THE OPTIMAL HOLE FOR THE BEST HÖLDER EXTENSION

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ABSTRACT. In this paper, we study the problem of finding the optimal hole among all subsets $A \subset \Omega$ that minimizes the $\alpha-$Hölder seminorm $[u]_\alpha$ among all functions u such that $u = 1$ on $\partial\Omega$ and $u = 0$ on A, plus a penalization on the volume of A. For a given set $A \subset \Omega$, we will also characterize the function that minimizes $[u]_\alpha$. In addition, we will study the limit when $p \to \infty$ of the "fractional" version of the Alt-Caffarelli problem.

1. INTRODUCTION

Let Ω be an open bounded domain in $\mathbb{R}^N$. In [2, 7, 6], the authors considered the problem of minimizing

$$\min\left\{\int_\Omega |\nabla u|^p + \lambda|\{u>0\}| \,:\, u \in W^{1,p}(\Omega),\ u \geq 0,\ \ u = 1 \ \text{ on } \ \partial\Omega\right\} \tag{1.1}$$

where $1 < p < \infty$ and $\lambda > 0$. Problems of this kind (known as Bernoulli-type problems) have several applications in heat flows [1, 3] and electrochemical machining [10].

In [9], the authors studied the limit when $p \to \infty$ of the minimizer u_p to the following problem:

$$\min\left\{\frac{1}{p}\int_\Omega \left[\frac{|\nabla u|}{\Lambda}\right]^p + \lambda|\{u>0\}| \,:\, u \in W^{1,p}(\Omega),\ u = 1 \ \text{ on } \ \partial\Omega\right\}. \tag{1.2}$$

More precisely, they show that up to a subsequence, u_p converges uniformly to a function u_∞ that solves

$$\min\left\{|\{u>0\}| \,:\, u \in \text{Lip}(\bar{\Omega}),\ |\nabla u| \leq \Lambda,\ u = 1 \ \text{ on } \ \partial\Omega\right\}. \tag{1.3}$$

In [5], the authors considered the following free boundary problem (which is the supremal version of the Alt-Caffarelli minimization problem (1.1)):

$$\min\{||\nabla u||_\infty + \lambda|\{u>0\}| \,:\, u \in \text{Lip}(\bar{\Omega}),\ u \geq 0,\ \ u = 1 \ \text{ on } \ \partial\Omega\}. \tag{1.4}$$

It is clear that

$$\min\,(1.4) = \min_{\Lambda>0}[\Lambda + \min\,(1.3)].$$

Notice that the minimizer u in Problem (1.4) will be constant (i.e. $u = 1$ on Ω) as soon as the parameter λ is sufficiently small. Otherwise, they show that there is a constant $r > 0$ such that $u_r := [1 - \frac{d(x,\partial\Omega)}{r}]_+$ is a minimizer (see [5, Theorem 1]).

In this paper, we will consider the fractional version of Problem (1.2), where the L^p norm of ∇u is replaced by the $W^{s,p}-$seminorm $[u]_{s,p}$ of u:

$$\min\left\{\frac{1}{p\Lambda^p}\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} + \lambda|\{u>0\}| \,:\, u \in W^{s,p}(\Omega),\ u = 1 \ \text{ on } \ \partial\Omega\right\}, \tag{1.5}$$

where $s = \alpha - \frac{N}{p}$ and

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega),\ [u]^p_{s,p} := \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} < \infty \right\}.$$

In Section 2, we will study the limit of Problem (1.5) as $p \to \infty$. We show that up to a subsequence, the minimizers u_p converge to u_∞ which turns out to be a solution to the following minimization problem (which is the $\alpha-$Hölder version of (1.3)):

$$\min\left\{ |\{u>0\}| \,:\, u \in C^{0,\alpha}(\Omega),\ [u]_\alpha \le \Lambda,\ u = 1 \text{ on } \partial\Omega \right\}$$

where

$$[u]_\alpha := \sup_{x,\, y\in\Omega,\, y\neq x} \frac{|u(y)-u(x)|}{|y-x|^\alpha}.$$

Moreover, we will see that u_∞ is a viscosity solution to the fractional infinity Laplacian $-L_\infty u_\infty = 0$ in the positivity set $\{u_\infty > 0\}$, where

$$L_\infty u := L^+_\infty u + L^-_\infty u$$

and

$$L^+_\infty u(x) := \sup_{y\in\Omega} \frac{u(y)-u(x)}{|y-x|^\alpha}, \qquad L^-_\infty u(x) := \inf_{y\in\Omega} \frac{u(y)-u(x)}{|y-x|^\alpha}.$$

We note that the fractional infinity Laplacian has been studied in [4] where the authors prove existence of a solution to $L_\infty u = 0$ using an approximation with the fractional $p-$Laplacian when $p \to \infty$.

In Section 3, we consider the problem of finding the best $\alpha-$Hölder extension (with $0 < \alpha < 1$) of the constant boundary condition ($u = 1$ on $\partial\Omega$) in the presence of a hole $A \subset\subset \Omega$, i.e. we minimize the $\alpha-$Hölder seminorm among all functions u that vanish on A while $u = 1$ on the boundary $\partial\Omega$:

$$\Lambda(A) := \min\{[u]_\alpha \,:\, u \in C^{0,\alpha}(\bar\Omega),\ u \ge 0,\ u = 1 \text{ on } \partial\Omega,\ u = 0 \text{ on } A\} \tag{1.6}$$

where

$$[u]_\alpha = \sup_{x,\, y\in\bar\Omega,\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

