INITIAL DATA IDENTIFICATION FOR CONSERVATION LAWS WITH SPATIALLY DISCONTINUOUS FLUX

FABIO ANCONA AND LUCA TALAMINI

Abstract. We consider a scalar conservation law with a spatially discontinuous flux at a single point $x = 0$, and we study the initial data identification problem for AB-entropy solutions associated to an interface connection (A, B). This problem consists in identifying the set of initial data driven by the corresponding AB-entropy solution to a given target profile ω^T, at a time horizon $T > 0$. We provide a full characterization of such a set in terms of suitable integral inequalities, and we establish structural and geometrical properties of this set. A distinctive feature of the initial set is that it is in general not convex, differently from the case of conservation laws with convex flux independent on the space variable. The results rely on the properties of the AB-backward-forward evolution operator introduced in [3], and on a proper concept of AB-genuine/interface characteristic for AB-entropy solutions provided in this paper.

Contents

1. Introduction

We are concerned with the initial value problem for a scalar conservation law in one space dimension

$$u_t + f(x, u)_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \tag{1.1}$$

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova, Italy, E-mail: ancona@math.unipd.it,
luca.talamini@math.unipd.it.

where $u = u(x,t)$ is the state variable, and the flux f is a space discontinuous function of the form

$$f(x,u) = \begin{cases} f_l(u), & x < 0, \\ f_r(u), & x > 0\,, \end{cases} \tag{1.3}$$

with $f_l, f_r : \mathbb{R} \to \mathbb{R}$ twice continuously differentiable, uniformly convex maps that satisfy

$$f_l''(u),\ f_r''(u) \geq a > 0\,, \tag{1.4}$$

and (up to a reparametrization)

$$f_l(0) = f_r(0), \quad f_l(1) = f_r(1)\,. \tag{1.5}$$

We assume also that the unique critical points θ_l, θ_r of f_l, f_r, respectively, satisfy

$$\theta_l \geq 0, \quad \theta_r \leq 1\,. \tag{1.6}$$

It is well known that, because of the nonlinearity of the equation, in order to achieve global in time existence and uniqueness results for problems of this type one has to consider weak distributional solutions that satisfy the classical Kružkov entropy inequalities away from the flux-discontinuity interface $x = 0$, augmented by an appropriate *interface entropy condition* at $x = 0$. Here, we will consider *entropy solutions of AB-type*, associated to a so-called *interface connection* (A, B) (cfr [1, 11] and see §2.1). Entropy solutions of AB-type form an $\mathbf{L}^1$-contractive semigroup on the domain of $\mathbf{L}^\infty$ initial data [11, 23]. Thus, we adopt the semigroup notation $u(x,t) \doteq \mathcal{S}_t^{[AB]+} u_0(x)$, $t \geq 0$, $x \in \mathbb{R}$, for the unique AB-entropy solution of (1.1)-(1.2), for every initial datum $u_0 \in \mathbf{L}^\infty(\mathbb{R})$.

In this paper, we study the initial data identification problem (or inverse design problem) for AB-entropy solutions of the equation (1.1). This problem consists in identifying the set of initial data for which the corresponding AB-entropy solution coincides with a given target profile ω^T, at a time horizon $T > 0$. Observe that we cannot expect to reach any desired profile $\omega^T \in \mathbf{L}^\infty(\mathbb{R})$. In fact, even in the case where $f_l = f_r$, since the work of Oleĭnik [32] it is well known that, because of the uniform convexity of the flux, the Kružkov entropy conditions imply that every entropy weak solution u of (1.1) must satisfy (in the sense of distributions) the one-sided Lipschitz estimate

$$\partial_x u(\,\cdot\,,t) \leq \frac{1}{at}, \qquad \text{in} \ \ \mathscr{D}', \quad \forall\, t > 0. \tag{1.7}$$

Essentially, the nonlinearity of the flux forces characteristic lines to intersect which, together with the entropy condition, produces a regularizing effect L^∞ to BV encoded in the Oleĭnik inequality (1.7). In the case of equation (1.1) with discontinuous flux (1.3) where $f_l \neq f_r$, we have shown in [2, 3] that the set of reachable profiles at a time $T > 0$:

$$\mathcal{A}^{[AB]}(T) \doteq \big\{ \mathcal{S}_T^{[AB]+} u_0 \quad \big| \quad u_0 \in \mathbf{L}^\infty(\mathbb{R}) \big\}, \tag{1.8}$$

is characterized in terms of suitable Oleĭnik-type estimates and unilateral pointwise constraints. Note that a "loss of information" takes place when characteristic lines intersect into a shock: there are infinitely many ways to create the same shock discontinuity at a given time T. Therefore the initial data identification problem for this type of equations is highly ill-posed: multiple initial data can be stirred by (1.1) into the same attainable profile $\omega^T \in \mathcal{A}^{[AB]}(T)$ at time T.

Our goal is to characterize and study the properties of the set of initial data leading to a given profile $\omega^T \in \mathcal{A}^{[AB]}(T)$ at time T:

$$\mathcal{I}_T^{[AB]}(\omega^T) \doteq \big\{u_0 \in \mathbf{L}^\infty(\mathbb{R}) \quad | \quad \mathcal{S}_T^{[AB]+} u_0 = \omega^T\big\}. \tag{1.9}$$

In the case of conservation laws with flux independent on the space variable, the initial data identification problem was firstly studied for the Burgers equation in [26, 29, 30], and next for general uniformly convex flux in [14], where it is fully characterized the initial set of data evolving to a given profile, and it is shown that such a set is convex. Similar results were obtained in [20, 21] for Hamilton-Jacobi equations with convex Hamiltonian, and in [15] for smoothly, space dependent, conservation laws or Hamilton-Jacobi equations.

When the flux is of the form (1.3) with $f_l \neq f_r$, the initial data reconstruction problem is more challenging because one has to deal with the richer and more complicated *near-interface* wave structure of AB-entropy solutions. This is due to the presence in the solution of waves that are reflected or refracted through the discontinuity interface $x = 0$, as well as of shock discontinuities emerging from the interface at positive times (see the analysis in [3]). Nonetheless, we are still able to provide a full characterization of the initial set (1.9) by suitable integral inequalities, for every given $\omega^T \in \mathcal{A}^{[AB]}(T)$, and we show that (1.9) shares almost the same geometric and topological properties of the initial set for conservation laws with uniformly convex flux independent on the space variable. Notably, a distinctive difference from the classical smooth case is the lack of convexity of the initial set (1.9) as shown in the Example 6.1. To establish these results we will rely on:

- a suitable definition of *AB-backward evolution operator* $\mathcal{S}_T^{[AB]-}$ given in [3], and on the structural properties of the range of $\mathcal{S}_T^{[AB]-}$ therein analized;
- a proper concept of *AB-genuine/interface characteristic* for AB-entropy solutions which can "travel" along the discontinuity interface $x = 0$ (see Definition 3.1).

Given an AB-entropy solution u, a time horizon $T > 0$, and a point $x \in \mathbb{R}$, we will let $\mathcal{C}_0(u, x)$ denote the set of the initial positions $\zeta(0)$ of the AB-genuine characteristics ζ for u that reach the point $x = \zeta(T)$ at time T (cfr. (3.3)). We recall that any element $\omega^T \in \mathcal{A}^{[AB]}(T)$ admits one-sided limits $\omega^T(x-), \omega^T(x+)$ at every point $x \in \mathbb{R}$, and that has at most countably many discontinuities (see [3]). Then, we summarize the main results of the paper in the following

Theorem 1.1. *Given $\omega^T \in \mathcal{A}^{[AB]}(T)$, set*

$$u_0^* \doteq \mathcal{S}_T^{[AB]-} \omega^T\,, \tag{1.10}$$

and

$$u^*(\cdot, t) \doteq \mathcal{S}_t^{[AB]+} u_0^* \qquad \forall\ t \in [0, T]\,. \tag{1.11}$$

Then, letting $\mathcal{I}_T^{[AB]}(\omega^T)$ be the set defined in (1.9), the following properties hold:

(i) $u_0 \in \mathcal{I}_T^{[AB]}(\omega^T)$ if and only if, for every point $\overline{x}$ of continuity of ω^T, there exists $\overline{y} \in \mathcal{C}_0(u^*, \overline{x})$ such that there hold

$$\int_y^{\overline{y}} u_0(x)\, \mathrm{d}x \le \int_y^{\overline{y}} u_0^*(x)\, \mathrm{d}x, \qquad \forall \quad y < \min \mathcal{C}_0(u^*, \overline{x}), \tag{1.12}$$

and

$$\int_{\overline{y}}^y u_0(x)\, \mathrm{d}x \ge \int_{\overline{y}}^y u_0^*(x)\, \mathrm{d}x, \qquad \forall \quad y > \max \mathcal{C}_0(u^*, \overline{x}). \tag{1.13}$$

(ii) *The set* $\mathcal{I}_T^{[AB]}(\omega^T)$ *is an infinite dimensional cone which has vertex* u_0^* *and is in general not convex.*

We will establish further geometric and topological properties of the initial set (1.9) besides the ones stated in Theorem 1.1-(ii), which are collected in Theorem 3.6 stated in § 4.

