - C\|\xi\|_{L^2(\partial E)}^2.$$

By the assumptions on E and by (2.15) it holds

$$(3.9) \quad - \int_{\partial E} H_E \xi d\mathcal{H}^2 \leq C\|\xi\|_{H^{-1}(\partial E)} = Cd_{H^{-1}}(F; E),$$

where the constant C depends on r_0 and K_0 .

By the interpolation inequality (2.14) and by (3.7) we have

$$\begin{aligned}\|\xi\|_{L^2(\partial E)}^2 &\leq \|\nabla_{\partial E}\xi\|_{L^2(\partial E)}\|\xi\|_{H^{-1}(\partial E)} \\ &\leq 2\|\nabla_{\partial E}\psi\|_{L^2(\partial E)}\|\xi\|_{H^{-1}(\partial E)} + \sqrt{\delta_1}\|\psi\|_{L^2(\partial E)}\|\xi\|_{H^{-1}(\partial E)} \\ &\leq 2\|\nabla_{\partial E}\psi\|_{L^2(\partial E)}\|\xi\|_{H^{-1}(\partial E)} + C\sqrt{\delta_1}\|\xi\|_{L^2(\partial E)}^2,\end{aligned}$$

where in the last inequality we used $\|\xi\|_{H^{-1}(\partial E)} \leq C\|\xi\|_{L^2(\partial E)}$ and $|\psi| \leq 2|\xi|$. Therefore when δ_1 is small enough we have

$$\|\xi\|_{L^2(\partial E)}^2 \leq 4\|\nabla_{\partial E}\psi\|_{L^2(\partial E)}\|\xi\|_{H^{-1}(\partial E)}.$$

It also holds

$$\|\xi\|_{H^{-1}(\partial E)} \leq C\|\xi\|_{L^2(\partial E)} \leq C\|\psi\|_{L^2(\partial E)} \leq C\delta_1.$$

Therefore when δ_1 is small, we have by the two above inequalities and by (2.15) that

$$\begin{aligned}(3.10) \quad \|\xi\|_{L^2(\partial E)}^2 &\leq 4\|\nabla_{\partial E}\psi\|_{L^2(\partial E)}\|\xi\|_{H^{-1}(\partial E)} \\ &\leq \varepsilon\|\nabla_{\partial E}\psi\|_{L^2(\partial E)}^2 + C_\varepsilon\|\xi\|_{H^{-1}(\partial E)}^2 \\ &\leq \varepsilon\|\nabla_{\partial E}\psi\|_{L^2(\partial E)}^2 + C_\varepsilon\delta_1 d_{H^{-1}}(F; E).\end{aligned}$$

Therefore we deduce from (3.8), (3.9) and (3.10) that

$$\left(\frac{1}{3} - \varepsilon\right)\|\nabla_{\partial E}\psi\|_{L^2(\partial E)}^2 + P(E) \leq P(F) + Cd_{H^{-1}}(F; E).$$

Thus we have (3.3) when ε is chosen small enough.

Let us then prove the first claim (3.1). We define the functional

$$(3.11) \quad \mathcal{J}(F) := P(F) + \Lambda_2 d_{H^{-1}}(F; E), \quad F \Delta E \subset \overline{n_\delta}(\partial E),$$

where Λ_2 is the constant from (3.3) and $\delta \in (0, \frac{r_0}{4})$. Using direct method of calculus of variations we deduce that \mathcal{G} has a minimizer, say F . The first claim (3.1) follows once we show that E is a minimizer of (3.11), when δ is chosen small enough.

We first claim that the minimizer F is a (Λ, α) -minimizer of the perimeter, for any $\alpha \in (0, \frac{1}{3})$, according to Definition 2.6, where Λ depends on r_0, R_0, K_0, Λ_2 and α . Since for F it clearly holds $d_{H^{-1}}(F; E) < \infty$, then by Lemma 2.4 we may divide the set F in pieces F_i according to the components Σ_i of ∂E , i.e., $F_i \Delta E \subset \overline{n_\delta(\Sigma_i)}$. By taking variations of each piece F_i separately, we may reduce to compare F with sets $G \subset \overline{n_\delta(E)}$ such that $G \Delta F \subset \overline{n_\delta(\Sigma_i)}$. For these sets we have by Lemma 2.4 that $d_{H^{-1}}(G; E) < \infty$ if $|G| = |E|$. Moreover, the supremum in the definition of $d_{H^{-1}}(G; E)$ is attained say by a function $f_G \in H^1(\partial E)$. By the condition (2.10) for $\xi_{G,E}$ we may assume that the function f_G has zero average on each component Σ_i . Therefore we have by the Sobolev embedding

$$(3.12) \quad \|f_G\|_{L^p(\partial E)} \leq C_p \|\nabla_{\partial E} f_G\|_{L^2(\partial E)} \leq C_p$$

for all $p > 2$. By the minimality of F it holds

$$(3.13) \quad P(F) - P(G) \leq \Lambda_2(d_{H^{-1}}(G; E) - d_{H^{-1}}(F; E))$$

and

$$(3.14) \quad \begin{aligned} & d_{H^{-1}}(G; E) - d_{H^{-1}}(F; E) \\ & \leq \int_{\mathbb{R}^3} f_G(\pi_{\partial E}(x))(\chi_G(x) - \chi_E(x))dx - \int_{\mathbb{R}^3} f_G(\pi_{\partial E}(x))(\chi_F(x) - \chi_E(x))dx \\ & = \int_{\mathbb{R}^3} f_G(\pi_{\partial E}(x))(\chi_G(x) - \chi_F(x))dx \\ & \leq \|f_G \circ \pi_{\partial E}\|_{L^p(n_\delta(\partial E))} \|\chi_G - \chi_F\|_{L^{\frac{p}{p-1}}(\mathbb{R}^3)}. \end{aligned}$$

The inequality (3.12) implies $\|f_G \circ \pi_{\partial E}\|_{L^p(n_\delta(\partial E))} \leq C$ and therefore we deduce from (3.13) and (3.14) that

$$P(F) \leq P(G) + C\Lambda_2|G \Delta F|^{1-\frac{1}{p}}.$$

Hence, the minimizer F is a constrained (Λ, α) -minimizer in the sense of Definition 2.9 for $\alpha = \frac{1}{p}$. By Lemma 2.10, F is also (Λ_1, α) -minimizer for Λ_1 depending on r_0, R_0, p .

By choosing δ small enough in (3.11), we may deduce by Lemma 2.8 that F is C^1 -regular and there is $\psi \in C^1(\partial E)$ with $\|\psi\|_{C^1(\partial E)} \leq \delta_1$, where δ_1 is as in (3.2), such that

$$\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}.$$

We then have from (3.3), which we proved above, that

$$P(E) \leq P(F) + \Lambda_2 d_{H^{-1}}(F; E).$$

This proves that E is a minimizer of (3.11), which implies the claim. □

4. REGULARITY ESTIMATES FOR THE FLOW

4.1. Definition of the flat flow solution. We begin by giving the construction of the flat flow solution for the surface diffusion flow, which is a slight modification of the one proposed by Cahn-Taylor in [9] by introducing an additional constraint. We first fix our constraint parameter $\delta > 0$ and then the time step $h > 0$, and given a bounded set $E \subset \mathbb{R}^3$ of finite perimeter, which coincides with its Lebesgue representative, we consider the minimization problem

$$(4.1) \quad \min \left\{ P(F) + \frac{d_{H^{-1}}(F; E)^2}{2h} : F \Delta E \subset \overline{n_\delta(\partial^* E)} \right\}$$

and note that the minimizer exists. The constraint in (4.1), which requires the sets to be in a neighborhood of E , simplifies the forthcoming analysis, and as we will see, it does not affect the actual construction if the set E is regular enough. Indeed, we will see that the minimizer F of (4.1) is also regular and satisfies $F \Delta E \subset n_{h\gamma}(\partial^* E)$ for $\gamma > 0$ when h is small enough. In other words, we need the constraint in (4.1) to overcome the lack of coerciveness with respect to $d_{H^{-1}}$.

Let $E_0 \subset \mathbb{R}^3$ be a bounded set of finite perimeter which coincides with its Lebesgue representative. We construct the discrete-in-time evolution $\{E_k^{h,\delta}\}_{k \in \mathbb{N}}$ recursively setting $E(0) = E_0^{h,\delta}$ and, assuming that $E_k^{h,\delta}$ is defined, we set $E_{k+1}^{h,\delta}$ to be a minimizer of (4.1) with $E = E_k^{h,\delta}$. We define the *approximate flat flow* $\{E^{h,\delta}(t)\}_{t \geq 0}$ by setting

$$E^{h,\delta}(t) = E_k^{h,\delta} \quad \text{for } t \in [kh, (k+1)h).$$

We define a flat flow solution $\{E^\delta(t)\}_{t \geq 0}$ of the surface diffusion as any cluster point when we let $h \rightarrow 0$.

4.2. The first regularity estimates. Let us study the minimization problem (4.1). Throughout the rest of this section we will assume that the set $E \subset \mathbb{R}^3$ is C^5 -regular and satisfies the assumptions of Proposition 3.1, i.e.,

$$(4.2) \quad E \text{ satisfies the UBC with radius } r_0, \quad E \subset B_{R_0} \quad \text{and} \quad \|\Delta_{\partial E} H_E\|_{L^2(\partial E)} \leq K_0,$$

for some $K_0 > 1$. Note that the above assumptions imply that the set E is uniformly $C^{2,\gamma}$ -regular for any $\gamma \in (0, 1)$. The first consequence of Proposition 3.1 is the following.

Proposition 4.1. *Assume $E \subset \mathbb{R}^3$ is C^5 -regular, satisfies the assumptions (4.2), and let $F \subset \mathbb{R}^3$ be a minimizer of (4.1) with $\delta < \delta_1$, where δ_1 is from Proposition 3.1. Then there is C_0 , depending on r_0, R_0, K_0 , such that*

$$(4.3) \quad d_{H^{-1}}(F; E) \leq C_0 h$$

for $h \leq h_0$, where h_0 depends on r_0, R_0, K_0 and on δ . Moreover, F is a (Λ, α) -minimizer of the perimeter for every $\alpha \in (0, \frac{1}{3})$, where Λ depends on r_0, R_0, K_0 and α .

In addition, if $\delta < \delta_2$, where $\delta_2 = \min\{\delta_0, \delta_1\}$ and δ_0 is from Lemma 2.8 with $\varepsilon = \delta_1$, then F is C^5 -regular, $\|H_F\|_{H^1(\partial F)} \leq C_0$, there is $\psi \in C^1(\partial E)$ such that $\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}$ and we have the estimates

$$(4.4) \quad \|\psi\|_{H^1(\partial E)} \leq C_0\sqrt{h} \quad \text{and} \quad \|\psi\|_{C^1(\partial E)} \leq h^{\frac{1}{9}},$$

when $h \leq h_0$. Moreover, for any $\varepsilon > 0$ there is C_ε such that $\|\psi\|_{L^\infty(\partial E)} \leq C_\varepsilon h^{\frac{1}{2}-\varepsilon}$.

Proof. The estimate (4.3) follows immediately by using the minimality of F against E , and from (3.1) stated in Proposition 3.1,

$$P(F) + \frac{d_{H^{-1}}(F; E)^2}{2h} \leq P(E) \leq P(F) + \Lambda_2 d_{H^{-1}}(F; E).$$

Next we prove that the minimizer F is a (Λ, α) -minimizer of the perimeter for any $\alpha \in (0, \frac{1}{3})$. Arguing as in the proof of Proposition 3.1, we may reduce to considering sets $G \subset \overline{\mathcal{N}_\delta(E)}$ such that $G\Delta F \subset \overline{\mathcal{N}_\delta(\Sigma_i)}$ where Σ_i is a component of ∂E . For these sets we have by Lemma 2.4 that $d_{H^{-1}}(G; E) < \infty$ if $|G| = |F|$. Moreover, the supremum in the definition of $d_{H^{-1}}(G; E)$ is attained say by a function $f_G \in H^1(\partial E)$. By the condition (2.10) for $\xi_{G,E}$ we may assume that the function f_G has zero average on every component Σ_i . Therefore we have by the Sobolev embedding $\|f_G\|_{L^p(\partial E)} \leq C\|\nabla_{\partial E} f_G\|_{L^2(\partial E)} \leq C$ for $p = \frac{1}{\alpha}$. We will show that

$$(4.5) \quad P(F) \leq P(G) + \Lambda|G\Delta F|^{1-\alpha}.$$

We remark that by Lemma 2.10, the estimate (4.5) then implies that F is a (Λ_1, α) -minimizer of the perimeter according to the Definition 2.6. We divide the proof of (4.5) in two cases.