The aim of this section is to give an explicit representation of the solution to Problem (1.6). More precisely, we show that the function

$$u(x) = \frac{d(x,A)^\alpha}{d(x,A)^\alpha + d(x,\partial\Omega)^\alpha}$$

minimizes (1.6). Moreover, we will show that this minimizer u solves the fractional infinity Laplacian

$$L_\infty u = 0 \qquad \text{in} \quad \Omega\backslash A. \tag{1.7}$$

In Section 4, we consider the $\alpha-$Hölder version of Problem (1.4). To be more precise, we study the problem of finding the optimal hole A that minimizes the functional $\Lambda(A)$ among all subsets $A \subset \Omega$, i.e. we consider the following shape optimization problem:

$$\min\{\Lambda(A) - \lambda|A| \,:\, A \subset\subset \Omega\}. \tag{1.8}$$

If $\lambda \le 0$, it is clear that the optimal set A in Problem (1.8) is empty. However, if $\lambda > 0$ the situation becomes much complicated since on one side we need to take A small as much as possible so that $\Lambda(A)$ will be small too but on the other side $|A|$ is increasing in A. Yet, we will show that there exists a constant $\lambda^\star > 0$ such that for all $\lambda \le \lambda^\star$ the optimal set A is empty, while if $\lambda \ge \lambda^\star$ the optimal set $A = \{x \in \Omega : d(x, \partial\Omega) \ge r_\lambda\}$, where the constant $r_\lambda > 0$ is such that

$$r_\lambda^{\alpha+1}\, \mathcal{H}^{N-1}(\partial A_{r_\lambda}) = \frac{\alpha}{\lambda}.$$

Moreover, $\lambda^\star$ satisfies

$$r_{\lambda^\star}^{\alpha}\, |A_{r_{\lambda^\star}}| = \frac{1}{\lambda^\star}.$$

2. The limit of the fractional $p-$Laplacian free boundary problem

Let Ω be a bounded Lipschitz set in $\mathbb{R}^N$ and λ, $\Lambda > 0$ are fixed. Then, we consider their minimization problem:

$$\min\left\{\frac{1}{p\Lambda^p}\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} + \lambda|\{u>0\}| \;:\; u \in W^{s,p}(\Omega),\ u = 1 \text{ on } \partial\Omega\right\}. \tag{2.1}$$

Proposition 2.1. *Problem (2.1) has a minimizer u_p. Moreover, we have $L_p u_p = 0$ (in the weak sense) inside the positivity set $\{u_p > 0\}$, where*

$$L_p u(x) := -\int_\Omega \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{\alpha p}}\frac{u(x)-u(y)}{|u(x)-u(y)|}\,\mathrm{d}y.$$

Proof. Let $\{u_n\}_n$ be a minimizing sequence in (2.1). So, there is a constant C (independent of n) such that

$$\frac{1}{p\Lambda^p}\int_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{\alpha p}} + \lambda|\{u_n>0\}| \le C, \qquad \text{for all } n.$$

Hence,

$$\left(\int_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{\alpha p}}\right)^{\frac{1}{p}} \le [Cp]^{\frac{1}{p}}\,\Lambda. \tag{2.2}$$

Yet, $u_n = 1$ on $\partial\Omega$. By [8, Theorem 8.2], we get

$$||u_n||_\infty \le C[u_n]_{s,p} + 1.$$

Thanks to (2.2), this implies that $\{u_n\}_n$ is bounded in $W^{s,p}(\Omega)$. Hence, up to a subsequence, u_n converges uniformly to u_p in $\bar{\Omega}$. Thanks to Fatou's Lemma, we see that

$$\frac{1}{p}\int_{\Omega\times\Omega}\left[\frac{\frac{|u_p(x)-u_p(y)|}{|x-y|^\alpha}}{\Lambda}\right]^p + \lambda|\{u_p>0\}| \le \liminf_n\left[\frac{1}{p}\int_{\Omega\times\Omega}\left[\frac{\frac{|u_n(x)-u_n(y)|}{|x-y|^\alpha}}{\Lambda}\right]^p + \lambda|\{u_n>0\}|\right].$$

In particular, we have $u_p = 1$ on $\partial\Omega$. Hence, u_p is a minimizer. Moreover, it is easy to see that $u_p \ge 0$. Assume this is not the case; consider the truncated function $\tilde{u}_p := \max\{u_p, 0\}$. Then, one has $\{u_p > 0\} = \{\tilde{u}_p > 0\}$ and

$$|\tilde{u}_p(x) - \tilde{u}_p(y)| \le |u_p(x) - u_p(y)|, \qquad \text{for all } x,\, y \in \Omega.$$

Fix $\varphi \in C_0^\infty(\Omega)$ such that $\mathrm{spt}(\varphi) \subset \{u_p > 0\}$. Then, we clearly have $u + \varepsilon\varphi \in W^{s,p}(\Omega)$,

$u + \varepsilon\varphi = 1$ on $\partial\Omega$ and $\{u_p + \varepsilon\varphi > 0\} = \{u_p > 0\}$ for ε small enough. From the optimality of u_p, we have

$$\int_{\Omega\times\Omega} \left[\frac{|u_p(x) - u_p(y)|}{|x-y|^\alpha}\right]^p \leq \int_{\Omega\times\Omega} \left[\frac{|(u_p+\varepsilon\varphi)(x) - (u_p+\varepsilon\varphi)(y)|}{|x-y|^\alpha}\right]^p.$$

Therefore, we get

$$\int_{\Omega\times\Omega} \frac{|u_p(x) - u_p(y)|^{p-1}}{|x-y|^{\alpha p}} \frac{u_p(x) - u_p(y)}{|u_p(x) - u_p(y)|} [\varphi(x) - \varphi(y)] = 0.$$

Then, we have

$$L_p u_p = 0 \quad \text{in} \ \{u_p > 0\} \quad \text{(in the weak sense).}$$

□

Proposition 2.2. *Up to a subsequence, $u_p \to u_\infty$ uniformly in Ω. Moreover, u_∞ minimizes the following problem:*

$$\min\left\{|\{u > 0\}| \,:\, u \in C^{0,\alpha}(\Omega),\, [u]_\alpha \leq \Lambda,\ u = 1 \ \ on\ \partial\Omega\right\}.$$

Proof. From the optimality of u_p in Problem (2.1), we have

$$\frac{1}{p}\int_{\Omega\times\Omega}\left[\frac{\frac{|u_p(x)-u_p(y)|}{|x-y|^\alpha}}{\Lambda}\right]^p + \lambda|\{u_p > 0\}| \leq \lambda|\Omega|, \qquad \text{for all } p.$$

Hence,

$$\left[\int_{\Omega\times\Omega}\frac{|u_p(x) - u_p(y)|^p}{|x-y|^{\alpha p}}\right]^{\frac{1}{p}} \leq [p\lambda|\Omega|]^{\frac{1}{p}}\Lambda, \qquad \text{for all } p.$$