Initial data identification problems are often formulated as least square optimization problems associated to observable states at a final time (also known in the literature as *data assimilation problems*). These type of problems arise naturally in environmental sciences [5, 6, 7, 8, 28], but also in life sciences (see [13] and references therein), to improve the forecast of a model or to refine numerical simulations. Similar issues, also related to parameter identification problems, arise in traffic flow modeling [27, 37, 38], in batch sedimentation [9, 19], or in petroleum reservoir engineering [33].

Conservation laws with spatially discontinuous flux have many relevant applications in physics and engineering including: porous media models with changing rock types (for oil reservoir simulation) [24, 25]; sedimentation in waste-water treatment plants [10, 18]; traffic flow dynamics with roads of variable width or surface conditions [31]; Saint Venant models of blood flow in endovascular treatments [22, 12]; radar shape-from-shading models [34].

The paper is organized as follows.

- In § 2 we collect the definitions of interface connection (A, B), of AB-entropy solution and of AB-backward solution operator.
- In § 3 we introduce the AB-genuine/interface characteristics and state the main results, Theorem 3.4 (integral inequalities) and Theorem 3.6 (structural and geometrical properties), which yield Theorem 1.1.
- In § 4 we establish some basic properties enjoyed by the AB-genuine/interface characteristics.
- In § 5 we prove Theorem 3.4.
- In § 6 we prove Theorem 3.6 and provide an example of non convex initial set $\mathcal{I}_T^{[AB]}(\omega^T)$.

2. Basic definitions and general setting

2.1. **Connections and AB-entropy solutions.** We recall here the definitions and properties of interface connection and of admissible solution satisfying an interface entropy condition introduced in [1]. Throughout the paper, for the one-sided limits of a function $u(x)$ we will use the notation

$$u(x\pm) \doteq \lim_{y\to x\pm} u(y). \tag{2.1}$$

Definition 2.1 (**Interface Connection).** Let f be a flux as in (1.3) satisfying the assumptions (1.4)-(1.6). A pair of values $(A, B) \in \mathbb{R}^2$ is called a *connection* if

(1) $f_l(A) = f_r(B)$,
(2) $f_l'(A) \le 0, \;\; f_r'(B) \ge 0$.

We will say that a connection (A, B) is *critical* if $f_l'(A) = 0$, or $f_r'(B) = 0$, i.e. if $A = \theta_l$ or $B = \theta_r$.

Clearly, condition (1) of Definition 2.1 is equivalent to $A \le \theta_l$, $B \ge \theta_r$. For sake of uniqueness, it is employed in [11] the *adapted entropy*

$$\eta_{AB}(x, u) = \big|u - c^{AB}(x)\big|, \qquad c^{AB}(x) \doteq \begin{cases} A, & x \le 0, \\ B, & x \ge 0, \end{cases} \tag{2.2}$$

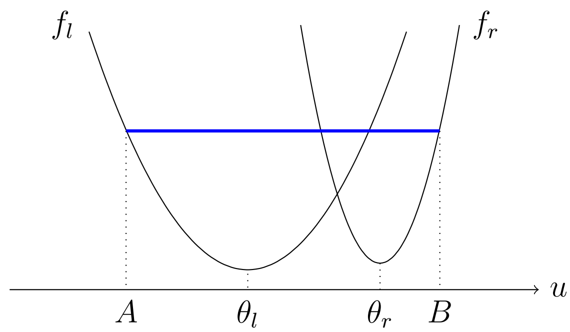

FIGURE 1. An example of connection (A, B) with f_l, f_r strictly convex fluxes

to select the unique solution of the Cauchy problem (1.1)-(1.2) that satisfies the Kružkov-type entropy inequality

$$\left|u - c^{AB}\right|_t + \left[\operatorname{sgn}(u - c^{AB})(f(x,u) - f(x, c^{AB}))\right]_x \le 0, \quad \text{in } \mathcal{D}', \tag{2.3}$$

in the sense of distributions, which leads to the following definition.

Definition 2.2 (*AB*-**entropy solution).** Let (A, B) be a connection and let c^{AB} be the function defined in (2.2). A function $u \in \mathbf{L}^\infty(\mathbb{R} \times [0, +\infty[)$ is said to be an *AB-entropy solution* of the problem (1.1),(1.2) if the following holds:

(1) u is a distributional solution of (1.1) on $\mathbb{R}\times\,]0, +\infty[$, that is, for all test functions $\phi \in \mathcal{C}_c^1$ with compact support contained in $\mathbb{R}\times\,]0, +\infty[$, there holds
$$\int_{-\infty}^{\infty}\int_0^{\infty} \left\{u\phi_t + f(x,u)\phi_x\right\} \mathrm{d}x\,\mathrm{d}t = 0\,.$$

(2) u is a Kružkov entropy weak solution of (1.1),(1.2) on $(\mathbb{R}\setminus\{0\})\times\,]0, +\infty[$, that is, $t \mapsto u(\cdot, t)$ is a continuous map from $[0, +\infty[$ to $\mathbf{L}^1_{\mathrm{loc}}(\mathbb{R})$, the initial condition (1.2) is satisfied almost everywhere, and:
   (2.a) for any non-negative test function $\phi \in \mathcal{C}_c^1$ with compact support contained in $]-\infty, 0[\times\,]0, +\infty[$, there holds
   $$\int_{-\infty}^{0}\int_0^{\infty} \left\{|u - k|\phi_t + \operatorname{sgn}(u-k)\,(f_l(u) - f_l(k))\,\phi_x\right\} \mathrm{d}x\,\mathrm{d}t \ge 0, \quad \forall k \in \mathbb{R}\,;$$
   (2.b) for any non-negative test function $\phi \in \mathcal{C}_c^1$ with compact support contained in $]0, +\infty[\times\,]0, +\infty[$, there holds
   $$\int_{0}^{\infty}\int_0^{\infty} \left\{|u - k|\phi_t + \operatorname{sgn}(u-k)\,(f_r(u) - f_r(k))\,\phi_x\right\} \mathrm{d}x\,\mathrm{d}t \ge 0, \quad \forall k \in \mathbb{R}\,.$$

(3) u satisfies the interface entropy inequality relative to the connection (A, B), that is, for any non-negative test function $\phi \in \mathcal{C}_c^1$ with compact support contained in $\mathbb{R}\times]0, +\infty[$, there holds
$$\int_{-\infty}^{\infty}\int_0^{\infty} \left\{\,\left|u - c^{AB}\right|\phi_t + \operatorname{sgn}(u - c^{AB})\left(f(x,u) - f(x, c^{AB})\right)\phi_x\right\} \mathrm{d}x\,\mathrm{d}t \ge 0\,.$$

Remark 2.3. Since the fluxes f_l, f_r in (1.3) are uniformly convex, by Property (2) of Definition 2.2 it follows that, if u is an AB-entropy solution, then $u(\cdot\,, t)$ is a function of locally bounded variation on $\mathbb{R}\setminus\{0\}$, for any $t > 0$. On the other hand, relying on [35, 36], and because of the $\mathbf{L}^1_{\mathrm{loc}}$-continuity of the map $t \mapsto u(\cdot, t)$, we deduce that u admits left and right strong traces at $x = 0$ for all $t > 0$, i.e. that there exist the one-sided limits

$$u_l(t) \doteq u(0-, t), \qquad u_r(t) \doteq u(0+, t), \qquad \forall\, t > 0\,. \tag{2.4}$$

Moreover, by properties (1), (3) of Definition 2.2, and thanks to the analysis in [3], we deduce that u must satisfy at almost any time $t > 0$ the interface conditions

$$\begin{aligned} & f_l(u_l(t)) = f_r(u_r(t)) \geq f_l(A) = f_r(B), \\ & \big(u_l(t) \leq \theta_l, \quad u_r(t) \geq \theta_r\big) \quad \Longrightarrow \quad u_l(t) = A, \quad u_r(t) = B\,. \end{aligned} \tag{2.5}$$

It was proved in [1, 11] (see also [4, 23]) that AB-entropy solutions of (1.1),(1.2) with bounded initial data are unique, and $\mathbf{L}^1$-contractive with respect to their initial data. Thus, one can define a semigroup map

$$\mathcal{S}^{[AB]+} : [0, +\infty[\, \times \mathbf{L}^\infty(\mathbb{R}) \to \mathbf{L}^\infty(\mathbb{R}), \qquad (t, u_0) \mapsto \mathcal{S}_t^{[AB]+} u_0\,, \tag{2.6}$$

where the function $u(x,t) \doteq \mathcal{S}_t^{[AB]+} u_0(x)$ provides the unique AB-entropy solution of the Cauchy problem (1.1), (1.2). Such a map is $\mathbf{L^1}$-stable also with respect to the time t and the values A, B of the connection, as shown in [3].

2.2. **Backward solution operator.** We review here the concept of backward solution operator associated to a connection (A,B) introduced in [3], referring to [3] for further details and properties.