Let us first assume that $d_{H^{-1}}(G; E) \leq 2C_0h$, where C_0 is the constant from (4.3). By the minimality of F we have

$$P(F) + \frac{d_{H^{-1}}(F; E)^2}{2h} \leq P(G) + \frac{d_{H^{-1}}(G; E)^2}{2h}.$$

If $d_{H^{-1}}(G; E) \leq d_{H^{-1}}(F; E)$, then (4.5) follows immediately. Let us then assume $d_{H^{-1}}(G; E) > d_{H^{-1}}(F; E)$. By (4.3) it holds

$$(4.6) \quad \begin{aligned} P(F) - P(G) &\leq \frac{1}{2h} (d_{H^{-1}}(G; E) + d_{H^{-1}}(F; E)) (d_{H^{-1}}(G; E) - d_{H^{-1}}(F; E)) \\ &\leq 2C_0 (d_{H^{-1}}(G; E) - d_{H^{-1}}(F; E)). \end{aligned}$$

Using (3.14) we deduce

$$(4.7) \quad d_{H^{-1}}(G; E) - d_{H^{-1}}(F; E) \leq \|f_G \circ \pi_{\partial E}\|_{L^p(\mathcal{N}_\delta(\partial E))} \|\chi_G - \chi_F\|_{L^{\frac{p}{p-1}}(\mathbb{R}^3)} \leq C|G\Delta F|^{1-\alpha},$$

where the last inequality follows from $\|f_G\|_{L^p(\partial E)} \leq C$ and $p = \frac{1}{\alpha}$. This together with (4.6) implies (4.5).

Let us then assume that $d_{H^{-1}}(G; E) > 2C_0h$. Note that by (4.3) it holds

$$(4.8) \quad d_{H^{-1}}(F; E) < \frac{1}{2}d_{H^{-1}}(G; E).$$

Using first the minimality of F against E , then (3.1) with F replaced by G , and finally (4.8), we obtain

$$\begin{aligned} P(F) &\leq P(E) \leq P(G) + \Lambda_2 d_{H^{-1}}(G; E) \\ &\leq P(G) + 2\Lambda_2 (d_{H^{-1}}(G; E) - d_{H^{-1}}(F; E)). \end{aligned}$$

The claim (4.5) then follows from (4.7).

We may use Lemma 2.8 to deduce that when δ is small enough, there is $\psi \in C^{1, \frac{1}{3}}(\partial E)$ with $\|\psi\|_{C^{1, \frac{1}{3}}(\partial E)} \leq \delta_1$, where δ_1 is from Proposition 3.1, such that $\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}$. Then (3.3), the minimality of F and (4.3) imply

$$\frac{1}{4} \|\nabla_{\partial E} \psi\|_{L^2(\partial E)}^2 + P(E) \leq P(F) + \Lambda_2 d_{H^{-1}}(F; E) \leq P(E) + Ch.$$

This yields

$$\|\nabla_{\partial E} \psi\|_{L^2(\partial E)} \leq C\sqrt{h}.$$

The estimate $\|\psi\|_{H^1(\partial E)} \leq C\sqrt{h}$ then follows from (3.7) and (3.10). In particular, we deduce by the Sobolev embedding and the C^1 -bound on ψ that $\|\psi\|_{L^\infty(\partial E)} \leq C_\varepsilon h^{\frac{1}{2}-\varepsilon}$ for any $\varepsilon > 0$ and therefore the minimizer F does not touch the constraint $\mathcal{N}_\delta(\partial E)$ when h is small enough. Hence, we may calculate the Euler-Lagrange equation for F which states that

$$(4.9) \quad H_F + \frac{d_{H^{-1}}(F; E)}{h} f \circ \pi_{\partial E} = \lambda \quad \text{on } \partial F,$$

where f is the function which attains the supremum in $d_{H^{-1}}(F; E)$. Note that since E is C^5 -regular and since f solves (2.12), we may bootstrap the regularity of F to $C^{5, \gamma}$ for any $\gamma \in (0, 1)$. We remark that the C^5 -regularity depends on h .

The estimate $\|H_F\|_{H^1(\partial F)} \leq C$ then follows from (4.9), from $d_{H^{-1}}(F; E) \leq C_0 h$, and from the fact that $\|\nabla_{\partial E} f\|_{L^2(\partial E)} \leq 1$ which in turn yields $\|\nabla_{\partial F}(f \circ \pi_{\partial E})\|_{L^2(\partial F)} \leq C$ by (2.19). Finally the estimate $\|\psi\|_{C^1(\partial E)} \leq h^{\frac{1}{9}}$ follows from $\|\psi\|_{L^\infty(\partial E)} \leq C_\varepsilon h^{\frac{1}{2}-\varepsilon}$, $\|\psi\|_{C^{1, \frac{1}{3}}(\partial E)} \leq \delta_1$ and the interpolation inequality (2.5). \square

As we already mentioned, an important consequence of Proposition 4.1 is that a minimizer of (4.1) does not touch the constraint $\mathcal{N}_\delta(\partial^* E)$ when $h \leq h_0$. We may then calculate the Euler-Lagrange equation for the minimizer F , and obtain by (2.12) and (4.9) that

$$(4.10) \quad \begin{cases} H_F + \frac{d_{H^{-1}}(F; E)}{h} f \circ \pi_{\partial E} = \lambda & \text{on } \partial F \\ -\Delta_{\partial E} f = \frac{\xi_{F, E}}{d_{H^{-1}}(F; E)} & \text{on } \partial E, \end{cases}$$

where $\xi_{F, E}$ is defined in (2.11). We remark that if E is $C^{5, \gamma}$ -regular for some $\gamma \in (0, 1)$, then by the elliptic regularity theory we may conclude that F is $C^{6, \gamma}$ -regular. This means that the minimizing movement scheme has a regularizing effect.

We may use Lemma 2.5 to combine the two above equations into the following

$$(4.11) \quad \frac{1}{h} \left(\psi + \frac{H_E}{2} \psi^2 + \frac{K_E}{3} \psi^3 \right) = \Delta_{\partial E} (-\Delta_{\partial E} \psi + H_E) + \Delta_{\partial E} R_0 \quad \text{on } \partial E,$$

where the error term R_0 is of the form (2.20). The estimates in Proposition 4.1 do not guarantee that we keep the uniform ball condition, because the bound $\|H_F\|_{H^1(\partial F)} \leq C$ does not imply a bound for $\|B_F\|_{L^\infty(\partial F)}$. Therefore we need to upgrade the regularity estimates from Proposition 4.1, which we do by using the equation (4.11).

Lemma 4.2. *Assume $E \subset \mathbb{R}^3$ is C^5 -regular, satisfies the assumptions (4.2) and $\|\Delta_{\partial E} H_E\|_{H^1(\partial E)} \leq K_0 h^{-\frac{1}{4}}$. Let $F \subset \mathbb{R}^3$ be a minimizer of (4.1) with $\delta < \delta_2$ where δ_2 is from Proposition 4.1. Then for the heightfunction in (4.4) it holds*

$$(4.12) \quad \|\psi\|_{L^2(\partial E)} \leq C_1 h \quad \text{and} \quad \|\psi\|_{H^4(\Sigma)} \leq C_1$$

for all $h \leq h_0$, where h_0 is from Proposition 4.1 and $\Sigma = \partial E$. The constant C_1 depends on r_0, R_0 and K_0 .

Proof. We begin by noticing that the assumptions on E and Lemma 2.3 imply

$$(4.13) \quad \|B_\Sigma\|_{L^\infty(\Sigma)} \leq \hat{C}, \quad \|B_\Sigma\|_{H^2(\Sigma)} \leq \hat{C} K_0 \quad \text{and} \quad \|B_\Sigma\|_{H^3(\Sigma)} \leq \hat{C} K_0 h^{-\frac{1}{4}},$$

for a constant \hat{C} , which depends on r_0 and R_0 .

We obtain the estimates by multiplying the Euler-Lagrange equation (4.11) with $\Delta_{\partial E}^2 \psi$. Integrating by parts and by re-organizing the terms we deduce

$$\begin{aligned} \frac{1}{h} \int_{\partial E} |\Delta_{\partial E} \psi|^2 d\mathcal{H}^2 + \int_{\partial E} |\Delta_{\partial E}^2 \psi|^2 d\mathcal{H}^2 &= \int_{\partial E} (\Delta_{\partial E}^2 \psi) (\Delta_{\partial E} H_E + \Delta_{\partial E} R_0) d\mathcal{H}^2 \\ &\quad - \frac{1}{h} \int_{\partial E} (\Delta_{\partial E} \psi) \Delta_{\partial E} \left(\frac{H_E}{2} \psi^2 + \frac{K_E}{3} \psi^3 \right) d\mathcal{H}^2. \end{aligned}$$

By Cauchy-Schwarz inequality this implies

$$(4.14) \quad \begin{aligned} \frac{1}{h} \|\Delta_{\partial E} \psi\|_{L^2}^2 + \|\Delta_{\partial E}^2 \psi\|_{L^2}^2 &\leq \|\Delta_{\partial E} H_E\|_{L^2}^2 + \|\Delta_{\partial E} R_0\|_{L^2}^2 \\ &\quad + \frac{1}{h} \|\Delta_{\partial E} \left(\frac{H_E}{2} \psi^2 \right)\|_{L^2}^2 + \frac{1}{h} \|\Delta_{\partial E} \left(\frac{K_E}{3} \psi^3 \right)\|_{L^2}^2. \end{aligned}$$

We need to estimate all four terms on the right-hand-side in (4.14).

By the assumptions (4.2) it holds

$$(4.15) \quad \|\Delta_{\partial E} H_E\|_{L^2}^2 \leq K_0^2.$$

The term $\|\Delta_{\partial E} R_0\|_{L^2}^2$ is the most difficult to deal with. In order to estimate it, we recall the form of R_0 in (2.20). Therefore we have by the Leibniz rule

$$\begin{aligned} \|\Delta_{\partial E} R_0\|_{L^2(\Sigma)} &\leq C \sum_{j+k=2} \|\bar{\nabla}^j (A_1(\psi B_\Sigma, \bar{\nabla} \psi))\| \|\bar{\nabla}^{2+k} \psi\|_{L^2(\Sigma)} \\ &\quad + C \sum_{j+k=2} \|\bar{\nabla}^j A_2(\psi B_\Sigma, \bar{\nabla} \psi)\| \|\bar{\nabla}^{1+k} (\psi B_\Sigma)\|_{L^2(\Sigma)} \\ &\quad + \|a_0(\psi, \bar{\nabla} \psi, B_\Sigma)\|_{H^2(\Sigma)}. \end{aligned}$$

For j, k with $j + k = 2$ we use Lemma 2.2 with $T_1 = A_1(\psi B_\Sigma, \bar{\nabla} \psi)$ and $T_2 = \bar{\nabla} \psi$ to estimate

$$(4.16) \quad \begin{aligned} & \| |\bar{\nabla}^j(A_1(\psi B_\Sigma, \bar{\nabla} \psi))| |\bar{\nabla}^{2+k} \psi| \|_{L^2} \\ & \leq C \|A_1(\psi B_\Sigma, \bar{\nabla} \psi)\|_{L^\infty} \|\psi\|_{H^4} + C \|\psi\|_{C^1} \|A_1(\psi B_\Sigma, \bar{\nabla} \psi)\|_{H^3} \end{aligned}$$

and similarly with $T_1 = A_2(\psi B_\Sigma, \bar{\nabla} \psi)$ and $T_2 = \psi B_\Sigma$

$$(4.17) \quad \begin{aligned} & \| |\bar{\nabla}^j(A_2(\psi B_\Sigma, \bar{\nabla} \psi))| |\bar{\nabla}^{1+k}(\psi B_\Sigma)| \|_{L^2} \\ & \leq C \|A_2(\psi B_\Sigma, \bar{\nabla} \psi)\|_{L^\infty} \|\psi B_\Sigma\|_{H^3} + C \|\psi B_\Sigma\|_{L^\infty} \|A_2(\psi B_\Sigma, \bar{\nabla} \psi)\|_{H^3}. \end{aligned}$$

Since A_i is smooth and $A_i(0, 0) = 0$, it holds

$$(4.18) \quad \|A_i(\psi B_\Sigma, \bar{\nabla} \psi)\|_{L^\infty} \leq C \|\psi\|_{C^1}.$$

Moreover, again by the smoothness of A_i we have by the chain rule that

$$\begin{aligned} |\bar{\nabla}^3(A_i(\psi B_\Sigma, \bar{\nabla} \psi))| & \leq C \sum_{|\alpha| \leq 3} (1 + |\bar{\nabla}^{\alpha_1}(\psi B_\Sigma)|)(1 + |\bar{\nabla}^{\alpha_2}(\psi B_\Sigma)|)(1 + |\bar{\nabla}^{\alpha_3}(\psi B_\Sigma)|) \cdots \\ & \quad \cdots (1 + |\bar{\nabla}^{1+\alpha_4} \psi|)(1 + |\bar{\nabla}^{1+\alpha_5} \psi|)(1 + |\bar{\nabla}^{1+\alpha_6} \psi|). \end{aligned}$$

Therefore Lemma 2.2 for $T_i = \psi B_\Sigma$ for $i = 1, 2, 3$ and $T_i = \bar{\nabla} \psi$ for $i = 4, 5, 6$ implies

$$(4.19) \quad \begin{aligned} \|A_i(\psi B_\Sigma, \bar{\nabla} \psi)\|_{H^3} & \leq C((1 + \|\psi B_\Sigma\|_{L^\infty})(1 + \|\psi\|_{H^4}) + (1 + \|\psi\|_{C^1})(1 + \|\psi B_\Sigma\|_{H^3})) \\ & \leq C(1 + \|\psi\|_{H^4} + \|\psi B_\Sigma\|_{H^3}). \end{aligned}$$