Fix $m < p$, one has

$$\left[\iint_{\Omega\times\Omega}\frac{|u_p(x) - u_p(y)|^m}{|x-y|^{\alpha m}}\right]^{\frac{1}{m}} \leq \left[\iint_{\Omega\times\Omega}\frac{|u_p(x) - u_p(y)|^p}{|x-y|^{\alpha p}}\right]^{\frac{1}{p}} |\Omega|^{2(1-\frac{m}{p})} \leq C.$$

Consequently, $(u_p)_p$ is bounded in $W^{s,m}(\Omega)$ (with $s = \alpha - \frac{N}{m}$). If $m > \frac{2N}{\alpha}$, then one has the following estimate

$$||u_p||_{C^{0,\gamma}(\overline{\Omega})} \leq C[u_p]_{W^{s,m}(\Omega)},$$

with $\gamma = s - \frac{N}{m} > 0$. Therefore, up to a subsequence, u_p converges uniformly to u_∞ in $\bar{\Omega}$. Moreover, we have $u_\infty \in C^{0,\alpha}(\Omega)$ and

$$[u_\infty]_\alpha \leq \Lambda.$$

Fix $u \in C^{0,\alpha}(\Omega)$ such that $[u]_\alpha \leq \Lambda$ and $u = 1$ on $\partial\Omega$. Thanks again to the optimality of u_p, we have

$$\lambda|\{u_p > 0\}| \leq \frac{1}{p}\int_{\Omega\times\Omega}\left[\frac{\frac{|u_p(x)-u_p(y)|}{|x-y|^\alpha}}{\Lambda}\right]^p + \lambda|\{u_p > 0\}| \leq \frac{|\Omega|^2}{p} + \lambda\,|\{u > 0\}|.$$

Passing to the limit when $p \to \infty$, we get

$$\lambda|\{u_\infty > 0\}| \leq \lambda\liminf_p |\{u_p > 0\}| \leq \liminf_p \frac{|\Omega|^2}{p} + \lambda\,|\{u > 0\}| = \lambda|\{u > 0\}|.$$

□

In order to show that u_∞ is a viscosity solution to the fractional infinity Laplacian in the positivity set, we need to introduce first the definition of a viscosity supersolution (resp. subsolution).

Definition 2.1. *We say that u is a viscosity supersolution of $-L_p u = 0$ in E if the following holds: for every $x_0 \in E$ and $\phi \in C^1(\Omega) \cap C(\bar{\Omega})$ such that*

$$\phi(x_0) = u(x_0) \ \text{ and } \ \phi(x) \le u(x) \ \text{ for all } x \in \bar{\Omega},$$

then we have

$$-L_p \phi(x_0) \ge 0.$$

The requirement for a viscosity subsolution is symmetric: the test function is touching from above and the inequality is reversed. Finally, a viscosity solution is defined as being both a viscosity supersolution and a viscosity subsolution.

Proposition 2.3. *For all $p > 1$, u_p is a viscosity solution to $-L_p u_p = 0$ in $\{u_p > 0\}$.*

Proof. In order to prove that u_p is a viscosity solution, we will use some technical points from the proof of [4, Proposition 6.4], where more details can be found there. First, we show that u_p is a viscosity subsolution. Assume that there is a function $\varphi \in C^1(\Omega) \cap C(\overline{\Omega})$ touching u_p from above at some point $x_0 \in \{u_p > 0\}$ such that

$$-L_p \varphi(x_0) > 0.$$

First, we claim that one can assume that $u_p < \phi$ in $\Omega \backslash \{x_0\}$. Fix $\delta > 0$ small enough and set $\varphi_\delta(x) := \varphi(x) + \delta |x - x_0|^2$, for every $x \in \Omega$. We have

$$\begin{aligned} &|\varphi_\delta(x) - \varphi_\delta(y)|^{p-2}[\varphi_\delta(x) - \varphi_\delta(y)] \\ &= |\varphi(x) - \varphi(y) + \delta[|x-x_0|^2 - |y-x_0|^2]|^{p-2}[\varphi(x) - \varphi(y) + \delta[|x-x_0|^2 - |y-x_0|^2]]. \end{aligned}$$

Yet,

$$\begin{aligned} &\Big| |\varphi_\delta(x) - \varphi_\delta(y)|^{p-2}[\varphi_\delta(x) - \varphi_\delta(y)] - |\varphi(x) - \varphi(y)|^{p-2}[\varphi(x) - \varphi(y)] \Big| \\ &= \left| (p-1)\left(\int_0^1 |\varphi(x) - \varphi(y) + \delta t[|x-x_0|^2 - |y-x_0|^2]|^{p-2}\, \mathrm{d}t \right) \delta[|x-x_0|^2 - |y-x_0|^2] \right| \\ &\le C\delta |x-y|^{p-1}. \end{aligned}$$

Then, we get

$$|L_p \varphi_\delta(x) - L_p \varphi(x)| \le C\delta.$$

Therefore, $-L_p \varphi_\delta(x_0) > 0$ provided that $\delta > 0$ is small enough. For $\varepsilon > 0$ small enough, we define

$$\varphi_\varepsilon := \min\{u_p, \varphi - \varepsilon\} \qquad \text{and} \qquad \varphi^\varepsilon := \max\{u_p, \varphi - \varepsilon\}.$$

Since $0 \le u_p < \phi$ in $\Omega \backslash \{x_0\}$, $\varphi(x_0) = u_p(x_0) > 0$ and $\varphi \in C(\overline{\Omega})$, then we have $\varphi_\varepsilon = u_p$ on $\partial\Omega$ and $\varphi - \varepsilon > 0$ on Ω, for all $\varepsilon > 0$ small enough. Yet, u_p is a minimizer and so, one has

$$\frac{1}{p} \int_{\Omega \times \Omega} \left[\frac{\frac{|u_p(x) - u_p(y)|}{|x-y|^\alpha}}{\Lambda} \right]^p + \lambda |\{u_p > 0\}| \le \frac{1}{p} \int_{\Omega \times \Omega} \left[\frac{\frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|}{|x-y|^\alpha}}{\Lambda} \right]^p + \lambda |\{\varphi_\varepsilon > 0\}|. \tag{2.3}$$

