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On the minimizers of some energies containing Riesz-like terms

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Abstract

In recent years, the study of Riesz-like energies has attracted much attention from the mathematical community. Given an interaction kernel $\bar{g} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$, the associated Riesz energy is defined on non-negative measures as

$$\mathcal{E}(\mu) = \iint \bar{g}(x-y)d\mu(x)d\mu(y),$$

and models self-interaction phenomena. Attractive-repulsive kernels, qualitatively similar to $\bar{g}(x) = |x|^2 + \frac{1}{|x|}$, have been used to model swarming phenomena, where minimizing the energy $\mathcal{E}$ is required. The choice of the kernel plays a crucial role in the analysis, and many different results are obtained by tweaking the kernel properly. We study the minimization problem in the class of probability measures, and we prove some regularity and symmetry results under very weak assumptions on $\bar{g}$. This topic has already been addressed by other authors for specific kernels, for example in the works [CDM16, BCT18, Lop19]. We address a related shape optimization problem, where we aim to find minimizers of $\mathcal{E}$ among subsets of $\mathbb{R}^N$ with a prescribed measure $m > 0$. We consider both the cases of small and large m, and we prove the existence of minimal sets in some cases. In the regime of small m, we consider weakly repulsive kernels, connecting the shape optimization problem with the minimization in the class of probability measures. In the opposite regime, when m is large, we follow the strategy outlined in [FL21] to show that balls uniquely solve the shape optimization problem. Finally, we consider a generalized Gamow model, which consists of minimizing the functional $\mathcal{G}_\gamma(E) = \mathsf{P}(E) + \gamma\mathcal{E}(\chi_E \mathscr{L}^N)$ among subsets of $\mathbb{R}^N$ with measure ω_N. In this case, it is natural to work with a completely repulsive kernel like $\frac{1}{|x|}$. We consider general kernels, and we characterize the balls as the unique minimizers of $\mathcal{G}_\gamma$ when γ is small, generalizing some previous works, such as [KM14, BC14, FFM+15]. Additionally, we study the optimal families of balls that minimize $\mathcal{G}_\gamma$, and their dependence on the total measure constraint.

Our contributions are contained in [CFP23, Car23, CP24, CPT24, CP25].

Contents

Introduction

In this thesis, we study the role of the Riesz-like energies in some variational problems. These objects have been intensively studied in the last years, as they appear in a variety of models coming from many different natural sciences. In general, a Riesz-like energy, associated with an interaction kernel $\bar{g} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$, is of the form

$$\mathcal{E}(\mu) = \iint \bar{g}(x-y)d\mu(x)d\mu(y),$$

where μ is a non-negative measure. The most significant instance of this kind of functional is probably the electrostatic energy: if $N = 3$, $\bar{g}(x) = 1/|x|$ and μ is a distribution of charges in the vacuum, then $\mathcal{E}(\mu)$ is the electrostatic energy of this system. Similar functionals have been used to model more complex phenomena, where the interaction between the agents in the system is not merely repulsive, as it is in the case electrostatic interaction for a system of electrons. In fact, there are some self-assembly phenomena, such as the swarming behaviour of large groups of animals, that are modeled through energies of Riesz type, with an attractive-repulsive kernel (see [BCT18, FL18, FL21]). A very popular choice for the interaction kernel is given by $\bar{g}(x) = |x|^\alpha + \frac{1}{|x|^\beta}$, with $\alpha > 0$ and $\beta \in (0, N)$. To be more precise, two objects that are very close to each other experience a repulsive force due to this interaction (exactly as two electrons would do), whereas two particles that are far away tend to get closer to one another. These kernels, despite being arguably the simplest possible choice for attractive-repulsive interactions, produce a large spectrum of results, which is likely one of the reasons why they have drawn so much attention from the mathematical community. In fact, the minimizers are absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^N$ when the repulsive part of the kernel, namely $\frac{1}{|x|^\beta}$, is singular enough in the origin (see [CDM16]). Instead, when the repulsion is weaker, there are examples of singular minimizers (for instance, they can contain "$(N-1)$-dimensional parts", like in [FM23]). When the kernel is smooth in the whole $\mathbb{R}^N$, there is nothing excluding the existence of Dirac masses in the minimizers, and in fact this happens in [DLM23]. Comparing these results with those of the work [DLM22], it is evident that this problem requires a very careful analysis. In fact, combining the two works, one obtains a family of C^2 kernels that vary smoothly, and which exhibit completely different minimizers. We also recall the existence of pathological examples presented in Sections 7, 8, 9 and 10 in [CS23], where the authors construct some kernels for which the local minimizers of the associated Riesz energy have a fractal-like behaviour. These serve as examples to justify our interest in the problem, and particularly in the underlying phenomena that drive the self-assembly behaviour modeled by these functionals. From the applied point of view, they show the importance of finding kernels as close as possible to the "true" interaction, since a tiny variation could result in completely unexpected results. With this overview in mind, our work is devoted to the investigation of (relatively) old questions under mild assumptions on the kernel to see which results are robust enough to hold in great generality. Even though the previous considerations regard only the minimization among probability measures, it is natural to consider the minimization within certain restricted classes, either to match some reasonable assumptions coming from the models, or to study a simplified problem as a starting point for our study. In fact, one could restrict oneself to work in the class of densities with an L^∞ upper bound, and a L^1 constraint, or in the class of sets with a given measure, and in both cases the concentration of mass is forbidden. Instead, as it will become clear later, it is also natural to treat the minimization problem among measures with radial symmetry, which arise spontaneously in some cases. We treat the following problems containing attractive-repulsive kernels:

$$\min\Big\{\mathcal{E}(\chi_E\mathscr{L}^N)\,:\,E\subseteq\mathbb{R}^N,\,|E|=m\Big\},\tag{P_S}$$

$$\min\Big\{\mathcal{E}(f\mathscr{L}^N)\,:\,f\in L^1(\mathbb{R}^N),0\le f\le 1,\|f\|_1=m\Big\},\tag{P_D}$$

$$\min\Big\{\mathcal{E}(\mu)\,:\,\mu\in\mathcal{M}_+(\mathbb{R}^N),\mu(\mathbb{R}^N)=1\Big\},\tag{P_M}$$

where $m>0$ is a given constraint. The three problems are obviously connected: the first two minimization classes are included one into the other, and they are both embedded in the space of probability measures after a rescaling. From this connection, we infer some results for the minimization among sets, i.e. (P_S), from the problem (P_M) set in the space of measures. The variational problem for measures contains intrinsically some difficulties, due to the possible singularities of the measures. The necessary analysis is interesting on its own, and it additionally sheds some light on the problem (P_D), identifying some relevant phenomena. We mention that the problem (P_M) attracted some attention also in the setting of potential theory, as one can see from [ST97]. We point out that similar problems, connected with the "discrete counterpart" dealing with particle systems, are considered in the recent works [Lew22, Section V] and [Ser24].
We remark that the analysis carried out in this thesis focuses on radially symmetric kernels, that represent an important example, providing a wide range of different phenomena. However, the problem makes perfect sense also for non-radial kernels, and this was treated in the literature (in a less extensive way). The research in [CS24a, CS24b, MMR$^+$23, Mor24] focused on the dimension 2 and 3, where the authors compute explicitly the minimizing measures. They also show the peculiar phenomenon of the change of dimensionality of the minimizers as they tweak a parameter involved in the anisotropy of the kernel. This phenomenon enters in the study of the density minimizers, that we treat in Chapter 2, as it is shown in [CMS24].

Returning to the electrostatic interaction, we mention another important model: the Gamow model. It describes atomic nuclei as charged liquid drops with constant density of charge, which amounts to minimizing the functional

$$\mathcal{G}_\gamma(E)=\mathsf{P}(E)+\gamma\iint_{E\times E}\frac{1}{|x-y|}dxdy$$

among subsets of $\mathbb{R}^3$ with a given measure, where $\gamma > 0$ is a parameter that encodes the strength of the electrostatic force. The physical model dates back to the 1930s, in Gamow's original paper [Gam30]. The mathematical community is still very active on such problems, with particular interest in the existence/non-existence of minimizers and on their characterization, when the electrostatic energy is replaced by more general Riesz energies. Again, we push the hypotheses to the minimum, and we characterize the balls as the unique minimizers when γ is small. Related to this, we also study the optimal splitting of the measure among families of balls in order to minimize $\mathcal{G}_\gamma$, providing a good description of the optimal families depending on their total measure. This is relevant because of the results in [KM13, BC14], and because it is believed that, at least in a wide range of power kernels, the minimizers of $\mathcal{G}_\gamma$ are necessarily balls. We refer to [Fra23] to have a broader view about some functionals containing a Riesz-like energy.

It is natural to consider also anisotropic versions of the Gamow model, as it is done in [CNT22], where they address the problem with anisotropy only on the perimeter term, or in combination with an anisotropic Riesz kernel. In this setting, the strategy is similar to the one outlined below, and in fact they use a strong quantitative isoperimetric inequality obtained in [Neu16]. However, there are important issues with the regularity theory when the Wulff shape is not smooth, as it is pointed out in [CNT22, Theorem 1.3].

On top of these applications, the Riesz-energies are independently studied in the fields of potential theory and measure theory (see for example [Lan72, Mat95, Mat15]). Some techniques coming from these abstract mathematical fields are borrowed to study the more applied problems. An important example is the notion of "$\bar{g}$-capacity" that is outlined in Remark 1.1.4.

Probability minimizers for Riesz-like energies

The goal of Chapter 1 is to study the problem (P_M) set in the space of probabilities, where we aim to prove some regularity results for the minimizers, as well as some symmetry properties. We begin by noting that the minimization problem is trivial if the kernel is radial and radially monotone because of the Riesz rearrangement inequality (see for example [LL01, Theorem 3.7]). In fact, we either have no existence, due to the complete loss of mass, or the unique minimizer is the Dirac delta. The existence for that problem is already investigated in the literature, as one can find out in [SST15, CnCP15], and for completeness we report a possible proof in a case of interest for us. Attempting to explicitly compute the minimizers is an extraordinarily difficult task, and there are only a few cases in which this is possible: for singular kernels, one can look at the works [Fra22, FM23] and [CS23, Section 5], while the case of locally bounded kernels (that are weakly repulsive) is treated in [CFP17, DLM22, DLM23, CPT24]. Because of this difficulty, the most reasonable questions regard qualitative properties of the minimizers, like their regularity and their symmetry. Section 1.2 is devoted to proving an L^∞ bound for some minimizers of (P_M), that can be considered a regularity result. In general, the expected regularity is very poor unless we assume some structure on the kernel $\bar{g}$. In fact, in [CDM16] the authors obtain some good regularity results via

PDE techniques because they consider kernels related to the fundamental solution of the (possibly fractional) Laplacian. In their setting, as well as in ours, the argument consists in studying the Euler-Lagrange conditions, that are written in terms of the so-called potential of a minimal measure μ, which is just the convolution of μ and the kernel $\bar{g}$: for every $x \in \mathbb{R}^N$

$$\psi_\mu(x) = \int \bar{g}(x - y) d\mu(y).$$

In fact, the potential turns out to be constant in the support of a minimizer, and takes larger values outside of the support. In [CDM16] the kernel satisfies a (possibly nonlocal) PDE, and they apply the theory of the (possibly fractional) obstacle problem to gain some regularity. Compared to those results, Theorem 1.2.1 shows that some minimizers are absolutely continuous with respect to $\mathscr{L}^N$, and their density is bounded, assuming a growth condition on the kernel $\bar{g}$, together with a differential *inequality*. A great strength of this result lies in the fact that we do not use any differential equation to infer the regularity of the minimizers. In fact, the differential inequality is sufficient to show that the Euler-Lagrange conditions associated to the minimality cannot be satisfied, unless we have an upper bound on the density of the minimizer. Heuristically, it is natural that we cannot have a strong concentration of the minimal measures when the kernel $\bar{g}$ has a very strong singularity in the origin. However, it is not so easy to justify this intuition. Even though this is a local statement, the functional is nonlocal, hence it could be convenient to concentrate the measure in a small region just to reduce the interaction with other far-away pieces of the measure. Our results in Theorem 1.2.1 follow in part this intuition, proving a concentration bound for some minimizers of (P_M). In fact, the general hypotheses collected in ($\mathbf{H}_p$) guarantee to work with a singular kernel (at least in dimension $N \geq 2$). However, some statements are conditional, meaning that we suppose a priori to work with minimizers possessing some additional properties. The first result in our Theorem 1.2.1 is the fundamental step, valid when the kernel is sufficiently repulsive, additionally requiring that the support of a minimizer is convex. The two subsequent results are obtained via an approximation argument with singular enough kernels, and restricting to the study of radial minimizers (in dimension $N \geq 2$). In fact, in those results, the global assumption $\Delta\bar{g} \geq 0$ (valid in $\mathbb{R}^N \setminus \{0\}$) allows to deduce that the support of a radial minimizer is a ball and, in particular, convex. In our fundamental step, the convexity is crucial to show that the minimal measure has little mass on the boundary of its support, and we do this in Lemma 1.2.9. Roughly speaking, this shows that we need to work only in the interior of the support of the minimizing measure μ, where ψ_μ is constant. Then, the differential inequality for $\bar{g}$ is used only in Lemma 1.2.7 to apply a simple maximum principle.

In Section 1.3 we strengthen the results obtained in the previous section, working on the minimization problem set in the real line. In fact, we prove in Theorem 1.3.1 a continuity result for critical points of the energy when we have some stronger hypotheses on the kernel $\bar{g}$. We stress that in [CDM16] the authors use the theory of the obstacle problem to obtain even the Hölder continuity of the minimizers, while we do not have any underlying PDE at our disposal. Our statement concerns critical points that are absolutely continuous with respect to $\mathscr{L}^1$, whose density is L^∞ and has compact and convex support, and we show that the density is continuous in the interior points of the support. Of course, the a priori assumptions on the density of the critical point are justified by the analysis carried out in Section 1.2, with particular focus on the one-dimensional statements in Proposition 1.2.2 and Theorem 1.2.1. These results prove that the class of measures that we are interested in is not empty, while our continuity result stated in Theorem 1.3.1 concerns critical points because we work solely with the Euler-Lagrange equation. Given a probability density f on $\mathbb{R}$, we say that it is a critical point of $\mathcal{E}$ if

$$\psi_f(x) = \int \bar{g}(x-y)f(y)dy = \mathcal{E}(f\mathscr{L}^1) \qquad \text{for } f-\text{a.e. } x \in \mathbb{R}.$$

If the density is sufficiently regular, and its support is an interval I, then we deduce the validity of that condition in the whole support. Therefore, the first and second derivatives of ψ_f are constantly equal to 0 inside the support of f. However, since proving the regularity is our goal, we justify these steps by approximating f via convolution $f_\delta = f * \varphi_\delta$, so that ψ_{f_δ} is still constant inside I, away from ∂I. The basic idea is that, when f is not continuous, we must see either some rapid oscillations or a jump. In the first case, also the smooth densities f_δ approximating f oscillate significantly. Our strategy consists in proving that, when the oscillations are too rapid, then the second derivative of ψ_{f_δ} cannot be 0 close to the oscillation points, and thus ψ_{f_δ} cannot be constant where f_δ oscillates. To obtain this result, we carry out a very precise analysis of the contribution given by each oscillation, using the cancellation lemmas collected in Subsection 1.3.2. Finally, the jump case can be artificially manipulated to obtain an oscillatory behaviour. In fact, it is possible to perform a finite number of simple operations, that do not interfere with the constancy of the potential, and construct an oscillatory behaviour. This is sufficient to conclude because the arguments used in the first case are quantitative, and we have a good enough control over the artificial oscillations that are constructed at a small, but not infinitesimal, lengthscale.

Our interest in radial minimizers in Section 1.2, and in particular in Theorem 1.2.1, is an important reason to work on the symmetry properties, together with the uniqueness question, and this is the content of Section 1.4. The symmetry is, up to our knowledge, always obtained as a consequence of the uniqueness of minimizers: if we know a priori the uniqueness (up to translations), then by considering rotations it is immediate to deduce the symmetry of the unique minimizer. This is our approach as well. Uniqueness is, on the other hand, always deduced from the convexity of the energy. To be more precise, there are kernels such that, for every pair of measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^N)$ with $\int x d\mu(x) = \int x d\nu(x)$, and for every $t \in (0,1)$, we have

$$\mathcal{E}(t\mu + (1-t)\nu) < t\mathcal{E}(\mu) + (1-t)\mathcal{E}(\nu).$$

Of course, when we have such *strict* inequality, the uniqueness of minimizers is trivial, and thus also its symmetry. This property, rewritten in a slightly different way, is often called *positive definiteness* of the kernel, and was investigated in [Lan72, LL01, Mat15] and in the more recent papers concerning attractive-repulsive interactions [BCT18, Lop19, NP21, CP24, CS23]. For decreasing kernels, like the negative powers $\bar{h}(x) = \frac{1}{|x|^\beta}$

appearing in the prototype kernel, this feature is usually obtained from a condition on the Fourier transform of the kernel $\bar{h}$. One can see this with a formal argument: calling $\mu_{\text{diff}} = \mu - \nu$, one uses Plancherel identity and the standard relation between the Fourier transform and the convolution operation to obtain that

$$\begin{aligned} t\mathcal{E}(\mu) + (1-t)\mathcal{E}(\nu) - \mathcal{E}(t\mu + (1-t)\nu) &= t(1-t)\mathcal{E}(\mu_{\text{diff}}) \\ &= t(1-t)\iint \bar{h}(x-y)d\mu_{\text{diff}}(x)d\mu_{\text{diff}}(y) \\ &= t(1-t)\int \hat{\bar{h}}(\xi)|\hat{\mu_{\text{diff}}}(\xi)|^2 d\xi. \end{aligned}$$

Thus, the sign of the Fourier transform of $\bar{h}$ determines the convexity of $\mathcal{E}$. Instead, to treat unbounded kernels, like the positive powers $\bar{h}(x) = |x|^\alpha$, there is no general approach. The only known cases are indeed the positive powers, with $\alpha \in [2, 4]$. The extremal powers can be treated with elementary considerations (see [BCT18, Lop19]), while the values in between require the more sophisticated argument contained in [Lop19].
We remark that the convexity of the energy $\mathcal{E}$ helps a lot in the task of finding explicitly the minimizers since it is equivalent to finding a measure that satisfies the Euler-Lagrange conditions. This step is present in many of the aforementioned papers, and we use this argument in Subsection 1.4.2 to prove that a sphere uniquely minimizes $\mathcal{E}$ in some cases (our result is already contained in [FM23, Theorem 1], while our proof is easier). Despite the usefulness of convexity to find minimizers, in some cases it is possible to identify them with different techniques. In particular, we mention [DLM23] and Theorem 2.2.13 where it is proven, in different contexts, that a collection of Dirac deltas is the unique minimizer of $\mathcal{E}$ up to translations and rotations. Of course, for this to be possible, the kernel $\bar{g}$ must be locally bounded also in the origin.
To conclude the introduction to the regularity theory, we mention some negative results, that partly confirm the sharpness of our results. In fact, also in presence of a singular kernel, it is possible to have concentration phenomena. It is possible to find minimizers that are absolutely continuous with respect to $\mathscr{L}^N$, while having unbounded density, as it is shown in [CS23, Theorem 5.1] and in [FM23, Theorem 2]. When the repulsion is slightly weaker, it is also possible to find minimal measures that are singular with respect to the Lebesgue measure, as it is shown in [FM23, Theorem 1] and in Theorem 1.4.2.

L^∞-constrained problem and minimizing sets

In Chapter 2 we study the minimization problem (P_S) in the class of the subsets of $\mathbb{R}^N$ with measure equal to m. In this case, of course, the extreme concentration of mass is automatically excluded, while the existence of minimizers is non-trivial. We focus on the existence problem, and depending on the situation we are able to deduce some additional information. In order to prove the existence of a minimizer, we rely on the usual variational toolbox. In a nutshell, the problem (P_D) is the relaxation of (P_S) with respect to the weak$*$ topology, as one can see the subsets of $\mathbb{R}^N$ as L^∞ functions.

From this consideration, in [BCT18] the authors proved that a set E is a minimizer for (P_S) if and only if χ_E solves (P_D). Because of this, we change perspective, and we see the existence of the minimal sets as a saturation phenomenon in (P_D). This is convenient since the existence of minimal densities is not difficult to prove, and we show this in Lemma 2.1.3, which is insensitive to the parameter $m > 0$. Additionally, the minimizers of the measure problem depend in a trivial way on the total mass constraint. In fact, for any $m > 0$, the minimizers of (P_M) and the minimizers of $\mathcal{E}$ in the class $\{\mu \in \mathcal{M}_+(\mathbb{R}^N) : \mu(\mathbb{R}^N) = m\}$ coincide up to a rescaling by a factor m. However, the value of that parameter is relevant when we study the problem (P_S). In fact, suppose that our model contains a kernel $\bar{g}$ for which we can apply our regularity result, namely Theorem 1.2.1. Then, loosely speaking, we have a uniform upper bound M on the L^∞ norm of the probability minimizers. Therefore, taking a probability minimizer of (P_M), which is of the form $\mu = f\mathscr{L}^N$, and multiplying it by m, we produce a density with L^∞ norm *strictly* smaller than 1 whenever $m < \frac{1}{M}$. In particular, mf is not the characteristic function of a set. However, mf is necessarily a minimizer of (P_D), because $mf\mathscr{L}^1$ is a minimizer in the larger class $\{\mu \in \mathcal{M}_+(\mathbb{R}^N) : \mu(\mathbb{R}^N) = m\}$. As we already pointed out, this is a consequence of strong repulsion, because highly concentrated mass tends to diffuse when we have an upper bound on the density of minimizing measures. With this picture in mind, it is reasonable to find minimal sets when there is *attraction* along the support of a minimal measure μ, that implicitly forces μ to be singular with respect to $\mathscr{L}^N$. This information is encoded into its potential ψ_μ. In fact, assuming that $\bar{g}$ is regular enough and that ψ_μ satisfies a differential inequality in $\operatorname{spt}\mu$, we deduce the existence of minimal sets with small measure m. This is obtained in Theorem 2.2.3, which is based on the observation that, if f_m is a minimizing density of (P_D) with mass m, then $m^{-1}f_m \overset{*}{\rightharpoonup} \mu$, where $\mu \in \mathcal{P}(\mathbb{R}^N)$ is a minimizer of (P_M). Since ψ_μ satisfies a differential inequality by hypothesis, we infer that the same holds for ψ_{f_m} with m small, and from this we deduce that $f_m = \chi_{E_m}$ for some set E_m. We devote Subsection 2.2.2 to the treatment of some special kernels, which have been investigated in [DLM22, DLM23]. They are of the form $\bar{g}(x) = |x|^\alpha - |x|^\beta$ with $\alpha > 2$ and $\beta \in [2, \alpha)$, which are of class $C^2(\mathbb{R}^N)$, and in particular they are very flat in the origin. This indicates a *weak* repulsion at short distance, and in principle this favours the concentration of mass. Instead, in Subsection 2.2.3 we find a much more generic class of kernels for which we get the existence of minimizing sets in (P_S) with small measure constraint. In both Subsection 2.2.2 and Subsection 2.2.3, it is possible to show the validity of the differential inequality for the potential ψ_μ when μ is a minimizer of (P_M). This is feasible because the minimizing probability measures are known, and they have a very simple structure: depending on the kernel, they are either a collection of Dirac deltas, or they coincide with the $(N-1)$-Hausdorff measure on a specific sphere. The general philosophy is: the presence of a singular minimizer among probabilities is a good hint of the existence of set minimizers with small measure.

In Section 2.3 we study the problem (P_S) with large measure constraint. Analogously to the previous discussion, we relax the problem in the space of densities with a given L^1 constraint, studying (P_D), where the existence of a minimal density f_m is guaranteed for every $m > 0$. The approach is very similar to the one used in [FL21]:

- prove that $f_m \to B$ in asymmetry sense when $m \to +\infty$ (basically, a scaling invariant L^1 distance) and that diam(spt f_m) is comparable with the natural scaling $m^{1/N}$;

- analyze the potential of a large ball $m^{1/N}B$, obtaining precise bounds on the local Lipschitz constant close to the boundary of the ball;
- pass from the convergence in asymmetry to the Hausdorff convergence of spt f_m to a ball (after a proper rescaling);
- conclude via a pair of quantitative inequalities applied to a minimizer f_m.

The first and second points are contained in Subsection 2.3.1 and Subsection 2.3.2 respectively. The third step is the most complex, and it is contained in Proposition 2.3.11. There, it is necessary to show that we can modify a density f_m in order to reduce the Hausdorff distance between spt f_m and a ball, and at the same time control the energy gained in this process. Finally, the last step is basically the contents of Theorem 2.3.12 which concludes Subsection 2.3.3. In the end, the fundamental quantitative inequality in this step concerns the Riesz term containing kernel $|x|^\alpha$ with $\alpha > 0$. This inequality is available because that kernel is "not flat at infinity" (the precise condition is (2.35)), and its proof is based on a quantitative rearrangement inequality due to Christ [Chr17] (we refer also to [FL21] and the discussion therein).

We conclude remarking that we treat the two extremal cases, namely we suppose that either m is very small or very large. In general, it is difficult to predict what happens for intermediate values of that parameter, even for those kernels $\bar{g}$ for which it is known that minimal sets exist both when m is small and when m is large. However, there are situations where the existence of minimal sets holds for *every* $m > 0$, and this is treated in Theorem 2.2.10. The basis of the proof consists in showing a global differential inequality for the potential of a minimal measure ψ_{f_m}. Once more, this is possible thanks to the rather simple structure of the minimizers, that are necessarily radially symmetric. In certain circumstances, it is particularly easy to infer some differential inequalities for the potential directly from the properties of the kernel, as we highlight in Remark 2.2.11.

Generalized Gamow model

There are a number of results related to this model in the literature, and we mention only the works that are more closely related to our study. To begin with, there are Knüpfer and Muratov's companion papers [KM13, KM14], where they prove some existence/non-existence results, together with a characterization of the minimizers with small measure in low space dimension, when the kernel is a suitable negative power. Instead, by choosing the power $2 - N$ (that, in particular, is the Coulombic one), Julin is able to prove in [Jul14] that balls minimize the Gamow functional for small measure constraint in every space dimension. Bonacini and Cristoferi extend the previous results to every dimension, and every negative power larger than $1 - N$, in their paper [BC14], where they also investigate the local minimality of the ball. Finally, the picture is completed in [FFM+15], at least for power kernels, since the authors characterize the minimizers with small measure in every space dimension, for every power greater than $-N$, and even treating the case of the fractional perimeter instead of the standard one. Remaining in the realm of negative power kernels, in dimension 2 it is possible to completely solve the problem under a restriction on the exponent (see [MZ15]). Moreover, there is a very recent work [CR24], where they give an explicit estimate of the values of γ for which the ball uniquely minimizes $\mathcal{G}_\gamma$. Novaga and Pratelli go beyond the power-like kernels in [NP21], characterizing the minimizers with small measure in dimension 2. We do not restrict the space dimension, and in Chapter 3 we treat a generalization of the Gamow model, making very mild assumptions on the interaction kernel $\bar{h}$ appearing in the Riesz energy. In any case, we recall that the Gamow model naturally comes with a kernel $\bar{h}$ that is repulsive and infinitesimal at infinity, and we keep this framework. In Section 3.2 we show that, when γ is small enough, balls are the unique minimizers of $\mathcal{G}_\gamma$ among sets with measure ω_N. Our approach is similar to the one exploited in [FFM+15], while we manage to prove some important estimates for a very general Riesz term. These estimates are so robust that we are able to treat also the model containing the fractional perimeter. In fact, in our presentation we denote either the standard perimeter or the fractional one by $\mathscr{P}$, since much of the discussion is not affected by that choice. We stress that, when the interaction kernel $\bar{h}$ is a negative power, the asymptotic analysis in the regime $\gamma \to 0$ is perfectly equivalent to the study of the minimal sets for $\mathcal{G}_1$ with measure constraint going to 0. Hence, our results generalize those already present in the literature. The strategy to characterize the minimizers for γ small is well understood by now, and can be summarized in the following steps:

1. prove that minimizers of $\mathcal{G}_\gamma$ are *almost minimizers of the perimeter*, which provides some regularity estimates;

2. prove that the minimizers converge to a ball in the L^1 sense as $\gamma \to 0$, which improves their regularity for small γ;

3. having at our disposal some quantitative inequalities for the perimeter and for the Riesz term of the form

$$P(E) - P(B) \geq C_1 d(E,B)^2, \qquad \mathcal{E}(B) - \mathcal{E}(E) \leq C_2 d(E,B)^2,$$

   we characterize the minimizers for small γ by combining these two inequalities and the minimality of E: we obtain that $C_1 d(E,B)^2 \leq \gamma C_2 d(E,B)^2$, which forces $E = B$ for $\gamma < C_1/C_2$.

The research on quantitative inequalities is still very active and, depending on the functional, obtaining such results can be far from trivial. Just to serve as examples, we mention some works on quantitative inequalities that are related to the topics of this thesis: the works focused on problems "of perimeter type" [CL12, BC14], and those addressing the Riesz energy alone [Chr17, FP20, FL21]. Moreover, we did not specify what the "distance" $d(E,B)$ is, and for which sets those inequalities hold. Our work in Section 3.2 is mainly focused on the quantitative inequality for $\mathcal{E}$ for regular sets, that are close enough to a ball in a suitable sense, and with a quite strong distance. The most

classical isoperimetric inequality is given in terms of the so-called asymmetry of E, that is a sort of L^1 distance from B. This, however, is too weak for our purposes because we are able to obtain the quantitative estimate for $\mathcal{E}$ only in terms of a distance "of higher order", and therefore we are forced to rely on the Fuglede inequality for the perimeter (see [Fug89, CL12], and the fractional counterpart present in [FFM+15]). In the end, our result is obtained by hand, with a careful analysis of the various cancellations, and we use a number of times that the kernel $\bar{h}$ is radial.
Of course, in the strategy outlined above we took for granted an important point: the existence of minimizers. There are some very powerful techniques to prove that result when γ is small, while the problem is much more subtle when the parameter γ is large. In fact, there are only a few results showing the *non-existence* of minimizers when $\gamma \gg 1$, and they are contained in [KM13, KM14, FKN16, FN21]. Those results have some restrictions on the kernel, that has to be of power-type, and the range of powers to which the arguments apply does not cover the full range of the locally integrable power kernels. To further stress the difficulty of this question, we mention two facts. First, when $\bar{h}$ is a locally integrable negative power, then a scaling argument shows immediately that a single ball cannot be a minimizer when γ is large because two balls with half of the measure do better. Second, there are some (non-homogeneous) kernels for which the existence for large γ is known, and sometimes it is possible to prove that the minimizer is precisely a ball, even for large γ (the references for these phenomena are [Peg21, MP22, NO22, GMP24]). In all of these cases, the authors crucially require that the interaction is extremely weak at large distances.

In Section 3.3 we study the energy profile of the families of balls, depending on their measure, and we give a precise characterization of the optimal subdivision of the total measure among different balls to minimize $\mathcal{G}_\gamma$. In general, this analysis provides some control on the isoperimetric profile of the Gamow functional $\mathcal{G}_\gamma$. However, in some special cases, we identify precisely the isoperimetric profile, thanks to [KM13, Theorem 2.7] and [BC14, Theorem 2.11]. Our result is purely one-dimensional and relies solely on a concavity-convexity property of the energy profile $\mathscr{G}(m) = \mathcal{G}_\gamma(B(0, m^{1/N}))$. This hypothesis, which is contained in ($\mathbf{H}_{1D}$), is satisfied when the kernel $\bar{h}$ is a negative power, while it does not hold if we consider the generalized Gamow energies presented in [MP22, GMP24]. In fact, in their case, a ball with large measure minimizes $\mathcal{G}_\gamma$, while our assumptions guarantee that splitting the measure into two masses is more convenient in that regime (mimicking the behaviour of the standard Gamow model). This splitting phenomenon naturally repeats, as it is convenient to have families with three, four, and more masses as the total measure increases. This was already observed in [BC14, Theorem 2.12], and a proof of this fact is contained in Corollary 3.3.10. In turn, we prove that the optimal families of balls are very simple, as they contain a certain number of equal masses, and at most one smaller mass. This simple structure allows us to prove a form of monotonicity for them, as stated in Theorem 3.3.15. We crucially use the concavity-convexity property of the profile $\mathscr{G}$ to deduce this result. In turn, this characterizes the regions where the isoperimetric profile for the Gamow functional is convex (for certain kernels). A remarkable feature is that, as the total measure grows, the optimal splitting of the mass eventually becomes trivial, containing only equal masses. This is the most relevant content of Proposition 3.3.18, which, like the rest of our analysis, is based on the optimality conditions obtained in Lemma 3.3.5. A much stronger result that compares to this was obtained in [FL15, Proposition 4.3], where the authors work with the classical Gamow's model, and they prove that the optimal splitting of the mass contains only equal masses for every $m > 0$. We conclude the section with a few examples that serve to show the sharpness of our results. In fact, the hypotheses ($\mathbf{H}_{1D}$) are very mild, and the examples highlight the limits of our approach. To obtain stronger results, one would need to assume some additional properties for the one-dimensional profile.

Notation

- $B(x, r) \subset \mathbb{R}^N$ denotes the euclidean ball, centered at x with radius r, moreover $B_r = B(0, r)$, and $B = B(0, 1)$
- $\mathcal{M}(\mathbb{R}^N)$ denotes the space of signed Borel measures in $\mathbb{R}^N$, and $\mathcal{M}_+(\mathbb{R}^N)$ is the subspace of the non-negative Borel measures in $\mathbb{R}^N$
- $\mathcal{P}(\mathbb{R}^N)$ denotes the space of probability measures in $\mathbb{R}^N$, and $\mathcal{P}_{\text{rad}}(\mathbb{R}^N)$ is the subspace of the probability measures that are invariant under rotations: $\mu \in \mathcal{P}_{\text{rad}}(\mathbb{R}^N)$ if for any $M \in O(N)$ and any Borel set $E \subset \mathbb{R}^N$, we have that $\mu(E) = \mu(M(E))$
- $\mathcal{P}_c(\mathbb{R}^N)$ is the space of probability measures with compact support, and we define the two subspaces $\mathcal{P}_{\bar{g},c}(\mathbb{R}^N) = \{\mu \in \mathcal{P}_c(\mathbb{R}^N) : \mathcal{E}(\mu) < +\infty\}$ and
$$\dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N) := \left\{\mu \in \mathcal{P}_{\bar{g},c}(\mathbb{R}^N) : \int x \, d\mu(x) = 0\right\}$$
- given $\mu \in \mathcal{M}(\mathbb{R}^N)$, we denote by $\|\mu\|$ the total variation of the measure, and $\|\mu\|_{\mathcal{M}} = \|\mu\| (\mathbb{R}^N)$
- $\mathcal{P}(\mathbb{R}^N)$ and its various subspaces are always endowed with the weak$*$ topology
- $\mathfrak{g}_\gamma$ denotes the power-law kernel $\mathfrak{g}_\gamma(x) = \frac{|x|^\gamma}{\gamma}$ with $\gamma \in \mathbb{R} \setminus \{0\}$, and $\mathfrak{g}_0(x) = \log |x|$ by definition; the power-law functions are the building blocks of a prototypical non-trivial kernel that often appears in the literature:
$$\bar{g}_p = \mathfrak{g}_\alpha - \mathfrak{g}_\beta + \mathfrak{g}_\beta(e_1) - \mathfrak{g}_\alpha(e_1) \qquad \alpha > \max\{\beta, 0\} \geq \min\{\beta, 0\} > -N \tag{1}$$
- $\mathcal{E}_{\bar{g}}$ simply denotes the energy $\mathcal{E}$ with an explicit reference to the interaction kernel. Instead, when $\mu, \nu \in \mathcal{M}_+(\mathbb{R}^N)$, we define the mutual interaction between them as
$$\mathcal{E}_{\bar{g}}(\mu, \nu) := \iint \bar{g}(x - y) d\mu(x) d\nu(y)$$
- $\mathsf{P}(E)$ denotes the standard perimeter of the set E, also called De Giorgi's perimeter, while for any $s \in (0, 1)$ we use the notation $\mathsf{P}_s(E)$ for the fractional perimeter of E

Chapter 1

Probability minimizers for Riesz-like energies

In this chapter we analyze the ground states of the Riesz-type functional $\mathcal{E}$, when it is defined in the space of probability measures $\mathcal{P}(\mathbb{R}^N)$. We recall that its expression is

$$\mathcal{E}(\mu) := \iint \bar{g}(x-y)d\mu(x)d\mu(y) \qquad \mu \in \mathcal{P}(\mathbb{R}^N),$$

where $\bar{g}$ is the kernel describing the interaction, and the minimization problem is stated in (P_M). Some important examples of kernels contain negative powers, and we will cover some of them. In this sense, one should think that the kernel is not everywhere regular, while we always assume that $\bar{g}$ is non-negative. We also stress that, in the setting of probability measures, we have to define the kernel in a pointwise sense. In particular, to define the energy $\mathcal{E}$ for every measure in a unique sense, $\bar{g}$ is not a class of Lebesgue functions, but we need to fix a precise representative. This setting has some good features: the existence is very easy, and there is a lot of flexibility concerning the manipulation of the competitors in the minimization process. A downside, however, is the possible singularity of the measures. In fact, this yields some issues when we do some computations, and certain steps are not easily justified when there is a combination of singular measure and singular kernel. In other words, we need some sort of *regularity theory*, and this chapter focuses precisely on this task. We stress that this possibility really occurs: it can happen that we find singular minimizers even when the kernel explodes in the origin, as we show in Theorem 1.4.2. We point out that, in this specific example, we are able to perform explicit computations because the kernel, despite being singular, is regular enough to justify every step. This lack of regularity is in contrast with the intuitive observation that a singular behaviour of the kernel $\bar{g}$ in the origin prevents the concentration of mass. From this rough picture, one can guess that, in order to obtain some regularity result, we should work with a kernel that is singular enough. This is the goal of Section 1.2, where we prove a L^∞ control on the minimizers of (P_M) when the kernel satisfies condition ($\mathbf{H}_p$). We stress that our approach requires that $\bar{g}$ be subharmonic close to the origin, that is a quite precise information about the singularity. It is not known if we can expect some similar results with a less precise understanding of $\bar{g}$. The 1-dimensional case is particularly simple, and allows us to obtain stronger results already in Theorem 1.2.1. Furthermore, the topic of Section 1.3 is precisely the investigation of finer regularity properties held by the minimizers of (P_M) in dimension 1. The study yields to Theorem 1.3.1 where we show that, under suitable hypotheses, the critical points of $\mathcal{E}$ are not only bounded densities, but even continuous ones (inside their support). To conclude this chapter, we collect in Section 1.4 some results concerning the question of uniqueness, that is related to their symmetry when the kernel is radial. In fact, Theorem 1.4.1 addresses precisely this matter, while Theorem 1.4.2 exploits the symmetry of the minimizers to perform some explicit computations, and characterize the sphere as the unique minimizer of (P_M) when the kernel has the expression (1), with some restrictions on the exponents. The study of symmetry properties, despite being interesting in itself, is tightly connected to the concentration bounds, as it is clear from Theorem 1.2.1. Concerning our contributions, Section 1.2 is based on [CP24], and Section 1.3 is based on [CP25]. Finally, Section 1.4 mainly contains the results of [CP24, Section 4], while the discussion about the minimality of the sphere is an unpublished contribution.

1.1 Preliminary results

We introduce some basic tools and results to study the variational problems associated to Riesz-like functionals. We establish the existence of optimal measures in Theorem 1.1.1 when the interaction kernel penalizes long range interactions. We also provide the Euler-Lagrange conditions satisfied by minimizers in Proposition 1.1.3, written in terms of the potential of the optimal measure. This is associated with the computation of the first variation of the energy, which indeed coincides with the potential generated by the minimizing measure.

Theorem 1.1.1 (Existence of optimal measures)**.** *Let $\bar{g} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a l.s.c. and L^1_{loc} function such that $\lim_{|x|\to+\infty} \bar{g}(x) = +\infty$. Then, there exists a minimizer of the energy $\mathcal{E}$ both in the class $\mathcal{P}(\mathbb{R}^N)$ and in the class $\mathcal{P}_{\mathrm{rad}}(\mathbb{R}^N)$. Moreover, the support of any optimal measure is contained in a ball of radius R, where R only depends on $\bar{g}$. More precisely, R needs only to be big enough so that, for every $|v| > R/4$, the quantity $\bar{g}(v)$ is larger than 24 times the energy of a ball of unit volume.*

Proof. We can assume without loss of generality that $\bar{g}$ is symmetric, since the energy does not change if we replace it by $v \mapsto \big(\bar{g}(v) + \bar{g}(-v)\big)/2$. Since $\bar{g} \in L^1_{\mathrm{loc}}$ and $\mathcal{P}_{\mathrm{rad}}(\mathbb{R}^N) \subset \mathcal{P}(\mathbb{R}^N)$, then we have $I \leq I' < +\infty$, having set

$$I := \inf\big\{\mathcal{E}(\mu) : \mu \in \mathcal{P}(\mathbb{R}^N)\big\}, \qquad I' := \inf\big\{\mathcal{E}(\mu) : \mu \in \mathcal{P}_{\mathrm{rad}}(\mathbb{R}^N)\big\}.$$

Let us call for brevity $C = 24I'$, and let $R > 0$ be such that $\bar{g}(v) > C$ for every $v \in \mathbb{R}^N$, $|v| > R/4$. Let us now take a measure μ, either in $\mathcal{P}(\mathbb{R}^N)$ or in $\mathcal{P}_{\mathrm{rad}}(\mathbb{R}^N)$, such that $\mathcal{E}(\mu) < 2I'$. We claim that there exists some $\bar{x} \in \mathbb{R}^N$ such that $\mu\big(B(\bar{x}, R/4)\big) > 1/2$. Indeed, otherwise we have

$$\begin{aligned}\mathcal{E}(\mu) &= \iint_{\mathbb{R}^N\times\mathbb{R}^N} \bar{g}(x-y)d\mu(y)d\mu(x)\\ &\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N\setminus B(x,R/4)} \bar{g}(x-y)d\mu(y)d\mu(x) > \frac{C}{2} > \mathcal{E}(\mu),\end{aligned}$$

which is absurd. Then, the existence of $\bar{x} \in \mathbb{R}^N$ so that $\mu\big(B(\bar{x}, R/4)\big) > 1/2$ follows. We can reduce ourselves to assume that

$$\mu\big(B_{R/2}\big) > \frac{1}{2}.$$

Indeed, in the general case when $\mu \in \mathcal{P}(\mathbb{R}^N)$ it is harmless to assume that $\bar{x} \equiv 0$, up to a translation, so there is even no need of passing from $R/4$ to $R/2$. Instead, in the radial case –where a translation is not possible– the above estimate is clearly true if $|\bar{x}| \leq R/4$. And in turn, we can exclude that $|\bar{x}| > R/4$, because if this happens then the balls $B(\bar{x}, R/4)$ and $B(-\bar{x}, R/4)$ are disjoint, and since μ is radial we obtain $\mu(\mathbb{R}^N) \geq \mu(B(\bar{x}, R/4)) + \mu(B(-\bar{x}, R/4)) = 2\mu(B(\bar{x}, R/4)) > 1$, which is absurd.

Let us now call $\eta = \mu\big(\mathbb{R}^N \setminus B_R\big) \in [0, 1/2]$, and let μ^- be the restriction of μ to B_R, that is a measure with mass $1 - \eta$. Then, we have

$$\begin{aligned}\mathcal{E}(\mu) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \bar{g}(x-y)d\mu(y)d\mu(x) &\geq \mathcal{E}(\mu^-) + 2\int_{B_{R/2}} \int_{\mathbb{R}^N \setminus B_R} \bar{g}(x-y)d\mu(y)d\mu(x) \\ &\geq \mathcal{E}(\mu^-) + 2C\mu\big(B_{R/2}\big)\mu\big(\mathbb{R}^N \setminus B_R\big) \geq \mathcal{E}(\mu^-) + C\eta.\end{aligned}$$

Keeping in mind that $\mathcal{E}$ is 2-homogeneous, that $C = 24I' > 12\mathcal{E}(\mu)$, and that $(1-\eta)^{-2} \leq 1 + 6\eta$ since $0 \leq \eta \leq 1/2$, we can estimate

$$\begin{aligned}\mathcal{E}\big((1-\eta)^{-1}\mu^-\big) = (1-\eta)^{-2}\mathcal{E}(\mu^-) &\leq (1+6\eta)\mathcal{E}(\mu^-) \\ &\leq (1+6\eta)\big(\mathcal{E}(\mu) - C\eta\big) \leq \mathcal{E}(\mu) - \frac{C}{2}\eta.\end{aligned}$$

Therefore, the measure $(1-\eta)^{-1}\mu^-$, which is a probability measure supported in B_R, and which is radial if so is μ, has energy lower than μ, and actually strictly lower unless μ itself is supported in B_R.

Summarizing, from any minimizing sequence for the energy (either in $\mathcal{P}(\mathbb{R}^N)$ or in $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$) we can construct another minimizing sequence, which is done by measures supported in the ball B_R. By lower semicontinuity of the energy, any weak limit of this latter minimizing sequence is a minimizer (observe that a weak limit of radial measures is clearly still radial). This gives the required existence of minimizer of the energy both in $\mathcal{P}(\mathbb{R}^N)$ and in $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$. Moreover, by the above calculation we obtain that *every* minimizer is supported in a ball of radius R. □

Once we have established the existence of minimizers, it is useful to provide the so-called Euler-Lagrange conditions associated to this problem. In our case, they are written in terms of the potential of the minimizing measure (the definition is standard, and it is provided below). Notice that, in order to define the potential, we assume the function $\bar{g}$ to be symmetric (but not necessarily radial). However, as already noticed in the proof of Theorem 1.1.1, this assumption can always be done without loss of generality, since the problem with the function $\bar{g}$ is completely equivalent to the problem with the function $v \mapsto \big(\bar{g}(v) + \bar{g}(-v)\big)/2$.

Definition 1.1.2 (Potential)**.** Given a l.s.c., symmetric and $L^1_{\rm loc}$ function $\bar{g} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$, for any positive measure μ we call *potential associated to* μ the function

$\psi_{\mu,\bar{g}} : \mathbb{R}^N \to \mathbb{R}^+$ defined as

$$\psi_{\mu,\bar{g}}(x) = \int_{\mathbb{R}^N} \bar{g}(x-y)d\mu(y).$$

When there is no ambiguity, we simply write ψ_μ, discarding the dependence on the kernel. Similarly, for any function $f \in L^1(\mathbb{R}^N)$, either positive or bounded and compactly supported, the *potential associated to* f is the function $\psi_{f,\bar{g}} : \mathbb{R}^N \to \mathbb{R}$ given by

$$\psi_{f,\bar{g}}(x) = \int_{\mathbb{R}^N} \bar{g}(x-y)f(y)dy.$$

As before, we use the notation ψ_f when there is no confusion about the kernel. Furthermore, if $f_E = \chi_E$, we use the shorthand notation $\psi_{E,\bar{g}}$, ψ_E, $\mathcal{E}_{\bar{g}}(E)$ and $\mathcal{E}(E)$ instead of the couterpart with f_E. It is immediate to see that, for every kernel $\bar{g} \geq 0$, any measure $\mu \in \mathcal{M}_+(\mathbb{R}^N)$ and every $\lambda > 0$, we have the following scaling properties:

$$\psi_{\lambda\mu,\bar{g}} = \lambda\psi_{\mu,\bar{g}}, \qquad \mathcal{E}_{\bar{g}}(\lambda\mu) = \lambda^2\mathcal{E}_{\bar{g}}(\mu).$$

Since $\bar{g}$ is non-negative, we apply Fubini's Theorem to the definition of $\mathcal{E}_{\bar{g}}(\mu)$, for any measure $\mu \in \mathcal{M}_+(\mathbb{R}^N)$, and we obtain the following remarkable property of the potential:

$$\mathcal{E}_{\bar{g}}(\mu) = \int_{\mathbb{R}^N} \psi_{\mu,\bar{g}}(x)d\mu(x). \tag{1.1}$$

As a consequence, it is easy to guess that, whenever μ is an optimal measure, its potential attains its minimum over the support of μ. A similar result has been already proved under different assumptions in many earlier papers, for instance [BCT18, CDM16]. We give a proof of this fact, which is instructive to work with the first variation of the energy $\mathcal{E}$.

Proposition 1.1.3. *Let $\bar{g} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a l.s.c., symmetric and L^1_{loc} function. Let μ be a minimizer of the energy, either in $\mathcal{P}(\mathbb{R}^N)$ or in $\mathcal{P}_{\mathrm{rad}}(\mathbb{R}^N)$ (in this latter case we also assume $\bar{g}$ to be radial). Then we have*

$$\begin{cases} \psi_\mu(x) = \mathcal{E}(\mu) & \text{for } \mu\text{-a.e. } x \in \operatorname{spt}\mu, \\ \psi_\mu(x) \geq \mathcal{E}(\mu) & \text{for } \mathscr{L}^N\text{-a.e. } x \in \mathbb{R}^N. \end{cases} \tag{EL_p}$$

Proof. We start by showing that ψ_μ is constant μ-a.e. in the support of μ. The fact that this constant is exactly $\mathcal{E}(\mu)$ will then be an obvious consequence of (1.1). If the claim is false, then there are two constants $\lambda_1 < \lambda_2$ and two positive measures $\mu', \mu'' \leq \mu$ with $\|\mu'\|_{\mathcal{M}} = \|\mu''\|_{\mathcal{M}} > 0$, radial if so are μ and $\bar{g}$, and such that

$$\psi_\mu(x) \leq \lambda_1 \quad \text{for } \mu\text{-a.e. } x \in \operatorname{spt}\mu', \qquad \psi_\mu(x) \geq \lambda_2 \quad \text{for } \mu\text{-a.e. } x \in \operatorname{spt}\mu''.$$

For any $0 < \varepsilon < 1$, the measure $\mu_\varepsilon = \mu + \varepsilon(\mu' - \mu'')$ is still a probability measure, and it is radial if so is μ. An easy calculation gives us that

$$\begin{aligned} \mathcal{E}(\mu_\varepsilon) - \mathcal{E}(\mu) &= 2\varepsilon \iint_{\mathbb{R}^N} \psi_\mu(x)d(\mu' - \mu'')(x) + \varepsilon^2\mathcal{E}(\mu' - \mu'') \\ &\leq 2\varepsilon \|\mu'\|_{\mathcal{M}} (\lambda_1 - \lambda_2) + \varepsilon^2\mathcal{E}(\mu' - \mu''), \end{aligned}$$

and then we derive that $\mathcal{E}(\mu_\varepsilon) < \mathcal{E}(\mu)$ for $\varepsilon \ll 1$, contradicting the minimality of μ. The first property in (EL_p) is then established.

Concerning the second one, let us assume that it is false. Then, there exists some $\lambda < \mathcal{E}(\mu)$ and some bounded Borel set $E \subseteq \mathbb{R}^N$, with strictly positive Lebesgue measure, such that

$$\psi_\mu(x) \leq \lambda \quad \text{for } \mathscr{L}^N\text{-a.e. } x \in E.$$

The set E can be taken radially symmetric if μ and $\bar{g}$ are radial. Notice that, by the first property in (EL_p), $\mu(E) = 0$. This time, for $0 < \varepsilon < |E|^{-1}$ we set

$$\mu_\varepsilon = (1 - \varepsilon|E|)\mu + \varepsilon \mathscr{L}^N \llcorner E = \mu + \varepsilon\big(\mathscr{L}^N \llcorner E - |E|\mu\big),$$

which is again a positive probability measure, radial if so are μ and $\bar{g}$ (and then E). We have

$$\begin{aligned}\mathcal{E}(\mu_\varepsilon) - \mathcal{E}(\mu) &= 2\varepsilon \int_{\mathbb{R}^N} \psi_\mu(x) d\big(\mathscr{L}^N \llcorner E(x) - |E|\mu\big)(x) + \varepsilon^2 \mathcal{E}\big(\mathscr{L}^N \llcorner E - |E|\mu\big) \\ &\leq 2\varepsilon(\lambda - \mathcal{E}(\mu))|E| + \varepsilon^2 \mathcal{E}\big(\mathscr{L}^N \llcorner E - |E|\mu\big).\end{aligned}$$

Since E is bounded and $\bar{g} \in L^1_{\text{loc}}$, we notice that

$$\begin{aligned}\big|\mathcal{E}(\mathscr{L}^N \llcorner E - |E|\mu)\big| &\leq \mathcal{E}(\mathscr{L}^N \llcorner E) + |E|^2 \mathcal{E}(\mu) + 2|E| \int_E \psi_\mu(x) dx \\ &\leq \mathcal{E}(\mathscr{L}^N \llcorner E) + |E|^2 \mathcal{E}(\mu) + 2\lambda |E|^2 < +\infty.\end{aligned}$$

By the fact that $\lambda < \mathcal{E}(\mu)$, we deduce that $\mathcal{E}(\mu_\varepsilon) < \mathcal{E}(\mu)$ for $\varepsilon \ll 1$, contradicting the minimality of μ. Also the second property in (EL_p) is then obtained. □

Remark 1.1.4. Keeping in mind that ψ_μ is l.s.c. on $\mathbb{R}^N$, we actually deduce that $\psi_\mu \leq \mathcal{E}(\mu)$ for every $x \in \operatorname{spt}\mu$. If $\bar{g}$ is also continuous in $\mathbb{R}^N \setminus \{0\}$, then ψ_μ is continuous on $\mathbb{R}^N \setminus \operatorname{spt}\mu$, and from ($EL_p$) we infer that $\psi_\mu(x) \geq \mathcal{E}(\mu)$ for every $x \notin \operatorname{spt}\mu$.
We expressed the optimality conditions in terms of the two measures that are naturally taken as reference: μ and $\mathscr{L}^N$. This makes the result more readable in some sense, but it is not the strongest statement that one can write. In fact, let us define the sublevel set $S_\mu = \{x \in \mathbb{R}^N : \psi_\mu(x) < \mathcal{E}(\mu)\}$, where μ is a minimizer of $\mathcal{E}$. Then, a more refined version of the Euler-Lagrange condition is the following (valid without the continuity assumption on $\bar{g}$): $\psi_\mu \leq \mathcal{E}(\mu)$ in $\operatorname{spt}\mu$, and $\mathcal{E}(\nu) = +\infty$ for every $\nu \in \mathcal{P}(\mathbb{R}^N)$ with $\nu(\mathbb{R}^N \setminus S_\mu) = 0$. To verify this condition one can check that, if we find a measure ν supported on S_μ and with finite energy, then

$$\mathcal{E}((1-t)\mu + t\nu) = (1-t)^2 \mathcal{E}(\mu) + t^2 \mathcal{E}(\nu) + 2t \int \psi_\mu d\nu < \mathcal{E}(\mu) \qquad \forall t \ll 1,$$

contradicting the minimality of μ. We refer to [Lan72] for the developments of the potential theory, where this arguments fits perfectly in the study of the so-called capacity (in our case, the kernel is not necessarily a negative power, differently from the theory developed in Landkof's book).

Remark 1.1.5. Notice that the lower-semicontinuity and the local integrability assumptions on $\bar{g}$ in Theorem 1.1.1 are very reasonable assumptions in this context. In fact, the first is related to the standard method, and is very common. The second one is natural whenever we expect to use characteristic functions of sets as competitors or, looking at the problem from the applied point of view, whenever there are meaningful objects that we want to include in our model which are naturally represented by characteristic functions of sets. In fact, if we suppose that $\bar{g} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ is *radial and radially decreasing* close to the origin, then we have the following implications:

$$\begin{aligned} \bar{g} \in L^1_{\text{loc}} \quad &\Rightarrow \quad \mathcal{E}(E) < +\infty \quad \forall E \subset \mathbb{R}^N \text{ bounded with } |E| > 0, \\ \bar{g} \notin L^1_{\text{loc}} \quad &\Rightarrow \quad \mathcal{E}(E) = +\infty \quad \forall E \subset \mathbb{R}^N \text{ bounded with } |E| > 0. \end{aligned}$$

If $\bar{g}$ is locally integrable, then the energy is finite since the potential $\psi_{\bar{g},\mu}$ is uniformly bounded in E, and the energy is simply the integral of the potential. For the other implication, instead, let us suppose by contradiction that $\bar{g} \notin L^1_{\text{loc}}$. We call g the radial profile of $\bar{g}$, i.e. $\bar{g}(x) = g(|x|)$. We aim to prove that $\psi_E = +\infty$ in any Lebesgue point of E. This, of course, concludes the argument. Without loss of generality, let us suppose that 0 is a point of density 1 for E, and let us take $r_k = 2^{-k}$. For any $k \in \mathbb{N}$ there exists $\varepsilon_k \in (0,1)$ such that $|E \cap B_{r_k}| \geq (1-\varepsilon_k)|B_{r_k}|$, and $\varepsilon_k \to 0$. We observe the following trivial inequality:

$$|E \cap (B_{r_k} \setminus B_{r_{k+1}})| \geq (1-\varepsilon_k)|B_{r_k}| - |B_{r_{k+1}}| = (1-\varepsilon_k - 2^{-N})|B_{r_k}|.$$

Since $\varepsilon_k \to 0$, then there exists $k_0 \in \mathbb{N}$ and $C > 0$ such that $1-\varepsilon_k - 2^{-N} > C$ for every $k \geq k_0$, and $\bar{g}$ is radially decreasing in $B_{r_{k_0}}$. Hence, we exploit the symmetry of $\bar{g}$ to see that

$$\begin{aligned} \psi_E(0) = \int_E \bar{g}(x)dx &\geq \sum_{k\geq k_0} \int_{E\cap(B_{r_k}\setminus B_{r_{k+1}})} \bar{g}(x)dx \\ &\geq \sum_{k\geq k_0} (1-\varepsilon_k - 2^{-N})|B_{r_k}|g(r_k) \geq \sum_{k\geq k_0} C|B_{r_k}|g(r_k) \\ &= \sum_{k\geq k_0} C2^{-N}|B_{r_{k-1}}|g(r_k) = \sum_{k\geq k_0} C\frac{2^{-N}}{1-2^{-N}}|B_{r_{k-1}} \setminus B_{r_k}|g(r_k) \\ &\geq \sum_{k\geq k_0+1} C\frac{2^{-N}}{1-2^{-N}} \int_{B_{r_{k-1}}\setminus B_{r_k}} \bar{g}(x)dx = C\frac{2^{-N}}{1-2^{-N}} \int_{B_{2^{-k_0}}} \bar{g}(x)dx, \end{aligned}$$

and the last expression is $+\infty$ by hypothesis.

If the kernel $\bar{g}$ is radial, locally integrable and decreasing, we infer also an integrability property for $\nabla\bar{g}$:

Lemma 1.1.6. *Let $\bar{g} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a radial function of class L^1_{loc}, with radial profile g. If $g \in C^1((0,+\infty))$ and $g' \leq 0$ in $(0,1)$, then*

$$-\int_0^1 g'(t)t^N < +\infty.$$

Proof. For any $s \in (0,1)$ we write $g(1) - g(s) = \int_s^1 g'(t)dt$. Then, since g' has constant sign in $(0,1)$, we can apply Fubini's theorem and see that

$$\int_0^1 (g(1) - g(s))s^{N-1}dt = \int_0^1 s^{N-1} \int_s^1 g'(t)dtds$$
$$= \int_0^1 dt \int_0^t g'(t)s^{N-1}ds = \int_0^1 \frac{t^N}{N} g'(t)dt.$$

This concludes the proof since $\bar{g} \in L^1(B_1)$, and thus $s \mapsto g(s)s^{N-1}$ is integrable in the interval $(0,1)$. □

1.2 Boundedness of optimal measures

In general, it is extremely hard to describe the minimizers of $\mathcal{E}$, even when the kernel $\bar{g}$ is one of the protorypes expressed in (1). We recall that, in those cases, some results are available to characterize minimizers: see for example [CS23, Section 5], [Fra22, FM23] and [DLM22, DLM23]. Therefore, some qualitative analysis is the most reasonable objective in the theoretical study of this problem. In this direction, our results are in some sense close to [CDM16], where they address the problem through some PDE techniques (when the kernel allows this approach). The space of probability measures is very convenient to prove the existence of minimizers due to the good compactness properties, but it is also very large. Therefore, it is natural to address the regularity problem for the minimizers. For a general kernel $\bar{g}$ we cannot hope for some strong regularity, but we aim to some form of concentration bound for the optimal measures. Indeed, we can prove a result of this sort when we restrict the class of interaction kernels: our fundamental assumption to give a positive answer to this problem is

($\mathbf{H}_p$) the function $\bar{g} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ is radial, L^1_{loc}, and its restriction to $\mathbb{R}^N \setminus \{0\}$ is C^2. In addition, calling $\bar{g}(x) = g(|x|)$, there is a small radius $r_{\bar{g}} > 0$ such that $\bar{g}$ is subharmonic in $B_{r_{\bar{g}}} \setminus \{0\}$, g and g' are respectively decreasing and increasing in $(0, r_{\bar{g}})$, and $g(0) = \lim_{t \searrow 0} g(t)$.