Given a flux f as in (1.3) satisfying the assumptions (1.4)-(1.6), and a connection (A,B), observe that, setting

$$\overline{B} \doteq \big(f_{r|\,]-\infty,\theta_r]}\big)^{-1} \circ f_r(B), \qquad \overline{A} \doteq \big(f_{l|[\theta_l,+\infty[}\big)^{-1} \circ f_l(A)\,, \tag{2.7}$$

where $f_{|\,I}$ denotes the restriction of the function f to the interval I, the pair $(\overline{B}, \overline{A}\,)$ provides a connection for the symmetric flux

$$\overline{f}(x,u) = \begin{cases} f_r(u), & x \leq 0, \\ f_l(u), & x \geq 0\,, \end{cases} \tag{2.8}$$

(see Figure 2). Then, letting $\overline{\mathcal{S}}_t^{[\overline{B}\,\overline{A}]+} u_0(x)$ denote the unique $\overline{B}\,\overline{A}$-entropy solution of

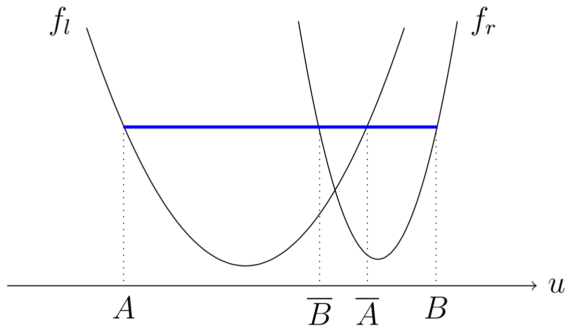

FIGURE 2. The connection $(\overline{B}, \overline{A})$ of the symmetric flux $\overline{f}(x,u)$ defined in (2.8).

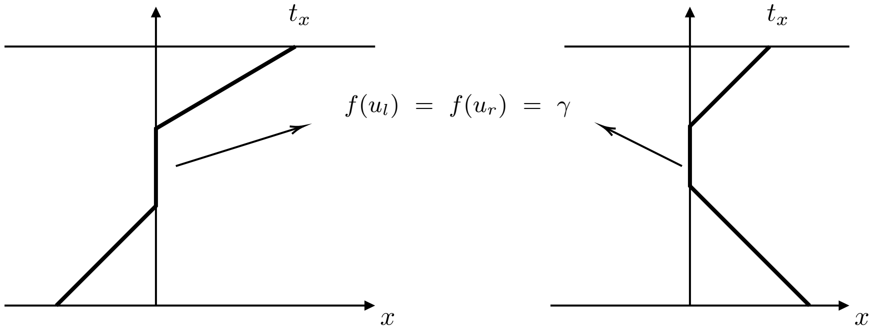

FIGURE 3. Example of a member of $\mathcal{C}(u, x)$.

$$\begin{cases} u_t + \overline{f}(x,u)_x = 0 & x \in \mathbb{R}, \quad t \geq 0, \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{2.9}$$

evaluated at (x,t), we shall define the *backward solution operator* associated to the connection (A,B) in terms of the operator $\overline{\mathcal{S}}_t^{[\overline{B}\,\overline{A}]+}$ as follows.

Definition 2.4 (AB**-Backward solution operator).** Given a connection (A,B), the *backward solution operator* associated to (A,B) is the map $\mathcal{S}_{(\cdot)}^{[AB]-} : [0,+\infty) \times \mathbf{L}^\infty(\mathbb{R}) \to \mathbf{L}^\infty(\mathbb{R})$, defined by

$$\mathcal{S}_t^{[AB]-}\omega(x) \doteq \overline{\mathcal{S}}_t^{[\overline{B}\,\overline{A}]+}\big(\omega(-\,\cdot\,)\big)(-x) \qquad x \in \mathbb{R},\ t \geq 0\,. \tag{2.10}$$

3. Statement of the main results

In this section we introduce the fundamental concept of *genuine/interface characteristic* for an AB-entropy solution, and we collect the statement of the main results of the paper.

3.1. **Genuine/interface characteristics.** The definition of *genuine/interface characteristic* extends to the setting of AB-entropy solutions the classical definition of genuine characteristic for a conservation law $u_t + f(u)_x = 0$ (see [16, 17]). Throughout the following we fix a time $T > 0$, we consider a fixed connection (A,B), and we set

$$\gamma \doteq f_l(A) = f_r(B). \tag{3.1}$$

Definition 3.1 (AB**-genuine/interface characteristics).** Let $u \in \mathbf{L}^\infty(\mathbb{R}\times[0,+\infty[\,;\mathbb{R})$ be an AB-entropy solution of (1.1). We say that a Lipschitz continuous function $\zeta : [0,T] \to \mathbb{R}$ is an *AB-genuine/interface characteristic* (*AB-gic*) for u if the following conditions hold:

(i) for a.e. $t \in [0,T]$ with $\zeta(t) \neq 0$ it holds
$$\dot{\zeta}(t) = f'(u(\zeta(t)-,t), \zeta(t)) = f'(u(\zeta(t)+,t), \zeta(t));$$

(ii) for a.e. $t \in [0,T]$ with $\zeta(t) = 0$, it holds
$$f_l(u(\zeta(t)-,t) = \gamma = f_r(u(\zeta(t)+,t)).$$

Remark 3.2. Applying the classical theory of generalized characteristics [16] it follows that any AB-gic $\zeta \in \mathrm{Lip}([0,T]\,;\mathbb{R})$ is a piecewise affine function for which there exist $0 \leq \tau_1 \leq \tau_2 \leq T$, such that:

- $\zeta(t) = 0$ and $f_l(u_l(t)) = f_r(u_r(t)) = \gamma$, for all $t \in [\tau_1, \tau_2]$;

- the restriction of ζ to $[0, \tau_1[$ and to $]\tau_2, T]$ is either a classical genuine characteristic for the conservation law $u_t + f_l(u)_x = 0$ on $\{x < 0\}$, or it is a classical genuine characteristic for the conservation law $u_t + f_r(u)_x = 0$ on $\{x > 0\}$.

Note that, if $\tau_1 = \tau_2 \in \{0, T\}$, then a curve defined by a function ζ satisfying the above conditions can cross the interface $x = 0$ only at its starting or terminal points. Thus, in this case ζ is a a classical genuine characteristic for $u_t + f_l(u)_x = 0$ on $\{x < 0\}$ or for $u_t + f_r(u)_x = 0$ on $\{x > 0\}$, in the whole interval $]0, T[$.

Remark 3.3. By Definition 3.1 an AB-gic can "travel" along the discontinuity interface $x = 0$ in an interval $[\tau_1, \tau_2]$ only if in such an interval the flux of the solution is the minimum possible, i.e. if $f_l(u_l(t)) = f_r(u_r(t)) = f_l(A) = f_r(B)$ for all $t \in [\tau_1, \tau_2]$, with u_l, u_r as in (2.4). This definition can be motivated by the following observation. Let f be a smooth convex flux. Then, relying on the inequality

$$f(u) - f(v) - f'(u)(v - u) \geq 0 \qquad \forall\, u, v \in \mathbb{R},$$

one can verify that a classical genuine characteristic $\zeta : [0, T] \to \mathbb{R}$ for a solution u of the conservation law $u_t + f(u)_x = 0$ satisfies at a.e. $t \in [0, T]$ the equality

$$f(u(\zeta(t), t)) - \dot{\zeta}(t) u(\zeta(t), t) = \min_{v \in \mathbb{R}} \big\{ f(v) - \dot{\zeta}(t) v \big\}.$$

Therefore if the characteristic is "vertical" (i.e. $\dot{\zeta} = 0$) we simply obtain

$$f(u(\zeta(t), t)) = \min_{v \in \mathbb{R}} f(v), \qquad \text{for a.e. } \ t \in [0, T].$$

In view of the interface constraint (2.5) for AB-entropy solutions, it is then natural to require in this setting that a "characteristic" lying on the interface $x = 0$ be called "genuine" only if it minimizes the admissible flux at the interface, i.e. if it satisfies condition (ii) of Definition 3.1.

Next, given an *AB-entropy solution* u of (1.1), we consider the set of AB-gic passing through a point $x \in \mathbb{R}$ at time $t = T$, and the set of the corresponding initial points at time $t = 0$, setting:

$$\mathcal{C}(u, x) \doteq \Big\{ \zeta \in \mathrm{Lip}([0, T]\,;\, \mathbb{R}) \;|\; \zeta(T) = x \ \text{ and } \ \zeta \text{ is an } \ AB\text{-gic for } u \Big\}, \tag{3.2}$$

and

$$\mathcal{C}_0(u, x) \doteq \Big\{ \zeta(0) \;\;|\;\; \zeta \in \mathcal{C}(u, x) \Big\}. \tag{3.3}$$

The set $\mathcal{C}_0(u, x)$ is a fundamental tool to analyze the set of initial data leading to an attainable profile ω^T. To this end, throughout this section we consider the initial datum u_0^* in (1.10) defined as the image of ω^T through the backward solution operator $\mathcal{S}_T^{[AB]-}$, and we let $u^*(x, t)$ denote the corresponding AB-entropy solution with initial datum u_0^*, defined in (1.11). Moreover, we let $\mathcal{A}^{[AB]}(T)$ denote the set of reachable profiles at time $T > 0$ defined in (1.8). We recall that any element of $\mathcal{A}^{[AB]}(T)$ has at most countably many discontinuities (see [3]).