For $p \ge 1$, we have the following convexity inequality (see [11]):

$$|\min\{a, c\} - \min\{b, d\}|^p + |\max\{a, c\} - \max\{b, d\}|^p \le |a - b|^p + |c - d|^p.$$

So, we have

$$\int_{\Omega\times\Omega}\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^p}{|x-y|^{\alpha p}}+\int_{\Omega\times\Omega}\frac{|\varphi^\varepsilon(x)-\varphi^\varepsilon(y)|^p}{|x-y|^{\alpha p}}\le\int_{\Omega\times\Omega}\frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}}+\int_{\Omega\times\Omega}\frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{\alpha p}}.$$

But, it is clear that

$$|\{\varphi_\varepsilon>0\}|=|\{u_p>0\}|.$$

Recalling (2.3), this yields that

$$\int_{\Omega\times\Omega}\frac{|\varphi^\varepsilon(x)-\varphi^\varepsilon(y)|^p}{|x-y|^{\alpha p}}\le\int_{\Omega\times\Omega}\frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{\alpha p}}.$$

Now, set

$$\mathcal{J}(t):=\int_{\Omega\times\Omega}\frac{|(1-t)\varphi(x)+t\varphi^\varepsilon(x)-(1-t)\varphi(y)-t\varphi^\varepsilon(y)|^p}{|x-y|^{\alpha p}},\qquad \text{for all } t\in[0,1].$$

Yet, we have

$$\begin{aligned}\mathcal{J}(t)&\le(1-t)\int_{\Omega\times\Omega}\frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{\alpha p}}+t\int_{\Omega\times\Omega}\frac{|\varphi^\varepsilon(x)-\varphi^\varepsilon(y)|^p}{|x-y|^{\alpha p}}\\&\le\int_{\Omega\times\Omega}\frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{\alpha p}}=\mathcal{J}(0).\end{aligned}$$

Hence, $t=0$ maximizes $\mathcal{J}(t)$. Therefore, $\mathcal{J}'(0)\le 0$. But, we have

$$\mathcal{J}'(0)=p\int_{\Omega\times\Omega}\frac{|\varphi(x)-\varphi(y)|^{p-1}}{|x-y|^{\alpha p}}\frac{\varphi(x)-\varphi(y)}{|\varphi(x)-\varphi(y)|}[\varphi^\varepsilon(x)-\varphi(x)+\varepsilon-(\varphi^\varepsilon(y)-\varphi(y)+\varepsilon)]\le 0.$$

So, we get

$$\int_\Omega[-L_p\varphi(x)][\varphi^\varepsilon(x)-\varphi(x)+\varepsilon]\le 0,$$

which yields to a contradiction as soon as $\varepsilon>0$ is small enough. In the same way we prove that u_p is a viscosity supersolution. □

Proposition 2.4. *The limit function u_∞ is a viscosity solution to $-L_\infty u_\infty=0$ in the positivity set $\{u_\infty>0\}$.*

Proof. Fix $x_0\in\{u_\infty>0\}$. Let $\phi\in C^1(\Omega)\cap C(\overline{\Omega})$ be such that $u_\infty\ge\phi$ on $\bar\Omega$ and $u_\infty(x_0)=\phi(x_0)$. Recall that one can assume that x_0 is the unique minimizer of $u_\infty-\phi$. Set $m_p=\min[u_p-\phi]$. So, one has $x_p\to x_0$ with $u_p(x_p)=\phi(x_p)+m_p$. Since u_p is a viscosity solution in $\{u_p>0\}$, then we have

$$-L_p\phi(x_p)\ge 0.$$

Hence,

$$\int_\Omega\frac{[\phi(y)-\phi(x_p)]_-^{p-1}}{|x-y|^{\alpha p}}\,\mathrm{d}y\ge\int_\Omega\frac{[\phi(y)-\phi(x_p)]_+^{p-1}}{|x-y|^{\alpha p}}\,\mathrm{d}y.$$

Letting $p\to\infty$ and thanks to [4, Lemma 6.5], we get the following inequality:

$$-L_\infty^-\phi(x_0)\ge L_\infty^+\phi(x_0).$$

Now, let us show that u_∞ is a viscosity subsolution (so, it is a viscosity solution). Fix $x_0 \in \{u_\infty > 0\}$. Let $\phi \in C^1(\Omega) \cap C(\bar{\Omega})$ be such that $u_\infty \leq \phi$ on $\bar{\Omega}$ and $u_\infty(x_0) = \phi(x_0)$. Assume again that x_0 is the unique minimizer of $u_\infty - \phi$. Set $M_p = \max[u_p - \phi]$ and $x_p \in \Omega$ is such that $u_p(x_p) = \phi(x_p) + M_p$. We note again that $x_p \to x_0$. Since u_p is a viscosity solution in $\{u_p > 0\}$, then one has

$$-L_p\phi(x_p) \leq 0.$$

Hence,

$$-\int_\Omega \frac{[\phi(y) - \phi(x_p)]_+^{p-1}}{|x-y|^{\alpha p}}\,\mathrm{d}y + \int_\Omega \frac{[\phi(y) - \phi(x_p)]_-^{p-1}}{|x-y|^{\alpha p}}\,\mathrm{d}y \leq 0.$$

Then, we get that

$$-L_\infty^+\phi(x_0) - L_\infty^-\phi(x_0) \leq 0. \quad \square$$

3. Characterization of the best Hölder extension

Let Ω be a bounded domain in $\mathbb{R}^N$. For a subset $A \subset\subset \Omega$, we consider the following minimization problem:

$$\Lambda(A) := \min\{[u]_\alpha : \ u \in C^{0,\alpha}(\bar{\Omega}),\ u \geq 0,\ \ u = 1 \ \text{ on } \ \partial\Omega,\ \ u = 0 \ \text{ on } \ A\}. \tag{3.1}$$

Proposition 3.1. *The following function* $u(x) = \frac{d(x,A)^\alpha}{d(x,A)^\alpha + d(x,\partial\Omega)^\alpha}$ *minimizes Problem* (3.1).