Finally, we estimate the third term similarly by the chain rule, by the regularity of a_0 , (4.13), Lemma 2.2 and by interpolation inequality in Proposition 2.1 as

$$(4.20) \quad \begin{aligned} & \|a_0(\psi, \bar{\nabla} \psi, B_\Sigma)\|_{H^2(\Sigma)} \\ & \leq C \sum_{|\alpha| \leq 2} \|(1 + |\bar{\nabla}^{1+\alpha_1} \psi|)(1 + |\bar{\nabla}^{1+\alpha_2} \psi|)(1 + |\bar{\nabla}^{\alpha_3} B_\Sigma|)(1 + |\bar{\nabla}^{\alpha_4} B_\Sigma|)\|_{L^2(\Sigma)} \\ & \leq C(1 + \|\psi\|_{H^3} + \|B_\Sigma\|_{H^2}) \\ & \leq C(1 + \|\psi\|_{H^3}) \leq \varepsilon \|\psi\|_{H^4} + C_\varepsilon \|\psi\|_{H^2} + C. \end{aligned}$$

We may thus estimate the term $\|\Delta_{\partial E} R_0\|_{L^2(\Sigma)}$ by (4.16), (4.17), (4.18), (4.19) and (4.20) as

$$\|\Delta_{\partial E} R_0\|_{L^2(\Sigma)} \leq C \|\psi\|_{C^1} (1 + \|\psi\|_{H^4} + \|\psi B_\Sigma\|_{H^3}) + \varepsilon \|\psi\|_{H^4} + C_\varepsilon \|\psi\|_{H^2} + C.$$

Using Lemma 2.2 once again we obtain $\|\psi B_\Sigma\|_{H^3} \leq C \|\psi\|_{L^\infty} \|B_\Sigma\|_{H^3} + C \|\psi\|_{H^3}$. We have by Proposition 4.1 that $\|\psi\|_{L^\infty} \leq C_\varepsilon h^{\frac{1}{2}-\varepsilon}$ for any $\varepsilon > 0$ and by (4.13) we have $\|B_\Sigma\|_{H^3(\Sigma)} \leq C_1 h^{-\frac{1}{4}}$, which yield

$$(4.21) \quad \|\psi\|_{L^\infty} \|B_\Sigma\|_{H^3} \leq C_\varepsilon h^{\frac{1}{4}-\varepsilon} \leq 1,$$

when $\varepsilon < \frac{1}{4}$ and h is small. Therefore we may bound $\|\Delta_{\partial E} R_0\|_{L^2(\Sigma)}$ by the previous estimates as

$$\|\Delta_{\partial E} R_0\|_{L^2(\partial E)} \leq C\|\psi\|_{C^1}\|\psi\|_{H^4} + \varepsilon\|\psi\|_{H^4} + C_\varepsilon\|\psi\|_{H^2} + C.$$

Finally, we use Lemma 2.3 with $k = 1, 2$ and (4.13) to deduce that

$$(4.22) \quad \|\psi\|_{H^2(\Sigma)} \leq C(1 + \|\Delta_{\partial E}\psi\|_{L^2(\partial E)}) \quad \text{and} \quad \|\psi\|_{H^4(\Sigma)} \leq C(1 + \|\Delta_{\partial E}^2\psi\|_{L^2(\partial E)}).$$

Hence, we obtain by (4.4) and choosing ε small enough

$$(4.23) \quad \|\Delta_{\partial E} R_0\|_{L^2(\partial E)} \leq \frac{1}{2}\|\Delta_{\partial E}^2\psi\|_{L^2(\partial E)} + C\|\Delta_{\partial E}\psi\|_{L^2(\partial E)} + C.$$

We are left with the two last terms in (4.14). We estimate only the second last term $\frac{1}{h}\|\Delta_{\partial E}(\frac{H_E}{2}\psi^2)\|_{L^2}^2$ as the final term is treated with the same argument. We first use Lemma 2.2, (4.13), the Sobolev embedding, interpolation inequalities and (2.6) to estimate

$$(4.24) \quad \begin{aligned} \|\Delta_{\partial E}(\frac{H_E}{2}\psi^2)\|_{L^2} &\leq C\|\psi\|_{L^\infty}^2\|H_\Sigma\|_{H^2} + C\|\psi\|_{L^\infty}\|H_\Sigma\|_{L^\infty}\|\psi\|_{H^2} + C\|H_\Sigma\|_{L^\infty}\|\nabla_{\partial E}\psi\|_{L^4}^2 \\ &\leq C\|\psi\|_{L^\infty}\|\psi\|_{H^2} \\ &\leq C\|\psi\|_{L^\infty}(\|\Delta_{\partial E}\psi\|_{L^2} + C\|\psi\|_{L^2}). \end{aligned}$$

Proposition 4.1 implies $\|\psi\|_{L^\infty} \leq C_\varepsilon h^{\frac{1}{2}-\varepsilon}$. We choose $\varepsilon = \frac{1}{4}$ and deduce

$$(4.25) \quad \frac{1}{h}\|\Delta_{\partial E}(\frac{H_E}{2}\psi^2)\|_{L^2}^2 \leq \frac{C}{\sqrt{h}}\|\Delta_{\partial E}\psi\|_{L^2}^2 + C.$$

The same argument also yields

$$(4.26) \quad \frac{1}{h}\|\Delta_{\partial E}(\frac{K_E}{3}\psi^3)\|_{L^2}^2 \leq \frac{C}{\sqrt{h}}\|\Delta_{\partial E}\psi\|_{L^2}^2 + C.$$

We obtain from (4.14), (4.15), (4.23), (4.25) and (4.26) that

$$\frac{1}{h}\|\Delta_{\partial E}\psi\|_{L^2}^2 + \|\Delta_{\partial E}^2\psi\|_{L^2}^2 \leq C + \frac{1}{2}\|\Delta_{\partial E}^2\psi\|_{L^2}^2 + \frac{C}{\sqrt{h}}\|\Delta_{\partial E}\psi\|_{L^2}^2$$

when h is small. This yields

$$\frac{1}{h}\|\Delta_{\partial E}\psi\|_{L^2}^2 + \|\Delta_{\partial E}^2\psi\|_{L^2}^2 \leq C,$$

when h is small. Therefore we obtain $\|\psi\|_{H^4} \leq C$ using (4.22).

In order to prove $\|\psi\|_{L^2} \leq Ch$, we remark that the previous argument, in fact, implies that the right-hand-side of (4.11) is bounded in $L^2(\partial E)$. Therefore we deduce that the left-hand-side of (4.11) is also bounded in $L^2(\partial E)$, i.e.,

$$\frac{1}{h}\|\xi_{F,E}\|_{L^2(\partial E)} = \frac{1}{h}\|\psi + \frac{H_E}{2}\psi^2 + \frac{K_E}{3}\psi^3\|_{L^2(\partial E)} \leq C.$$

The estimate $\|\psi\|_{L^2} \leq Ch$ then follows from (3.7).

□

The previous lemma implies regularity for the heighfunction. We also need an opposite result which states that regularity for the heighfunction implies regularity for the minimizing set F .

Lemma 4.3. *Assume $E \subset \mathbb{R}^3$ is C^5 -regular, satisfies the assumptions (4.2) and $\|\Delta_{\partial E} H_E\|_{H^1(\partial E)} \leq K_0 h^{-\frac{1}{4}}$ for $K_0 \geq 1$. Let $F \subset \mathbb{R}^3$ be a minimizer of (4.1) with $\delta < \delta_2$, where δ_2 is from Proposition 4.1. Assume that the heightfunction in (4.4) satisfies*

$$\|\psi\|_{L^2(\partial E)} \leq L_0 h \quad \text{and} \quad \|\Delta_{\partial E} \psi\|_{L^2(\partial E)} \leq L_0 \sqrt{h}.$$

Then there is a constant C_2 , depending on r_0, R_0 and L_0 , such that

$$\|\Delta_{\partial F} H_F\|_{L^2(\partial F)} \leq C_2(1 + \sqrt{K_0}) \quad \text{and} \quad \|\nabla_{\partial F} \Delta_{\partial F} H_F\|_{L^2(\partial F)} \leq C_2 h^{-\frac{1}{4}}$$

for $h \leq h_0$, where h_0 depends on r_0, R_0, K_0 and δ . Moreover, F satisfies the UBC with radius $r_0/2$.

Proof. Below we denote by C a generic constant that depends on r_0, R_0, L_0 . We need only to be careful to trace the dependence on K_0 .

Note first that by (2.6) and by interpolation from Proposition 2.1 it holds

$$(4.27) \quad \|\psi\|_{H^1(\partial E)} \leq C h^{\frac{3}{4}}.$$

Moreover, by Lemma 4.2 it holds $\|\psi\|_{H^4(\Sigma)} \leq C_1$ for $C_1 = C_1(r_0, R_0, K_0)$, which together with (4.27), with Sobolev embedding and interpolation inequality (2.5) yields that given $0 < \alpha < 1$ there exists $\gamma > 0$ such that $\|\psi\|_{C^{2,\alpha}(\Sigma)} \leq C_1 h^\gamma$ where $C_1 = C_1(r_0, R_0, K_0, \alpha)$. From here we infer, see e.g. [13, Theorem 2.6] and [15, Remark 3.4.6], that F satisfies the UBC with radius $r_0/2$ when h is small. In particular,

$$(4.28) \quad \|B_F\|_{L^\infty(\partial F)} \leq \frac{2}{r_0}.$$

We will obtain the estimates again from the Euler-Lagrange equations (4.10). We define $\tilde{f} = \frac{d_{H-1}(F;E)}{h} f$ and write (4.10) as

$$(4.29) \quad \begin{cases} H_F + (\tilde{f} \circ \pi_{\partial E}) = \lambda & \text{on } \partial F \\ -\Delta_{\partial E} \tilde{f} = \frac{\xi_{F,E}}{h} & \text{on } \partial E, \end{cases}$$

where $\xi_{F,E}$ is defined in (2.11).

By (3.7), by the assumption $\|\psi\|_{L^2(\partial E)} \leq L_0 h$ and by (4.27) it holds

$$(4.30) \quad \|\xi_{F,E}\|_{L^2(\partial E)} \leq C h \quad \text{and} \quad \|\xi_{F,E}\|_{H^1(\partial E)} \leq C h^{\frac{3}{4}}.$$

We have by (2.6), by Lemma 2.3, by (4.13) and by the second equation in (4.29) that

$$(4.31) \quad \begin{aligned} \|\tilde{f}\|_{H^2(\Sigma)} &\leq C(1 + \|\Delta_{\partial E} \tilde{f}\|_{L^2(\partial E)}) \leq C(1 + \frac{\|\xi_{F,E}\|_{L^2(\partial E)}}{h}) \leq C \quad \text{and} \\ \|\tilde{f}\|_{H^3(\Sigma)} &\leq C(1 + \|B_\Sigma\|_{H^2(\Sigma)} + \|\Delta_{\partial E} \tilde{f}\|_{H^1(\partial E)}) \leq C(K_0 + \frac{\|\xi_{F,E}\|_{H^1(\partial E)}}{h}) \leq C h^{-\frac{1}{4}}, \end{aligned}$$

when $h \leq h_0(r_0, R_0, K_0, \delta)$ is small. Above we also used the fact that \tilde{f} has zero average in every component of ∂E and Sobolev embedding.

Next we need to relate the derivative of the function $\tilde{f} \circ \pi_{\partial E}$ on $\Gamma = \partial F$ to the derivatives of \tilde{f} on $\Sigma = \partial E$. First, we have by (2.19) that

$$|\nabla_{\partial F}(\tilde{f} \circ \pi_{\partial E})(x)| \leq C|\nabla_{\partial E}\tilde{f}(\pi_{\partial E}(x))|$$

for all $x \in \partial F$. We write this in terms of covariant derivatives as

$$(4.32) \quad |\bar{\nabla}_{\Gamma}(\tilde{f} \circ \pi_{\partial E})(x)| \leq C|\bar{\nabla}_{\Sigma}\tilde{f}(\pi_{\partial E}(x))|$$

for all $x \in \Gamma$. The higher order version of (4.31) is technically more involved and to that aim, we proceed by using [25, Lemma 5.3] with $\Sigma = \partial E$ to deduce for $k = 2, 3$ and for all $x \in \Gamma = \partial F$ that

$$(4.33) \quad |\nabla^k(\tilde{f} \circ \pi_{\partial E})(x)| \leq C \sum_{|\alpha| \leq k} (1 + |\bar{\nabla}_{\Sigma}^{\alpha_1} B_{\Sigma}(\pi_{\partial E}(x))|) \cdots (1 + |\bar{\nabla}_{\Sigma}^{\alpha_{k-1}} B_{\Sigma}(\pi_{\partial E}(x))|) |\bar{\nabla}_{\Sigma}^{\alpha_k} \tilde{f}(\pi_{\partial E}(x))|,$$

where $\bar{\nabla}_{\Sigma}^j$ denotes j -th order covariant derivative on Σ and ∇^j denotes the j -th differential in \mathbb{R}^3 . Even if it is not explicitly stated in [25, Lemma 5.3], it is clear that in (4.33) the last index is never zero, i.e., $\alpha_k \geq 1$.