3.2. **Examples.** We consider here different examples of AB-entropy solutions u that reach the same attainable profile $\omega^T \in \mathcal{A}^{[AB]}(T)$ at time T, which illustrate various structures and properties of the sets $\mathcal{C}(u,x)$, $\mathcal{C}_0(u,x)$. Although we choose a relatively simple profile, it gives already the possibility to capture the essence and the key points of Definition 3.1. Namely, given $L_0 < 0$, we define

$$\omega_1(x) = \begin{cases} p & x < L_0, \\ A & L_0 < x < 0, \\ \overline{B} & 0 < x, \end{cases} \tag{3.4}$$

choosing

$$p > \boldsymbol{v} \doteq \boldsymbol{v}[\,L_0, A, f_l], \tag{3.5}$$

where $\boldsymbol{v}[\,L_0, A, f_l]$ denotes the quantity defined in [3, § 3.2], that satisfies

$$A < \boldsymbol{v} < \overline{A} \tag{3.6}$$

and $\overline{A}, \overline{B}$ are defined as in (2.7). Here we are assuming that the connection (A, B) is not critical. Moreover, we assume that

$$f_l'(A) < L_0/T < f_l'(\boldsymbol{v})\,. \tag{3.7}$$

Note that, since $f_r'(\overline{B}) \leq 0$ it follows that $\mathsf{R} = \mathsf{R}[\omega_1, f_r] = 0$, while (3.5), (3.7) imply $\mathsf{L} = \mathsf{L}[\omega_1, f_l] = L_0$. One can readly verify that ω_1 fulfills the conditions (i)-(ii) of [3, Theorem 4.7] characterizing a class of attainable profiles in $\mathcal{A}^{[AB]}(T)$. By the analysis in [3, see Remark 4.5] it follows that, because of (3.7), any AB-entropy solution reaching the profile ω_1 at time T must necessary contain at least one shock, located in $\{x \leq 0\}$, that produces at time T the discontinuity occurring at $x = L_0$. We shall now briefly describe four different AB-entropy solutions driving (1.1), (1.3) to ω_1 at time T, that are represented in Figures 4-7, with the shock curves coloured in red.

- In Figure 4 it is represented the solution u^* defined as in (1.10)-(1.11) by $u^*(\cdot,t) = \mathcal{S}_t^{[AB]+} \circ \mathcal{S}_T^{[AB]-}\omega_1$. This solution contains in particular a compression wave that creates a shock discontinuity at (L_0, T), which is located on the left of a rarefaction wave centered at the point $(L_0 - T \cdot f_l'(\boldsymbol{v}), 0)$. This rarefaction impinges (from the left) on a shock curve emerging from the interface $x = 0$, at some time $t = \boldsymbol{\sigma}$, which has right state equal to A. The left trace of u^* at $x = 0$ is equal to $\overline{A}$ in the interval $[0, \boldsymbol{\sigma}[$, and it is equal to A in the interval $]\boldsymbol{\sigma}, T]$. Instead the right trace of u^* at $x = 0$ is always equal to $\overline{B}$. At any point (x, T), $x \in]L_0, 0[$, we can trace a unique backward genuine characteristic with slope $f_l'(A)$, which meets the interface $x = 0$ at time $t = T - x/f_l'(A)$. We can then define an AB-gic prolonging this characteristic on the side $\{x > 0\}$ with slope $f_r'(\overline{B})$. Another possible choice to backward define an AB-gic is to travel along the interface $x = 0$ until some time τ, and then to prolong it either on the right (again with slope $f_r'(\overline{B})$), or on the left if $\tau \leq \boldsymbol{\sigma}$ (with slope $f_l'(\overline{A})$). Therefore we have two distinct minimal and maximal polygonal lines in the set $\mathcal{C}(u^*, x)$, represented by the blue polygonal lines in Figure 4, while all the other blue dashed lines are the segmens of the other elements in $\mathcal{C}(u^*, x)$. A more detailed description of these sets for a profile similar to ω_1 is given in Remark 3.8.
- The solution u_1 represented in Figure 5 contains a shock located in $\{x < 0\}$ which has left state p and right state A. Here we are assuming that the corresponding Rankine-Hugoniot speed $\lambda_l(p, A)$ satisfy $L_0 - T\lambda_l(p, A) < 0$, which is certainly true

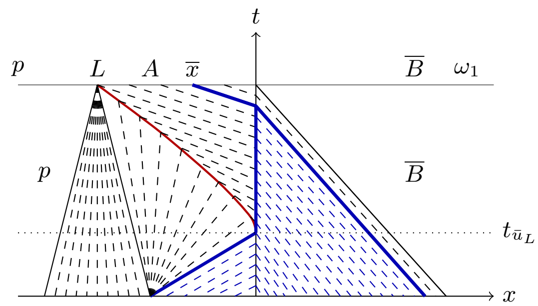

FIGURE 4. The solution u^*.

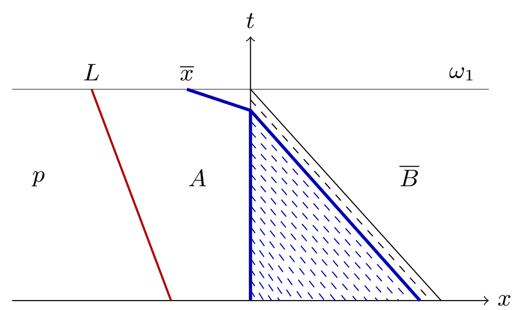

FIGURE 5. The solution u_1.

if we take p sufficiently close to $\overline{A}$. In this case we cannot have AB-gics starting at (x, T), $x \in]L_0, 0[$, that are backward prolonged on the side $\{x < 0\}$ since the left trace of u_1 at $x = 0$ is always equal to A, and $f_l'(A) < 0$ by Definition 2.1. Hence, the set $\mathcal{C}(u_1, x)$ is smaller than in the previous case and we have $\mathcal{C}(u_1, x) \subset \mathcal{C}(u^*, x)$.

- In the case of the solution u_2 represented in Figure 6, two rarefaction waves coming from both sides impinge on the interface $x = 0$ in the time interval $[0, \boldsymbol{\sigma}[$. Therefore, in this interval the left trace of u_2 at $x = 0$ has values $u_{2,l} > \overline{A}$, while the right trace at $x = 0$ has values $u_{2,r} < \overline{B}$. As a consequence, the only AB-gic starting at (x, T), $x \in]L_0, 0[$, that can be backward prolonged on the side $\{x < 0\}$ after traveling on the interface is the one that remains on the interface in the time interval $[\boldsymbol{\sigma}, T - x/f_l'(A)]$, and then continues with slope $f_l'(\overline{A})$ on the side $\{x < 0\}$ in the time interval $[0, \boldsymbol{\sigma}]$. Similarly, the leftmost AB-gic starting at (x, T), $x \in]L_0, 0[$, that is backward prolonged on the side $\{x > 0\}$, is the one that remains on the interface in the time interval $[\boldsymbol{\sigma}, T - x/f_l'(A)]$, and then continues with slope $f_r'(\overline{B})$ on the side $\{x > 0\}$ in the time interval $[0, \boldsymbol{\sigma}]$. We deduce from this analysis that, differently from the other cases, here the set $\mathcal{C}_0(u_2, x)$ is not an interval.
- Finally, we consider the solution u_3 represented in Figure 6, where besides the shock located in $\{x \leq 0\}$ reaching the point (x, T), there is another a shock located in $\{x \geq 0\}$. This shock emerges from the interface $x = 0$ at some time τ_1, and is then reabsorbed by the interface at some later time $\tau_2 > \tau_1$, due to the interaction with rarefaction and compression waves coming from the right. Here we see that, differently from the previous cases, we have $\max \mathcal{C}_0(u_3, x) < \max \mathcal{C}_0(u^*, x)$.

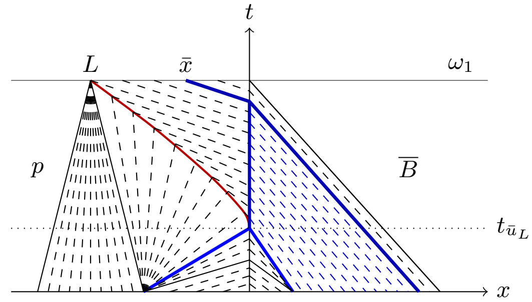

FIGURE 6. The solution u_2.

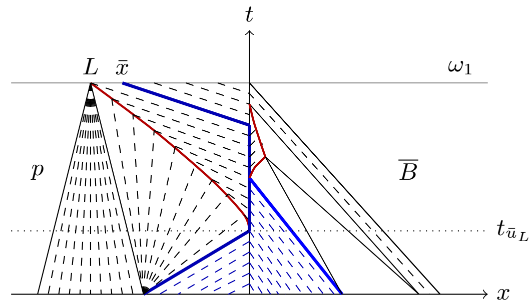

FIGURE 7. The solution u_3.