Proof. Clearly, this function u satisfies the boundary conditions $u = 0$ on A and $u = 1$ on $\partial\Omega$. Moreover, we have the following:

$$\sup_{y\in\partial\Omega\cup A} \frac{u(y)-u(x)}{|y-x|^\alpha} = \sup_{y\in\partial\Omega} \frac{1-u(x)}{|y-x|^\alpha} = \sup_{y\in\partial\Omega} \frac{1 - \frac{d(x,A)^\alpha}{d(x,A)^\alpha + d(x,\partial\Omega)^\alpha}}{|y-x|^\alpha}$$

$$= \frac{d(x,\partial\Omega)^\alpha}{d(x,A)^\alpha + d(x,\partial\Omega)^\alpha} \sup_{y\in\partial\Omega} \frac{1}{|y-x|^\alpha} = \frac{1}{d(x,A)^\alpha + d(x,\partial\Omega)^\alpha}$$

and

$$\inf_{y\in\partial\Omega\cup A} \frac{u(y)-u(x)}{|y-x|^\alpha} = \inf_{y\in A} \frac{-u(x)}{|y-x|^\alpha} = -\sup_{y\in A} \frac{\frac{d(x,A)^\alpha}{d(x,A)^\alpha + d(x,\partial\Omega)^\alpha}}{|y-x|^\alpha}$$

$$= -\frac{d(x,A)^\alpha}{d(x,A)^\alpha + d(x,\partial\Omega)^\alpha} \sup_{y\in A} \frac{1}{|y-x|^\alpha} = -\frac{1}{d(x,A)^\alpha + d(x,\partial\Omega)^\alpha}.$$

Hence,

$$\sup_{y\in\partial\Omega\cup A} \frac{u(y)-u(x)}{|y-x|^\alpha} + \inf_{y\in\partial\Omega\cup A} \frac{u(y)-u(x)}{|y-x|^\alpha} = 0.$$

For every $x \in \Omega$, we claim that

$$\sup_{y\in\bar{\Omega},\, y\neq x} \frac{u(y)-u(x)}{|y-x|^\alpha} = \sup_{y\in\partial\Omega} \frac{u(y)-u(x)}{|y-x|^\alpha}$$

and

$$\inf_{y\in\bar{\Omega},\, y\neq x} \frac{u(y)-u(x)}{|y-x|^\alpha} = \inf_{y\in A} \frac{u(y)-u(x)}{|y-x|^\alpha}.$$

Fix $x_1,\, x_2 \in \Omega$. Assume that $S^+_{x_1} := \sup_{y\in\partial\Omega} \frac{u(y)-u(x_1)}{|y-x_1|^\alpha} \geq \sup_{y\in\partial\Omega} \frac{u(y)-u(x_2)}{|y-x_2|^\alpha} := S^+_{x_2}$. Let $y_1^+ \in \partial\Omega$ be such that

$$S^+_{x_1} = \frac{1-u(x_1)}{|y_1^+ - x_1|^\alpha}.$$

Yet, we have

$$S^+_{x_2} \geq \frac{1-u(x_2)}{|y_1^+ - x_2|^\alpha}.$$

Hence,

$$u(x_1) - u(x_2) \leq S^+_{x_2}\,|y_1^+ - x_2|^\alpha - S^+_{x_1}|y_1^+ - x_1|^\alpha \leq S^+_{x_2}|x_2 - x_1|^\alpha.$$

Consequently,

$$\frac{u(x_1)-u(x_2)}{|x_1-x_2|^\alpha} \leq S^+_{x_2} \qquad \text{if } \; S^+_{x_1} \geq S^+_{x_2}.$$

On the other side, we denote by $S^-_x := \inf_{y\in A} \frac{u(y)-u(x)}{|y-x|^\alpha}$. Since $S^+_{x_1} \geq S^+_{x_2}$ and $S^+_{x_1} + S^-_{x_1} = S^+_{x_2} + S^-_{x_2} = 0$ then we have $S^-_{x_1} \leq S^-_{x_2} \leq 0$. Let $y_1^- \in A$ be such that

$$S^-_{x_1} = -\frac{u(x_1)}{|y_1^- - x_1|^\alpha}.$$

One has

$$S^-_{x_2} \leq -\frac{u(x_2)}{|y_1^- - x_1|^\alpha}.$$

Then, we get

$$u(x_1) - u(x_2) \geq -S^-_{x_1}\,|y_1^- - x_1|^\alpha + S^-_{x_2}\,|y_1^- - x_2|^\alpha \geq S^-_{x_2}[|y_1^- - x_2|^\alpha - |y_1^- - x_1|^\alpha] \geq S^-_{x_2}|x_2 - x_1|^\alpha.$$

Therefore,

$$u(x_2) - u(x_1) \leq -S^-_{x_2}|x_2 - x_1|^\alpha = S^+_{x_2}|x_2 - x_1|^\alpha.$$

Interchanging x_1 and x_2, we get

$$\frac{u(x_1)-u(x_2)}{|x_1-x_2|^\alpha} \leq S^+_{x_1} \qquad \text{if } \; S^+_{x_1} \leq S^+_{x_2}.$$

Hence,

$$\sup_{x_1\in\bar\Omega,\, x_1\neq x_2} \frac{u(x_1)-u(x_2)}{|x_1-x_2|^\alpha} \leq \sup_{y\in\partial\Omega} \frac{u(y)-u(x_2)}{|y-x_2|^\alpha}.$$

In the same way, we show that

$$\inf_{y\in\bar\Omega,\, y\neq x} \frac{u(y)-u(x)}{|y-x|^\alpha} = \inf_{y\in A} \frac{u(y)-u(x)}{|y-x|^\alpha}.$$

Thus, we infer that

$$\sup_{y\in\bar\Omega,\, y\neq x} \frac{u(y)-u(x)}{|y-x|^\alpha} + \inf_{y\in\bar\Omega,\, y\neq x} \frac{u(y)-u(x)}{|y-x|^\alpha} = 0. \tag{3.2}$$