Notice that these hypotheses provide a lower bound on the rate at which the kernel $\bar{g}$ diverges in the origin. One can expect that this feature plays some role in the analysis: a strong penalization of the short range interaction prevents the concentration of mass, going in the direction of proving some control on the density of the minimizer μ with respect to $\mathscr{L}^N$.

Theorem 1.2.1 (L^∞ bound for optimal measures)**.** *If $\bar{g} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ satisfies assumption ($\mathbf{H}_p$) and $\lim_{t \to \infty} g(t) = +\infty$, there exists a constant $M = M(N, g)$ such that the L^∞ bound $\|\mu\|_{L^\infty} \leq M$ is true in the following cases:*

1. *for any minimizer μ, either in $\mathcal{P}(\mathbb{R}^N)$ or in $\mathcal{P}_{\text{rad}}(\mathbb{R}^N)$, if the support of μ is convex and*

$$\limsup_{t \searrow 0} |g'(t)|t^N > 0; \tag{1.2}$$

2. *for any minimizer μ in $\mathcal{P}_{\text{rad}}(\mathbb{R}^N)$ (and also in $\mathcal{P}(\mathbb{R})$ if $N = 1$), if $\bar{g}$ is subharmonic in $\mathbb{R}^N \setminus \{0\}$, strictly subharmonic in $B_{r_{\bar{g}}} \setminus \{0\}$, and (1.2) holds;*

3. *for at least a minimizer μ in the class $\mathcal{P}_{\text{rad}}(\mathbb{R}^N)$ (and also in $\mathcal{P}(\mathbb{R})$ if $N = 1$) if $\bar{g}$ is subharmonic in $\mathbb{R}^N \setminus \{0\}$.*

We point out that the dimension 1 is different from the others already at this level. The situation in dimension 1 is much easier, since there is no geometry involved, leading even to the continuity results presented in Section 1.3. We point out that condition (1.2) does not require that the left hand side is finite.
Concerning the negative results, one can seek for minimizers that are not absolutely continuous with respect to $\mathscr{L}^N$. In this direction, one can see for example [CFP17, DLM22, DLM23, FM23] where the kernel is isotropic, and [CS24b, CS24a, MMR+23, Mor24] where $\bar{g}$ is not radial.

1.2.1 Convexity of the support of optimal measures

We show that the support of an optimal measure is convex in some cases. As in Proposition 1.1.3, our assumptions are not strong enough to guarantee the existence of optimal measures, hence what we prove is that every minimizing measure, if any, has convex support. This kind of result was already present in the proof of [CS23, Theorem 4.1], but they made some different hypotheses on the kernel $\bar{g}$ and some a-priori regularity assumption on the potential generated by an optimal measure. On the other hand, they work with local minimizers with respect to the ∞-Wasserstein distance, while we are interested only in the global minimizers of $\mathcal{E}$. We start with the 1-dimensional case.

Proposition 1.2.2. *Let $\bar{g} : \mathbb{R} \to \mathbb{R}^+ \cup \{+\infty\}$ be a l.s.c., symmetric and L^1_{loc} function, whose restriction to $(0, +\infty)$ is convex, and strictly convex in a right neighborhood of 0. Let μ be a measure which minimizes the energy either in $\mathcal{P}(\mathbb{R})$ or in $\mathcal{P}_{\text{rad}}(\mathbb{R})$. Then the support of μ is a closed segment.*

Proof. Let us assume that μ is a minimal measure, either in $\mathcal{P}(\mathbb{R})$ or in $\mathcal{P}_{\text{rad}}(\mathbb{R})$, and that its support is not a segment. As a consequence, there is an open segment $(a, b) \subseteq \mathbb{R}$ such that $\text{spt}(\mu)$ does not intersect (a, b), but it contains both $\{a\}$ and $\{b\}$.

By construction, the function ψ_μ is convex in the interval (a, b). Moreover, it is *strictly* convex in $(a, a + \varepsilon)$ and in $(b - \varepsilon, b)$ for some $\varepsilon > 0$, much smaller than $b - a$. By Proposition 1.1.3 and Remark 1.1.4, we deduce that $\psi_\mu \geq \mathcal{E}(\mu)$ in the whole open segment (a, b), and that, up to possibly decreasing the value of $\varepsilon > 0$, the inequality is strict in $(a, a + \varepsilon) \cup (b - \varepsilon, b)$. Consequently, and again up to further decreasing ε, the function ψ_μ is either strictly decreasing in $(a, a+\varepsilon)$, or strictly increasing in $(b-\varepsilon, b)$, or both. By symmetry, we assume without loss of generality that ψ_μ is strictly decreasing in $(a, a + \varepsilon)$. Let us now notice that

$$\psi_\mu(a)+\psi_\mu(a+\varepsilon)-2\psi_\mu(a+\varepsilon/2) = \int \bar{g}(a-y)+\bar{g}(a+\varepsilon-y)-2\bar{g}(a+\varepsilon/2-y)d\mu(y). \quad (1.3)$$

Since $\bar{g}$ is convex in $(0,+\infty)$ and symmetric, and since $\varepsilon < b-a$, for μ-a.e. y we have that

$$\bar{g}(a-y)+\bar{g}(a+\varepsilon-y)-2\bar{g}(a+\varepsilon/2-y)\geq 0.$$

Inserting this estimate in ((1.3) we deduce that

$$\psi_\mu(a)\geq 2\psi_\mu(a+\varepsilon/2)-\psi_\mu(a+\varepsilon)>\psi_\mu(a+\varepsilon/2),$$

where we have also used that ψ_μ is strictly decreasing in $(a,a+\varepsilon)$. And finally, this is absurd since $\psi_\mu(a+\varepsilon/2)>\mathcal{E}(\mu)$, as already noticed, while $\psi_\mu(a)\leq\mathcal{E}(\mu)$ by Proposition 1.1.3 and since ψ_μ is l.s.c. by construction. □

The idea of the proof in the general case when $N\geq 2$ is similar, one only needs more care in the construction. A geometric property that we are going to use is the following one.

Lemma 1.2.3. *For any $N\geq 2$, there exists a geometric constant $C_N>1$ such that, if δ, η, d, r are four positive numbers such that*

$$\eta>C_N\delta, \qquad d>C_N\eta, \qquad r>C_N d,$$

then one has

$$\frac{\mathscr{H}^{N-1}\Big(\Big\{x\in\partial B(0,r):\,\big|x-(r-\eta)\mathrm{e}_1\big|\in(d,d+\delta)\Big\}\Big)}{(N-1)\omega_{N-1}d^{N-2}\delta}\in\Big[\frac{1}{2},2\Big]. \tag{1.4}$$

Proof. This is an elementary geometric property, easy to establish with the aid of Figure 1.1. Let us consider four constants $\delta<\eta<d<r$, each quite smaller than the following one. Let us call $P=(r-\eta)\mathrm{e}_1$ as in the figure. The points of $\partial B(0,r)$ having distance exactly d from P are the intersection between the spheres $\partial B(0,r)$ and $\partial B(P,d)$, hence they are a $(N-2)$-dimensional sphere contained in a hyperplane orthogonal to the direction e_1. The radius of this sphere, call it ρ_0, is smaller than d,

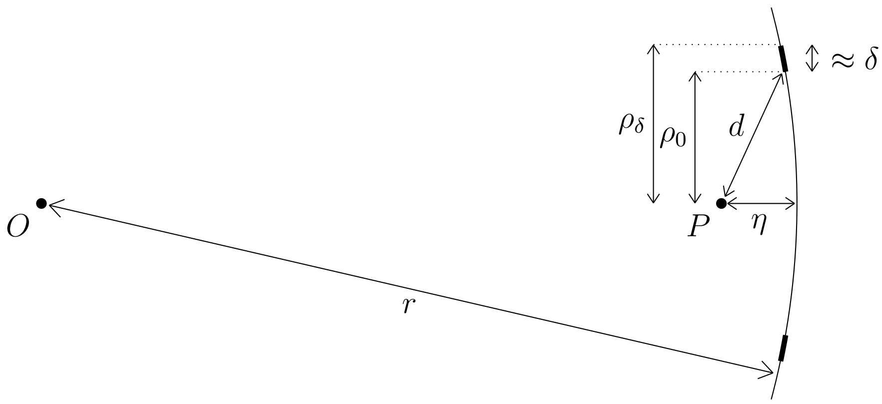


Figure 1.1: The situation in Lemma 1.2.3.

but the ratio between ρ_0 and d becomes arbitrarily close to 1 if the ratii d/η and r/d are both large enough. In the very same way, for any $0\leq t\leq\delta$, the points of $\partial B(0,r)$

having distance exactly $d+t$ from P are a $(N-2)$-dimensional sphere, with radius ρ_t very close to $d+t$. Moreover, $\rho_\delta - \rho_0 \approx \delta$, that is, the ratio between $\rho_\delta - \rho_0$ and δ is arbitrarily close to 1 as soon as η/δ, d/η, r/d are large enough. In addition, the centers of all these spheres are all on the line $\mathbb{R}e_1$, and they are almost coincident with respect to δ. More formally, if we call C_t the center of the sphere corresponding to any $0 \leq t \leq \delta$, we have that the ratio $|C_t - C_s|/|t-s|$ is arbitrarily close to 0 as soon as η/δ, d/η, r/d are large enough.

Summarizing, the $\mathscr{H}^{N-1}$-measure of the union of these spheres is arbitrarily close to the measure of a $(N-1)$-dimensional annulus contained between two concentric spheres of radii d and $d+\delta$, which in turn is arbitrarily close to $(N-1)\omega_{N-1}d^{N-2}\delta$ if d/δ is large enough. This completes the proof (in particular, instead of $1/2$ and 2 we could have used a and $1/a$ for any $a<1$). □

Proposition 1.2.4. *Let $\bar{g} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a radial, l.s.c. and $L^1_{\rm loc}$ function. Let us suppose that it is of class C^2 in $\mathbb{R}^N \setminus \{0\}$, it is subharmonic in that domain, and it is strictly subharmonic in $B_r \setminus \{0\}$ for some $r>0$. Then, the support of any measure which minimizes the energy in $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$ is a closed ball.*

Proof. Let μ be a measure minimizing the energy in $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$. Since both $\bar{g}$ and μ are radial, then so is also the potential ψ_μ. Let us define for brevity $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ the function such that $\psi_\mu(x) = f(|x|)$. Let us assume that the support of μ is not a closed ball, and let us look for a contradiction. Among all the open bounded intervals I in $(0,+\infty)$ such that the annulus $\{x \in \mathbb{R}^N : |x| \in I\}$ does not intersect $\operatorname{spt}(\mu)$, there is at least one, say (a,b), which is maximal with respect to the inclusion.

Notice that ψ_μ is a subharmonic radial function on $\mathbb{R}^N \setminus \operatorname{spt}(\mu)$, hence in particular we have

$$f''(t) + \frac{N-1}{t} f'(t) \geq 0 \qquad \text{in } (a,b). \tag{1.5}$$

We subdivide our proof in few steps. In the first one, we show that f cannot be flat close to both a and b, and in the following steps we reach a contradiction in each of the possible cases.

<u>Step I.</u> *There is some $\varepsilon > 0$ such that either $f' > \varepsilon$ in $(b-\varepsilon, b)$, or $f' < -\varepsilon$ in $(a, a+\varepsilon)$.*

First of all, we want to show the existence of a small $\varepsilon > 0$ such that either $f' > \varepsilon$ in $(b-\varepsilon, b)$ or $f' < -\varepsilon$ in $(a, a+\varepsilon)$. Since by construction $b>0$, it is clear by (1.5) that, if $f'(t) > 0$ for some $t < b$ close enough to b, then the value of f' is at least $f'(t)/2$ in the whole interval (t,b), and then we have already concluded this step. On the other hand, let us assume that $f'(t) \leq 0$ for every $t<b$ close enough to b. Since by construction the sphere $\partial B(0,b)$ belongs to $\operatorname{spt}(\mu)$, then ψ_μ is *strictly* subharmonic in the annulus $\{x \in \mathbb{R}^N : b-\eta < |x| < b\}$ for $\eta \ll 1$, and this means that $f'(t) < 0$ for some $t<b$ close to b. But then, (1.5) implies that $f'(s) < f'(t)$ for every $a<s<t$, and then the step is concluded.

<u>Step II.</u> *Proof if $f' < -\varepsilon$ in $(a, a+\varepsilon)$ and $a=0$.*

We first assume that $f' < -\varepsilon$ in $(a, a+\varepsilon)$ for some small ε. As a consequence, we can deduce that the sphere $\partial B(0,a)$ belongs to $\operatorname{spt}(\mu)$, but only if $a>0$. Let us instead suppose in this step that $a=0$. The fact that $f' < -\varepsilon$ in a right neighborhood of 0 implies that ψ_μ is not regular at the origin, having a cusp point. However, by construction ψ_μ is regular in $\mathbb{R}^N \setminus \mathrm{spt}(\mu)$, and then we deduce that the origin belongs to $\mathrm{spt}(\mu)$. Since the annulus $\{x \in \mathbb{R}^N : 0 < |x| < b\}$ does not intersect $\mathrm{spt}(\mu)$, this means that the origin is an isolated point of $\mathrm{spt}(\mu)$. But since μ minimizes the energy, so in particular $\mathcal{E}(\mu) < +\infty$, the presence of an isolated point is only possible if $\bar{g}(0) < +\infty$. And finally, if $\bar{g}(0)$ is finite, then ψ_μ is clearly continuous, and we find a contradiction because we should have $\lim_{t\searrow 0} f(t) > \mathcal{E}(\mu)$ since f is strictly decreasing in a right neighborhood of 0 and $f \geq \mathcal{E}(\mu)$ a.e. in (a,b) by Proposition 1.1.3. And again by Proposition 1.1.3, we have $f(0) = \mathcal{E}(\mu)$, obtaining the searched contradiction.

Step III. Proof if $f' < -\varepsilon$ in $(a, a+\varepsilon)$ and $a > 0$.

We now assume again that $f' < -\varepsilon$ in $(a, a+\varepsilon)$, but $a > 0$. As already noticed before, this implies that $\partial B(0,a) \subseteq \mathrm{spt}(\mu)$, and by Proposition 1.1.3 and the lower semicontinuity of ψ_μ we deduce that $f(a) \leq \mathcal{E}(\mu)$. On the other hand, $\lim_{t\searrow a} f(t) > \mathcal{E}(\mu)$, and then f has a jump point at a, with $f(a) < \lim_{t\searrow a} f(t)$. We can easily show that this is impossible. Indeed, the discontinuity of ψ_μ implies that $\bar{g}$ is not bounded around the origin, and since $\bar{g}$ is subharmonic this implies that $\bar{g}$ is a radial, decreasing function in a neighborhood of the origin. In other words, calling $g : \mathbb{R} \to \mathbb{R}^+$ the function such that $\bar{g}(x) = g(|x|)$, up to possibly decreasing the value of ε we have that g is strictly decreasing in $(0, 2\varepsilon)$ and $\lim_{t\searrow 0} g(t) = +\infty$.

Let us now call $\bar{x} = a\mathrm{e}_1$, and $w = (a+\delta)\mathrm{e}_1$ for some $\delta \ll \varepsilon$. Since as noticed before $\lim_{t\searrow a} f(t) > \mathcal{E}(\mu) \geq f(a)$, up to taking δ small enough we have that $\psi_\mu(w) \geq \psi_\mu(\bar{x}) + J$ for some $J > 0$. Let us also write $\psi_\mu = \psi_1 + \psi_2$, where

$$\psi_1(x) = \int_{y \in B(\bar{x},\varepsilon)} \bar{g}(x-y)\,d\mu(y), \qquad \psi_2(x) = \int_{y \in \mathbb{R}^N \setminus B(\bar{x},\varepsilon)} \bar{g}(x-y)\,d\mu(y).$$

Since the function ψ_2 is clearly continuous in a small neighborhood of $\bar{x}$, up to further decreasing δ we must have

$$\psi_1(w) \geq \psi_1(\bar{x}) + \frac{J}{2}. \tag{1.6}$$

And finally, we find the contradiction since as already noticed g must be strictly decreasing in $(0, 2\varepsilon)$, and since by construction μ-a.e. $y \in B(\bar{x},\varepsilon)$ satisfies $|w-y| \geq |\bar{x}-y|$ then

$$\begin{aligned}\psi_1(\bar{x}) &= \int_{B(\bar{x},\varepsilon)} \bar{g}(\bar{x}-y)\,d\mu(y) = \int_{B(\bar{x},\varepsilon)} g(|\bar{x}-y|)\,d\mu(y)\\ &\geq \int_{B(\bar{x},\varepsilon)} g(|w-y|)\,d\mu(y) = \psi_1(w),\end{aligned}$$

against (1.6).

Step IV. Proof if $f' > \varepsilon$ in $(b-\varepsilon, b)$.

We are left with the last possible case to exclude, namely, that $f' > \varepsilon$ in $(b-\varepsilon, b)$. Our argument will be similar to the one of Step III, we just need this time a little more care to deal with the geometry.

As in the previous case, we have a jump discontinuity at b, since $f(b) \leq \mathcal{E}(\mu)$ by lower semicontinuity of ψ_μ and Proposition 1.1.3, while $J := \lim_{t\nearrow b} f(t) - \mathcal{E}(\mu) > 0$ by Proposition 1.1.3 and by assumption. Let now $\ell \ll b-a$ be a positive quantity, to be specified in a moment. This time, we write $\psi_\mu = \psi_1 + \psi_2$ with

$$\psi_1(x) = \int_{y\in B(0,b+\ell)\cap B(x,2C_N\ell)} \bar g(x-y)d\mu(y), \qquad \psi_2(x) = \psi_\mu(x) - \psi_1(x),$$

where C_N is the constant of Lemma 1.2.3. The value of ℓ is so small that

$$\ell < \frac{b}{2C_N^2}, \qquad \psi_1(be_1) < \frac{J}{6}. \tag{1.7}$$

It is again clear by construction that ψ_2 is continuous in a neighborhood of $\bar x = b\mathrm{e}_1$. As a consequence, up to further decreasing $\varepsilon \ll \ell$ and calling this time $w = (b-\varepsilon)\mathrm{e}_1$, we have again (1.6). We claim now that, for any $b \le r \le b+\ell$, we have

$$\int_{y\in\partial B(0,r)\cap B(\bar x,2C_N\ell)} \bar g(\bar x - y)d\mathscr{H}^{N-1}(y) \ge \frac14 \int_{y\in\partial B(0,r)\cap B(w,2C_N\ell)} \bar g(w-y)d\mathscr{H}^{N-1}(y). \tag{1.8}$$

Since μ is radial, by integration this will give $\psi_1(w) \le 4\psi_1(\bar x)$, and this provides us with the searched contradiction thanks to (1.7) and (1.6). Therefore, to conclude we only have to establish (1.8).

Let us then fix $b \le r \le b+\ell$, and let us call $\xi = r - b + \varepsilon \le \ell + \varepsilon < 2\ell$, which is the distance between w and $\partial B(0,r)$. It is immediate to observe that, since $\ell \ll 1$, for any $y \in \operatorname{spt}\mu$ which belongs to the ball $B(\bar x, 3C_N\ell)$ (which contains $B(w,2C_N\ell)$ since $\varepsilon \ll \ell$), the implication

$$|w-y| \le C_N\xi \quad\Longrightarrow\quad |\bar x - y| \le |w-y|$$

holds. As a consequence, for any such y we have $\bar g(\bar x - y) \ge \bar g(w-y)$ –indeed, as before we have that $\bar g$ is a radially strictly decreasing in a neighborhood of the origin, because otherwise ψ_μ could not be discontinuous. We deduce

$$\begin{aligned}\int_{y\in\partial B(0,r)\cap B(w,C_N\xi)} \bar g(w-y)d\mathscr{H}^{N-1}(y) &\le \int_{y\in\partial B(0,r)\cap B(w,C_N\xi)} \bar g(\bar x-y)d\mathscr{H}^{N-1}(y)\\ &\le \int_{y\in\partial B(0,r)\cap B(\bar x,C_N\xi)} \bar g(\bar x-y)d\mathscr{H}^{N-1}(y).\end{aligned}$$

Consequently, to conclude the validity of (1.8) we can limit ourselves to show

$$\begin{aligned}&\int_{\partial B(0,r)\cap B(\bar x,2C_N\ell)\setminus B(\bar x,C_N\xi)} \bar g(\bar x-y)d\mathscr{H}^{N-1}(y)\\ &\qquad\ge \frac14 \int_{\partial B(0,r)\cap B(w,2C_N\ell)\setminus B(w,C_N\xi)} \bar g(w-y)d\mathscr{H}^{N-1}(y).\end{aligned}$$

And in turn, this is clearly true if for any $C_N\xi < d < 2C_N\ell$ and for any $\delta \ll \varepsilon$ we have

$$\mathscr{H}^{N-1}\Big(\partial B(0,r)\cap B(\bar x,d+\delta)\setminus B(\bar x,d)\Big) \ge \frac14 \mathscr{H}^{N-1}\Big(\partial B(0,r)\cap B(w,d+\delta)\setminus B(w,d)\Big). \tag{1.9}$$

Finally, this last inequality is a consequence of Lemma 1.2.3. Indeed, take any $C_N\xi < d < 2C_N\ell$, and call $\eta' = \xi$ and $\eta'' = r - b$. By construction, $d > C_N\eta' > C_N\eta''$, and $r > C_N d$ by (1.7). Therefore, for any $\delta \ll 1$ we can apply Lemma 1.2.3 with constants r, d, η', δ as well as with constants r, d, η'', δ, and then (1.4) gives (1.9). As noticed before, this establishes (1.8) and then the proof is concluded. □

We conclude observing an important consequence of the convexity of the support of an optimal measure, that is, the potential is continuous. We point out that the same conclusion is obtained in [CDM16, Proposition 3.2], but that result requires a specific control on the Laplacian of the potential in the origin, while we need some additional geometric information about the minimizer.

Lemma 1.2.5 (Continuity of ψ_μ). *Let $\bar g : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a radial, l.s.c., $L^1_{\rm loc}$ function such that, calling $g(|x|) = \bar g(x)$, the function g is continuous in $(0, +\infty)$ and decreasing in a right neighborhood of 0. Let also μ be an optimal measure, either in $\mathcal{P}(\mathbb{R}^N)$ or in $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$, with support bounded and convex. Then the function ψ_μ is continuous. More precisely, there exists a continuous function $\tilde\psi : \mathbb{R}^N \to \mathbb{R}$ such that the set $\{\psi_\mu \neq \tilde\psi\}$ is negligible with respect to both μ and $\mathscr{L}^N$.*

Proof. Calling for brevity $\Gamma = \operatorname{spt}(\mu)$, we simply define $\tilde\psi$ as the function which equals ψ_μ on $\mathbb{R}^N \setminus \Gamma$ and $\mathcal{E}(\mu)$ on Γ. The set where $\tilde\psi \neq \psi_\mu$ is the set of the points in Γ where $\psi_\mu < \mathcal{E}(\mu)$, and this set is both μ- and $\mathscr{L}^N$-negligible by Proposition 1.1.3 and Remark 1.1.4. As a consequence, we only have to show that $\tilde\psi$ is continuous.

Since in the open set $\mathbb{R}^N \setminus \Gamma$ we have that $\tilde\psi = \psi_\mu$ is continuous by construction, all we have to do is to show the continuity of $\tilde\psi$ at points of $\partial\Gamma$. Let us call $0 < r < R$ two constants such that g is decreasing in $(0, 2r)$ and the diameter of Γ is less than $R - r$, and let ω be the modulus of continuity of g in the closed interval $[r, R]$. Let $y \notin \operatorname{spt}(\mu)$ be any point with $\operatorname{dist}(y, \Gamma) < r$, and let $x \in \Gamma$ be the point which minimizes the distance from y. We claim that

$$\psi_\mu(y) - \psi_\mu(x) \leq \omega(|y - x|), \tag{1.10}$$

which will clearly conclude the thesis. By minimality of x, for every $z \in \Gamma$ we have $|y - z| \geq |x - z|$, thus $g(|y - z|) \leq g(|x - z|)$ if $|y - z| \leq 2r$. If, instead, $z \in \Gamma$ but $|y - z| > 2r$, then we have also $|x - z| > r$, and then $g(|y - z|) - g(|x - z|) \leq \omega(|y - x|)$. As a consequence,

$$\begin{aligned}
\psi_\mu(y) - \psi_\mu(x) &= \int_{B(y,2r)} g(|y - z|) - g(|x - z|) d\mu(z) \\
&\quad + \int_{\mathbb{R}^N \setminus B(y,2r)} g(|y - z|) - g(|x - z|) d\mu(z) \\
&\leq \omega(|y - x|)\mu\big(\mathbb{R}^N \setminus B(y, 2r)\big) \leq \omega(|y - x|),
\end{aligned}$$

which proves (1.10) and thus the thesis. □

1.2.2 The main geometric estimates

We collect here three geometric estimates, that we will use to get the proof of Theorem 1.2.1. We start with an elementary calculation.

Lemma 1.2.6. *Let $\bar g$ be a function satisfying condition ($\mathbf{H}_p$), let $r_{\bar g} > 0$ be given by ($\mathbf{H}_p$), and let $\tilde r \le r_{\bar g}$. There exists $c = c(g, N, \tilde r) > 0$ such that, defining $\tilde f : \mathbb{R}^N \to \mathbb{R}^+$ as $\tilde f(x) = 1$ if $|x| < \tilde r$ and $\tilde f(x) = 0$ otherwise, one has $\psi_{\tilde f}(z) \le \psi_{\tilde f}(0) - c|z|^2$ for every $z \in \mathbb{R}^N$ with $|z| \ll 1$, depending on g and N. The constant c actually depends only on N, $\tilde r$ and $g'(\tilde r)$.*

Proof. Let $z \in \mathbb{R}^N$ be a point with $\eta = |z|$ sufficiently small. For every $w \in \partial B(0, \tilde r)$, call $\Gamma(w)$ the segment joining w and $w + z$, and $\theta = \theta(w) \in \mathbb{S}^1$ the angle between w and z, that is, $w \cdot z = |w||z| \cos\theta = \tilde r \eta \cos\theta$. We can then evaluate

$$\begin{aligned}
\psi_{\tilde f}(z) - \psi_{\tilde f}(0) &= \int_{B(z,\tilde r)} \bar g(y)dy - \int_{B(0,\tilde r)} \bar g(y)dy \\
&= \int_{w\in\partial B(0,\tilde r)} \int_{x\in\Gamma(w)} \bar g(x) d\mathscr{H}^1(x) \cos\theta d\mathscr{H}^{N-1}(w) \\
&= \int_{w\in\partial B(0,\tilde r)} \int_{t=0}^{\eta} \Big(g(\tilde r) + t\cos\theta g'(\tilde r) + o(\eta)\Big) dt \cos\theta d\mathscr{H}^{N-1}(w) \\
&= \frac{\eta^2}{2} g'(\tilde r) \int_{w\in\partial B(0,\tilde r)} \cos^2\theta d\mathscr{H}^{N-1}(w) + o(\eta^2) \\
&= \frac{\eta^2}{2} g'(\tilde r) C_N \tilde r^{N-1} + o(\eta^2).
\end{aligned}$$

Notice that C_N is a purely geometric constant, only depending on N. Its exact value, though elementary to calculate, is not important. Here, by $o(\eta)$ and $o(\eta^2)$ we denote a quantity which becomes arbitrarily smaller than η, or η^2, if η is small enough, depending on g, N and $\tilde r$. Since $g'(\tilde r) < 0$, we obtain the thesis with $c = |g'(\tilde r)| C_N \tilde r^{N-1}/3$. □

We now pass to give an L^∞ estimate in a very peculiar case. We will obtain the proof of Theorem 1.2.1 basically reducing ourselves to this case.

Lemma 1.2.7 (L^∞ estimate)**.** *Let us assume that $\bar g$ satisfies ($\mathbf{H}_p$), and let $f : \mathbb{R}^N \to \mathbb{R}$ be a positive, radial, C^2 function, with unit L^1 norm, supported in $B_{\bar R}$ for some $\bar R > 0$ and such that ψ_f is constant in a neighborhood of 0 and $f(0) = \max\{f(x)\}$. Then $f(0) \le M_0$ for some constant $M_0 = M_0(\bar g, N, \bar R)$, which actually only depends on N, $\bar R$ and on the restriction of g to $[r_{\bar g}, \bar R]$.*

Proof. Let $r = r_{\bar g}/2$. First of all, we subdvide $\psi_f = \psi_1 + \psi_2$, where

$$\psi_1(x) = \int_{B_r} \bar g(x-y) f(y) dy, \qquad \psi_2(x) = \int_{\mathbb{R}^N\setminus B_r} \bar g(x-y) f(y) dy.$$

We start considering the function ψ_2, which is easier to deal with around 0. In fact, of course ψ_2 is radial and of class C^2 in B_r, so in particular $D\psi_2(0) = 0$. Moreover, also keeping in mind that $\|f\|_{L^1} = 1$, we have that

$$\|D^2\psi_2(0)\| = \left\| \int_{\mathbb{R}^N \setminus B_r} D^2\bar{g}(y)f(y)dy \right\| \leq C(\bar{g}, N, \bar{R}), \tag{1.11}$$

where the norm symbol denotes the operator norm, considering the Hessian as a linear map from $\mathbb{R}^N$ into itself. Notice that the constant C actually only depends on N and on $\max_{t\in[r,\bar{R}]} \{|g'(t)| + |g''(t)|\}$.

We now pass to consider ψ_1. Notice that, for every $x \in B_r$, we have

$$\begin{aligned}\psi_1(x) - \psi_1(0) &= \int_{B_r} \big(\bar{g}(x-y) - \bar{g}(y)\big) f(y)dy \\ &= \int_{B_r} \big(\bar{g}(x-y) - \bar{g}(y)\big)\big(f(y) - f(0)\big)dy \\ &\quad + f(0) \int_{B_r} (\bar{g}(x-y) - \bar{g}(y))\, dy.\end{aligned}$$

The last integral in the above equation is nothing else than $\psi_{\tilde{f}}(x) - \psi_{\tilde{f}}(0)$ if we call $\tilde{f}$ the characteristic function of the ball B_r. We can then apply Lemma 1.2.6 and deduce from the above equation that, whenever $|x|$ is small enough, we have the bound

$$\psi_1(x) - \psi_1(0) \leq -cf(0)|x|^2 + \int_{B_r} \big(\bar{g}(x-y) - \bar{g}(y)\big)\big(f(y) - f(0)\big)dy, \tag{1.12}$$

where $c = c(\bar{g}, N)$ is the constant given by Lemma 1.2.6 –notice that c depends on N, r and $g'(r)$, and in turn $r = r_{\bar{g}}/2$ depends only on $\bar{g}$ by assumption ($\mathbf{H}_p$). We can now subdivide the last integral in two parts, namely, the integral in the smaller ball $B_{|x|}$, and the integral in $B_r \setminus B_{|x|}$. Since f is radial and of class C^2, we have that $|f(y) - f(0)| \leq \|D^2 f\|_{L^\infty}|x|^2$ for every $y \in B_{|x|}$, thus since $\bar{g}$ is integrable we deduce

$$\begin{aligned}&\left| \int_{B_{|x|}} \big(\bar{g}(x-y) - \bar{g}(y)\big)\big(f(y) - f(0)\big)dy \right| \\ &\qquad \leq \|D^2 f\|_{L^\infty}|x|^2 \int_{B_{|x|}} \big|\bar{g}(x-y) - \bar{g}(y)\big|dy \leq 2\|\bar{g}\|_{L^1(B_{2|x|})}\|D^2 f\|_{L^\infty}|x|^2.\end{aligned} \tag{1.13}$$

We finally use that $\bar{g}$ is subharmonic in $B_{r_{\bar{g}}} \setminus \{0\}$, that coincides with $B_{2r} \setminus \{0\}$. Indeed, this implies that for every $0 < s < r$ the function

$$z \mapsto \int_{\partial B_s} \bar{g}(z-y)d\mathscr{H}^{N-1}(y)$$

is also subharmonic in B_s, and since this function is also radial by construction then its minimum is at $z = 0$, i.e.

$$\int_{\partial B_s} (\bar{g}(z-y) - \bar{g}(y))\, d\mathscr{H}^{N-1}(y) \geq 0 \qquad \forall z \in B_s.$$

By assumption we have that the maximum of f in B_r is attained in 0, therefore by integration we immediately deduce that

$$\int_{B_r \setminus B_{|x|}} \big(\bar{g}(x-y) - \bar{g}(y)\big)\big(f(y) - f(0)\big) dy \leq 0 \qquad \forall x \in B_r.$$

Putting this inequality together with (1.13) into (1.12), we obtain

$$\psi_1(x) - \psi_1(0) \leq \Big(2\|\bar{g}\|_{L^1(B_{2|x|})} \|D^2 f\|_{L^\infty} - cf(0)\Big)|x|^2.$$

Since $\bar{g} \in L^1_{\text{loc}}(\mathbb{R}^N)$, for $|x|$ small enough we deduce $\psi_1(x) - \psi_1(0) \leq -cf(0)|x|^2/2$. Finally, combining this inequality with (1.11), keeping in mind that $\psi_1 + \psi_2 = \psi_f$ is constant in a neighborhood of 0, we deduce that $f(0) \leq M_0$ with

$$M_0 = \frac{C(\bar{g}, N, \bar{R})}{c(\bar{g}, N)}.$$

The proof is then concluded. We underline that the constant $C(\bar{g}, N, \bar{R})$ only depends on N and on the restriction of g to $[r, \bar{R}]$, while $c(\bar{g}, N)$ only depends on N, r and $g'(r)$ (and we recall that $r = r_{\bar{g}}/2$ depends only on $\bar{g}$). □

Remark 1.2.8. Notice that the above estimate is true also if the origin is only a *local* maximum of f. More precisely, if 0 is a maximum of f in the ball $B_{\hat{r}}$, then the above proof works substituting r with $\tilde{r} := \min\{r, \hat{r}\}$ (in fact, Lemma 1.2.6 is proved with $\tilde{r}$). Therefore, the L^∞ bound in this more general case also depends on $\hat{r}$.

We can now show an estimate on the mass of a small ball around the boundary of the support of an optimal measure.

Lemma 1.2.9 (Estimate near the boundary)**.** *Let $\bar{g}$ be a function satisfying (**H**$_p$), and let μ be an optimal measure, either in $\mathcal{P}(\mathbb{R}^N)$ or in $\mathcal{P}_{\text{rad}}(\mathbb{R}^N)$, with support convex and contained in $B_{\bar{R}}$. Then there exists a constant $C = C(\bar{g}, N, \bar{R})$ such that, for every $x \in \partial\big(\text{spt}(\mu)\big)$ and every $\rho \ll 1$, one has*

$$\mu\big(B(x, \rho)\big) \leq \frac{C}{|g'(2\rho)|}. \tag{1.14}$$

Proof. Let us call for brevity $\Gamma = \text{spt}(\mu)$, and let $x \in \partial\Gamma$ be a given point. Since Γ is convex, we can take an external direction $v \in \mathbb{S}^{N-1}$ to Γ at x, that is, for every $y \in \Gamma$ one has $(x - y) \cdot v \leq 0$. We now pick $\rho < \frac{1}{3}\min\{1, r_{\bar{g}}\}$, and we observe that

$$\psi_\mu(x + 2\rho v) - \psi_\mu(x) = \int_\Gamma \left(\bar{g}(x + 2\rho v - y) - \bar{g}(x - y)\right) d\mu(y). \tag{1.15}$$

The convexity of Γ implies that, for every $y \in \Gamma$,

$$|x + 2\rho v - y| \geq |x - y|. \tag{1.16}$$

As a consequence,

$$\bar{g}(x + 2\rho v - y) \leq \bar{g}(x - y) + \rho C, \tag{1.17}$$

where $C = 2\max\big\{g'(t) : 0 < t < \bar R + 1\big\}$. Notice carefully that $C = C(\bar g, N, \bar R)$ is a well-defined real number thanks to the fact the we are maximizing $g'(t)$ instead of $|g'(t)|$, and this is possible thanks to (1.16), which in turn is a consequence of the convexity of Γ.

While the estimate (1.17) is true for every $y \in \Gamma$, let us now take $y \in \Gamma \cap B(x, \rho)$. For such a y, not only we have (1.16), but we also have

$$|x + 2\rho v - y| - |x - y| \geq \rho,$$

and then since g is decreasing and g' increasing in $(0, r_{\bar g})$, we have

$$\begin{aligned}\bar g(x + 2\rho v - y) &= g(|x + 2\rho v - y|) \leq g(|x - y| + \rho) \leq g(|x - y|) + \rho g'(|x - y| + \rho)\\ &\leq g(|x - y|) + \rho g'(2\rho) = \bar g(x - y) - \rho\big|g'(2\rho)\big|.\end{aligned}$$

Insterting in (1.15) this estimate for $y \in \Gamma \cap B(x, \rho)$, and the estimate (1.17) for points $y \in \Gamma \setminus B(x, \rho)$, we obtain

$$\frac{\psi_\mu(x + 2\rho v) - \psi_\mu(x)}{\rho} \leq -\big|g'(2\rho)\big|\mu\big(B(x, \rho)\big) + C.$$

By Proposition 1.1.3, also keeping in mind Remark 1.1.4, we know that $\psi_\mu(x) \leq \mathcal{E}(\mu) \leq \psi_\mu(x + 2\rho v)$, and then the above inequality implies (1.14). □

1.2.3 Proof Theorem 1.2.1

This section is devoted to present the proof of Theorem 1.2.1. We start with the first case.

Lemma 1.2.10 (L^∞ bound for a rapidly exploding $\bar g$)**.** *Let $\bar g : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a function satisfying ($\mathbf{H}_p$) and condition (1.2), with $\lim_{t\to\infty} g(t) = +\infty$. Let moreover $\bar\mu$ be a minimizer of the energy, either in $\mathcal{P}(\mathbb{R}^N)$ or in $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$, with convex support. Then $\bar\mu \in L^\infty$, and in particular $\|\bar\mu\|_{L^\infty} \leq M(\bar g, N)$.*

Proof. First of all, we apply Theorem 1.1.1, obtaining a constant $R = R(\bar g, N)$ such that $\Gamma = {\rm spt}(\bar\mu)$ is contained in a ball of radius R. We set then $\bar R = 2R + 2$, we let $M_0(\bar g, N, \bar R)$ and $C = C(\bar g, N, \bar R)$ be the constants given by Lemma 1.2.7 and Lemma 1.2.9 respectively, and we define

$$M = \max\left\{M_0, \frac{2 \cdot 4^N C}{\omega_N \alpha}\right\}, \tag{1.18}$$

where $\alpha := \limsup_{t\searrow 0} |g'(t)|t^N$ is the quantity appearing in (1.2). Notice that, if $\alpha = +\infty$, then the constant M coincides with M_0, and we stress that M depends only on $\bar g$ and N since $\bar R = \bar R(\bar g, N)$.

For every $\rho < 1$, we consider a standard mollifier $\varphi_\rho : \mathbb{R}^N \to \mathbb{R}^+$, that is, a smooth, radial function supported in B_ρ such that $\|\varphi_\rho\|_{L^1} = 1$ and

$$\|\varphi_\rho\|_{L^\infty} \leq \frac{2}{\omega_N \rho^N}. \tag{1.19}$$

We let then $\mu_\rho = \bar\mu * \varphi_\rho$, which is a positive, smooth function supported in a ball of radius $R + \rho$, and we claim that

$$\|\mu_\rho\|_{L^\infty} \leq \max\left\{M_0, \frac{2}{\omega_N \rho^N} \cdot \sup_{x\in\partial\Gamma} \bar\mu\big(B(x,2\rho)\big)\right\}. \tag{1.20}$$

We can easily show that this estimate concludes the proof. Indeed, we can take a sequence $\rho_j \searrow 0$ such that $g'(4\rho_j)(4\rho_j)^N \to \alpha$, and up to subsequences we can suppose that $\mu_{\rho_j} \stackrel{*}{\rightharpoonup} \bar\mu$. This last fact guarantees that $\|\bar\mu\|_\infty \leq \liminf_{j\to+\infty} \left\|\mu_{\rho_j}\right\|_\infty$, and combining Lemma 1.2.9 with (1.20) we obtain that

$$\limsup_{j\to\infty} \frac{2}{\omega_N \rho_j^N} \cdot \sup_{x\in\partial\Gamma} \bar\mu\big(B(x,2\rho_j)\big) \leq \limsup_{j\to\infty} \frac{2C}{\omega_N g'(4\rho_j)\rho_j^N} = \frac{2\cdot 4^N C}{\omega_N \alpha},$$

showing the desired estimate $\|\bar\mu\|_\infty \leq M$. To conclude the thesis, we are then reduced to show the validity of (1.20). Let y be a maximum point for the smooth function μ_ρ. Suppose first that y is contained in a ρ-neighborhood of $\partial\Gamma$, thus there exists some $x \in \partial\Gamma \cap B(y,\rho)$. In this case, also by (1.19) we have

$$\|\mu_\rho\|_{L^\infty} = \mu_\rho(y) = \int_{B(y,\rho)} \varphi_\rho(y-z)\,d\bar\mu(z) \leq \frac{2}{\omega_N\rho^N}\bar\mu\big(B(y,\rho)\big) \leq \frac{2}{\omega_N\rho^N}\bar\mu\big(B(x,2\rho)\big),$$

hence (1.20) is established.

Let us now assume that the distance between y and $\partial\Gamma$ is strictly greater than ρ, say $\rho + d$ with $d > 0$. In this case, the ball $B(y, \rho + d)$ is entirely contained either in Γ, or in $\mathbb{R}^N \setminus \Gamma$. However, this second case is impossible because it would imply $\mu_\rho(y) = 0$, against the fact that y is a maximum point for μ_ρ, so we deduce $B(y, \rho + d) \subseteq \Gamma$. Now, we observe that $\psi_{\mu_\rho} = \psi_{\bar\mu} * \varphi_\rho$. Since $\psi_{\bar\mu}(z) = \mathcal{E}(\bar\mu)$ for $\mathscr{L}^N$-a.e. $z \in \Gamma$ by Proposition 1.1.3, we deduce that $\psi_{\mu_\rho}(z) = \mathcal{E}(\bar\mu)$ for every $z \in B(y,d)$. Finally, we define f as the radial average of μ_ρ around y, that is,

$$f(x) = \fint_{\partial B(y,|x-y|)} \mu_\rho(w)\,d\mathscr{H}^{N-1}(w).$$

Notice that f is a smooth, radial function, with unit L^1 norm, supported in the ball $B_{2(R+\rho)} \subseteq B_{\bar R}$, and 0 is a maximum point for f. Moreover, ψ_f is constantly equal to $\mathcal{E}(\bar\mu)$ in the ball B_d. As a consequence, we can apply Lemma 1.2.7, obtaining that $f(0) \leq M_0$, and since by construction $f(0) = \mu_\rho(y) = \|\mu_\rho\|_{L^\infty}$, we have obtained (1.20) also in this case and the proof is concluded. □

Remark 1.2.11. Notice that in the above lemma the assumption that $\lim_{t\to\infty} g(t) = +\infty$ has been used only to be able to apply Theorem 1.1.1, and in turn this was needed only to be sure that the support of $\bar\mu$ was contained in some ball. As a consequence, if $\bar g$ does not explode at ∞ but a minimizer $\bar\mu$ has support which is convex and bounded, then it is still true that $\bar\mu$ is in L^∞ (and in this case, the L^∞ bound also depends on the diameter of the support).

We want now to extend the L^∞ bound in order to cover also cases when (1.2) does not hold. To do so, we will perturb the function $\bar g$ so to satisfy (1.2) and we will use the above lemma. It is simple to notice that the argument only works if we can approximate any measure with smooth functions in such a way that the energy converges. Therefore, we first have to show the following result. We point out that some approximations of similar flavour are present in the works concerning Ginzburg-Landau energies, where it is useful to approximate in energy by means of collections of Dirac deltas. See [SS07, Proposition 7.4] and [Ser15, Proposition 2.8].

Lemma 1.2.12. *Assume that $g:(0,+\infty)\to\mathbb{R}^+$ is a continuous function, decreasing in a right neighborhood of 0 and such that $\lim_{t\searrow 0} g(t)t^\gamma = 0$ for some $0<\gamma<N$, and let $\bar g:\mathbb{R}^N\to\mathbb{R}^+\cup\{+\infty\}$ be given by $\bar g(x)=g(|x|)$ for $x\neq 0$, and $\bar g(0)=\lim_{t\to 0} g(t)$. Then, for any probability measure $\mu\in\mathcal{P}(\mathbb{R}^N)$ with compact support there exists a sequence of smooth measures $\mu_j\in\mathcal{P}(\mathbb{R}^N)\cap C^\infty_0(\mathbb{R}^N)$ weakly* converging to μ and such that*

$$\lim_{j\to\infty}\mathcal{E}(\mu_j)=\mathcal{E}(\mu). \tag{1.21}$$

Moreover, each measure μ_j belongs to $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$ if so does μ.

Proof. Let g, $\bar g$ and μ be as in the claim. As in the proof of Lemma 1.2.10, for any $\rho>0$ we denote by $\varphi_\rho:\mathbb{R}^N\to\mathbb{R}$ a smooth, radial function supported in B_ρ, with unit L^1 norm and such that (1.19) holds. By lower semicontinuity of $\bar g$, for any sequence μ_j which weakly* converges to μ one has $\mathcal{E}(\mu)\le\liminf\mathcal{E}(\mu_j)$. As a consequence, we can limit ourselves to consider the case when $\mathcal{E}(\mu)<+\infty$, since otherwise the claim is trivial. We start assuming that for some fixed $0<\varepsilon<1$ one has

$$\iint \bar g\left(\frac{x-y}{1+\varepsilon}\right)d\mu(y)d\mu(x)<+\infty. \tag{1.22}$$

Let us call $u(t)=g(t)t^\gamma$, which is a positive and continuous function which goes to 0 when $t\searrow 0$. For any $j\in\mathbb{N}$, let us call $\rho_j=\min\big\{t>0\,:\,u(3t/\varepsilon)=1/j\big\}$. Keeping in mind that $\varepsilon>0$ is small but fixed, we have that $\rho_j>0$, and that $\lim_{j\to\infty}\rho_j=0$. We claim that

$$g(\rho_j)<\frac{3^\gamma g(3\rho_j/\varepsilon)}{\varepsilon^\gamma},\qquad g(3\rho_j/\varepsilon)>\frac{\varepsilon^\gamma(N-\gamma)}{3^\gamma 2^{N-\gamma}N}\iint_{B_{\rho_j}\times B_{\rho_j}}\bar g(x-y)dydx. \tag{1.23}$$

The left inequality simply follows from the definition of ρ_j and the fact that $\varepsilon<1$:

$$g(\rho_j)=\frac{u(\rho_j)}{\rho_j^\gamma}<\frac{1}{j\rho_j^\gamma}=\frac{u(3\rho_j/\varepsilon)}{\rho_j^\gamma}=\frac{3^\gamma g(3\rho_j/\varepsilon)}{\varepsilon^\gamma}.$$

Concerning the inequality on the right, for any $0<t<2\rho_j$ one has $g(t)<1/jt^\gamma$, thus

$$\begin{aligned}\iint_{B_{\rho_j}\times B_{\rho_j}}\bar g(x-y)dydx&\le\omega_N\rho_j^N\int_{B_{2\rho_j}}\bar g(w)dw=N\omega_N^2\rho_j^N\int_{t=0}^{2\rho_j}g(t)t^{N-1}dt\\&<\frac{N\omega_N^2\rho_j^N}{j}\int_{t=0}^{2\rho_j}t^{N-1-\gamma}dt=\frac{2^{N-\gamma}N\omega_N^2\rho_j^{2N-\gamma}}{j(N-\gamma)}\\&=\frac{3^\gamma 2^{N-\gamma}N\omega_N^2\rho_j^{2N}}{\varepsilon^\gamma(N-\gamma)}g(3\rho_j/\varepsilon).\end{aligned}$$

We have then proved also the right inequality in (1.23). We can now define $\mu_j = \mu * \varphi_{\rho_j}$, which is by construction a smooth probability measure with compact support, and which is radial if so is μ. We can start calculating the energy of μ_j as

$$\begin{aligned}\mathcal{E}(\mu_j) &= \iint \bar{g}(x-y)d\mu_j(y)d\mu_j(x)\\ &= \iint \bigg(\iint \bar{g}(y'-x')\varphi_{\rho_j}(y-y')\varphi_{\rho_j}(x-x')dy'dx'\bigg)d\mu(y)d\mu(x).\end{aligned} \tag{1.24}$$

Let us denote for brevity $\xi_j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^+$ as

$$\xi_j(x,y) = \iint \bar{g}(y'-x')\varphi_{\rho_j}(y-y')\varphi_{\rho_j}(x-x')dy'dx',$$

and let $r > 0$ be such that g is decreasing in $(0, r)$. If j is large enough, we can assume that $6\rho_j < \varepsilon r$. Let us then estimate $\xi_j(x, y)$ in three possible cases. First of all, assume that $|x-y| < 3\rho_j/\varepsilon$. Then, by Riesz inequality, (1.19) and both inequalities in (1.23) we have

$$\begin{aligned}\xi_j(x,y) &\leq \iint \bar{g}(y'-x')\varphi_{\rho_j}(y')\varphi_{\rho_j}(x')dy'dx' \leq \frac{4}{\omega_N^2\rho_j^{2N}}\iint_{B_{\rho_j}\times B_{\rho_j}} \bar{g}(y'-x')dy'dx'\\ &\leq \frac{3^\gamma 2^{N+2-\gamma}N}{\varepsilon^\gamma(N-\gamma)} g(3\rho_j/\varepsilon) \leq \frac{3^\gamma 2^{N+2-\gamma}N}{\varepsilon^\gamma(N-\gamma)}\bar{g}(x-y).\end{aligned}$$

Second, assume that $3\rho_j/\varepsilon \leq |x-y| < r/2$. In this case, we simply have

$$\xi_j(x,y) \leq g(|x-y|-2\rho_j) \leq \bar{g}\left(\frac{x-y}{1+\varepsilon}\right).$$

Finally, assume that $r/2 \leq |x-y| \leq 2R$, where R is a constant such that the support of μ is contained in a ball of radius R. In this case, we have

$$\xi_j(x,y) \leq \max\big\{g(t) : |x-y|-2\rho_j \leq t \leq |x-y|+2\rho_j\big\} \leq 2\bar{g}(x-y),$$

where the last inequality is true by the continuity of g as soon as ρ_j is small enough, hence again for any j large enough. Putting together the last three estimates, we derive the existence of a constant $C = C(N, \gamma, \varepsilon)$ such that for any $x, y \in \operatorname{spt}\mu$

$$\xi_j(x,y) \leq C\left(\bar{g}(x-y) + \bar{g}\left(\frac{x-y}{1+\varepsilon}\right)\right).$$

Since we are assuming that $\mathcal{E}(\mu) < +\infty$, as well as (1.22), the right hand side of the above inequality is integrable with respect to $\mu \otimes \mu$. Since the sequence $\xi_j(x, y)$ pointwise converges to $\bar{g}(x-y)$ when $j \to +\infty$, by the Dominated Converge Theorem and (1.24) we deduce that $\mathcal{E}(\mu_j) \to \mathcal{E}(\mu)$. In other words, $\{\mu_j\}$ is a sequence of smooth probability measures, radial if so is μ, which weakly* converge to μ and which satisfy (1.21). The proof is then concluded under the additional assumption (1.22).

Let us now assume that μ is a generic probability measure, not necessarily satisfying (1.22) for some $\varepsilon > 0$. For every $\varepsilon > 0$, let us now call $\tau_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ the function $\tau_\varepsilon(x) = (1+\varepsilon)x$, and let us set $\mu^\varepsilon = (\tau_\varepsilon)_{\#}\mu$, that is

$$\int \eta(x)d\mu^\varepsilon(x) = \int \eta\big((1+\varepsilon)x\big)d\mu(x) \qquad \forall \eta \in C_b(\mathbb{R}^N).$$

Notice that also μ^ε is a probability measure, radial if so is μ, and that the sequence $\{\mu^\varepsilon\}$ weakly* converge to μ when $\varepsilon \searrow 0$. Moreover, since g is decreasing in a right neighborhood of 0 and μ has compact support, then again by the Dominated Convergence Theorem we have that

$$\mathcal{E}(\mu^\varepsilon) = \iint \bar{g}(x-y)d\mu^\varepsilon(y)d\mu^\varepsilon(x) = \iint \bar{g}\big((1+\varepsilon)(x-y)\big)d\mu(y)d\mu(x) \xrightarrow[\varepsilon \searrow 0]{} \mathcal{E}(\mu).$$

In addition, μ^ε satisfies (1.22) since

$$\iint \bar{g}\left(\frac{x-y}{1+\varepsilon}\right)d\mu^\varepsilon(y)d\mu^\varepsilon(x) = \iint \bar{g}(x-y)d\mu(y)d\mu(x) < +\infty.$$

As a consequence, for every $\varepsilon > 0$ we can find a sequence $\{\mu_j^\varepsilon\}$ of smooth probability measures, radial if so is μ, which weakly* converge to μ^ε and so that $\mathcal{E}(\mu_j^\varepsilon) \to \mathcal{E}(\mu^\varepsilon)$. The thesis follows then by a standard diagonal argument. □

Thanks to the above approximation result, we are able to show the following L^∞ bound without assuming the validity of (1.2).

Lemma 1.2.13 (L^∞ bound for a more general $\bar{g}$)**.** *Let us assume that $\bar{g}$ satisfies* ($\mathbf{H}_p$) *and it is subharmonic on $\mathbb{R}^N \setminus \{0\}$, as well that $\lim_{t\to\infty} g(t) = +\infty$. Then there exist a constant $M = M(\bar{g}, N)$ and a measure $\bar{\mu} \in L^\infty$ which minimizes $\mathcal{E}$ in $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$, with $\|\bar{\mu}\|_{L^\infty} \leq M$. If $N = 1$, then there exists also a measure $\hat{\mu} \in L^\infty$ which minimizes $\mathcal{E}$ in the whole class $\mathcal{P}(\mathbb{R})$, again with $\|\hat{\mu}\|_{L^\infty} \leq M$.*

Proof. Notice that, under our assumptions, we can apply Theorem 1.1.1, and we know that there exists a minimizing measure $\bar{\mu}$ in $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$, and in $\mathcal{P}(\mathbb{R})$ if $N = 1$. If $\bar{g}$ is strictly subharmonic in $B_{r_{\bar{g}}} \setminus \{0\}$ and (1.2) holds, then *every* minimal measure in $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$, as well as *every* minimal measure in $\mathcal{P}(\mathbb{R})$ if $N = 1$, has bounded and convex support by Proposition 1.2.2 or Proposition 1.2.4, and then it satisfies the L^∞ bound by Lemma 1.2.10. Therefore, we have only to consider the case when (1.2) does not hold, or $\bar{g}$ is not *strictly* subharmonic on $B_{r_{\bar{g}}} \setminus \{0\}$.

Let us first suppose that (1.2) does not hold (the case when (1.2) holds and $\bar{g}$ is not strictly subharmonic on $B_{r_{\bar{g}}} \setminus \{0\}$ is much simpler, and it will be discussed at the end of the proof). We can define $h : \mathbb{R}^+ \to \mathbb{R}^+$ as the function such that $h(t) = 0$ for $t \geq r_{\bar{g}}$, while for $0 < t < r_{\bar{g}}$

$$h(t) = t^{-N+\frac{1}{2}} + \frac{-4N^2-8N-3}{8} r_{\bar{g}}^{-N+\frac{1}{2}} + \frac{4N^2+4N-3}{4} r_{\bar{g}}^{-N-\frac{1}{2}} t - \frac{4N^2-1}{8} r_{\bar{g}}^{-N-\frac{3}{2}} t^2.$$

An elementary calculation ensures that the function $\bar{h} : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ given by $\bar{h}(x) = h(|x|)$ is C^2, radial, subharmonic in $\mathbb{R}^N \setminus \{0\}$, strictly subharmonic in $B_{r_{\bar{g}}} \setminus \{0\}$, and that $\limsup_{t\searrow 0}|h'(t)|t^N = +\infty$. We define then $g_\varepsilon = g+\varepsilon h$, and consistently $\bar g_\varepsilon = \bar g + \varepsilon\bar h$. Notice that $\bar g_\varepsilon$ clearly satisfies assumption ($\mathbf{H}_p$), is *strictly* subharmonic in $B_{r_{\bar g}}\setminus\{0\}$, and satisfies (1.2). For brevity of notations, we write $\mathcal{E}_\varepsilon$ to denote the energy corresponding to the function $\bar g_\varepsilon$, so in particular $\mathcal{E}_\varepsilon = \mathcal{E} + \varepsilon\mathcal{E}_h$.

We can apply Theorem 1.1.1 with the function $\bar g_\varepsilon$ in place of $\bar g$, and this ensures the existence of a minimizer of the energy $\mathcal{E}_\varepsilon$ both in $\mathcal{P}(\mathbb{R}^N)$ and in $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$. Notice that the map $\varepsilon\mapsto \mathcal{E}_\varepsilon(\omega_N^{-1}\chi_B)$ is continuous, converging to $\mathcal{E}(\omega_N^{-1}\chi_B)\in(0,+\infty)$ when $\varepsilon\searrow 0$. As a consequence, Theorem 1.1.1 ensures the existence of some $R>0$, depending on $\bar g$ and N but not on ε, such that if ε is small enough then the support of every measure minimizing the energy $\mathcal{E}_\varepsilon$ in $\mathcal{P}(\mathbb{R}^N)$ or in $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$ is contained in the ball $B(0,R)$.

Let us then call μ_ε a minimizer of $\mathcal{E}_\varepsilon$ in $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$. If $N=1$, we can also call μ_ε a minimizer in $\mathcal{P}(\mathbb{R})$. Every μ_ε has support in the ball $B(0,R)$, and it has convex support. Indeed, if $N>1$ (hence μ_ε minimizes the energy $\mathcal{E}_\varepsilon$ in $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$) then Proposition 1.2.4 ensures the support to be a closed ball. Instead, if $N=1$, then Proposition 1.2.2 ensures that the support is a closed segment, both if μ_ε minimizes in $\mathcal{P}(\mathbb{R})$ and in $\mathcal{P}_{\rm rad}(\mathbb{R})$. We can the apply Lemma 1.2.10 to $\bar g_\varepsilon$, obtaining that $\mu_\varepsilon\in L^\infty$ and $\|\mu_\varepsilon\|_{L^\infty}\le M(\bar g_\varepsilon,N)=M(\bar g,\varepsilon,N)$.

We want to show that M actually does not depend on ε. To do so, we recall that by (1.18) the value of M coincides with M_0, since $\limsup_{t\searrow 0}|g_\varepsilon'(t)|t^N=+\infty$. And in turn, by Lemma 1.2.7, the value of M_0 depends on N, $\bar R$ and on the restriction of g_ε to $[r_{\bar g},\bar R]$, where $\bar R = 2R+1$. Since all the functions g_ε coincide with g in $[r_{\bar g},\bar R]$, we have shown that M only depends on $\bar g$ and N, and not on ε.

We can then find a sequence $\varepsilon_j\searrow 0$ such that the measures μ_{ε_j} weakly* converge to some $\bar\mu$, which is a probability measure since the measures μ_ε are all supported in a same ball $B(0,R)$, and which is radially symmetric if so are all the measures μ_ε. By construction, we have that $\bar\mu$ is actually in L^∞, and that $\|\bar\mu\|_{L^\infty}\le M$. The proof will then be concluded once we show that $\bar\mu$ is a minimizer for the energy $\mathcal{E}$ in the class $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$, or in the class $\mathcal{P}(\mathbb{R})$ if $N=1$ and we are considering the minimization in $\mathcal{P}(\mathbb{R})$.