Let us first prove the estimate $\|\Delta_{\partial F} H_F\|_{L^2(\partial F)} \leq C_2(1 + \sqrt{K_0})$. We use (4.33) for $k = 2$ and (4.28) to estimate point-wise on $x \in \partial F$ (recall that in (4.33) $\alpha_2 \geq 1$)

$$|\nabla^2(\tilde{f} \circ \pi_{\partial E})(x)| \leq C|\bar{\nabla}_{\Sigma}^2 \tilde{f}(\pi_{\partial E}(x))| + C(1 + |\bar{\nabla}_{\Sigma} B_{\Sigma}(\pi_{\partial E}(x))|) |\bar{\nabla}_{\Sigma} \tilde{f}(\pi_{\partial E}(x))|.$$

By Cauchy-Schwarz inequality, by the Sobolev embedding and by (4.31) it holds

$$\begin{aligned} \|\nabla^2(\tilde{f} \circ \pi_{\partial E})\|_{L^2(\Gamma)} &\leq C(\|\tilde{f}\|_{H^2(\Sigma)} + (1 + \|\bar{\nabla}_{\Sigma} B_{\Sigma}\|_{L^4(\Sigma)}) \|\bar{\nabla}_{\Sigma} \tilde{f}\|_{L^4(\Sigma)}) \\ &\leq C(\|\tilde{f}\|_{H^2(\Sigma)} + (1 + \|\bar{\nabla}_{\Sigma} B_{\Sigma}\|_{L^4(\Sigma)}) \|\tilde{f}\|_{H^2(\Sigma)}) \\ &\leq C(1 + \|\bar{\nabla}_{\Sigma} B_{\Sigma}\|_{L^4(\Sigma)}). \end{aligned}$$

By the interpolation inequality in Proposition 2.1, by (4.13) and by the fact that $|B_{\Sigma}| \leq C$ it holds

$$\|\bar{\nabla}_{\Sigma} B_{\Sigma}\|_{L^4(\Sigma)} \leq C\|B_{\Sigma}\|_{H^2(\Sigma)}^{\frac{1}{2}} \|B_{\Sigma}\|_{L^{\infty}(\Sigma)}^{\frac{1}{2}} \leq C\sqrt{K_0}.$$

Therefore we have

$$(4.34) \quad \|\nabla^2(\tilde{f} \circ \pi_{\partial E})\|_{L^2(\Gamma)} \leq C(1 + \sqrt{K_0}).$$

We may write the Laplacian of $\tilde{f} \circ \pi_{\partial E}$ on ∂F as

$$(4.35) \quad \Delta_{\partial F}(\tilde{f} \circ \pi_{\partial E}) = \Delta_{\mathbb{R}^3}(\tilde{f} \circ \pi_{\partial E}) - \langle \nabla^2(\tilde{f} \circ \pi_{\partial E}) \nu_F, \nu_F \rangle - H_F(\nabla(\tilde{f} \circ \pi_{\partial E}) \cdot \nu_F).$$

Therefore we have by (4.28) and (4.34) that

$$(4.36) \quad \|\Delta_{\partial F}(\tilde{f} \circ \pi_{\partial E})\|_{L^2(\partial F)} \leq C(1 + \sqrt{K_0}).$$

The first equation in (4.29) then implies $\|\Delta_{\partial F} H_F\|_{L^2(\partial F)} \leq C_2(1 + \sqrt{K_0})$ and the claim follows.

We need yet to prove $\|\nabla_{\partial F} \Delta_{\partial F} H_F\|_{L^2(\partial F)} \leq C_2 h^{-\frac{1}{4}}$. We use (4.33) for $k = 3$ and (4.28) to estimate point-wise on $x \in \partial F$

$$\begin{aligned} |\nabla^3(\tilde{f} \circ \pi_{\partial E})(x)| &\leq C|\bar{\nabla}_\Sigma^3 \tilde{f}(\pi_{\partial E}(x))| + C(1 + |\bar{\nabla}_\Sigma B_\Sigma(\pi_{\partial E}(x))|)|\bar{\nabla}_\Sigma^2 \tilde{f}(\pi_{\partial E}(x))| \\ &\quad + C(1 + |\bar{\nabla}_\Sigma^2 B_\Sigma(\pi_{\partial E}(x))| + |\bar{\nabla}_\Sigma B_\Sigma(\pi_{\partial E}(x))|^2)|\bar{\nabla}_\Sigma \tilde{f}(\pi_{\partial E}(x))|. \end{aligned}$$

We use Lemma 2.2 with $k = 2$ for $T_1 = T_2 = B_\Sigma$ and $T_3 = \bar{\nabla}_\Sigma \tilde{f}$, and (4.28) to infer

$$\|\nabla^3(\tilde{f} \circ \pi_{\partial E})\|_{L^2(\Gamma)} \leq C(\|\tilde{f}\|_{H^3(\Sigma)} + C(1 + \|B_\Sigma\|_{H^2(\Sigma)})\|\bar{\nabla} \tilde{f}\|_{L^\infty(\Sigma)}).$$

We use Sobolev embedding, interpolation in Proposition 2.1 and (4.31)

$$\|\bar{\nabla} \tilde{f}\|_{L^\infty(\Sigma)} \leq C\|\tilde{f}\|_{W^{2,4}(\Sigma)} \leq C\|\tilde{f}\|_{H^3(\Sigma)}^{\frac{1}{2}}\|\tilde{f}\|_{H^2(\Sigma)}^{\frac{1}{2}} \leq Ch^{-\frac{1}{8}}.$$

Therefore the two inequalities above together with (4.31) and (4.13) yield

$$(4.37) \quad \|\nabla^3(\tilde{f} \circ \pi_{\partial E})\|_{L^2(\Gamma)} \leq C(h^{-\frac{1}{4}} + K_0 h^{-\frac{1}{8}}) \leq Ch^{-\frac{1}{4}}$$

for $h \leq h_0 = h_0(r_0, R_0, K_0, \delta)$.

Recall the formula (4.35) for the Laplacian of $\tilde{f} \circ \pi_{\partial E}$ on ∂F , and that by the first equation in (4.29) it holds $\nabla_{\partial F} H_F = -\nabla_{\partial F}(\tilde{f} \circ \pi_{\partial E})$. Therefore we may estimate point-wise on ∂F using (4.28)

$$|\nabla_{\partial F} \Delta_{\partial F}(\tilde{f} \circ \pi_{\partial E})| \leq C(|\nabla^3(\tilde{f} \circ \pi_{\partial E})| + |\nabla^2(\tilde{f} \circ \pi_{\partial E})| + |\nabla(\tilde{f} \circ \pi_{\partial E})| + |\nabla(\tilde{f} \circ \pi_{\partial E})|^2).$$

We infer by the Sobolev embedding, by (4.34) and (4.37) that

$$\begin{aligned} \|\nabla_{\partial F} \Delta_{\partial F}(\tilde{f} \circ \pi_{\partial E})\|_{L^2(\partial F)} &\leq C\|\nabla^3(\tilde{f} \circ \pi_{\partial E})\|_{L^2(\partial F)} + C\|\nabla^2(\tilde{f} \circ \pi_{\partial E})\|_{L^2(\partial F)} \\ &\quad + C\|\nabla(\tilde{f} \circ \pi_{\partial E})\|_{L^2(\partial F)} + C\|\nabla(\tilde{f} \circ \pi_{\partial E})\|_{L^4(\partial F)}^2 \\ &\leq C\|\nabla^3(\tilde{f} \circ \pi_{\partial E})\|_{L^2(\partial F)} + C\|\tilde{f} \circ \pi_{\partial E}\|_{H^2(\Gamma)} + C\|\tilde{f} \circ \pi_{\partial E}\|_{H^2(\Gamma)}^2 \\ &\leq Ch^{-\frac{1}{4}} + C(1 + K_0) \leq Ch^{-\frac{1}{4}}, \end{aligned}$$

for $h \leq h_0 = h_0(r_0, R_0, K_0)$. This concludes the proof. \square

5. PROOF OF THE MAIN THEOREMS

5.1. Iteration Lemma. We consider three sets E, F and G such that $E \subset \mathbb{R}^3$ satisfies the assumptions (4.2) and $\|\Delta_{\partial E} H_E\|_{H^1(\partial E)} \leq K_0 h^{-\frac{1}{4}}$, and F is a minimizer of (4.1). It follows from Lemma 4.2 and Lemma 4.3 that also F satisfies the assumptions (4.2) with different constants, i.e., it satisfies the UBC with radius $r_0/2$,

$$\|\Delta_{\partial F} H_F\|_{L^2(\partial F)} \leq K_1 \quad \text{and} \quad \|\Delta_{\partial F} H_F\|_{H^1(\partial F)} \leq K_1 h^{-\frac{1}{4}}$$

for some K_1 depending on r_0, R_0, K_0 and $F \subset B_{2R_0}$. We then proceed and consider a minimizer G of the problem

$$(5.1) \quad \min \left\{ P(G) + \frac{d_{H^{-1}}(G; F)^2}{2h} : G\Delta F \subset \overline{n_\delta}(\partial F) \right\}.$$

We may again apply Proposition 4.1, Lemma 4.2 and Lemma 4.3 to deduce that when $\delta \leq \delta_3$, where δ_3 depends on r_0, K_1, R_0 and δ_2 from Proposition 4.1 and $h \leq h_1$, we may write the boundary of G as a normal deformation with a heightfunction $\psi_F : \partial F \rightarrow \mathbb{R}$,

$$\partial G = \{x + \psi_F(x)\nu_F(x) : x \in \partial F\}$$

and $\|\psi_F\|_{L^2(\partial F)} \leq C_3 h$, $\|\psi_F\|_{H^4(\Gamma)} \leq C_3$, where $\Gamma = \partial F$. In order to avoid confusion we denote the heightfunction of ∂F over ∂E as ψ_E , i.e.,

$$\partial F = \{x + \psi_E(x)\nu_E(x) : x \in \partial E\}.$$

Since K_1 depends on r_0, R_0 and K_0 , then eventually δ_3 also depends on r_0, R_0, K_0 and δ_2 .

Lemma 5.1. *Assume $E \subset \mathbb{R}^3$ is C^5 -regular, satisfies the assumptions (4.2) and $\|\Delta_{\partial E} H_E\|_{H^1(\partial E)} \leq K_0 h^{-\frac{1}{4}}$. Assume $F \subset \mathbb{R}^3$ is a minimizer of (4.1) with $\delta < \delta_3$, where δ_3 depends on r_0, R_0, K_0 and δ_2 from Proposition 4.1, and assume $G \subset \mathbb{R}^3$ is a minimizer of (5.1) and denote the associated heightfunctions by ψ_E, ψ_F and let $\xi_{F,E}, \xi_{G,F}$ as in (2.11). Then, there exists a constant C_4 , depending on r_0, R_0 and K_0 , such that*

$$(5.2) \quad \int_{\partial F} \left(\xi_{G,F}^2 + \frac{h}{2} |\Delta_{\partial F} \psi_F|^2 \right) d\mathcal{H}^2 \leq (1 + C_4 h) \int_{\partial E} \left(\xi_{F,E}^2 + \frac{h}{8} |\Delta_{\partial E} \psi_E|^2 \right) d\mathcal{H}^2,$$

for $h \leq h_1$, where $h_1 = h_1(r_0, R_0, K_0, \delta)$.