Finally, assume that there is a function $v \in C^{0,\alpha}(\Omega)$ such that $v = 0$ on A and $v = 1$ on $\partial\Omega$ with

$$[v]_\alpha < [u]_\alpha.$$

This implies that there is a point $x_0 \in \Omega$ such that

$$[v]_\alpha < \sup_{y\in\bar{\Omega},\, y\neq x_0} \frac{u(y)-u(x_0)}{|y-x_0|^\alpha} = \sup_{y\in\partial\Omega} \frac{u(y)-u(x_0)}{|y-x_0|^\alpha} = \sup_{y\in\partial\Omega} \frac{v(y)-u(x_0)}{|y-x_0|^\alpha}.$$

If $u(x_0) \geq v(x_0)$, we get that

$$[v]_\alpha < \sup_{y\in\partial\Omega} \frac{v(y)-u(x_0)}{|y-x_0|^\alpha} \leq \sup_{y\in\partial\Omega} \frac{v(y)-v(x_0)}{|y-x_0|^\alpha},$$

which is a contradiction. Finally, assume that $u(x_0) < v(x_0)$. Then, thanks to (3.2), we have

$$[v]_\alpha < \sup_{y\in\bar{\Omega},\, y\neq x_0} \frac{u(y)-u(x_0)}{|y-x_0|^\alpha} = -\inf_{y\in\bar{\Omega},\, y\neq x_0} \frac{u(y)-u(x_0)}{|y-x_0|^\alpha} = -\inf_{y\in A} \frac{u(y)-u(x_0)}{|y-x_0|^\alpha}$$

$$= -\inf_{y\in A} \frac{v(y)-u(x_0)}{|y-x_0|^\alpha} \leq -\inf_{y\in A} \frac{v(y)-v(x_0)}{|y-x_0|^\alpha}.$$

But, this is again a contradiction. □

4. Optimal hole

Fix $\lambda > 0$. Then, we consider the problem of minimizing the optimal $\alpha-$Hölder semi-norm $\Lambda(A)$, among all nonnegative functions u in $C^{0,\alpha}(\bar{\Omega})$ with a prescribed Dirichlet boundary condition $u = 1$ on $\partial\Omega$ and $u = 0$ on A, plus a penalization on the volume of the set A. More precisely, we study the following shape optimization problem:

$$\min\{\Lambda(A) - \lambda|A| \,:\, A \subset\subset \Omega\}, \tag{4.1}$$

where

$$\Lambda(A) := \min\{[u]_\alpha : \, u \in C^{0,\alpha}(\bar{\Omega}),\, u \geq 0,\; u = 1 \text{ on } \partial\Omega,\; u = 0 \text{ on } A\}.$$

First, we start by proving the existence of a solution to Problem (4.1).

Proposition 4.1. *There exists an optimal set $A^\star$ that minimizes Problem* (4.1)*.*

Proof. Let $\{A_n\}_n$ be a minimizing sequence in Problem (4.1). For every $n \in \mathbb{N}$, we may assume that A_n is closed since it is clear that $\Lambda(\overline{A_n}) = \Lambda(A_n)$ while $|A_n| \leq |\overline{A_n}|$. For all $n \in \mathbb{N}$, let u_n be a minimizer of Problem (3.1), i.e. $\Lambda(A_n) = [u_n]_\alpha$. Then, there is a constant $C < \infty$ such that for all $n \in \mathbb{N}$,

$$\Lambda(A_n) - \lambda|A_n| = [u_n]_\alpha - \lambda|A_n| \leq C.$$

In particular, $[u_n]_\alpha \leq C + \lambda|\Omega|$. Yet, $u_n = 1$ on $\partial\Omega$. Hence, $\{u_n\}_n$ is uniformly bounded since

$$|u_n(x)| \leq 1 + C\,\mathrm{diam}(\Omega)^\alpha, \qquad \text{for all } \; x \in \bar{\Omega}.$$

So, up to subsequences, $u_n \to u$ uniformly in $\bar{\Omega}$ and A_n converges to A in the Hausdorff distance. Clearly, $u = 1$ on $\partial\Omega$ and $u = 0$ on A. But, one has

$$|u_n(x) - u_n(y)| \leq [u_n]_\alpha\, |x-y|^\alpha, \qquad \text{for all } \; x,\, y \in \bar{\Omega}. \tag{4.2}$$

Passing to the limit in (4.2) when $n \to \infty$, this yields that

$$|u(x) - u(y)| \leq \liminf_n [u_n]_\alpha\, |x-y|^\alpha, \qquad \text{for all } \; x,\, y \in \bar{\Omega}.$$

Thus,

$$\Lambda(A) \leq [u]_\alpha \leq \liminf_n [u_n]_\alpha = \liminf_n \Lambda(A_n). \tag{4.3}$$

On the other hand, if $x \in A_n$ for all n large enough then we also have $x \in A$ and so, we get that

$$\limsup_n |A_n| \leq |A|. \tag{4.4}$$

Combining (4.3) & (4.4), we get

$$\Lambda(A) - \lambda|A| \leq \liminf_n [\Lambda(A_n) - \lambda|A_n|]. \tag{4.5}$$

This implies that A is an optimal set. □

In the next proposition, we show that we can always find an explicit optimal set to Problem (4.1). Let R_Ω be the inradius of Ω (the radius of the largest ball that can be contained in Ω), i.e.

$$R_\Omega = \max_{x\in\Omega} d(x, \partial\Omega).$$

Proposition 4.2. *For every $\lambda > 0$, at least one of the following two statements holds: $A = \emptyset$ is an optimal set or there exists $r \in (0, R_\Omega]$ such that $A_r := \{x : d(x, \partial\Omega) \geq r\}$ is an optimal set. Moreover, the function*

$$u_r(x) = \left[1 - \frac{d(x,\partial\Omega)^\alpha}{r^\alpha}\right]_+$$

minimizes $\Lambda(A_r)$. More precisely, if there is an optimal set A such that $\Lambda(A) < \frac{1}{R_\Omega^\alpha}$ then $A = \emptyset$, while if $\Lambda(A) \geq \frac{1}{R_\Omega^\alpha}$ then for $r := \frac{1}{\Lambda(A)^{1/\alpha}} \in (0, R_\Omega]$, A_r is an optimal set.