To do so, let μ be any other probability measure, in $\mathcal{P}(\mathbb{R}^N)$ or in $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$, and let us try to show that $\mathcal{E}(\bar\mu)\le\mathcal{E}(\mu)$. First of all, we observe that, since (1.2) does not hold for g, then $\lim_{t\searrow 0}g'(t)t^N=0$, and this immediately implies that $\lim_{t\searrow 0}g(t)t^{N-1}=0$. As a consequence, Lemma 1.2.12 gives us a sequence $\{\mu^n\}$ of smooth probability measures which converge weakly* to μ and so that $\mathcal{E}(\mu^n)\to\mathcal{E}(\mu)$. For any fixed $n\in\mathbb{N}$, by lower semicontinuity of the cost, the fact that $g\le g_{\varepsilon_j}$, and that μ_{ε_j} minimizes the energy $\mathcal{E}_{\varepsilon_j}$ in its class, we have

$$\begin{aligned}\mathcal{E}(\bar\mu) &\le \liminf_{j\to\infty}\mathcal{E}(\mu_{\varepsilon_j}) \le \liminf_{j\to\infty}\mathcal{E}_{\varepsilon_j}(\mu_{\varepsilon_j})\\ &\le \liminf_{j\to\infty}\mathcal{E}_{\varepsilon_j}(\mu^n) = \mathcal{E}(\mu^n) + \liminf_{j\to\infty}\varepsilon_j\mathcal{E}_h(\mu^n) = \mathcal{E}(\mu^n).\end{aligned}\tag{1.25}$$

It is important to notice that the last equality holds because $\varepsilon_j\to 0$ and because $\mathcal{E}_{\bar h}(\mu^n)<+\infty$. This last fact is true because μ^n is a smooth function with compact support, and $\bar h\in L^1_{\rm loc}$. Using directly the measure μ in place of μ^n in the above chain of inequalities would clearly work if $\mathcal{E}_{\bar h}(\mu)<+\infty$, but there is in general no guarantee that this is true, and this is why we had to use the regular measures μ^n. Having observed that $\mathcal{E}(\bar\mu) \leq \mathcal{E}(\mu^n)$ for any generic $n \in \mathbb{N}$, and keeping in mind that $\mathcal{E}(\mu^n) \to \mathcal{E}(\mu)$ when $n \to \infty$, we conclude that $\mathcal{E}(\bar\mu) \leq \mathcal{E}(\mu)$ as required, and this ends the proof under the assumption that (1.2) does not hold.

To conclude the proof, we have only to consider the case when (1.2) holds and $\bar g$ fails to be strictly subharmonic on some $B_{r_{\bar g}} \setminus \{0\}$ (the case when (1.2) holds and $\bar g$ is strictly subharmonic on some $B_{r_{\bar g}} \setminus \{0\}$ has been considered at the beginning). This case is much simpler than the one already studied. Indeed, this time it is enough to define $g_\varepsilon = g + \varepsilon t^2$ and to argue as before. Everything works without difficulties except for the fact that this time it is not necessarily true that $g(t)t^{N-1} \to 0$, so we are not allowed to use Lemma 1.2.12 to obtain the sequence $\{\mu^n\}$. However, there is no need to do so; indeed, as observed above, the only reason to use the sequence $\{\mu^n\}$ was that their regularity guaranteed that $\mathcal{E}_{\bar h}(\mu^n) < +\infty$, while in general $\mathcal{E}_{\bar h}(\mu)$ could have been $+\infty$. But this time, the function $\bar h$ has radial profile $h(t) = t^2$, hence the fact that $\mathcal{E}_{\bar h}(\mu) < +\infty$ is surely true because μ is compactly supported. Then one can directly use μ in place of μ^n in (1.25) without using Lemma 1.2.12. The proof is then finished. □

The proof of Theorem 1.2.1 is then concluded. Indeed, case (1) is considered in Lemma 1.2.10; case (2) follows from case (1) since the assumptions guarantee that the optimal measures have convex support by Proposition 1.2.2 and Proposition 1.2.4; and case (3) is considered in Lemma 1.2.13.

1.3 Continuity of bounded critical points in $1D$

This section is devoted to the study of the 1-dimensional minimizers of (P_M), and we aim to improve the results of Section 1.2 showing that the minimizers are actually continuous inside their support. The fundamental hypotheses concerning the kernel $\bar g : \mathbb{R} \to \mathbb{R}^+ \cup \{+\infty\}$ for our result are collected in

($\mathbf{H}_c$) $\bar g(x) = g(|x|)$, with $g \in C^1((0,+\infty)) \cap L^1_{\rm loc}(\mathbb{R})$, $\lim_{x\to\infty} \bar g(x) = +\infty$, and also $\lim_{x\to 0} \bar g(x) = \bar g(0)$. Moreover, there exists a length-scale $r_{\bar g} > 0$ such that g is decreasing in $(0, r_{\bar g})$, g is convex in $(0, r_{\bar g})$, $g' \in BV_{\rm loc}((0,+\infty))$, g' is concave in $(0, r_{\bar g})$, and there exists $\Lambda \in (1,+\infty]$ such that

$$\liminf_{x\to 0^+} \frac{|g'(x/2)|}{|g'(x)|} = \Lambda.$$

Our theorem is the following:

Theorem 1.3.1 (Continuity in dimension 1). *Let $\bar g : \mathbb{R} \to \mathbb{R}^+ \cup \{+\infty\}$ be a function satisfying conditions ($\mathbf{H}_c$). If $f\mathscr{L}^1 \in \mathcal{P}(\mathbb{R})$ is a probability measure with* spt $f = [a,b]$ *for some $a, b \in \mathbb{R}$, $\|f\|_\infty = M < +\infty$ and ψ_f is constant $\mathscr{L}^1$-a.e. in (a,b), then there exists a representative in the Lebesgue class of f that is continuous in (a,b).*

We point out that we address only the interior regularity, while we do not treat the points on the boundary of the support of the measures that we work with. In higher dimension, it can actually happen that a minimizer is continuous inside the support, but not on the boundary. For example, if the space dimension is $N \geq 3$, and $\bar{g} = \mathfrak{g}_2 - \mathfrak{g}_{2-N}$, it is well known that the minimizer of (P_M) is a multiple of the characteristic function of a ball (see for example [CDM16, BCT18]).

Remark 1.3.2. We recall again the reference [CDM16], where the authors show that the minimizers are Hölder continuous in some cases using PDE techniques. It would be interesting to understand whether it is possible to push our result further. In particular, the last part of our proof might yield some improvement if we manage to adapt the construction and produce artificial oscillations when, roughly speaking, we see two very different moduli of continuity on the left and on the right of a point $x \in (a, b)$.

In Subsection 1.3.1 we collect the very basic concepts needed for our analysis like the notion of essential limits and the second order approach to this problem. Then, in Subsection 1.3.2 we provide some lemmas that show the cancellation phenomena due to the convexity properties that we assume on the kernel. Subsection 1.3.3 serves to fix some parameters, and identifies a convenient starting point for the proof of the main result of this section, i.e. Theorem 1.3.1, that is contained in the last subsection. The proof is divided in various steps, depending on the different situations that are identified in Subsection 1.3.3. The overall strategy can be summarized as follows: if a minimizer $f\mathscr{L}^1$ is not continuous, then it should oscillate, and therefore also $f * \varphi_\delta$ has the same property when φ_δ is a smooth mollifier very concentrated around the origin. This, however, is not compatible with the constancy of the potential ψ_{f_δ} inside the support. In the end, the convolution is very convenient because ψ_f behaves well with respect to this operation. Additionally, most of our computations work well with smooth functions, and thus the convolution is a good choice to make our procedure rigorous.

1.3.1 Setting and preliminary results

In this chapter we suppose that the interaction kernel $\bar{g} : \mathbb{R} \to \mathbb{R}^+ \cup \{+\infty\}$ satisfies the conditions listed in ($\mathbf{H}_c$). We notice that any prototypical kernel presented in (1), which are of the form

$$\bar{g} = \bar{g}_p = \mathfrak{g}_\alpha - \mathfrak{g}_\beta + \mathfrak{g}_\beta(1) - \mathfrak{g}_\alpha(1),$$

with $\alpha > 0$ and $-1 < \beta < \min\{1, \alpha\}$, satisfies our hypotheses, and we recall that $\mathfrak{g}_0(x) = \log|x|$.

The starting point for this more refined analysis relies, of course, on the results presented in Section 1.2. We collect in Theorem 1.3.3 the minimal information that we need in the sequel:

Theorem 1.3.3 (Theorem 1.2.1 and Proposition 1.2.2)**.** *Let $\bar{g} : \mathbb{R} \to \mathbb{R}^+ \cup \{+\infty\}$ be a symmetric function such that $\lim_{x\to+\infty} \bar{g}(x) = +\infty$. If, in addition, it satisfies ($\mathbf{H}_p$), it is strictly convex in $(0, r_{\bar{g}})$, and it is convex in the whole half-line $(0, +\infty)$, then any minimizer $\mu \in \mathcal{P}(\mathbb{R})$ of (P_M) is of class L^∞ and $\operatorname{spt}\mu = [a, b]$ for some $a, b \in \mathbb{R}$. Moreover, $b - a$ and $\|\mu\|_\infty$ are controlled from above by a constant depending only on the kernel $\bar{g}$.*

Definition 1.3.4 (Essential directional limits)**.** Given a function $F:\mathbb{R}\to\mathbb{R}$ and $\bar{x}\in\mathbb{R}$, we say that $l\in\mathbb{R}$ is the *essential liminf from the left* of F at $\bar{x}$ if for every $\varepsilon>0$ there exists $\eta>0$ such that

$$\begin{cases}\mathscr{L}^1(\{F<l-\varepsilon\}\cap(\bar{x}-\eta,\bar{x}))=0\\ \mathscr{L}^1(\{F<l+\varepsilon\}\cap(\bar{x}-\eta,\bar{x}))>0\end{cases}.$$

In this case, we write that $l=\operatorname{ess-lim\,inf}_{t\to\bar{x}^-}F(t)$. Similar definitions can be given for the essential limsup and for the limits from the right. If all of the essential limits coincide, then we say that F admits essential limit at $\bar{x}$.

Second order argument

Roughly speaking, we are going to study the second derivative of ψ_f, and we will see that it cannot be 0 when f is not continuous. More precisely, we will regularize f in order to work with a smooth potential. In order to do that, we fix a symmetric mollifier $\varphi\in C_c^\infty(\mathbb{R})$ such that $\operatorname{spt}\varphi=[-1,1]$, $\|\varphi\|_1=1$, $\|\varphi'\|_\infty\le 3$, and $\varphi'\le 0$ in $(0,1)$. Then, we consider the smooth function $f_\delta=f*\varphi_\delta$, which has compact support, and it has constant (and smooth) potential ψ_{f_δ} in $(a+\delta,b-\delta)$. The idea is that, if f is not continuous, then ψ''_{f_δ} is not 0 at some critical points of f_δ. The role of the critical points is just technical. In fact, we will use an alternative formula for ψ''_{f_δ} obtained integrating by parts, that can be justified only if the point where we compute that derivative is a critical point of f_δ. Of course, this contradicts the constancy of ψ_{f_δ} in $(a+\delta,b-\delta)$, and provides the thesis.
The integration by parts can be performed for a general smooth function F. In fact, let $F\in C_c^\infty(\mathbb{R})$ be given, and let x be a critical point for F. Then we can justify an integration by parts and arrive to an alternative expression for $\psi''_F(x)$:

$$\begin{aligned}\psi''_F(x)&=\int F''(t)\bar{g}(x-t)dt=-\int_{\mathbb{R}}F'(t)\frac{d}{dt}\bar{g}(t-x)dt\\&=\int_{\mathbb{R}}\operatorname{sgn}(x-t)F'(t)\bar{g}'(|x-t|)dt.\end{aligned}\tag{1.26}$$

Notice that the expressions containing only first derivatives are well defined since x is a critical point of F, because in this case $|F'(t)|\lesssim|t-x|$ and we can apply Lemma 1.1.6.

1.3.2 Cancellation lemmas

We are going to manipulate the expression (1.26), and we obtain some inequalities for the contribution due to the integral between two critical points in that expression.

Lemma 1.3.5. *Let $\bar{g}:\mathbb{R}\to\mathbb{R}^+\cup\{+\infty\}$ be a function satisfying ($\mathbf{H}_c$). Let $\alpha,\beta\in\mathbb{R}$ be given, with $\alpha<\beta$ and $\beta-\alpha<r_{\bar{g}}$. Let $F\in C^2([\alpha,\beta])$ be a function with $F(\alpha)=F(\beta)$, and such that α and β are absolute minimum points of F in $[\alpha,\beta]$. If $F'(\beta)=0$, then for every $x\ge\beta$ with $x-\alpha<r_{\bar{g}}$ we have that*

$$\int_\alpha^\beta F'(t)\bar{g}'(x-t)dt\ge 0.$$

If, instead, $F'(\alpha) = 0$*, then for every* $x \le \alpha$ *with* $\beta - x < r_{\bar{g}}$ *we have that*

$$\int_{\alpha}^{\beta} -F'(t)\bar{g}'(t-x)dt \ge 0.$$

Proof. First of all, we observe that Lemma 1.1.6 guarantees that the integral is finite. It is also easy to check that the second inequality can be deduced from the first one considering the function $G(t) = F(-t)$ defined in the interval $[-\beta, -\alpha]$, so we will just prove the first one. Moreover, it is sufficient to prove the result when F' chages sign only once: the general result can be obtained approximating F with functions whose derivative has a finite number of sign changes (see the proof of Lemma 1.3.7, where this procedure is slightly more complex). Therefore, we need to prove the result when there exists $\xi \in (\alpha, \beta)$ such that $F_1 = F|_{(\alpha,\xi)}$ is monotone increasing, $F_2 = F|_{(\xi,\beta)}$ is monotone decreasing, and $F' \neq 0$ in $(\alpha, \beta) \setminus \{\xi\}$. With this reduction, we use the change of variables $z = F_1(t)$ and $w = F_2(t)$ to get that

$$\begin{aligned}\int_{\alpha}^{\beta} F'(t)\bar{g}'(x-t)dt &= \int_{\alpha}^{\xi} F_1'(t)\bar{g}'(x-t)dt + \int_{\xi}^{\beta} F_2'(t)\bar{g}'(x-t)dt \\ &= \int_{F_1(\alpha)}^{F_1(\xi)} \bar{g}'(x - F_1^{-1}(z))dz - \int_{F_2(\beta)}^{F_2(\xi)} \bar{g}'(x - F_2^{-1}(w))dw \\ &= \int_{F(\alpha)}^{F(\xi)} \left[\bar{g}'(x - F_1^{-1}(z)) - \bar{g}'(x - F_2^{-1}(z))\right] dz.\end{aligned}$$

For every $z \in [F(\alpha), F(\xi)]$ we have that $x - F_2^{-1}(z) \le x - F_1^{-1}(z) < r$, and since g is convex in $(0, r_{\bar{g}})$, then the function inside the integral is non-negative, concluding the proof. □

Lemma 1.3.6. *Let* $\bar{g} : \mathbb{R} \to \mathbb{R}^+ \cup \{+\infty\}$ *satisfy* ($\mathbf{H}_c$)*. Let* $\alpha, \beta \in \mathbb{R}$ *be given, with* $\alpha < \beta$ *and* $\beta - \alpha < r_{\bar{g}}$*. If* $F \in C^2([\alpha, \beta])$ *and* α *is an absolute minimum point of* F *in* $[\alpha, \beta]$*, then for every* $x \le y < \alpha$ *with* $\beta - x < r_{\bar{g}}$ *we have that*

$$\int_{\alpha}^{\beta} F'(t)(\bar{g}'(t-x) - \bar{g}'(t-y))dt \ge 0.$$

Proof. We use a similar cancellation principle compared to Lemma 1.3.5, this time relying on the concavity of g'. As before, by approximation it is sufficient to prove the result when there exists $\xi \in (\alpha, \beta)$ such that $F_1 = F|_{(\alpha,\xi)}$ is monotone increasing, $F_2 = F|_{(\xi,\beta)}$ is monotone decreasing, and $F' \neq 0$ in $(\alpha, \beta) \setminus \{\xi\}$. Using the change of varibles $z = F_1(t)$ and $w = F_2(t)$ we arrive to

$$\begin{aligned}\int_{\alpha}^{\beta} F'(t)(\bar{g}'(t-x) - \bar{g}'(t-y))dt &= \int_{F_1(\alpha)}^{F_1(\xi)} \left[\bar{g}'(F_1^{-1}(z) - x) - \bar{g}'(F_1^{-1}(z) - y)dz\right] \\ &\quad - \int_{F_2(\beta)}^{F_2(\xi)} \left[\bar{g}'(F_2^{-1}(w) - x) - \bar{g}'(F_2^{-1}(w) - y)dw\right] \\ &= \int_{F(\alpha)}^{F(\xi)} \left[\bar{g}'(F_1^{-1}(z) - x) - \bar{g}'(F_2^{-1}(z) - x)\right] dz \\ &\quad - \int_{F(\alpha)}^{F(\xi)} \left[\bar{g}'(F_1^{-1}(z) - y) - \bar{g}'(F_2^{-1}(z) - y)\right] dz.\end{aligned}$$

For any $z \in [F(\alpha), F(\xi)]$ we have that $F_1^{-1}(z) - x \leq F_2^{-1}(z) - x$ and $F_1^{-1}(z) - y \leq F_2^{-1}(z) - y$, and since g' is concave in $(0, r_{\bar{g}})$, with $\beta - x < r_{\bar{g}}$, then

$$g'(F_1^{-1}(z)-x)-g'(F_2^{-1}(z)-x) \geq g'(F_1^{-1}(z)-y)-g'(F_2^{-1}(z)-y) \qquad \forall z \in [F(\alpha), F(\xi)],$$

and the inequality is proved. □

Lemma 1.3.7. *Let $\bar{g} : \mathbb{R} \to \mathbb{R}^+ \cup \{+\infty\}$ satisfy ($\mathbf{H}_c$). Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ and $\beta - \alpha = \gamma < r_{\bar{g}}$. If $F \in C^2([\alpha, \beta])$ with $F'(\alpha) = F'(\beta) = 0$ and α and β are respectively an absolute minimum and an absolute maximum of F in that interval, then*

$$\int_\alpha^\beta F'(t)|\bar{g}'(t-\alpha)|dt + \int_\alpha^\beta F'(t)|\bar{g}'(\beta - t)|dt \geq (F(\beta) - F(\alpha))\left(|\bar{g}'(\gamma)| + |\bar{g}'(\gamma/2)|\right). \tag{1.27}$$

Therefore, also the following weaker inequality holds:

$$\max\left\{\int_\alpha^\beta F'(t)|\bar{g}'(\alpha - t)|dt, \int_\alpha^\beta F'(t)|\bar{g}'(\beta - t)|dt\right\} \\ \geq \frac{F(\beta) - F(\alpha)}{2}\left(|\bar{g}'(\gamma)| + |\bar{g}'(\gamma/2)|\right).$$

Proof. First we observe that, since $F' = 0$ in α, β and F' is Lipschitz, then the integrals are finite thanks to Lemma 1.1.6. If $F' \geq 0$ in $[\alpha, \beta]$, then the proof is trivial: since g is convex in $(0, \gamma)$, then

$$\int_\alpha^\beta F'(t)|\bar{g}'(\alpha - t)|dt \geq \int_\alpha^{\frac{\alpha+\beta}{2}} F'(t)|\bar{g}'(\gamma/2)|dt + \int_{\frac{\alpha+\beta}{2}}^\beta F'(t)|\bar{g}'(\gamma)|dt,$$
$$\int_\alpha^\beta F'(t)|\bar{g}'(\beta - t)|dt \geq \int_\alpha^{\frac{\alpha+\beta}{2}} F'(t)|\bar{g}'(\gamma)|dt + \int_{\frac{\alpha+\beta}{2}}^\beta F'(t)|\bar{g}'(\gamma/2)|dt.$$

Adding up these two inequalities one gets the thesis. In general, if F is not monotone, then we are able to show that the contributions due to segments where F is not monotone are non-negative. This proves that the inequality holds for any function F satisfying our hypotheses. For technical reasons, it is better to approximate F by means of functions with controlled oscillations. In fact, we approximate it with a sequence of functions $Q_n \in C^\infty([\alpha, \beta])$ with the following properties:

(a) $Q_n' \to F'$ in $L^\infty([\alpha, \beta])$;

(b) there exists a constant $L < +\infty$ such that $Q_n'(t)|t - \alpha|^{-1}|t - \beta|^{-1} \leq L$ for every $t \in [\alpha, \beta]$ and for every $n \in \mathbb{N}$;

(c) $Q_n(\alpha) \leq Q_n(t) \leq Q_n(\beta)$ for every $t \in [\alpha, \beta]$;

(d) Q_n' has a finite number of sign changes.

If we prove the inequality (1.27) for the functions Q_n, then we get the desired one applying the Dominated Convergence Theorem (conditions *(a)-(b)* guarantee that we can do that). To simplify the notation, we will not work with Q_n, and we directly assume that F' has a finite number of sign changes. We define the function $\tilde{F}$ represented in Figure 1.2, whose expression is

$$\tilde{F}(t) := \begin{cases} \inf\left\{F(s) : s \in \left[t, \frac{\alpha+\beta}{2}\right]\right\} & \text{if } t \in \left[\alpha, \frac{\alpha+\beta}{2}\right] \\ \sup\left\{F(s) : s \in \left[\frac{\alpha+\beta}{2}, t\right]\right\} & \text{if } t \in \left[\frac{\alpha+\beta}{2}, \beta\right] \end{cases},$$

that is non-decreasing, Lipschitz, and $\tilde{F}(t) = F(t)$ for $t = \alpha, \beta, (\alpha+\beta)/2$. We aim to show that

$$\int_\alpha^\beta F'(t)(-\bar{g}'(t-\alpha) - \bar{g}'(\beta-t))dt \geq \int_\alpha^\beta \tilde{F}'(t)(-\bar{g}'(t-\alpha) - \bar{g}'(\beta-t))dt, \tag{1.28}$$

where we replaced $|\bar{g}'|$ with $-\bar{g}'$ since $\bar{g}$ is decreasing in $(0, r_{\bar{g}})$ and $\beta - \alpha = \gamma < r_{\bar{g}}$. In fact, suppose for a moment that this inequality holds. Then, we observe that our initial argument for monotone functions used only the Fundamental Theorem of Calculus, that is available also for $\tilde{F}$ since it is Lipschitz continuous. In other words, (1.27) holds for $\tilde{F}$. Finally, $F = \tilde{F}$ in α and β, hence using (1.28) we obtain (1.27) also for F. Therefore, it is sufficient to prove (1.28) to conclude the proof. Notice that, since F' has a fininte number of sign changes, then $\tilde{F}$ can be obtained applying the following operations a finite number of times:

1. find (if they exist) $\alpha' < \xi' < \beta'$ three points in $[\alpha, (\alpha+\beta)/2]$ with $F(\alpha') = F(\beta') < F(\xi')$, $F' \geq 0$ in $[\alpha', \xi']$ and $F' \leq 0$ in $[\xi', \beta']$, and either α' or β' is a local minimum for F. Then replace F with $\bar{F}$ defined as

$$\bar{F}(t) = \begin{cases} F(\alpha') & \text{if } t \in [\alpha', \beta'] \\ F(t) & \text{otherwise} \end{cases};$$

2. find (if they exist) $\alpha'' < \xi'' < \beta''$ three points in $[(\alpha+\beta)/2, \beta]$ with $F(\alpha'') = F(\beta'') > F(\xi'')$, $F' \leq 0$ in $[\alpha'', \xi'']$ and $F' \geq 0$ in $[\xi'', \beta'']$, and either α'' or β'' is a local maximum for F. Then replace F with $\bar{F}$ defined as

$$\bar{F}(t) = \begin{cases} F(\alpha'') & \text{if } t \in [\alpha'', \beta''] \\ F(t) & \text{otherwise} \end{cases};$$

Thanks to this observation, it is sufficient to prove that each step does not ruin the inequality. We are going to show this only for the first step, since the successive one are perfectly similar, and then (1.28) follows iterating the following inequality:

$$\int_\alpha^\beta F'(t)(-\bar{g}'(t-\alpha) - \bar{g}'(\beta-t))dt \geq \int_\alpha^\beta \bar{F}'(t)(-\bar{g}'(t-\alpha) - \bar{g}'(\beta-t))dt.$$

That inequality is a consequence of the concavity of g', and we prove it under the additional assumption that $\bar{F} = F$ in $[(\alpha+\beta)/2, \beta]$ since this does not affect our argument. Of course, the integral outside of $[\alpha', \beta']$ is not altered, so we consider only the smaller interval. With this reduction, we define $F_1 = F|_{[\alpha',\xi']}$ and $F_2 = F|_{[\xi',\beta']}$ that are monotone functions, and we use the change of variables $z = F_1(t)$ and $w = F_2(t)$ to see that

$$\begin{aligned}\int_{\alpha'}^{\beta'} F'(t)(-\bar{g}'(t-\alpha) - \bar{g}'(\beta - t))dt &= \int_{F(\alpha')}^{F(\xi')} \left[-\bar{g}'(F_1^{-1}(z) - \alpha) - \bar{g}'(\beta - F_1^{-1}(z))\right] dz \\ &\quad + \int_{F(\xi')}^{F(\beta')} \left[-\bar{g}'(F_2^{-1}(w) - \alpha) - \bar{g}'(\beta - F_2^{-1}(w))\right] dw \\ &= \int_{F(\alpha')}^{F(\xi')} \left[\bar{g}'(F_2^{-1}(z) - \alpha) - \bar{g}'(F_1^{-1}(z) - \alpha)\right] dz \\ &\quad - \int_{F(\alpha')}^{F(\xi')} \left[\bar{g}'(\beta - F_1^{-1}(z)) - \bar{g}'(\beta - F_2^{-1}(z))\right] dz\end{aligned} \tag{1.29}$$

Since $\alpha', \beta', \xi' \in [\alpha, (\alpha+\beta)/2]$, then

$$F_1^{-1}(z) - \alpha \le F_2^{-1}(z) - \alpha \le \beta - F_2^{-1}(z) \le \beta - F_1^{-1}(z) \qquad \forall z \in [F(\alpha'), F(\xi')],$$

hence the expression in (1.29) is non-negative thanks to the concavity of g'. $\square$

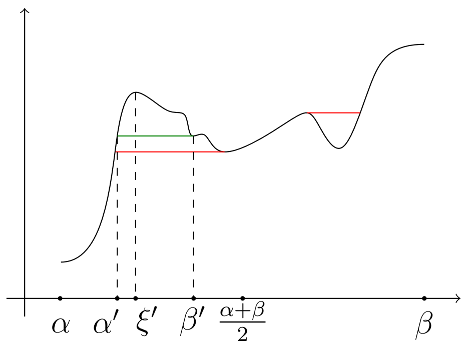


Figure 1.2: In black we represent the function F, while we draw in red the parts of $\tilde{F}$ that are different from F. The points α', β', ξ' are used to build $\bar{F}$, that is represented in green here.

Remark 1.3.8. It is immediate to notice that Lemma 1.3.7 holds also when α is a maximum point, and β is a minimum, rewriting the first inequality as

$$\begin{aligned}(F(\beta) - F(\alpha)) &\left(\int_\alpha^\beta F'(t)|\bar{g}'(t-\alpha)|dt + \int_\alpha^\beta F'(t)|\bar{g}'(\beta - t)|dt\right) \\ &\ge (F(\beta) - F(\alpha))^2 \left(|\bar{g}'(\gamma)| + |\bar{g}'(\gamma/2)|\right),\end{aligned}$$

that is valid for any function $F \in C^2([\alpha, \beta])$ which attains its extremal values in α and β. The second inequality in the statement of that lemma can be adapted in a similar way. We additionally remark that, if the second inequality in Lemma 1.3.7 holds with basepoint $p \in \{\alpha, \beta\}$ that is a minimum for F, then we can rewrite it as

$$-\int_{\alpha}^{\beta} F'(t) \frac{d}{dt} \bar{g}(|p-t|) dt \geq \frac{|F(\beta) - F(\alpha)|}{2} \left(|\bar{g}'(\gamma)| + |\bar{g}'(\gamma/2)| \right).$$

When p is a maximum point, instead, the inequality holds after changing the sign to the left hand side.

1.3.3 Selection of critical points

We are going to prove that f admits essential limit at every point in (a, b), and to simplify the notation we prove this only for the point 0 (that belongs to (a, b) up to translations). To further simplify our expressions, we use the notation

$$l_L^-(f) = \operatorname{ess-}\liminf_{t \to 0^-} f(t) \qquad l_L^+(f) = \operatorname{ess-}\limsup_{t \to 0^-} f(t),$$

and $l_R^\pm(f)$ have analogous definitions, while $h_L(f) = l_L^+(f) - l_L^-(f)$ and $h_R(f) = l_R^+(f) - l_R^-(f)$. We define the auxiliary quantity $\bar{\Lambda} = \min\{2, (1+\Lambda)/2\} > 1$ and the non-negative function $V : [0, 1] \to [0, 1]$ defined as

$$V(t) := 1 - 2 \int_t^1 \varphi(s) ds = \int_{-1}^t \varphi(s) ds - \int_t^1 \varphi(s) ds = \int_{-t}^t \varphi(s) ds. \tag{1.30}$$

If the essential limit from the left does not exist, then f has to oscillate when we approach 0 from the left, and if we take δ small enough then f_δ oscillates as well. With this observation in mind, we fix a sequence of alternating critical points for f_δ, and among those we fix a good pair that will play the role of base points in our proof. We fix these points only in two specific (more symmetric) cases, since we are able to reduce to those situations. We introduce some additional parameters, that are redundant for now, because they will play a role at the very end of the proof of Theorem 1.3.1: $h_L := h_L(f)$, $l_L^\pm := l_L^\pm(f)$ and $l_R^\pm := l_R^\pm(f)$. In fact, our construction does not necessarily need that they coincide with the limits of f, while it is easier to understand the procedure thinking about them as limits. In the sequel we are going to impose a lot of constraints on a number of parameters. Even though this part is technical, and it is not clear at a first sight how we are going to use them, it is important to show that we can impose the bounds a priori, and then let the convolution parameter go to 0.

1. In the first case, we suppose that $f(x) = f(-x)$ and that $h_L > 0$. We fix the parameters $\varepsilon > 0$ and $\eta \in (0, r_{\bar{g}}/4)$ such that

$$\varepsilon < \frac{(\bar{\Lambda} - 1) h_L}{16 \bar{\Lambda} + 10} < \frac{h_L}{4}, \qquad \begin{cases} \mathscr{L}^1(\{f < l_L^- - \varepsilon\} \cap (-\eta, 0)) = 0 \\ \mathscr{L}^1(\{f < l_L^- + \varepsilon\} \cap (-\eta, 0)) > 0 \end{cases}.$$

   We choose any point $p_1 \in (-\eta/8, 0)$ with $f_\delta(p_1) < l_L^- + \varepsilon$. Then we define p_2 as the largest $x < p_1$ such that $f_\delta(x) > l_L^+ - \varepsilon$. Notice that, if we choose the convolution parameter δ small enough, we can guarantee that p_1 exists and that $p_2 \in (-\eta/4, 0)$. Now we replace p_1 with any minimum point of f_δ in $[p_2, 0]$. Then we define p_3 as the largest point $x < p_2$ such that $f_\delta(x) < l_L^- + \varepsilon$. Now we substitute p_2 with any maximum point of f_δ in $[p_3, p_1]$. We go on in this way, alternating maximum and minimum points for f_δ, until we reach $p_{H+1} < -\eta$. Of course, by construction we have that p_{2i} is a maximum for f_δ in $[p_{2i+1}, p_{2i-1}]$ and p_{2i+1} is a minimum for f_δ in $[p_{2i+2}, p_{2i}]$ for every index i where the previous points are defined. Finally, we define the points $q_i = -p_{i+1}$, so that the sequence $\{p_H, \dots, p_1, q_1, \dots, q_{H-1}\}$ is an alternating sequence of maximum and minimum points.

2. In the second case we suppose that $f(x) = -f(-x)$, $h_L > 0$ and $l_L^- < \min\{0, l_R^-\}$. We fix $\varepsilon > 0$ and $\eta \in (0, r_{\bar{g}}/4)$ such that

$$\begin{aligned} &\varepsilon < \min\left\{\frac{(\bar{\Lambda}-1)h_L}{16\bar{\Lambda}+10}, \frac{l_R^- - l_L^-}{3}, \frac{|l_L^-|}{2}\right\} < \frac{h_L}{4}, \\ &\frac{1-\varepsilon/|l_L^-|}{1+\varepsilon/|l_L^-|} > V\left(\max\left\{1-\frac{h_L}{6}, 0\right\}\right), \\ &\begin{cases} \mathscr{L}^1(\{f < l_L^- - \varepsilon\} \cap (-\eta, 0)) = 0 \\ \mathscr{L}^1(\{f < l_L^- + \varepsilon\} \cap (-\eta, 0)) > 0 \end{cases}. \end{aligned} \tag{1.31}$$

Since the function is antisymmetric and φ is symmetric, then f_δ is antisymmetric and we can define p_1 as the largest $x < 0$ such that $f_\delta(x) < l_L^- + \varepsilon$. Then define p_2 as the largest $x < p_1$ such that $f_\delta(x) > l_L^+ - \varepsilon$. Now redefine p_1 as an absolute minimum of f_δ in $[p_2, 0]$. Successively, define p_3 as the largest point $x < p_2$ such that $f_\delta(x) < l_L^- + \varepsilon$ and redefine p_2 as an absolute maximum of f_δ in $[p_3, p_1]$. Continue in this way, alternating maximum and minimum points, until $p_{H+1} < -\eta$. Then we call $q_i \coloneqq -p_i$, where the maximality and minimality properties are reverted with respect to p_i since f_δ is antisymmetric.

Up to now, δ is essentially a free parameter, and we choose it small enough in order to have those critical points close to each other. In particular, we take it so small that the previous procedure goes on at least up to $H = 10$. Moreover, if $D = b - a$, we define the constant $C(\eta, D, M, \bar{g})$ as

$$C(\eta, D, M, \bar{g}) \coloneqq 20M\left\{\|\bar{g}''\|\,([\eta/2, 2D]) + 2\sup_{\eta/2 \le t \le 2D} |\bar{g}'(t)|\right\}, \tag{1.32}$$

and we additionally impose that $\delta < \frac{1}{4}\min\{|a|, |b|\}$ is so small that

$$\bar{\gamma} = \max\{q_1 - p_1, p_1 - p_2\} < \frac{\eta}{8} \qquad \text{and} \qquad \varepsilon|\bar{g}'(\bar{\gamma}/2)| \ge \varepsilon|\bar{g}'(\bar{\gamma})| \ge C(\eta, D, M, \bar{g}). \tag{1.33}$$

We select a good pair of consecutive critical points among $\{p_H, \dots, p_1, q_1, \dots, q_{H-1}\}$, so we implicitly suppose that f falls into one of the previous two categories. We claim that there exists an index $j < H$ such that the following conditions hold:

$$p_{j+1} - p_{j+2} \geq \frac{p_j - p_{j+1}}{2} \tag{1.34}$$

$$p_{j-1} - p_j \geq \frac{p_j - p_{j+1}}{2} \qquad \text{if } j > 1. \tag{1.35}$$

We begin considering $j = 1$. If $p_2 - p_3 \geq (p_1 - p_2)/2$ we are done, otherwise we pass to consider $j = 2$. In this case $p_1 - p_2 > 2(p_2 - p_3) > (p_2 - p_3)/2$, and therefore condition (1.35) is satisfied and we need to check only (1.34). This, however, is precisely what we did for $j = 1$, and thus we can proceed inductively. We notice that we cannot arrive to consider an index j such that $p_{j+2} < -\eta/2$: if we did, then

$$\frac{\eta}{2} \leq p_1 - p_{j+2} + |p_1| = \sum_{l=1}^{j+1}(p_l - p_{l+1}) + |p_1| \leq (p_1 - p_2)\sum_{l=1}^{j+1} 2^{-l} + \frac{\eta}{8} \leq 2\bar{\gamma} + \frac{\eta}{8}, \tag{1.36}$$

and this is impossible since $\bar{\gamma} < \eta/8$. Hence, the inductive procedure has to stop at some $j < H$ and $p_{j+1} > -\eta/2$. If the chosen j is strictly larger than 1, then the good pair is $\{p_{j+1}, p_j\}$. If instead $j = 1$, then there are two possibilities: either f is symmetric, and in that case we choose $\{p_{j+1}, p_j\}$ as before, or f is antisymmetric, and in this case we choose the pair which defines the shortest segment between $[p_2, p_1]$ and $[p_1, q_1]$. Through this construction we spotted a segment (whose endpoints form the chosen pair of points) that is not longer than twice the length of its neighbouring segments in the family that we are considering. Notice that this is true also if the good pair is $\{p_2, p_1\}$ and f is even. In fact, in this case we defined $q_i = -p_{i+1}$, and thus $q_1 - p_1 \geq p_1 - p_2 \geq \frac{1}{2}(p_1 - p_2)$. Finally, we define γ as the length of the segment whose endpoints form the chosen good pair, i.e. $\gamma = p_j - p_{j+1}$ in the usual case, or $\gamma = q_1 - p_1$ when we choose the pair $\{p_1, q_1\}$.

1.3.4 Proof of Theorem 1.3.1

We are going to see that f admits essential limit at *every* point $x \in (a, b)$. It is necessary to use this weaker definition of limits since f is not pointwise defined. This, however, is sufficient to prove that there exists a continuous representative of f. In fact, if we suppose that f admits essential limit at every $x \in (a, b)$, and we define $\bar{f}(x) = \text{ess-lim}_{t\to x} f(t)$, then $\bar{f}$ is continuous in (a, b). To check this, let us take $x \in (a, b)$, and for every $\varepsilon_1 > 0$ there exists $\eta_1 > 0$ such that $\bar{f}(x) - \varepsilon_1 < f(y) < \bar{f}(x) + \varepsilon_1$ for a.e. $y \in (x - \eta_1, x + \eta_1)$. Therefore, for every y in that interval we have that

$$\bar{f}(y) = \underset{t\to y}{\text{ess-lim}}\, f(t) \in [\bar{f}(x) - \varepsilon_1, \bar{f}(x) + \varepsilon_1],$$

and this is the definition of continuity of $\bar{f}$. It is also easy to check that

$$\bar{f}(x) = \lim_{s\to 0} \frac{1}{2s} \int_{x-s}^{x+s} f(y)dy,$$

and the Lebesgue differentiation theorem guarantees that $f = \bar{f}$ a.e. in (a, b).

As we already pointed out in Subsection 1.3.3, the point x does not play any role, and thus we simply prove that f admits essential limit at $x = 0 \in (a, b)$. Moreover, if f has constant potential in a neighborhood of 0, then the same holds for its symmetric and antisymmetric part, i.e. $\frac{f(x)+f(-x)}{2}$ and $\frac{f(x)-f(-x)}{2}$, so we just need to prove the existence of the essential limit under the additional hypothesis that f is either even or odd. To simplify the notation later on, we denote by $D = b - a$ the diameter of spt f.

When f is even In this case we can suppose without loss of generality that $h_L(f) > 0$: otherwise $l_L^+(f) = l_L^-(f) = l_R^-(f) = l_R^+(f)$ and we already know that the essential limit exists. We recall that, in Subsection 1.3.3, we introduced the simplified notation $h_L := h_L(f)$, $l_L^\pm := l_L^\pm(f)$ and $l_R^\pm := l_R^\pm(f)$. We fix the parameters ε, η and δ as we did in Subsection 1.3.3, and we consider the critical points $\{p_H, \dots, p_1, q_1, \dots, q_{H-1}\}$ selected there, together with the good pair $\{p_{j+1}, p_j\}$. We suppose that p_{j+1} is a minimum point for f_δ (or, equivalently, that j is even), and that the second part of Lemma 1.3.7 applied to f_δ in the interval $[p_{j+1}, p_j]$ tells us that

$$\begin{aligned} -\int_{p_{j+1}}^{p_j} f_\delta'(t)\frac{d}{dt}\bar{g}(t-p)dt &= \int_{p_{j+1}}^{p_j} f_\delta'(t)|\bar{g}'(t-p)|dt \\ &\geq \frac{|f_\delta(p_j) - f_\delta(p_{j+1})|}{2}(|\bar{g}'(\gamma)| + |\bar{g}'(\gamma/2)|), \end{aligned} \tag{1.37}$$

where $p = p_{j+1}$ and $\gamma = p_j - p_{j+1}$. Notice that Lemma 1.3.7 can be applied since, by construction, $\gamma \leq \bar{\gamma} < \eta/8 < r_{\bar{g}}$. Now we consider the quantity (1.26) with $x = p_{j+1}$ and $F = f_\delta$ and we split the integral in three parts:

$$\begin{aligned} -\int_a^b f_\delta'(t)\frac{d}{dt}\bar{g}(|p_{j+1}-t|)dt &= \int_a^{p_{j+1}} f_\delta'(t)\bar{g}'(p_{j+1}-t)dt - \int_{p_{j+1}}^{p_j} f_\delta'(t)\bar{g}'(t-p_{j+1})dt \\ &\quad - \int_{p_j}^b f_\delta'(t)\bar{g}'(t-p_{j+1})dt = I + J + K. \end{aligned} \tag{1.38}$$

Applying the aforementioned lemma we obtain the leading term in our estimate:

$$J \geq \frac{h_L - 2\varepsilon}{2}(|\bar{g}'(\gamma)| + |\bar{g}'(\gamma/2)|). \tag{1.39}$$

We treat the first term in (1.38), namely I, introducing the closed set $\mathcal{Z} \subset [-\eta, p_{j+1}]$ defined as

$$\mathcal{Z} = \{t \in [-\eta, p_{j+1}] : \ f_\delta(t) \leq f_\delta(s) \ \forall s \in [t, p_{j+1}]\}, \tag{1.40}$$

that is depicted in Figure 1.3. We point out that $f_\delta|_{\mathcal{Z}}$ is increasing. Writing $(-\eta, p_{j+1}) \setminus \mathcal{Z} = \bigcup_{k\in\mathbb{N}} A_k$, where $A_k = (a_k, b_k)$ are open segments, it is easy to see that $f_\delta(a_k) = f_\delta(b_k)$ and $f_\delta'(b_k) = 0$. Therefore, we apply Lemma 1.3.5 to each segment A_k and obtain that

$$\int_{(-\eta,p_{j+1})\setminus\mathcal{Z}} f_\delta'(t)\bar{g}'(p_{j+1}-t)dt \geq 0.$$

Roughly speaking, the relevant domain is then $\tilde{\mathcal{Z}} = \mathcal{Z} \cap \{f'_\delta > 0\}$, and we use the change of variables induced by the monotone function $\bar{F} = f_\delta|_{\tilde{\mathcal{Z}}}$ to get that

$$\begin{aligned} I &\geq \int_a^{-\eta} f'_\delta(t)\bar{g}'(p_{j+1}-t)dt + \int_{\mathcal{Z}} f'_\delta(t)\bar{g}'(p_{j+1}-t)dt \\ &= \int_a^{-\eta} f'_\delta(t)\bar{g}'(p_{j+1}-t)dt + \int_{\tilde{\mathcal{Z}}} f'_\delta(t)\bar{g}'(p_{j+1}-t)dt \\ &= \int_a^{-\eta} f'_\delta(t)\bar{g}'(p_{j+1}-t)dt + \int_{\bar{F}(\tilde{\mathcal{Z}})} \bar{g}'(p_{j+1}-\bar{F}^{-1}(y))dy. \end{aligned} \tag{1.41}$$

The first integral is controlled in absolute value by the constant $C(\eta, D, M, \bar{g})$ defined in (1.32) because g'' is a locally finite measure and $|f_\delta| \leq M$. In fact, we know that $p_{j+1} \geq -\eta/2$ thanks to (1.36), and denoting by $h = \bar{g}'\chi_{p_{j+1}+[\eta,-a]}$, with h' its distributional derivative, we have that

$$\begin{aligned} \left| \int_a^{-\eta} f'_\delta(t)\bar{g}'(p_{j+1}-t)dt \right| &= \left| \int_{\mathbb{R}} f'_\delta(t)h(p_{j+1}-t)dt \right| \\ &\leq (\sup f_\delta) \int_{\mathbb{R}} d|h'| \\ &\leq 20M \left\{ \|\bar{g}''\|\left([\eta/2, 2D]\right) + 2 \sup_{\eta/2 \leq t \leq 2D} |\bar{g}'(t)| \right\} \\ &= C(\eta, D, M, \bar{g}), \end{aligned} \tag{1.42}$$

where the additional factor 20 appears in the definition of $C(\eta, D, M, \bar{g})$ to use the same constant here and at the end of the proof. In the second one, instead, we observe that $\tilde{\mathcal{Z}} \subset [-\eta, p_{j+2}]$, and this happens because we supposed that p_{j+1} was a minimum point for f_δ and, thanks to our construction in Subsection 1.3.3, it is an absolute minimum in $[p_{j+2}, p_j]$. With this observation, we immediately obtain that $p_{j+1} - \bar{F}^{-1}(y) \geq p_{j+1} - p_{j+2}$ for every $y \in \bar{F}(\tilde{\mathcal{Z}})$. We combine various ingredients to bound I from below: $\tilde{\mathcal{Z}} \subset [-\eta, 0]$, where $\eta < r_{\bar{g}}$ and g is convex in $(0, r_{\bar{g}})$, moreover (1.34) guarantees that $p_{j+1} - p_{j+2} \geq (p_j - p_{j+1})/2 = \gamma/2$, and thus

$$I \geq -C(\eta, D, M, \bar{g}) + \int_{\bar{F}(\tilde{\mathcal{Z}})} \bar{g}'(p_{j+1} - p_{j+2})dy \geq -C(\eta, D, M, \bar{g}) + |\bar{F}(\tilde{\mathcal{Z}})|\bar{g}'(\gamma/2). \tag{1.43}$$

Moreover, since the domain of $\bar{F}$ is contained in $[-\eta, 0]$, then $\bar{F}(\tilde{\mathcal{Z}}) \subset [l_L^- - \varepsilon, l_L^- + \varepsilon]$, and thus $I \geq -3\varepsilon|\bar{g}'(\gamma/2)|$, where we used (1.33) to absorb the constant term in the one containing γ and ε. The last term, namely K, can be treated in a similar way. In fact, we can apply the second part of Lemma 1.3.5, showing that the same cancellation phenomenon happens outside of the set

$$\tilde{\mathcal{W}} = \{t \in [p_j, \eta] : \ f_\delta(t) \leq f_\delta(s) \ \forall s \in [p_j, t]\} \cap \{f'_\delta < 0\}, \tag{1.44}$$

yielding to the inequality

$$K \geq -\int_{\tilde{\mathcal{W}}} f'_\delta(t)\bar{g}'(t - p_{j+1})dt - \int_\eta^b f'_\delta(t)\bar{g}'(t - p_{j+1})dt.$$

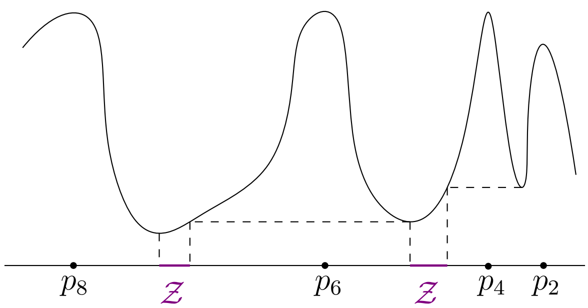


Figure 1.3: We depict f_δ and we emphasize in violet the set $\mathcal{Z}$ defined in (1.40), that coincides with $\tilde{\mathcal{Z}}$ in this case, supposing for simplicity that $j = 2$.

With a computation similar to (1.42), we control the second term in absolute value with the constant $C(\eta, D, M, \bar{g})$. Defining the strictly decreasing function $\hat{F} = f_\delta|_{\tilde{\mathcal{W}}}$, we combine again the change of variable $y = \hat{F}(t)$ with the convexity of $\bar{g}$ in $(0, r_{\bar{g}})$ to see that

$$K \geq \int_{\hat{F}(\tilde{\mathcal{W}})} \bar{g}'(\hat{F}^{-1}(y) - p_{j+1})dy - C(\eta, D, M, \bar{g}) \geq -|\hat{F}(\tilde{\mathcal{W}})||\bar{g}'(\gamma)| - C(\eta, D, M, \bar{g}), \tag{1.45}$$

where we used that $\hat{F}^{-1}(y) - p_{j+1} \geq p_j - p_{j+1} = \gamma$ for every $y \in \hat{F}(\tilde{\mathcal{W}})$ in the second inequality. In the end, we observe that $|\hat{F}(\tilde{\mathcal{W}})| \leq h_L + 2\varepsilon$, and thus we can absorb the constant as we did before and arrive to

$$\begin{aligned} -\int_a^b f_\delta'(t)\frac{d}{dt}\bar{g}(|p_{j+1} - t|)dt &\geq -3\varepsilon|\bar{g}'(\gamma/2)| + \frac{h_L - 2\varepsilon}{2}(|\bar{g}'(\gamma)| + |\bar{g}'(\gamma/2)|) \\ &\quad - (h_L + 3\varepsilon)|\bar{g}'(\gamma)| \\ &= \frac{h_L - 8\varepsilon}{2}|\bar{g}'(\gamma/2)| - \frac{h_L + 8\varepsilon}{2}|\bar{g}'(\gamma)|. \end{aligned}$$

We immediately obtain a contradiction since we chose the parameters ε and δ (and, consequently, γ) so small that $|\bar{g}'(\gamma/2)| > \bar{\Lambda}|\bar{g}'(\gamma)|$ and $(h_L - 8\varepsilon)\bar{\Lambda} - h_L - 8\varepsilon > 0$, while the left hand side is 0.

This almost concludes the proof when f is even. In fact, at the beginning we supposed that p_{j+1} was a minimum point for f_δ and that (1.37) holds with base point p_{j+1}, while this might not be the case in general. Concerning the first issue, if p_{j+1} is a maximum point and (1.37) holds with $p = p_{j+1}$, it is sufficient to consider $-f$ and the procedure works in the same way, even though the points p_{2i+1} become maximum points and those of the form p_{2i} become minimum points. Therefore, we only need to treat the case when p_{j+1} is a minimum point, but (1.37) holds with $p = p_j$ instead of p_{j+1}. Indeed, the previous proof works in a very similar way also in this situation: we consider $-\psi''_{f_\delta}(p_j)$ instead of $\psi''_{f_\delta}(p_{j+1})$, i.e.

$$\begin{aligned}\int_a^b f'_\delta(t)\frac{d}{dt}\bar{g}(|t-p_j|)dt = -\int_a^{p_{j+1}} f'_\delta(t)\bar{g}'(p_j-t)dt - \int_{p_{j+1}}^{p_j} f'_\delta(t)\bar{g}'(p_j-t)dt \\ + \int_{p_j}^b f'_\delta(t)\bar{g}'(t-p_j)dt = \bar{I}+\bar{J}+\bar{K},\end{aligned} \tag{1.46}$$

and the estimate for $\bar{J}$ is exactly (1.37) with $p = p_j$. Instead, one can see that the estimates for $\bar{I}$ and $\bar{K}$ are swapped with respect to the bounds for I and K, and we have that

$$\bar{I} \geq -(h_L+3\varepsilon)|\bar{g}'(\gamma)| \qquad \text{and} \qquad \bar{K} \geq -3\varepsilon|\bar{g}'(\gamma/2)|. \tag{1.47}$$

Plugging these inequalities into (1.46) we arrive to the same contradiction as before.

When f is odd In this second case, for now, we suppose that $h_L(f) > 0$. Moreover, up to changing sign to f, we can suppose that $l_L^-(f) < \min\{0, l_R^-(f)\}$. In fact, if $\min\{l_L^-(f), -l_L^+(f)\} = 0$, then $l_L^-(f) = l_L^+(f) = 0$, and by antisymmetry we know that all the limits are 0, so there is nothing to do. Instead, if $l_L^-(f) = l_R^-(f)$, then one can use the same techniques exploited when f is even to arrive to a contradiction. As before, we use the notation $h_L = h_L(f)$, $l_L^\pm = l_L^\pm(f)$ and $l_R^\pm = l_R^\pm(f)$. We pick the critical points $\{p_H, \ldots, p_1, q_1, \ldots, q_H\}$ defined in Subsection 1.3.3. In many cases, there are no significant differences compared to the situation where f is even, and in those cases we just give a sketch of how to adjust the previous arguments. In the end we will present a more refined analysis that is necessary to complete the proof when f is odd.

If the pair of consecutive critical points is $\{p_1, q_1\}$, then the antisymmetry of f guarantees that

$$\int_{p_1}^{q_1} f'_\delta(t)|\bar{g}'(t-p_1)|dt = \int_{p_1}^{q_1} f'_\delta(t)|\bar{g}'(q_1-t)|dt,$$

and applying Lemma 1.3.7 to f_δ in $[p_1, q_1]$ we obtain that

$$\begin{aligned}\int_{p_1}^{q_1} f'_\delta(t)|\bar{g}'(t-p_1)|dt &\geq \frac{f_\delta(q_1)-f_\delta(p_1)}{2}(|\bar{g}'(\gamma/2)|+|\bar{g}'(\gamma)|) \\ &\geq \frac{l_R^+-l_L^--2\varepsilon}{2}(|\bar{g}'(\gamma/2)|+|\bar{g}'(\gamma)|),\end{aligned}$$

where $\gamma = q_1 - p_1$. Then everything goes on exactly as in the case of $f(x) = f(-x)$ when p_{j+1} is a minimum point and (1.37) holds with $p = p_{j+1}$: notice that our estimate for $\bar{J}$ is stronger than (1.39) since $l_R^+ \geq l_L^+$. If, instead, the good pair is $\{p_{j+1}, p_j\}$ and the estimate for the leading term

$$-\int_{p_{j+1}}^{p_j} f'_\delta(t)\frac{d}{dt}\bar{g}(|p-t|)dt \geq \frac{|f(p_j)-f(p_{j+1})|}{2}(|\bar{g}'(\gamma)|+|\bar{g}'(\gamma/2)|)$$

holds for a minimum point $p \in \{p_{j+1}, p_j\}$, then again there are no significant differences compared to the symmetric case (this bound is deduced from Lemma 1.3.7, as we discuss in Remark 1.3.8). In fact, only the estimates concerning $(0, b)$ are different from before, but we observe that, since $\varepsilon < (l_R^- - l_L^-)/3$, then $f_\delta > l_L^- + \varepsilon$ in $[0, \eta]$. As a consequence, the set $\tilde{\mathcal{W}}$ defined in (1.44) is contained in $[p_j, 0]$, and the estimates work as before.

We have to consider the only remaining possibility: the selected pair is $\{p_{j+1}, p_j\}$ for some $j \geq 1$, but the estimate for the leading term holds for a maximum point $p \in \{p_{j+1}, p_j\}$, and takes the form

$$\int_{p_{j+1}}^{p_j} f_\delta'(t) \frac{d}{dt} \bar{g}(|p-t|) dt \geq \frac{|f(p_j) - f(p_{j+1})|}{2} \left(|\bar{g}'(\gamma)| + |\bar{g}'(\gamma/2)| \right). \tag{1.48}$$

In this case, our previous estimates cannot be repeated in the same way since $l_R^+ > l_L^+ + 3\varepsilon$, and the term analogous to the quantity $|\hat{F}(\tilde{\mathcal{W}})|$ appearing in (1.45) cannot be controlled by $h_L + 2\varepsilon$. In principle, it is even possible that there exist some points

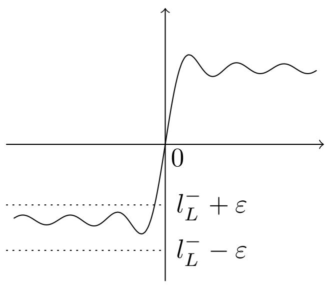


Figure 1.4: Picture of f_δ when f is odd, showing that the contribution on one side can very tiny compared to the contribution around 0.

in $(-\eta, 0)$ where $f_\delta > l_L^+ + \varepsilon$: if $x \in (-\delta, 0)$, then the value of $f_\delta(x)$ is influenced by $f|_{(0,\delta)}$, that is larger than l_L^+ in a set of positive measure. These phenomena cause some trouble when we seek for bounds similar to (1.47), and we refer to Figure 1.4 to show qualitatively why we need a more refined approach.

First we exclude that there exists a critical point p_{2i}, which is a maximum point for f_δ, where $f_\delta(p_{2i}) > l_L^+ + \varepsilon$. We do this showing that, thanks to our choice of the parameters, $p_2 \leq -\delta$: in this case, $p_{2i} \leq p_2$ for every $i \geq 1$, hence $f_\delta(p_{2i})$ is an average of $f|_{(-\eta,0)}$, and thus $f_\delta(p_{2i}) \leq l_L^+ + \varepsilon$ thanks to the definition of l_L^+. If $p_1 \leq -\delta$, then we automatically have that $p_2 < p_1 \leq -\delta$, and so we can suppose without loss of generality that $p_1 > -\delta$. Here we are going to use the bound on δ given in terms of the function V defined in (1.30), and we observe that

$$V(t/\delta) = 1 - 2 \int_t^\delta \varphi_\delta(s) ds = \int_{-\delta}^t \varphi_\delta(s) ds - \int_t^\delta \varphi_\delta(s) ds \qquad \forall t \in [0, \delta].$$

Using the explicit expression of f_δ and the antisymmetry of f we arrive to

$$\begin{aligned} f_\delta(p_1) = \int \varphi_\delta(p_1 - t) f(t) dt &= \int_{p_1-\delta}^{0} \varphi_\delta(p_1 - t) f(t) dt + \int_{0}^{p_1+\delta} \varphi_\delta(p_1 - t) f(t) dt \\ &= \int_{p_1-\delta}^{0} \varphi_\delta(p_1 - t) f(t) dt - \int_{0}^{p_1+\delta} \varphi_\delta(p_1 - t) f(-t) dt \\ &= \int_{p_1-\delta}^{0} \varphi_\delta(p_1 - t) f(t) dt - \int_{-p_1-\delta}^{0} \varphi_\delta(p_1 + t) f(t) dt \\ &= \int_{p_1-\delta}^{0} (\varphi_\delta(p_1 - t) - \varphi_\delta(p_1 + t)) f(t) dt, \end{aligned}$$

where we used that $\operatorname{spt} \varphi_\delta \subset [-\delta, \delta]$ in the last equality. It is immediate to see that, since both p_1 and t are negative, then $|p_1 - t| \le |p_1 + t| = -p_1 - t$ for every $t \in [p_1 - \delta, 0]$. The convolution kernel is symmetric and it satisfies $\varphi' \le 0$ in $(0, 1)$, hence $\varphi_\delta(p_1 - t) - \varphi_\delta(p_1 + t) \ge 0$ for every $t \in [p_1 - \delta, 0]$. Therefore, we can control $f_\delta(p_1)$ from below:

$$\begin{aligned} f_\delta(p_1) &\ge (l_L^- - \varepsilon) \left(\int_{p_1-\delta}^{0} \varphi_\delta(p_1 - t) dt - \int_{-p_1-\delta}^{0} \varphi_\delta(p_1 + t) dt \right) \\ &= (l_L^- - \varepsilon) \left(\int_{p_1}^{\delta} \varphi_\delta(t) dt - \int_{-\delta}^{p_1} \varphi_\delta(t) dt \right) = (l_L^- - \varepsilon) V(|p_1|/\delta). \end{aligned}$$

By definition of p_1, we have that $f_\delta(p_1) \le l_L^- + \varepsilon$, and combining this with the previous inequality, recalling that $l_L^- - \varepsilon < 0$, we have that $V(|p_1|/\delta) \ge \frac{l_L^- + \varepsilon}{l_L^- - \varepsilon}$. Since V is monotone increasing, we combine this inequality with condition (1.31) to obtain that $|p_1|/\delta > 1 - h_L/6$. We chose the kernel φ so that $|\varphi'| \le 3$, and thus $|f_\delta'| \le 3/\delta$. Using that $f_\delta(p_2) \ge l_L^+ - \varepsilon$ and $f_\delta(p_1) \le l_L^- + \varepsilon$, then

$$|p_2 - p_1| \ge \frac{\delta}{3}(l_L^+ - \varepsilon - l_L^- - \varepsilon) = \frac{\delta}{3}(h_L - 2\varepsilon) \ge \frac{\delta}{3}(h_L - h_L/2) = \frac{\delta}{6} h_L.$$

Combining this with the lower bound on $|p_1|/\delta$ we get that

$$|p_2| = |p_2 - p_1| + |p_1| \ge \frac{\delta}{6} h_L + \delta \left(1 - \frac{h_L}{6} \right) = \delta,$$

as we wanted. Since p_2 is an absolute maximum for f_δ in $[p_3, p_1]$, then the previous argument shows that $f_\delta \le l_L^+ + \varepsilon$ in $[-\eta, p_1]$.