Proof. By the discussion at the beginning of the section, we may deduce that Proposition 4.1, Lemma 4.2 and Lemma 4.3 hold also for F and G with possibly smaller uniform ball radius and smaller threshold for δ which we denote by $\delta_3 \leq \delta_2$. In particular, we may write the Euler-Lagrange equation (4.11) for G , which reads as

$$\frac{1}{h} \xi_{G,F} = \Delta_{\partial F} (-\Delta_{\partial F} \psi_F + H_F) + \Delta_{\partial F} R_0 \quad \text{on } \partial F,$$

where the error term is of the form (2.20). We multiply this by $\xi_{G,F}$ and integrate by parts to infer

$$(5.3) \quad \frac{1}{h} \int_{\partial F} \xi_{G,F}^2 d\mathcal{H}^2 = \int_{\partial F} (-\Delta_{\partial F} \psi_F + H_F) (\Delta_{\partial F} \xi_{G,F}) d\mathcal{H}^2 + \int_{\partial F} R_0 (\Delta_{\partial F} \xi_{G,F}) d\mathcal{H}^2.$$

Recall that by (2.11)

$$\xi_{G,F} = \psi_F + \frac{H_F}{2} \psi_F^2 + \frac{K_F}{3} \psi_F^3.$$

Arguing as in (4.24) we deduce that for every $\varepsilon > 0$ it holds by the smallness of $\|\psi_F\|_{L^\infty(\partial F)}$ that

$$\|\Delta_{\partial F} \left(\frac{H_F}{2} \psi_F^2 \right)\|_{L^2(\partial F)} \leq C \|\psi_F\|_{L^\infty(\partial F)} \|\psi_F\|_{H^2(\Gamma)} \leq \varepsilon \|\psi_F\|_{H^2(\Gamma)}$$

for $\Gamma = \partial F$ when h is small. A similar argument yields

$$\|\Delta_{\partial F}(\frac{K_F}{3}\psi_F^3)\|_{L^2(\partial F)} \leq C\|\psi_F\|_{L^\infty(\partial F)}\|\psi_F\|_{H^2(\Gamma)} \leq \varepsilon\|\psi_F\|_{H^2(\Gamma)}.$$

Therefore we have by (5.3) using the above and (2.6)

$$\begin{aligned} (5.4) \quad & \frac{1}{h} \int_{\partial F} \xi_{G,F}^2 d\mathcal{H}^2 + \int_{\partial F} (\Delta_{\partial F} \psi_F)^2 d\mathcal{H}^2 \\ & \leq \int_{\partial F} H_F(\Delta_{\partial F} \xi_{G,F}) d\mathcal{H}^2 + \varepsilon \|\Delta_{\partial F} \psi_F\|_{L^2(\partial F)}^2 + C\varepsilon \|\psi_F\|_{L^2(\partial F)}^2 \\ & \quad + C(\|\Delta_{\partial F} \psi_F\|_{L^2(\partial F)} + \|\psi_F\|_{L^2(\partial F)}) \|R_0\|_{L^2(\partial F)}. \end{aligned}$$

We need to bound $\|R_0\|_{L^2(\partial F)}$. We use the form in (2.20) to bound the error term point-wise on $\Gamma = \partial F$ as

$$|R_0| \leq C(|\psi_F| + |\bar{\nabla} \psi_F|)(1 + |\bar{\nabla}^2 \psi_F| + |\bar{\nabla}(\psi_F B_\Gamma)|).$$

By Lemma 4.3, Lemma 2.3 and by Sobolev embedding it holds

$$(5.5) \quad \|B_\Gamma\|_{L^\infty(\Gamma)}, \|B_\Gamma\|_{H^2(\Gamma)} \leq C \quad \text{and} \quad \|\bar{\nabla} B_\Gamma\|_{L^\infty(\Gamma)} \leq Ch^{-\frac{1}{4}}.$$

Moreover Proposition 4.1 with $\varepsilon = \frac{1}{4}$ implies $\|\psi_F\|_{L^\infty(\partial F)} \leq Ch^{\frac{1}{4}}$. Therefore we have

$$|\bar{\nabla}(\psi_F B_\Gamma)| \leq |\bar{\nabla} \psi_F| |B_\Gamma| + |\psi_F| |\bar{\nabla} B_\Gamma| \leq C.$$

Thus it holds by interpolation from Proposition 2.1, by the smallness of $\|\psi_F\|_{C^1}$ and by (2.6)

$$\begin{aligned} \|R_0\|_{L^2(\partial F)} & \leq C\|\psi_F\|_{C^1}\|\psi_F\|_{H^2(\Gamma)} + C\|\psi_F\|_{H^1(\Gamma)} \leq \varepsilon\|\psi_F\|_{H^2(\Gamma)} + C_\varepsilon\|\psi_F\|_{L^2(\partial F)} \\ & \leq \varepsilon\|\Delta_{\partial F} \psi_F\|_{L^2(\partial F)} + C_\varepsilon\|\psi_F\|_{L^2(\partial F)}. \end{aligned}$$

Recall also that (3.7) implies $|\psi_F(x)| \leq 2|\xi_{G,F}(x)|$ for all $x \in \partial F$. Hence we may estimate (5.4) as

$$(5.6) \quad \left(\frac{1}{h} - C\right) \int_{\partial F} \xi_{G,F}^2 d\mathcal{H}^2 + \frac{1}{2} \int_{\partial F} (\Delta_{\partial F} \psi_F)^2 d\mathcal{H}^2 \leq \int_{\partial F} H_F(\Delta_{\partial F} \xi_{G,F}) d\mathcal{H}^2$$

when ε is chosen small enough.

Let us next analyze the right-hand-side of (5.6). We integrate by parts and use a change of variables with the diffeomorphism $\Psi_E : \partial E \rightarrow \partial F$, $\Psi_E(x) = x + \psi_E(x)\nu_E(x)$

$$\begin{aligned} \int_{\partial F} H_F(\Delta_{\partial F} \xi_{G,F}) d\mathcal{H}^2 & = - \int_{\partial F} (\nabla_{\partial F} H_F \cdot \nabla_{\partial F} \xi_{G,F}) d\mathcal{H}^2 \\ & = - \int_{\partial E} ((\nabla_{\partial F} H_F \cdot \nabla_{\partial F} \xi_{G,F}) \circ \Psi_E) J_{\partial E} \Psi_E d\mathcal{H}^2. \end{aligned}$$

We use (2.30) for the function $f = H_F \circ \Psi_E : \partial E \rightarrow \mathbb{R}$. Notice that for this function it holds $f \circ \pi_{\partial E} = H_F$ on ∂F . We deduce by (2.30) that for $x \in \partial E$ it holds

$$(\nabla_{\partial F} H_F)(\Psi_E(x)) = (I + A(\psi_E(x), \nabla_{\partial E} \psi_E(x), \nu_E(x), B_E(x))) \nabla_{\partial E} (H_F \circ \Psi_E)(x),$$

for a matrix which satisfies $A(0, 0, \cdot, \cdot) = 0$. Similarly it holds

$$(\nabla_{\partial F} \xi_{G,F})(\Psi_E(x)) = (I + A(\psi_E(x), \nabla_{\partial E} \psi_E(x), \nu_E(x), B_E(x))) \nabla_{\partial E} (\xi_{G,F} \circ \Psi_E)(x).$$

Therefore by the above and by the formula for the Jacobian (3.5) we have

$$(5.7) \quad \int_{\partial F} H_F(\Delta_{\partial F} \xi_{G,F}) d\mathcal{H}^2 \leq - \int_{\partial E} ((\nabla_{\partial E}(H_F \circ \Psi_E) \cdot \nabla_{\partial E}(\xi_{G,F} \circ \Psi_E)) \sqrt{J_{\partial E} \Psi_E} d\mathcal{H}^2 + \|R_1\|_{L^1(\partial E)},$$

where the error term satisfies the point-wise bound $x \in \partial E$

$$|R_1(x)| \leq C(|\psi_E(x)| + |\nabla_{\partial E} \psi_E(x)|) |\nabla_{\partial F} H_F(\Psi_E(x))| |\nabla_{\partial F} \xi_{G,F}(\Psi_E(x))|.$$

By Hölder's inequality

$$\|R_1\|_{L^1(\partial E)} \leq C \|\psi_E\|_{W^{1,4}(\partial E)} \|\nabla_{\partial F} H_F\|_{L^4(\partial F)} \|\nabla_{\partial F} \xi_{G,F}\|_{L^2(\partial F)}.$$

The estimate (3.7) implies $\|\nabla_{\partial F} \xi_{G,F}\|_{L^2(\partial F)} \leq C \|\psi_F\|_{H^1(\partial F)}$, while (5.5) and Sobolev embedding give $\|\nabla_{\partial F} H_F\|_{L^4(\partial F)} \leq \|B_\Gamma\|_{H^2(\Gamma)} \leq C$. Therefore we may bound by interpolation from Proposition 2.1, by (2.6) and by (3.7)

$$(5.8) \quad \begin{aligned} \|R_1\|_{L^1(\partial E)} &\leq C \|\psi_E\|_{W^{1,4}(\partial E)} \|\psi_F\|_{H^1(\partial F)} \\ &\leq \varepsilon \|\Delta_{\partial E} \psi_E\|_{L^2(\partial E)}^2 + C_\varepsilon \|\xi_{F,E}\|_{L^2(\partial E)}^2 + \varepsilon \|\Delta_{\partial F} \psi_F\|_{L^2(\partial F)}^2 + C_\varepsilon \|\xi_{G,F}\|_{L^2(\partial F)}^2. \end{aligned}$$

We integrate (5.7) by parts and have

$$(5.9) \quad \begin{aligned} \int_{\partial F} H_F(\Delta_{\partial F} \xi_{G,F}) d\mathcal{H}^2 &\leq \int_{\partial E} (\Delta_{\partial E}(H_F \circ \Psi_E))(\xi_{G,F} \circ \Psi_E) \sqrt{J_{\partial E} \Psi_E} d\mathcal{H}^2 \\ &\quad + \|R_1\|_{L^1(\partial E)} + \|R_2\|_{L^1(\partial E)}, \end{aligned}$$

where the first error term satisfies (5.8) and the second error term satisfies the following point-wise bound on $\Sigma = \partial E$ by the formula for $J_{\partial E} \Psi_E$ in (3.5)

$$|R_2(x)| \leq C |\psi_F(\Psi_E(x))| |\nabla_{\partial F} H_F(\Psi_E(x))| (|\psi_E(x)| + |\nabla_{\partial E} \psi_E(x)|) (1 + |\bar{\nabla}_\Sigma^2 \psi_E(x)| + |\bar{\nabla}_\Sigma B_\Sigma(x)|).$$

By Hölder's inequality

$$\|R_2\|_{L^1(\partial E)} \leq C \|\psi_F\|_{L^\infty(\partial F)} \|\nabla_{\partial F} H_F\|_{L^4(\partial F)} \|\psi_E\|_{W^{1,4}(\partial E)} (1 + \|\psi_E\|_{H^2(\Sigma)} + \|B_\Sigma\|_{H^1(\Sigma)}).$$

By (5.5) it holds $\|\nabla_{\partial F} H_F\|_{L^4(\partial F)} \leq \|B_\Gamma\|_{H^2(\Gamma)} \leq C$ and Lemma 4.2 implies $\|\psi_E\|_{H^2(\Sigma)} \leq C$. We use the inequality (2.6), Sobolev embedding and interpolation from Proposition 2.1 to get

$$(5.10) \quad \begin{aligned} \|R_2\|_{L^1(\partial E)} &\leq C (\|\psi_E\|_{W^{1,4}(\partial E)}^2 + \|\psi_F\|_{L^\infty(\partial F)}^2) \\ &\leq \varepsilon \|\Delta_{\partial E} \psi_E\|_{L^2(\partial E)}^2 + C_\varepsilon \|\xi_{F,E}\|_{L^2(\partial E)}^2 + \varepsilon \|\Delta_{\partial F} \psi_F\|_{L^2(\partial F)}^2 + C_\varepsilon \|\xi_{G,F}\|_{L^2(\partial F)}^2. \end{aligned}$$

Since F is a minimizer it satisfies the Euler-Lagrange equation (4.10), which we write as a one equation on ∂E

$$\frac{1}{h} \xi_{F,E} = \Delta_{\partial E}(H_F \circ \Psi_E) \quad \text{on } \partial E.$$

We use this in (5.9) and have by Young's inequality and change of variables

$$\begin{aligned}
& \int_{\partial F} H_F(\Delta_{\partial F} \xi_{G,F}) d\mathcal{H}^2 \\
& \leq \frac{1}{h} \int_{\partial E} \xi_{F,E} (\xi_{G,F} \circ \Psi_E) \sqrt{J_{\partial E} \Psi_E} d\mathcal{H}^2 + \|R_1\|_{L^1(\partial E)} + \|R_2\|_{L^1(\partial E)} \\
& \leq \frac{1}{2h} \int_{\partial E} (\xi_{G,F} \circ \Psi_E)^2 J_{\partial E} \Psi_E d\mathcal{H}^2 + \frac{1}{2h} \int_{\partial E} \xi_{F,E}^2 d\mathcal{H}^2 + \|R_1\|_{L^1(\partial E)} + \|R_2\|_{L^1(\partial E)} \\
& = \frac{1}{2h} \int_{\partial F} \xi_{G,F}^2 d\mathcal{H}^2 + \frac{1}{2h} \int_{\partial E} \xi_{F,E}^2 d\mathcal{H}^2 + \|R_1\|_{L^1(\partial E)} + \|R_2\|_{L^1(\partial E)}.
\end{aligned}$$

The above, (5.6), (5.8) and (5.10) yield

$$\begin{aligned}
& \left(\frac{1}{2h} - C \right) \int_{\partial F} \xi_{G,F}^2 d\mathcal{H}^2 + \frac{1}{3} \int_{\partial F} |\Delta_{\partial F} \psi_F|^2 d\mathcal{H}^2 \\
& \leq \left(\frac{1}{2h} + C \right) \int_{\partial E} \xi_{F,E}^2 d\mathcal{H}^2 + \frac{1}{20} \int_{\partial E} |\Delta_{\partial E} \psi_E|^2 d\mathcal{H}^2.
\end{aligned}$$

when ε is chosen small enough, and the claim follows. \square

5.2. Proof of Theorem 1.1 and Theorem 1.2. We remark that under the assumptions $\|\psi\|_{L^2} \leq L_0 h$ and $\|\psi\|_{H^4} \leq L_0$ in Lemma 4.3, when K_0 in the assumption (4.2) is chosen large enough, we know that the minimizer F satisfies

$$\|\Delta_{\partial F} H_F\|_{L^2(\partial F)} \leq K_0 \quad \text{and} \quad \|\Delta_{\partial F} H_F\|_{H^1(\partial F)} \leq K_0 h^{-\frac{1}{4}}.$$

This means that we are able to keep the a priori regularity estimates assumed in (4.2) except possibly the radius for the uniform ball condition which may decrease from r_0 to $r_0/2$. We may ensure that we also keep the uniform ball condition using the following lemma.