3.3. **Main results.** The first main result of the paper provides a characterization of the set $\mathcal{I}_T^{[AB]}(\omega^T)$ in (1.9). By the analysis in [3] we know that if $\omega^T \in \mathcal{A}^{[AB]}(T)$, then the AB-entropy solution u^* defined by (1.10)-(1.11) satisfies $u^*(\cdot, T) = \omega^T$, which means that $u_0^* \in \mathcal{I}_T^{[AB]}(\omega^T)$. Our next Theorem gives a characterization of the possible elements $u_0 \in \mathcal{I}_T^{[AB]}(\omega^T)$ which are different from u_0^*.

Theorem 3.4. *Given $\omega^T \in \mathcal{A}^{[AB]}(T)$, let $\mathcal{C}_0(u^*, x)$ denote the set defined in (3.3) for the AB-entropy solution u^* defined by (1.10)-(1.11), and let $u_0 \in \mathbf{L}^\infty(\mathbb{R})$. Then $u_0 \in \mathcal{I}_T^{[AB]}(\omega^T)$ if and only if for every point $\overline{x} \in \mathbb{R}$ there exists $\overline{y} \in \mathcal{C}_0(u^*, \overline{x})$ such that there hold*

$$\int_y^{\overline{y}} u_0(x)\,\mathrm{d}x \le \int_y^{\overline{y}} u_0^*(x)\,\mathrm{d}x, \qquad \forall\ y < \min \mathcal{C}_0(u^*, \overline{x}) \tag{3.8}$$

and

$$\int_{\overline{y}}^{y} u_0(x)\,\mathrm{d}x \ge \int_{\overline{y}}^{y} u_0^*(x)\,\mathrm{d}x, \qquad \forall\ y > \max \mathcal{C}_0(u^*, \overline{x}) \tag{3.9}$$

Remark 3.5. Given $\omega^T \in \mathcal{A}^{[AB]}(T)$, one can verify that the set of initial data $\mathcal{I}_T^{[AB]}(\omega^T)$ shares the same topological properties enjoyed by the set of initial data leading at time T to an attainable profile for a conservation laws with uniformly convex flux independent on the space variable (see [14, Proposition 5.1]). Namely, with respect to the $\mathbf{L}^1_{\mathrm{loc}}$ topology, we have:

(i) for every $M > 0$, the set $\mathcal{I}_T^{[AB]}(\omega^T) \cap \{u_0 : \|u_0\|_{\mathbf{L}^\infty} \le M\}$ is closed, and $\mathcal{I}_T^{[AB]}(\omega^T)$ is an F_σ set;

(ii) the set $\mathcal{I}_T^{[AB]}(\omega^T)$ has empty interior.

The first property follows immediately by the $\mathbf{L^1}$-contractivity of the semigroup of AB-entropy solutions. Concerning property (ii), consider two points $0 < x_1 < x_2$ of continuity for ω^T, such that the classical genuine characteristics $\vartheta_{x_1}, \vartheta_{x_2} : [0,T] \to \mathbb{R}$ for $u_t + f_r(u)_x = 0$, passing at time T through x_1, x_2, respectively, never cross the interface $x = 0$. Let u^* be the AB-entropy solution defined in (1.11). Then, by Remark 3.2, $\vartheta_{x_1}, \vartheta_{x_2}$ are the unique AB-gic for u^* that reach at time T the points x_1, x_2, respectively. By definition (3.3) this means that $\mathcal{C}_0(u^*, x_i) = \{\vartheta_{x_i}(0)\}$, $i = 1, 2$. Note that, by the non crossing property of genuine characteristics, we have $\theta_{x_1}(0) < \theta_{x_2}(0)$. Next, applying the inequality (1.12) for $\overline{x} = x_2, y = \vartheta_{x_1}(0)$, we find that any element $u_0 \in \mathcal{I}_T^{[AB]}(\omega^T)$ satisfies

$$\int_{\vartheta_{x_1}(0)}^{\vartheta_{x_2}(0)} u_0(x)\,\mathrm{d}x \le \int_{\vartheta_{x_1}(0)}^{\vartheta_{x_2}(0)} u_0^*(x)\,\mathrm{d}x \tag{3.10}$$

On the other hand, applying the inequality (1.13) for $\overline{x} = x_1, y = \vartheta_{x_2}(0)$, we find that any element $u_0 \in \mathcal{I}_T^{[AB]}(\omega^T)$ satisfies

$$\int_{\vartheta_{x_1}(0)}^{\vartheta_{x_2}(0)} u_0(x)\,\mathrm{d}x \ge \int_{\vartheta_{x_1}(0)}^{\vartheta_{x_2}(0)} u_0^*(x)\,\mathrm{d}x\,. \tag{3.11}$$

The inequalities (3.10), (3.11) together imply that every element $u_0 \in \mathcal{I}_T^{[AB]}(\omega^T)$ satisfies

$$\int_{\vartheta_{x_1}(0)}^{\vartheta_{x_2}(0)} u_0(x)\,\mathrm{d}x = \int_{\vartheta_{x_1}(0)}^{\vartheta_{x_2}(0)} u_0^*(x)\,\mathrm{d}x. \tag{3.12}$$

Then, letting $G : \mathbf{L}^\infty(\mathbb{R}) \to \mathbb{R}$ be the linear map defined by $G(u_0) = \int_{\vartheta_{x_1}(0)}^{\vartheta_{x_2}(0)} u_0(x)\,\mathrm{d}x$, we deduce from (3.12) that

$$\mathcal{I}_T^{[AB]}(\omega^T) \subset \{u_0 \in \mathbf{L}^\infty(\mathbb{R}) \mid G(u_0) = G(u_0^*),\}$$

which shows that $\mathcal{I}_T^{[AB]}(\omega^T)$ has an empty interior since is is contained in an hyperplane of $\mathbf{L}^\infty(\mathbb{R})$).

The second main contribution of this paper establishes some structural and geometrical properties of the set $\mathcal{I}_T^{[AB]}(\omega^T)$.

Theorem 3.6. *Given $\omega^T \in \mathcal{A}^{[AB]}(T)$, with the same notations of Theorem 3.4 the following properties hold.*

(i) The set $\mathcal{I}_T^{[AB]}(\omega^T)$ reduces to the singleton $\{u_0^*\}$ if and only if $|\mathcal{C}_0(u^*, x)| = 1$ for every $x \in \mathbb{R}$. In particular, if $\mathcal{I}_T^{[AB]}(\omega^T) = \{u_0^*\}$ then ω^T is continuous on $\mathbb{R} \setminus \{0\}$.

(ii) The set $\mathcal{I}_T^{[AB]}(\omega^T)$ is an affine cone having u_0^* as its vertex (i.e. the set $\mathcal{I}_T^{[AB]}(\omega^T) - u_0^*$ is a linear cone). Moreover, u_0^* is the unique extremal point of the set $\mathcal{I}_T^{[AB]}(\omega^T)$.

(iii) If, setting

$$\begin{aligned} \mathsf{L} &\doteq \mathsf{L}[\omega^T, f_l] \doteq \sup\big\{L < 0 \;:\; x - T\cdot f_l'(\omega^T(x)) \le 0 \quad \forall\, x \le L\big\},\\ \mathsf{R} &\doteq \mathsf{R}[\omega^T, f_r] \doteq \inf\big\{R > 0 \;:\; x - T\cdot f_r'(\omega^T(x)) \ge 0 \quad \forall\, x \ge R\big\}, \end{aligned} \tag{3.13}$$

and

$$\mathcal{X} \doteq \mathcal{X}(\omega^T) \doteq \Big\{x \in \mathbb{R} \;\Big|\; |\mathcal{C}_0(u^*, x)| = 1\Big\}, \tag{3.14}$$

for every point $\overline{x} \in]\mathsf{L}, \mathsf{R}[$ of continuity of ω^T there holds

$$\mathcal{C}_0(u^*, \bar{x}) \cap \mathrm{cl}\left(\bigcup_{x \in \mathcal{X}} \mathcal{C}_0(u^*, x)\right) \neq \emptyset, \tag{3.15}$$

then, the set $\mathcal{I}_T^{[AB]}(\omega^T)$ is convex.

Theorem 3.4, together with Theorem 3.6-(ii) and Example in § 6.1 (showing that the set $\mathcal{I}_T^{[AB]}(\omega^T)$ can well be non convex if condition (3.15) is not verified), yield Theorem 1.1 stated in the Introduction.

Remark 3.7. Note that the stronger condition

$$|\mathcal{C}_0(u^*, \overline{x})| = 1 \qquad \text{for every point } \overline{x} \in]\mathsf{L}, \mathsf{R}[\text{ of continuity of } \omega^T, \tag{3.16}$$

clearly implies (3.15) and thus ensures the convexity of $\mathcal{I}_T^{[AB]}(\omega^T)$. Actually, we will first show that the set $\mathcal{I}_T^{[AB]}(\omega^T)$ is convex under condition (3.16). Next, we will extend the result to the case where (3.15) is verified at every point $\overline{x} \in]\mathsf{L}, \mathsf{R}[$ of continuity of ω^T.