We have just established that f_δ is well controlled in $[-\eta, p_1]$ and we are ready to proceed. The idea is that, if an estimate does not work as we need when $p \in \{p_{j+1}, p_j\}$ is a maximum point, then it has the right sign when we choose as a basepoint p_1, which is a minimum for f_δ. In fact, notice that the quantity considered in (1.46) when the base point p_j is a maximum point is $-\psi''_{f_\delta}(p_j)$, differently from the one expressed in (1.38), that coincides with $\psi''_{f_\delta}(p_{j+1})$, and our argument provides a lower bound for one of those quantities. We recall the bound (1.48) valid when the basepoint is a maximum point (as it is in our situation). If, by chance, we have that

$$\int_{p_1}^{\eta} f_\delta'(t)\bar{g}'(t-p)dt \geq -5\varepsilon|\bar{g}'(\gamma/2)|, \tag{1.49}$$

then the proof is not very different from before. In fact, the contributions coming from $[a,b]\setminus[-\eta,\eta]$ are still controlled by a constant, while the remaining part is divided in various pieces where the usual cancellation phenomena help to obtain the desired lower bound. Differently from the previous computations, we need to take care of the point p_1, that plays a special role now: we apply (1.48), arriving to

$$\begin{aligned}
\int_a^b f_\delta'(t)\frac{d}{dt}\bar{g}(|t-p|)dt &\geq \int_{-\eta}^{\eta} f_\delta'(t)\frac{d}{dt}\bar{g}(|t-p|)dt - 2C(\eta,D,M,\bar{g})\\
&\geq -\int_{-\eta}^{p_{j+1}} f_\delta'(t)\bar{g}'(p-t)dt + \int_{p_j}^{p_1} f_\delta'(t)\bar{g}'(t-p)dt\\
&\quad + \int_{p_1}^{\eta} f_\delta'(t)\bar{g}'(t-p)dt + \frac{h_L-2\varepsilon}{2}(|\bar{g}'(\gamma)|+|\bar{g}'(\gamma/2)|)\\
&\quad - 2C(\eta,D,M,\bar{g})\\
&\geq -\int_{-\eta}^{p_{j+1}} f_\delta'(t)\bar{g}'(p-t)dt + \int_{p_j}^{p_1} f_\delta'(t)\bar{g}'(t-p)dt - 5\varepsilon|\bar{g}'(\gamma/2)|\\
&\quad + \frac{h_L-2\varepsilon}{2}(|\bar{g}'(\gamma)|+|\bar{g}'(\gamma/2)|) - 2C(\eta,D,M,\bar{g}).
\end{aligned}$$

Now our discussion on the position of the maximum points p_{2i} pays off. In fact, we already know that $l_L^- - \varepsilon \leq f_\delta \leq l_L^+ + \varepsilon$ in $[-\eta,p_1]$, and therefore we control the two remaining integrals as we did when f is even:

$$-\int_{-\eta}^{p_{j+1}} f_\delta'(t)\bar{g}'(p-t)dt + \int_{p_j}^{p_1} f_\delta'(t)\bar{g}'(t-p)dt \geq -2\varepsilon|\bar{g}'(\gamma/2)| - (h_L+2\varepsilon)|\bar{g}'(\gamma)|.$$

Notice that this inequality is valid both when $p=p_j$ and when $p=p_{j+1}$, and one can observe that this happens also when f is even. Combining it with the previous inequalities we arrive to a contradiction:

$$0 = \int_a^b f_\delta'(t)\frac{d}{dt}\bar{g}(t-p)dt \geq |\bar{g}'(\gamma/2)|\frac{h_L-16\varepsilon}{2} - |\bar{g}'(\gamma)|\frac{h_L+10\varepsilon}{2} > 0,$$

where the last inequality holds because γ is so small that $|\bar{g}'(\gamma/2)| \geq \bar{\Lambda}|\bar{g}'(\gamma)|$, and ε satisfies (1.31). We need to consider one last case to conclude the proof when $h_L(f)>0$, i.e. when (1.49) does not hold. We observe that, applying Lemma 1.3.6 to $F=f_\delta$ in the interval $[p_1,\eta]$ with $x=p$ and $y=p_1$, we obtain that

$$-\int_{p_1}^{\eta} f_\delta'(t)\bar{g}'(t-p_1)dt \geq -\int_{p_1}^{\eta} f_\delta'(t)\bar{g}'(t-p)dt > 5\varepsilon|\bar{g}'(\gamma/2)|. \tag{1.50}$$

By construction of the critical points, we have that $\gamma = p_j - p_{j+1} \leq p_1 - p_2 =: \gamma'$, and one could argue as we did when f is even (in fact, p_1 is a minimum, and we already noticed that the argument in this case is analogous to that exploited when f is even). Since the estimates are very similar, we do not go through them again, but we highlight only the differences. In fact, we have that

$$\begin{aligned}\psi''_{f_\delta}(p_1) &= -\int f'_\delta(t)\frac{d}{dt}\bar{g}(p_1-t)dt \\ &= -\int_{\mathbb{R}\setminus[-\eta,\eta]} f'_\delta(t)\frac{d}{dt}\bar{g}(p_1-t)dt + \int_{-\eta}^{p_1} f'(t)\bar{g}'(p_1-t)dt - \int_{p_1}^{\eta} f'_\delta(t)\bar{g}'(t-p_1)dt \\ &\geq -2C(\eta,D,M,\bar{g}) - 2\varepsilon|\bar{g}'(\gamma')| + 5\varepsilon|\bar{g}'(\gamma/2)| \\ &\geq -2C(\eta,D,M,\bar{g}) - 2\varepsilon|\bar{g}'(\gamma)| + 5\varepsilon|\bar{g}'(\gamma/2)|,\end{aligned}$$

where we used that $\gamma' \geq \gamma$ and that g is convex in $(0, r_{\bar{g}})$ to obtain the last inequality. Instead, to pass from the second to the third line we control the last term with (1.50), and the inequality for the second term is an adaptation of the inequality for I in (1.43). We arrive to a contradiction thanks to our choice of parameters: $C(\eta,D,M,\bar{g}) < \varepsilon|\bar{g}'(\gamma)|$, and thus the last expression is larger than $5\varepsilon|\bar{g}'(\gamma/2)| - 4\varepsilon|\bar{g}'(\gamma)| \geq 5\varepsilon|\bar{g}'(\gamma)| - 4\varepsilon|g'(\gamma)| > 0$.

To finally conclude the proof we deal with the possiblity of having $h_L(f) = 0$. Here it is beneficial to use a different notation for the parameters h_L, $l_L^\pm$ and $l_R^\pm$ appearing in the estimates and for the actual limits of the function f under examination. The issue is that, if $h_L(f) = 0$, then there is no guarantee that the critical points selected in Subsection 1.3.3 exist, and we have to construct them artificially. To be more clear, we observe that our arguments do not actually need that $l_L^\pm$ and $l_R^\pm$ are the limits, we just care about the existence of the critical points, where the value of the function has to lie close enough to the bounds l_L^- and l_L^+, and that $l_L^- - \varepsilon < f < l_L^+ + \varepsilon$ in $(-\eta, 0)$. With this observation in mind, if $h_L(f) = 0$, then we fix the parameters $\varepsilon' > 0$ and $\eta' \in (0, \frac{\eta}{100})$ such that

$$\varepsilon' < \frac{1}{40}\cdot\frac{(\bar{\Lambda}-1)|l_L^-(f)|}{16\bar{\Lambda}+10}, \qquad \begin{cases}\mathscr{L}^1(\{f > l_L^+(f)-\varepsilon'\}\cap(-100\eta',0)) > 0\\ \mathscr{L}^1(\{f > l_L^+(f)+\varepsilon'\}\cap(-100\eta',0)) = 0\\ \mathscr{L}^1(\{f < l_L^-(f)-\varepsilon'\}\cap(-100\eta',0)) = 0\\ \mathscr{L}^1(\{f < l_L^-(f)+\varepsilon'\}\cap(-100\eta',0)) > 0\end{cases}.$$

We define the auxiliary function $u(x) = f(x) - f(x-\eta')$ and we observe that, since f has constant potential in $(-\eta,\eta)$, then ψ_u is constant in $(-\eta+\eta', \eta-\eta')$. Hence, we construct a bounded and integrable function with several oscillations close to 0 and with constant potential in $(-\eta/2, \eta/2)$:

$$\bar{u}(x) = \sum_{i=0}^{9} u(x - 2i\eta').$$

Therefore, $\tilde{u}(x) = \bar{u}(x) + \bar{u}(-x)$ satisfies $l_L^- - \varepsilon < \tilde{u} < l_L^+ + \varepsilon$ in $(-\eta/2, \eta/2)$, where $\varepsilon = 40\varepsilon'$, $l_L^- = 0$ and $l_L^+ = 2|l_L^-(f)|$, and it oscillates between the upper and lower bound several times. Of course, if we take a convolution parameter small enough, then the function $\tilde{u}_\delta = \tilde{u} * \varphi_\delta$ is smooth and with the desired critical points. Notice also that, if $\delta < \eta'/8$, then we can run the selection of the good pair of critical points performed in Subsection 1.3.3, and that procedure stops because each critical point p_i belongs to the segment $(-(i+1)\eta' - \eta'/8, -i\eta' + \eta'/8)$, and thus the conditions (1.34) and (1.35) are automatically satisfied for some $j < 9$. Now we run our argument for the function $\tilde{u}_\delta$, that is even, and we arrive to the desired contradiction since we chose accurately the parameters ε, η and δ. □

1.4 Uniqueness and symmetry of minimizing measures

This last section is devoted to discuss the question of the uniqueness (of course, up to translations) and radiality of the optimal measures, and in particular to prove Theorem 1.4.1 below. We start noticing that, for a general function $\bar{g}$, there is no reason why there should be a unique optimal measure, and it is easy to build examples in which the uniqueness fails. Less clear is the question whether optimal measures should be radial. Indeed, on one side, the problem is rotationally invariant; but on the other side, in most of the examples the radial profile g has a unique minimum point, say $d_{\min}$, and then, roughly speaking, points would like to stay at distance $d_{\min}$ from each other, and this somehow pushes against the radiality. The two questions are related, because obviously if there is a unique optimal measure then it must be radial, since all the rotations of this measure are also optimal, and then they have to coincide with it. Our results in the positive direction work for strongly positive definite kernels (see Definition 1.4.4 and the discussion preceding Lemma 1.4.9). Finally, in Theorem 1.4.2 we combine some explicit computations and the convexity of the energy to show that a sphere minimizes $\mathcal{E}$, even in the presence of a singular repulsion in the origin.

Theorem 1.4.1 (Uniqueness and symmetry of optimal measures)**.** *Let $\bar{g} = \mathfrak{g}_\alpha + \bar{h}$ be a function satisfying ($\mathbf{H}_p$), with $2 \leq \alpha \leq 4$. If $\bar{h}$ is strongly positive definite in $\dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$, then there is some minimal measure $\bar{\mu} \in \mathcal{P}(\mathbb{R}^N)$ which belongs to $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$, and if $\bar{h}$ is strictly strongly positive definite in $\dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$, then $\bar{\mu}$ is the unique minimal measure up to translations. If $\bar{h}$ is subharmonic in $\mathbb{R}^N \setminus \{0\}$ and radially decreasing, then it is also strongly positive definite in $\mathcal{P}(\mathbb{R}^n)$, so in particular there is some minimal measure $\bar{\mu} \in \mathcal{P}(\mathbb{R}^N)$ which belongs to $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$. Moreover, in this case there exists a radius R_1 such that the support of every minimal measure in $\mathcal{P}(\mathbb{R}^N)$ is a ball of radius R_1.*

Theorem 1.4.2. *Let $N \geq 5$, and let us fix the parameters $\alpha \in [2,4]$ and $\beta \in (0, N-4)$. If $\bar{g} = \mathfrak{g}_\alpha - \mathfrak{g}_{-\beta}$, then there exists a radius $r = r(N,\alpha,\beta) > 0$ such that $\mu = \frac{1}{|\partial B_r|}\mathscr{H}^{N-1} \llcorner \partial B_r$ is the unique minimizer of $\mathcal{E}$ in $\mathcal{P}(\mathbb{R}^N)$.*

The results are, in some measure, sharp:

Remark 1.4.3. A kernel $\bar{g} = \mathfrak{g}_\alpha - \mathfrak{g}_{-\beta}$, with $\alpha > 0$ and $\beta \in (-N, \alpha)$, induces a convex energy only if $\alpha \in [2,4]$ and $\beta \in (-N, 2]$. In fact, the term $\mu \mapsto \iint \mathfrak{g}_\alpha(x-y)d\mu(x)d\mu(y)$ is convex for $\alpha \in [2,4]$ and strictly concave for $\alpha \in (-N, 2)$ (see [Lop19] and [CS23, Theorem 5.3]). For $\alpha > 4$, instead, that term is not convex because the minimizers of $\mathcal{E}$ with $\beta = 2$ are not radially symmetric (see [DLM23]).

Finally, since the two terms $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are homogeneous, it is sufficient that one of them induces a non-convex energy to exclude that $\mathcal{E}$ is convex.

1.4.1 Positive definite functions and convexity of $\mathcal{E}$

The question of uniqueness is of fundamental importance for this problem, and it is already treated in some papers. More precisely, in [BCT18] it is shown that there is a unique optimal measure whenever $\bar{g} = \mathfrak{g}_2 - \mathfrak{g}_{-\beta}$ with any $0 < \beta < N$. Later on, the same result was obtained in [Lop19] for $\bar{g} = \mathfrak{g}_\alpha - \mathfrak{g}_{-\beta}$ with any $2 \leq \alpha \leq 4$, and again $0 < \beta < N$. We can now further generalize the result by substituting the term $-\mathfrak{g}_{-\beta}$ with any positive definite function, basically using the same approach of [Lop19] and the properties of such functions. We start with the definition.

Definition 1.4.4 (Strongly positive definite functions)**.** Given a $L^1_{\rm loc}$ function $\bar{h} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$, we say that $\bar{h}$ *is strongly positive definite* if for any $\mu, \nu \in \mathcal{P}(\mathbb{R}^N)$ one has

$$\mathcal{E}_{\bar{h}}(\mu,\mu) + \mathcal{E}_{\bar{h}}(\nu,\nu) \geq 2\mathcal{E}_{\bar{h}}(\mu,\nu), \tag{1.51}$$

and that $\bar{h}$ *is strictly strongly positive definite* if the inequality is strict whenever $\mu \neq \nu$.

The definition of positive definite functions is standard, see for instance [Rud91, LL01, Mat15], and it simply consists in asking the validity of the property (1.51) when μ and ν are L^1 functions with unit L^1 norm, or characteristic functions of sets of unit volume –these two choices are equivalent by a simple approximation argument. We add the word "strongly" to remember that our assumption is in principle stronger, since we want to test with every probability measure. However, it is simple to see that the two notions are in fact equivalent, at least for kernels for which the measures can be approximated *in energy* by functions, as we show now.

Lemma 1.4.5. *Assume that $\bar{h} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ is a $L^1_{\rm loc}$ function with the property that for any measure $\mu \in \mathcal{P}(\mathbb{R}^N)$ there is a sequence of smooth functions $\mu_j \in \mathcal{P}(\mathbb{R}^N) \cap C^\infty_c(\mathbb{R}^N)$ weakly* converging to μ and such that $\mathcal{E}_{\bar{h}}(\mu_j) \to \mathcal{E}_{\bar{h}}(\mu)$ for $j \to \infty$. Then, $\bar{h}$ is strongly positive definite if and only if it is positive definite.*

Proof. Of course, whenever $\bar{h}$ is strongly positive definite, then it is also positive definite, so we only have to prove the opposite implication. Let us then assume that $\bar{h}$ is positive definite, and let μ and ν be two probability measures. By assumption, we can take two sequences $\{\mu_j\}$ and $\{\nu_n\}$ in $\mathcal{P}(\mathbb{R}^N) \cap C^\infty_c(\mathbb{R}^N)$ which weakly* converge to μ and ν respectively, and such that

$$\lim_{j\to\infty} \mathcal{E}_{\bar{h}}(\mu_j) = \mathcal{E}_{\bar{h}}(\mu), \qquad \lim_{n\to\infty} \mathcal{E}_{\bar{h}}(\nu_n) = \mathcal{E}_{\bar{h}}(\nu).$$

Since (1.51) is true for any pair (μ_j, ν_n), by lower semicontinuity of the energy we have

$$\begin{aligned} 2\mathcal{E}_{\bar{h}}(\mu,\nu) &\leq \liminf_{j\to\infty} 2\mathcal{E}_{\bar{h}}(\mu_j,\nu) \leq \liminf_{j\to\infty}\liminf_{n\to\infty} 2\mathcal{E}_{\bar{h}}(\mu_j,\nu_n) \\ &\leq \liminf_{j\to\infty}\liminf_{n\to\infty} \mathcal{E}_{\bar{h}}(\mu_j,\mu_j) + \mathcal{E}_{\bar{h}}(\nu_n,\nu_n) = \mathcal{E}_{\bar{h}}(\mu,\mu) + \mathcal{E}_{\bar{h}}(\nu,\nu). \end{aligned}$$

Therefore, $\bar{h}$ is strongly positive definite. $\square$

As said above, the notion of positive definiteness is well known. In particular, the following sufficient conditions for the positive definiteness are known, see for instance [NP21].

Theorem 1.4.6. *Let $\bar{h}:\mathbb{R}^N\to\mathbb{R}^+\cup\{+\infty\}$ be a L^1_{loc} function. If $\lim_{|x|\to\infty}\bar{h}(x)=\inf\bar{h}$ and $\bar{h}$ is subharmonic in $\mathbb{R}^N\setminus\{0\}$, then $\bar{h}$ is positive definite. Instead, if we suppose that $\limsup_{|x|\to\infty}\bar{h}(x)<+\infty$ and the Fourier transform of $\bar{h}$ is a positive Borel measure, then $\bar{h}$ is positive definite.*

In particular, the function $-\mathfrak{g}_{-\beta}$ is strictly positive definite for every $0<\beta<N$, as well as the Gaussian function $e^{-|x|^2/2}$. An important example is given by the kernel $\bar{h}_{\log}(x)=-\mathfrak{g}_0(x)=-\log|x|$ in dimension $N=2$, that is positive definite among densities with compact support: for every pair of densities $f_1,f_2\in\mathcal{P}_c(\mathbb{R}^2)$ we have that

$$\mathcal{E}_{\bar{h}_{\log}}(f_1,f_1)+\mathcal{E}_{\bar{h}_{\log}}(f_2,f_2)\geq 2\mathcal{E}_{\bar{h}_{\log}}(f_1,f_2).$$

Remark 1.4.7. We point out that, on top of the aforementioned examples of positive definite kernels, it is possible to build other ones with a simple operation: the double convolution. In fact, given a positive definite kernel $\bar{g}:\mathbb{R}^N\to\mathbb{R}^+\cup\{+\infty\}$, and any non-negative and symmetric kernel $\varphi\in L^1$, we define in a pointwise sense the new kernel $\bar{h}:=\bar{g}*\varphi*\varphi$. With formal computations, one sees immediately why this is again positive definite: for every pair of probability densities $f_1,f_2\in\mathcal{P}(\mathbb{R}^N)$

$$\begin{aligned}\mathcal{E}_{\bar{h}}(f_1,f_2)&=\iint\bar{h}(x-y)f_1(x)f_2(y)dxdy=\int f_1(x)\cdot(\bar{h}*f_2)(x)dx\\&=\int f_1(x)\cdot(\bar{g}*\varphi*\varphi*f_2)(x)dx=\int(f_1*\varphi)(x)\cdot(\bar{g}*\varphi*f_2)(x)dx\\&=\mathcal{E}_{\bar{g}}(f_1*\varphi,f_2*\varphi).\end{aligned}$$

Plugging this formal identities into (1.51), with $\mu=f_1\mathscr{L}^N$ and $\nu=f_2\mathscr{L}^N$, we immediately obtain that $\bar{h}$ is positive definite if so is $\bar{g}$. Of course, some assumptions are necessary to justify the previous steps. For example, if $\varphi\in L^\infty$ and it has compact support, then the previous steps are correct also for any pair of measures (even for those not absolutely continuous with respect to $\mathscr{L}^N$). In this case, we even obtain that $\bar{h}$ is strongly positive definite. Instead, using Young's inequality [LL01, Theorem 4.2], it is immediate to see that, when $\varphi\in L^p$ and $\bar{g}\in L^q$, then $\bar{g}*\varphi*\varphi\in L^s$ with $\frac{1}{s}=\frac{2}{p}+\frac{1}{q}-2$, so in particular it is locally integrable.

An important corollary of Lemma 1.4.5, following directly from Lemma 1.2.12, is that a radial function $\bar{h}$ with the property that its radial profile h, defined by $\bar{h}(x)=h(|x|)$, is continuous, decreasing in a right neighborhood of 0, and so that $\lim_{t\searrow 0}h(t)t^\gamma=0$ for some $0<\gamma<N$, is strongly positive definite as soon as it is positive definite. If $\bar{h}$ is subharmonic in $\mathbb{R}^N\setminus\{0\}$, there is not even need of asking the existence of some γ as above. More precisely, the following result holds.

Lemma 1.4.8. *Every radial function $\bar{h}:\mathbb{R}^N\to\mathbb{R}^+\cup\{+\infty\}$ which is radially decreasing and subharmonic in $\mathbb{R}^N\setminus\{0\}$ is strongly positive definite.*

Proof. Let $\bar{h} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a radial, radially decreasing and subharmonic function and, as usual, let us call h its radial profile. By Theorem 1.4.6 we know that $\bar{h}$ is positive definite. Therefore, as noticed above, Lemma 1.4.5 and Lemma 1.2.12 readily give that $\bar{h}$ is strongly positive definite if there is some $0 < \gamma < N$ such that $\lim_{t\searrow 0} h(t)t^\gamma = 0$.

If this is not the case, we proceed as follows. For any small $\varepsilon > 0$, we call $\bar{h}_\varepsilon \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R}^+)$ the radial function which coincides with $\bar{h}$ in $\mathbb{R}^N \setminus B_\varepsilon$ and which is harmonic in $B_\varepsilon \setminus \{0\}$. In particular, the corresponding radial profile h_ε is defined in $(0, \varepsilon)$ by the formula

$$h_\varepsilon(t) = \begin{cases} \dfrac{\big(t^{2-N} - \varepsilon^{2-N}\big)h'(\varepsilon)}{(2-N)\varepsilon^{1-N}} + h(\varepsilon) & \text{if } N \neq 2, \\ \varepsilon\big(\log(t) - \log(\varepsilon)\big)h'(\varepsilon) + h(\varepsilon) & \text{if } N = 2. \end{cases}$$

Notice that by construction $\bar{h}_\varepsilon$ satisfies all of our hypotheses (it is subharmonic in distributional sense), and moreover $\lim_{t\searrow 0} h_\varepsilon(t)t^{N-1/2} = 0$. As a consequence, $\bar{h}_\varepsilon$ is strongly positive definite. Given then any two probability measures μ and ν, we know that

$$\mathcal{E}_\varepsilon(\mu, \mu) + \mathcal{E}_\varepsilon(\nu, \nu) \geq 2\mathcal{E}_\varepsilon(\mu, \nu), \tag{1.52}$$

where we write $\mathcal{E}_\varepsilon$ in place of $\mathcal{E}_{\bar{h}_\varepsilon}$ for simplicity of notations. We observe that, since $\bar{h}$ is subharmonic, then $\bar{h}_\varepsilon \leq \bar{h}$ for every $\varepsilon > 0$. In fact, the function $\bar{h} - \bar{h}_\varepsilon$ is subharmonic in $B_\varepsilon \setminus \{0\}$, it is 0 on ∂B_ε and its gradient vanishes in the same sphere by construction. Then, taking any $r \in (0, \varepsilon)$ we apply the divergence theorem in the annulus $A = B_\varepsilon \setminus B_r$ and we see that

$$\begin{aligned} 0 \leq \int_A \Delta(\bar{h} - \bar{h}_\varepsilon) &= \int_{\partial B_\varepsilon} \frac{x}{|x|}\nabla(\bar{h} - \bar{h}_\varepsilon)d\mathscr{H}^{N-1}(x) - \int_{\partial B_r} \frac{x}{|x|}\nabla(\bar{h} - \bar{h}_\varepsilon)d\mathscr{H}^{N-1}(x) \\ &= -(h'(r) - h_\varepsilon'(r))\mathscr{H}^{N-1}(\partial B_r). \end{aligned}$$

In turn, this shows that the difference between the radial profile is decreasing in $(0, \varepsilon)$ because r was arbitrary. Since $h(\varepsilon) - h_\varepsilon(\varepsilon) = 0$, this proves that $\bar{h}_\varepsilon \leq \bar{h}$ in $B_\varepsilon \setminus \{0\}$. We notice, additionally, that the kernels $\bar{h}_\varepsilon$ pointwise converge to $\bar{h}$ when $\varepsilon \to 0$, and thus the Dominated Convergence Theorem allows us to pass to the limit in (1.52), establishing the validity of (1.51) for $\bar{h}$. □

In the results presented before, we tried to formulate in the most general setting, working in the whole space of probability measures $\mathcal{P}(\mathbb{R}^N)$. Keeping an eye on the applications that we have in mind, however, it is also natural to work with the restricted class $\dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$. In fact, when we try to obtain information about minimizers of (P_M), it is clear that we can restrict to the class of measures with finite energy, and up to a translation we can suppose that their barycenter lies in the origin. Moreover, we proved in Theorem 1.1.1 that the support of the minimizers if uniformly bounded. Collecting all these properties, we can directly work in the restricted class $\dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$. Hence, for our purposes, we can work with any kernel $\bar{g}$ that is strongly positive definite in this subclass, i.e.

$$\mathcal{E}_{\bar{g}}(\mu, \mu) + \mathcal{E}_{\bar{g}}(\nu, \nu) \geq 2\mathcal{E}_{\bar{g}}(\mu, \nu) \qquad \forall \mu, \nu \in \dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N).$$

This fact, despite being trivial, allows us to treat in a unified framework some kernels that are not positive definite in $\mathcal{P}(\mathbb{R}^N)$, while they are in $\dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$. This might be due to a ill posedness of the interaction functional for measures with unbounded support, as in the case that we already mentioned of $\bar{h}_{\log}(x) = -\log|x|$. On the other hand, it may happen that a kernel is positive definite only when we fix the barycenter, and this is the case of the quadratic kernel $\mathfrak{g}_2$ for instance (see [Lop19, Theorem 2.1]).
The importance in this context of the notion of positive definiteness is mainly given by the following elementary observation, applied to the convex set $\mathcal{C} = \dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$:

Lemma 1.4.9. *Let $\bar{h} : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function in the class L^1_{loc}, and let $\mathcal{C} \subset \mathcal{P}(\mathbb{R}^N)$ be a convex set. If $\bar{h}$ is strongly positive definite in $\mathcal{C}$, i.e.*

$$\mathcal{E}_{\bar{h}}(\mu,\mu) + \mathcal{E}_{\bar{h}}(\nu,\nu) \geq 2\mathcal{E}_{\bar{h}}(\mu,\nu) \qquad \forall \mu,\nu \in \mathcal{C}, \tag{1.53}$$

then the energy $\mathcal{E}$ is convex in $\mathcal{C}$. If $\bar{h}$ is strictly strongly positive definite in $\mathcal{C}$, then $\mathcal{E}_{\bar{h}}$ is strictly convex in that subspace of measures.

Proof. Let μ, ν be two measures in $\mathcal{C}$, and let $0 \leq \lambda \leq 1$. Then,

$$\begin{aligned}\mathcal{E}_{\bar{h}}\big(\lambda\mu + (1-\lambda)\nu\big) &= \mathcal{E}_{\bar{h}}\big(\lambda\mu + (1-\lambda)\nu, \lambda\mu + (1-\lambda)\nu\big) \\ &= \lambda^2\mathcal{E}_{\bar{h}}(\mu) + (1-\lambda)^2\mathcal{E}_{\bar{h}}(\nu) + 2\lambda(1-\lambda)\mathcal{E}_{\bar{h}}(\mu,\nu),\end{aligned}$$

and then by (1.53)

$$\begin{aligned}&\mathcal{E}_{\bar{h}}\big(\lambda\mu + (1-\lambda)\nu\big) - \big(\lambda\mathcal{E}_{\bar{h}}(\mu) + (1-\lambda)\mathcal{E}_{\bar{h}}(\nu)\big) \\ &\qquad = \lambda(\lambda-1)\Big(\mathcal{E}_{\bar{h}}(\mu,\mu) + \mathcal{E}_{\bar{h}}(\nu,\nu) - 2\mathcal{E}_{\bar{h}}(\mu,\nu)\Big) \leq 0,\end{aligned}$$

which gives the required convexity of $\mathcal{E}_{\bar{h}}$. If $\bar{h}$ is strictly strongly positive definite, then the above inequality is strict whenever $0 < \lambda < 1$ and $\mu \neq \nu$, thus $\mathcal{E}_{\bar{h}}$ is strictly convex in $\mathcal{C}$. □

We conclude with the remarkable symmetry of the minimizers of $\mathcal{E}_{\bar{g}}$ in $\mathcal{P}(\mathbb{R}^N)$ when the kernel is strictly positive definite in $\dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$. If, instead, the kernel is just positive definite in that subset of measures, then we obtain only the existence of a symmetric minimizer, whereas this is not a feature shared by every ground state of $\mathcal{E}_{\bar{g}}$ in $\mathcal{P}(\mathbb{R}^N)$. We stress that this scenario is not unrealistic: in Section 2.2 we are guided by the works [DLM22, DLM23], and the second one is mainly devoted to kernels of the form $\bar{g} = \mathfrak{g}_\alpha - \mathfrak{g}_\beta$ with $\alpha \geq 4$, $\beta \geq 2$ and $\alpha > \beta$. In general, these kernels are not positive definite, exept for the choice of parameters $\alpha = 4$ and $\beta = 2$. In fact, in this case [DLM23, Theorem 1.1] shows that there is a plethora of different minimizers (characterized by a second moment condition), and among them only the sphere is radially symmetric.

Proof of Theorem 1.4.1. The function $\mathfrak{g}_\alpha$ is positive definite in $\dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$ for $2 \leq \alpha \leq 4$, as proved in [Lop19, Theorem 2.1], and then it is also strongly positive definite in $\dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$ since Lemma 1.4.5 can be applied to this kernel. As a consequence, if $\bar{h}$ is strongly positive definite in $\dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$, so is also $\bar{g} = \mathfrak{g}_\alpha + \bar{h}$. Our assumptions are largely sufficient to apply Theorem 1.1.1, hence we know the existence of an optimal measure $\mu \in \mathcal{P}(\mathbb{R}^N)$ for the energy $\mathcal{E}$, which is compactly supported. Up to a translation, we can assume that μ has baricenter in the origin. For every $\theta \in \mathbb{S}^{N-1}$, we call μ_θ the measure obtained by rotating μ of an angle θ. By radiality of $\bar{g}$, each measure μ_θ has the same energy as μ, so they are all optimal. The energy $\mathcal{E}$ is convex in the space $\dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$ thanks to Lemma 1.4.9. Of course, $\mu_\theta \in \dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$ for every $\theta \in \mathbb{S}^{N-1}$, and the convexity of $\mathcal{E}$ ensures that the measure

$$\bar{\mu} = \fint_{\mathbb{S}^{N-1}} \mu_\theta d\mathscr{H}^{N-1}(\theta),$$

is itself optimal (and belongs to $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$). The existence of an optimal measure in $\mathcal{P}_{\rm rad}(\mathbb{R}^N)$ is then established.

If $\bar{h}$ is strictly strongly positive definite in $\dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$, then so is $\bar{g}$, and then the energy $\mathcal{E}$ is strictly convex in $\dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$. As a consequence, all the measures μ_θ have to coincide, and this means that $\mu = \bar{\mu}$ is radial. We claim that μ is actually the only optimal measure with baricenter in the origin. This is in fact obvious: if there is another such optimal measure $\nu \neq \mu$, the strict convexity of $\mathcal{E}$ gives that $\mathcal{E}\big((\mu+\nu)/2\big) < \mathcal{E}(\mu)$, which is absurd.

Let us now assume that $\bar{h}$ is subharmonic in $\mathbb{R}^N \setminus \{0\}$ and radially decreasing. The fact that $\bar{h}$ is strongly positive definite is given by Lemma 1.4.8, so $\mathcal{E}$ is convex in $\dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$. As we already pointed out, this feature guarantees that, for every minimizer $\mu \in \dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$, its symmetrization $\bar{\mu}$ is a minimizer as well. It is immediate to see that, since $\Delta\bar{g} > 0$ out of the origin, and applying either Proposition 1.2.2 (in dimension $N = 1$) or Proposition 1.2.4 (in higher dimensions) to $\bar{\mu}$, we obtain that $\operatorname{spt}\bar{\mu} = \bar{B}_{R_1}$ for some $R_1 > 0$. Exploiting once more the convexity of $\mathcal{E}$, we know that $\frac{1}{2}(\mu + \bar{\mu})$ is again a minimizer, and thus

$$\mathcal{E}(\mu) = \mathcal{E}\left(\frac{\mu+\bar{\mu}}{2}\right) = \frac{1}{4}\mathcal{E}(\mu) + \frac{1}{2}\mathcal{E}(\mu,\bar{\mu}) + \frac{1}{4}\mathcal{E}(\bar{\mu}) = \frac{\mathcal{E}(\mu)}{2} + \frac{1}{2}\int \psi_\mu(x) d\bar{\mu}(x). \quad (1.54)$$

Thanks to the Euler-Lagrange conditions (EL_p), and their refined version discussed in Remark 1.1.4, we know that $\psi_\mu \geq \mathcal{E}(\mu)$ in $\mathbb{R}^N \setminus \operatorname{spt}\mu$, and also that $\bar{\mu}(\{\psi_\mu < \mathcal{E}(\mu)\}) = 0$. Combining this information with (1.54), we get that $\psi_\mu(x) = \mathcal{E}(\mu)$ for $\bar{\mu}$-a.e. x. Additionally, since $\bar{g}$ is continuous in $\mathbb{R}^N \setminus \{0\}$, we know that ψ_μ is continuous in $\mathbb{R}^N \setminus \operatorname{spt}\mu$. There are two possibilities: either $\operatorname{spt}\mu$ is dense in $\operatorname{spt}\bar{\mu} = \bar{B}_{R_1}$, or there exists $x \in B_{R_1}$ and $\varepsilon > 0$ such that $B(x,\varepsilon) \subset B_{R_1} \setminus \operatorname{spt}\mu$. In the first case the two supports must coincide since they are closed sets. The second one, instead, cannot occur because the potential ψ_μ should be at the same time constant and strictly subharmonic in $B(x,\varepsilon)$, that is impossible. This argument shows that, assuming $\Delta\bar{h} \geq 0$ out of the origin, the support of any minimizer $\mu \in \dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$ coincides with a ball centerd in the origin. Let $\nu \in \dot{\mathcal{P}}_{\bar{g},c}(\mathbb{R}^N)$ be any other minimizer. The previous considerations show that $\operatorname{spt}\nu = \bar{B}_{R_2}$ for some $R_2 > 0$. Repeating the previous argument with ν in place of the symmetrized measure $\bar{\mu}$ we obtain that the two balls necessarily coincide, concluding the proof.

□

1.4.2 Minimality of the spheres

We aim to characterize the spheres as the unique minimizers of $\mathcal{E}$ for some choice of the kernel $\bar{g}$. The basic observation is that, whenever the energy is convex, it is sufficient to find a measure μ that satisfies the Euler-Lagrange conditions. For some choice of the kernel, we can prove that the $(N-1)$-Hausdorff measure restricted to a certain sphere (that is unique) is a critical point of the energy $\mathcal{E}$. Whenever this is coupled with the convexity of the energy, we are able to characterize that sphere as the unique minimizer. We stress that our result is not new, as it is contained in [FM23], where they characterize the spheres as minimizers for a range of parameters that contains ours. However, the approach presented here has the advantage of being extremely simple, relying on a trivial application of the maximum principle, and does not depend on any prior result concerning special functions. This yields to a less complete statement, since we do not compute explicitly the potential of the optimal sphere, while this is part of the work made in [FM23]. We highlight that our approach shows that, for any $\alpha \geq 2$ and any $\beta \in (0, N-4)$, there exists a radius $r > 0$ such that the probability measure $\mu^r = \frac{1}{|\partial B_r|}\mathscr{H}^{N-1} \llcorner \partial B_r$ satisfies the Euler-Lagrange conditions (EL_p). This, however, guarantees that the measure μ^r is a minimizer only when $\mathcal{E}$ is convex. As we discussed in Remark 1.4.3, the energy is convex only when $\alpha \in [2, 4]$, and this is why we restrict to this range in Theorem 1.4.2.

Proof of Theorem 1.4.2. We begin with some basic computations, valid for power-law kernels $\mathfrak{g}_\gamma$ with $\gamma > -N+2$ and $\gamma \neq 0$. For any $\rho > 0$, we call $\mu^\rho = \frac{1}{|\partial B_\rho|}\mathscr{H}^{N-1} \llcorner \partial B_\rho$, and a simple computation shows that for every $x \in \mathbb{R}^N$

$$\begin{aligned}
\psi_{\mathfrak{g}_\gamma,\mu^\rho}(\rho x) &= \frac{\rho^\gamma}{|\partial B_1|}\int_{\partial B_1} \frac{|x-y|^\gamma}{\gamma} d\mathscr{H}^{N-1}(y),\\
\partial_1\psi_{\mathfrak{g}_\gamma,\mu^\rho}(\rho x) &= \frac{\rho^{\gamma-1}}{|\partial B_1|}\int_{\partial B_1} (x_1-y_1)|x-y|^{\gamma-2} d\mathscr{H}^{N-1}(y), \\
\Delta\psi_{\mathfrak{g}_\gamma,\mu^\rho}(\rho x) &= \frac{\rho^{\gamma-2}}{|\partial B_1|}\int_{\partial B_1} (N+\gamma-2)|x-y|^{\gamma-2} d\mathscr{H}^{N-1}(y).
\end{aligned} \tag{1.55}$$

Notice that the first, second and third expression defines a continuous function on the whole $\mathbb{R}^N$ when γ is larger than $1-N$, $2-N$ and $3-N$ respectively. Of course, the symmetry of the chosen kernels and the symmetry of μ^ρ guarantee that also $\psi_{\mathfrak{g}_\gamma,\mu^\rho}$ and $\Delta\psi_{\mathfrak{g}_\gamma,\mu^\rho}$ enjoy spherical symmetry. Given $\bar{g} = \mathfrak{g}_\alpha - \mathfrak{g}_{-\beta}$, with $\alpha > 0$ and $0 < \beta < N-2$, then the special structure of μ^ρ automatically forces $\psi_{\bar{g},\mu^\rho} = \mathcal{E}(\mu^\rho)$ in $\partial B_\rho = \operatorname{spt}\mu^\rho$. Hence, to identify the correct sphere we need only to find a radius $r > 0$ for which the inequality $\psi_{\bar{g},\mu^r} \geq \mathcal{E}(\mu^r)$ in (EL_p) holds everywhere. The computations in (1.55) show that

$$\begin{aligned}\partial_1\psi_{\bar{g},\mu^\rho}(\rho e_1) &= \frac{\rho^{\alpha-1}}{|\partial B_1|}\int_{\partial B_1}(1-y_1)|e_1-y|^{\alpha-2}d\mathscr{H}^{N-1}(y)\\ &\quad-\frac{\rho^{-\beta-1}}{|\partial B_1|}\int_{\partial B_1}\frac{1-y_1}{|e_1-y|^{\beta+2}}d\mathscr{H}^{N-1}(y)\\ &=\frac{\rho^{-\beta-1}}{|\partial B_1|}\bigg(\rho^{\alpha+\beta}\int_{\partial B_1}(1-y_1)|e_1-y|^{\alpha-2}d\mathscr{H}^{N-1}(y)\\ &\quad-\int_{\partial B_1}\frac{1-y_1}{|e_1-y|^{\beta+2}}d\mathscr{H}^{N-1}(y)\bigg).\end{aligned} \tag{1.56}$$

Hence, there exists a unique radius $\rho > 0$ such that $\nabla\psi_{\bar{g},\mu^\rho} = 0$ on $\partial B_\rho = \operatorname{spt}\mu^\rho$, and we call this radius $r = r(\alpha,\beta,N)$. In order to find a measure μ^ρ satisfying (EL_p), every point in $\operatorname{spt}\mu^\rho$ needs to be a critical point for $\psi_{\bar{g},\mu^r}$, and thus we fix that specific radius. Notice that the computations are valid because the kernel $\bar{g}$ is not too singular in the origin (in this particular case, because $\beta < N-2$, ensuring that $\nabla\bar{g}\in L^1_{\text{loc}}(\mathbb{R}^{N-1})$). We observe that, whenever $\alpha\geq 2$ and $\beta\in(0,N-4)$, we have that $\Delta^2\bar{g} > 0$ in $\mathbb{R}^N\setminus\{0\}$. Hence, also the potential $\psi_{\bar{g},\mu^\rho}$ is bi-subharmonic in $\mathbb{R}^N\setminus\partial B_\rho$ for every choice of $\rho > 0$, with the parameters α,β in the aforementioned range. We claim that $\Delta\psi_{\bar{g},\mu^r}\geq 0$ in ∂B_r. Let us suppose by contradiction that the function $u = \Delta\psi_{\bar{g},\mu^r}$ verifies the opposite inequality $u(re_1) < 0$. Since u is symmetric and subharmonic in B_r, then it is radially increasing. Combining this information with the sign of u in ∂B_r we obtain that $u = \Delta\psi_{\bar{g},\mu^r} < 0$ in the whole B_r. Applying the divergence theorem, we see that this is not compatible with the condition $\nabla\psi_{\bar{g},\mu^r} = 0$ on ∂B_r, that is guaranteed by the specific choice of the radius, and we arrive to a contradition. Using again that u is subharmonic in B_r, we know that there exists $r_0\in[0,r)$ such that $u < 0$ in B_{r_0} and $u\geq 0$ in $B_r\setminus B_{r_0}$. If $r_0 = 0$, then $\Delta\psi_{\bar{g},\mu^r}\geq 0$ in B_r. Using once more the divergence theorem in B_r, and the vanishing condition of $\nabla\psi_{\bar{g},\mu^r}$ on ∂B_r, we obtain that $\psi_{\bar{g},\mu^r}$ is harmonic in B_r. However, this is not possible since $\Delta^2\psi_{\bar{g},\mu^r} > 0$ in B_r, so this phenomenon does not happen. Instead, if $r_0 > 0$, then we apply the divergence theorem in the annulus $A = B_r\setminus\bar{B}_{r_1}$, with $r_1\in(r_0,r)$, using that $\nabla\psi_{\bar{g},\mu^r} = 0$ on ∂B_r:

$$0\leq\int_A\Delta\psi_{\bar{g},\mu^r} = \int_{\partial B_r}\frac{x}{r}\cdot\nabla\psi_{\bar{g},\mu^r}(x) - \int_{\partial B_{r_1}}\frac{x}{r_1}\cdot\nabla\psi_{\bar{g},\mu^r}(x) = -\partial_\nu\psi_{\bar{g},\mu^r}(r_1e_1)|\partial B_{r_1}|.$$

Since $r_1\in(r_0,r)$ is arbitrary, this shows that $\psi_{\bar{g},\mu^r}$ is radially decreasing in $B_r\setminus B_{r_0}$. In particular, $\psi_{\bar{g},\mu^r}\geq\mathcal{E}(\mu^r)$ in that annulus. Now it is easy to infer the same inequality in the whole ball B_r. In fact, $\psi_{\bar{g},\mu^r}$ is superharmonic in B_{r_0}, and thus it is also radially decreasing. Since we have just proved that $\psi_{\bar{g},\mu^r}\geq\mathcal{E}(\mu^r)$ in ∂B_{r_0}, then the same inequality holds also in the interior of that ball.

We treat the region outside B_r, and we write down explicitly the expression of u using (1.55):

$$\begin{aligned}u(rx) = \Delta\psi_{\bar{g},\mu^r}(rx) &= \frac{r^{\alpha-2}}{|\partial B_1|}\int_{\partial B_1}(N+\alpha-2)|x-y|^{\alpha-2}d\mathscr{H}^{N-1}(y)\\ &\quad-\frac{r^{-\beta-2}}{|\partial B_1|}\int_{\partial B_1}\frac{(N-\beta-2)}{|x-y|^{\beta+2}}d\mathscr{H}^{N-1}(y).\end{aligned}$$

From this expression it is evident that u is radially strictly increasing in $\mathbb{R}^N \setminus \bar{B}_r$. Since we have already proved that $u(re_1) \geq 0$, we deduce that $\Delta\psi_{\bar{g},\mu^r} \geq 0$ out of $\bar{B}_r$. For any $r_1 > r$, we apply the divergence theorem in the annlus $A = B_{r_1} \setminus \bar{B}_r$. Similarly to what we did before, this argument shows that $\psi_{\bar{g},\mu^r}$ is radially increasing in that annulus. Since $r_1 > r$ is arbitrary, this is sufficient to prove that $\psi_{\bar{g},\mu^r} \geq \psi_{\bar{g},\mu^r}(re_1) = \mathcal{E}(\mu^r)$, yielding the validity of the Euler-Lagrange conditions (EL_p) for μ^r. □

Remark 1.4.10. We highlight that the previous argument shows that, for every $\alpha \geq 2$ and $\beta \in (0, N-4)$, the sphere is a local minimizer of $\mathcal{E}$ in the class of the symmetric measures $\mathcal{P}_{\text{rad}}(\mathbb{R}^N)$ since we have the strict inequality $\psi_{\bar{g},\mu^r} > \mathcal{E}(\mu^r)$ in $\mathbb{R}^N \setminus \partial B_r$.
For the same reason, to prove that the measure μ^r is a local minimizer in the whole class $\mathcal{P}(\mathbb{R}^N)$ one would need to prove that μ^r is a minimizer in the restricted class $\mathcal{P}(\partial B_r) \subset \mathcal{P}(\mathbb{R}^N)$. Unfortunately this information is not known to us, hence we cannot state this result in the extended range of parameters $\alpha \geq 2$, $\beta \in (0, N-4)$, but we only obtain the global minimality when $\alpha \in [2, 4]$ exploiting the convexity of the energy.

Chapter 2

L^∞-constrained problem and minimizing sets

In this chapter we address the minimization of the energy $\mathcal{E}$ among the subsets of $\mathbb{R}^N$ with a given mass, namely problem (P_S). This is a non-trivial shape optimization problem and, similarly to the case of measures (P_M), there are very different results depending on the interaction kernel $\bar{g}$ chosen in the energy. In fact, there is not only a formal analogy between the two problems, but there is a precise connection between them. First notice that, when $m = 1$, there is a strict inclusion between the classes considered in the minimization process: the characteristic functions of sets with unit volume belong to the class of densities with unitary L^1 norm and with image in $[0, 1]$, that is itself a subclass of the probability measures. Additionally, from the energetic point of view there is not so much difference in considering sets or densities. To be more precise, [BCT18, Theorem 4.5] shows that the infima in (P_D) and (P_S) have the same value, and that a set E minimizes (P_S) if and only if χ_E minimizes (P_D). From the perspective of addressing the shape optimization problem, the class of densities is considered just to relax the problem, and to easily have the existence result stated in Lemma 2.1.3.

The connection with (P_M) is related to the approximation procedure contained in Lemma 1.2.12. To be more clear, we notice that we can always rescale the densities, and work in a class with fixed L^1 norm equal to 1, while the height constraint becomes a parameter, studying the equivalent problem

$$\inf\left\{\mathcal{E}(f) : f : \mathbb{R}^N \to \left[0, m^{-1}\right], \|f\|_1 = 1\right\}.$$

In this way it is more transparent that, when the mass m is going to 0, we are relaxing the L^∞ constraint (that is the case treated in Section 2.2). However, the limit problem is well posed in $\mathcal{P}(\mathbb{R}^N)$, and not in $L^1(\mathbb{R}^N) \cap \mathcal{P}(\mathbb{R}^N)$, because this smaller space is not closed with respect to the weak$*$ topology of $\mathcal{P}(\mathbb{R}^N)$. This clarifies the intuition that, whenever we have an approximation result like Lemma 1.2.12, we can expect that the minimizers of (P_D) are very close to the minimal probabilities when $m \ll 1$. Essentially, when the parameter m is small our strategy consists in solving the problem (P_M), and then infer some properties about the minimizing densities that force them to attain only the values $\{0, 1\}$, i.e. they are characteristic functions of some sets. This analysis is carried out in [CPT24], and Section 2.2 is based on that work.

On the other hand, the situation in the large mass case is totally different. In fact, for some choices of the kernel $\bar{g}$, we are in the very peculiar situation of showing that minimizers of (P_S) exist because the only solutions of the density problem (P_D) are the characteristic functions of balls. This is obtained via a careful analysis of the energy dissipation in a iterative concentration procedure, that modifies a density in order to make it closer to the characteristic function of a ball. Additionally, a fundamental ingredient consists in proving that, roughly speaking, the Hausdorff distance between a ball and the support of a minimal density is controlled by the asymmetry of the density (when the mass is large). This is the content of Section 2.3, which is based on [Car23, Section 4].

The mechanism behind these phenomena at different scales (with respect to the mass) is similar: heuristically, when we see minimizing sets, there is a lot of attraction that keeps the density packed, saturating the L^∞ constraint in the problem (P_D). In the large mass case the attraction is due directly to the shape of the kernel $\bar{g}$ at large distance, and keeps the mass as close as possible to the origin. In the small mass case, instead, the attraction in encoded in the potential of the minimizing density f, i.e. ψ_f, since the support of a minimizing density is contained in a sublevel of its own potential. For many kernels these two phenomena cohexist, and some examples can be found among the power-law kernels $\bar{g}_p$ defined in (1). In general, it is very challenging to understand what happens for intermediate values of tha mass constraint. A peculiar example, where it is not easy to imagine what is going on for intermediate values of m, is the case of $\bar{g}_p$ with powers $\alpha > 4$, $\beta > 2$ and $\alpha > \beta$. In fact, in this situation, the minimizers with small mass are sets supported around a finite number of distant points (see Theorem 2.2.8), while the only minimizer with $m \gg 1$ is a large ball, as we discuss in Remark 2.3.13. In some way, there should be an interpolation between these two sets when m passes from being small to being large, and it is unclear how this process happens, and whether the minimizing densities of (P_D) continue to be sets or not. However, there are some cases in which we can treat all the mass constraints at once, as we do in Theorem 2.2.10. This is a very special situation where, enlarging the mass, the minimizer of the problem (P_S) passes from being an annulus to being a ball (that inflates at $m \nearrow +\infty$).

2.1 Basic results

This section is devoted to collect a few useful results. We state here the main hypotheses for the small mass case (concerning the full kernel) and for the large mass case (dealing only with the repulsive part). They are respectively:

($\mathbf{H}_s$) $\bar{g} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ is a radial, l.s.c. and locally integrable function, with $\lim_{|x|\to\infty} \bar{g}(x) = +\infty$ and $\bar{g}(0) = \lim_{x\to 0} \bar{g}(x)$. Moreover, the radial profile $g(|x|) = \bar{g}(x)$ is non-decreasing in $(L_g, +\infty)$ for some constant $L_g > 0$.

($\mathbf{H}_l$) $\bar{h} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ is of the form $\bar{h}(x) = h(|x|)$, where $h : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is of class C^1 away from the origin, with $h' \leq 0$. Moreover, the map $t \to h(t)t^{N-2}$ is locally integrable.

Proposition 2.1.1. *Let $\bar{g} \in L^1_{\text{loc}}(\mathbb{R}^N)$ be a given kernel. If f is a minimizer of (P_D), then*

$$\begin{cases} \psi_f = \lambda & \mathscr{L}^N\text{-a.e. in } \{0 < f < 1\}, \\ \psi_f \geq \lambda & \mathscr{L}^N\text{-a.e. in } \{f = 0\}, \\ \psi_f \leq \lambda & \mathscr{L}^N\text{-a.e. in } \{f = 1\}, \end{cases} \tag{EL_d}$$

for some constant $\lambda \in (-\infty, +\infty]$.

Sketch of the proof. The proof is very similar to Proposition 1.1.3, and we are very brief now. Given any function $\eta : \mathbb{R}^N \to [-1, 1]$ with compact support and satisfying $\int \eta = 0$ and $0 \leq f + \eta \leq 1$, for any $t \in [0, 1]$ we have

$$\mathcal{E}(f + t\eta) = \mathcal{E}(f) + 2t \int \psi_f(x)\eta(x)dx + t^2\mathcal{E}(\eta).$$

For any $0 \leq t \leq 1$ the function $f + t\eta$ is an admissible competitor in (P_D), thus by minimality we have $\mathcal{E}(f) \leq \mathcal{E}(f+t\eta)$. We notice that, since η has compact support, and $\bar{g} \in L^1_{\text{loc}}$, then $\mathcal{E}(\eta) < +\infty$. Hence, we deduce that $\psi_f(x) \leq \psi_f(y)$ for any two points x, y such that $f(x) > 0$ and $f(y) < 1$. This is stronger than (EL_d), and concludes the proof. □

Another standard result is the existence of minimizers for the problem (P_D). The proof in our general setting can be easily adapted from those already available in the literature, see for instance [CnCP15, SST15]. Apart from the specifics of our problem, we use the general scheme, often called concentration-compactness principle, that we recall here:

Lemma 2.1.2 (Concentration-compactness, [Str08])**.** *Let $\mu_n \in \mathcal{P}(\mathbb{R}^N)$ be a given sequence of probability measures. Then there exists a subsequence (not relabelled) such that one of the following holds:*

1. (Compactness) *There exists a sequence of points $x_n \in \mathbb{R}^N$ such that, for every $\varepsilon > 0$, there exists $L > 0$ large enough such that $\mu_n(B(x_n, L)) > 1 - \varepsilon$.*
2. (Vanishing) *For every $\varepsilon > 0$ and every $L > 0$ there exists $\bar{n} \in \mathbb{N}$ such that*
$$\mu_n(B(x, L)) < \varepsilon \qquad \forall x \in \mathbb{R}^N, \forall n > \bar{n}.$$
3. (Dichotomy) *There exist $\lambda \in (0, 1)$ and a sequence of points $x_n \in \mathbb{R}^N$ with the following property: for any $\varepsilon > 0$, there exists $L > 0$ such that, for any $L' > L$ there exist two non-negative measures μ_n^1 and μ_n^2 that satisfy, for every n large enough, the following conditions*
$$\begin{aligned} &\mu_n^1 + \mu_n^2 \leq \mu_n, \\ &\operatorname{spt} \mu_n^1 \subset B(x_n, L), \quad \operatorname{spt} \mu_n^2 \subset \mathbb{R}^N \setminus B(x_n, L'), \\ &\left|\mu_n^1(\mathbb{R}^N) - \lambda\right| + \left|\mu_n^2(\mathbb{R}^N) - (1 - \lambda)\right| < \varepsilon. \end{aligned}$$

Lemma 2.1.3. *Assume that $\bar{g} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ is lower semicontinuous, and that* $\lim_{|x| \to +\infty} \bar{g}(x) = +\infty$. *Then, for any $m > 0$ there exist a minimizer of (P_D).*

Proof. Let $C_m = C(\bar{g}, N, m)$ be the energy of a ball with mass m. We consider a competitor $f : \mathbb{R}^N \to [0,1]$ in the minimization of (P_D) with $\|f\|_1 = m$, and without loss of generality we can suppose that $\mathcal{E}(f) \le C_m$. We fix $\tilde{R} = \tilde{R}(\bar{g}, N, m) > 0$ so large that

$$\bar{g}(x) > \frac{5C_m}{m^2} \qquad \forall x \notin B_{\tilde{R}}. \tag{2.1}$$

We claim that, up to translations, we have $\int_{B_{\tilde{R}}} f \ge \frac{4m}{5}$. In fact, if this is not the case, then we apply the lower bound (2.1) to obtain

$$\begin{aligned}
\mathcal{E}(f) &\ge \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus B(x,\tilde{R})} \bar{g}(x-y) f(y) f(x) dy dx \\
&> \int_{\mathbb{R}^N} \frac{5C_m}{m^2} \left(\int_{\mathbb{R}^N \setminus B(x,\tilde{R})} f(y) dy \right) f(x) dx \\
&\ge \frac{5C_m}{m^2} \int_{\mathbb{R}^N} \frac{m}{5} f(x) dx = \frac{5C_m}{m^2} \cdot \frac{m^2}{5} = C_m,
\end{aligned}$$

and this contradicts the initial assumption $\mathcal{E}(f) \le C_m$. Let us define the auxiliary function $G : (0, +\infty) \to \mathbb{R}^+$ as $G(s) = \inf\{\bar{g}(x) : |x| > s\}$, that explodes at infinity. We observe that for any $\tilde{R}^+ > \tilde{R}$ we have the following estimate:

$$\begin{aligned}
\mathcal{E}(f) &\ge \int_{B_{\tilde{R}}} \int_{\mathbb{R}^N \setminus B_{\tilde{R}^+}} \bar{g}(x-y) f(y) f(x) dy dx \\
&\ge G(\tilde{R}^+ - \tilde{R}) \int_{B_{\tilde{R}}} f(y) dy \int_{\mathbb{R}^N \setminus B_{\tilde{R}^+}} f(x) dx \\
&\ge G(\tilde{R}^+ - \tilde{R}) \frac{4m}{5} \int_{\mathbb{R}^N \setminus B_{\tilde{R}^+}} f(x) dx.
\end{aligned}$$

This estimate shows at once that, for any competitor f such that $\mathcal{E}(f) \le C_m$, we have a uniform decay of its mass at infinity (since G explodes at infinity). Since $\mathcal{E}$ is lower semicontinuous with respect to the weak$*$ convergence, the existence of a minimizer is an easy application of the concentration compactness principle that we recalled in Lemma 2.1.2. □

The last result that we present is an a-priori bound on the diameter of the support of a minimizing density, and this deserves a quick comment. When dealing with minimizing measures, the boundedness of the support is a quite standard result, and it has been proved in several different contexts (see for instance [CnCP15, BCT18]). As we have already noticed, for many properties (for instance the existence given by the above lemma) working with measures or with densities does not make much difference. However, the compactness of the support of minimizers is more delicate for the case of densities due to the fact that the Euler–Lagrange condition (EL_p) for densities has an additional constraint. As a consequence, the proof of the result below does not follow by a simple generalization of the proofs available for the case of measures. Therefore we provide a complete proof.

Proposition 2.1.4. *Let $\bar g : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a function satisfying ($\mathbf{H}_s$). Then, there exists a constant $\tilde D = \tilde D(\bar g, N, m) > 0$ such that* $\operatorname{diam}\operatorname{spt} f_m \le \tilde D$ *for any minimizer f_m of (P_D). Finally, if $\bar g$ is also locally bounded in $\mathbb{R}^N$, then $D(\bar g, N, m)$ is uniformly bounded when $m \le 1$.*

Proof. The hypotheses allow to apply Lemma 2.1.3, hence we have a minimizer f for every $m > 0$. Let us denote by $\kappa = \kappa(N) > 0$ a purely geometric constat, whose value will be determined later in the proof. We fix the constant $\tilde R = \tilde R(\bar g, N, m) > L_g$ so large that (2.1) holds, and such that $|B_{\tilde R}| > \kappa m$. Notice that $\int_{B_{\tilde R}} f \ge \frac{4m}{5}$ thanks to the same argument of Lemma 2.1.3. We denote by C_m the energy of a ball of mass m, as we did in the existence lemma, so $\mathcal{E}(f) \le C_m$. Let now $\tilde R^+ = \tilde R^+(\bar g, m) \ge 50\tilde R$ be another constant satisfying

$$g(\tilde R^+ - \tilde R) \ge 2g(6\tilde R) + \frac{5}{2m}\int_{B_{11\tilde R}} \bar g(x)dx, \tag{2.2}$$

and we aim to prove that f is supported in $B_{\tilde R^+}$, so that the proof of the first part will be concluded with $\tilde D = 2\tilde R^+$. Let us call $f_2 = f\chi_{B_{\tilde R^+}\setminus B_{\tilde R}}$ and $f_3 = f\chi_{\mathbb{R}^N\setminus B_{\tilde R^+}}$, so that $f = f_1 + f_2 + f_3$. Calling now $\delta = \|f_2\|_1 + \|f_3\|_1$, and $\varepsilon = \|f_3\|_{L^1} \le \delta \le m/5$, our claim can be rewritten as $\varepsilon = 0$, thus we assume $\varepsilon > 0$ and we look for a contradiction. We point out that $\delta \le m/5$ because we already noticed that $\|f_1\|_1 = \int_{B_{\tilde R}} f \ge \frac{4m}{5}$.