Proof. Let $A \subset\subset \Omega$ and u be a minimizer in Problem (3.1). Let us denote by $P(x)$ any projection point of x onto the boundary. Since $u = 1$ on $\partial\Omega$, then we have that

$$|u(x) - 1| = |u(x) - u(P(x))| \leq [u]_\alpha \, |x - P(x)|^\alpha = [u]_\alpha \, d(x, \partial\Omega)^\alpha.$$

For every $x \in A$, one has $u(x) = 0$. Then, we must have

$$d(x, \partial\Omega) \geq \frac{1}{[u]_\alpha^{1/\alpha}}.$$

This implies that

$$\left\{x \, : \, d(x,\partial\Omega) < \frac{1}{[u]_\alpha^{1/\alpha}}\right\} \subset \{u > 0\}. \tag{4.6}$$

If $R_\Omega < \frac{1}{\Lambda(A)^{1/\alpha}} = \frac{1}{[u]_\alpha^{1/\alpha}}$ then $\{u > 0\} = \Omega$, since for every $x \in \Omega$ one has

$$d(x, \partial\Omega) \leq R_\Omega < \frac{1}{[u]_\alpha^{1/\alpha}}$$

and then, $A = \emptyset$ (so, $u = 1$).

Finally, assume that $R_\Omega \geq \frac{1}{\Lambda(A)^{1/\alpha}} = \frac{1}{[u]_\alpha^{1/\alpha}}$. So, there is a $r \in (0, R_\Omega]$ such that $[u]_\alpha = \frac{1}{r^\alpha}$. Now, consider the function u_r. It is clear that $u_r \in C^{0,\alpha}(\bar{\Omega})$ with $[u_r]_\alpha = \frac{1}{r^\alpha}$. From the definition of A_r, we also have $u_r = 1$ on $\partial\Omega$ and $u_r = 0$ on A_r. Hence, $\Lambda(A) = [u]_\alpha = [u_r]_\alpha$. Recalling (4.6), one has

$$\Omega \backslash A_r = \left\{x \, : \, d(x, \partial\Omega) < r\right\} \subset \{u > 0\}. \tag{4.7}$$

Yet, $u = 0$ on A. Hence, $A \subset A_r$. Thus, we have the following:

$$|A| \leq |A_r|.$$

From the optimality of A, we infer that

$$\Lambda(A) - \lambda|A| \leq \Lambda(A_r) - \lambda|A_r| \leq [u_r]_\alpha - \lambda|A_r| \leq [u_r]_\alpha - \lambda|A| = \Lambda(A) - \lambda|A|.$$

This yields that A_r is an optimal set in Problem (4.1), u_r is also a minimizer in $\Lambda(A_r)$ and, $A = A_r$ almost everywhere. □

Proposition 4.3. *Assume Ω is convex. Then, there exists $\lambda^\star > 0$ such that for all $\lambda < \lambda^\star$, $A = \emptyset$ is the unique optimal set. For $\lambda = \lambda^\star$, there exists a unique $r_{\lambda^\star} \in (0, R_\Omega)$ such that $A_{r_{\lambda^\star}}$ is an optimal set (while $A = \emptyset$ is always optimal). And for $\lambda > \lambda^\star$, there is a $r_\lambda \in (0, R_\Omega)$ such that A_{r_λ} is optimal. Moreover, we have the following characterization:*

$$r_\lambda^{\alpha+1}\, \mathcal{H}^{N-1}(\partial A_{r_\lambda}) = \frac{\alpha}{\lambda}, \qquad \textit{for all } \lambda \geq \lambda^\star,$$

and

$$r_{\lambda^\star}^{\alpha}\, |A_{r_{\lambda^\star}}| = \frac{1}{\lambda^\star}.$$

Proof. First, we define

$$f_\lambda(r) := \Lambda(A_r) - \lambda|A_r|.$$

Thanks to Proposition 4.2, we have

$$f_\Lambda(r) = [u_r]_\alpha + \lambda|\Omega\backslash A_r| - \lambda|\Omega|, \qquad \text{for all } r \in (0, R_\Omega].$$

Hence, one has

$$f_\Lambda(r) = \frac{1}{r^\alpha} + \lambda|\{x : d(x, \partial\Omega) < r\}| - \lambda|\Omega|.$$

Using the coarea formula, we have

$$f_\lambda(r) = \frac{1}{r^\alpha} + \lambda \int_o^r \mathcal{H}^{N-1}(\{x : d(x, \partial\Omega) = t\})\,\mathrm{d}t - \lambda|\Omega|.$$

Then,

$$f'_\lambda(r) = -\frac{\alpha}{r^{\alpha+1}} + \lambda\mathcal{H}^{N-1}(\{x : d(x, \partial\Omega) = r\}).$$

So,

$$f'_\lambda(r) \leq 0 \Longleftrightarrow \frac{\alpha}{r^{\alpha+1}} \geq \lambda\, \mathcal{H}^{N-1}(\partial A_r) \Longleftrightarrow \frac{\alpha^{\frac{1}{N-1}}}{r^{\frac{\alpha+1}{N-1}}} \geq \lambda^{\frac{1}{N-1}}\, \mathcal{H}^{N-1}(\partial A_r)^{\frac{1}{N-1}}.$$

For simplicity of notation, we set

$$\Psi(r) = \mathcal{H}^{N-1}(\partial A_r)^{\frac{1}{N-1}} \qquad \text{and} \qquad G(r) = \frac{\alpha^{\frac{1}{N-1}}}{r^{\frac{\alpha+1}{N-1}}}.$$

Since Ω is convex, thanks to the Brunn-Minkowski inequality, Ψ is decreasing and concave. However, G is decreasing and strictly convex. Moreover, one has $\lim_{r\to 0^+} G(r) = +\infty$. First, assume that $\Psi(R_\Omega) = 0$ (we note that this is not the case in general; consider the case when Ω is a stadium or simply a rectangle). Then, thanks to these properties on G and Ψ, it is clear that there is a constant λ_i such that the following statements hold:

- $G(r) > \lambda^{\frac{1}{N-1}}\Psi(r)$ on $[0, R_\Omega]$, for all $\lambda < \lambda_i$.