Lemma 5.2. *Assume that $E_0 \subset \mathbb{R}^3$ satisfies the UBC with radius r_0 , $E_0 \subset B_{R_0}$ and is uniformly C^3 -regular. Assume further that $F \subset \mathbb{R}^n$ is a (Λ, α) -minimizer of the perimeter according to Definition 2.6 and*

$$\|B_F\|_{L^\infty(\partial F)} \leq K \quad \text{and} \quad \|\Delta_{\partial F} H_F\|_{L^2(\partial F)} \leq K,$$

for some $K > 0$. Then there is δ , depending on $r_0, R_0, \Lambda, \alpha$ and K , such that if $F \Delta E_0 \subset n_\delta(\partial E_0)$ then F satisfies the UBC with radius $r_0/2$.

Proof. Lemma 2.8 implies that when δ is small then there is $\psi \in C^{1,\gamma}(\partial E_0)$ such that

$$\partial F = \{x + \psi(x)\nu_{E_0}(x) : x \in \partial E_0\}$$

with $\|\psi\|_{C^{1,\gamma}(\partial E_0)}$ small. Note that from $F \Delta E_0 \subset n_\delta(\partial E_0)$ it follows $\|\psi\|_{L^\infty} \leq \delta$. The assumptions on F together with Lemma 2.3 and Sobolev embedding yield for all $\beta \in [0, 1)$, $\|B_\Gamma\|_{C^{0,\beta}(\Gamma)} \leq C$, where $\Gamma = \partial F$. By straightforward calculation we then deduce that $\|\psi\|_{C^{2,\beta}(\Gamma)} \leq C$. By the interpolation inequality (2.5) we then have $\|\psi\|_{C^2(\Gamma)} \leq C\delta^{\frac{\beta}{2+\beta}}$.

We deduce by [13, Theorem 2.6] and [15, Remark 3.4.6] that F satisfies the UBC with radius $r_0/2$ when δ is small. \square

Proof of Theorem 1.1. Assume $E_0 \subset B_{R_0/2} \subset \mathbb{R}^3$ is a C^5 -regular set which satisfies the uniform ball condition with radius $r = 2r_0$ and $\|\Delta_{\partial E_0} H_{E_0}\|_{L^2(\partial E_0)} \leq K$. We fix a large constant $K_0 = K_0(r_0, R_0, K)$, whose choice will be clear later. Let us then choose δ' , which is from Lemma 5.2, and $\delta < \min\{\delta_3, \delta'\}$, where δ_3 is from Lemma 5.1. We consider an approximate sequence $E_k^{h,\delta}$ minimizing the energy (4.1) and simplify the notation by setting $E_k = E_k^{h,\delta}$. We obtain from Proposition 4.1, Lemma 4.2 and Lemma 4.3 that we may write

$$(5.11) \quad \partial E_1 = \{x + \psi_0(x)\nu_{E_0}(x) : x \in \partial E_0\}$$

and it holds $\|\psi_0\|_{L^2(\partial E_0)} \leq L_0 h$ and $\|\psi_0\|_{H^4(\Sigma_0)} \leq L_0$, for $\Sigma_0 = \partial E_0$ and $L_0 = L_0(r_0, R_0, K)$.

We thus have obtained that E_0 and ψ_0 satisfy by the above and by interpolation from Proposition 2.1

$$\|\psi_0\|_{L^2(\partial E_0)} \leq L_0 h \quad \text{and} \quad \|\Delta_{\partial E_0} \psi_0\|_{L^2(\Sigma_0)} \leq L_0 \sqrt{h},$$

by possibly increasing the original constant L_0 , and

$$\|\Delta_{\partial E_0} H_{E_0}\|_{L^2(\partial E_0)} \leq K_0 \quad \text{and} \quad \|\Delta_{\partial E_0} H_{E_0}\|_{H^1(\partial E_0)} \leq K_0 h^{-\frac{1}{4}}.$$

We deduce by Lemma 4.3 that there is a constant $C_2 = C_2(r_0, R_0, L_0)$ such that

$$\|\Delta_{\partial E_1} H_{E_1}\|_{L^2(\partial E_1)} \leq C_2(1 + \sqrt{K_0}) \leq K_0 \quad \text{and} \quad \|\Delta_{\partial E_1} H_{E_1}\|_{H^1(\partial E_1)} \leq C_2 h^{-\frac{1}{4}} \leq K_0 h^{-\frac{1}{4}},$$

when K_0 is chosen large enough, and that E_1 satisfies the UBC with radius r_0 .

We denote by $k_0 \in \mathbb{N}$ the largest index such that $\sigma_0 = k_0 h \leq 1$ and for all $k \leq k_0$ it holds

$$\partial E_k \subset n_\delta(\partial E_0).$$

We claim that for every $k \leq k_0$, E_k satisfies the UBC with radius r_0 and it holds

$$(5.12) \quad \|\Delta_{\partial E_k} H_{E_k}\|_{L^2(\partial E_k)} \leq K_0 \quad \text{and} \quad \|\Delta_{\partial E_k} H_{E_k}\|_{H^1(\partial E_k)} \leq K_0 h^{-\frac{1}{4}}.$$

We prove (5.12) by induction. Recall that it holds for $k = 1$, and assume it holds for $k - 1$. By applying again Proposition 4.1, Lemma 4.2 and Lemma 4.3 we may write the set E_k as

$$\partial E_k = \{x + \psi_{k-1}(x)\nu_{E_{k-1}}(x) : x \in \partial E_{k-1}\}$$

with $\|\psi_{k-1}\|_{L^2(\partial E_{k-1})} \leq C_1 h$ and $\|\psi_{k-1}\|_{H^4(\Sigma_{k-1})} \leq C_1$ for $C_1 = C_1(r_0, R_0, K_0)$, where $\Sigma_{k-1} = \partial E_{k-1}$. We denote $\xi_{k-1} = \xi_{E_k, E_{k-1}}$, the function defined in (2.11), and have by Lemma 5.1

$$\int_{\partial E_j} \left(\xi_j^2 + \frac{h}{2} |\Delta_{\partial E_j} \psi_j|^2 \right) d\mathcal{H}^2 \leq (1 + C_4 h) \int_{\partial E_{j-1}} \left(\xi_{j-1}^2 + \frac{h}{8} |\Delta_{\partial E_{j-1}} \psi_{j-1}|^2 \right) d\mathcal{H}^2$$

for every $j \leq k$. By iterating the above estimate and using $\|\xi_0\|_{L^2(\partial E_0)} \leq 2\|\psi_0\|_{L^2(\partial E_0)} \leq 2L_0h$ and $\|\Delta_{\partial E_0}\psi_0\|_{L^2(\partial E_0)} \leq L_0\sqrt{h}$, we obtain for a constant $C_4 = C_4(r_0, R_0, K_0)$

$$\begin{aligned} \int_{\partial E_k} \left(\xi_k^2 + \frac{h}{4} \sum_{j=1}^k |\Delta_{\partial E_j}\psi_j|^2 \right) d\mathcal{H}^2 &\leq (1 + C_4h)^k \int_{\partial E_0} \left(\xi_0^2 + \frac{h}{8} |\Delta_{\partial E_0}\psi_0|^2 \right) d\mathcal{H}^2 \\ &\leq 5e^{C_4kh} L_0^2 h^2 \\ &\leq 5e^{C_4k_0h} L_0^2 h^2 \leq 10L_0^2 h^2, \end{aligned}$$

provided that $k_0h = \sigma_0 = \sigma_0(r_0, R_0, K_0)$ is small. Therefore, by possibly increasing the value of L_0 we deduce by the above and (3.7)

$$(5.13) \quad \|\psi_k\|_{L^2(\partial E_k)}^2 + h \sum_{j=0}^k \|\Delta_{\partial E_j}\psi_j\|_{L^2(\partial E_j)}^2 \leq L_0^2 h^2.$$

Lemma 4.3 implies that it holds for $C_2 = C_2(r_0, R_0, L_0)$

$$\|\Delta_{\partial E_k} H_{E_k}\|_{L^2(\partial E_k)} \leq C_2(1 + \sqrt{K_0}) \leq K_0 \quad \text{and} \quad \|\Delta_{\partial E_k} H_{E_k}\|_{H^1(\partial E_k)} \leq C_2 h^{-\frac{1}{4}} \leq K_0 h^{-\frac{1}{4}},$$

when K_0 is large. Finally we deduce by Lemma 5.2 that the set E_k satisfies the UBC with radius r_0 , when δ' is chosen small enough. This proves (5.12).

Let us then show that if we denote $\sigma_0 = k_0h$, then σ_0 is uniformly bounded from below. To this aim we assume that for the set E_{k_0} there is a point $x \in \partial E_{k_0}$ such that $|d_{E_0}(x)| \geq \delta/2$. Proposition 4.1 implies that E_{k_0} is a (Λ, α) -minimizer of the perimeter for any $\alpha \in (0, \frac{1}{3})$, where Λ depends on r_0, R_0, K_0 and α . Then by $|d_{E_0}(x)| \geq \delta/2$ and by standard density estimate we have

$$(5.14) \quad |E_{k_0} \Delta E_0| \geq c\delta^3.$$

Let $k \leq k_0$. Following the calculations in the proof of Lemma 2.4 and using (3.7) we may write

$$|E_k \Delta E_{k-1}| = \int_{\partial E_{k-1}} |\xi_{k-1}| d\mathcal{H}^2 \leq 2 \int_{\partial E_{k-1}} |\psi_{k-1}| d\mathcal{H}^2.$$

Then we may estimate by (5.13)

$$\begin{aligned} |E_{k_0} \Delta E_0| &\leq \sum_{j=1}^{k_0} |E_j \Delta E_{j-1}| \leq 2 \sum_{j=0}^{k_0-1} \|\psi_j\|_{L^1(\partial E_j)} \\ &\leq C \sum_{j=0}^{k_0-1} \|\psi_j\|_{L^2(\partial E_j)} \leq C k_0 h = C \sigma_0. \end{aligned}$$

The inequality (5.14) proves that σ_0 is uniformly bounded from below.

We have thus proved a uniform H^4 -regularity estimate (5.12) for the sets E_k for $k \leq k_0$ where $k_0h = \sigma_0$ is uniformly bounded from below. Again Proposition 4.1 implies that the sets

E_k are (Λ, α) -minimizers of the perimeter for constants that are independent of h . Therefore by Lemma 2.8 we deduce that we may write

$$\partial E_k = \{x + u_k(x)\nu_{E_0}(x) : x \in \partial E_0\} \quad \text{for } u_k : \partial E_0 = \Sigma_0 \rightarrow \mathbb{R},$$

when σ_0 is sufficiently small, $\|u_k\|_{C^{1,\gamma}(\Sigma_0)} \leq \delta^\gamma$ for some $\gamma > 0$, and by the uniform bounds from (5.12) we deduce that $\|u_k\|_{H^4(\Sigma_0)} \leq C$. In particular, u_k are uniformly $C^{2,\alpha}$ -regular. Let us define the discrete normal velocity on ∂E_k as $v_k : \partial E_k \rightarrow \mathbb{R}$,

$$v_k(y) := \frac{\psi_k(y)}{h}.$$

Define $\Phi_k : \partial E_0 \rightarrow \partial E_k$, $\Phi_k(x) = x + u_k(x)\nu_{E_0}(x)$ and denote its tangential Jacobian as $J_{\partial E_0} \Phi_k$. We claim that it holds

$$(5.15) \quad \left\| v_k \circ \Phi_k - \frac{u_{k+1} - u_k}{|N_k| h} \right\|_{L^2(\partial E_0)} \leq \varepsilon(h)$$

for $\varepsilon(h) \rightarrow 0$ as $h \rightarrow 0$, where

$$|N_k(x)| = \frac{J_{\partial E_0} \Phi_k(x)}{1 + H_{E_0} u_k(x) + K_{E_0} u_k^2(x)}.$$

The notation is justified by the fact that $|N_k(x)|$ defined above is in fact the norm of the vector $N_k(x)$ defined in (2.22) for $E = E_0$ and $F = E_k$, which is not obvious but follows from standard calculation. We note that by the estimates from the previous section we have $\|\psi_k \circ \Phi_k\|_{C^1} \leq h^\gamma$ for $\gamma > 0$ and $\|\psi_k \circ \Phi_k\|_{L^2} \leq Ch$. Moreover, as $\|u_k\|_{C^{1,\gamma}(\Sigma_0)} \leq \delta^\gamma$ then it is easy to see that $|u_{k+1}(x) - u_k(x)| \leq C|\psi_k(\Phi_k(x))|$ for all $x \in \partial E_0$ and $\|u_{k+1} - u_k\|_{C^1} \leq h^\gamma$ by possibly decreasing the value of γ if needed. We will use these repeatedly below.