Let z^+ be a minimum point of the potential $\psi_{f_2}(z) = \int_{\mathbb{R}^N} \bar g(z-y) f_2(y)dy$ within the support of f_3. Notice that such a minimizer exists. Indeed, by assumption the support of f_3 is a non-empty closed set, and the above function is either constantly 0 if $f_2 \equiv 0$ (and in such a case any point of the support is a minimizer), or it is a lower semicontinuous function which explodes for $|z| \to \infty$. The minimality property of z^+ ensures that

$$\mathcal{E}(f_2, f_3) = \int_{\mathbb{R}^N} \psi_{f_2}(z) f_3(z)dz \ge \psi_{f_2}(z^+)\|f_3\|_{L^1} = \psi_{f_2}(z^+)\varepsilon. \tag{2.3}$$

Let us now define the set

$$\mathcal{C} = \left\{ z \in \mathbb{R}^N :\, 4\tilde R \le |z| \le 5\tilde R,\, \frac{z\cdot z^+}{|z|\cdot|z^+|} \ge \cos(\pi/15) \right\},$$

which is the portion of cone highlighted in Figure 2.1. We call then $\kappa = \omega_N \tilde R^N/|\mathcal{C}|$, which is a purely geometrical constant only depending on N. Then, since by construction $|B_{\tilde R}| > \kappa m$, we have $|\mathcal{C}| = |B_{\tilde R}|/\kappa > m$. Since $\|f\|_{L^1(B_{\tilde R^+})} = m - \varepsilon$, there exists a positive density $\tilde f_3$, supported in $\mathcal{C}$, such that

$$\|\tilde f_3\|_{L^1} = \varepsilon, \qquad\qquad 0 \le \tilde f := f_1 + f_2 + \tilde f_3 \le 1.$$

In particular, the fact that $\tilde f_3$ is supported in $\mathcal{C}$ gives

$$\frac{z\cdot z^+}{|z|\cdot|z^+|} \ge \cos(\pi/15) \qquad \forall z \in \operatorname{spt}\tilde f_3. \tag{2.4}$$

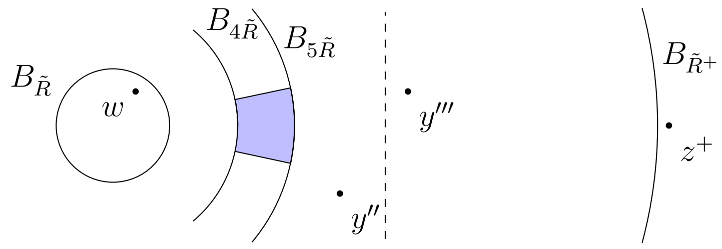


Figure 2.1: The construction in Proposition 2.1.4.

We will conclude our proof by showing that $\mathcal{E}(f) > \mathcal{E}(\tilde f)$, which will contradict the minimality of f since by construction $\tilde f$ is a competitor for problem (P_D). Notice that

$$\mathcal{E}(f) - \mathcal{E}(\tilde f) = 2\Big(\mathcal{E}(f_1, f_3) - \mathcal{E}(f_1, \tilde f_3) + \mathcal{E}(f_2, f_3) - \mathcal{E}(f_2, \tilde f_3)\Big) + \mathcal{E}(f_3) - \mathcal{E}(\tilde f_3). \quad (2.5)$$

Let us evaluate separately the different pieces. First of all, by construction

$$\mathcal{E}(f_1, f_3) \geq g(\tilde R^+ - \tilde R)\|f_3\|_{L^1}\|f_1\|_{L^1} = g(\tilde R^+ - \tilde R)\varepsilon(m-\delta),$$
$$\mathcal{E}(f_1, \tilde f_3) \leq g(6\tilde R)\|\tilde f_3\|_{L^1}\|f_1\|_{L^1} = g(6\tilde R)\varepsilon(m-\delta),$$

thus by (2.2) and since $\delta \leq m/5$ and (2.1) we have

$$\mathcal{E}(f_1, f_3) - \mathcal{E}(f_1, \tilde f_3) \geq \frac{4}{5}m\varepsilon\bigg(g(6\tilde R) + \frac{5}{2m}\int_{B_{11\tilde R}} \bar g(x)dx\bigg) > 4\varepsilon\frac{C_m}{m} + 2\varepsilon\int_{B_{11\tilde R}} \bar g(x)dx. \quad (2.6)$$

To estimate $\mathcal{E}(f_2, f_3) - \mathcal{E}(f_2, \tilde f_3)$, it is convenient to subdivide $\mathbb{R}^N$ into three pieces. The first one is the ball $H' = B_{6\tilde R}$, and the other two are

$$H'' = \bigg\{x \notin H' : \frac{x\cdot z^+}{|z^+|} \leq \frac{1}{2}\tilde R^+\bigg\}, \qquad H''' = \bigg\{x \notin H' : \frac{x\cdot z^+}{|z^+|} > \frac{1}{2}\tilde R^+\bigg\},$$

which are respectively on the left and on the right of the dashed hyperplane in the figure. We call then f_2', f_2'' and f_2''' the restrictions of f_2 to H', H'' and H''', so that $f_2 = f_2' + f_2'' + f_2'''$. We now observe that

$$\begin{aligned}\mathcal{E}(f_2', \tilde f_3) &= \int_{\mathbb{R}^N}\int_{H'} \bar g(y'-z)\tilde f_3(z)f_2(y')dy'dz \leq \int_{\mathbb{R}^N}\int_{B_{6\tilde R}} \bar g(y'-z)\tilde f_3(z)dy'dz \\ &\leq \int_{\mathbb{R}^N}\int_{B_{11\tilde R}} \bar g(x)\tilde f_3(z)dxdz = \varepsilon\int_{B_{11\tilde R}} \bar g(x)dx.\end{aligned} \quad (2.7)$$

Next, we pass to f_2''. For any $y'' \in H'' \cap B_{\tilde R^+}$ and $z \in \operatorname{spt}\tilde f_3$, by construction and using (2.4) we have $\tilde R < |y'' - z| \leq |y'' - z^+|$. Since g is non-decreasing on $[\tilde R, +\infty)$, also by (2.3) we can evaluate

$$\begin{aligned}\mathcal{E}(f_2'', \tilde f_3) &= \int_{H''}\int_{\mathbb{R}^N} \bar g(y''-z)f_2(y'')\tilde f_3(z)dzdy'' \\ &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \bar g(y''-z^+)f_2(y'')\tilde f_3(z)dzdy'' = \varepsilon\int_{\mathbb{R}^N} \bar g(y''-z^+)f_2(y'')dy'' \\ &= \varepsilon\psi_{f_2}(z^+) \leq \mathcal{E}(f_2, f_3).\end{aligned} \quad (2.8)$$

The argument to estimate $\mathcal{E}(f_2''', \tilde f_3)$ is similar. Since for any $w \in B_{\tilde R}$, any $y''' \in H''' \cap B_{\tilde R^+}$, and any z in the support of $\tilde f_3$ we have, by construction and an elementary trigonometric calculation, $|y''' - w| \geq |y''' - z| > \tilde R$, we evaluate

$$\mathcal{E}(f_2''', \tilde f_3) = \int_{H'''} \int_{\mathbb{R}^N} \bar g(y''' - z) f_2(y''') \tilde f_3(z) dz dy''' \leq \varepsilon \int_{H'''} \bar g(y''' - w) f_2(y''') dy'''.$$

Since this is true for every $w \in B_{\tilde R}$, and f_1 is supported on $B_{\tilde R}$, we obtain

$$\begin{aligned}(m-\delta)\mathcal{E}(f_2''', \tilde f_3) &= \int_{B_{\tilde R}} \mathcal{E}(f_2''', \tilde f_3) f_1(w) dw \\ &\leq \varepsilon \int_{B_{\tilde R}} \int_{H'''} \bar g(y''' - w) f_2(y''') f_1(w) dy''' dw \\ &= \varepsilon \mathcal{E}(f_1, f_2''') \leq \varepsilon \mathcal{E}(f) \leq \varepsilon C_m.\end{aligned}$$

Since $\delta \leq m/5$, this implies that

$$\mathcal{E}(f_2''', \tilde f_3) \leq 2\varepsilon \frac{C_m}{m}. \tag{2.9}$$

Putting together (2.7), (2.8) and (2.9), we have

$$\mathcal{E}(f_2, f_3) - \mathcal{E}(f_2, \tilde f_3) > -2\varepsilon \frac{C_m}{m} - \varepsilon \int_{B_{11\tilde R}} \bar g(x) dx \tag{2.10}$$

Lastly, since the support of $\tilde f_3$ is contained in $\mathcal{C}$, whose diameter is much smaller than $11\tilde R$, we can readily estimate

$$\mathcal{E}(\tilde f_3) = \mathcal{E}(\tilde f_3, \tilde f_3) \leq \int_{\mathbb{R}^N} \int_{B(z, 11\tilde R)} \bar g(y - z) \tilde f_3(z) dy dz = \varepsilon \int_{B_{11\tilde R}} \bar g(x) dx. \tag{2.11}$$

Inserting (2.6), (2.10) and (2.11) into (2.5), and minding also $\mathcal{E}(f_3) \geq 0$, we finally obtain

$$\mathcal{E}(f) - \mathcal{E}(\tilde f) \geq 4\varepsilon \frac{C_m}{m} + \varepsilon \int_{B_{11\tilde R}} \bar g(x) dx > 0,$$

thus the contradiction $\mathcal{E}(f) > \mathcal{E}(\tilde f)$ is established and this concludes the first part of the proof.

Assume now that $\bar g$ is locally bounded, and let us notice that a simple modification of the proof provides the same constant $\tilde D(\bar g, N, m)$ for every $m \leq 1$. Notice first that any ball with volume $m \leq 1$ has radius less than $\omega_N^{-1/N}$, thus $C_m \leq C m^2$, where $C = \sup\{\bar g(x), |x| \leq 2\omega_N^{-1/N}\}$. As a consequence, one can take the same radius $\tilde R$ in (2.1) for every $m \leq 1$. Up to enlarging $\tilde R$, we can also suppose that $g(s) \leq g(\tilde R)$ for every $s \in (0, \tilde R)$. The radius $\tilde R^+$ defined in (2.2) explodes when $m \searrow 0$, but the local boundedness of $\bar g$ allows for a much simpler definition of $\tilde R^+$. More precisely, using that g is non-decreasing in $(\tilde R, +\infty)$, we can replace (2.7) with the simpler inequality

$$\mathcal{E}(f_2', \tilde f_3) \leq \frac{\varepsilon m}{5} g(11\tilde R),$$

and then the proof works with no other modification replacing the definition (2.2) of $\tilde{R}^+$ by

$$g(\tilde{R}^+ - \tilde{R}) \geq 2g(6\tilde{R}) + g(11\tilde{R}),$$

which does not depend on m. Since $\tilde{D}(\bar{g}, N, m) = 2\tilde{R}^+$, the proof is concluded. □

2.2 Existence of minimal sets with small mass

We are going to consider the minimization problem (P_D) with generic interaction kernels which are *weakly repulsive* (in the origin). Roughly speaking, for us this means that $\bar{g}$ is bounded in the origin, and $|\nabla \bar{g}| \ll 1$ in a small neighborhood of 0. In particular, while for strongly repulsive kernels, like $\bar{g}_p$ defined in (1) with $\beta < 0$, a measure containing some atom has always infinite energy, for weakly repulsive kernels atomic measures have finite energy, and hence they are possible candidates for the minimization problem (P_M). This is not just a theoretical possibility; in fact, Carrillo, Figalli and Patacchini showed in [CFP17] that global minimizers of (P_M) among probability measures are supported on finitely many points if $\bar{g}(x) = g(|x|)$, $\bar{g}(0) = 0$, and there exist $C > 0$ and $\beta > 2$ such that

$$\lim_{t \to 0} g'(t) t^{1-\beta} = -C.$$

An important example of a weakly repulsive kernel is given by the power-like kernels defined in (1) with $\alpha > \beta > 2$. This kind of kernel has been studied by Davies, Lim and McCann in a series of papers [LM21, DLM22, DLM23], and they are able to precisely characterize the solutions of (P_M) in some cases.

Theorem 2.2.1 (Davies–Lim–McCann, [DLM22, DLM23])**.** *Let $N \geq 2$, and $\bar{g} = \bar{g}_p$ be given by (1). If $\beta = 2 < \alpha < 4$, then the unique minimizer of (P_M), up to rigid motion, is given by the uniform distribution over a sphere, that is, $\mu = c\mathscr{H}^{N-1} \llcorner \partial B_r$ for a suitable choice of c and r. If $\alpha \geq 4$, $\beta \in [2, \alpha)$ and $(\alpha, \beta) \neq (4, 2)$, then the unique minimizer is given by a purely atomic measure uniformly distributed over the vertices of the unit regular $(N+1)$-gon Δ_N.*

The minimizers have been investigated also in dimension $N = 1$. In this case, the unique minimizer is given by two equal masses at distance 1 as soon as $\alpha \geq 3$, $\beta \in [2, \alpha)$ (see [DLM23]), while for $2 < \alpha < 3$, $\beta = 2$ the minimizer, which is computed explicitly in [Fra22], is absolutely continuous and supported on an interval.

As we already mentioned, in this section we study the question of existence of optimal *sets*, that is, minimizers of (P_S). The euristics behind our results consists of two parts. First, in order to have a minimizer μ of (P_M) that is singular with respect to the Lebesgue measure, we need to have a lot of attraction nearby the support of μ. On top of that, for small mass, the minimizers of (P_D) are close (in some sense) to the probability minimizers. Therefore, we can transfer some properties of the minimal measures to the minimal densities, and this is enough to obtain the existence of minimizing sets in (P_S). This is, in few words, the heart of our argument, that is contained in Subsection 2.2.1, and working with weakly repulsive kernels ensures that we see some attraction close to the support of a minimizing measure. Some more precise results are available for a special choice of the kernel, and we detail these cases in Subsection 2.2.2. We also provide, in Subsection 2.2.3, some sufficient conditions on the kernel which guarantee that the conclusions of Theorem 2.2.8 hold true. We stress that these conditions are quite generic, and do not require a precise structure of the kernel, but rather a quantitative control on its shape. In particular, they are stable with respect to small perturbations of the kernel.

We remark that the existence of minimal sets for similar energies was also investigated, with different techniques, by Clark in her Ph.D. thesis [Cla22].

2.2.1 General strategy for the small mass minimizers

A minimizer of the problem (P_D) exists under mild hypotheses on $\bar g$, as we recalled in Lemma 2.1.3. In general, however, the minimization problem among sets (P_S) does not necessarily admit a minimizer (see for example [BCT18, FL18]). Here we show that, under certain rather general conditions on $\bar g$, the solutions to the problem (P_D) are characteristic functions of sets when m is small enough (and thus they coincide with the solutions of (P_S), as proved in [BCT18, Theorem 4.5]). Here we focus on the general results, while we specialize to the cases considered by [DLM22, DLM23] in Subsection 2.2.2.

Lemma 2.2.2. *Let $\bar g \in C(\mathbb{R}^N)$ be a function satisfying ($\mathbf{H}_s$). Let f_j be a minimizer of (P_D) with $\|f_j\|_1 = m_j \searrow 0$. Then, up to translations and up to taking a subsequence, $m_j^{-1} f_j \stackrel{*}{\rightharpoonup} \mu$ for some $\mu \in \mathcal{P}(\mathbb{R}^N)$ minimizing (P_M). Moreover, if $\psi_\mu > \mathcal{E}(\mu)$ in $\mathbb{R}^N \setminus \operatorname{spt} \mu$, then for any $\varepsilon > 0$ there is $\bar j$ such that*

$$\operatorname{spt} f_j \subseteq B_\varepsilon + \operatorname{spt} \mu \qquad \forall\, j > \bar j\,. \tag{2.12}$$

Proof. Since $\bar g$ is continuous, thus locally bounded, Proposition 2.1.4 ensures that the supports of the densities f_j are uniformly bounded. Therefore, the probability measures $\mu_j = m_j^{-1} f_j$ have uniformly bounded supports and then, up to subsequences and translations, we have $\mu_j \stackrel{*}{\rightharpoonup} \mu$ for some probability measure μ with bounded support.

Let now $\bar\mu$ be any minimizer of (P_M), and for any j let $\mathscr{P}_j$ be a partition of $\mathbb{R}^N$ made by pairwise disjoint cubes of volume m_j. We define the measure $\nu_j \in \mathcal{P}(\mathbb{R}^N)$ with density

$$\nu_j(x) = \frac{1}{m_j}\bar\mu(Q) \qquad \forall\, Q \in \mathscr{P}_j, \forall\, x \in Q\,,$$

and the density $\tilde f_j = m_j \nu_j$. By construction, $0 \le \tilde f_j \le 1$ and $\|\tilde f_j\|_1 = m_j$, so by the minimality of f_j we have $\mathcal{E}(f_j) \le \mathcal{E}(\tilde f_j)$. The continuity of $\bar g$ easily guarantees that $\mathcal{E}(\nu_j) \to \mathcal{E}(\bar\mu)$, and then also by the lower semicontinuity of $\mathcal{E}$ we deduce

$$\mathcal{E}(\mu) \le \liminf_j \mathcal{E}(\mu_j) = \liminf_j m_j^{-2}\mathcal{E}(f_j) \le \liminf_j m_j^{-2}\mathcal{E}(\tilde f_j) = \liminf_j \mathcal{E}(\nu_j) = \mathcal{E}(\bar\mu)\,.$$

Hence, μ is a minimizer of $\mathcal{E}$ in $\mathcal{P}(\mathbb{R}^N)$.

Suppose now that $\psi_\mu > \mathcal{E}(\mu)$ outside of $\operatorname{spt}\mu$, and keep in mind that $\psi_\mu = \mathcal{E}(\mu)$ on $\operatorname{spt}\mu$ by Proposition 1.1.3. Since $\bar{g}$ is continuous and explodes at infinity, and since μ has bounded support, we deduce that the potential ψ_μ is continuous and explodes at infinity as well. Combining these properties with the inequality $\psi_\mu > \mathcal{E}(\mu)$ valid out of $\operatorname{spt}\mu$ by assumption, we see that for any $\varepsilon > 0$ there exists $\gamma > 0$ such that $\psi_\mu(x) \geq \mathcal{E}(\mu) + \gamma$ whenever $\operatorname{dist}(x, \operatorname{spt}\mu) \geq \varepsilon$. Let us now call $U = \operatorname{spt}\mu + B_\varepsilon$ and $V = \operatorname{spt}\mu + B_\delta$, with δ so small that $\psi_\mu(x) < \psi_\mu(y) - \gamma/2$ for any $x \in V$ and any $y \in U^c$. Since $\bar{g}$ is continuous and $\operatorname{spt}\mu_j$ are uniformly bounded, then ψ_{μ_j} are locally uniformly continuous with a common modulus of continuity. The convergence $\mu_j \stackrel{*}{\rightharpoonup} \mu$ guarantees then that ψ_{μ_j} converge pointwise to ψ_μ, and thanks to the common modulus of continuity this convergence is locally uniform. Therefore, if j is large enough we have that

$$\psi_{\mu_j}(x) < \psi_{\mu_j}(y) - \frac{\gamma}{3} \qquad \forall x \in V,\ y \in U^c. \tag{2.13}$$

Suppose now by contradiction that (2.12) does not hold true. In other words, lets suppose that f_j is not supported in U for arbitrarily large indexes j. Then, for every $\eta_j \ll m_j$, we can define a modified function $0 \leq \hat{f}_j \leq 1$ by "moving a mass η_j from U^c to V". Formally speaking, $\hat{f}_j$ is a function such that $0 \leq \hat{f}_j \leq f_j$ on U^c while $f_j \leq \hat{f}_j \leq 1$ on V, and so that

$$\int_V \hat{f}_j - f_j = \int_{U^c} f_j - \hat{f}_j = \eta_j.$$

The existence of such a function $\hat{f}_j$ is obvious as soon as $m_j \leq |V|$, which is certainly true for j large enough. Then we call $\hat{\mu}_j = m_j^{-1}\hat{f}_j$, and $\hat{\nu}_j = \mu_j - \hat{\mu}_j = m_j^{-1}(f_j - \hat{f}_j)$, so that $\|\hat{\nu}_j\|_{\mathcal{M}} = 2m_j^{-1}\eta_j$. Thus, we estimate

$$\begin{aligned}
\mathcal{E}(\hat{\mu}_j) - \mathcal{E}(\mu_j) &= \mathcal{E}(\hat{\nu}_j) + 2\mathcal{E}(\mu_j, \hat{\nu}_j) \\
&= \mathcal{E}(\hat{\nu}_j) + 2\int_{\mathbb{R}^N} \psi_{\mu_j}(x) d\hat{\nu}_j(x) \leq C\,\|\hat{\nu}_j\|^2_{\mathcal{M}} - \frac{\gamma}{3}\,\|\hat{\nu}_j\|_{\mathcal{M}},
\end{aligned}$$

where we have used (2.13), the continuity of $\bar{g}$, and that the support of $\hat{\nu}_j$ is bounded. For $\frac{\eta_j}{m_j} \ll 1$ this gives $\mathcal{E}(\hat{\mu}_j) < \mathcal{E}(\mu_j)$, thus $\mathcal{E}(\hat{f}_j) < \mathcal{E}(f_j)$, and this is impossible since f_j is a minimizer of (P_D) and $\hat{f}_j$ is a competitor. □

Theorem 2.2.3. *Let $\bar{g} \in C^2(\mathbb{R}^N)$ be a function satisfying $(\mathbf{H}_s)$. Let f_j be a minimizer of (P_D) with $\|f_j\|_1 = m_j$ and any sequence $m_j \searrow 0$, and assume that $m_j^{-1} f_j \stackrel{*}{\rightharpoonup} \mu \in \mathcal{P}(\mathbb{R}^N)$. If $\psi_\mu > \mathcal{E}(\mu)$ in $\mathbb{R}^N \setminus \operatorname{spt}\mu$ and $D^2\psi_\mu \neq 0$ in every point of $\operatorname{spt}\mu$, then f_j is the characteristic function of a set when j is large enough.*

Proof. By Lemma 2.2.2 we know that μ is a minimizer of (P_M), and that (2.12) holds. The potential ψ_μ is of class C^2 because the kernel is regular, and conditions (EL_p) guarantee that ψ_μ attains its minimum in $\operatorname{spt}\mu$. Therefore, the hessian of the potential is non-negative definite. Since by hypothesis $D^2\psi_\mu \neq 0$ in $\operatorname{spt}\mu$, then for every $x \in \operatorname{spt}\mu$ there exits $v \in \mathbb{S}^{N-1}$ such that $\partial_v^2\psi_\mu(x) > 0$. By compactness, there are finitely many points $x_1, x_2, \dots, x_k \in \operatorname{spt}\mu$, corresponding directions $v_1, v_2, \dots, v_k \in \mathbb{S}^{N-1}$, and two constants $\delta, r > 0$ such that the balls $B_r(x_i)$ cover the whole $\operatorname{spt}\mu$, and one has

$$\partial^2_{v_i}\psi_\mu(y) > 2\delta \qquad \forall i \in \{1, \dots, k\}, \forall y \in B_r(x_i). \tag{2.14}$$

Since $\operatorname{spt}\mu$ is covered by the finitely many balls $B_r(x_i)$, there exists some $\varepsilon > 0$ such that the balls cover also $\operatorname{spt}\mu + B_\varepsilon$, thus also $\operatorname{spt} f_j$ for any j large enough, by (2.12). Moreover, we have that $m_j^{-1} D^2\psi_{f_j}$ converges to $D^2\psi_\mu$ locally uniformly because $m_j^{-1} f_j \overset{*}{\rightharpoonup} \mu$ and $\bar{g} \in C^2(\mathbb{R}^N)$. Therefore, (2.14) holds also replacing 2δ with δ and ψ_μ with $m_j^{-1}\psi_{f_j}$ for every j large enough. This condition clearly implies that each level set of ψ_{f_j} has zero measure. But the Euler-Lagrange conditions (EL_d) ensure that $\{0 < f_j < 1\}$ is contained in a single level set. We deduce then that the function f_j has value 0 or 1 almost everywhere, thus it is the characteristic function of a set. □

Corollary 2.2.4. *Let $\bar{g} \in C^2(\mathbb{R}^N)$ satisfy ($\mathbf{H}_s$). Suppose that, for any minimizer μ of (P_M), we have*

1. *$\psi_\mu > \mathcal{E}(\mu)$ in $\mathbb{R}^N \setminus \operatorname{spt}\mu$;*
2. *$D^2\psi_\mu \neq 0$ everywhere in $\operatorname{spt}\mu$.*

Then, there exists $\overline{m} > 0$ such that any f_m minimizing (P_D) with $\|f_m\|_1 = m$ is the characteristic function of a set when $m < \overline{m}$.

Proof. We proceed by contradiction. If the thesis is false, there exists some sequence $m_j \searrow 0$ and densities f_j which minimize (P_D) with mass m_j that are not characteristic functions. Since, as already noticed in the proof of Lemma 2.2.2, their supports are uniformly bounded, up to a translation and a subsequence we have that $m_j^{-1} f_j \overset{*}{\rightharpoonup} \mu \in \mathcal{P}(\mathbb{R}^N)$. Since μ is a minimizer of $\mathcal{E}$ in $\mathcal{P}(\mathbb{R}^N)$ by Lemma 2.2.2, our assumption ensures that we can apply Theorem 2.2.3, clearly obtaining a contradiction. □

Remark 2.2.5. We observe that the proofs of Theorem 2.2.3 and Corollary 2.2.4 work also if we have a function $\bar{g} \in C^{2k}(\mathbb{R}^N)$ and, for a given μ that minimizes $\mathcal{E}$ in $\mathcal{P}(\mathbb{R}^N)$ (or any minimizer, in the corollary), we have that for every $x \in \operatorname{spt}\mu$ there exist $v \in \mathbb{S}^{N-1}$ and $j \in \{1, \dots, k\}$ with $\partial_v^{2j}\psi_\mu(x) > 0$.

Remark 2.2.6. This approach is similar to [FL18], where they deduce the existence of minimizing sets with large mass. We stress that our assumptions avoid the technical problems that are addressed in that paper concerning the regularity of the potential.
It is worth to point out the general idea behind this approach. Frank and Lieb work with a specific choice for the kernel, that naturally involves subharmonic functions, so in particular the Laplace operator. However, loosely speaking, one can obtain some results when the kernel $\bar{g}$ satisfies a differential inequality with respect to a differential operator $\mathcal{L}$ of order k with constant coefficients, and $\psi_f \in W^{k,1}_{\mathrm{loc}}$ for any bounded density f with compact support. An example of this phenomenon, with $\mathcal{L} = \Delta$, is [FL18, Proposition 5.3].

2.2.2 More precise results for power-law kernels

This section is devoted to discuss the situation in the special case of a function $\bar{g}$ of power-law type defined in (1). Let us start with a couple of definitions. We define by $\Delta_N := \{x_1, \dots, x_{N+1}\} \subset \mathbb{R}^N$ the vertices of the standard regular $(N+1)$-gon centered at the origin and with mutual distance 1. We call $H_N = \sqrt{\frac{N+1}{2N}}$ its height, and $C_N = \sqrt{\frac{N}{2N+2}}$ its circumradius. Moreover, we define

$$\mu_{\Delta_N} := \frac{1}{N+1} \sum_{i=1}^{N+1} \delta_{x_i} \tag{2.15}$$

the probability measure which is uniformly distributed over the points of Δ_N. We present now a geometric result which will provide us a positive bound on the second derivative of the potential.

Lemma 2.2.7. *The constant*

$$K_N := \min \left\{ \sum_{i=1}^{N+1} \langle v, x_i - x_1 \rangle^2 \,:\, v \in \mathbb{S}^{N-1} \right\} \tag{2.16}$$

satisfies $K_N = 1$ if $N = 1$ and $K_N = 1/2$ if $N \geq 2$.

Proof. First of all, we claim that for every $N \geq 2$ and every $v \in \mathbb{S}^{N-1}$

$$\sum_{i=1}^{N+1} \langle v, x_i \rangle^2 = \frac{1}{2}. \tag{2.17}$$

To do so, we decompose $v = v_1 + v_2$, where v_2 is the projection of v onto the hyperplane Π parallel to the face containing $x_2, \dots, x_{N+1}$ and passing through the origin. We can write

$$\begin{aligned}
\sum_{i=1}^{N+1} \langle v, x_i \rangle^2 &= \sum_{i=1}^{N+1} \big(\langle v_1, x_i \rangle + \langle v_2, x_i \rangle \big)^2 \\
&= \sum_{i=1}^{N+1} \langle v_1, x_i \rangle^2 + \sum_{i=1}^{N+1} \langle v_2, x_i \rangle^2 + 2 \sum_{i=1}^{N+1} \langle v_1, x_i \rangle \langle v_2, x_i \rangle.
\end{aligned}$$

Notice now that by definition $\langle v_2, x_1 \rangle = 0$, and $\langle v_1, x_i \rangle$ has the same value for each $i \geq 2$. Since $\sum_{i=1}^{N+1} x_i = 0$, we deduce that the last sum vanishes. Moreover, notice that $|x_i| = C_N$, and the distance of any x_i with $i \geq 2$ from the hyperplane Π is $H_N - C_N$. Therefore

$$\begin{aligned}
\sum_{i=1}^{N+1} \langle v, x_i \rangle^2 &= |v_1|^2 C_N^2 + \sum_{i=2}^{N+1} \langle v_1, x_i \rangle^2 + \sum_{i=2}^{N+1} \langle v_2, x_i \rangle^2 \\
&= |v_1|^2 \Big(C_N^2 + N \big(H_N - C_N \big)^2 \Big) + \big(1 - |v_1|^2 \big) \sum_{i=2}^{N+1} \langle \frac{v_2}{|v_2|}, x_i \rangle^2 \\
&= \frac{|v_1|^2}{2} + \big(1 - |v_1|^2 \big) \sum_{i=2}^{N+1} \langle \frac{v_2}{|v_2|}, x_i \rangle^2.
\end{aligned}$$

The last expression is linear with respect to $|v_1|^2$. Therefore, either it is constant, or it is minimized for $|v_1| = 0$ or $|v_1| = 1$. This means that, if the sum in (2.17) is not constant, then it is minimized only if v is either parallel or orthogonal to x_1. However, the same should be true also with any other x_i, and this is clearly impossible. We deduce then that the sum in (2.17) is constant, and then it is enough to choose $|v_1| = 1$ to deduce that the constant value is $1/2$, that is, (2.17) is proved.

Let us now consider the sum in (2.16). We can assume that $N \geq 3$, since the cases $N = 1, 2$ are elementary computations. Arguing similarly as before, we get

$$\begin{aligned}\sum_{i=1}^{N+1} \langle v, x_i - x_1\rangle^2 &= \sum_{i=2}^{N+1} \langle v_1, x_i - x_1\rangle^2 + \langle v_2, x_i - x_1\rangle^2 + 2\langle v_1, x_i - x_1\rangle\langle v_2, x_i - x_1\rangle \\ &= NH_N^2|v_1|^2 + \sum_{i=2}^{N+1}\langle v_2, x_i\rangle^2 - 2H_N\langle v_1, \frac{x_1}{|x_1|}\rangle \sum_{i=2}^{N+1}\langle v_2, x_i\rangle \\ &= \frac{N+1}{2}|v_1|^2 + \sum_{i=2}^{N+1}\langle v_2, x_i\rangle^2 + 2H_N\langle v_1, \frac{x_1}{|x_1|}\rangle\langle v_2, x_1\rangle \\ &= \frac{N+1}{2}|v_1|^2 + \sum_{i=2}^{N+1}\langle v_2, x_i\rangle^2 \\ &= \frac{N+1}{2}|v_1|^2 + \big(1 - |v_1|^2\big)\sum_{i=2}^{N+1}\langle \frac{v_2}{|v_2|}, x_i\rangle^2.\end{aligned}$$

Notice now that the projections of the points x_i with $2 \leq i \leq N$ on the $(N-1)$-dimensional hyperplane Π are the vertices of the standard regular N-gon centered at the origin. Therefore, the property (2.17) in dimension $N - 1 \geq 2$ ensures us that the value of the last sum in the above estimate is $1/2$, regardless of what v_2 is. Therefore, we have

$$\sum_{i=1}^{N+1}\langle v, x_i - x_1\rangle^2 = \frac{N+1}{2}|v_1|^2 + \frac{1-|v_1|^2}{2} = \frac{N|v_1|^2 + 1}{2},$$

and the minimum of this expression among all $v \in \mathbb{S}^{N-1}$ is clearly $1/2$. Therefore, the proof is completed. □

We can now present our main results for the power-law kernel $\bar{g}$ given by (1).

Theorem 2.2.8. *Let $N \geq 2$ and let $\bar{g} = \bar{g}_p$ be defined by (1), with $\alpha > \beta \geq 2$, $\alpha \geq 4$ and $(\alpha, \beta) \neq (4, 2)$. Then, if m is small enough, every minimizer of (P_D) is the characteristic function of some set E_m which is then a minimizer of (P_S). Moreover, E_m consists of $N+1$ convex components, each of which is contained in a small neighborhood of a vertex of Δ_N.*

Proof. With this choice of powers α, β, we know by [DLM23, Theorem 1.2, Corollary 1.4] that the measure μ_{Δ_N} defined in (2.15) is the only minimizer of $\mathcal{E}$ in $\mathcal{P}(\mathbb{R}^N)$ (up to rotations and translations), and that $\psi_{\mu_{\Delta_N}} > \mathcal{E}(\mu_{\Delta_N})$ outside of $\operatorname{spt}\mu_{\Delta_N} = \Delta_N$.

We now want to compute the first and second derivatives of the function

$$\psi_{\mu_{\Delta_N}} = \frac{1}{N+1}\sum_{i=1}^{N+1}\psi_{\delta_{x_i}}$$

at the point x_1. First of all, we do some computations that rely only on the symmetry of the kernel: for any choice of $x \in \mathbb{R}^N$, $v \in \mathbb{S}^{N-1}$, and $t > 0$ one has

$$\frac{d}{dt}\bar{g}(x+tv) = g'(|x+tv|)\frac{\langle x, v\rangle + t}{|x+tv|},$$
$$\left.\frac{d^2}{dt^2}\right|_{t=0}\bar{g}(x+tv) = g''(|x|)\frac{\langle x, v\rangle^2}{|x|^2} + \frac{g'(|x|)}{|x|}\left(1 - \frac{\langle x, v\rangle^2}{|x|^2}\right).$$

So, with the kernel $\bar{g} = \bar{g}_p$ defined in (1), keeping in mind that $g'(1) = 0$ and $g''(1) = \alpha - \beta$, we have for every $i \geq 2$ that

$$\begin{aligned}\partial_v^2\psi_{\delta_{x_i}}(x_1) &= g''(|x_i - x_1|)\frac{\langle x_i - x_1, v\rangle^2}{|x_i - x_1|^2} + \frac{g'(|x_i - x_1|)}{|x_i - x_1|}\left(1 - \frac{\langle x_i - x_1, v\rangle^2}{|x_i - x_1|^2}\right)\\ &= \big(\alpha - \beta\big)\langle x_i - x_1, v\rangle^2,\end{aligned}$$

while of course

$$\partial_v^2\psi_{\delta_{x_1}}(x_1) = g''(0).$$

We have now to distinguish the cases $\beta = 2$ and $\beta > 2$. If $\beta > 2$, then $g''(0) = 0$ and then by Lemma 2.2.7

$$\partial_v^2\psi_{\mu_{\Delta_N}}(x_1) = \frac{1}{N+1}\sum_{i=2}^{N+1}\partial_v^2\psi_{\delta_{x_i}}(x_1) \geq \frac{(\alpha-\beta)K_N}{N+1} \geq \frac{\alpha-\beta}{2(N+1)},$$

so $\partial_v^2\psi_{\mu_{\Delta_N}} > 0$ for every $v \in \mathbb{S}^{N-1}$. Instead, if $\beta = 2$, then $g''(0) = -1$, and then

$$\partial_v^2\psi_{\mu_{\Delta_N}} = \frac{1}{N+1}\left(-1 + \sum_{i=2}^{N+1}\partial_v^2\psi_{\delta_{x_i}}(x_1)\right) \geq \frac{-1 + (\alpha-2)K_N}{N+1}. \tag{2.18}$$

Since $K_N = 1/2$ by Lemma 2.2.7 and $\alpha > 4$ because we are considering $\beta = 2$, then $-1 + (\alpha - 2)K_N > 0$, hence again $\partial_v^2\psi_{\mu_{\Delta_N}} > 0$ for every $v \in \mathbb{S}^{N-1}$. The fact that any minimizer of problem (P_D) with $\|f_m\|_1 = m$ is given by a characteristic function $f_m = \chi_{E_m}$ is then ensured by Corollary 2.2.4. Moreover, we know that the sets E_m converge to Δ_N in the Hausdorff sense by Lemma 2.2.2, and $m^{-1}D^2\psi_{f_m}$ converges to $D^2\psi_{\mu_{\Delta_N}}$ locally uniformly as noticed in Theorem 2.2.3. As a consequence, $D^2\psi_{f_m}$ is strictly positive definite in a neighborhood of each point x_i when m is sufficiently small, and so the set $E_m \cap B(x_i, 1/2)$ is convex for each i because it coincides with the sublevel set of a convex function. □

Remark 2.2.9. The same result is true also if $N = 1$ for $\alpha > \beta \geq 2, \alpha \geq 3$ and $(\alpha, \beta) \neq (3, 2)$. The proof is exactly the same, the only difference is that the term in (2.18) was strictly positive since $\alpha - 2 > 2$ and $K_N = 1/2$, while now it is strictly positive since $\alpha - 2 > 1$ and $K_N = 1$.

Differently from before, the next theorem shows that for certain choices of the parameters α and β the minimizer of (P_D) is the characteristic function of a set *for all* values of m.

Theorem 2.2.10. *Let $\bar{g}$ be defined as in (1) with $\beta = 2$, $N \geq 2$ and $\alpha \in (2, 4)$, or $\beta = 2$, $N = 1$ and $\alpha \in (3, 4)$. Then, for every $m > 0$ the minimizer f_m of (P_D) is the characteristic function of a radial set, which is either an annulus or a ball. In particular, as $m \searrow 0$, the set is an annulus which converges to a sphere in Hausdorff distance.*

Proof. Let us consider any $m > 0$. As we detail in Subsection 1.4.1, the choice of the parameters $\alpha \in (2, 4)$ and $\beta = 2$ ensures that the energy $\mathcal{E}$ is strictly convex among the functions with barycenter in the origin. This implies that there is only a single minimizer among the densities with barycenter in the origin, and thanks to the invariance of the energy by rotation we obtain that this minimal function has to be spherically symmetric. Since f_m is spherically symmetric, and since $\alpha > 2$ for $N \geq 2$ or $\alpha > 3$ for $N = 1$, [DLM22, Theorem 2.2] ensures that the potential ψ_{f_m} has positive third derivative, that is, calling $\Upsilon(s) = \psi_{f_m}(se_1)$, one has $\Upsilon'''(s) > 0$ for every $s > 0$. Moreover, $\Upsilon'(0) = 0$ because ψ_{f_m} is regular and radial. This implies that all level sets of Υ are given by either one or two points, hence for every $\lambda \in \mathbb{R}$ the set $\{x \in \mathbb{R}^N : \Upsilon(|x|) = \lambda\}$ is negligible with respect to $\mathscr{L}^N$. Proposition 1.1.3 ensures that the potential attains a constant value λ in the set where $0 < f_m < 1$, hence the previous observation shows that the set $\{0 < f_m < 1\}$ is $\mathscr{L}^N$-negligible, and this precisely means that f_m is the characteristic function of some set E_m, which, in turn, is radial because so is f_m. Moreover, calling $I \subseteq \mathbb{R}$ the set such that $E_m = \{x \in \mathbb{R}^N : |x| \in I\}$, again Proposition 1.1.3 ensures that $I = \{s \in \mathbb{R} : \Upsilon(s) < \lambda\}$ for some $\lambda \in \mathbb{R}$ (up to $\mathscr{L}^1$-negligible sets). Keeping again in mind that $\Upsilon'(0) = 0$ and $\Upsilon'''(s) > 0$ for all $s > 0$, we have that the sublevel sets of Υ are all intervals, either of the form (a, b) for some $0 \leq a < b$, or of the form $[0, b)$ for some $b > 0$. This means that E_m is either an annulus or a ball. In particular, Lemma 2.2.2 ensures that E_m is an annulus for $m \ll 1$, since it must converge in the Hausdorff sense to a sphere for $m \searrow 0$. On the other hand, E_m is surely a full ball for $m \gg 1$ (and the large mass asymptotics is treated in broader generality in Section 2.3). □

Remark 2.2.11. In certain circumstances, it is easy to prove that minimizing densities are necessarily characteristic functions of some sets, without any information about the minimal measure, and without assuming that the mass constraint is small or large. This is related to Remark 2.2.6, since in some cases we have a *global* differential inequality for the kernel $\bar{g}$, as we noticed in Subsection 1.4.2. To be more precise, an immediate computation shows that whenever $N \geq 2$, $\alpha \geq 2$ and $4 - N < \beta < 0$, and we consider the power-law kernel $\bar{g} = \bar{g}_p$ defined in (1), then $\Delta^2 \bar{g}_p > 0$ outside of the origin, and $\bar{g}_p \in W^{4,1}_{\text{loc}}(\mathbb{R}^N)$. These features guarantee that, for every density $f : \mathbb{R}^N \to [0, 1]$ with compact support, we have $\psi_f \in C^4(\mathbb{R}^N)$ and $\Delta^2 \psi_f > 0$ everywhere. Since the potential is regular, then $\Delta^2 \psi_f = 0$ almost everywhere in the level sets of ψ_f, and hence each level set must have 0 Lebesgue measure. This information, combined with (EL_d), shows that any minimizing density (for any mass constraint) must be the characteristic function of a set. However, we highlight that this does not give any information about

the geometry of that set.
We finally remark that this approach works for any differential operator with constant coefficients, and that we do not need any symmetry information about the kernel.

2.2.3 The study of general kernels

We know that, by Theorem 2.2.1, in the special case when $\bar{g}$ is a power-law kernel of the form (1) for a suitable choice of the parameters α, β, the unique minimizing measure is the purely atomic measure $\bar{\mu}$ uniformly distributed over the vertices of the regular $(N+1)$-gon Δ_N. The goal of this last section is to show that minimality of such a measure does not necessarily require the particular form (1), but it can also be a consequence of more geometrical, general properties of $\bar{g}$. Let us be more precise. If $\bar{g}(x) = g(|x|)$ and we assume, just to fix the ideas, that $g(0) = 1$, $g(1) = 0$ and $g(t) \gtrsim 1$ for $t \neq 1$, then pairs of points in the support of an optimal measure have convenience to stay at distance 1, but it is impossible that all pair of points have distance 1 since every point of the support has distance 0 from itself. It is reasonable to guess that in some cases the most convenient choice could be to have as many points as possible with mutual distance 1, hence, with $N+1$ points in the vertices of a unit regular $(N+1)$-gon. In particular, one can imagine that this could happen whenever $g \approx 0$ only in a small neighborhood of 1, and g is flat enough close to the origin. In this section, we are going to prove that it is indeed so. In order to present a simple proof with geometric flavour, we use highly non-sharp assumptions, and we write the proof in the planar case $N = 2$ for simplicity of notations. The general case $N \geq 3$ does not require any different ideas. The only caveat is notational complication due to several indices. The final Remark 2.2.14 discusses the case of higher dimensions with slight improvements of the constants.

The first result we present is a "confinement result", which says that if the value of g is close to 0 only at points close to 1, and $g \geq 1 - \eta$ everywhere else for some small η, then an optimal measure must be supported in a union of 3 small balls around the vertices of Δ_2. We represent in Figure 2.2, on the left, the shape of g.

Lemma 2.2.12 (Confinement around Δ_2)**.** *Let $\bar{g} \in C(\mathbb{R}^2; \mathbb{R}^+)$ be a radial kernel, with radial profile g such that $g(0) = 1$, $g(1) = \min g = 0$, and for some $\eta < 1/64$ and $\xi < 1/165$ one has*

$$g(t) > 1 - \eta \quad \text{for } t \in [0, 3/2] \setminus (1 - \xi, 1 + \xi), \qquad g(t) > 1 \quad \text{for } t \geq 3/2.$$

Then, every minimizing measure $\mu \in \mathcal{P}(\mathbb{R}^2)$ is supported in the union of three sets with diameter less than 5ξ and mutual distance between $1 - 6\xi$ and $1 + \xi$. More precisely, given any point in any of the three sets, its distance with each of the other two sets is between $1 - 6\xi$ and $1 + \xi$.

Proof. The assumptions on g imply that its graph must be in the shaded region in Figure 2.2, left (a possible choice of g is depicted just as an example). Let μ be an optimal measure for the problem (P_M). We divide the proof in few steps.

Step I. The diameter of $\operatorname{spt}\mu$ *is at most* $3/2$.
Let us call $\bar{\mu}$ the measure which is uniformly distributed over the vertices of an equilateral triangle of side 1. Then by minimality of μ we have

$$\mathcal{E}(\mu) \leq \mathcal{E}(\mu_{\Delta_2}) = \frac{1}{3}. \tag{2.19}$$

Assume now the existence of $x_1, x_2 \in \operatorname{spt}\mu$ with $|x_1 - x_2| > 3/2$. For a small $r \ll 1$, that will be specified in few lines, we can take two measures $\mu_1, \mu_2 \leq \mu$ so that $\gamma := \|\mu_1\|_{\mathcal{M}} = \|\mu_2\|_{\mathcal{M}} > 0$ and $\operatorname{spt}\mu_i \subset B_{r/2}(x_i)$. For every $-1 < \varepsilon < 1$ we define $\mu_\varepsilon = \mu + \varepsilon(\mu_1 - \mu_2)$, which is still a probability measure. We have

$$\mathcal{E}(\mu_\varepsilon) = \mathcal{E}(\mu) + 2\varepsilon\Big(\mathcal{E}(\mu, \mu_1) - \mathcal{E}(\mu, \mu_2)\Big) + \varepsilon^2\Big(\mathcal{E}(\mu_1) + \mathcal{E}(\mu_2) - 2\mathcal{E}(\mu_1, \mu_2)\Big).$$

However, keeping in mind Proposition 1.1.3 and the fact that $\mu_1 \leq \mu$, we have

$$\mathcal{E}(\mu, \mu_1) = \iint \bar{g}(x-y)d\mu_1(x)d\mu(y) = \int \psi_\mu(x)d\mu_1(x) = \gamma\mathcal{E}(\mu),$$

and similarly $\mathcal{E}(\mu, \mu_2) = \gamma\mathcal{E}(\mu)$. Therefore, the above expression becomes

$$\mathcal{E}(\mu_\varepsilon) = \mathcal{E}(\mu) + \varepsilon^2\Big(\mathcal{E}(\mu_1) + \mathcal{E}(\mu_2) - 2\mathcal{E}(\mu_1, \mu_2)\Big). \tag{2.20}$$

Since by assumption $g(0) = 1 < C := \bar{g}(x_1 - x_2)/2 > 0$, we can pick $r > 0$ so small that

$$g(s) < C < g(t) \qquad \text{for every } 0 \leq s \leq r \text{ and } |x_1 - x_2| - 2r \leq t \leq |x_1 - x_2| + 2r.$$

We have then

$$\mathcal{E}(\mu_1) = \iint \bar{g}(x-y)d\mu_1 d\mu_1 < C\gamma^2 \quad \text{ and } \quad \mathcal{E}(\mu_2) = \iint \bar{g}(x-y)d\mu_2 d\mu_2 < C\gamma^2,$$

while

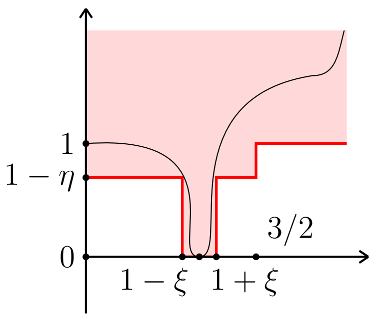
$$\mathcal{E}(\mu_1, \mu_2) = \iint \bar{g}(x-y)d\mu_1 d\mu_2 > C\gamma^2.$$

This ensures that the term in parentheses in (2.20) is strictly negative, giving $\mathcal{E}(\mu_\varepsilon) < \mathcal{E}(\mu)$ which contradicts the minimality of μ. This concludes the step.
Step II. The sets A_x, A_y *and* $Q_{x,y}$.
Let us now fix any point $x \in \operatorname{spt}\mu$, and call

$$A_x = \big\{y : 1 - \xi < |y - x| < 1 + \xi\big\}$$

the annulus centered at x with radii $1 - \xi$ and $1 + \xi$. By (2.19) and minding (EL_p), we have

$$\begin{aligned}\frac{1}{3} \geq \mathcal{E}(\mu) = \psi_\mu(x) &= \int_{A_x} \bar{g}(x-y)d\mu(y) + \int_{\mathbb{R}^2 \setminus A_x} \bar{g}(x-y)d\mu(y)\\ &\geq (1-\eta)(1 - \mu(A_x)) \geq 1 - \eta - \mu(A_x),\end{aligned}$$



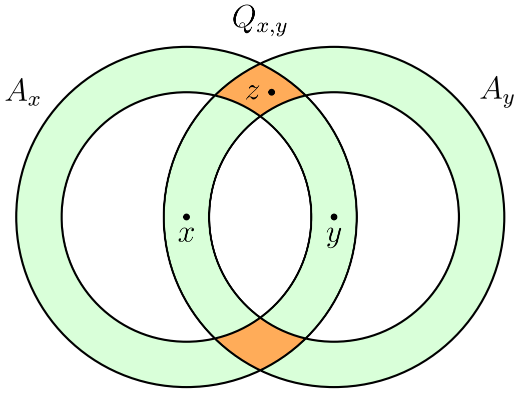


Figure 2.2: Left: the graph of g must be in the shaded region. Right: the points x, y and z and the sets A_x, A_y and $Q_{x,y}$ in Step II.

which can be rewritten as

$$\mu(A_x) \geq \frac{2}{3} - \eta. \tag{2.21}$$

We can then take a second point y in $\operatorname{spt}\mu$ so that $y \in A_x$, and then also $x \in A_y$. The intersection $A_x \cap A_y$ is made by two different connected pieces, orange in Figure 2.2, right. A trivial computation ensures that, by the assumption on ξ, the diameter of each piece is less than 5ξ and the distance between the two pieces is more than $3/2$. Step I implies then that at least one connected piece of $A_x \cap A_y$ is μ-negligible. On the other hand, applying (2.21) both to x and y we obtain that $\mu(A_x \cap A_y) \geq 1/3 - 2\eta > 0$, and then exactly one connected piece of $A_x \cap A_y$ has positive μ-measure. We call $Q_{x,y}$ this piece, so that, as just observed,

$$\mu(Q_{x,y}) \geq \frac{1}{3} - 2\eta. \tag{2.22}$$

Step III. The point z and the conclusion.
We can now define a third point $z \in \operatorname{spt}\mu \cap Q_{x,y}$, so that each of the annuli A_x, A_y and A_z centered at one of the points x, y, z contains the other two points. Moreover, keeping the same notation as in Step II, we call $Q_1 = Q_{x,y}$, $Q_2 = Q_{x,z}$ and $Q_3 = Q_{y,z}$. Let now w_1 be any point in $\operatorname{spt}\mu$; since by Step I we know that the distance between w_1 and any of the points x, y, z is at most $3/2$, an immediate computation ensures that, thanks to the bound on ξ, the distance between w_1 and at least one of the points x, y, z is less than $1 - 6\xi$. To fix the ideas, we can assume that

$$|z - w_1| < 1 - 6\xi. \tag{2.23}$$

We assume then the existence of a point $w_2 \in Q_1$ such that

$$|w_2 - w_1| > 5\xi, \tag{2.24}$$

and we look for a contradiction. Notice that this contradiction will conclude the proof; indeed, if (2.24) is false for every $w_2 \in Q_1$, there are some consequences. The first one is that the whole $\operatorname{spt}\mu$ is contained in the three balls of radius 5ξ centered at x, y

and z. Then, a second consequence is that the intersection of any of these balls with $\operatorname{spt}\mu$ has diameter at most 5ξ, and thus μ is supported in the union of three sets with diameter less than 5ξ. Moreover, by construction, for every point a in one of these sets, the annulus A_a intersects both the other two sets, and as a consequence the distance between a and each of the other two sets is between $1-6\xi$ and $1+\xi$. Therefore, we only have to get a contradiction.

Let us write $\psi_\mu = \psi_1 + \psi_2 + \psi_3 + \psi_\infty$, where we define

$$\psi_i(a) = \int_{Q_i} \bar{g}(a-b)d\mu(b) \quad \forall i \in \{1,2,3\}, \quad \psi_\infty(a) = \int_{\mathbb{R}^2\setminus(Q_1\cup Q_2\cup Q_3)} \bar{g}(a-b)d\mu(b).$$

Notice now that, for every $b \in \operatorname{spt}\mu \cap Q_1$, since the diameter of Q_1 is less than 5ξ and by (2.23) we have $|b-w_2| < 5\xi < 1-\xi$ and $|b-w_1| < 1-\xi$, and by the assumption on g this implies

$$\psi_1(w_1) > (1-\eta)\mu(Q_1), \qquad \psi_1(w_2) > (1-\eta)\mu(Q_1). \tag{2.25}$$

These inequalities, combined with the trivial bound $\psi_1(w_2) \leq \psi_\mu(w_2) \leq \mathcal{E}(\mu) \leq 1/3$ that follows from (EL_p), we obtain

$$\mu(Q_1) \leq \frac{1}{3} + \eta, \tag{2.26}$$

Take now any two points $p \in Q_2$, $q \in Q_3$. As seen before, $A_p \cap A_q$ has diameter less than 5ξ, so by the assumption (2.24) at least one between w_1 and w_2 does not belong to $A_p \cap A_q$, hence

$$\bar{g}(p-w_2) + \bar{g}(p-w_1) + \bar{g}(q-w_2) + \bar{g}(q-w_1) \geq 1-\eta.$$

Consequently, also by the bound in (2.22) with Q_2 and Q_3 in place of $Q_{x,y} = Q_1$, we have

$$\begin{aligned}
\mu(Q_2)\big(\psi_3(w_1) + \psi_3(w_2)\big) &+ \mu(Q_3)\big(\psi_2(w_1) + \psi_2(w_2)\big) \\
&= \int_{Q_2}\int_{Q_3} \bar{g}(w_2-p) + \bar{g}(w_1-p) + \bar{g}(w_2-q) + \bar{g}(w_1-q)d\mu(p)d\mu(q) \\
&\geq (1-\eta)\mu(Q_2)\mu(Q_3) \geq (1-\eta)\left(\frac{1}{3} - 2\eta\right)^2.
\end{aligned} \tag{2.27}$$

Again by (2.19) and (EL_p), using the inequalities (2.22), (2.25) and (2.27) in combination with the trivial lower bound $\psi_\infty(w_1) + \psi_\infty(w_2) \geq 0$, we have

$$\begin{aligned}
\frac{2}{3} &\geq \psi_\mu(w_1) + \psi_\mu(w_2) \\
&= \psi_1(w_1) + \psi_2(w_1) + \psi_3(w_1) + \psi_\infty(w_1) + \psi_1(w_2) + \psi_2(w_2) + \psi_3(w_2) + \psi_\infty(w_2) \\
&> 2(1-\eta)\mu(Q_1) \\
&\quad + \left(\frac{1}{3} + \eta\right)^{-1} \left[\mu(Q_2)\big(\psi_3(w_1) + \psi_3(w_2)\big) + \mu(Q_3)\big(\psi_2(w_1) + \psi_2(w_2)\big)\right] \\
&\geq (1-\eta)\left[\left(\frac{2}{3} - 4\eta\right) + \left(\frac{1}{3} + \eta\right)^{-1}\left(\frac{1}{3} - 2\eta\right)^2\right],
\end{aligned}$$

and we arrive to the desired contradiction since this inequality is impossible for $\eta < 1/64$. □

The main result of this subsection is that, under suitable assumptions on the second derivative of g around 0 and around 1, the unique optimal measure is purely atomic and uniformly distributed over the vertices of a triangle of side 1. More precisely, we have the following result:

Theorem 2.2.13. *Let $\bar g \in C(\mathbb{R}^2;\mathbb{R}^+)$ be a radial kernel, with radial profile g such that $g(0) = 1$, $g(1) = \min g = 0$, and for some $\eta < 1/64$ and $\xi < 1/165$ one has*

$$g(t) > 1-\eta \quad \text{for } t \in [0,3/2] \setminus (1-\xi, 1+\xi), \qquad g(t) > 1 \quad \text{for } t \geq 3/2.$$

Additionally, let us assume that

$$g''(t) > -12 g''(s) \qquad \forall t \in (0, 5\xi), s \in (1-6\xi, 1+6\xi).$$

Then, the unique optimal measure (up to translations and rotations) is the purely atomic one, uniformly distributed over the vertices of Δ_2.

Proof. Let μ be an optimal measure. By Lemma 2.2.12, μ is supported on three sets B_1, B_2, B_3, with diameter less than 5ξ and mutual distance between $1-6\xi$ and $1+\xi$. Moreover, by (2.22) and (2.26), each of them has measure between $\frac{1}{3} - 2\eta$ and $\frac{1}{3} + \eta$. Let us call $C' = -\min\{g''(t) : 0 \leq t \leq 5\xi\}$ and $C'' = \min\{g''(t) : 1-6\xi \leq t \leq 1+6\xi\}$.