Remark 3.8. In [3, Theorems 4.2, 4.7, 4.9] it is shown that the attainable set $\mathcal{A}^{[AB]}(T)$ can be partitioned in classes of attainable profiles ω^T which depend on the quantities L, R defined in (3.13) and on the relative positions of $f_l'(A)/T$ with respect to L, or of $f_r'(B)/T$ with respect to R. These classes of profiles do not provide a finer partition than the one given by the two sets

$$\big\{\omega \in \mathcal{A}^{[AB]}(T) \mid \mathcal{I}_T^{[AB]}(\omega) \ \text{ is convex}\big\}, \qquad \big\{\omega \in \mathcal{A}^{[AB]}(T) \mid \mathcal{I}_T^{[AB]}(\omega) \ \text{ is not convex}\big\}.$$

In fact, there are profiles $\omega_2, \omega_3 \in \mathcal{A}^{[AB]}(T)$ that belong to one same class of attainable profiles described in [3], but such that $\mathcal{I}_T^{[AB]}(\omega_2)$ is convex while $\mathcal{I}_T^{[AB]}(\omega_3)$ is not convex. For example, we consider the profile defined in § 3.2, but replacing p with $\boldsymbol{v}$, i.e. setting

$$\omega_2(x) = \begin{cases} \boldsymbol{v} & x < L_0, \\ A & x \in]\, L_0, 0[\,, \\ \overline{B} & x > 0. \end{cases} \tag{3.17}$$

As observed in § 3.2 we have $\mathsf{R} = \mathsf{R}[\omega_2, f_r] = 0$, and $\mathsf{L} = \mathsf{L}[\omega_2, f_l] = L_0$. One can readly verify that ω_2 fulfills the conditions (i)-(ii) of [3, Theorem 4.7], as does the profile ω_3 in (6.21) considered in Example of § 6.1. We will show in § 6.1 that the set of initial data $\mathcal{I}_T^{[AB]}(\omega_3)$ is not convex. On the other hand, we will see here that, setting

$$u_0^* \doteq \mathcal{S}_T^{[AB]-}\omega_2, \qquad u^*(\cdot, t) \doteq \mathcal{S}_t^{[AB]+}u_0^* \qquad \forall\, t \in [0, T]\,, \tag{3.18}$$

at every point $\overline{x} \in]L_0, 0[$ there holds (3.15). Thus, the set $\mathcal{I}_T^{[AB]}(\omega_2)$ is convex because of Theorem 3.6-(iii).

In order to determine the sets $\mathcal{C}_0(u^*, x)$, $x \in \mathbb{R}$ (and then check (3.15)), we construct explicitly the AB-entropy solution u^* defined in (3.18), following the procedure described in [3, § 5.4]. Namely, because of condition (3.6) the solution u^* contains a shock curve starting at the interface $x = 0$, and then lying in the semiplane $\{x < 0\}$, which reaches the point $x = L_0$ at the time T. In fact, according with the analysis in [3, § 3.5], there exist a constant $\boldsymbol{\sigma} \doteq \boldsymbol{\sigma}[\,L_0, A, f_l]$, and a map $\gamma : [\boldsymbol{\sigma}, T] \to]-\infty, 0]$, with the properties that $\gamma(\boldsymbol{\sigma}) = 0$, $\gamma(T) = L_0$, and that $t \to (\gamma(t), t)$ is a shock curve for the conservation law $u_t + f_l(u)_x = 0$, which connects the left states $(f_l')^{-1}\big(\big(\gamma(t) - L_0 + T \cdot f_l'(\boldsymbol{v})\big)/t\big)$, $t \in [\boldsymbol{\sigma}, T]$,

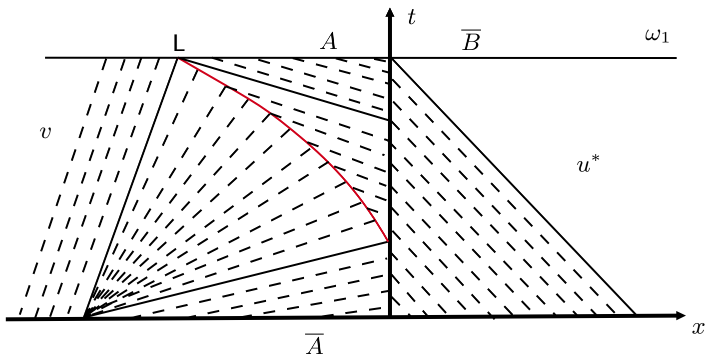

FIGURE 8. The profile ω_1 with its solution u^* constructed in Remark 3.8

with the right state A. On the left of $\gamma(t)$ there is a rarefaction wave, connecting the left state $\boldsymbol{v}$ with the right state $\overline{A}$, and centered at the point $(L_0 - T \cdot f_l'(\boldsymbol{v}), 0)$. Moreover, there holds

$$\boldsymbol{\sigma} = \frac{T \cdot f_l'(\boldsymbol{v}) - L_0}{f_l'(\overline{A})}. \tag{3.19}$$

Then, setting

$$\eta_-(t) \doteq L_0 - (T-t) \cdot f_l'(\boldsymbol{v}), \quad t \in [0,T],$$

$$\eta_+(t) \doteq L_0 - T \cdot f_l'(\boldsymbol{v}) + t \cdot f_l'(\overline{A}), \quad t \in [0,\boldsymbol{\sigma}],$$

we find that the function u^* in (3.18) is given by (see Figure 8)

$$u^*(x,t) = \begin{cases} \boldsymbol{v} & \text{if} \quad x < \eta_-(t), \quad t \in [0,T], \\ (f_l')^{-1}\Big(\dfrac{x - L_0 + T \cdot f_l'(\boldsymbol{v})}{t}\Big) & \text{if} \quad \begin{cases} \eta_-(t) < x < \gamma(t), \quad t \in [\boldsymbol{\sigma}, T], \\ \eta_-(t) < x < \eta_+(t), \quad t \in]0, \boldsymbol{\sigma}], \end{cases} \\ A & \text{if} \quad \gamma(t) < x < 0, \quad t \in [\boldsymbol{\sigma}, T], \\ \overline{A} & \text{if} \quad \eta_+(t) < x < 0, \quad t \in [0,\boldsymbol{\sigma}], \\ \overline{B} & \text{if} \quad x > 0, \quad t \in [0,T]. \end{cases} \tag{3.20}$$

Observe that every AB-gic for u^* that reaches a point $x \in]L_0, 0[$ at time T has either the expression

$$\eta_{\tau_1}(t) = \begin{cases} x - (T-t) \cdot f_l'(A) & \text{if} \quad t \in [\tau_2, T], \\ 0 & \text{if} \quad t \in [\tau_1, \tau_2], \\ (t - \tau_1) \cdot f_r'(\overline{B}) & \text{if} \quad t \in [0, \tau_1], \end{cases}$$

with $\tau_2 \doteq T - x/f_l'(A)$, and $\tau_1 \in [0, \tau_2]$, or the expression

$$\widetilde{\eta}_{\widetilde{\tau}_1}(t) = \begin{cases} x - (T-t) \cdot f_l'(A) & \text{if} \quad t \in [\tau_2, T], \\ 0 & \text{if} \quad t \in [\widetilde{\tau}_1, \tau_2], \\ (t - \widetilde{\tau}_1) \cdot f_l'(\overline{A}) & \text{if} \quad t \in [0, \widetilde{\tau}_1], \end{cases}$$

with τ_2 as above and $\widetilde{\tau}_1 \in [0, \boldsymbol{\sigma}]$. By definition (3.2) this means that

$$\mathcal{C}(u^*, x) = \big\{ \eta_{\tau_1} \mid \tau_1 \in [0, \tau_2] \big\} \cup \big\{ \widetilde{\eta}_{\widetilde{\tau}_1} \mid \widetilde{\tau}_1 \in [0, \boldsymbol{\sigma}] \big\}.$$

Since we have

$$\big\{\eta_{\tau_1}(0)\,|\,\tau_1\in[0,\tau_2]\big\}=\big[0,\,(x/f_l'(A)-T)\cdot f_r'(\overline{B})\big],$$

$$\big\{\widetilde{\eta}_{\widetilde{\tau}_1}(0)\,|\,\widetilde{\tau}_1\in[0,\,\boldsymbol{\sigma}]\big\}=\big[-\boldsymbol{\sigma}\cdot f_l'(\overline{A}),0\big],$$

by definition (3.3) and by virtue of (3.19) we then find that

$$\mathcal{C}_0(u^*,x)=\big[\,L_0-T\cdot f_l'(\boldsymbol{v}),\,(x/f_l'(A)-T)\cdot f_r'(\overline{B})\big]\qquad\forall\,x\in\,]\,L_0,0[\,. \tag{3.21}$$

On the other hand, since ω_2 is constant for $x<L_0$, there exists a unique AB-gic for u^* that reaches a point $x<L_0$ at time T, which is a classical genuine characteristic

$$\vartheta_x(t)=x-(T-t)\cdot f_l'(\boldsymbol{v}),\qquad t\in[0,T],$$

because it never crosses the interface $x=0$. Hence, we have

$$\mathcal{C}_0(u^*,x)=\{x-T\cdot f_l'(\boldsymbol{v})\}\qquad\forall\,x<L_0\,. \tag{3.22}$$

Therefore, from (3.21), (3.22), we deduce

$$\mathcal{C}_0(u^*,\overline{x})\cap\mathrm{cl}\left(\bigcup_{x<L_0}\mathcal{C}_0(u^*,x)\right)=\big\{\,L_0-T\cdot f_l'(\boldsymbol{v})\big\}\qquad\forall\,\overline{x}\in\,]\,L_0,0[\,,$$

which proves (3.15), and thus concludes the proof of the convexity of $\mathcal{I}_T^{[AB]}(\omega_2)$.