In order to show (5.15) we fix $f : \partial E_0 \rightarrow \mathbb{R}$ such that $\|f\|_{C^1} \leq Ch^\gamma$, define $\Phi_\tau : \partial E_0 \rightarrow \mathbb{R}^3$ as $\Phi_\tau(x) = x + \tau\nu_{E_0}(x)$, and calculate the following integral as in the proof of Lemma 2.4

$$\begin{aligned} & \int_{\mathbb{R}^3} f(\pi_{\partial E_0}(x))(\chi_{E_{k+1}} - \chi_{E_k}) dx \\ &= \int_{\partial E_0} f(x) \int_{-r_0}^{r_0} (\chi_{E_{k+1}}(\Phi_\tau(x)) - \chi_{E_k}(\Phi_\tau(x))) J_{\partial E_0} \Phi_\tau(x) d\tau d\mathcal{H}^2(x) \\ &= \int_{\partial E_0} f(x) \int_{u_k(x)}^{u_{k+1}(x)} J_{\partial E_0} \Phi_\tau(x) d\tau d\mathcal{H}^2(x) \\ &= \int_{\partial E_0} f(x)(u_{k+1} - u_k)(1 + H_{E_0} u_k + K_{E_0} u_k^2) d\mathcal{H}^2(x) + o(h^2) \\ &= \int_{\partial E_0} f(x) J_{\partial E_0} \Phi_k \frac{u_{k+1} - u_k}{|N_k|} d\mathcal{H}^2(x) + o(h^2), \end{aligned}$$

where the last equality follows from the definition of $|N_k|$. Above the integral $\int_{u_k(x)}^{u_{k+1}(x)}$ is understood to be $-\int_{u_{k+1}(x)}^{u_k(x)}$ in the case $u_{k+1}(x) < u_k(x)$. We compute the same integral

in a different way by defining $\Psi_k(\cdot, \tau) : \partial E_k \rightarrow \mathbb{R}^3$ as $\Psi_k(x, \tau) = x + \tau \nu_{E_k}(x)$, recall that $\Phi_k(x) = x + u_k(x) \nu_{E_0}$,

$$\begin{aligned}
& \int_{\mathbb{R}^3} f(\pi_{\partial E_0}(x)) (\chi_{E_{k+1}} - \chi_{E_k}) dx \\
&= \int_{\partial E_k} \int_{-r_0}^{r_0} f(\pi_{\partial E_0}(\Psi_k(x, \tau))) (\chi_{E_{k+1}}(\Psi_k(x, \tau)) - \chi_{E_k}(\Psi_k(x, \tau))) J_{\partial E_k} \Psi_k(\cdot, \tau) d\tau d\mathcal{H}^2(x) \\
&= \int_{\partial E_k} \int_0^{\psi_k(x)} f(\pi_{\partial E_0}(\Psi_k(x, \tau))) J_{\partial E_k} \Psi_k(\cdot, \tau) d\tau d\mathcal{H}^2(x) \\
&= \int_{\partial E_k} \psi_k(x) f(\pi_{\partial E_0}(x)) d\mathcal{H}^2(x) + o(h^2) \\
&= \int_{\partial E_0} \psi_k(\Phi_k(x)) f(x) J_{\partial E_0} \Phi_k d\mathcal{H}^2(x) + o(h^2).
\end{aligned}$$

We compare the two formulas above and choose

$$f(x) = \frac{1}{J_{\partial E_0} \Phi_k} \left(\psi_k \circ \Phi_k - \frac{u_{k+1} - u_k}{|N_k|} \right)$$

and obtain the estimate (5.15).

Using the estimates from above we deduce that we may write the approximate flat flow $(E^{h,\delta}(t))_{t \geq 0}$ starting from E_0 as

$$\partial E^{h,\delta}(t) = \{x + u^{h,\delta}(x, t) \nu_{E_0}(x) : x \in \partial E_0\}$$

and the heightfunctions $u^{h,\delta}$ satisfy

$$\|u^{h,\delta}(\cdot, t)\|_{H^4(\Sigma_0)} \leq C \quad \text{and} \quad \frac{1}{h} \|u^{h,\delta}(\cdot, t) - u^{h,\delta}(\cdot, t-h)\|_{L^2(\Sigma_0)} \leq C$$

for all $t \in [h, \sigma_0]$. We may thus pass to the limit as $h \rightarrow 0$, up to a subsequence, and deduce that the limit function $u(x, t) = \lim_{h \rightarrow 0} u^{h,\delta}(x, t)$ exists for every $t \in [0, \sigma_0]$. By (5.15) the discrete normal velocity converges, up to a subsequence, to

$$\lim_{h \rightarrow 0} v_k(\Phi_k(x)) = \frac{\partial_t u}{|N(x, t)|} \quad \text{in } L^2(E_0), \quad t \in [0, \sigma_0],$$

where $|N(x, t)| = \frac{J_{\partial E_0} \Phi_t}{1 + H_{E_0} u(x, t) + K_{E_0} u^2(x, t)}$ and $\Phi_t(x) = x + u(x, t) \nu_{E_0}$. Let us show that the limit function solves the surface diffusion equation.

To this aim we fix $\varphi \in C^2(\mathbb{R}^3)$, multiply the Euler-Lagrange equation (4.11) with φ and integrate by parts

$$\int_{\partial E_k^h} \frac{\psi_k}{h} \varphi d\mathcal{H}^2(x) = \int_{\partial E_k^h} \Delta_{\partial E_k^h} \varphi (H_{E_k^h} - \Delta_{\partial E_k^h} \psi_k + R_0) d\mathcal{H}^2(x).$$

Passing to the limit as $h \rightarrow 0$ and using the bounds $\|\psi_k\|_{H^2} \leq h^\gamma$ and $\|R_0\|_{L^2} \leq h^\gamma$, which follow from the results in the previous section, yield

$$\begin{aligned} \int_{\partial E_0} \frac{\partial_t u}{|N|} \varphi(\Phi_t(x)) J_{\partial E_0}(\Phi_t(x)) d\mathcal{H}^2(x) &= \int_{\partial E^\delta(t)} \Delta_{\partial E^\delta(t)} \varphi H_{E^\delta(t)} d\mathcal{H}^2(x) \\ &= \int_{\partial E^\delta(t)} \varphi \Delta_{\partial E^\delta(t)} H_{E^\delta(t)} d\mathcal{H}^2(x) = \int_{\partial E_0} \varphi(\Phi_t(x)) (\Delta_{\partial E^\delta(t)} H_{E^\delta(t)}) J_{\partial E_0}(\Phi_t(x)) d\mathcal{H}^2(x). \end{aligned}$$

Hence, we conclude that the limit flow $E^\delta(t)$, parametrized by the family of diffeomorphisms $\Phi_t(x) = x + u(x, t)\nu_{E_0}$, is a strong solution to the surface diffusion equation. To be more precise, using the expansion of the mean curvature from Lemma 2.5 and the expansion of the Laplace-Beltrami operator from e.g. [20, formula (3.4)] we deduce that $u : \Sigma_0 \times [0, \sigma_0] \rightarrow \mathbb{R}$, where $\Sigma_0 = \partial E_0$, is a strong solution of the equation (see [15, formula (3.6)])

$$(5.16) \quad \partial_t u = -\Delta_{\Sigma_0}^2 u + \langle A(x, u, \bar{\nabla} u), \bar{\nabla}^4 u \rangle + J(x, u, \bar{\nabla} u, \bar{\nabla}^2 u, \bar{\nabla}^3 u) \quad \text{on } \Sigma_0 \times [0, \sigma_0],$$

with initial datum $u(\cdot, 0) = 0$, where the tensor field satisfies $A(x, 0, 0) = 0$ and J is a smooth function with sublinear growth on $\bar{\nabla}^3 u$. By a strong solution we mean that $u \in W^{1,\infty}(0, \sigma_0; L^2(\Sigma_0)) \cap L^\infty(0, \sigma_0; H^4(\Sigma_0))$ and it satisfies the equation (5.16) almost everywhere. By standard Grönwall argument one may deduce that the strong solution to (5.16) with zero initial datum is unique. This proves that the limiting flat flow coincides with the classical solution for the time interval $[0, \sigma_0]$ and the claim follows. \square

Remark 5.3. We may quantify the statement of Theorem 1.1 more precisely as follows. Assume that E_0 is C^5 regular, $E_0 \subset B_{R_0/2}$, satisfies the UBC with radius $2r_0$, and the heightfunction in (5.11) satisfies

$$\|\psi_0\|_{L^2(\partial E_0)} \leq L_0 h \quad \text{and} \quad \|\Delta_{\partial E_0} \psi_0\|_{L^2(\Sigma_0)} \leq L_0 \sqrt{h}.$$

Then there is $K_0 = K_0(r_0, R_0, L_0)$ such that if

$$\|\Delta_{\partial E_0} H_{E_0}\|_{L^2} \leq K_0 \quad \text{and} \quad \|\Delta_{\partial E_0} H_{E_0}\|_{H^1} \leq K_0 h^{-\frac{1}{4}}$$

then the approximate flat flow $(E^{(\delta,h)}(t))_{t \geq 0}$, where $\delta = \delta(r_0, R_0, K_0)$ is small, also satisfies the UBC with radius r_0 , $E^{(\delta,h)}(t) \subset B_{R_0}$ and

$$\|\Delta_{\partial E^{(\delta,h)}(t)} H_{E^{(\delta,h)}(t)}\|_{L^2} \leq K_0 \quad \text{and} \quad \|\Delta_{\partial E^{(\delta,h)}(t)} H_{E^{(\delta,h)}(t)}\|_{H^1} \leq K_0 h^{-\frac{1}{4}}$$

for all $t \in [0, \sigma_0]$, where $\sigma_0 = \sigma_0(r_0, R_0, K_0)$.

Remark 5.4. We note that the strong solution of (5.16) with a zero initial datum $\psi(x, 0) = 0$ is always unique. The argument in the proof of Theorem 1.1 implies that if the approximate flat flow $(E^{(\delta,h)}(t))_{t \geq 0}$ is such that $E^{(\delta,h)}(0)$ is uniformly C^5 -regular and $E^{(\delta,h)}(t)$ satisfies the UBC with radius r_0 and $\|\Delta_{\partial E^{(\delta,h)}(t)} H_{E^{(\delta,h)}(t)}\|_{L^2} \leq K_0$ for all $t \in [0, \sigma_0]$, then the limiting flat flow coincides with the classical solution for the time interval $[0, \sigma_0]$.

Proof of Theorem 1.2. We assume that the classical solution $(E_t)_{t \in [0, T_0]}$ exists for the time interval $[0, T_0)$ and fix $T < T_0$. Since the classical solution is regular in $[0, T]$, there are $r_0 > 0$ and $L_0 > 1$, depending not only on E_0 but also on T , such that E_t satisfies the UBC with radius $4r_0$ and

$$(5.17) \quad \|\Delta_{\partial E_t} H_{E_t}\|_{L^2} \leq \frac{L_0}{2} \quad \text{for all } t \in [0, T].$$

Also it holds $E_t \subset B_{R_0/2}$ for some $R_0 = R_0(T) > 0$ for all $t \leq T$. As in the proof of Theorem 1.1, the heightfunction ψ_0 as in (5.11) satisfies

$$\|\psi_0\|_{L^2(\partial E_0)} \leq L_0 h \quad \text{and} \quad \|\Delta_{\partial E_0} \psi_0\|_{L^2(\Sigma_0)} \leq L_0 \sqrt{h},$$

for L_0 depending on E_0 . We choose $K_0 = K_0(r_0, R_0, L_0)$, $\sigma_0 = \sigma_0(r_0, R_0, K_0)$ and $\delta_0 = \sigma_0(r_0, R_0, K_0)$ as in Remark 5.3, and denote $k_0 \in \mathbb{N}$ the number such that $\sigma_0 \in [k_0 h, (k_0 + 1)h)$. Note that ultimately K_0 , σ_0 and δ_0 depend on T .

Let $(E^{(\delta, h_i)}(t))_{t \geq 0}$ be the approximate flat flow associated with the chosen flat flow $(E^\delta(t))_{t \geq 0}$, with $\delta < \delta_0$, and denote the associated sequence by $E_k^{(\delta, h_i)}$. We simplify the notation by $E^{h_i}(t) = E^{(\delta, h_i)}(t)$ and $E_k = E_k^{(\delta, h_i)}$. We claim that for every $t \in [0, T]$ it holds

$$(5.18) \quad \begin{aligned} &E^{h_i}(t) \text{ satisfies the UBC with radius } r_0, \quad E^{h_i}(t) \subset B_{R_0} \\ &\|\Delta_{\partial E^{h_i}(t)} H_{E^{h_i}(t)}\|_{L^2} \leq K_0 \quad \text{and} \\ &\|\Delta_{\partial E^{h_i}(t)} H_{E^{h_i}(t)}\|_{H^1} \leq K_0 h^{-\frac{1}{4}}, \end{aligned}$$

for h_i small. We point out that here we do not quantify the smallness of h_i and we may have to pass to another subsequence of h_i . Once we have proven (5.18), the consistency follows from Remark 5.4.

Note that by Remark 5.3, (5.18) holds for all $t \in [0, \sigma_0]$. Let us choose any $t_1 \in [\sigma_0, T]$ for which (5.18) holds for all $t \leq t_1$. We will show that then (5.18) continues to hold for all $t \leq t_1 + \frac{\sigma_0}{2}$. This will imply that (5.18) holds for all $t \leq T$.