Let us take any four points x, y, z, w in $\operatorname{spt}\mu$, in particular $x, y \in B_1$, $z \in B_2$ and $w \in B_3$. By construction and by Lemma 2.2.12, we have that

$$|x-y| \leq 5\xi, \qquad |x-z| \leq 1+6\xi, \qquad |x-w| \leq 1+6\xi, \qquad |z-w| \geq 1-6\xi.$$

Calling for brevity $\theta_{a,b}$ the direction of the vector $a-b$ for any two points $a \neq b \in \mathbb{R}^2$, the above estimates give

$$\sin\big(|\theta_{x,z} - \theta_{y,z}|\big) \leq \frac{5\xi}{1-6\xi}, \qquad \sin\left(\frac{|\theta_{x,z} - \theta_{x,w}|}{2}\right) \geq \frac{1}{2} \cdot \frac{1-6\xi}{1+6\xi}. \tag{2.28}$$

Two elementary trigonometric estimates tell that, for a generic direction $v \in \mathbb{S}^{N-1}$,

$$\begin{aligned} |\theta_{x,z} \cdot v|^2 + |\theta_{x,w} \cdot v|^2 &\geq 2\sin^2\left(\frac{|\theta_{x,z} - \theta_{x,w}|}{2}\right), \\ \Big||\theta_{x,z} \cdot v|^2 - |\theta_{y,z} \cdot v|^2\Big| &\leq \sin\big(|\theta_{x,z} - \theta_{y,z}|\big). \end{aligned} \tag{2.29}$$

In particular, we set $v = \theta_{x,y}$. Let us now consider the difference $\big||y-z| - |x-z|\big|$. By convexity of the distance, we can estimate this difference from below by $|y-x|$ multiplied either by $|\theta_{x,z} \cdot v|$ or by $|\theta_{y,z} \cdot v|$, unless the projection of z onto the line passing through x and y is contained inside the segment xy, which means that $\theta_{x,z}$ and v are very close to be perpendicular (and we discuss this case, which is in fact simpler, in a moment). We then have that

$$\Big||y-z| - |x-z|\Big| \geq \min\Big\{|\theta_{x,z} \cdot v|, |\theta_{y,z} \cdot v|\Big\}|y-x|,$$

which in turn yields

$$\bar{g}(x-z)+\bar{g}(y-z) \geq \frac{C''}{4} \min\Big\{|\theta_{x,z}\cdot v|, |\theta_{y,z}\cdot v|\Big\}^2 |y-x|^2. \tag{2.30}$$

We can now repeat the very same argument with w in place of z. Again, unless $\theta_{x,w}$ is very close to be perpendicular to v, we have

$$\bar{g}(x-w)+\bar{g}(y-w) \geq \frac{C''}{4} \min\Big\{|\theta_{x,w}\cdot v|, |\theta_{y,w}\cdot v|\Big\}^2 |y-x|^2. \tag{2.31}$$

Putting together (2.28) and (2.29), and in particular observing that the second estimate in (2.28) holds also with y in place of x since x and y are generic points in B_1, we get that

$$\begin{aligned}
\min\Big\{|\theta_{x,z}\cdot v|, |\theta_{y,z}\cdot v|\Big\}^2 &+ \min\Big\{|\theta_{x,w}\cdot v|, |\theta_{y,w}\cdot v|\Big\}^2 \\
&\geq 2\sin^2\left(\frac{|\theta_{x,z}-\theta_{x,w}|}{2}\right) - \Big||\theta_{x,z}\cdot v|^2 - |\theta_{y,z}\cdot v|^2\Big| \\
&\geq \frac{1}{2}\left(\frac{1-6\xi}{1+6\xi}\right)^2 - \frac{5\xi}{1-6\xi} \geq \frac{2}{5},
\end{aligned}$$

where the last estimate is true by the assumption in Lemma 2.2.12 that $\xi < 1/165$. This last estimate together with (2.30) and (2.31) gives

$$\bar{g}(x-z)+\bar{g}(y-z)+\bar{g}(x-w)+\bar{g}(y-w) \geq \frac{C''}{10}|y-x|^2. \tag{2.32}$$

Recall that (2.32) holds under the assumption that v is not very close to be perpendicular to either $\theta_{x,z}$ or $\theta_{x,w}$. However, if this is the case then an even stronger estimate holds; in fact, if for instance v is almost perpendicular to $\theta_{x,w}$, then we simply have

$$\Big||y-z|-|x-z|\Big|+\Big||y-w|-|x-w|\Big| \geq \Big||y-z|-|x-z|\Big| \geq \min\Big\{|\theta_{x,z}\cdot v|, |\theta_{y,z}\cdot v|\Big\}|y-x|,$$

and since the minimum is close to $\sqrt{3}/2$ because the triangle xyz is nearly equilateral, the resulting estimate is stronger than (2.32). Hence, the validity of (2.32) is established in any case. Concerning $\bar{g}(y-x)$, on the other hand, we have

$$\bar{g}(y-x) \geq 1 - \frac{C'}{2}|y-x|^2. \tag{2.33}$$

Let us write $\psi^- = \psi_{\mu\llcorner(B_2\cup B_3)}$, that is, $\psi^-(a) = \int_{B_2\cup B_3} \bar{g}(a-b)d\mu(b)$. Using (2.32), and recalling that $\bar{g}(x-z)+\bar{g}(y-z)$ and $\bar{g}(x-w)+\bar{g}(y-w)$ are both non-negative, we obtain

$$\begin{aligned}
\psi^-(x)+\psi^-(y) &= \int_{B_2} \bar{g}(x-z)+\bar{g}(y-z)d\mu(z) + \int_{B_3} \bar{g}(x-w)+\bar{g}(y-w)d\mu(w) \\
&\geq \frac{C''}{10}|y-x|^2 \min\big\{\mu(B_2), \mu(B_3)\big\} \\
&\geq \frac{C''}{10}|y-x|^2\left(\frac{1}{3}-2\eta\right) \geq \frac{C''}{34}|y-x|^2.
\end{aligned} \tag{2.34}$$

We now evaluate $\mathcal{E}(\mu\llcorner B_1,\mu)$, which by ($EL_p$) coincides with $\mu(B_1)\mathcal{E}(\mu)$. We have

$$\mathcal{E}(\mu\llcorner B_1,\mu)=\int_{B_1}\int_{B_1}\bar{g}(y-x)d\mu(y)d\mu(x)+\int_{B_1}\int_{B_2\cup B_3}\bar{g}(y-x)d\mu(y)d\mu(x)=:\mathcal{E}_1+\mathcal{E}_2.$$

By (2.33), we get

$$\mathcal{E}_1\geq\mu(B_1)^2-\frac{C'}{2}\int_{B_1}\int_{B_1}|y-x|^2d\mu(y)d\mu(x).$$

Instead, concerning $\mathcal{E}_2$, by (2.34) we have

$$\begin{aligned}\mathcal{E}_2&=\int_{B_1}\psi^-(x)d\mu(x)=\frac{1}{2\mu(B_1)}\int_{B_1}\int_{B_1}\psi^-(x)+\psi^-(y)d\mu(x)d\mu(y)\\&\geq\frac{1}{2\mu(B_1)}\int_{B_1}\int_{B_1}\frac{C''}{34}|y-x|^2d\mu(x)d\mu(y)\\&=\frac{C''}{68\mu(B_1)}\int_{B_1}\int_{B_1}|y-x|^2d\mu(x)d\mu(y).\end{aligned}$$

Now, the assumptions imply that $C''\geq 12C'\geq 34\mu(B_1)C'$. Hence, from the two estimates above we get that $\mathcal{E}(\mu\llcorner B_1,\mu)\geq\mu(B_1)^2$, with strict inequality unless $\mu\llcorner B_1$ is supported in a single point. Since the same estimate clearly works with $\mu\llcorner B_2$ and $\mu\llcorner B_3$ in place of $\mu\llcorner B_1$, calling $\mathfrak{m}_i=\mu(B_i)$ for $i\in\{1,2,3\}$ and keeping in mind that $\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{m}_3=1$, we get

$$\mathcal{E}(\mu)\geq\mathfrak{m}_1^2+\mathfrak{m}_2^2+\mathfrak{m}_3^2\geq\frac{1}{3}.$$

Since we already noticed that $\mathcal{E}(\mu)\leq\frac{1}{3}$, we finally deduce that necessarily $\mathfrak{m}_1=\mathfrak{m}_2=\mathfrak{m}_3=\frac{1}{3}$ and each of the three measures $\mu\llcorner B_i$ is supported in a single point. In addition, all the distances between any two of these three points must be equal to 1, as we claimed. □

Remark 2.2.14. In the general case of dimension $N\geq 3$, one can perform the very same construction as in Lemma 2.2.12 and Theorem 2.2.13, and obtain the very same results. More precisely, there are explicitly computable constants $\bar{\eta}$, $\bar{\xi}$, c_1 and c_2, only depending on the dimension, such that the following holds. If $\bar{g}\in C(\mathbb{R}^N;\mathbb{R}^+)$ is a radial function with radial profile g such that $g(0)=1$, $g(1)=\min g=0$, and for some $\eta<\bar{\eta}$ and $\xi<\bar{\xi}$ one has $g(t)>1-\eta$ for $t\in[0,\sqrt{3}H_N]\setminus(1-\xi,1+\xi)$ and $g(t)>1$ for $t\geq\sqrt{3}H_N$, then every minimizing measure is supported over the union of $N+1$ sets with diameter less than $c_1\xi$ and mutual distance between $1-(1+c_1)\xi$ and $1+\xi$. In addition, if $g''(t)\geq -c_2g''(s)$ for every $t\in(0,c_1\xi)$ and $s\in\Big(1-(1+c_1)\xi,1+(1+c_1)\xi\Big)$, then the unique optimal measure is the purely atomic one, uniformly distributed over the vertices of Δ_N.

2.3 Large mass case

We devote this section to the study of the minimization problem (P_D) with large mass constraint m. We aim to generalize the result in [FL21], and we highlight the key features that make their proof work. In this situation, the result clearly depends strongly on the confining term $\mathfrak{g}_\alpha$ that appears in the energy (with $\alpha > 0$). The fundamental notion to measure the distance between a density and a ball is the asymmetry, define as:

Definition 2.3.1. Given a non-negative and integrable density $f : \mathbb{R}^N \to [0,1]$, we call *Frankel asymmetry* of f, or simply *asymmetry*, the quantity

$$A(f) := \inf\left\{ \frac{\left\| f - \chi_{B(x,R)} \right\|_1}{\|f\|_1} : x \in \mathbb{R}^N, |B(x,R)| = \|f\|_1 \right\}.$$

Our program starts with some geometric estsimates contained in Subsection 2.3.1, where we show that the asymmetry of a minimizer f_m is infinitesimal when the mass constraint m is large. Additionally, we prove a diamenter bound for spt f_m with the natural scaling, namely $m^{1/N}$. This control reveals to be important in Proposition 2.3.11. In Subsection 2.3.2, instead, we provide a precise control on the potential when the density satisfies some geometric hypotheses. Finally, we prove the main result of this section, namely Theorem 2.3.12, in Subsection 2.3.3, exploiting a clever construction developed in [FL21] to promote a control on the asymmetry to a bound on the Hausdorff distance between the support of an optimal density and a ball.
We preliminarly show that, whenever we have a part of the kernel that is locally integrable in $\mathbb{R}^N$ and locally bounded away from the origin, we can control its contibution to the potential of a density f in a linear way with respect to $\|f\|_1$:

Lemma 2.3.2. *Let $\bar{h} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a kernel of class L^1_{loc} that is locally bounded away from the origin. Then, there exists a constant $K_{\bar{h},N} > 0$, such that, for any density $f : \mathbb{R}^N \to [0,1]$ with $\|f\|_1 \geq \omega_N$, we have the following bound on the potential:*

$$\left\|\psi_{f,\bar{h}}\right\|_\infty \leq K_{\bar{h},N} \|f\|_1 .$$

Proof. We split the contributions at short range (strong interaction, but bounded mass) and the long range ones (with weak interaction). In fact, we have that

$$\begin{aligned}
\psi_{f,\bar{h}}(x) &= \int_{B(x,1)} \bar{h}(x-y)f(y)dy + \int_{\mathbb{R}^N\setminus B(x,1)} \bar{h}(x-y)f(y)dy \\
&\leq \int_{B(x,1)} \bar{h}(x-y)dy + \left(\sup_{|x|\geq 1} \bar{h}(x)\right) \int_{\mathbb{R}^N\setminus B(x,1)} f(y)dy \\
&\leq \frac{\|f\|_1}{\omega_N} \int_{B(x,1)} \bar{h}(x-y)dy + \left(\sup_{|x|\geq 1} \bar{h}(x)\right) \int_{\mathbb{R}^N\setminus B(x,1)} f(y)dy.
\end{aligned}$$

Because of our assumptions on $\bar{h}$, the final expression is clearly bounded by the L^1 norm of f, up to a multiplicative constant that depends only on $\bar{h}$ and N. □

2.3.1 Geometric properties of minimizers with large mass

Even if Theorem 2.3.12 requires the aforementioned structure for $\bar{g}$, i.e. $\bar{g} = \mathfrak{g}_\alpha + \bar{h}$, we state some results in a more general framework. In particular, Lemma 2.3.3 and Lemma 2.3.5 do not require that particular splitting of the kernel.

Lemma 2.3.3. *Let $\bar{g} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be locally integrable, radial and differentiable outside of the origin. Let us also suppose that its radial profile g, i.e. the function satisfying $\bar{g}(x) = g(|x|)$, fullfills the following requirements: $\{g' < 0\} \subset \mathbb{R}$ is bounded with $\int_{\{g'<0\}} g'(t)t^N dt > -\infty$ and that there exists $\lambda > 1$ such that*

$$\liminf_{|x|\to+\infty} [\bar{g}(\lambda x) - \bar{g}(x)] > 0. \tag{2.35}$$

If f_m is a minimizer of the problem (P_D) with mass m, then $\lim_{m\to+\infty} A(f_m) = 0$.

Proof. We mimic the proof of [FL21, Theorem 1.2], which uses a slice decomposition of the kernel and a sort of quantitative Riesz inequality. Let us take any $f : \mathbb{R}^N \to [0,1]$ with $\|f\|_1 = m$, and let us take $R > 0$ such that $|B_R| = m$. We compare the energy of f with the energy of χ_{B_R}:

$$\begin{aligned}
\mathcal{E}(f) - \mathcal{E}(B_R) &= \iint \bar{g}(x-y)f(x)f(y)dxdy - \iint_{B_R\times B_R} \bar{g}(x-y)dxdy \\
&= \int_{\mathbb{R}^+} drg'(r) \iint_{B_R\times B_R} \chi_{B_r}(x-y)dxdy \\
&\quad - \int_{\mathbb{R}^+} drg'(r) \iint \chi_{B_r}(x-y)f(x)f(y)dxdy.
\end{aligned}$$

Let $\delta = (1+\lambda)^{-1}$ and let $\mathcal{I}_m = \{r \geq 0 : \ \delta \leq (|B_r|/m)^{1/N} \leq 1-\delta\}$, and notice that for m large enough we have that $g' \geq 0$ in $\mathcal{I}_m$. We are going to split the integral on $\mathbb{R}^+$ in three pieces: $\mathcal{I}_m$, $\{g' < 0\}$ and $\{g' \geq 0\} \setminus \mathcal{I}_m$. For the first part we use [FL21, Theorem 2.1], that provides a constant $C = C(\bar{g}, N) > 0$ such that

$$\iint_{B_R\times B_R} \chi_{B_r}(x-y)dxdy - \iint \chi_{B_r}(x-y)f(x)f(y)dxdy \geq Cm^2A(f)^2 \qquad \forall r \in \mathcal{I}_m,$$

hence

$$\begin{aligned}
&\int_{\mathcal{I}_m} drg'(r) \left[\iint_{B_R\times B_R} \chi_{B_r}(x-y)dxdy - \iint \chi_{B_r}(x-y)f(x)f(y)dxdy\right] \\
&\qquad \geq Cm^2A(f)^2 \int_{\mathcal{I}_m} g'(r)dr = Cm^2A(f)^2 \left(g\left(\frac{(1-\delta)m^{1/N}}{\omega_N^{1/N}}\right) - g\left(\frac{\delta m^{1/N}}{\omega_N^{1/N}}\right)\right).
\end{aligned}$$

The integral in $\{g' \geq 0\} \setminus \mathcal{I}_m$ is simply non-negative thanks to the Riesz inequality.

The remaining domain can be treated forgetting about the density f:

$$\begin{aligned}\int_{\{g'<0\}} drg'(r)\left[\iint_{B_R\times B_R}\chi_{B_r}(x-y)dxdy-\iint\chi_{B_r}(x-y)f(x)f(y)dxdy\right]\\ \geq\int_{\{g'<0\}} drg'(r)\iint_{B_R\times B_R}\chi_{B_r}(x-y)dxdy\\ \geq\int_{\{g'<0\}} drg'(r)\int_{B_R}|B_r|dy\geq -C'm\end{aligned}$$

where $C'=C'(\bar g,N)=-\omega_N\int_{\{g'<0\}}g'(r)r^N dr$. Adding up all the inequalities and using the definition of δ we obtain that

$$\mathcal{E}(f)-\mathcal{E}(B_R)\geq m^2\left[CA(f)^2\left(g\left(\lambda\frac{\delta m^{1/N}}{\omega_N^{1/N}}\right)-g\left(\frac{\delta m^{1/N}}{\omega_N^{1/N}}\right)\right)-\frac{C'}{m}\right]. \tag{2.36}$$

Thanks to (2.35), any competitor f with energy smaller than $\mathcal{E}(B_R)$ must satisfy a bound for the asymmetry: $A(f)^2\leq\frac{C''}{m}$ for some constant $C''(\bar g,N)>0$, concluding the proof. □

Remark 2.3.4. Arguing as we did in Lemma 1.1.6, one can see that the integrability hypothesis on g' is automatically satisfied when $\bar g(x)=g(|x|)$ is of class $C^1(\mathbb{R}^N\setminus\{0\})\cap L^1_{\rm loc}(\mathbb{R}^N)$, and g' changes sign a finite number of times. Hence, if the kernel $\bar g$ is of the type defined in (1), the result holds. Also notice that the condition (2.35) is very mild but it is not satisfied by every function that diverges at infinity: an example is $\bar g(x)=\log|\log|x||$.

Lemma 2.3.5. *Let $\bar g:\mathbb{R}^N\to\mathbb{R}^+\cup\{+\infty\}$ be a function satisfying ($\mathbf{H}_s$), and additionally*

$$\liminf_{\lambda\to+\infty}\liminf_{|x|\to+\infty}\frac{\bar g(\lambda x)}{\bar g(x)}=+\infty. \tag{2.37}$$

Then, there exists a constant $D=D(\bar g,N)>0$ such that, for every m large enough and for every f_m minimizer of (P_D) with $\|f_m\|_1=m$, we have the diameter estimate

$$\operatorname{diam}(\operatorname{spt} f_m)\leq DR$$

where $|B_R|=m$.

Proof. The hypotheses for this lemma are stronger than the assumptions in Proposition 2.1.4, and thus we can apply that result, obtaining a diameter bound $\tilde D$ for the support of a minimal density for the problem (P_D). We claim that, under the additional requirement (2.37), the dependence on the mass is the natural one, i.e. there exists $D=D(\bar g,N)$ such that $\tilde D\leq DR$ when m is large enough, where $|B_R|=m$. Without loss of generality, we can suppose that $R>L_g$, where L_g is the constant appearing in ($\mathbf{H}_s$). For us, it is sufficient to show that we can choose the values of $\tilde R$ and $\tilde R^+$ in the proof of Proposition 2.1.4 with the natural growth, that is $R<\tilde R\leq\tilde R^+\lesssim R$. We recall that, in the aforementioned proof, we defined a purely geometric constant κ, and we can suppose without loss of generality that $|B_{\tilde R}|>\kappa m$. It is not difficult to see that we can choose that radius in order to satisfy also (2.1). In fact, the constant C_m appearing in that condition corresponds to $\mathcal{E}(B_R)$, and thus we need to estimate that energy:

$$\begin{aligned}\frac{1}{m^2}\int_{B_R}\int_{B_R}\bar{g}(x-y)dxdy &= \frac{1}{m^2}\int_{B_R}\int_{B(x,L_g)\cap B_R}\bar{g}(x-y)dxdy \\ &\quad + \frac{1}{m^2}\int_{B_R}\int_{B_R\setminus B(x,L_g)}\bar{g}(x-y)dxdy \\ &\le \frac{1}{m^2}\int_{B_{L_g}}\bar{g}(y)dy\int_{B_R}dx + \frac{1}{m^2}\int_{B_R}\int_{B_R}g(2R)dxdy \\ &= \frac{C}{m} + g(2R),\end{aligned}$$

and the inequality follows from the definition of L_g, and the fact that $R > L_g$. The constant appearing in the last expression depends only on $\bar{g}$ and N, but not on m. Using (2.37) it is clear that there exists $\lambda > 3$ such that, taking $\tilde{R} > \lambda R$, we have the inequality $\bar{g}(x) > g(2R) + C/m$ for every $m > \omega_N$ and every $x \notin B_{\tilde{R}}$. The constant λ depends only on $\bar{g}$ and N, and so does $\tilde{R}$, and (2.1) holds for such radius. Finally, we need to provide a bound on the radius $\tilde{R}^+$ appearing in (2.2) that is linear in $\tilde{R}$. In turn, this provides the required result since in Proposition 2.1.4 we suppose that $\tilde{R}^+ \ge 50\tilde{R}$, and $\tilde{R}$ has a linear bound with respect to R as we showed before. Similarly to our previous estimate, we notice that

$$\frac{1}{m}\int_{B_{11\tilde{R}}}\bar{g}(x)dx \le \frac{1}{m}\int_{B_{L_g}}\bar{g}(x)dx + \frac{1}{m}\int_{B_{11\tilde{R}}\setminus B_{L_g}}\bar{g}(x)dx \le \frac{C}{m} + g(22\tilde{R}).$$

Exploiting once more the growth condition (2.37), it is immediate to see that, up to enlarging λ, we can guarantee that

$$g(\lambda\tilde{R} - \tilde{R}) \ge 2g(6\tilde{R}) + \frac{5}{2}\left(\frac{C}{m} + g(22\tilde{R})\right) \ge 2g(6\tilde{R}) + \frac{5}{2m}\int_{B_{11\tilde{R}}}\bar{g}(x)dx.$$

To conclude, we just need to choose $\tilde{R}^+ = \lambda\tilde{R}$ since $\tilde{D} = 2\tilde{R}^+$.

□

Remark 2.3.6. We highlight that the growth assumption in Lemma 2.3.5 is mild, but it is not satisfied for example by the kernel $\bar{g}(x) = \log|x|$.

2.3.2 Fine analysis of the potential

Differently from the previous section, here we need a kernel $\bar{g}$ with attractive term of the form $\mathfrak{g}_\alpha$. This is required because we need a good control on its derivative at large scales. One could obtain a similar result (with a different control in (2.40)) assuming that the radial profile g has properties that mimic the power law kernel at large distance. Roughly speaking, the key features are that $\lim_{t\to\infty} tg'(t) = +\infty$ and $g'(\lambda t)/g'(t) \ge C$ for some constants $\lambda > 1$, $C > 0$, and for every t large enough.

We will often use some balls to build competitors, and their potential will come into play. For this reason, we define the auxiliary radial functions

$$\Phi(r,R) := \int_{B_R} \bar{g}(re_1 - x)dx \qquad \Phi_{\tilde{g}}(r,R) := \int_{B_R} \tilde{g}(re_1 - x)dx \tag{2.38}$$

where $\tilde{g}$ is a generic radial function, that in our applications will be $\mathfrak{g}_\alpha$ or $\bar{h}$. Of course, notice that $\Phi(\cdot, R)$ is exactly the radial profile of ψ_{B_R}, and $\Phi_{\tilde{g}}(\cdot, R)$ is the radial profile of $\psi_{B_R,\tilde{g}}$. We have the following result concerning the function Φ:

Lemma 2.3.7. *Let $\bar{g} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a kernel of the form $\bar{g} = \mathfrak{g}_\alpha + \bar{h}$, with $\alpha > 0$ and $\bar{h}$ satisfying ($\mathbf{H}_l$). Then, there exist $m_1 = m_1(\alpha, \bar{h}, N) > 0$ and $C_1 = C_1(\alpha, \bar{h}, N) > 0$ such that for every $R > 0$ atisfying $|B_R| > m_1$, we have that*

$$\Phi(r,R) \le \Phi(R,R) \quad \text{if } r \le R \qquad \text{and} \qquad \Phi(r,R) \ge \Phi(R,R) \quad \text{if } r \ge R, \tag{2.39}$$

$$|\Phi(r,R) - \Phi(R,R)| \ge C_1 R^{N+\alpha-1} \min\{|r-R|, R\} \qquad \forall r \ge 0. \tag{2.40}$$

Proof. First of all we change variable in the definition of Φ:

$$\Phi(r,R) = \alpha^{-1} R^{N+\alpha} \int_B \left|\frac{r}{R}e_1 - x\right|^\alpha dx + R^N \int_B \bar{h}\,(re_1 - Rx)\, dx.$$

Now we take $m_1 \ge \omega_N$ (thus $R \ge 1$) and, since h is locally bounded in $(0, +\infty)$, then we apply Lemma 2.3.2 to obtain that

$$\begin{aligned} \Phi(r,R) - \Phi(R,R) &= R^{N+\alpha}(\Phi_{\mathfrak{g}_\alpha}(r/R,1) - \Phi_{\mathfrak{g}_\alpha}(1,1)) \\ &\quad + R^N \int_B \left[\bar{h}\,(re_1 - Rx) - \bar{h}(Re_1 - Rx)\right] dx \\ &\ge R^{N+\alpha}(\Phi_{\mathfrak{g}_\alpha}(r/R,1) - \Phi_{\mathfrak{g}_\alpha}(1,1)) - K_{\bar{h},N} R^N, \end{aligned} \tag{2.41}$$

and in the same way

$$\Phi(R,R) - \Phi(r,R) \ge R^{N+\alpha}(\Phi_{\mathfrak{g}_\alpha}(1,1) - \Phi_{\mathfrak{g}_\alpha}(r/R,1)) - K_{\bar{h},N} R^N.$$

Moreover, using the change of variables $y = te_1 - x$, it is easy to obtain an expression for the derivative of $\Phi_{\mathfrak{g}_\alpha}$ with respect to the first variable:

$$\partial_1 \Phi_{\mathfrak{g}_\alpha}(t,R) = \int_B (t - \langle y, e_1\rangle)|te_1 - x|^{\alpha-2} dx = \int_{te_1 - B} \langle y, e_1\rangle |y|^{\alpha-2} dy,$$

and therefore using the symmetry of B we have that $\Phi_{\mathfrak{g}_\alpha}$ is of class C^1, with strictly positive derivative at each point $t > 0$. We will denote by $C(\alpha, N) > 0$ a constant such that $\partial_1 \Phi_{\mathfrak{g}_\alpha}(t,R) > C(\alpha, N)$ for all $t \in [2/3, 4/3]$. From the previous observations it follows immediately that both (2.39) and (2.40) are valid for $|r - R| \ge R/3$ if we take m_1 big enough to have that

$$K_{\bar{h}} \le \frac{1}{2} R^\alpha \min\{\Phi_{\mathfrak{g}_\alpha}(4/3,1) - \Phi_{\mathfrak{g}_\alpha}(1,1), \Phi_{\mathfrak{g}_\alpha}(1,1) - \Phi_{\mathfrak{g}_\alpha}(2/3,1)\}.$$

Now we concentrate ourselves on the case $|r-R| \le R/3$. We treat more carefully the repulsive terms in (2.41), that coincide with

$$\gamma = \int_{re_1 - B_R} \bar h(x)dx - \int_{Re_1 - B_R} \bar h(x)dx.$$

Let us define $\tau = (r-R)/R$, $E = Re_1 - B_R$ and let $l = \mathrm{Span}\{e_1\}$. It is immediate to see that

$$\mathscr{H}^1((E\Delta(\tau Re_1 + E)) \cap (x+l)) \le 2|\tau|R \qquad \forall x \in \mathbb{R}^N.$$

Thus, we use that h is decreasing and that $\bar h \in L^1_{\mathrm{loc}}(\mathbb{R}^{N-1})$ to get

$$\begin{aligned} |\gamma| &\le (2N-2)\omega_{N-1}|\tau|R\int_0^1 h(s)s^{N-2}ds + h(1)|E\Delta(\tau Re_1 + E)| \\ &\le C(\bar h, N)|\tau|(R+R^N), \end{aligned} \tag{2.42}$$

where we used the cylindrical coordinates around the e_1 axis. Hence we obtain both (2.39) and (2.40) if we plug this inequality for $|\gamma|$ into the first line of (2.41) and use that $\partial_1\Phi_{\mathfrak{g}_\alpha}(t,R) \ge C(\alpha,N) > 0$ for $t \in [2/3, 4/3]$. □

When a density f has some special structure, we can deduce a bound on the potential also when $\|f\|_1$ is small, differently from the general bound given in Lemma 2.3.2. In fact, we have the following:

Lemma 2.3.8. *Let $\bar h : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a function satisfying ($\mathbf{H}_l$). There exists a positive constant $C_2 = C_2(\bar h, N)$ such that, for any $R \ge 1$, any $\tau \in [0,1]$, and any function $f : \mathbb{R}^N \to [0,1]$ that satisfies* $\mathrm{spt}\, f \subset B(0,(1+\tau)R) \setminus B(0,(1-\tau)R)$, *we have that*

$$\left\|\psi_{f,\bar h}\right\|_\infty = \sup_{x\in\mathbb{R}^N} \int \bar h(x-y)f(y)dy \le C_2\tau R^N. \tag{2.43}$$

Remark 2.3.9. Our proof goes on quite like [FL21, Lemma 3.6], but we provide a very rough estimate, where f does not appear explicitly in the right hand side. Besides this inequality might seem very bad, notice that if we take $f = \chi_{B(0,(1+\tau)R)} - \chi_{B(0,(1-\tau)R)}$ then the bound must be linear in τ for $\tau \to 0$: the left hand side of (2.43) is larger than $\int \bar h(y)f(y)dy$, that is larger than $C_N h((1+\tau)R)\tau R^N$ for some dimensional constant $C_N > 0$.

Proof. We define the annulus $A_{R,\tau} := B(0,(1+\tau)R)\setminus B(0,(1-\tau)R)$, and since $|A_{R,\tau}| = \omega_N R^N((1+\tau)^N - (1-\tau)^N)$, then it is immediate to see that $N\omega_N\tau R^N \le |A_{R,\tau}| \le 2^N N\omega_N \tau R^N$. Without loss of generality we can suppose that $|A_{R,\tau}| \le \varepsilon_N$ for every fixed $\varepsilon_N < \omega_N$: if the other case holds, then let $r_N, r_A > 0$ be such that $|B_{r_N}| = \varepsilon_N$ and $|B_{r_A}| = |A_{R,\tau}|$, and we provide a easy bound:

$$\begin{aligned} \sup_{x\in\mathbb{R}^N} \int \bar h(x-y)f(y)dy &\le \int_{B_{r_A}} \bar h(y)dy = \int_{B_{r_N}} \bar h(y)dy + \int_{B_{r_A}\setminus B_{r_N}} \bar h(y)dy \\ &\le \int_B \bar h(y)dy + h(r_N)|A_{R,\tau}| = \left(\frac{1}{|A_{R,\tau}|}\int_B \bar h(y)dy + h(r_N)\right)|A_{R,\tau}| \\ &\le \left(\frac{K_{\bar h,N}\omega_N}{\varepsilon_N} + h(r_N)\right)|A_{R,\tau}|, \end{aligned} \tag{2.44}$$

that is the desired result since $|A_{R,\tau}| \le 2^N N\omega_N \tau R^N$. The value of ε_N will be fixed later, but it is important to keep in mind that $|A_{R,\tau}|$ can be taken arbitrarily small. Thus, we need to prove (2.43) exploiting the particular shape of $A_{R,\tau}$. In the end, it is sufficient to estimate the contribution of a slab:

$$S := \int_{[-R/2,R/2]^{N-1}} \int_{[-C_N\tau R, C_N\tau R]} \bar{h}((y',t))dy'dt,$$

where $C_N > 0$ is a geometric constant. In fact, by compactness there exist a constant $K_N > 0$ and a family $\{q_1, \dots, q_{K_N}\}$ of $(N-1)$-dimensional cubes embedded in $\mathbb{R}^N$ such that

- the center c_j of q_j belongs to $\partial B(0,R)$ for all j;
- their sides have length $R/2$;
- for every $1 \le j \le K_N$ we have that $q_j \cap B(0,R) = \emptyset$;
- if $D_j = \{tc_j + y : t > -1, y \in q_j\}$ and $\pi_j^\perp$ is the orthogonal projection onto $\mathrm{Span}\{c_j\}^\perp$, then we define the map $\pi_j : D_j \to \mathbb{R}^N$ as

$$\pi_j(x) = \pi_j^\perp(x - c_j) + c_j\sqrt{1 - \frac{|\pi^\perp(x-c_j)|^2}{R^2}},$$

so that $\bigcup_{j=1}^{K_N} \pi_j(q_j) = \partial B(0,R)$, namely they "cover" $\partial B(0,R)$. Notice that the map π_j is just pushing the points of D_j onto $\partial B(0,R)$ as shown in the left picture in Figure 2.3.

Then, thanks to the positivity of $\bar{h}$, we can replace the "curved slabs" $A_{R,\tau} \cap D_j$ with some flat slabs F_j (the smallest N-dimensional rectangle containing $A_{R,\tau} \cap D_j$ with sides parallel or orthogonal to q_j):

$$\begin{aligned} \int \bar{h}(x-y)f(y)dy &\le \sum_{j=1}^{K_N} \int_{D_j \cap A_{R,\tau}} \bar{h}(x-y)f(y)dy \le \sum_{j=1}^{K_N} \int_{F_j} \bar{h}(x-y)dy \\ &\le K_N \int_{[-R/2,R/2]^{N-1}} \int_{[-C_N\tau R, C_N\tau R]} \bar{h}((y',t))dy'dt, \end{aligned} \tag{2.45}$$

where we used the monotonicity of h to pass from the second to the third line. We also highlight that F_j has thickness smaller than $C_N\tau R$ for some constant C_N (see Figure 2.3, on the right). In fact, if $\tau \le 1/10$ this is clearly true, and we know that $\tau \le |A_{R,\tau}|/(N\omega_N R^N)$. Since $R \ge 1$ and $|A_{R,\tau}| \le \varepsilon_N$, we can choose ε_N so that $\tau \le 1/10$. Then from (2.45) it is clear that we need only to control the quantity S defined before. The kernel $\bar{h}$ is radial, with profile h that is decreasing. Therefore we have that

$$S \le \int_{[-R/2,R/2]^{N-1}} \int_{[-C_N\tau R, C_N\tau R]} \bar{h}((y',0))dy'dt = 2C_N\tau R \int_{[-R/2,R/2]^{N-1}} \bar{h}((y',0))dy'.$$

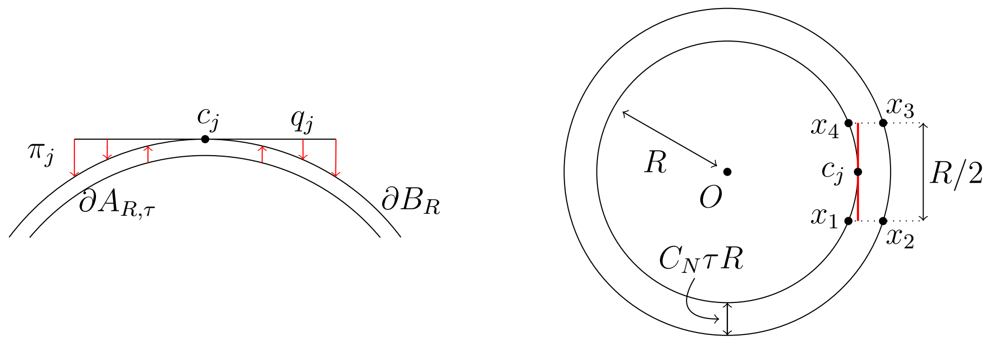


Figure 2.3: The image on the left represents the map π_j with the red arrows, the cube q_j that is the horizontal segment. On the right, the cube q_j is represented by the red vertical segment, while the points x_1, x_2, x_3, x_4 denote the corners (since the figure is in 2D) of the outer part of what we call "curved slab" $A_{R,\tau} \cap D_j$.

As we did in (2.42), we use that $\bar{h} \in L^1_{\mathrm{loc}}(\mathbb{R}^{N-1})$ and $R \geq 1$ to continue the previous L^∞ bound, using Lemma 2.3.2:

$$S \leq 2C_N \tau R \cdot K_{\bar{h},N-1} R^{N-1} = C\tau R^N,$$

where of course the constant C depends only on $\bar{h}$ and the space dimension N. In the end, the statement is proved with

$$C_2 = \max\left\{K_N C, 2^N N\omega_N \left(\frac{K_{\bar{h},N}\omega_N}{\varepsilon_N} + h(r_N)\right)\right\}.$$

□

2.3.3 Characterization of large-mass minimizers

For convenience, we report here a lemma needed for the next proposition. The proof of this lemma can be found in [FL21].

Lemma 2.3.10. *Let $f : \mathbb{R}^N \to [0,1]$ be a function with $\|f\|_1 = m$, and let $\tau \in [0,1]$. Then there exists $\bar{f} : \mathbb{R}^N \to [0,1]$ with the following properties*

$$\|\bar{f}\|_1 = \|f\|_1, \tag{2.46}$$

$$\chi_{(1-\tau)B_R} \leq \bar{f} \leq \chi_{(1+\tau)B_R}, \tag{2.47}$$

$$\bar{f}(x) \leq f(x) \text{ for } x \notin B_R, \quad \bar{f}(x) \geq f(x) \text{ for } x \in B_R, \tag{2.48}$$

$$\int |\bar{f} - \chi_{B_R}| dx \leq \int |f - \chi_{B_R}| dx, \tag{2.49}$$

$$\int |f - \bar{f}| dx \leq 2 \int_E |f - \bar{f}| dx \qquad \text{where } E = (1-\tau)B_R \cup (\mathbb{R}^N \setminus (1+\tau)B_R), \tag{2.50}$$

where $|B_R| = m$.

Proposition 2.3.11. *Let $\alpha > 0$ be given, let $\bar{h} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a function satisfying ($\mathbf{H}_l$), and let $\bar{g} = \mathfrak{g}_\alpha + \bar{h}$. Then, there exist two constants $C_3 = C_3(\alpha, \bar{h}, N) > 0$ and $m_3 = m_3(\alpha, \bar{h}, N) > 0$ such that, any minimizer f of the problem (P_D) with $\|f\|_1 = m > m_3$ satisfies the following condition:*

$$\chi_{B(x_A, R - C_3 A(f) R)} \le f \le \chi_{B(x_A, R + C_3 A(f) R)},$$

where $B(x_A, R)$ is an optimal ball to compute $A(f)$.

Proof. We repeat quickly the strategy exposed in [FL21, Proposition 3.4]. To do this, we build a family of competitors starting from the given minimizer f and applying interatively Lemma 2.3.10. Without loss of generality we can suppose that $x_A = 0$ and that $m_3 \ge m_1$, where m_1 is the constant in the statement of Lemma 2.3.7. We define $f_0 := f$, and by induction we define f_{k+1} for any $k \in \mathbb{N}$ applying Lemma 2.3.10 to $f = f_k$ with parameter $\tau = 2^{-k}$. We use the quadratic structure of $\mathcal{E}$ to rewrite the difference in energy between two consecutive densities f_k and f_{k+1}:

$$\begin{aligned}\mathcal{E}(f_{k+1}) - \mathcal{E}(f_k) &= \mathcal{E}(f_{k+1} - f_k, f_{k+1}) + \mathcal{E}(f_k, f_{k+1}) + \mathcal{E}(f_{k+1} - f_k, f_k) - \mathcal{E}(f_{k+1}, f_k) \\ &= \mathcal{E}(f_{k+1} - f_k, f_{k+1} - \chi_{B_R}) + \mathcal{E}(f_{k+1} - f_k, f_k - \chi_{B_R}) \\ &\quad + 2\mathcal{E}(f_{k+1} - f_k, \chi_{B_R}).\end{aligned}$$

We treat the last term using Lemma 2.3.7. In fact, we notice that $f_{k+1} - f_k$ has the opposite sign with respect to $\Phi(\cdot, R) - \Phi(R, R)$, and $\int f_{k+1} - f_k = 0$, so

$$\begin{aligned}\mathcal{E}(f_{k+1} - f_k, \chi_{B_R}) &= \int (f_{k+1}(x) - f_k(x))\Phi(|x|, R)dx \\ &= \int (f_{k+1}(x) - f_k(x))(\Phi(|x|, R) - \Phi(R, R))dx \\ &\le -\int_{\{||x| - R| \ge 2^{-k}R\}} |f_{k+1}(x) - f_k(x)||\Phi(|x|, R) - \Phi(R, R)|dx \\ &\le -C_1 2^{-k} R^{N+\alpha} \|f_{k+1}(x) - f_k(x)\|_1 .\end{aligned} \tag{2.51}$$

For brevity we define $a_k = 2^k R^{-N} \left\| f_k - \chi_{B_R} \right\|_1$, and thanks to the diameter bound for spt f (see Lemma 2.3.5) we estimate the following quantities:

$$\begin{aligned}|\mathcal{E}_{\mathfrak{g}_\alpha}(f_{k+1} - f_k, f_{k+1} - \chi_{B_R})| &\le \alpha^{-1}(DR)^\alpha \|f_{k+1} - f_k\|_1 \cdot \left\| f_{k+1} - \chi_{B_R} \right\|_1 \\ &\le \alpha^{-1} D^\alpha R^{N+\alpha} 2^{-k} a_k \|f_{k+1} - f_k\|_1 , \\ |\mathcal{E}_{\mathfrak{g}_\alpha}(f_{k+1} - f_k, f_k - \chi_{B_R})| &\le \alpha^{-1} D^\alpha R^{N+\alpha} 2^{-k} a_k \|f_{k+1} - f_k\|_1 ,\end{aligned}$$

where we used also (2.49). We treat the contribution given by the kernel $\bar{h}$ applying Lemma 2.3.8 to the function $|f_{k+1} - \chi_{B_R}|$:

$$\int \bar{h}(x - y)|f_{k+1}(y) - \chi_{B_R}(y)|dy \le C_2 2^{-k} R^N \qquad \forall x \in \mathbb{R}^N,$$

and an analogous estimate holds for $|f_k - \chi_{B_R}|$, therefore

$$|\mathcal{E}_{\bar h}(f_{k+1} - f_k, f_{k+1} - \chi_{B_R})| + |\mathcal{E}_{\bar h}(f_{k+1} - f_k, f_k - \chi_{B_R})| \le 2C_2 2^{-k} R^N \|f_{k+1} - f_k\|_1 .$$

Hence, combining the leading term inequality (2.51) with the remainder terms that we just obtained we arrive to the following estimate:

$$\begin{aligned}\mathcal{E}(f_{k+1}) - \mathcal{E}(f_k) &\le -R^{N+\alpha} 2^{-k} \|f_{k+1} - f_k\|_1 \left(C_1 - \frac{2D^\alpha a_k}{\alpha} - \frac{2C_2}{R^\alpha} \right) \\ &\le -R^{N+\alpha} 2^{-k} \|f_{k+1} - f_k\|_1 \left(\frac{C_1}{2} - \frac{2D^\alpha a_k}{\alpha} \right),\end{aligned} \tag{2.52}$$

where the last inequality holds when $m \ge m_3$ is so large that $2C_2 R^{-\alpha} < C_1/2$. Note that a_0 coincides with $A(f)$ up to multiplicative constants, therefore $\alpha^{-1} D^\alpha a_0 < C_1/4$ if m is large enough because of Lemma 2.3.3. Two possibilities may occour: either there exists $k_0 \ge 1$ such that $\alpha^{-1} D^\alpha a_{k_0} \ge C_1/4$ or $\alpha^{-1} D^\alpha a_k < C_1/4$ for every $k \in \mathbb{N}$. In the first case we take k_0 the minimum index displaying this phenomenon, and we have that

$$\mathcal{E}(f_{k_0}) - \mathcal{E}(f) \le -R^{N+\alpha} \sum_{k=0}^{k_0-1} 2^{-k} \|f_{k+1} - f_k\|_1 \left(\frac{C_1}{2} - \frac{2D^\alpha a_k}{\alpha} \right) \le 0. \tag{2.53}$$

Since f is a minimizer of $\mathcal{E}$ and $\|f_{k_0}\|_1 = \|f\|_1$, we have that $\mathcal{E}(f_{k_0}) - \mathcal{E}(f) \ge 0$. This is compatible with (2.53) only if $f_k = f_0$ for any $k \le k_0$. Using our construction of f_{k_0}, and in particular the property (2.47), we have that

$$\chi_{\left(1-2^{1-k_0}\right)B_R} \le f_0 \le \chi_{\left(1+2^{1-k_0}\right)B_R}.$$

We can control the quantity 2^{-k_0} in term of $A(f)$. In fact, by definition of k_0 we have that $a_{k_0} \ge \alpha C_1/(4D^\alpha)$, and thus

$$2^{-k_0} \le \frac{4D^\alpha}{\alpha C_1} R^{-N} \left\| f_{k_0} - \chi_{B_R} \right\|_1 \le \frac{4D^\alpha}{\alpha C_1} R^{-N} \left\| f_0 - \chi_{B_R} \right\|_1 = \frac{4D^\alpha \omega_N}{\alpha C_1} A(f_0),$$

where we applied (2.49) to obtain the second inequality. Hence, the proof is concluded in this situation with $C_3 = \frac{8D^\alpha \omega_N}{\alpha C_1}$ because $f_0 = f$ and that constant depends only on α, $\bar h$ and N. If instead $\alpha^{-1} D^\alpha a_k < C_1/4$ for every k, we can apply (2.53) with any natural index k_0 and see that $f_{k_0} = f_0$. This implies directly that $f_0 = \chi_{B_R}$, concluding the proof also in the second case. □

Theorem 2.3.12. *Let $\bar g : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a kernel of the form $\bar g = \mathfrak{g}_\alpha + \bar h$, with $\alpha > 0$ and $\bar h$ satisfying ($\mathbf{H}_l$). Then there exists $m_4 = m_4(\alpha, \bar h, N) > 0$ such that the only minimizer (up to translations) of (P_D) with mass $m > m_4$ is the characteristic function of a ball B_R with $|B_R| = m$.*

Proof. We fix any minimizer f_m of (P_D) with mass m. Without loss of generality, we can suppose that $A(f_m)$ is realized by B_R and that $m_4 > \max\{m_1, m_3\}$. We observe that, despite the statement of Lemma 2.3.3 concerns minimizers, the method of proof can be applied to any density, and the inequality (2.36) is valid in any case. Therefore, we apply that proof with the radially increasing kernel $\mathfrak{g}_\alpha$ and density f_m, obtaining the quantitative inequality

$$C(\alpha, N)m^{2+\alpha/N}A(f_m)^2 \le \mathcal{E}_{\mathfrak{g}_\alpha}(f_m) - \mathcal{E}_{\mathfrak{g}_\alpha}(B_R).$$

Exploiting the minimality of f_m for $\mathcal{E}_{\bar{g}}$ we continue the previous inequality:

$$C(\alpha, N)m^{2+\alpha/N}A(f_m)^2 \le \mathcal{E}_{\mathfrak{g}_\alpha}(f_m) - \mathcal{E}_{\mathfrak{g}_\alpha}(B_R) \le \mathcal{E}_{\bar{h}}(B_R) - \mathcal{E}_{\bar{h}}(f_m). \tag{2.54}$$

Writing $f_m = (f_m - \chi_{B_R}) + \chi_{B_R}$, we use the quadratic structure of $\mathcal{E}_{\bar{h}}$ to see that

$$\mathcal{E}_{\bar{h}}(B_R) - \mathcal{E}_{\bar{h}}(f_m) = 2\mathcal{E}_{\bar{h}}(\chi_{B_R} - f_m, \chi_{B_R}) + \mathcal{E}_{\bar{h}}(\chi_{B_R} - f_m).$$

We apply Proposition 2.3.11 to f_m, obtaining that $\mathrm{spt}(\chi_{B_R} - f_m) \subset B_{R+C_3A(f_m)R} \setminus B_{R-C_3A(f_m)R}$. Taking m_4 large enough, we can use Lemma 2.3.3 and we obtain that $C_3A(f_m) < 1$ for every $m > m_4$. In this range of masses we can apply Lemma 2.3.8 to $\chi_{B_R} - f_m$, arriving to the estimate

$$\begin{aligned}\left|\mathcal{E}_{\bar{h}}(\chi_{B_R} - f_m)\right| &\le \left\|\chi_{B_R} - f_m\right\|_1 \cdot \sup_x \int \bar{h}(x-y)\left|\chi_{B_R}(y) - f_m(y)\right| dx \\ &\le mA(f_m) \cdot C_2C_3A(f_m)\frac{m}{\omega_N} = Cm^2A(f_m)^2.\end{aligned}$$

Finally, it is not hard to see that $\Phi_{\bar{h}}(\cdot, R)$ is decreasing, and using the integrability hypothesis on $\bar{h}$ one can obtain a Lipschitz bound on $\Phi_{\bar{h}}(\cdot, R)$ that is analogous to estimate (2.42), arriving to:

$$\begin{aligned}\left|\mathcal{E}_{\bar{h}}(\chi_{B_R} - f_m, \chi_{B_R})\right| &= \left|\int \Phi_{\bar{h}}(|x|, R)(\chi_{B_R}(x) - f_m(x))dx\right| \\ &\le |\Phi_{\bar{h}}(R + C_3RA(f_m), R) - \Phi_{\bar{h}}(R - C_3RA(f_m), R)| \left\|\chi_{B_R} - f_m\right\|_1 \\ &\le C'R^N A(f_m)\left\|\chi_{B_R} - f_m\right\|_1 = C''m^2A(f_m)^2.\end{aligned}$$

Combining these last inequalities with (2.54) we get

$$C(\alpha, N)m^{2+\alpha/N}A(f_m)^2 \le C'''m^2A(f_m)^2,$$

with $C''' > 0$ being a constant depending only on α, $\bar{h}$ and N. If m_4 is large enough, the above inequality is valid only when $A(f_m) = 0$, that is equivalent to saying that $f_m = \chi_{B_R}$. □

Remark 2.3.13. Since the power-like kernels are very popular, and they constitute important examples in this field, it is worth to mention that all the arguments to obtain Theorem 2.3.12 work also when $\bar{g} = \bar{g}_p$ defined in (1) with $\alpha > \beta > 0$. A few adjustments are needed when we choose $\bar{h} = -\mathfrak{g}_\beta$. First, Lemma 2.3.7 is still valid when $m \gg 1$ because $|\nabla \mathfrak{g}_\beta| \ll |\nabla \mathfrak{g}_\alpha|$ far away from the origin. Second, in Lemma 2.3.8, the upper bound contains an additional factor R^β, and this is still good enough. In fact, going through the proof of Proposition 2.3.11, we notice that the additional factor transforms the iterative bound (2.52) into

$$\mathcal{E}(f_{k+1}) - \mathcal{E}(f_k) \leq -R^{N+\alpha}2^{-k} \|f_{k+1} - f_k\|_1 \left(C_1 - \frac{D^\alpha a_k}{\alpha} - \frac{2C_2}{R^{\alpha-\beta}} \right),$$

that can be treated exactly as before since $\alpha > \beta$.

Chapter 3

Generalized Gamow model

In this chapter we study a generalized Gamow liquid drop model, which classically consists in the minization of the functional

$$\mathsf{P}(E) + \iint_{E\times E} \frac{1}{|x-y|} dxdy$$

among sets $E \subset \mathbb{R}^3$ with a given measure constraint. The two terms compete since the perimeter is minimized by the ball, while the Riesz term is maximized by the ball. The two terms scale differently, and heuristically the perimeter is more important when the measure constraint is small, while the main contribution is given by the Riesz term when the measure is large. For this reason, it is convenient (and equivalent, in this case) to work with a fixed measure constraint, equal to ω_N, and consider a functional with a factor in front of the Riesz term that accounts for the strength of the interaction. We aim to study a generalization of the classical Gamow functional. In fact, we replace the Riesz term with the analogous functional $\mathcal{E}_{\bar{h}}$ for an appropriate kernel $\bar{h}$, and we simply write $\mathscr{P}$ to denote the perimeter, that represents either P or the fractional counterpart P_s. This is meant to stress that our approach works in a general framework. We mainly focus on the situation where γ is small, and we characterize the balls as the unique minimizers of $\mathcal{G}_\gamma$ (in some cases). The argument exploits a quantitative inequality for the perimeter, and a stability inequality for the Riesz term, to control the energy gap between a set and ball, as we do in (3.24). Therefore, one could obtain analogous results whenever the functional contains a perimeter term with suitable properties and inequalities (as those presented in Section 3.1), and when the Riesz term enjoys an estimate similar to Lemma 3.2.2 that is compatible with the perimeter $\mathscr{P}$. In the end, for any $\gamma > 0$ we consider the functional $\mathcal{G}_\gamma(E) = \mathscr{P}(E) + \gamma\mathcal{E}_{\bar{h}}(E)$, and we address the minimization problem

$$\min\left\{\mathcal{G}_\gamma(E) : E \subset \mathbb{R}^N, |E| = \omega_N\right\}, \tag{P_G}$$

where the kernel $\bar{h}$ satisfies the very mild assumption

($\mathbf{H}_g$) $\bar{h} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ is radial, radially decreasing, and of class $L^1_{\text{loc}}(\mathbb{R}^N)$.

In this setting, we study the aforementioned variational problem when γ is small, and we characterize the balls as the unique minimizers when that parameter is small enough.

This is the content of Section 3.2, where we exploit the regularity results collected in Section 3.1 to prove Theorem 3.2.6. The careful analysis of the Riesz term is contained in [CFP23], while the minor adjustments required to deal with the fractional perimeter can be found in [Car23, Section 3].

In Section 3.3, instead, we study the optimal way to subdivide the given measure among balls (thought to be at infine distance from each other) in order to minimize $\mathcal{G}_\gamma$. This approach is compatible with the notion of existence of generalized minimizers, as presented in [KMN16, Definition 4.3] and [NP21, Proposition 1.2]. Differently from before, the parameter γ is fixed during this analysis because it is more natural to consider the measure parameter when we actually split that measure into smaller masses. This is relevant in view of [KM13, Theorem 2.7] and [BC14, Theorem 2.12], which state that the minimizers of (P_G) coincide with collections of balls "at infinite distance" for certain Riesz kernels (thus, strictly speaking, the minimizers do not exist in the standard sense). In this case, finding the ground states of (P_G) basically amounts to solving a one-dimensional variational problem, where we consider the energy profile $m \mapsto \mathcal{G}_\gamma(B(0, m^{1/N}))$. Our approach is elementary, and we exploit only a very simple concavity-convexity property, summarized in ($\mathbf{H}_{1D}$), to understand the dependence of the optimal splitting of the total measure depending on the parameter m. In fact, our argument relies only on the hypothesis ($\mathbf{H}_{1D}$), and not on the Gamow model that originates the problem, and we show in Subsection 3.3.3 the limits of this approach. The main result of this section is Theorem 3.3.15, that is based on the optimality conditions derived in Lemma 3.3.5, and also on Lemma 3.3.9 that yields to a precise structure of the optimal splitting of the total measure. Our work gives a more precise result compared to [BC14, Theorem 2.12] and it is unpublished.

3.1 Basic tools for the Gamow model

We recall the definitions of fractional perimeter and fractional Sobolev norm, together with some important classes of sets that we will make use of.

Definition 3.1.1 (Fractional perimeter, [CRS10])**.** The fractional perimeter of order $s \in (0,1)$ is denoted by P_s, and it is defined as

$$\mathsf{P}_s(E) := \int_E \int_{E^c} \frac{1}{|x-y|^{N+s}} dxdy$$

for every measurable set $E \subset \mathbb{R}^N$ (of course, it could possibly be $+\infty$).

Definition 3.1.2. Given an open set $\Omega \subset \mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$, its fractional Sobolev seminorm of order s (and exponent 2) is defined as

$$[u]_s := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}.$$

The fractional Sobolev norm of u is, naturally, $\|u\|^2_{W^{s,2}} := \|u\|^2_{L^2(\Omega)} + [u]_s^2$.
Moreover, we use an analogous definition if $M^n \subset \mathbb{R}^N$ is a compact n-dimensional

submanifold embedded in $\mathbb{R}^N$: given a function $u : M \to \mathbb{R}$, we define its fractional Sobolev seminorm as

$$[u]_s := \left(\int_M \int_M \frac{|u(z) - u(w)|^2}{|z - w|^{n+2s}} d\mathscr{H}^n(z) d\mathscr{H}^n(w) \right)^{1/2},$$

where $|z - w|$ is the distance between z and w measured in the ambient space $\mathbb{R}^N$, and $\mathscr{H}^n$ is the Hausdorff measure induced by this distance on M. As before, we define $\|u\|^2_{W^{s,2}} := \|u\|^2_{L^2(M;\mathscr{H}^n)} + [u]^2_s$. In order to simplify the notation, we will often omit the set where we compute the various norms/seminorms when it coincides with the domain of the function u.

Remark 3.1.3. From the definitions it is clear that, for every set $E \subset \mathbb{R}^N$, we have the ideantity $2\mathsf{P}_s(E) = [\chi_E]^2_{s/2}$, with $\Omega = \mathbb{R}^N$ in Definition 3.1.2.

The next definition appears in [Fug89, CL12]. Its importance in our problem is due to the so-called Fuglede inequality, valid for nearly spherical sets. The $W^{1,\infty}$ bound in our definition is different from the one present in the aforementioned papers because Theorem 3.1.6 already contains the suitable bound for the Sobolev norm.

Definition 3.1.4. An open set $E \subset \mathbb{R}^N$ is *nearly spherical* if $|E| = \omega_N$, its barycenter is 0 and there exists a C^1 function $u : \partial B \to (-1, 1)$ such that

$$E = \{(1 + u(z))tz : z \in \partial B, t \in [0, 1)\},$$

with $\|u\|_\infty + \|\nabla u\|_\infty \leq 1/4$.

The following are two different versions of the quantitative isoperimetric inequality. The first is a general quantitative isoperimetric inequality, valid without any a-priori assumption on the set E. The second one, instead, is stronger, but it is valid only for nearly spherical sets. We refer to [FMP08, Theorem 1.1], [CL12, Theorem 4.1] and [FFM+15, Theorem 1.1, Theorem 2.1] for the proof of these results. The general quantitative isoperimetric inequality is written in terms of the asymmetry of a set, that coincides with the quantity present in Definition 2.3.1.

Theorem 3.1.5 (Quantitative isoperimetric inequality)**.** *Let $N \geq 2$ and $s \in (0, 1)$. There exists a constant $C_Q = C_Q(N, \mathscr{P}) > 0$ such that, for every $E \subset \mathbb{R}^N$ with $|E| = \omega_N$, we have*

$$\mathscr{P}(E) - \mathscr{P}(B) \geq C_Q A(E)^2.$$

Theorem 3.1.6 (Fuglede inequality)**.** *There exist $\delta_0 < 1/2$ and $C_F > 0$ that depend only on N with the following property: if $E \subset \mathbb{R}^N$ is a nearly spherical set, and if ∂E is parametrized by $u : \partial B \to (-1, 1)$ with $\|u\|_{W^{1,\infty}(\partial B)} < \delta_0$, then the following inequalities hold:*

$$\mathsf{P}_s(E) - \mathsf{P}_s(B) \geq C_F \left([u]^2_{\frac{1+s}{2}} + s\mathsf{P}_s(B) \|u\|^2_{L^2(\partial B)} \right) \qquad \forall s \in (0, 1),$$
$$\mathsf{P}(E) - \mathsf{P}(B) \geq C_F \|u\|^2_{W^{1,2}(\partial B)}.$$

Definition 3.1.7. Let $E \subset \mathbb{R}^N$ be a measurable set. Given $\Lambda > 0$, we say that E is a Λ-minimizer of $\mathscr{P}$ if for every bounded set $F \subset \mathbb{R}^N$ we have that

$$\mathscr{P}(E) \leq \mathscr{P}(F) + \Lambda|E\Delta F|.$$

Notice that the notion of Λ-minimizer of the perimeter is stronger than its localized (and more classical) version of (Λ, r)-minimizer of the perimeter (at least for bounded sets). One can find the more classical definition in [Mag12, Chapter 21].

Lemma 3.1.8. *Let $\bar{h} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a given kernel satisfying ($\mathbf{H}_g$). Let $E, F \subset \mathbb{R}^N$ be two sets with $|F| \leq |E| < +\infty$. Then, there exists a constant $C = C(N, \bar{h}, |E|)$ such that*

$$|\mathcal{E}_{\bar{h}}(E) - \mathcal{E}_{\bar{h}}(F)| \leq C|E\Delta F|.$$

Moreover, for any $\lambda > 1$ we have that $\mathcal{E}_{\bar{h}}(\lambda E) \leq \lambda^{2N}\mathcal{E}_{\bar{h}}(E)$.

Proof. We use the quadratic structure of the energy to expand the expression in the statement:

$$\mathcal{E}_{\bar{h}}(E) - \mathcal{E}_{\bar{h}}(F) = \mathcal{E}_{\bar{h}}(E \setminus F) + 2\mathcal{E}_{\bar{h}}(E \setminus F, E \cap F) - \mathcal{E}_{\bar{h}}(F \setminus E) - 2\mathcal{E}_{\bar{h}}(F \setminus E, E \cap F).$$

Each term in the right hand side coincides with the integral on the set $E \setminus F$, or on the set $F \setminus E$, of the potential generated by a set with measure smaller than $|E|+|F| \leq 2|E|$. Using the Riesz inequality, it is immediate to see that for any set $E' \subset \mathbb{R}^N$ with finite measure

$$\psi_{E',\bar{h}} \leq \int_{\tilde{B}} \bar{h}(x)dx = C(N, \bar{h}, |E'|),$$

where $\tilde{B}$ is the ball centered at the origin with measure $|E'|$. Plugging this inequality in the expression obtained expanding the energy, one obtains the desired Lipschitz control of $\mathcal{E}_{\bar{h}}$.