- There exists $r_i \in (0, R_\Omega]$ such that $G(r_i) = \lambda_i^{\frac{1}{N-1}} \Psi(r_i)$ and $G(r) > \lambda_i^{\frac{1}{N-1}} \Psi(r)$ for all $r \neq r_i$.
- There exist $r_\lambda,\ R_\lambda \in (0, R_\Omega]$ such that $G(r_\lambda) = \lambda^{\frac{1}{N-1}} \Psi(r_\lambda),\ G(R_\lambda) = \lambda^{\frac{1}{N-1}} \Psi(R_\lambda)$ and,

$$G(r) < \lambda^{\frac{1}{N-1}} \Psi(r) \Longleftrightarrow r_\lambda < r < R_\lambda.$$

For all $\lambda < \lambda_i$, we have $f'_\lambda(r) < 0$ for every $0 < r \leq R_\Omega$. Hence, we get the following:

$$\min_{r \in (0, R_\Omega]} f_\lambda(r) = f_\lambda(R_\Omega) = \frac{1}{R_\Omega^\alpha} > 0.$$

Then,

$$\Lambda(A_r) - \lambda |A_r| > 0, \qquad \text{for all } r \in (0, R_\Omega].$$

Recalling Proposition 4.2, this implies that for any optimal set A, we must have $\Lambda(A) < \frac{1}{R_\Omega^\alpha}$ and so, $A = \emptyset$. If $\lambda = \lambda_i$, $f'_\lambda(r) \leq 0$ and so, we get again that

$$\min_{r \in (0, R_\Omega]} f_\lambda(r) = f_\lambda(R_\Omega) = \frac{1}{R_\Omega^\alpha} > 0$$

and so, $A = \emptyset$ is the unique minimizer.

Now, assume $\lambda > \lambda_i$. Then, one has

$$\min_{r \in (0, R_\Omega]} f_\lambda(r) = \min\{f_\lambda(r_\Lambda), f_\lambda(R_\Omega)\} = \min\left\{f_\lambda(r_\lambda), \frac{1}{R_\Omega^\alpha}\right\}.$$

Yet, $\min_{r \in (0, R_\Omega]} f_0(r) = \frac{1}{R_\Omega^\alpha} > 0$ while $\min_{r \in (0, R_\Omega]} f_\lambda(r) < 0$ for $\lambda > \lambda_i$ sufficiently large. Moreover, it is easy to see that $\lambda \mapsto \min_{r \in (0, R_\Omega]} f_\lambda(r)$ is continuous and decreasing. Hence, there is a unique $\lambda^\star > \lambda_i$ such that $\min_{r \in (0, R_\Omega]} f_{\lambda^\star}(r) = 0$. Thus, $f_{\lambda^\star}(r_{\lambda^\star}) = 0$ for some $r_{\lambda^\star} \in (0, R_\Omega]$ and so, we get that

$$\Lambda(A_{r_{\lambda^\star}}) - \lambda^\star |A_{r_{\lambda^\star}}| = 0 < \Lambda(A_r) - \lambda^\star |A_r|, \qquad r \neq r_{\lambda^\star}.$$

This implies that $A_{r_{\lambda^\star}}$ is an optimal set (while $A = \emptyset$ is always optimal). If $\lambda > \lambda^\star$, then there exists r_λ such that $f_\lambda(r_\lambda) = \min_{r \in (0, R_\Omega]} f_\lambda(r_\lambda) < 0$. Hence, A_{r_Λ} is a minimizer.

Finally, we note that if $\lambda_i < \lambda < \lambda^\star$ then we have

$$\min_{r \in (0, R_\Omega]} f_\lambda(r) = \min\left\{f_\lambda(r_\lambda), \frac{1}{R_\Omega^\alpha}\right\} > 0.$$

Hence,

$$\Lambda(A_r) - \lambda |A_r| > 0, \qquad \text{for all } r \in (0, R_\Omega].$$

So, $A = \emptyset$ is again the unique minimizer.

In the same way, we treat the case when $\Psi(R_\Omega) > 0$; the only difference now is that for λ large there exists a unique $r_\lambda \in (0, R_\Omega]$ such that $G(r_\lambda) = \lambda^{\frac{1}{N-1}} \Psi(r_\lambda)$ and $G(r) < \lambda^{\frac{1}{N-1}} \Psi(r)$ when $r > r_\lambda$. This yields that $\min_{r \in (0, R_\Omega]} f_\lambda(r) = f_\lambda(r_\lambda) < 0$ and so, A_{r_λ} is an optimal set. □

References

[1] Acker A., Heat flow inequalities with applications to heat flow optimization problems, *SIAM J. Math. Anal.* 4, 604-618 (1977).

[2] H. W. Alt and L. A. Caffarelli, Existence and regularity for a minimum problem with free boundary, *J. Reine Angew. Math.* 325 (1981), 105-144.

[3] Beurling A., On free-boundary problems for the Laplace equation, *Seminars on analytic functions*, vol. I, pp. 248-263.

[4] Chambolle A., Lindgren E., Monneau R., A Hölder infinity Laplacian, *ESAIM: Control, Optimisation and Calculus of Variations* 18.3 (2012): 799-835.

[5] G. Crasta and I. Fragalà, On the supremal version of the Alt-Caffarelli minimization problem, *Advances in Calculus of Variations*, vol. 14, no. 3, 2021, pp. 327-341.

[6] D. Daners and B. Kawohl, An isoperimetric inequality related to a Bernoulli problem, *Calc. Var. Partial Differential Equations* 39 (2010), no. 3-4, 547-555.

[7] D. Danielli and A. Petrosyan, A minimum problem with free boundary for a degenerate quasilinear operator, *Calc. Var. Partial Differential Equations* 23 (2005), no. 1, 97-124.

[8] Di Nezza E., Palatucci G., Valdinoci E.: Hitchhikers guide to the fractional Sobolev spaces, *Bull. Sci. Math.* 136(5), 521-573 (2012).

[9] B. Kawohl and H. Shahgholian, Gamma limits in some Bernoulli free boundary problem, *Arch. Math.* (Basel) 84 (2005), no. 1, 79-87.

[10] Lacey, A. A., Shillor M., Electrochemical and electro-discharge machining with a threshold current. IMA J. Appl. Math. 39(2), 121-142 (1987).

[11] K. Murota, Discrete convex analysis. SIAM Monographs on Discrete Mathematics and Applications, *Society for Industrial and Applied Mathematics (SIAM)*, Philadelphia, PA (2003).

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