Assume thus that (5.18) holds for all $t \leq t_1$ and denote k_1 the index for which $t_1 - \frac{\sigma_0}{2} \in [k_1 h_i, (k_1 + 1)h_i)$. Since E_{k_1} satisfies the estimates in (5.18), we may use Proposition 4.1 and Remark 5.3 to deduce that there is $\tilde{\sigma} = \tilde{k} h_i \geq \tilde{c} > 0$, with $\tilde{\sigma} \leq \sigma_0$, such that for all $k \in [k_1, k_1 + \tilde{k}]$ we may write the set E_k as

$$\partial E_k = \{x + \psi_{k-1}(x) \nu_{E_{k-1}}(x) : x \in \partial E_{k-1}\}.$$

We use (5.13) to conclude

$$\|\psi_k\|_{L^2(\partial E_k)}^2 + h \sum_{j=k_1}^{k_1 + \tilde{k}} \|\Delta_{\partial E_j} \psi_j\|_{L^2(\partial E_j)}^2 \leq C h_i^2 \quad \text{for all } k \in [k_1, k_1 + \tilde{k}],$$

for some constant C . Recall that $\tilde{\sigma} = \tilde{k} h_i \geq \tilde{c} > 0$ for all i . Therefore we obtain from above that there is an index $k_i \in [k_1, k_1 + \tilde{k}/4]$ such that

$$(5.19) \quad \|\psi_{k_i}\|_{L^2(\partial E_{k_i})} + \|\Delta_{\partial E_{k_i}} \psi_{k_i}\|_{L^2(\partial E_{k_i})} \leq C h_i.$$

Next we recall that since the estimates in (5.18) hold for all $t \leq t_1$ and $k_i h_i \leq (k_1 + \tilde{k}/4)h_i \leq t_1$, the set E_{k_i} satisfies the UBC with radius r_0 and is, in fact, uniformly $C^{2,\alpha}$ -regular. Let $\hat{t}_i = k_i h_i$ and define

$$v^{h_i}(x, \hat{t}_i) = \frac{\psi_{k_i}(x)}{h_i}.$$

Then (5.19) together with Lemma 2.3 and Sobolev embedding imply that

$$\|v^{h_i}(\cdot, \hat{t}_i)\|_{C^\alpha(\partial E^{h_i}(\hat{t}_i))} \leq C.$$

By choosing a subsequence of h_i if needed, we may assume that

$$\lim_{h_i \rightarrow 0} \hat{t}_i = \hat{t} \in [t_1 - \frac{\sigma_0}{2}, t_1].$$

Therefore we deduce by compactness that

$$(5.20) \quad \lim_{h_i \rightarrow 0} \|v^{h_i}(\cdot, \hat{t}_i)\|_{L^2(\partial E^{h_i}(\hat{t}_i))} = \|v(\cdot, \hat{t})\|_{L^2(\partial E^\delta(t))},$$

by possibly choosing another subsequence. Since we assume that (5.18) holds for all $t \leq t_1$, and $\hat{t} \leq t_1$, then by Remark 5.4 the flat flow agrees with the classical solution up to t_1 . Then using (5.15) with E_0 replaced by $E_{t_1 - \frac{\sigma_0}{2}}$, we deduce that $v(\cdot, \hat{t})$ agrees with the normal velocity V_t of the classical solution (E_t) at time \hat{t} , i.e.,

$$\|v(\cdot, \hat{t})\|_{L^2(\partial E^\delta(\hat{t}))} = \|V_{\hat{t}}\|_{L^2(\partial E_{\hat{t}})}.$$

Since $V_t = \Delta_{\partial E_t} H_{E_t}$ we deduce by (5.17) that

$$\|V_{\hat{t}}\|_{L^2(\partial E_{\hat{t}})} = \|\Delta_{\partial E_{\hat{t}}} H_{E_{\hat{t}}}\|_{L^2(\partial E_{\hat{t}})} \leq \frac{L_0}{2}.$$

Therefore we conclude that it holds

$$\|\psi_{k_i}\|_{L^2(\partial E_{k_i})} \leq L_0 h_i,$$

when h_i is small. By (5.19) we also have

$$\|\Delta_{\partial E_{k_i}} \psi_{k_i}\|_{L^2(\partial E_{k_i})} \leq C h_i \leq L_0 \sqrt{h_i}.$$

when h_i is small. Finally, since $E^\delta(\hat{t}) = E_{\hat{t}}$ satisfies the UBC with radius $4r_0$, then by the uniform $C^{2,\alpha}$ -regularity we also deduce that $E_{k_i} = E^{h_i}(\hat{t}_i)$ satisfies the UBC with radius $2r_0$, when h_i is small.

We apply Remark 5.3 for $E_{k_i} = E^{h_i}(\hat{t}_i)$ in place of E_0 and deduce that the approximate flat flow satisfies

$$\|\Delta_{\partial E^{h_i}(t)} H_{E^{h_i}(t)}\|_{L^2} \leq K_0 \quad \text{and} \quad \|\Delta_{\partial E^{h_i}(t)} H_{E^{h_i}(t)}\|_{H^1} \leq K_0 h^{-\frac{1}{4}}$$

for all $t \in [\hat{t}_i, \hat{t}_i + \sigma_0]$. Since $\hat{t}_i \in [t_1 - \frac{\sigma_0}{2}, t_1]$, this means that the sets $E^{h_i}(t)$ satisfy the bounds in (5.18) for all $t \leq t_1 + \frac{\sigma_0}{2}$, when h_i is small.

We note that by showing that the estimate (5.18) continues to hold from t_1 to $t_1 + \frac{\sigma_0}{2}$ we might have to pass to another subsequence from the original sequence h_i , but this is of minor importance as we need to repeat the argument only finitely many times. Hence, the

approximate flat flow $E^{h_i}(t)$ satisfies the bounds in (5.18) for all $t \leq T$ and the claim follows from Remark 5.4. \square

ACKNOWLEDGMENTS

During the completion of this paper V.J. and A.K were supported by the Academy of Finland grant 314227, while N.F. was supported by the Alexander von Humboldt Foundation.

REFERENCES

- [1] E. ACERBI, N. FUSCO, V. JULIN, M. MORINI, *Nonlinear stability results for the modified Mullins-Sekerka and the surface diffusion flow*. J. Differential Geom. **113** (2019), 1–53.
- [2] E. ACERBI, N. FUSCO, M. MORINI, *Minimality via second variation for a nonlocal isoperimetric problem*. Comm. Math. Phys. **322** (2013), 515–557.
- [3] N.D. ALIKAKOS, P.W. BATES, X. CHEN, *Convergence of the Cahn-Hilliard equation to the Hele-Shaw model*, Arch. Ration. Mech. Anal. **128** (1994), 165–205.
- [4] F. ALMGREN, J.E. TAYLOR, L. WANG, *Curvature-driven flows: a variational approach*. SIAM J. Control Optim. **31** (1993), 387–438.
- [5] T. AUBIN, *Some nonlinear problems in Riemannian geometry*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.
- [6] P. BARAS, J. DUCHON, R. ROBERT, *Évolution d’une interface par diffusion de surface*. Comm. Partial Differential Equations **9** (1984), 313–335 .
- [7] K. BRAZDA, M. KRUIK, U. STEFANELLI, *Generalized minimizing movements for the varifold Cahn-Helfrich flow*. Advances in Calculus of Variations **17** (2024), 727–751.
- [8] J.W. CAHN, C.M. ELLIOTT, A. NOVICK-COHEN, *The Cahn-Hilliard equation with a concentration dependent mobility: motion by minus the Laplacian of the mean curvature*, European J. Appl. Math. **7** (1996), 287–301.
- [9] J.W. CAHN, J.E. TAYLOR, *Surface motion by surface diffusion*. Acta Metallurgica et Materialia **42** (1994), 1045–1063
- [10] A. CHAMBOLE, *An algorithm for mean curvature motion*. Interfaces Free Bound., **6** (2004), 195–218.
- [11] A. CHAMBOLE, M. MORINI, M. PONSIGLIONE, *Minimizing movements and level set approaches to non-local variational geometric flows*. Geometric partial differential equations, 93–104, CRM Series, 15, Ed. Norm., Pisa, 2013.
- [12] K.Y. CHUNG, *On variational schemes modeling surface diffusion*. Ph.D. Thesis (1997). Princeton University.
- [13] J. DALPHIN, *Uniform ball property and existence of optimal shapes for a wide class of geometric functionals*, Interfaces Free Bound. **20** (2018), 211–260.
- [14] F. DAVÌ, M.E. GURTIN, *On the motion of a phase interface by surface diffusion*, Z. Angew. Math. Phys. **41** (1990), 782–811.
- [15] A. DIANA, *The Surface Diffusion Flow. Long-Time Behaviour and Asymptotic Stability*. Ph.D. Thesis (2024). Università degli Studi di Napoli Federico II & Scuola Superiore Meridionale.
- [16] A. DIANA, N. FUSCO, C. MANTEGAZZA, *Stability for the surface diffusion flow*, arXiv:2304.04011 (2023).
- [17] C.M. ELLIOTT, H. GARCKE, *Existence results for geometric interface models for surface diffusion*. Adv. Math. Sci. Appl. **7** (1997), 467–490.
- [18] J. ESCHER, U.F. MAYER, G. SIMONETT, *The surface diffusion flow for immersed hypersurfaces*. SIAM J. Math. Anal. **29** (1998), 1419–1433.

- [19] J. FISCHER, S. HENSEL, T. LAUX, T.M. SIMON, *A weak-strong uniqueness principle for the Mullins-Sekerka equation*. arXiv:2404.02682 (2024).
- [20] N. FUSCO, V. JULIN, M. MORINI, *The Surface Diffusion Flow with Elasticity in Three Dimensions*. Arch. Ration. Mech. Anal. **237** (2020), 1325–1382.
- [21] Y. GIGA, K. ITO, *On pinching of curves moved by surface diffusion*. Commun. Appl. Anal. **2** (1998), 393–405.
- [22] M.E. GURTIN, M.E. JABBOUR, *Interface evolution in three dimensions with curvature-dependent energy and surface diffusion: interface-controlled evolution, phase transitions, epitaxial growth of elastic films*, Arch. Ration. Mech. Anal. **163** (2002), 171–208.
- [23] K. HOPF, J. KING, A. MÜNCH, B. WAGNER, *Interface dynamics in a degenerate Cahn–Hilliard model for viscoelastic phase separation*, arXiv:2412.06762 (2024).
- [24] V. JULIN, D.A. LA MANNA, *A priori estimates for the motion of charged liquid drop: A dynamic approach via free boundary Euler equations*. J. Math. Fluid Mech. **26** (2024), Paper No. 48.
- [25] V. JULIN, J. NIINIKOSKI, *Consistency of the flat flow solution to the volume preserving mean curvature flow*. Arch. Ration. Mech. Anal. **248** (2024), Paper No. 1.
- [26] J.M. LEE, *Riemannian manifolds. An introduction to curvature*. Graduate Texts in Mathematics, **176**. Springer-Verlag, New York, 1997.
- [27] S. LUCKHAUS, T. STÜRZENHECKER, *Implicit time discretization for the mean curvature flow equation*. Calc. Var. Partial Differential Equations **3** (1995), 253–271.
- [28] A. LUNARDI *Interpolation theory*. Lecture Notes, Ed. Norm., Pisa, 2018.
- [29] C. MANTEGAZZA, *Smooth geometric evolutions of hypersurfaces*. Geom. Funct. Anal. **12** (2002), 138–182.
- [30] C. MANTEGAZZA, *Lecture notes on Mean Curvature Flow*. Progress in Mathematics **290**. Birkhäuser/Springer, Basel, 2011.
- [31] M.G. MORA, M. MORINI, *Functionals depending on curvatures with constraints*, Rend. Sem. Mat. Univ. Padova, **104** (2000), 173–199.
- [32] L. MUGNAI, C. SEIS, E. SPADARO, *Global solutions to the volume-preserving mean-curvature flow*. Calc. Var. Partial Differential Equations **55** (2016), Art. 18.
- [33] W.W. MULLINS, *Theory of thermal grooving*. J. Appl. Phys. **28** (1957), 333–339.
- [34] R.L. PEGO, *Front migration in the nonlinear Cahn–Hilliard equation*, Proc. Roy. Soc. London Ser. A **422** (1989), 261–278.
- [35] M. RÖGER, *Existence of weak solutions for the Mullins-Sekerka flow*, SIAM J. Math. Anal. **37** (2005), 291–301.
- [36] R. SCHÄTZLE, *Hypersurfaces with mean curvature given by an ambient Sobolev function*. J. Differential Geom. **58** (2001), 371–420.
- [37] I. TAMANINI, *Regularity results for almost minimal oriented hypersurfaces in R^N* . Quaderni del Dipartimento di Matematica dell’Università di Lecce (1984).
- [38] J.E. TAYLOR, J.W. CAHN, *Linking anisotropic sharp and diffuse surface motion laws via gradient flows*, J. Statist. Phys. **77** (1994), 183–197.
- [39] M.E. TAYLOR, *Partial Differential Equations III*. Applied Mathematical Sciences **117**, Springer, Berlin 2011.

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