The second part of the statement is simply due to the change of variables in the expression of the Riesz energy, recalling that $\bar{h}$ is radially decreasing:

$$\mathcal{E}_{\bar{h}}(\lambda E) = \int_{\lambda E}\int_{\lambda E} \bar{h}(x-y)dxdy = \lambda^{2N}\int_E\int_E \bar{h}(\lambda(x'-y'))dx'dy' \leq \lambda^{2N}\mathcal{E}_{\bar{h}}(E).$$

□

Proposition 3.1.9. *Let $\gamma \in (0,1)$ be given, and let $\bar{h} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a kernel satisfying ($\mathbf{H}_g$). Then, any set $E \subset \mathbb{R}^N$ that minimizes $\mathcal{G}_\gamma$ with measure ω_N is a Λ-minimizer of $\mathscr{P}$ for some constant $\Lambda = \Lambda(N, \bar{h}, \mathscr{P}) > 0$.*

Proof. Suppose by contradiction that there exists a sequence of bounded sets $F_k \subset \mathbb{R}^N$ with $\mathscr{P}(F_k) \leq \mathscr{P}(E)$, $|E\Delta F_k| \neq 0$ and

$$\Lambda_k := \frac{\mathscr{P}(E) - \mathscr{P}(F_k)}{|E\Delta F_k|} \to +\infty.$$

Since $\mathscr{P}$ is either P or P_s, we use the isoperimetric inequality for those perimeters, and that $\mathscr{P}(F_k) \leq \mathscr{P}(E)$, to see that $|F_k|$ is bounded by a constant that depends on N, $\bar{h}$ and $\mathscr{P}$. Now we can estimate $\mathcal{G}_\gamma(F_k)$ as

$$\begin{aligned}\mathcal{G}_\gamma(F_k) &= \mathscr{P}(E) - \Lambda_k|E\Delta F_k| + \gamma\mathcal{E}_{\bar{h}}(F_k)\\ &= \mathscr{P}(E) - \Lambda_k|E\Delta F_k| + \gamma(\mathcal{E}_{\bar{h}}(F_k) - \mathcal{E}_{\bar{h}}(E) + \mathcal{E}_{\bar{h}}(E))\\ &\leq \mathcal{G}_\gamma(E) - \Lambda_k|E\Delta F_k| + \gamma C|E\Delta F_k|,\end{aligned} \tag{3.1}$$

where we used Lemma 3.1.8 in the last inequality since $|F_k|$ is controlled by a constant, as we pointed out above. Since $\Lambda_k \to +\infty$ and $\mathcal{G}_\gamma(F_k) \geq 0$, then (3.1) guarantees that $|E\Delta F_k| \to 0$. Let us take k so large that $\Lambda_k > C > \gamma C$, and we claim that $|F_k| < |E|$. In fact, when $\Lambda_k > \gamma C$, we have that $\mathcal{G}_\gamma(F_k) < \mathcal{G}_\gamma(E)$. If $|F_k| \geq |E|$, then we can cut F_k with an hyperplane to obtain a new set F'_k with $|F'_k| = |E|$, and both $\mathscr{P}$ and $\mathcal{E}$ decrease after this operation. Then $\mathcal{G}_\gamma(F'_k) \leq \mathcal{G}_\gamma(F_k) < \mathcal{G}_\gamma(E)$, but this is not possible since E is a minimizer of $\mathcal{G}_\gamma$.
The only possibility remaining is that $|F_k| < |E|$, and we can rescale the sets F_k in order to have the right measure. Notice that $|F_k| = |E| + |F_k \setminus E| - |E \setminus F_k|$, hence we define

$$\begin{aligned}\lambda_k = \left(\frac{|E|}{|F_k|}\right)^{1/N} &= \left(\frac{|E|}{|E| + |F_k \setminus E| - |E \setminus F_k|}\right)^{1/N}\\ &\leq \left(1 - \frac{|E\Delta F_k|}{|E|}\right)^{-1/N}.\end{aligned} \tag{3.2}$$

From the scaling properties of $\mathscr{P}$ and from the second part of Lemma 3.1.8 we know that $\mathcal{G}_\gamma(\lambda_k F_k) \leq \lambda_k^{2N}\mathcal{G}_\gamma(F_k)$. If we combine this estimate with (3.1), we take k large enough to have that $\Lambda_k > 2\gamma C$ and expand the rightmost formula in (3.2) with $\frac{|E\Delta F_k|}{|E|} < 1/2$ to get

$$\begin{aligned}\mathcal{G}_\gamma(\lambda_k F_k) &< \left(1 - \frac{|E\Delta F_k|}{|E|}\right)^{-2}(\mathcal{G}_\gamma(E) - \Lambda_k|E\Delta F_k| + \gamma C|E\Delta F_k|)\\ &\leq \left(1 - \frac{|E\Delta F_k|}{|E|}\right)^{-2}\left(\mathcal{G}_\gamma(E) - \frac{\Lambda_k}{2}|E\Delta F_k|\right)\\ &\leq \mathcal{G}_\gamma(E) - \frac{\Lambda_k}{2}|E\Delta F_k| + 2\frac{|E\Delta F_k|}{|E|}\mathcal{G}_\gamma(E).\end{aligned}$$

This yields to a contradiction since Λ_k is going to $+\infty$ as $k \to \infty$, so $\mathcal{G}_\gamma(\lambda_k F_k) < \mathcal{G}_\gamma(E)$. In the end, notice that the threshold for Λ_k depends only on N, $\bar{h}$ and $\mathscr{P}$. □

An important step in our approach is to show that the minimizers of $\mathcal{G}_\gamma$ are regular when γ is small. More precisely, they are of class $C^{1,\beta}$ and they converge in $C^{1,\beta}$ to the ball, up to translations. This is a fairly standard result, based on the fact that minimizers of $\mathcal{G}_\gamma$ are Λ-minimizers of $\mathscr{P}$ (see for instance [CL12, Proposition 2.2 and Lemma 3.6] and [FFM+15, Corollary 3.5 and Corollary 3.6]).

Lemma 3.1.10 (Existence and regularity of minimizers)**.** *Let $\bar{h} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a kernel satisfying ($\mathbf{H}_g$). Then, there exists $\gamma_0 > 0$, only depending on N, $\bar{h}$ and $\mathscr{P}$, such that for every $0 < \gamma < \gamma_0$ there exists a minimizer for $\mathcal{G}_\gamma$ with measure constraint ω_N. Additionally, there exists a (possibly) smaller $\gamma_1 \in (0, \gamma_0)$ such that, for any $0 < \gamma < \gamma_1$ and any minimizer E of $\mathcal{G}_\gamma$ with measure ω_N, there exists a function $u \in C^1(\partial B)$ such that, up to a translation,*

$$E = E(u) = \Big\{\rho z : \, z \in \mathbb{S}^{N-1}, 0 \le \rho < 1 + u(z)\Big\}, \qquad \int_E x dx = 0. \tag{3.3}$$

Furthermore, the function u belongs to $C^{1,\beta}$ for some $0 < \beta < 1/2$, and its norm can be taken arbitrarily small, up to decreasing the value of γ_1.

Proof. There exists a threshold $\gamma_0 > 0$, depending only on N, $\bar{h}$ and $\mathscr{P}$ such that, for any $\gamma < \gamma_0$ there exists a minimizer for $\mathcal{G}_\gamma$ with measure constraint ω_N. This is a well know fact, proved in slightly different settings in some previous works (see for instance [FFM+15, Lemma 5.1]), so we will not prove it here. In a nutshell, one uses the quantitative isoperimetric inequality present in Theorem 3.1.5 and the "cutting lemma" [FFM+15, Lemma 4.5] to localize the competitors, concluding with the standard compactness arguments valid for $\mathscr{P}$.

We already know from Proposition 3.1.9 that, for any $\gamma < 1$, the minimizers of $\mathcal{G}_\gamma$ are Λ-minimizers of $\mathscr{P}$ for some constant Λ. Using the quantitative isoperimetric inequality contained in Theorem 3.1.5 and the Lipschitz bound present in Lemma 3.1.8, we obtain the following chain of inequalities for a set $E_\gamma \subset \mathbb{R}^N$ that minimizes $\mathcal{G}_\gamma$ with measure constraint ω_N:

$$C_Q A(E_\gamma)^2 \le \mathscr{P}(E_\gamma) - \mathscr{P}(B) \le \gamma \left(\mathcal{E}_{\bar{h}}(B) - \mathcal{E}_{\bar{h}}(E_\gamma)\right) \le \gamma \omega_N A(E_\gamma).$$

Therefore, up to translations we have that $\mathrm{bar}(E_\gamma) = \int_{E_\gamma} x dx = 0$ and $E_\gamma \to B$ in L^1 as $\gamma \to 0$. Since the limit set is smooth, and all the E_γ are Λ-minimizers of $\mathscr{P}$, then we can apply the standard regularity theory (see [Tam84], [Mag12, Theorem 26.3] and [FFM+15, Corollary 3.6]), that provides a threshold γ_1 such that, for every $\gamma < \gamma_1$, the set E_γ has the structure represented in (3.3): $E_\gamma = E(u_\gamma)$ for some function $u_\gamma \in C^1(\partial B)$. Additionally, there exists $\beta \in (0, 1/2)$ such that $u_\gamma \in C^{1,\beta}(\partial B)$, and $u_\gamma \to 0$ in $C^1(\partial B)$. □

3.2 Gamow model with general repulsion

Here we provide a simple growth property for the function $\bar{h}$ that is a special case of [BL83, Lemma A.IV]. It is exploited in the successive result, namely Lemma 3.2.2, which in turn is the key estimate to obtain the energy bound in Lemma 3.2.4. Notice the close relation between this result and the "growth properties" present in Remark 1.1.5 and Lemma 1.1.6.

Lemma 3.2.1. *If $\bar{h} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ satisfies ($\mathbf{H}_g$), then there exists a constant $C(N, \bar{h}) > 0$ such that*

$$\bar{h}(x) \le \frac{C(N, \bar{h})}{|x|^N} \qquad \forall x \in B \setminus \{0\}.$$

More precisely, we must have that $\limsup_{x\to 0} \bar{h}(x)|x|^N = 0$.

Proof. We argue by contradiction. As usual, we denote by h the radial profile of $\bar{h}$. Let us suppose that there exists a sequence $r_k \to 0^+$ such that $\limsup_k h(r_k)r_k^N = \lim_k h(r_k)r_k^N > 0$. Without loss of generality we can assume that $r_k < 1$ and $r_{k+1} < r_k/2$ for all $k \in \mathbb{N}$. The monotonicity of h implies that

$$\int_B \bar{h}(x)dx \geq \omega_N \sum_{k=1}^{+\infty} h(r_k)(r_k^N - r_{k+1}^N) \geq \omega_N \sum_{k=1}^{+\infty} h(r_k)r_k^N \left(1 - \frac{1}{2^N}\right).$$

Since $\bar{h} \in L^1(B)$ we have that the last series converges, so its terms have to be infinitesimal, but this is not compatible with the fact that $\lim_k h(r_k)r_k^N > 0$.
We proved only the second part of the statement, but the first part can be proved reasoning in an analogous way. Indeed, it is sufficient to take two sequences $r_k \in (0,1)$ and $C_k \to +\infty$ with $h(r_k)r_k^N > C_k$. Then notice that r_k must converge to 0^+ (otherwise we would reach immediately a contradiction with the integrability of $\bar{h}$), so that the previous argument works again. □

Lemma 3.2.2. *Let* $u : \partial B \to \mathbb{R}$ *be a measurable function, and let* $\bar{h} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ *be a kernel satisfying* ($\mathbf{H}_g$). *Then, there exists a constant* $C_W = C_W(N, \bar{h}) > 0$ *such that*

$$\int_{\partial B}\int_{\partial B} \bar{h}(z-w)|u(z)-u(w)|^2 d\mathscr{H}^{N-1}(z)d\mathscr{H}^{N-1}(w) \leq C_W[u]_s^2 \qquad \forall s \in [1/2, 1),$$

and the same holds with the local seminorm $[u]_{W^{1,2}(\partial B)}$ *(i.e.* $\|\nabla_\tau u\|_{L^2(\partial B)}$*):*

$$\int_{\partial B}\int_{\partial B} \bar{h}(z-w)|u(z)-u(w)|^2 d\mathscr{H}^{N-1}(z)d\mathscr{H}^{N-1}(w) \leq C_W[u]_{W^{1,2}(\partial B)}^2.$$

Proof. Thanks to Lemma 3.2.1, for any $s_0 \in [0,1)$ we have that

$$\begin{aligned}
&\int_{\partial B}\int_{\partial B} \bar{h}(z-w)|u(z)-u(w)|^2 d\mathscr{H}^{N-1}(z)d\mathscr{H}^{N-1}(w) \\
&\quad \leq C(N,\bar{h}) \int_{\partial B}\int_{\partial B} \frac{|u(z)-u(w)|^2}{|z-w|^N} d\mathscr{H}^{N-1}(z)d\mathscr{H}^{N-1}(w) \\
&\quad \leq 2^{s_0} C(N,\bar{h}) \int_{\partial B}\int_{\partial B} \frac{|u(z)-u(w)|^2}{|z-w|^{N+s_0}} d\mathscr{H}^{N-1}(z)d\mathscr{H}^{N-1}(w) \leq 2C(N,\bar{h})[u]_{\frac{1+s_0}{2}}^2,
\end{aligned}$$

where clearly the fractional Sobolev seminorm is relative to the hypersurface $\partial B \subset \mathbb{R}^N$. This is the desired inequality since we can take $s = (1+s_0)/2$. Finally, the inequality with the seminorm $[u]_{W^{1,2}(\partial B)}^2$ on the right hand side is obtained applying [DNPV12, Proposition 2.2] in two charts that cover ∂B. □

Remark 3.2.3. It is possible to prove the inequality for the local Sobolev seminorm $[u]_{W^{1,2}}(\partial B)$ without relying on the Sobolev embedding [DNPV12, Proposition 2.2], as it was originally accomplished in [CFP23, Lemma 2.2].

For a given function $u \in C^1(\mathbb{S}^{N-1})$ with $u > -1$ everywhere, denoting by E the set given by (3.3), we define

$$E^+ = E \setminus B, \qquad\qquad E^- = B \setminus E, \tag{3.4}$$

so that

$$\begin{aligned} E^+ &= \Big\{\rho z : \; z \in \mathbb{S}^{N-1}, 1 \le \rho < 1 + u^+(z)\Big\}, \\ E^- &= \Big\{\rho z : \; z \in \mathbb{S}^{N-1}, 1 - u^-(z) < \rho < 1\Big\}, \end{aligned}$$

calling as usual $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. We point out that, in our applications, the set E has barycenter in the origin (see (3.3)), so the ball B is not necessarily the optimal ball to compute the asymmetry of E. Thanks to the above result, we deduce the following estimate.

Lemma 3.2.4 ($\mathcal{E}_{\bar h}(E^+, E^-)$ is "negligible")**.** *Let $\bar h : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a kernel satisfying ($\mathbf{H}_g$), and let $u \in C^1(\mathbb{S}^{N-1})$, with $|u| < 1/2$. Then for every $s \in [1/2, 1)$*

$$\mathcal{E}_{\bar h}(E^+, E^-) \le C[u]_s^2 \qquad \textit{and} \qquad \mathcal{E}_{\bar h}(E^+, E^-) \le C[u]_{W^{1,2}}^2,$$

where C is a constant, only depending on N and on $\bar h$.

Proof. For every $x \in E^+$ and every $y \in E^-$, we write $z = x/|x|$ and $w = y/|y|$. We define for brevity the auxiliary function $\tilde h(v) = \bar h(v/2)$ for every $v \in \mathbb{R}^N$, that is integrable:

$$\int_B \tilde h(x)dx \le 2^N \int_B \bar h(x)dx < +\infty, \tag{3.5}$$

where we used that $\bar h$ is radial and radially decreasing. This definition is convenient because we notice that $|x - y| \ge |z - w|/2$, and thus

$$\bar h(x - y) \le \tilde h(z - y).$$

Calling $\pi : \mathbb{R}^N \setminus \{0\} \to \mathbb{S}^{N-1}$ the projection on the unit sphere, we can then evaluate

$$\begin{aligned} \mathcal{E}_{\bar h}(E^+, E^-) &= \iint_{E^+ \times E^-} \bar h(x - y)dxdy \\ &\le \iint_{\pi(E^+)\times\pi(E^-)} \int_{\rho=1}^{1+u^+(z)} \int_{\sigma=1-u^-(w)}^{1} \tilde h(w - z)\rho^{N-1}\sigma^{N-1} d\rho d\sigma d\mathscr{H}^{N-1}(w) d\mathscr{H}^{N-1}(z) \\ &\le 2^{N-1} \iint_{\pi(E^+)\times\pi(E^-)} u^+(z)u^-(w)\tilde h(w - z) d\mathscr{H}^{N-1}(w) d\mathscr{H}^{N-1}(z) \end{aligned}$$

Notice that, for every $z \in \pi(E^+)$ and $w \in \pi(E^-)$, we have $u^+(z) > 0$ and $u^-(w) > 0$, hence

$$u^+(z)u^-(w) \le \big(u^+(z) + u^-(w)\big)^2 = (u(z) - u(w))^2.$$

Thus the above estimate can be continued as

$$\begin{aligned}\mathcal{E}_{\bar{h}}(E^+,E^-) &\le 2^{N-1}\iint_{\pi(E^+)\times\pi(E^-)}(u(w)-u(z))^2\tilde{h}(w-z)d\mathscr{H}^{N-1}(w)d\mathscr{H}^{N-1}(z)\\ &\le 2^{N-1}\iint_{\mathbb{S}^{N-1}\times\mathbb{S}^{N-1}}(u(w)-u(z))^2\tilde{h}(w-z)d\mathscr{H}^{N-1}(w)d\mathscr{H}^{N-1}(z).\end{aligned}$$

Thanks to (3.5), we can apply Lemma 3.2.2 to the last expression, obtaining the desired result with a constant depending only on N and $\bar{h}$. □

Since we will need to calculate integrals of $\bar{h}$ over translated balls, it is useful to use again the function $\Phi_{\bar{h}}$ defined in (2.38), and the auxiliary function $\mathscr{I} : (-1/2, 1/2) \to \mathbb{R}$ as

$$\mathscr{I}(\sigma) = \Phi_{\bar{h}}(1+\sigma, 1) - \Phi_{\bar{h}}(1,1). \tag{3.6}$$

It is simple to observe that $\Phi_{\bar{h}}$ is locally Lipschitz continuous outside the diagonal, but this is not helpful since we will need to use $\Phi_{\bar{h}}(a,b)$ with $a \approx b \approx 1$. However, the following weaker property will play a crucial role in our construction.

Lemma 3.2.5. *Let $\bar{h} : \mathbb{R}^N \to \mathbb{R}^+ \cup \{+\infty\}$ be a function satisfying ($\mathbf{H}_g$). Then, there exists a constant $C = C(\bar{h}, N)$ such that, for every $3/4 \le \rho \le 5/4$ and every $-1/4 \le \tau \le 1/4$ one has*

$$\begin{gathered}\left|\Phi_{\bar{h}}(\rho+\tau,\rho) - \Phi_{\bar{h}}(\rho,\rho) - \mathscr{I}(\tau)\right| \le C|\rho-1|,\\ \left|\Phi_{\bar{h}}(1,1+\tau) - \Phi_{\bar{h}}(1,1) + \mathscr{I}(\tau)\right| \le C|\tau|,\\ |\mathscr{I}(\tau) + \mathscr{I}(-\tau)| \le C|\tau|.\end{gathered} \tag{3.7}$$

Proof. In this proof, we will not keep track of the constants, and the same letter can represent different constants, changing even from line to line. In any case, when we do not explicitly assert the dependence of such constants, it is intended that they depend on N and $\bar{h}$. Similarly to the previous chapters, since $\bar{h}$ is radial by hypothesis, we denote by h its radial profile.
The thesis will follow from three main estimates. To start, we take $1/2 \le r, r' \le 3/2$, and we show that $|r - r'|$ controls $|\Phi_{\bar{h}}(r,r) - \Phi_{\bar{h}}(r',r')|$. Without loss of generality we assume that $r > r'$. Notice that $\Phi_{\bar{h}}(r,r) - \Phi_{\bar{h}}(r',r')$, by definition, is the integral of $\bar{h}$ on the set $A(r,r')$ given by the difference of two balls, a bigger one with radius r and a smaller one with radius r', being the smaller one contained in the bigger one and internally tangent. Figure 3.1 shows the set $A(r,r')$, coloured, close to the point of tangency, that we consider to be the origin O. We also consider the exterior normal to the two balls in the tangency point to be horizontal (i.e., parallel to the first vector of a given orthonormal basis). Let us assume for a moment that $N = 2$, just for simplicity in the figure. As shown in the figure, we fix $0 < t < 1/4$, and we call $Q_1 = te_1$ the point having distance t from O in the horizontal direction. We consider then the circle $S_2(t)$ with radius t centered at O, we call Q_2 one of the two points of intersection of $S_2(t)$ with the larger ball, and we denote by θ the angle $\angle Q_1OQ_2$. In the very same way, we call Q_3 a point of intersection between $S_2(t)$ and the smaller ball, and we call θ' the angle $\angle Q_1OQ_3$. One readily has that $\cos\theta = -t/2r$, and similarly $\cos\theta' = -t/2r'$.

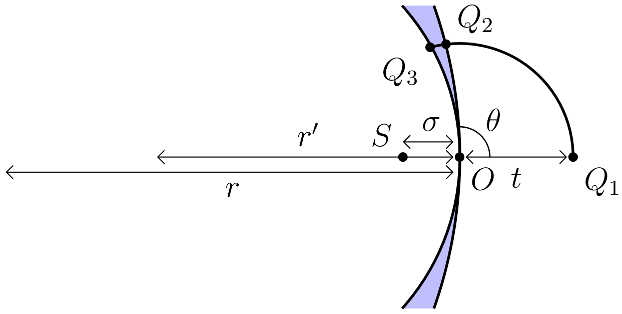


Figure 3.1: The (coloured) set $A(r,r')$ and the angle θ in the proof of (3.9) and (3.10).

Since by geometric reasons $\frac{\pi}{2} < \theta < \theta' < \frac{3}{4}\pi$ because we are considering $0 < t < 1/4$, we get

$$\theta' - \theta \le 2(\cos\theta - \cos\theta') = \frac{t(r-r')}{rr'} \le 4t(r-r').$$

Therefore, in the 2-dimensional case, we can estimate for every $0 < t < 1/4$

$$\mathscr{H}^1\big(A(r,r') \cap S_2(t)\big) = 2t(\theta' - \theta) \le 8t^2(r-r').$$

Let us pass to the general N-dimensional case. Calling $A_2(r,r')$ the 2-dimensional set already studied, we have in general

$$A(r,r') = \Big\{(x_1,x') \in \mathbb{R} \times \mathbb{R}^{N-1} : \, (x_1,|x'|) \in A_2(r,r')\Big\}.$$

Calling then $S_N(t) = t\mathbb{S}^{N-1} \subset \mathbb{R}^N$ the sphere with radius t centered at 0, an immediate integration in cylindrical coordinates gives, for every $0 < t < 1/4$,

$$\begin{aligned}\mathscr{H}^{N-1}\big(A(r,r') \cap S_N(t)\big) &\le (N-1)\omega_{N-1} t^{N-2} \mathscr{H}^1\big(A_2(r,r') \cap S_2(t)\big)\\ &\le 8(N-1)\omega_{N-1} t^N (r-r').\end{aligned}$$

With this estimate at hand, we obtain that

$$\begin{aligned}\Phi_{\bar h}(r,r) - \Phi_{\bar h}(r',r') &= \int_{A(r,r')} \bar h(x)dx = \int_{t=0}^3 h(t)\mathscr{H}^{N-1}\big(A(r,r') \cap S_N(t)\big)dt\\ &\le \int_0^{1/4} h(t)\mathscr{H}^{N-1}\big(A(r,r') \cap S_N(t)\big)dt + h(1/4)\int_{1/4}^3 \mathscr{H}^{N-1}\big(A(r,r') \cap S_N(t)\big)dt\\ &\le 8(N-1)\omega_{N-1}(r-r')\int_0^{1/4} h(t)t^N dt + h(1/4)\big|A(r,r')\big| \le C(r-r'),\end{aligned} \tag{3.8}$$

where C is a constant only depending on N and on $\bar h$. For every $1/2 \le r, r' \le 3/2$ we have then

$$\big|\Phi_{\bar h}(r,r) - \Phi_{\bar h}(r',r')\big| \le C|r-r'|. \tag{3.9}$$

We pass now to the second main estimate. Let us take $-1/4 \le \sigma \le 1/4$ and let us show that $|r-r'|$ controls also $|\Phi_{\bar h}(r,r-\sigma) - \Phi_{\bar h}(r',r'-\sigma)|$. As before, without loss of generality we can assume that $r > r'$. The value of the difference $\Phi_{\bar h}(r, r-\sigma) - \Phi_{\bar h}(r', r'-\sigma)$ is then exactly as before given by an integral over the set $A(r,r')$. The only difference is that this time the function to integrate is not $\bar h(x)$, but $\bar h(x-S)$, where $S = -\sigma e_1$ is the point having distance σ from O in the horizontal, negative direction. Figure 3.1 shows the point S in the case when $\sigma > 0$. Notice that the points of $A(r,r')$ close to O are much closer to O than to S. More in general, a trivial geometric argument ensures that for every $x \in A(r,r')$ one has

$$|x| = |x - O| \leq 2|x - S|,$$

the constant 2 is actually not needed if $\sigma < 0$. As a consequence, we have

$$\Phi_{\bar h}(r, r-\sigma) - \Phi_{\bar h}(r', r'-\sigma) = \int_{A(r,r')} \bar h(x-S)dx \leq \int_{A(r,r')} \tilde h(x)dx,$$

where we write $\tilde h(x) = \bar h(x/2)$ as in the proof of Lemma 3.2.4. The same calculation made in (3.8), keeping in mind (3.5), gives that for every $1/2 \leq r, r' \leq 3/2$ and every $-1/4 \leq \sigma \leq 1/4$

$$\big|\Phi_{\bar h}(r, r-\sigma) - \Phi_{\bar h}(r', r'-\sigma)\big| \leq C|r - r'|. \tag{3.10}$$

Let us finally pass to the third and last main estimate, which consists in taking again $-1/4 \leq \sigma \leq 1/4$, and showing that $|\sigma|$ controls $|\mathscr{I}(\sigma) + \mathscr{I}(-\sigma)|$. Without loss of generality let us assume that $\sigma > 0$. Observe that $\mathscr{I}(\sigma) = \Phi_{\bar h}(1+\sigma, 1) - \Phi_{\bar h}(1,1)$ is

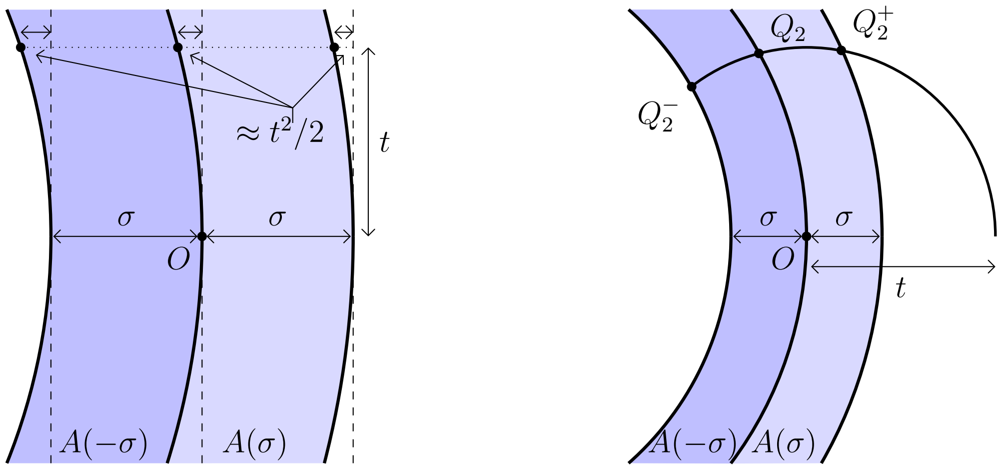


Figure 3.2: The (coloured) sets $A(\sigma)$ and $A(-\sigma)$ and the situation in the proof of (3.18).

the integral of $\bar h$ over an annulus $A(\sigma)$ with radii 1 and $1+\sigma$, the origin being at the internal boundary, while $-\mathscr{I}(-\sigma) = \Phi_{\bar h}(1,1) - \Phi_{\bar h}(1-\sigma, 1)$ is the integral of $\bar h$ over an annulus $A(-\sigma)$ with radii 1 and $1-\sigma$, the origin being at the external boundary. Figure 3.2 shows the annuli $A(\sigma)$ and $A(-\sigma)$ near O with two different magnifications. Let us start near the origin O, noticing that $A(\sigma)$ and $A(-\sigma)$ are close to the slabs

$$\mathscr{C}^+ = \big\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : 0 < x_1 < \sigma\big\},$$
$$\mathscr{C}^- = \big\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : -\sigma < x_1 < 0\big\}.$$

More precisely, fix any $0 < t < 2\sigma$, and set $S(t) = B_{2\sigma} \cap \{(x_1, x') : |x'| = t\}$. Since of course

$$\int_{\mathscr{C}^+ \cap S(t)} \bar{h}(x) d\mathscr{H}^{N-1}(x) = \int_{\mathscr{C}^- \cap S(t)} \bar{h}(x) d\mathscr{H}^{N-1}(x),$$

keeping in mind that h is decreasing and, by an immediate geometric argument (see Figure 3.2, left), we can estimate

$$\begin{aligned}
\bigg| \int_{A(\sigma)\cap S(t)} \bar{h}(x) d\mathscr{H}^{N-1}(x) - \int_{A(-\sigma)\cap S(t)} \bar{h}(x) d\mathscr{H}^{N-1}(x) \bigg| \\
\leq h(t)\Big(\mathscr{H}^{N-1}\big(\big(A(\sigma)\Delta\mathscr{C}^+\big) \cap S(t)\big) + \mathscr{H}^{N-1}\big(\big(A(-\sigma)\Delta\mathscr{C}^-\big) \cap S(t)\big)\Big) \\
\leq 4(N-1)\omega_{N-1} t^N h(t).
\end{aligned}$$

By integrating in t, then, we have

$$\begin{aligned}
\bigg| \int_{A(\sigma)\cap B_{2\sigma}} \bar{h}(x)dx - \int_{A(-\sigma)\cap B_{2\sigma}} \bar{h}(x)dx \bigg| &\leq \int_0^{2\sigma} 4(N-1)\omega_{N-1} t^N h(t) dt \\
&\leq 8(N-1)\omega_{N-1}\sigma \int_0^{2\sigma} h(t) t^{N-1} dt \leq C\sigma,
\end{aligned} \tag{3.11}$$

and the last inequality uses the local integrability of $\bar{h}$. Let us now pass to consider the situation outside the ball $B_{2\sigma}$. As in the proof of (3.9), we call $S_N(t)$ the sphere with radius t centered at 0, and we start considering the situation in the 2-dimensional case, with circle $S_2(t)$ and annuli $A_2(\pm\sigma)$. Let us fix any $2\sigma < t < 1/4$. As depicted in the right part of Figure 3.2, the circle $S_2(t)$ intersects $A_2(\sigma)$ in two symmetric arcs, and the same is true for the intersection with $A_2(-\sigma)$. Let us call Q_2, Q_2^+ and Q_2^- three intersection points, as in the figure, and let us call θ, θ^+ and θ^- the directions of the segments OQ_2, OQ_2^+ and OQ_2^-. Notice that $\theta^+ < \theta < \theta^-$, and the three directions are close to $\pi/2$ when $\sigma \ll t \ll 1$. A very simple trigonometric calculation ensures that

$$\cos\theta = -\frac{t}{2}, \qquad \cos\theta^+ = \frac{-t^2 + 2\sigma + \sigma^2}{2t}, \qquad \cos\theta^- = \frac{-t^2 - 2\sigma + \sigma^2}{2t}, \tag{3.12}$$

and since $2\sigma < t < 1/4$ this implies

$$\theta < \theta^- < \frac{3}{4}\pi, \qquad \frac{\pi}{2} < \theta < \frac{7}{12}\pi, \qquad \frac{\pi}{3} < \theta^+ < \theta,$$

in particular $\theta^- - \theta$ and $\theta - \theta^+$ are both smaller than $\pi/4$, so that

$$\begin{aligned}
\big|\theta^+ + \theta^- - 2\theta\big| &= \big|(\theta^- - \theta) - (\theta - \theta^+)\big| \leq \sqrt{2}\big|\sin(\theta^- - \theta) - \sin(\theta - \theta^+)\big| \\
&\leq 2\sin\theta\big|\sin(\theta^- - \theta) - \sin(\theta - \theta^+)\big| \\
&= 2\Big|\cos\theta^+ + \cos\theta^- - 2\cos\theta - \cos\theta\Big(\cos(\theta^- - \theta) + \cos(\theta - \theta^+) - 2\Big)\Big| \\
&\leq 2\frac{\sigma^2}{t} + t\Big|\big(\cos(\theta^- - \theta) + \cos(\theta - \theta^+) - 2\big)\Big| \\
&\leq 2\frac{\sigma^2}{t} + \frac{t}{2}\Big((\theta^- - \theta)^2 + (\theta - \theta^+)^2\Big) \\
&\leq 2\frac{\sigma^2}{t} + t\Big((\cos\theta^- - \cos\theta)^2 + (\cos\theta - \cos\theta^+)^2\Big) \leq 5\frac{\sigma^2}{t} \leq 3\sigma.
\end{aligned} \tag{3.13}$$

We can now calculate

$$\begin{aligned}\mathscr{H}^{N-1}\big(A(\sigma)\cap S_N(t)\big) &= \int_{\theta^+}^{\theta}(N-1)\omega_{N-1}(t\sin\alpha)^{N-2}t\,d\alpha\\ &= (N-1)\omega_{N-1}t^{N-1}\int_{\theta^+}^{\theta}(\sin\alpha)^{N-2}d\alpha\\ &= (N-1)\omega_{N-1}t^{N-1}\bigg((\sin\theta)^{N-2}(\theta-\theta^+)+\int_{\theta^+}^{\theta}(\sin\alpha)^{N-2}-(\sin\theta)^{N-2}d\alpha\bigg),\end{aligned}$$

and similarly

$$\frac{\mathscr{H}^{N-1}\big(A(-\sigma)\cap S_N(t)\big)}{(N-1)\omega_{N-1}} = t^{N-1}\bigg((\sin\theta)^{N-2}(\theta^- -\theta)+\int_{\theta}^{\theta^-}(\sin\alpha)^{N-2}-(\sin\theta)^{N-2}d\alpha\bigg),$$

so that

$$\frac{\Big|\mathscr{H}^{N-1}\big(A(\sigma)\cap S_N(t)\big)-\mathscr{H}^{N-1}\big(A(-\sigma)\cap S_N(t)\big)\Big|}{(N-1)\omega_{N-1}} \leq t^{N-1}\Big(\big|\theta^+ +\theta^- -2\theta\big|+K\Big), \tag{3.14}$$

where

$$K=\bigg|\int_{\theta^+}^{\theta}(\sin\alpha)^{N-2}-(\sin\theta)^{N-2}d\alpha-\int_{\theta}^{\theta^-}(\sin\alpha)^{N-2}-(\sin\theta)^{N-2}d\alpha\bigg|.$$

We claim that

$$K\leq 9(N-2)\sigma. \tag{3.15}$$

To show this inequality, we first observe that by (3.12) we have

$$|\theta^+-\theta|\leq\sqrt{2}|\cos\theta^+-\cos\theta|\leq 2\frac{\sigma}{t},\quad |\theta^--\theta|\leq\sqrt{2}|\cos\theta^--\cos\theta|\leq 2\frac{\sigma}{t}. \tag{3.16}$$

We distinguish then two cases. If $t\geq\sqrt{\sigma}$, then again by (3.12) we have $|\cos\theta^+|\leq 2t$ and $|\cos\theta^-|\leq 2t$, thus for every $\theta^+<\alpha<\theta^-$ by (3.16) one has

$$|(\sin\alpha)^{N-2}-(\sin\theta)^{N-2}|\leq(N-2)|\sin\alpha-\sin\theta|\leq 2(N-2)t|\alpha-\theta|\leq 4(N-2)\sigma,$$

so that

$$K\leq 16(N-2)\frac{\sigma^2}{t}\leq 8(N-2)\sigma,$$

and then (3.15) is proved in the case $t\geq\sqrt{\sigma}$. Suppose insted that $2\sigma<t<\sqrt{\sigma}$. In this case, for every $\theta^+<\alpha<\theta^-$ by (3.16) we have that

$$|(\sin\alpha)^{N-2}-(\sin\theta)^{N-2}|\leq(N-2)|\sin\alpha-\sin\theta|\leq(N-2)|\alpha-\theta|\leq 2(N-2)\frac{\sigma}{t}. \tag{3.17}$$

Let us now call $\hat\theta$ and $\hat\theta^+$ the directions obtained by a vertical mirror symmetry of θ and θ^+, that is, $\hat\theta=\pi-\theta$ and $\hat\theta^+=\pi-\theta^+$. Observe that, again by (3.12) and since $t<\sqrt{\sigma}$, we have $\theta^+<\hat\theta<\pi/2<\theta<\hat\theta^+<\theta^-$. Since by symmetry we have

$$\int_{\theta^+}^{\hat\theta}(\sin\alpha)^{N-2}-(\sin\theta)^{N-2}d\alpha=\int_{\theta}^{\hat\theta^+}(\sin\alpha)^{N-2}-(\sin\theta)^{N-2}d\alpha,$$

by (3.17) and (3.13) we have

$$\begin{aligned} K &\leq 2(N-2)\frac{\sigma}{t}\Big((\theta-\hat\theta)+(\theta^- -\hat\theta^+)\Big) = 2(N-2)\frac{\sigma}{t}\Big(2(\theta-\hat\theta)+(\theta^+ +\theta^- -2\theta)\Big) \\ &\leq 2(N-2)\frac{\sigma}{t}\Big(2\sqrt{2}(\cos\hat\theta-\cos\theta)+3\sigma\Big) = 2(N-2)\frac{\sigma}{t}\Big(2\sqrt{2}t+3\sigma\Big) \leq 9(N-2)\sigma, \end{aligned}$$

thus (3.15) is proved also in the case $t<\sqrt{\sigma}$. Inserting (3.15) into (3.14) and keeping in mind (3.13), we have then for every $2\sigma<t<1/4$

$$\Big|\mathscr{H}^{N-1}\big(A(\sigma)\cap S_N(t)\big)-\mathscr{H}^{N-1}\big(A(-\sigma)\cap S_N(t)\big)\Big| \leq Ct^{N-1}\sigma.$$

Putting together this inequality and (3.11), we obtain the third main estimate, that is,

$$\begin{aligned} \big|\mathscr{I}(\sigma)+\mathscr{I}(-\sigma)\big| &= \left|\int_{A(\sigma)}\bar h(x)dx - \int_{A(-\sigma)}\bar h(x)dx\right| \\ &\leq C\sigma + \int_{t=2\sigma}^{3} h(t)\Big|\mathscr{H}^{N-1}\big(A(\sigma)\cap S_N(t)\big)-\mathscr{H}^{N-1}\big(A(-\sigma)\cap S_N(t)\big)\Big|dt \qquad (3.18)\\ &\leq C\sigma + C\int_{t=2\sigma}^{1/4} h(t)t^{N-1}\sigma dt + h(1/4)\Big(\big|A(\sigma)\big|+\big|A(-\sigma)\big|\Big) \leq C\sigma. \end{aligned}$$

Thanks to the main estimates (3.9), (3.10) and (3.18), it is immediate to prove (3.7). The third estimate in (3.7) is simply (3.18) with $\sigma=|\tau|$. The first estimate in (3.7) comes by putting together (3.10) with $r=\rho+\tau$, $r'=1+\tau$ and $\sigma=\tau$, and (3.9) with $r=\rho$ and $r'=1$, getting

$$\begin{aligned} \big|\Phi_{\bar h}(\rho+\tau,\rho)-\Phi_{\bar h}(\rho,\rho)-\mathscr{I}(\tau)\big| &= \big|\Phi_{\bar h}(\rho+\tau,\rho)-\Phi_{\bar h}(\rho,\rho)-\Phi_{\bar h}(1+\tau,1)+\Phi_{\bar h}(1,1)\big| \\ &\leq \big|\Phi_{\bar h}(\rho+\tau,\rho)-\Phi_{\bar h}(1+\tau,1)\big| + |\Phi_{\bar h}(\rho,\rho)-\Phi_{\bar h}(1,1)| \\ &\leq C|\rho-1|. \end{aligned}$$

Finally, the second estimate in (3.7) comes by putting together (3.10) with $r=1$, $r'=1-\tau$ and $\sigma=-\tau$, and (3.18) with $\sigma=|\tau|$, obtaining

$$\big|\Phi_{\bar h}(1,1+\tau)-\Phi_{\bar h}(1,1)+\mathscr{I}(\tau)\big| \leq \big|\Phi_{\bar h}(1,1+\tau)-\Phi_{\bar h}(1-\tau,1)\big|+\big|\mathscr{I}(-\tau)+\mathscr{I}(\tau)\big| \leq C|\tau|.$$

The proof is then concluded. □

We are now ready to give the proof of our main result, namely

Theorem 3.2.6 (Balls are unique minimizers for small γ)**.** *Let $\bar h:\mathbb{R}^N\to\mathbb{R}^+\cup\{+\infty\}$ be a kernel satisfying* ($\mathbf{H}_g$)*. Then, there exists $\bar\gamma>0$ such that, for every $0<\gamma<\bar\gamma$, the unique minimizer (up to translations) of $\mathcal{G}_\gamma$ among sets of measure ω_N is B.*

Proof. Let $\gamma>0$ be given, and let E be a minimizer of $\mathcal{G}_\gamma$ among sets of measure ω_N. We write the proof only in the case of the standard perimeter (namely, $\mathscr{P}=\mathsf{P}$) since everything can be repeated verbatim for the fractional perimeter using the corresponding version of Theorem 3.1.6, Lemma 3.1.10, Lemma 3.2.2, Lemma 3.2.4. We already know by Lemma 3.1.10 that, if γ is small enough, then up to a translation

E is of the form $E(u)$ given by (3.3) for a uniformly small function $u \in C^1(\mathbb{S}^{N-1})$. In particular, up to reducing γ, we can suppose without loss of generality that $\|u\|_\infty \leq 1/2$. Let E^+ and E^- be defined as in (3.4). We notice that the sets $E^+ \subseteq \mathbb{R}^N \setminus B$ and $E^- \subseteq B$ have the same measure, and they are uniformly close to the sphere $\mathbb{S}^{N-1}$. We can write

$$\mathscr{E}_{\bar{h}}(E) - \mathscr{E}_{\bar{h}}(B) = 2\mathscr{E}_{\bar{h}}(B, E^+) - 2\mathscr{E}_{\bar{h}}(B, E^-) + \mathscr{E}_{\bar{h}}(E^+) + \mathscr{E}_{\bar{h}}(E^-) - 2\mathscr{E}_{\bar{h}}(E^+, E^-). \quad (3.19)$$

Using the notation introduced in (3.6), we can also calculate

$$\begin{aligned}
\mathscr{E}_{\bar{h}}(B, E^+) - \mathscr{E}_{\bar{h}}(B, E^-) &= \iint_{B\times E^+} \bar{h}(x-y)dxdy - \iint_{B\times E^-} \bar{h}(x-y)dxdy \\
&= \int_{E^+} \Phi_{\bar{h}}(1,|x|)dx - \int_{E^-} \Phi_{\bar{h}}(1,|x|)dx \\
&= \int_{E^+} \Phi_{\bar{h}}(1,|x|) - \Phi_{\bar{h}}(1,1)dx - \int_{E^-} \Phi_{\bar{h}}(1,|x|) - \Phi_{\bar{h}}(1,1)dx.
\end{aligned} \quad (3.20)$$

Let us now observe that, also by Lemma 3.2.5,

$$\begin{aligned}
&\int_{E^+} \Phi_{\bar{h}}(1,|x|) - \Phi_{\bar{h}}(1,1)dx \\
&\quad = \int_{z\in\partial B} \int_0^{u^+(z)} (1+t)^{N-1}\Big(\Phi_{\bar{h}}(1,1+t) - \Phi_{\bar{h}}(1,1)\Big)dt\mathscr{H}^{N-1}(z) \\
&\quad = \int_{\partial B} \int_0^{u^+(z)} \Phi_{\bar{h}}(1,1+t) - \Phi_{\bar{h}}(1,1)dt\mathscr{H}^{N-1}(z) + O(\|u\|^2_{L^2}) \\
&\quad = -\int_{\partial B} \int_0^{u^+(z)} \mathscr{I}(t)dt\mathscr{H}^{N-1}(z) + O(\|u\|^2_{L^2}),
\end{aligned}$$

and in the very same way

$$\int_{E^-} \Phi_{\bar{h}}(1,|x|) - \Phi_{\bar{h}}(1,1)dx = -\int_{\partial B} \int_0^{u^-(z)} \mathscr{I}(-t)dtd\mathscr{H}^{N-1}(z) + O(\|u\|^2_{L^2}).$$

The equality (3.20) becomes then

$$\begin{aligned}
\mathscr{E}_{\bar{h}}(B, E^+) - \mathscr{E}_{\bar{h}}(B, E^-) = &-\int_{\partial B} \int_0^{u^+(z)} \mathscr{I}(t)dtd\mathscr{H}^{N-1}(z) \\
&+ \int_{\partial B} \int_0^{u^-(z)} \mathscr{I}(-t)dtd\mathscr{H}^{N-1}(z) + O(\|u\|^2_{L^2}),
\end{aligned}$$

and inserting it in (3.19), recalling Lemma 3.2.4, we obtain that

$$\begin{aligned}
\mathscr{E}_{\bar{h}}(E) - \mathscr{E}_{\bar{h}}(B) = \mathscr{E}_{\bar{h}}(E^+) &- 2\int_{\partial B} \int_0^{u^+(z)} \mathscr{I}(t)dtd\mathscr{H}^{N-1}(z) \\
&+ \mathscr{E}_{\bar{h}}(E^-) + 2\int_{\partial B} \int_0^{u^-(z)} \mathscr{I}(-t)dtd\mathscr{H}^{N-1}(z) + O(\|u\|^2_{W^{1,2}}).
\end{aligned} \quad (3.21)$$

In order to evaluate $\mathcal{E}_{\bar h}(E^+)$ and $\mathcal{E}_{\bar h}(E^-)$, we call for brevity

$$\varphi(z,w,s,t) = (1+t)^{N-1}(1+s)^{N-1}\bar h\left((1+t)z-(1+s)w\right),$$

so that by definition

$$\begin{aligned}
\mathcal{E}_{\bar h}(E^+) &= \int_{z\in\partial B}\int_{t=0}^{u^+(z)}\int_{w\in\partial B}\int_{s=0}^{u^+(w)} \varphi(z,w,s,t)\,ds\,d\mathscr{H}^{N-1}(w)\,dt\,d\mathscr{H}^{N-1}(z)\\
&= \int_{\partial B}\int_0^{u^+(z)}\int_{\partial B}\int_0^{u^+(z)} \varphi(z,w,s,t)\,ds\,d\mathscr{H}^{N-1}(w)\,dt\,d\mathscr{H}^{N-1}(z)\\
&\qquad + \int_{\partial B}\int_0^{u^+(z)}\int_{\partial B}\int_{u^+(z)}^{u^+(w)} \varphi(z,w,s,t)\,ds\,d\mathscr{H}^{N-1}(w)\,dt\,d\mathscr{H}^{N-1}(z)\\
&= K_1+K_2,
\end{aligned}$$

where K_1 and K_2 denote the two terms of the last equality.

Let us start working on K_2. As in the proof of Lemma 3.2.4, we define $\tilde h(v)=\bar h(v/2)$ for every $v\in\mathbb{R}^N$, and we observe that for every $z,w\in\partial B$ and $s,t\in(-1/2,1/2)$ one has

$$\bar h\big((1+t)z-(1+s)w\big)\le \tilde h(w-z).$$

As a consequence, for every pair $z,w\in\partial B$, we can estimate

$$\begin{aligned}
\int_0^{u^+(z)}\int_{u^+(z)}^{u^+(w)} \varphi(z,w,s,t)\,ds\,dt + \int_0^{u^+(w)}\int_{u^+(w)}^{u^+(z)} \varphi(z,w,s,t)\,ds\,dt\\
= -\int_{u^+(z)}^{u^+(w)}\int_{u^+(z)}^{u^+(w)} \varphi(z,w,s,t)\,ds\,dt \qquad\qquad (3.22)\\
\ge -\left(\frac{3}{2}\right)^{2N-2}\int_{u^+(z)}^{u^+(w)}\int_{u^+(z)}^{u^+(w)} \tilde h(w-z)\,ds\,dt.
\end{aligned}$$

Inserting this estimate in the definition of K_2, and applying again Lemma 3.2.2 with $\tilde h$ in place of $\bar h$, which is admissible by (3.5), we have

$$K_2 \ge -\frac{3^{2N-2}}{2^{2N-1}}\int_{\partial B}\int_{\partial B}\big(u^+(z)-u^+(w)\big)^2\tilde h(z-w)\,d\mathscr{H}^{N-1}(z)\,d\mathscr{H}^{N-1}(w) \ge -C\,\|u\|^2_{W^{1,2}}, \qquad (3.23)$$

where as usual C is a constant depending only on N and $\bar h$. In particular, from the first equality in (3.22), we know that $K_2\le 0$, and (3.23) guarantees that $K_2=O(\|u\|^2_{W^{1,2}})$. Let us now pass to evaluate K_1, which can be rewritten as

$$\begin{aligned}
K_1 &= \int_{\partial B}\int_0^{u^+(z)}(1+t)^{N-1}\int_{B(0,1+u^+(z))\setminus B}\bar h\left((1+t)z-y\right)dy\,dt\,d\mathscr{H}^{N-1}(z)\\
&= \int_{\partial B}\int_0^{u^+(z)}(1+t)^{N-1}\Big(\Phi_{\bar h}(1+u^+(z),1+t)-\Phi_{\bar h}(1,1+t)\Big)dt\,d\mathscr{H}^{N-1}(z)\\
&= \int_{\partial B}\int_0^{u^+(z)}\Phi_{\bar h}(1+u^+(z),1+t)-\Phi_{\bar h}(1,1+t)\,dt\,d\mathscr{H}^{N-1}(z)+O(\|u\|^2_{L^2}).
\end{aligned}$$

Rewriting $\Phi_{\bar{h}}(1+u^+(z),1+t)-\Phi_{\bar{h}}(1,1+t)$ as

$$\Phi_{\bar{h}}(1+u^+(z),1+t)-\Phi_{\bar{h}}(1+t,1+t)+\Phi_{\bar{h}}(1+t,1+t)-\Phi_{\bar{h}}(1,1+t)$$

and keeping in mind Lemma 3.2.5, we obtain the following estimates:

$$\begin{aligned}\Phi_{\bar{h}}(1+u^+(z),1+t)-\Phi_{\bar{h}}(1+t,1+t)&=\mathscr{I}(u^+(z)-t)+O(t),\\ \Phi_{\bar{h}}(1,1+t)-\Phi_{\bar{h}}(1+t,1+t)&=\mathscr{I}(-t)+O(t).\end{aligned}$$

Using once more Lemma 3.2.5, we arrive to the following lower bound, where we brutally substitute the above remainder terms of order t with $u^+(z)$ (up a constant factor):

$$\begin{aligned}K_1&\geq\int_{\partial B}\int_0^{u^+(z)}\left[\mathscr{I}(u^+(z)-t)+\mathscr{I}(t)-Cu^+(z)\right]dtd\mathscr{H}^{N-1}(z)+O(\|u\|_{L^2}^2)\\ &=\int_{\partial B}\int_0^{u^+(z)}\mathscr{I}(u^+(z)-t)+\mathscr{I}(t)dtd\mathscr{H}^{N-1}(z)+O(\|u\|_{L^2}^2)\\ &=2\int_{\partial B}\int_0^{u^+(z)}\mathscr{I}(t)dtd\mathscr{H}^{N-1}(z)+O(\|u\|_{L^2}^2).\end{aligned}$$

Since $\mathcal{E}_{\bar{h}}(E^+)=K_1+K_2$, this equality and (3.23) give

$$\mathcal{E}_{\bar{h}}(E^+)\geq 2\int_{\partial B}\int_0^{u^+(z)}\mathscr{I}(t)dtd\mathscr{H}^{N-1}(z)+O(\|u\|_{W^{1,2}}^2).$$

The very same calculations with E^- in place of E^+ give

$$\mathcal{E}_{\bar{h}}(E^-)\geq -2\int_{\partial B}\int_0^{u^-(z)}\mathscr{I}(-t)dtd\mathscr{H}^{N-1}(z)+O(\|u\|_{W^{1,2}}^2).$$

Putting these last two estimates into (3.21), we have then $\mathcal{E}_{\bar{h}}(E)-\mathcal{E}_{\bar{h}}(B)\geq -C\|u\|_{W^{1,2}}^2$, where C is a constant depending only on N and $\bar{h}$. Using the quantitative isoperimetric inequality stated in Theorem 3.1.6, we have a lower bound for $\mathcal{G}_\gamma(E)$:

$$\mathcal{G}_\gamma(E)\geq\mathcal{G}_\gamma(B)+\big(C_F-\gamma C\big)\|u\|_{W^{1,2}}^2, \tag{3.24}$$

hence of course the unique minimizer of the energy $\mathcal{G}_\gamma$ is the ball B as soon as $\gamma<\bar{\gamma}$, with $\bar{\gamma}=C_F/C$. □

3.3 A concave-convex problem in Gamow's model

We study a one-dimensional minimization problem of "partition type" of very simple nature: we are given an energy profile and a measure constraint $m>0$, we subdivide the measure in smaller masses and we optimize the sum of the energies of the small masses. The hypothesis on the profile, namely ($\mathbf{H}_{1D}$), is due to our interest in the Gamow model. In fact, it is easy to see that the function $m\to\mathcal{G}_\gamma(B(0,m^{1/N}))$ satisfies ($\mathbf{H}_{1D}$) for every $\gamma>0$ when the interaction kernel in the Riesz energy is $-\mathfrak{g}_\beta$ for any $\beta \in (-N, 0)$. This is important because of the great interest of the mathematical community in such a kernel, and because of the particular role played by the balls in this problem, as we already pointed out at the beginning of the chapter. It is therefore natural to consider such a partition problem, resulting from a dichotomy phenomenon, and the presence of generalized minimizers, as presented in [KMN16,NP21]. We will see that the optimal families possess some structure when hypothesis ($\mathbf{H}_{1D}$) is in force, and successively we will understand their dependence on the total measure. The problem can be presented as follows: let $\mathscr{G} : \mathbb{R}^+ \to \mathbb{R}^+$ be a given energy profile, satisfying

($\mathbf{H}_{1D}$) $\mathscr{G} \in C^0(\mathbb{R}^+) \cap C^1((0, +\infty))$ is strictly increasing, $\mathscr{G}(0) = 0$ and $\mathscr{G}$ is superlinear, meaning that $\lim_{m \to +\infty} \mathscr{G}(m)/m = +\infty$. Moreover, we suppose that there exists a unique "flex point" m_F such that $\mathscr{G}$ is stricly concave in $[0, m_F]$ and stricly convex in $[m_F, +\infty)$.

Then, for any $k \in \mathbb{N}$ we want to study the function

$$G_k(m) := \inf \left\{ \sum_{i=1}^{k} \mathscr{G}(m_i) \, : m_i \geq 0 \ \forall i, \ m_1 + \cdots + m_k = m \right\}, \tag{3.25}$$

and their infimum $G(m) := \inf_{k \in \mathbb{N}} G_k(m)$. Our aim is to better understand the dependence on m of the so-called optimal families of masses, i.e. those collections $\mathbf{F} = (m_1, \ldots, m_k)$ for which $G(m) = \sum_1^k \mathscr{G}(m_i)$, where $m = \sum_1^k m_i$. Our main result is contained in Theorem 3.3.15, where we show a sort of monotonicity property of the optimal families with respect to their total measure. To avoid redundancy, we will omit in the whole section the hypotheses on the energy profile, that are collected in ($\mathbf{H}_{1D}$), and that will be always assumed to be true. We stress that this problem has already been studied in [BC14, Theorem 2.12] and [FL15, Section 4.2]. In particular, in the second reference, the authors work with the profile associated to the classical Gamow's model, and they provide a stronger result, that we can only obtain for large m: in their setting, the optimal families contain always equal masses.

3.3.1 Optimality conditions

We collect some simple observations that are crucial for our study. From the hypothesis ($\mathbf{H}_{1D}$) it is immediate to deduce that there exist a unique point $m_T > 0$ and a unique coefficient $\alpha_T > 0$ such that the line $l_T := \{(s, \alpha_T s) : s \in [0, +\infty)\}$ is tangent to the graph of $\mathscr{G}$ at the point m_T. Moreover, we notice that necessarily $\mathscr{G}'(m_T) = \alpha_T = \mathscr{G}(m_T)/m_T$ and that m_T minimizes the energy/mass ratio, i.e.

$$\frac{\mathscr{G}(m_T)}{m_T} \leq \frac{\mathscr{G}(m)}{m} \qquad \forall m > 0,$$

with equality only if $m = m_T$. These properties are immediately inferred from Figure 3.3. We remark that the quantity m_T was already present in the literature: it plays a central role in [FL15, Theorem 3.2]. To simplify the exposition, we will always suppose $m_1 \leq m_2 \leq \ldots \leq m_k$ in the definition of G_k, i.e. (3.25), since their order is irrelevant. We will use the bold font for a family of masses: if $\mathbf{F} = (m_1, \dots, m_k)$, then

$$|\mathbf{F}| := \sum_{i=1}^{k} m_i \qquad \#\mathbf{F} := k,$$

and we write $\mathscr{G}(\mathbf{F})$ to denote $\mathscr{G}(m_1) + \ldots + \mathscr{G}(m_k)$. We call *total measure* of $\mathbf{F}$ (or simply measure of $\mathbf{F}$) the quantity $|\mathbf{F}|$.

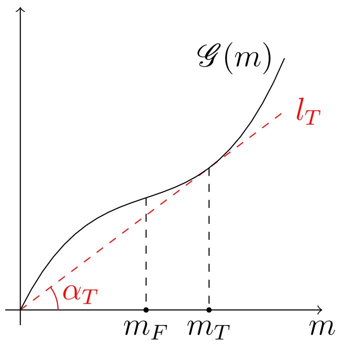


Figure 3.3: This is an example of energy satisfying ($\mathbf{H}_{1D}$), where α_T denotes the slope of the line l_T passing through the origin and the point $(m_T, \mathscr{G}(m_T))$.

Remark 3.3.1. For every $m \geq 0$ and for every $k \in \mathbb{N}$ it is clear that the infimum in the definition of $G_k(m)$ is actually a minimum. We also have that G_k is non-decreasing for any $k \in \mathbb{N}$: if $\mathbf{F} = (m_1, \dots, m_k)$ is a family such that $\bar{m} = |\mathbf{F}|$ and $G_k(\bar{m}) = \mathscr{G}(\mathbf{F})$, then for every $h \in [0, \bar{m}]$ we have that

$$G_k(\bar{m} - h) \leq \sum_{i=1}^{k} \mathscr{G}(m_i - h_i) \leq \sum_{i=1}^{k} \mathscr{G}(m_i) = G_k(\bar{m}) \quad \forall h_i \in [0, m_i],\ h_1 + \cdots + h_k = h.$$

Definition 3.3.2. A family $\mathbf{F} = (m_1, \dots, m_k)$ of non-negative numbers is said to be an *optimal family* if

$$G(|\mathbf{F}|) = \mathscr{G}(\mathbf{F}).$$

It is not difficult to see that $G_{k+1} \leq G_k$ for any $k \in \mathbb{N}$: to compute $G_{k+1}(m)$ we can always choose a family with $m_1 = 0$, and the successive k masses form an optimal family to compute $G_k(m)$. In order to proceed into the study of the profile G, it is convenient to determine precisely the value of the parameter k inside the definition of G itself (depending on m). We will denote by $K(m)$ the minimum number of masses that are needed to achieve the minimum in the definition of $G(m)$, namely

$$K(m) := \min \{k \in \mathbb{N} : G(m) = G_k(m)\}. \tag{3.26}$$

That definition is well posed. In fact, since $\mathscr{G}$ is strictly concave in $[0, m_F]$, then it is not convenient to have $k > 2\lceil m/m_F \rceil$ when we compute $G(m)$: otherwise we would have a family $\mathbf{F} = (m_1, \dots, m_k)$ with $m_1 > 0$, $m_2 > 0$ and $m_1 + m_2 < m_F$, and then

$$\mathscr{G}(m_1 + m_2) < \mathscr{G}(m_1) + \mathscr{G}(m_2)$$

since $\mathscr{G}$ is strictly concave in $[0, m_F]$. Therefore, $\mathbf{F}$ is not optimal since the family $(m_1 + m_2, m_3, \ldots, m_k)$ has less energy. Having an upper bound on k, we immediately deduce that $K(m)$ is well defined for every $m > 0$.

Remark 3.3.3. With the previous concavity argument we also see that $G(m) = G_1(m)$ if $m \leq m_F$, and therefore $K(m) = 1$ for $m \leq m_F$.

Remark 3.3.4. For every $m > 0$, if $\mathbf{F} = (m_1, \ldots, m_k)$ is an optimal family with $|\mathbf{F}| = m$, then for every $1 \leq p \leq k$ and for every choice of the indices $1 \leq l_1 < \ldots < l_p \leq k$ we have that the subfamily $\tilde{\mathbf{F}} = (m_{l_1}, \ldots, m_{l_p})$ is optimal for its own total measure $\tilde{m} = |\tilde{\mathbf{F}}|$. In fact, if this was not the case, then we could reduce $G(m)$ substituting the subfamily with a better one. Additionally, if $k = K(m)$, then $p = K(\tilde{m})$: if $p > K(\tilde{m})$, then we can substitute the selected subfamily with another one containing fewer masses.