4. Properties of genuine/interface characteristics

In this section we establish some basic properties enjoyed by the AB-genuine/interface characteristics for an AB-entropy solution u, and by the sets $\mathcal{C}(u,x)$, $\mathcal{C}_0(u,x)$, introduced in § 3.

Proposition 4.1. *Let u be an AB-entropy solution to (1.1). Then the following properties hold.*

(i) $\mathcal{C}(u,x)\neq\emptyset$ *for all* $x\in\mathbb{R}$*;*
(ii) the map $x\mapsto\mathcal{C}(u,x)$ *has closed graph as a set-valued map from* $\mathbb{R}$ *into the power set of the space* $\mathrm{Lip}([0,T]\,;\mathbb{R})$ *with the topology of uniform convergence;*
(iii) the map $x\mapsto\mathcal{C}_0(u,x)$ *has closed graph as a set-valued map from* $\mathbb{R}$ *into the power set of* $\mathbb{R}$*;*
(iv) the maps $x\mapsto\min\mathcal{C}_0(u,x)$, $x\mapsto\max\mathcal{C}_0(u,x)$ *are monotone nondecreasing.*

Proof. Throughout the proof we set $\omega^T(x)\doteq u(x,T)$, $x\in\mathbb{R}$, and we let $u_l(t),u_r(t)$ denote the one-sided traces of $u(t,\cdot)$ at $x=0$.

1. Proof of (i). Given $x>0$, consider the minimal backward characteristic for the conservation law $u_t+f_r(u)_x=0$, in the semiplane $\{x>0\}$, starting from (x,T), defined by $\vartheta_{x,-}(t)=x-(T-t)\cdot f_r'(\omega^T(x-))$. If $x-T\cdot f_r'(\omega^T(x-))\geq 0$, then $\vartheta_{x,-}$ is a classical genuine characteristic for u in the whole interval $[0,T]$, since it never crosses the interface $x=0$ but at most at $t=0$. Therefore, according with Definition 3.1, the map

$$\zeta(t)=x-(T-t)\cdot f_r'(\omega^T(x-)),\qquad t\in[0,T],$$

is an AB-genuine/interface characteristic, and hence by (3.2) it holds $\zeta\in C(u,x)$. Otherwise, we have $x-T\cdot f_r'(\omega^T(x-))<0$, and thus $\vartheta_{x,-}$ impacts the interface at the time:

$$\tau_-(x)\doteq T-\frac{x}{f_r'(\omega^T(x-))}>0. \tag{4.1}$$

Then, consider the set

$$E \doteq \Big\{ t \in [0, \tau_-(x)] \;\Big|\; \text{either } u_l(t) > \theta_l \;\text{ or }\; u_r(t) < \theta_r \Big\}, \tag{4.2}$$

and let

$$\overline{\tau} \doteq \sup E, \tag{4.3}$$

where we understand that $\overline{\tau} = 0$ when $E = \emptyset$. Because of the non-intersection property of classical genuine characteristics in the domains $\{x < 0\}, \{x > 0\}$, and since uniform limit of classical genuine characteristics is a classical genuine characteristic as well (e.g. cfr. [3, proof o Lemma C.1]), we deduce that

$$\overline{\tau} \in E \qquad \text{if} \qquad E \neq \emptyset. \tag{4.4}$$

Thus, when $E \neq \emptyset$, if $u_l(\overline{\tau}) > \theta_l$ we can consider the minimal backward characteristic for the conservation law $u_t + f_l(u)_x = 0$, in the semiplane $\{x < 0\}$, starting from $(0, \overline{\tau})$, defined by $\vartheta_{\overline{\tau},-}(t) = (t - \overline{\tau}) \cdot f_l'(u_l(\overline{\tau}))$. Otherwise, if $u_r(\overline{\tau}) < \theta_r$ we can consider the maximal backward characteristic for the conservation law $u_t + f_r(u)_x = 0$, in the semiplane $\{x > 0\}$, starting from $(0, \overline{\tau})$, defined by $\vartheta_{\overline{\tau},+}(t) = (t - \overline{\tau}) \cdot f_r'(u_r(\overline{\tau}))$. On the other hand, by definition of E, and recalling the interface condition (2.5), we find that

$$u_l(t) = A, \quad u_r(t) = B \qquad \forall\, t \in \,]\,\overline{\tau}, \tau_-(x)]. \tag{4.5}$$

Note in particular that

$$\overline{\tau} < \tau_-(x) \quad \Longrightarrow \quad u_r(\tau_-(x)) = \omega^T(x-) = B. \tag{4.6}$$

Therefore, the piecewise affine map

$$\zeta(t) = \begin{cases} x - (T - t) \cdot f_r'(\omega^T(x-)), & t \in [\tau_-(x), T], \\ 0, & t \in]\overline{\tau}, \tau_-(x)[, \\ (t - \overline{\tau}) \cdot f_l'(u_l(\overline{\tau})), & t \in [0, \overline{\tau}], \quad \text{if } u_l(\overline{\tau}) > \theta_l, \\ (t - \overline{\tau}) \cdot f_r'(u_r(\overline{\tau})), & t \in [0, \overline{\tau}], \quad \text{if } u_r(\overline{\tau}) < \theta_r, \end{cases} \tag{4.7}$$

satisfy the conditions of Definition 3.1, and thus it is an AB-gic belonging to the set $\mathcal{C}(u, x)$. Note that it may well happen that $\overline{\tau} = \tau_-(x)$, in which case there will be in (4.7) no nontrivial interval where the characteristic is travelling along the interface $x = 0$. Instead, in the case $\overline{\tau} = 0$, the AB-gic in (4.7) lies on the interface $x = 0$ in the whole interval $[0, \tau_-(x)]$.

Clearly, the same analysis can be carried out to show that $\mathcal{C}(u, x) \neq \emptyset$ also for $x < 0$. It remains to consider the case $x = 0$. Notice that this case would follow from (ii) and from (i) for $x \neq 0$, however for clarity we write the construction explicitly. If we assume that $\omega^T(0-) > \theta_l$, then the minimal backward characteristic for $u_t + f_l(u)_x = 0$, in the semiplane $\{x < 0\}$, starting from $(0, T)$, is a classical genuine characteristic for u in the whole interval $]0, T]$, and hence it it is an AB-gic belonging to the set $\mathcal{C}(u, 0)$. Similarly, if $\omega^T(0+) < \theta_r$, then the maximal backward characteristic for $u_t + f_r(u)_x = 0$, in the semiplane $\{x > 0\}$, starting from $(0, T)$, is a classical genuine characteristic for u in the whole interval $[0, T]$, and hence it is an AB-gic belonging to the set $\mathcal{C}(u, 0)$. Finally, if $\omega^T(0-) \leq \theta_l$ and $\omega^T(0+) \geq \theta_r$, by the interface condition (2.5), we deduce that $\omega^T(0-) = A$, $\omega^T(0+) = B$. Then, set

$$\overline{\tau} = \sup E, \qquad E \doteq \Big\{ t \in [0, T] \;\Big|\; \text{either } u_l(t) > \theta_l \;\text{ or }\; u_r(t) < \theta_r \Big\}.$$

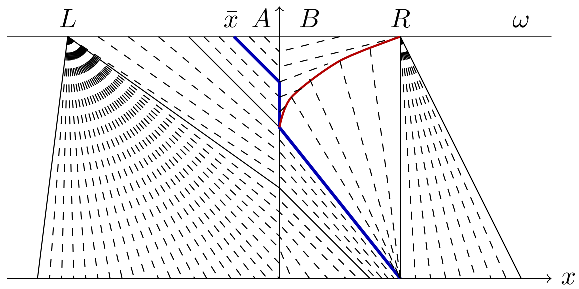

Figure 9. There are no backward generalized characteristics with time of existence $[0,T]$ from the point $\bar{x}$ which reach time $t=0$ if we use the classical definition. If, instead, we consider elements in $\mathcal{C}(u,\bar{x})$ (the blue line), we see that, also if at the time at which the characteristic reaches the interface it cannot be prolonged on the other side in classical sense, there is at least an element in $\mathcal{C}(u,\bar{x})$ that is defined on the whole $[0,T]$.