Lemma 3.3.5. *Let $\mathbf{F} = (m_1, \ldots, m_k)$ be an optimal family with $m_1 > 0$. For every $1 \leq i, j \leq k$ we have that $\mathscr{G}'(m_i) = \mathscr{G}'(m_j)$. Moreover, we also have that $m_F \leq m_2 = \ldots = m_k$.*

Proof. We first prove the condition on the derivative, and we just consider the case $i = 1$ and $j = 2$ because the choice of the indices is irrelevant. We suppose by contradiction that $\mathscr{G}'(m_1) < \mathscr{G}'(m_2)$ (the other inequality case is completely analogous), and we take $\varepsilon < m_2$. Then

$$\mathscr{G}(m_1 + \varepsilon) + \mathscr{G}(m_2 - \varepsilon) = \mathscr{G}(m_1) + \mathscr{G}(m_2) + \varepsilon(\mathscr{G}'(m_1) - \mathscr{G}'(m_2)) + o(\varepsilon),$$

and thus $\mathscr{G}(m_1 + \varepsilon) + \mathscr{G}(m_2 - \varepsilon) < \mathscr{G}(m_1) + \mathscr{G}(m_2)$ for $\varepsilon \ll 1$. This is not possible, since $\tilde{\mathbf{F}} = (m_1, m_2)$ is optimal for its own total measure thanks to Remark 3.3.4, and therefore $\mathscr{G}'(m_1) = \mathscr{G}'(m_2)$.
Since $\mathscr{G}$ is strictly concave in $[0, m_F]$, then $\mathscr{G}'$ is strictly decreasing in that interval, and thus the only way to have $0 < m_1, m_2 \leq m_F$ and $\mathscr{G}'(m_1) = \mathscr{G}'(m_2)$ is that $m_1 = m_2$. If $m_1 = m_2 = m_F$, then there is nothing to prove, so we suppose that $m_1 = m_2 < m_F$. In this situation, we observe that the strict concavity guarantees that $\mathscr{G}(m_1 - \varepsilon) + \mathscr{G}(m_1 + \varepsilon) < 2\mathscr{G}(m_1)$ for any $\varepsilon < m_F - m_1$, and thus the family $\tilde{\mathbf{F}}$ cannot be optimal. This proves that we cannot find two masses in $[0, m_F)$, and using that $\mathscr{G}$ is strictly convex in $[m_F, +\infty)$ we also obtain that $m_2 = \ldots = m_k$ since we need to have that $\mathscr{G}'(m_2) = \ldots = \mathscr{G}'(m_k)$. □

Remark 3.3.6. The previous lemma holds also when some of the masses are zero: the condition on the derivatives holds only for the non-zero masses, and we can have at most one mass in $(0, m_F)$. The equality of the large masses then follows again by convexity.

Remark 3.3.7. We notice that for every $k \geq 1$ the function G_k is of class $\mathrm{Lip}_{loc}((0, +\infty))$. For $k = 1$ this is trivial since $G_1 = \mathscr{G}$ and we know that $\mathscr{G}$ is of class $C^1((0, +\infty))$, so we suppose that $k \geq 2$. For any $m > 0$ we take a family $\mathbf{F} = (m_1, \ldots, m_k)$ with $|\mathbf{F}| = m$ and $G_k(m) = \mathscr{G}(\mathbf{F})$. If we consider $\tilde{\mathbf{F}} = (m_1, \ldots, m_{k-1})$, then

$$G_k(m + h) \leq \mathscr{G}(\tilde{\mathbf{F}}) + \mathscr{G}(m_k + h) = G_k(m) + \mathscr{G}(m_k + h) - \mathscr{G}(m_k) \qquad \forall h > 0. \quad (3.27)$$

Thanks to Lemma 3.3.5 we know that $m_k \geq m_F$, and by definition $m_k \leq m$, thus we exploit (3.27) and the fact that G_k and $\mathscr{G}$ are monotone to locally control the Lipschitz constant of G_k with the local Lipschitz constant of $\mathscr{G}$. This ensures that G_k is differentiable almost everywhere and that the fundamental theorem of calculus holds using the standard derivative.

Remark 3.3.8. We collect the necessary optimality conditions obtained so far. Given $m > 0$ and an optimal family $\mathbf{F} = (m_1, \ldots, m_k)$ with $m_1 > 0$ and $|\mathbf{F}| = m$, then

1. $m_2 = m_3 = \ldots = m_k$ and $m_2 \geq m_F$;
2. any subfamily of $\mathbf{F}$ is optimal for its own total measure;
3. $\mathscr{G}'(m_i) = \mathscr{G}'(m_j)$ for every $i, j \in \{1, \ldots, k\}$;
4. if $k \geq 2$, then for any $\tilde{m}_2 \leq \frac{m}{k-1}$ we have that

   $$\mathscr{G}(\mathbf{F}) = \mathscr{G}(m - (k-1)m_2) + (k-1)\mathscr{G}(m_2) \leq \mathscr{G}(m - (k-1)\tilde{m}_2) + (k-1)\mathscr{G}(\tilde{m}_2).$$

   Hence, if $\mathscr{G}$ is twice differentiable in m_1 and m_2, then

   $$(k-1)\mathscr{G}''(m_1) + \mathscr{G}''(m_2) \geq 0.$$

3.3.2 Main results

Lemma 3.3.9. *Let $m > 0$ be given, and suppose that we have two optimal families $\mathbf{F} = (m_1, \ldots, m_{k_1})$ and $\tilde{\mathbf{F}} = (\tilde{m}_1, \ldots, \tilde{m}_{k_2})$ with $m_1, \tilde{m}_1 > 0$, $|\mathbf{F}| = |\tilde{\mathbf{F}}| = m$ and $k_1 < k_2$. Then $G_{k_1}(s) > G_{k_2}(s)$ for every $s > m$.*

Proof. We want to obtain the thesis simply comparing the derivatives of the two functions. Of course, if we show that $G'_{k_1} > G'_{k_2}$ in $(m, +\infty)$, then we obtain the desired result since by definition we have that $G_{k_1}(m) \geq G_{k_2}(m)$, and we already observed that the fundamental theorem of calculus works for G_{k_1} and G_{k_2} because they are locally Lipschitz (see Remark 3.3.7). In order to proceed with this plan, we express those derivatives in terms of $\mathscr{G}'$. In fact, for every $k \geq 1$ and every $s > 0$ that is a differentiability point of G_k we have that

$$G'_k(s) = \mathscr{G}'(\bar{m}_k), \tag{3.28}$$

where $\bar{\mathbf{F}} = (\bar{m}_1, \ldots, \bar{m}_k)$ is a family with $|\bar{\mathbf{F}}| = s$ and $G_k(s) = \mathscr{G}(\bar{\mathbf{F}})$. To prove this, we fix $h > 0$ arbitrarily small and, the family $(\bar{m}_1, \ldots, \bar{m}_{k-1}, \bar{m}_k + h)$ to estimate $G_k(s+h)$ and the family $(\bar{m}_1, \ldots, \bar{m}_{k-1}, \bar{m}_k - h)$ to estimate $G_k(s-h)$, we obtain the following inequalities:

$$\begin{aligned}\frac{G_k(s+h) - G_k(s)}{h} &\leq \sum_{i=1}^{k-1} \frac{\mathscr{G}(\bar{m}_i) - \mathscr{G}(\bar{m}_i)}{h} + \frac{\mathscr{G}(\bar{m}_k + h) - \mathscr{G}(\bar{m}_k)}{h} = \frac{\mathscr{G}'(\bar{m}_k)h + o(h)}{h},\\ \frac{G_k(s) - G_k(s-h)}{h} &\geq \sum_{i=1}^{k-1} \frac{\mathscr{G}(\bar{m}_i) - \mathscr{G}(\bar{m}_i)}{h} + \frac{\mathscr{G}(\bar{m}_k) - \mathscr{G}(\bar{m}_k - h)}{h} = \frac{\mathscr{G}'(\bar{m}_k)h + o(h)}{h}.\end{aligned} \tag{3.29}$$

Since s is a differentiability point for G_k, then the inequalities in (3.29) imply (3.28). For any $s > m$ we take a family $\mathbf{F}(s) = (m_1(s), \dots, m_{k_1}(s))$ such that

$$s = |\mathbf{F}(s)| \qquad G_{k_1}(s) = \mathscr{G}(\mathbf{F}(s)).$$

Notice that, since up to now we do not know that K is monotone, we may have $m_j(s) = 0$ for some j and some s. Moreover, we fix another family of masses $\tilde{\mathbf{F}}(s) = (\tilde{m}_1(s), \dots, \tilde{m}_{k_2}(s))$ defined as

$$\tilde{m}_1(s) = \tilde{m}_1 \qquad \tilde{m}_2(s) = \dots = \tilde{m}_{k_2}(s) = \frac{s - \tilde{m}_1}{k_2 - 1},$$

with $|\tilde{\mathbf{F}}(s)| = s$. By definition of G_k (with $k = k_1, k_2$) we have that $G_{k_2}(s) \le \mathscr{G}(\tilde{\mathbf{F}}(s))$ and $G_{k_1}(m) \ge G_{k_2}(m)$. As a consequence, we take $s > m$ and we see that

$$\begin{aligned} G_{k_1}(s) - G_{k_2}(s) &\ge G_{k_1}(s) - G_{k_2}(s) - G_{k_1}(m) + G_{k_2}(m) \\ &\ge \int_m^s G'_{k_1}(r)dr - \Big(\mathscr{G}(\tilde{\mathbf{F}}(s)) - G_{k_2}(m)\Big) \\ &= \int_m^s \left[\mathscr{G}'(m_2(r)) - \mathscr{G}'\left(\tilde{m}_2(r)\right)\right] dr. \end{aligned} \tag{3.30}$$

The formula (3.28) and the fundamental theorem of calculus applied to the function $s \mapsto \mathscr{G}(\tilde{m}_2(s))$, combined with the fact that $G_{k_2}(m) = \mathscr{G}(\tilde{\mathbf{F}}(m))$, prove the validity of the last equality in (3.30). Thanks to Lemma 3.3.5 we have that $m_F \le \tilde{m}_2$. Furthermore, by construction $\tilde{m}_2(r) \ge \tilde{m}_2$ for every $r \ge m$, hence $m_F \le \tilde{m}_2(r)$ for every $r \ge m$. We claim that $m_2(r) > \tilde{m}_2(r) \ge m_F$ for every $r > m$. With this inequality we conclude because $\mathscr{G}$ is strictly convex in $[m_F, +\infty)$, and thus the last line in (3.30) is strictly positive. We need to justify our claim, and to do that we take $p_1, p_2 \in \mathbb{N}$ with $0 < p_1 < p_2$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $0 < x_1 \le x_2$, $0 < y_1 \le y_2$ and $x_1 + p_1 x_2 = y_1 + p_2 y_2$. Then we have that $x_2 > y_2$. In fact, if we suppose by contradiction that $x_2 \le y_2$, then

$$x_1 + p_1 x_2 = y_1 + p_2 y_2 \ge y_1 + p_1 y_2 + y_2 \ge y_1 + y_2 + p_1 x_2.$$

Hence, $y_2 < y_1 + y_2 \le x_1 \le x_2$, and this is not compatible with the fact that $x_2 \le y_2$. From the inequality $x_2 > y_2$ we obtain the desired inequality choosing $p_i = k_i$, $x_i = m_i(s)$ and $y_i = \tilde{m}_i(s)$ for $i = 1, 2$.

□

Corollary 3.3.10. *The function $K : [0, +\infty) \to \mathbb{N}^+$ is non-decreasing, it has countably many discontinuity points that are denoted by $0 < J_1 < J_2 < \dots < J_k < \dots$ and $J_k \in [k m_T, (k+1) m_T)$ for each $k \in \mathbb{N}^+$. Moreover, $K(J_k) = k$ for every $k \in \mathbb{N}^+$.*

Proof. The monotonicity of the function K is a direct consequence of Lemma 3.3.9, so its discontinuity points form a countable set. Thanks to hypothesis ($\mathbf{H}_{1D}$) on the profile $\mathscr{G}$, we know that $G(m) \ge \alpha_T m$ for every $m \ge 0$, and the equality is achieved only for $m = p\alpha_T$ with $p \in \mathbb{N}$. Therefore, we have that $G(p\alpha_T) = G_p(p\alpha_T)$ and $K(p\alpha_T) = p$ for every $p \in \mathbb{N}$, concluding the proof. □

Remark 3.3.11. By the continuity of $\mathscr{G}$, we have that $G(J_k) = G_k(J_k) = G_{k+1}(J_k)$ for every $k \geq 1$, and that K is lower semicontinuous. From Corollary 3.3.10 it follows that we can characterize the discontinuity points as

$$J_k = \sup\{m > 0 \,:\, K(m) = k\}.$$

Moreover, we clearly have that $G(m) = G_k(m)$ for $m \in [J_{k-1}, J_k]$.

Lemma 3.3.12. *For every $k \geq 1$ and every $m > 0$ that is a differentiability point for G_k, there exists a unique family $\mathbf{F} = (m_1, \ldots, m_k)$ with $|\mathbf{F}| = m$ and $G_k(m) = \mathscr{G}(\mathbf{F})$.*

Proof. If $k = 1$ then there is nothing to prove since $G_1 = \mathscr{G}$ and it is of class $C^1((0, +\infty))$. Instead, if $k > 1$, Lemma 3.3.5 guarantees that $m_k \geq m_F$, but $\mathscr{G}'$ is strictly increasing in $[m_F, +\infty)$, and thus it can exist only one value $t \geq m_F$ such that $\mathscr{G}'(t) = \mathscr{G}'(m_k) = G_k'(m)$. This of course characterizes the family $\mathbf{F}$. □

We point out that the previous lemma does not imply that $G_k \in C^1((J_{k-1}, J_k))$, as one can see analyzing G_2 in Example 3.3.21. In fact, in general there is no hope to find a continuous map that associates $m \mapsto \mathbf{F}(m)$, where $\mathbf{F}(m)$ is an optimal family with $|\mathbf{F}(m)| = m$, not even when we restrict to (J_k, J_{k+1}).

Remark 3.3.13. We notice that the asymptotic density of energy converges to $\alpha_T = \mathscr{G}(m_T)/m_T$ as $m \to +\infty$, that is to say that $G(m)/m \to \mathscr{G}(m_T)/m_T$ as $m \to +\infty$. In fact, we already know that $G(m) \geq \alpha_T m$ for every $m \geq 0$, and for every $m \in [km_T, (k+1)m_T]$ we have that $G(m) \leq k\mathscr{G}(m_T) + \mathscr{G}(2m_T)$. Therefore

$$\frac{G(m)}{m} \leq \frac{k\mathscr{G}(m_T)}{m} + \frac{\mathscr{G}(2m_T)}{m} = \frac{km_T}{m}\frac{\mathscr{G}(m_T)}{m_T} + \frac{\mathscr{G}(2m_T)}{m},$$

and thus $\limsup_{m\to+\infty} G(m)/m \leq \mathscr{G}(m_T)/m_T$ since $k = \lfloor m/m_T \rfloor$.

We want to show a monotonicity property for the optimal families with respect to their total measure. We have already seen in Lemma 3.3.5 that for any $k \in \mathbb{N}$, $k \geq 2$, we have that any optimal family $\mathbf{F} = (m_1, \ldots, m_k)$ with measure $|\mathbf{F}| = m \in (J_{k-1}, J_k)$ is of the form $m_1 = a$ and $m_2 = \ldots = m_k = b$ for some $0 < a \leq b$. We call a_{min} the minimum possible value of a in order to find an optimal family with $m_1 = a$, and a_{max}, b_{min}, b_{max} have obvious analogous definitions. All of these quantities depend on m, and whenever we need to explicitly write it, we will denote them by $a_{min}(m)$, $a_{max}(m)$, etc. It is easy to see that

$$a_{max}(m) + (k-1)b_{min}(m) = a_{min}(m) + (k-1)b_{max}(m) = m \qquad \forall m \in (J_{k-1}, J_k),$$

and additionally the "extremal" families $\mathbf{F}_1 = (a_{max}(m), b_{min}(m), \ldots, b_{min}(m))$ and $\mathbf{F}_2 = (a_{min}(m), b_{max}(m), \ldots, b_{max}(m))$ are optimal with measure $m = |\mathbf{F}_1| = |\mathbf{F}_2|$. Exploiting completely the inequalities in (3.29) we obtain that

$$\begin{aligned} \limsup_{h\to0^+} \frac{G_k(m+h) - G_k(m)}{h} &\leq \mathscr{G}'(b_{min}(m)) \\ &\leq \mathscr{G}'(b_{max}(m)) \leq \liminf_{h\to0^+} \frac{G_k(m) - G_k(m-h)}{h}. \end{aligned} \tag{3.31}$$

The quantities $a_{min}, a_{max}, b_{min}$ and b_{max} enjoy a semicontinuity result, that is presented in the following lemma. Their definition is also convenient in the proof of our main result, contained in Theorem 3.3.15.

Lemma 3.3.14. *Let us fix any $k \geq 2$ and any $m \in [J_{k-1}, J_k)$ and let $m^n > m$ with $m^n \searrow m$. Then*

$$\limsup_{n\to+\infty} b_{max}(m^n) \leq b_{min}(m).$$

If instead $m \in (J_{k-1}, J_k]$, $m^n < m$ with $m^n \nearrow m$, then

$$\liminf_{n\to+\infty} b_{min}(m_n) \geq b_{max}(m).$$

Proof. We will prove only the first part of the statement, since the second one follows the same lines. Up to taking a subsequence, we can suppose that $m^n < J_k$ and that $\bar{b} = \lim_n b_{max}(m^n) = \limsup_n b_{max}(m^n)$.
We suppose by contradiction that $\bar{b} > b_{min}(m)$, and we let $\varepsilon_n = m^n - m \searrow 0$. Then

$$\begin{aligned} G_k(m^n) &= \mathscr{G}(a_{min}(m^n)) + (k-1)\mathscr{G}(b_{max}(m^n)) \\ &= \mathscr{G}(a_{min}(m^n)) + (k-2)\mathscr{G}(b_{max}(m^n)) + \mathscr{G}(b_{max}(m^n) - \varepsilon_n) + \varepsilon_n \mathscr{G}'(x_n) \\ &\geq G_k(m) + \varepsilon_n \mathscr{G}'(x_n), \end{aligned}$$

where $x_n \in (b_{max}(m^n) - \varepsilon_n, b_{max}(m^n))$. We provide an opposite inequality containing $b_{min}(m)$:

$$\begin{aligned} G_k(m^n) &\leq \mathscr{G}(a_{max}(m)) + (k-2)\mathscr{G}(b_{min}(m)) + \mathscr{G}(b_{min}(m) + \varepsilon_n) \\ &= \mathscr{G}(a_{max}(m)) + (k-2)\mathscr{G}(b_{min}(m)) + \mathscr{G}(b_{min}(m)) + \varepsilon_n \mathscr{G}'(y_n) \\ &= G_k(m) + \varepsilon_n \mathscr{G}'(y_n), \end{aligned}$$

with $y_n \in (b_{min}(m), b_{min}(m) + \varepsilon_n)$. Combining the two inequalities, we deduce that $\mathscr{G}'(x_n) \leq \mathscr{G}'(y_n)$ for every $n \in \mathbb{N}$. Since $x_n \to \bar{b}$ and $y_n \to b_{min}(m)$, we pass to the limit in this last inequality, obtaining that $\mathscr{G}'(\bar{b}) \leq \mathscr{G}'(b_{min}(m))$. This, however, is not compatible with the assumption $\bar{b} > b_{min}(m)$ because $\mathscr{G}$ is strictly convex in $[m_F, +\infty)$, and $b_{min}(m) \geq m_F$ thanks to Remark 3.3.8. □

Theorem 3.3.15. *Let us fix $k \geq 2$, any $m \in [J_{k-1}, J_k)$ and let us suppose to have an optimal family $\mathbf{F} = (m_1, \ldots, m_k)$ with $0 < m_1 < m_2$ and $|\mathbf{F}| = m$. Let $M = \min\{km_2, J_k\}$, then for any $\tilde{m} \in (m, M)$ and for any optimal family $\tilde{\mathbf{F}} = (\tilde{m}_1, \ldots, \tilde{m}_k)$ with $|\tilde{\mathbf{F}}| = \tilde{m}$ we have that*

$$m_1 < \tilde{m}_1 \leq \tilde{m}_2 < m_2. \tag{3.32}$$

Moreover, if $\mathbf{F}$ is an optimal family with $|\mathbf{F}| = m$, $\#\mathbf{F} = k$ and it is made of equal masses $m_1 = \ldots = m_k$, then for any $\tilde{m} \in (m, J_k)$ there exists only one optimal family $\tilde{\mathbf{F}}$ with $|\tilde{\mathbf{F}}| = \tilde{m}$ and it is made of equal masses.

Proof. In order to obtain (3.32) with non-strict inequalities it is sufficient to prove that $\tilde{m}_2 \leq m_2$. In fact, suppose by contradiction that $\tilde{m}_2 \leq m_2$ and $\tilde{m}_1 < m_1$. Thanks to Lemma 3.3.5 we know that $\tilde{m}_2 \geq m_F$, and the convexity property of

$\mathscr{G}$ shows that $\mathscr{G}'(\tilde{m}_2) \geq \mathscr{G}'(m_2)$, while the concavity of $\mathscr{G}$ in $[0, m_F]$ implies that $\mathscr{G}'(\tilde{m}_1) < \mathscr{G}'(m_1)$. These inequalities, however, are not compatible with the optimality condition $\mathscr{G}'(\tilde{m}_1) = \mathscr{G}'(\tilde{m}_2)$ obtained in Lemma 3.3.5, and thus $\tilde{m}_1 \geq m_1$. Moreover, the two strict inequalities in (3.32) hold true because $\tilde{m} \neq m$: at least one of the strict inequalities must hold, and then the first order optimality condition exploited before guarantees that also the other inequality is strict.

With these preliminary observations, we are ready to prove the first part of the thesis, showing only the non-strict inequality $m_2 \leq \tilde{m}_2$, and we remark that M is the largest possible measure for which the conclusion holds. We also notice that, thanks to the second part of Lemma 3.3.14, we just need to prove that there exists a dense set of total measures $\tilde{m} \in (m, M)$ where (3.32) holds. We argue by contradiction, and we suppose that there exists an open interval $(l, r) \subset (m, M)$ where the conclusion does not hold and that is maximal with respect to the inclusion. If $l = m$, then we fix $t < \min\{r, m_F + l - m_1\}$ and we use the concavity-convexity of $\mathscr{G}$ to see that

$$\begin{aligned} G_k(t) - G_k(l) &= \int_l^t \mathscr{G}'(b_{max}(s))ds > \int_l^t \mathscr{G}'(m_2)ds = \int_l^t \mathscr{G}'(m_1)ds \\ &\geq \int_l^t \mathscr{G}'(m_1 + s - l)ds \\ &= \mathscr{G}(m_1 + t - l) - \mathscr{G}(m_1) \\ &= \mathscr{G}(m_1 + t - l) + (k-1)\mathscr{G}(m_2) - G_k(l). \end{aligned}$$

This however is in contradiction with the definition of G_k since $m_1 + t - l + (k-1)m_2 = m + t - l = t$.

If instead $l > m$, then we claim that $b_{max}(l) \leq m_2$. In fact, since (l, r) is a maximal interval such that (3.32) does not hold for $\tilde{m} \in (l, r)$, then there exists a sequence $m^n \nearrow l$ such that $b_{max}(m^n) \leq m_2$. Therefore, using the second part of Lemma 3.3.14 we obtain that $b_{max}(l) \leq m_2$. If $a_{min}(l) < b_{max}(l)$, then we repeat the previous argument substituting m with l, m_2 with $b_{min}(l)$ and m_1 with $a_{max}(l)$. If instead $a_{min}(l) = b_{max}(l)$, then we apply the second part of our statement (that we are going to prove in a moment), and we automatically get that $b_{max}(t) < m_2$ for every $t \in (l, M)$.

We treat now the second case, namely the one with $\mathbf{F}$ made of equal masses. Again, thanks to Lemma 3.3.14 we need only to prove that the set of total measures $\tilde{m} \in (m, J_k)$ for which the thesis holds is dense. One can argue pretty much in the same way, using the formula (3.28) for the derivative of G_k and that, if we have an optimal family $\tilde{\mathbf{F}} = (\tilde{m}_1, \ldots, \tilde{m}_k)$ with $\tilde{m}_1 < \tilde{m}_2$ and $|\tilde{\mathbf{F}}| = \tilde{m}$, then $\tilde{m}_2 > \tilde{m}/k$. In fact, we take a maximal interval $(l, r) \subset (m, J_k)$ such that for every $\tilde{m} \in (l, r)$ the thesis does not hold, and we fix any $t \in (l, r)$. Then, we exploit the usual convexity of $\mathscr{G}$ in $[m_F, +\infty)$

$$\begin{aligned} G_k(t) - G_k(l) &= \int_l^t \mathscr{G}'(b_{max}(s))ds > \int_l^t \mathscr{G}'(s/k)ds \\ &= k\mathscr{G}(t/k) - k\mathscr{G}(l/k) \\ &= k\mathscr{G}(t/k) - G_k(l), \end{aligned}$$

where we used that $s/k \geq m_2 \geq m_F$ for every $s \in (l, t)$. Moreover, we used again the maximality of the interval to infer that $G_k(l) = k\mathscr{G}(l/k)$ and pass from the second to the third line. This is impossible since it is always true that $G_k(t) \le k\mathscr{G}(t/k)$, and thus the proof is complete.

□

Proposition 3.3.16. *For every $k \ge 2$ it is well defined the point*

$$C_k := \inf\left\{m \in [J_{k-1}, J_k] : G(m) = k\mathscr{G}\left(\frac{m}{k}\right)\right\}. \tag{3.33}$$

Moreover, the infimum is a minimum and $C_k < J_k$.

In view of Theorem 3.3.15, one can think about C_k as the point where the optimal families with total measure in $[J_{k-1}, J_k]$ collapse, becoming a family with k equal masses.

Proof. The point C_k exists because $J_{k-1} \le km_T \le J_k$ (as we showed in Corollary 3.3.10) and $G(km_T) = k\mathscr{G}(m_T)$ as we noticed in Corollary 3.3.10. The infimum is actually a minimum thanks to the continuity of $\mathscr{G}$, so $G(C_k) = k\mathscr{G}(C_k/k)$.
It remains to prove only the inequality $C_k < J_k$. To do that, let us suppose that $C_k = J_k$, and let us take an optimal family $\mathbf{F} = (m_1, \dots, m_{k+1})$ with total measure $|\mathbf{F}| = m \in (J_k, J_{k+1})$. Thanks to Remark 3.3.8 we know that any subfamily of $\mathbf{F}$ is optimal for its own total measure, and thus the family composed by k masses equal to m_2 is optimal (here we also use Lemma 3.3.5, which says that $m_2 = \ldots = m_{k+1}$). Since $C_k = J_k$, the only possibility is that $m_2 = J_k/k$. The optimality conditions guarantee that $\mathscr{G}'(m_1) = \mathscr{G}'(m_2)$, and the concavity-convexity property $\mathscr{G}$ contained in ($\mathbf{H}_{1D}$) imposes that m_1 can assume at most two values. This is impossible since $m \in (J_k, J_{k+1})$ was arbitrary and $m = m_1 + km_2$, and thus $C_k < J_k$. □

The collapsing points C_k determine the convexity of G, as we present in the following lemma:

Lemma 3.3.17 (concavity-convexity of G)**.** *For every $k \ge 2$ we have that G is strictly concave in $[J_{k-1}, C_k]$ and it is strictly convex in $[C_k, J_k]$. Moreover, for every $m > 0$ we have that*

$$\limsup_{h\to 0^+} \frac{G(m+h) - G(m)}{h} \le \liminf_{h\to 0^+} \frac{G(m) - G(m-h)}{h}.$$

Proof. The concavity-convexity of G is a direct consequence of Theorem 3.3.15. In fact, for $m \in [J_{k-1}, C_k]$, we know from that theorem that the families shrink, in the sense that $b_{max}(m)$ is striclty decreasing in that interval. Since $b_{max}(m) \ge m_F$ (because of Lemma 3.3.5) and $\mathscr{G}$ is strictly convex in $[m_F, +\infty)$, then (3.28) gives that $G'(m) = \mathscr{G}'(b_{max}(m))$, which is strictly decreasing in $[J_{k-1}, C_k]$. The same theorem also says that, for $m \in [C_k, J_k]$, we have that $b_{max}(m) = m/k$, and thus $G'(m) = \mathscr{G}'(m/k)$ is increasing since $m/k \ge m_F$.
The inequality between the incremental ratios is valid in $(0, J_1)$ and in (J_{k-1}, J_k) for every $k \ge 2$ thanks to (3.31) since $G = G_k$ in (J_{k-1}, J_k) for every $k \ge 2$. Therefore, we need to prove it only in the points J_k for $k \ge 1$. Since $G = G_k$ to the left of J_k and

$G = G_{k+1}$ to the right of J_k, the incremental ratios can be written in terms of G_k and G_{k+1}, and proving the desired result is equivalent to showing that

$$\limsup_{h\to 0+} \frac{G_{k+1}(J_k+h) - G_{k+1}(J_k)}{h} \leq \liminf_{h\to 0^+} \frac{G_k(J_k) - G_k(J_k-h)}{h}.$$

Let us take $\mathbf{F} = (m_1, \ldots, m_k)$ an optimal family with $|\mathbf{F}| = J_k$, and $\#\mathbf{F} = k$, and $\mathbf{F}_1 = (\tilde{m}_1, \ldots, \tilde{m}_{k+1})$ another optimal family with $|\mathbf{F}_1| = J_k$, but $\#\mathbf{F}_1 = k+1$. Using again the inequalities present in (3.31), we know that the left hand side is smaller than $\mathscr{G}'(\tilde{m}_{k+1})$ and the right hand side is larger than $\mathscr{G}'(m_k)$. Lemma 3.3.5 guarantees that both m_k and $\tilde{m}_{k+1}$ are larger than m_F, and thus the thesis is proved if we show that $\tilde{m}_{k+1} \leq m_k$ since $\mathscr{G}'$ is striclty increasing in $[m_F, +\infty)$. This, however, is a direct consequence of the structure of the optimal families, and of the different number of masses that compose them. In fact, it is sufficient to repeat the argument used in the conclusion of Lemma 3.3.9 to deduce the inequality $\tilde{m}_{k+1} \leq m_k$ and finish the proof. □

It is possible to have a certain control the ratio J_{k+1}/J_k, as we show in the next proposition. Moreover, we prove that the minimal families trivialize when the total measure is large enough, in the sense that they are necessarily made of equal masses.

Proposition 3.3.18. *The following properties hold true:*

1. *the sequence J_k/k is decreasing and converges from above to m_T as $k \to +\infty$;*
2. *there exists an integer $\bar{k}$ such that any optimal family with total measure $m \geq J_{\bar{k}}$ is made of equal masses. In other words, $C_k = J_{k-1}$ for every $k > \bar{k}$;*
3. *the sequence $J_k/(k+1)$ is increasing for every $k \geq \bar{k}$ and converges from below to m_T.*

Proof. We successively prove the three points:

1. We fix $k \geq 1$. Thanks to the second part of Theorem 3.3.15 we know that any optimal family with total measure $m \in (C_{k+1}, J_{k+1})$ is made of equal masses. Thus, the family $\mathbf{F} = (m_1, \ldots, m_k)$ with $m_1 = \ldots = m_k = m/(k+1)$ is optimal, and $|\mathbf{F}| = \frac{mk}{k+1}$. As a consequence, $\frac{mk}{k+1} \leq J_k$, and we can take the limit $m \to J_{k+1}$ to see that $J_{k+1}/(k+1) \leq J_k/k$, that is the required monotonicity. Necessarily $J_k/k \geq m_T$ because J_k is the largest measure for which there exists a optimal family with k masses, and for $m = km_T$ the only optimal family with measure m is made of k masses equal to m_T (see Corollary 3.3.10). It is also easy to see that J_k/k converges to m_T: if this was not the case, then we could find a sequence $k_n \to +\infty$ such that $J_{k_n}/k_n > m_T + \delta$ for some $\delta > 0$. Since m_T is the unique minimizer of the energy/mass ratio $\mathscr{G}(m)/m$, and since we already noticed that $G(J_k) = k\mathscr{G}(J_k/k)$ for any $k \geq 1$, then there exists $\varepsilon > 0$ such that

$$\frac{G(J_{k_n})}{J_{k_n}} = \frac{k_n \mathscr{G}(J_{k_n}/k_n)}{J_{k_n}} = \frac{\mathscr{G}(J_{k_n}/k_n)}{J_{k_n}/k_n} > \frac{\mathscr{G}(m_T)}{m_T} + \varepsilon.$$

   This is in contradiction with Remark 3.3.13, deducing that $\lim_{k\to+\infty} J_k/k = m_T$.

2. Let us suppose by contradiction that the thesis does not hold. Then we can find a sequence of families $\mathbf{F}_n$ with $k_n + 1 = \#\mathbf{F}_n \to +\infty$, $m^n = |\mathbf{F}_n| \in [J_{k_n}, J_{k_n+1}]$, that contain the masses $m_1^n < m_2^n$ and such that

$$G(m^n) = \mathscr{G}(\mathbf{F}_n). \tag{3.34}$$

Since $m_1^n \in [0, m_F]$, then $\mathscr{G}(m_1^n)$ is bounded. Therefore, using the previous point and the structure of the optimal families obtained in Lemma 3.3.5 to see that $m_2^n \to m_T$ for $n \to +\infty$. The strict concavity of $\mathscr{G}$ in $[0, m_F]$ ensures that there exists a unique point $m_T^* \in [0, m_F]$ with $\mathscr{G}'(m_T^*) = \mathscr{G}'(m_T)$. Thus, using the condition $\mathscr{G}'(m_1^n) = \mathscr{G}'(m_2^n)$ obtained in Lemma 3.3.5, we also have that $m_1^n \to m_T^*$. Hence, we can find $\varepsilon > 0$ such that $\mathscr{G}(m_1^n) > (\alpha_T + \varepsilon) m_1^n$ for every n large enough, and therefore

$$\mathscr{G}(\mathbf{F}_n) = \mathscr{G}(m_1^n) + k_n \mathscr{G}(m_2^n) > (\alpha_T + \varepsilon)\, m_1^n + k_n \mathscr{G}(m_2^n) \qquad \text{for } n \gg 1.$$

But we also observe that

$$\begin{aligned}\mathscr{G}\left(m_2^n + \frac{m_1^n}{k_n}\right) &= \mathscr{G}(m_2^n) + \frac{m_1^n}{k_n}\mathscr{G}'(m_2^n) + o(1/k_n)\\ &= \mathscr{G}(m_2^n) + \frac{m_1^n}{k_n}(\alpha_T + o_n(1)) + o(1/k_n),\end{aligned}$$

and multiplying by k_n we get that the family $\tilde{\mathbf{F}}_n$ containing $\tilde{m}_1 = 0$ and $\tilde{m}_2 = \ldots = \tilde{m}_{k_n+1} = m^n/k_n$ is strictly better that $\mathbf{F}_n$ if n is large enough. Hence, $\mathbf{F}_n$ was not an optimal family, contradicting (3.34). Therefore it must exist $\bar{k} \in \mathbb{N}$ for which the thesis holds.

3. We use a similar argument compared to the first point of this proposition. In fact, we fix any $k \geq \bar{k}$ and we know that any optimal family $\mathbf{F}$ with $|\mathbf{F}| = J_{k+1}$ is made of equal masses. Thanks to Remark 3.3.11, we also know that

$$G(J_{k+1}) = (k+1)\mathscr{G}\left(\frac{J_{k+1}}{k+1}\right) = (k+2)\mathscr{G}\left(\frac{J_{k+1}}{k+2}\right),$$

Since any subfamily of an optimal family is itself optimal, as we expressed in Remark 3.3.4, we consider $\tilde{\mathbf{F}}$ containing $k+1$ masses equal to $\frac{J_{k+1}}{k+2}$, and we know that it is optimal. Clearly $|\tilde{\mathbf{F}}| = \frac{k+1}{k+2} J_{k+1}$ and $\#\tilde{\mathbf{F}} = k+1$. Using the definition of J_k, it is immediate to see that $J_k \leq |\tilde{\mathbf{F}}| = \frac{k+1}{k+2} J_{k+1}$, that is the desired inequality. Finally, the convergence follows directly from the first point of the statement.

□

Concerning the second point of Proposition 3.3.18, one can find $\bar{k}$ just looking at the first moment when $G(J_k)$ can be realized only by families containing equal masses. In fact, the following proposition holds:

Proposition 3.3.19. *If $k \geq 1$ is an integer, and if $G(J_k) = G_{k+1}(J_k) = (k+1)\mathscr{G}(\frac{J_k}{k+1})$, then any optimal family $\mathbf{F}$ with total measure $|\mathbf{F}| = m > J_k$ is made of equal masses.*

Proof. It is enough to prove the result for $m = J_{k+1}$: for $m \in (J_k, J_{k+1})$ it has been proved in Theorem 3.3.15, and for $m > J_{k+1}$ it is sufficient to argue by induction on k. Hence, we will prove that, if $G(J_k) = (k+1)\mathscr{G}(\frac{J_k}{k+1})$, then every optimal family for J_{k+1} is made of equal masses. We need to consider only the families with $k+2$ masses, because we already noticed that the only optimal family $\mathbf{F}$ with $|\mathbf{F}| = J_{k+1}$ and $\#\mathbf{F} = k+1$ is made of equal masses (see Proposition 3.3.16). We argue by contradiction, and we suppose that there exists an optimal family $\mathbf{F}$ containing $0 < m_1 < m_2$ with $|\mathbf{F}| = J_{k+1}$ and $\#\mathbf{F} = k+2$. If this is the case, then the family $\mathbf{F}_1$ made of m_1 and k copies of m_2 is also optimal. By definition of J_k, we have that $J_k \leq |\mathbf{F}_1| < J_{k+1}$. Since by hypothesis $G(J_k) = (k+1)\mathscr{G}(\frac{J_k}{k+1})$, then Theorem 3.3.15 guarantees that any optimal family $\mathbf{F}_2$ with $|\mathbf{F}_2| = \bar{m} \in (J_k, J_{k+1})$ must be made of equal masses. Thus, the only possibility is that $|\mathbf{F}_1| = m_1 + km_2 = J_k$ and $G(J_k) = \mathscr{G}(m_1) + k\mathscr{G}(m_2)$. Therefore, we have that

$$G(J_{k+1}) = \mathscr{G}(m_1) + (k+1)\mathscr{G}(m_2) = G(J_k) + \mathscr{G}(m_2) = (k+1)\mathscr{G}\left(\frac{J_k}{k+1}\right) + \mathscr{G}(m_2).$$

As a consequence, the family $\mathbf{F}_3$ containing m_2 and $k+1$ copies of $J_k/(k+1)$ has total measure J_{k+1} and is optimal. This, however, is not possible: since $m_1 < m_2$, then $J_k/(k+1) < m_2$, and this goes against the optimality condition provided in Lemma 3.3.5 because there is more than one "small mass". Since we obtained a contradiction, this argument proves that $\mathbf{F}$ must contain equal masses, concluding the proof. □

3.3.3 Examples

We finally remark that some phenomena cannot be ruled out with the sole hypotheses ($\mathbf{H}_{1D}$). In general, we cannot exclude that the smaller mass is present in the optimal families. Moreover, we emphasize that it could happen that, even if we have the monotonicity result shown in Theorem 3.3.15, the optimal families may be non continuous with respect to their total measure. These two phenomena are showed respectively in Example 3.3.20 and Example 3.3.21.

Example 3.3.20. We provide an energy function for which we can find an optimal family made of two different masses. The aim of this example is to actually construct a function with that property, and the idea is to impose a sudden growth right after the flex point in order to favor the division of the measure. Apart from the specific function, in Example 3.3.21 we provide a simple condition that guarantees this behaviour. A possible example is the function

$$\mathscr{G}(m) = \begin{cases} \sqrt{m} & \text{for } m \in [0,1] \\ 100(m-1)^2 + \frac{1}{2}(m-1) + 1 & \text{for } m > 1 \end{cases},$$

that is of class $C^1((0,+\infty))$, as one can check with an easy computation, and has flex point in $m_F = 1$. Moreover, we have that $\mathscr{G}(1.1) > 2$, while $2\mathscr{G}(0.55) \cong 1.48$. If we fix $m = 1.1$, then these numbers tell us that $K(m) \geq 2$. Given an optimal family $\mathbf{F}$ with $|\mathbf{F}| = m$, if it contains two or more masses greater than m_F, then their sum exceeds $2 > m$, and this is impossible. Exploiting the optimality conditions stated in Lemma 3.3.5, we obtain that we cannot have an optimal family with 3 or more masses (because two of them should coincide and be larger than m_F), and we cannot find an optimal family of two equal masses because again their sum would exceed 2. In conclusion, $\mathbf{F}$ is necessarily made of two different masses.

Example 3.3.21. As we anticipated, this example provides a function for which the families made of two masses collapse in a non-continuous way. This also clarifies why we do not prove that G_k of class $C^1((J_{k-1}, J_k))$. We notice that we can enforce the presence of a 2-masses optimal configuration with the very simple requirement that

$$\mathscr{G}\,(7m_F/4) > 2\mathscr{G}(m_F).$$

In fact, if this condition holds, then clearly $G(m) < G_1(m)$ for every $m \in (7m_F/4, 2m_F]$. Moreover, for $m \in (7m_F/4, 2m_F]$ we cannot have $G_3(m) < G_2(m)$: any family $\mathbf{F}$ with $\#\mathbf{F} = 3$ and $|\mathbf{F}| = m$ would contain at least two masses smaller than m_F, and thus it cannot be optimal thanks to Lemma 3.3.5. In the end, any optimal family $\mathbf{F}$ with $|\mathbf{F}| = m \in (7m_F/4, 2m_F)$ is made of two different masses $m_1 < m_2$: if the two masses coincide, then $m_1 = m_2 = m/2 < m_F$, and this is ruled out again by Lemma 3.3.5. Furthermore, there exists a small parameter $\eta > 0$ such that $K(m) = 2$ for all $m \in [2m_F, 2m_F + \eta]$. In fact, let us take $m = 2m_F + \eta$ and let us take a family $\mathbf{F} = (m_1, m_2, m_3)$ with $|\mathbf{F}| = m$. If $\mathbf{F}$ is optimal, then $m_F \le m_2 = m_3 < m_F + \eta/2$, and thus $m_1 = m - 2m_2 < \eta$. Thanks to the monotonicity properties of $\mathscr{G}'$ we have that

$$\mathscr{G}'(m_F) \le \mathscr{G}'(m_2) \le \mathscr{G}'(m_F + \eta/2), \qquad \mathscr{G}'(m_F) < \mathscr{G}'(\eta) \le \mathscr{G}'(m_1). \tag{3.35}$$

If we take $\eta > 0$ small enough, the previous inequalities are not compatible with the optimality condition $\mathscr{G}'(m_1) = \mathscr{G}'(m_2)$. Therefore the family $\mathbf{F}$ is not optimal and $K(m) = 2$.

We take the function $\tilde{\mathscr{G}} : \mathbb{R}^+ \to \mathbb{R}^+$ defined as $\tilde{\mathscr{G}}(m) = (m-1)^3 + 1$. It satisfies ($\mathbf{H}_{1D}$), and it has a flex point in $m_F = 1$. Then we define the function

$$g(m) = \begin{cases} (m - 3/2)^2 & \text{if } m \ge 3/2 \\ 0 & \text{if } m \in [0, 3/2) \end{cases},$$

and take $\mathscr{G}(m) = \tilde{\mathscr{G}}(m) + \lambda g(m)$ for some constant $\lambda > 0$. It still satisfies ($\mathbf{H}_{1D}$), and the flex point is again in $m_F = 1$. Moreover, we clearly have that $\mathscr{G}(7m_F/4) > 2\mathscr{G}(m_F)$ if λ is large enough, and therefore $K(m) = 2$ for $m \in (7/4, 2)$ thanks to the previous considerations. One can define $\tilde{G}$ as in (3.25) using the profile $\tilde{\mathscr{G}}$ instead of $\mathscr{G}$. The symmetry of $\tilde{\mathscr{G}}$ with respect to the point 1 guarantees that $\tilde{\mathscr{G}}(m) + \tilde{\mathscr{G}}(2-m) = \tilde{\mathscr{G}}(2)$ for every $m \in (0, 2)$, and thus $\tilde{G}(2) = 2\tilde{\mathscr{G}}(1) = 2\mathscr{G}(1) = 2$. Moreover, since $\mathscr{G} \ge \tilde{\mathscr{G}}$, then $G \ge \tilde{G}$. But one can notice that, by definition,

$$G(2) \le 2\mathscr{G}(1) = \tilde{G}(2) \le G(2), \tag{3.36}$$

and so $G(2) = 2\mathscr{G}(1)$. Now we take $t \nearrow 2$ and an optimal family $\mathbf{F}(t)$ with $|\mathbf{F}(t)| = t$ and $G(t) = \mathscr{G}(\mathbf{F}(t))$. By the previous considerations, we know that $\mathbf{F}(t)$ is made of two different masses $m_1(t) < m_2(t)$ for every $t \in (7/4, 2)$ if λ is large enough (and from now on it is fixed). Additionally, there exists a parameter $\eta > 0$ such that $K(t) = 2$ for $t \in [2m_F, 2m_F+\eta]$. If $t < 2$ and $m_2(t) < 3/2$, then the condition $\mathscr{G}'(m_1(t)) = \mathscr{G}'(m_2(t))$ forces

$$m_1(t) + m_2(t) = 2,$$

thanks to the particular choice of f. Of course this is not possible since $m_1(t) + m_2(t) = t < 2$, and therefore it cannot happen that $m_2(t) \to 1$ as $t \to 2$. Arguing as we did to show the existence of the parameter η out of the inequalities (3.35), we also get that $\tilde{G}(t) = 2\tilde{\mathscr{G}}(t/2)$ for $t \in (2, 2+\eta']$, where $\eta' < \eta$ is a small enough parameter. Thus, using a comparison between G and $\tilde{G}$ analogous to (3.36), we obtain that $G(t) = \tilde{G}(t) = 2\mathscr{G}(t/2)$ for $t \in (2, 2+\eta']$, so that the optimal families $\mathbf{F}(t)$ for the energy $\mathscr{G}$ are actually discontinuous when the total measure is $t = 2$.

The next example deals with the optimal families of two consecutive jump points, and establishes the inequality between the masses belonging to them (when it is possible).

Example 3.3.22. Let $k \geq 2$ be given. If $G(J_{k-1}) = k\mathscr{G}(\frac{J_{k-1}}{k})$ and also $G(J_k) = (k+1)\mathscr{G}(\frac{J_k}{k+1})$, then $\frac{J_k}{k+1} \geq \frac{J_{k-1}}{k}$ thanks to Proposition 3.3.18 and Proposition 3.3.19. If, instead, J_{k-1} admits an optimal family $\mathbf{F}_1$ containing different masses m_1 and m_2, and J_k admits a optimal family $\mathbf{F}_2$ containing masses $\bar{m}_1 < \bar{m}_2$, then we apply the second point in Remark 3.3.8 to the subfamily $\mathbf{F}_3 \subset \mathbf{F}_2$ that contains $\bar{m}_1$ and $k-1$ copies of $\bar{m}_2$. Since $\mathbf{F}_3$ is optimal, we know that $J_{k-1} \leq |\mathbf{F}_3|$. Applying the first part of Theorem 3.3.15 we obtain that $\bar{m}_2 \leq m_2$. The last case remaining is that J_{k-1} admits an optimal family with different masses, while J_k does not. We focus on the pair J_1, J_2, and we will see that there is no general inequality between the masses that compose the aforementioned optimal families. We take the function

$$\tilde{\mathscr{G}}(m) = 1 + \frac{(m-1)^3}{|m-1|},$$

that has $\tilde{\mathscr{G}}'' \equiv -2$ in $(0,1)$ and $\tilde{\mathscr{G}}'' \equiv 2$ in $(1,+\infty)$. Clearly $m_F = 1$, while one can easily see that $m_T = \sqrt{2}$. In this example we call $\tilde{G}_k$ and $\tilde{G}$ the functions defined in (3.25) with $\tilde{\mathscr{G}}$ in place of $\mathscr{G}$, while $\tilde{J}_k$ and $\tilde{C}_k$ with $k \in \mathbb{N}$ are analogous quantities to those defined in Corollary 3.3.10 and in (3.33). Similarly to the function $\mathscr{G}$ in Example 3.3.21, we have that $\tilde{\mathscr{G}}(m) = \tilde{\mathscr{G}}(2-m)$ for every $m \in [0,2]$. Therefore, if $\mathbf{F} = (m_1, m_2)$ is an optimal family with $0 < m_1 < m_2$ and $|\mathbf{F}| = m \in [0,2]$, then the condition $\tilde{\mathscr{G}}'(m_1) = \tilde{\mathscr{G}}'(m_2)$ imposes that $1 - m_1 = m_2 - 1$. As a consequence, $m = m_1 + m_2 = 2$. Let $\tilde{\mathbf{F}} = (\tilde{m}_1, \tilde{m}_2)$ be an optimal family with $|\tilde{\mathbf{F}}| = \tilde{J}_1$. If $\tilde{m}_1 < \tilde{m}_2$, then $\tilde{J}_1 = 2$ thanks to the previous argument. If instead $\tilde{m}_1 = \tilde{m}_2$, then $\tilde{m}_2 \geq m_F$ and $\tilde{\mathscr{G}}(2\tilde{m}_2) = 2\tilde{\mathscr{G}}(\tilde{m}_2)$. One can easily compute the value $\tilde{m}_2$ for which that equation admits a solution and see that $\tilde{m}_2 = 1 = m_F$. This proves that $\tilde{J}_1 = 2$. Since $\tilde{G}(\tilde{J}_1) = 2\tilde{\mathscr{G}}(\tilde{J}_1/2)$, then Proposition 3.3.19 tells us that $\tilde{G}(\tilde{J}_2) = 3\tilde{\mathscr{G}}(\tilde{J}_2/3)$. Moreover, thanks to the properties of $\tilde{C}_2$ we also know that $\tilde{J}_2$ satisfies the condition $2\tilde{\mathscr{G}}(\tilde{J}_2/2) = 3\tilde{\mathscr{G}}(\tilde{J}_2/3)$, and this forces the value of $\tilde{J}_2$ to be $\tilde{J}_2 = 2\sqrt{3} \cong 3.46$.
Our strategy now is to obtain $\mathscr{G}$ adding a small perturbation to $\tilde{\mathscr{G}}$. In this process,

we want to keep some good control on the optimal families $\mathbf{F}_1$ and $\mathbf{F}_2$ with $\#\mathbf{F}_1 = 2$, $\#\mathbf{F}_2 = 3$, $|\mathbf{F}_1| = J_1$ and $|\mathbf{F}_2| = J_2$. In particular, we need that $\mathbf{F}_1$ contains two different masses, while $\mathbf{F}_2$ needs to contain three equal masses. Imposing these conditions we are sure to be in an interesting case, as we pointed out before the example. Moreover, if we modify $\tilde{\mathscr{G}}$ only in $(0, m_F)$, we are sure that $J_2 = \tilde{J}_2 = 2\sqrt{3}$ since J_2 must satisfy $2\mathscr{G}(J_2/2) = 3\mathscr{G}(J_2/3)$. To construct the function $\mathscr{G}$ we fix $t_1, t_2 \in (0,1)$ two arbitrary points with $t_1 < t_2$, we take a smooth function $\phi \geq 0$ with compact support contained in (t_1, t_2) and we take $\mathscr{G} = \tilde{\mathscr{G}} - \varepsilon\phi$ with $\varepsilon > 0$ small. Thus $\mathscr{G} \leq \tilde{\mathscr{G}}$, while if we choose ε small enough we have also that $\mathscr{G}$ is increasing and $\mathscr{G}'' < -4/3$ in $(0,1)$. Of course $G(2) \leq \mathscr{G}(m_1) + \mathscr{G}(m_2)$ for any $m_1 \in (t_1, t_2)$ with $\mathscr{G}(m_1) < \tilde{\mathscr{G}}(m_1)$ and $m_2 = 2 - m_1$. Since we decreased $\tilde{\mathscr{G}}$, this implies that $G(2) < \mathscr{G}(2) = \tilde{\mathscr{G}}(2) = 2\mathscr{G}(1)$. This proves that $J_1 \leq 2$, and combining Theorem 3.3.15 with the inequality $\mathscr{G}(2) < 2\mathscr{G}(1)$ we get that an optimal family $\mathbf{F}_2$ with $|\mathbf{F}_2| = J_1$ and $\#\mathbf{F}_2 = 2$ is made of different masses. We also see that an optimal family $\mathbf{F}_3$ with $|\mathbf{F}_3| = J_2$ and $\#\mathbf{F}_3 = 3$ cannot contain different masses. In fact, if $\mathbf{F}_3 = (m_1, m_2, m_3)$ is that optimal family with $0 < m_1 < m_2 = m_3$, then Remark 3.3.8 guarantees that

$$2\mathscr{G}''(m_1) + \mathscr{G}''(m_2) \geq 0.$$

But this is not possible since $m_1 < m_F$, and $\mathscr{G}'' < -4/3$ in $(0, m_F)$, while $\mathscr{G}'' \equiv 2$ in $(1, +\infty)$.
The previous considerations are valid for any choice of $t_1, t_2 \in (0,1)$ and any (suitably small) $\varepsilon > 0$, and we remind that any optimal family $\mathbf{F}_2$ with $|\mathbf{F}_2| = J_1$ and $\#\mathbf{F}_2 = 2$ contains a mass $m_1 \in (t_1, t_2)$. Therefore we can choose t_1 and t_2 very close to 1 to construct a function $\mathscr{G}$ for which $G(J_1) = \mathscr{G}(m_1) + \mathscr{G}(m_2)$ and $m_2 < J_2/3$, while if we choose them close to 0 we obtain that $m_2 > J_2/3$. The position of m_2 here is determined by the condition $\mathscr{G}'(m_1) = \mathscr{G}'(m_2)$ and the fact that $m_1 \in (t_1, t_2)$.

With the last two examples we see that no relationship can be expected between m_T and the masses m_1, m_2 of an optimal family with total measure J_1. In particular, we prove that m_2 can be smaller or larger than m_T, depending on the choice of $\mathscr{G}$.

Example 3.3.23. We provide an energy $\mathscr{G}$ such that $G(J_1) = \mathscr{G}(m_1) + \mathscr{G}(m_2)$ with $0 < m_1 < m_T < m_2$. In this example we argue for the first time by approximation of $\mathscr{G}$: it is easier to construct some piecewise smooth function $\tilde{\mathscr{G}}$, find some optimal families (this can be done with a barely lower semicontinuous and coercive profile) and only in the end approximate it with C^1 functions satisfying ($\mathbf{H}_{1D}$). We take as function $\tilde{\mathscr{G}}$ the one depicted on the left side of Figure 3.4. It is easy to find $\mathscr{G} \in C^1((0, +\infty))$ satisfying ($\mathbf{H}_{1D}$) such that $|\mathscr{G} - \tilde{\mathscr{G}}| \leq \varepsilon$ in $[0, 20]$ for an arbitrarily small $\varepsilon \in (0, 1/2)$, and such that $m_F = 9$, $m_T = 10$, $\mathscr{G}'(m) \in [1, 1.1]$ for $m \in [0, 19/10]$ and $\mathscr{G}' \leq 1/2$ in $[9, 10.5]$. Then we consider the first jumping point J_1 and an optimal family $\mathbf{F} = (m_1, m_2)$ with $|\mathbf{F}| = J_1$. We first notice that $J_1 \leq 12$, because $\mathscr{G}(12) \geq 40$, while $\mathscr{G}(10) + \mathscr{G}(2) \leq 5$. Thanks to Lemma 3.3.5 we know that $m_2 \geq m_F = 9$. If $m_1 \leq 19/10$, then m_2 cannot lie in $[9, 10.5]$: otherwise we would have that $\mathscr{G}'(m_2) \leq 1/2 < 1 \leq \mathscr{G}'(m_1)$, and this is in contrast with the first order optimality conditions stated in Remark 3.3.8. Therefore, $m_2 \geq 10.5 > m_T$, and this was the desired result.

This is actually the only possible scenario: if $m_1 > 19/10$, then $\mathscr{G}(m_1) + \mathscr{G}(m_2) \geq 2\mathscr{G}(19/10) \geq 19/5$, while

$$G(J_1) \leq G(12) \leq \mathscr{G}(1) + \mathscr{G}(11) \leq \tilde{\mathscr{G}}(1) + \tilde{\mathscr{G}}(11) + 2\varepsilon = 1 + \frac{5}{2} + 2\varepsilon = \frac{7}{2} + 2\varepsilon.$$

This chain of inequalities shows that $\mathbf{F}$ cannot be optimal if $m_1 > 19/10$ whenever we take $\varepsilon < 3/20$, hence this case does not occur choosing ε properly.

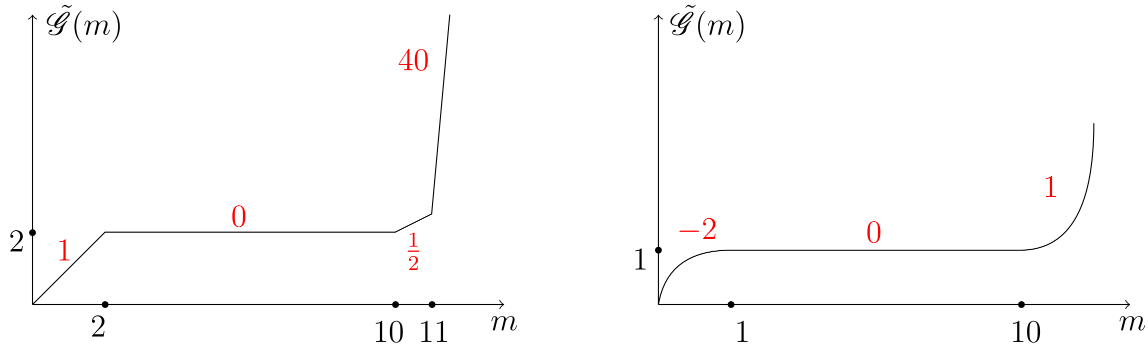


Figure 3.4: On the left we depicted a piecewise linear energy, whose slope is written in red in each linear part. On the right, instead, it is represented a piecewise quadratic function $\tilde{\mathscr{G}}$ such that $\tilde{\mathscr{G}}'(0) = 2$, and its second derivative is written in red above the graph.

Example 3.3.24. This example is complementary to the previous one: we show that there exists an energy $\mathscr{G}$ such that $G(J_1) = \mathscr{G}(m_1) + \mathscr{G}(m_2)$ with $0 < m_1 < m_2 < m_T$. In fact, we fix the function $\tilde{\mathscr{G}}$ that is represented on the right side of Figure 3.4. For $\tilde{\mathscr{G}}$ it is well defined the point m_T which minimizes $\tilde{\mathscr{G}}(m)/m$, and clearly $m_T \geq 10$. It is also easy to see that $m_T \leq 12$: by construction $\tilde{\mathscr{G}}'(m_T) \leq \mathscr{G}(10)/10 = 1/10$, while $\tilde{\mathscr{G}}'(m) \geq \tilde{\mathscr{G}}'(12) = 2$ for every $m \geq 12$. Moreover, it is also well defined the point m_T^* as the unique point $m \in (0,1)$ with $\tilde{\mathscr{G}}'(m) = \tilde{\mathscr{G}}'(m_T)$ (we recall that this point was used also in Proposition 3.3.18). Then we take any function $\mathscr{G} \in C^2((0,+\infty))$ satisfying ($\mathbf{H}_{1D}$), that coincides with $\tilde{\mathscr{G}}$ in $[0, m_T^*] \cup [m_T, +\infty)$, and with $m_F = 9$. By construction, the point $m > 0$ that minimizes $\mathscr{G}(m)/m$ is still m_T, and since $\mathscr{G}$ coincides with $\tilde{\mathscr{G}}$ in $(m_T, +\infty)$ we also have that $J_1 \leq 14$: it is sufficient to notice that $\mathscr{G}(14) = 9$ while $2\mathscr{G}(7) \leq 2\mathscr{G}(12) = 6$. In this way, the family $\tilde{\mathbf{F}} = (J_1/2, J_1/2)$ is not optimal because it contains two masses smaller than the flex point m_F. We take an optimal family $\mathbf{F} = (m_1, m_2)$ with $|\mathbf{F}| = J_1$, and the previous observation shows that $m_1 < m_2$. If $m_2 \geq m_T$, then m_1 must be smaller than m_T^* because of the condition on the first derivatives. Now the fourth point in Remark 3.3.8 comes into play: since $m_1 \leq m_T^*$ and $m_2 \geq m_T$, we have that $\mathscr{G}''(m_1) = -2$ and $\mathscr{G}''(m_2) = 1$, so we contradict the condition $\mathscr{G}''(m_1) + \mathscr{G}''(m_2) \geq 0$. This proves that any optimal family for J_1 with masses $0 < m_1 \leq m_2$ must satisfy $m_2 \leq m_T$, as we claimed.

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