With the same type of analysis as above we find that $\overline{\tau} \in E$ and that the map

$$\zeta(t) = \begin{cases} 0, & t \in \,]\overline{\tau}, T], \\ (t-\overline{\tau}) \cdot f_l'(u_l(\overline{\tau})), & t \in [0,\overline{\tau}], \quad \text{if } u_l(\overline{\tau}) > \theta_l, \\ (t-\overline{\tau}) \cdot f_r'(u_r(\overline{\tau})), & t \in [0,\overline{\tau}], \quad \text{if } u_r(\overline{\tau}) < \theta_r, \end{cases} \tag{4.8}$$

is an AB-gic belonging to the set $\mathcal{C}(u,0)$, thus completing the proof of (i).

2. Proof of (ii). The closed graph property of the map $x \mapsto \mathcal{C}(u,x)$ is equivalent to

$$\Big(x_n \to x, \quad \zeta_n \in \mathcal{C}(u,x_n), \quad \zeta_n \to \zeta \quad \text{uniformly}\Big) \quad \Longrightarrow \quad \zeta \in \mathcal{C}(u,x). \tag{4.9}$$

Then, let $\{x_n\}_n$ be a sequence converging to $x \geq 0$, and consider a sequence of AB-gic $\zeta_n \in \mathcal{C}(u,x_n)$, that converge uniformly to some $\zeta \in \mathrm{Lip}([0,T]\ ;\mathbb{R})$. By Remark 3.2, for every n there will be $0 \leq \tau_{1,n} \leq \tau_{2,n} \leq T$, such that

$$\zeta_n(t) = 0, \qquad f_l(u_l(t)) = f_r(u_r(t)) = \gamma \qquad \forall\, t \in [\tau_{1,n},\, \tau_{2,n}], \tag{4.10}$$

and such that the restriction of ζ_n to $]0,\tau_{1,n}[$ and to $]\tau_{2,n},T[$ is either a classical genuine characteristic for $u_t + f_l(u)_x = 0$ on $\{x<0\}$, or it is a classical genuine characteristic for $u_t + f_r(u)_x = 0$ on $\{x>0\}$. This, in particular, implies that

$$\dot{\zeta}_n(t) = \begin{cases} \dfrac{x_n}{T-\tau_{2,n}} & \forall\, t \in \,]\tau_{2,n}, T[\,, \\[2ex] -\dfrac{\zeta_n(0)}{\tau_{1,n}} & \forall\, t \in \,]0, \tau_{1,n}[\,. \end{cases} \tag{4.11}$$

Possibly considering a subsequence we can assume that $\{\tau_{i,n}\}_n$ converge to some $\tau_i \in [0,T]$, $i=1,2$, with $\tau_1 \leq \tau_2$. Suppose that $\tau_1 > 0$, $\tau_2 < T$. The cases where $\tau_1 = 0$, or/and $\tau_2 = T$ can be treated with entirely similar and simpler arguments. Up to extracting a further subsequence we may also assume that $x_n > 0$ for all n, and that

$$\zeta_n(t) < 0 \qquad \forall\, t \in [0,\tau_{1,n}[\,, \qquad\qquad \zeta_n(t) > 0 \qquad \forall\, t \in \,]\tau_{2,n},\, T], \quad \forall\, n\,. \tag{4.12}$$

Again, the cases where $\zeta_n(t) > 0$ for all $t \in [0,\tau_{1,n}[\,$, or/and $x_n < 0$, $\zeta_n(t) < 0$ for all $t \in \,]\tau_{2,n},\, T]$, can be analyzed in an entirely similar way. By the uniform convergence of ζ_n

to ζ and since $\tau_{i,n} \to \tau_i$, $i = 1, 2$, it follows from (4.10) that

$$\zeta(t) = 0, \qquad f_l(u_l(t)) = f_r(u_r(t)) = \gamma \qquad \forall\, t \in]\tau_1,\, \tau_2[\,. \tag{4.13}$$

and

$$\zeta(t) \leq 0 \quad \forall\, t \in [0, \tau_1], \qquad\qquad \zeta(t) \geq 0 \quad \forall\, t \in [\tau_2,\, T]. \tag{4.14}$$

Moreover, we have

$$\zeta(T) = x, \tag{4.15}$$

since $x_n \to x$ and $x_n = \zeta_n(T) \to \zeta(T)$. Note also that, because of (4.11), there holds

$$\dot\zeta(t) = \lim_n \dot\zeta_n(t) = \begin{cases} \dfrac{x}{T - \tau_2} & \forall\, t \in]\tau_2, T[\,, \\[2ex] -\dfrac{\zeta(0)}{\tau_1} & \forall\, t \in]0, \tau_1[\,. \end{cases} \tag{4.16}$$

Now, if we assume that $x > 0$, it follows from (4.16) that that $\zeta(t) > 0$ for all $t \in]\tau_2, T]$. On the other hand, since uniform limit of classical genuine characteristics is a classical genuine characteristic as well, we deduce that the restriction of ζ to $]\tau_2, T]$ is a classical genuine characteristic for $u_t + f_r(u)_x$.

Next, if we assume that $x = 0$, then the uniform convergence of ζ_n to ζ, together with (4.12), (4.16), imply that

$$\zeta(t) = \dot\zeta(t) = 0 \qquad \forall\, t \in]\tau_2, T[\,, \tag{4.17}$$

and

$$f_r'(u_r(t)) = \lim_n f_r'(u(\zeta_n(t), t)) = \lim_n \dot\zeta_n(t) = 0 \qquad \forall\, t \in]\tau_2, T[\,. \tag{4.18}$$

In turn, (4.18) implies that $u_r(t) = \theta_r = B$ for all $t \in]\tau_2, T[$, and that (A, B) is a critical connection. On the other hand, because of the interface condition (2.5), it follows that $f_l(u_l(t)) = f_r(u_r(t)) = \gamma$ for all $t \in]\tau_2, T[$, which proves that the restriction of ζ to the interval $[\tau_2, T]$ satisfies the condition (ii) of Definition 3.1. With entirely similar arguments one can show that the restriction of ζ to the interval $[0, \tau_1]$ satisfies the condition (i) or (ii) of Definition 3.1, which, together with (4.13), completes the proof that ζ is an AB-gic belonging to the set $\mathcal{C}(u, x)$. This completes the proof of (4.9) whenever $\{x_n\}_n$ is a sequence converging to $x \geq 0$. The case where the limit point x of $\{x_n\}_n$ is non positive can be treated in entirely similar way.

3. Proof of (iii). Let $\{x_n\}_n$ be a sequence converging to $x \in \mathbb{R}$, and let $\{y_n\}_n$ be a sequence of elements of $\mathcal{C}_0(u, x_n)$ converging to some point $y \in \mathbb{R}$. Then, there will be a sequence of AB-gic $\zeta_n \in \mathcal{C}(u, x_n)$, such that $y_n = \zeta_n(0)$ for all n. Observe that by Definition 3.1 it follows that

$$|\zeta_n(t)| \leq |x_n| + LT, \qquad |\dot\zeta_n(t)| \leq L, \qquad \forall\, t \in [0, T], \quad \forall\, n\,, \tag{4.19}$$

for some constant $L > 0$ depending on $\|u\|_{\mathbf{L}^\infty}$. Hence, applying Ascoli-Arzelà Theorem, we deduce that up to a subsequence $\{\zeta_n\}_n$ converges uniformly to some $\zeta \in \mathrm{Lip}([0, T]\,; \mathbb{R})$. Thus, in particular we have

$$\zeta(0) = \lim_m \zeta_n(0) = \lim_n y_n = y. \tag{4.20}$$

Then, in view of property (ii) established at previous point, we find that $\zeta \in \mathcal{C}(u, x)$, and (4.20) implies $y \in \mathcal{C}_0(u, x)$, completing the proof of (iii).

4. Proof of (iv). Given $x_1 < x_2$, let $y_1 = \max \mathcal{C}_0(u, x_1)$, and consider $\zeta_1 \in \mathcal{C}(u, x_1)$ such that $\zeta_1(0) = y_1$. Choose any $\zeta_2 \in \mathcal{C}(u, x_2)$ and define

$$\zeta(t) \doteq \max\{\zeta_1(t),\, \zeta_2(t)\} \qquad t \in [0, T].$$

Observe that, by definition the maximum of two AB-gic is still an AB-gic, and $\zeta(T) = x_2$, so that one has $\zeta \in (u, x_2)$. Moreover:

$$\max \mathcal{C}_0(u, x_1) = y_1 = \zeta_1(0) \leq \zeta(0) \leq \max \mathcal{C}_0(u, x_2).$$

This proves (iv), and thus concludes the proof of the proposition. □

The next Proposition states that the AB-entropy solution u^* defined in (1.11) has always at least an AB-gic in common with every AB-entropy solutions u satisfying $u(\cdot\,, T) = u^*(\cdot\,, T)$.

Proposition 4.2. *Given $\omega^T \in \mathcal{A}^{[AB]}(T)$, let u^* be the AB-entropy solution defined by (1.10)-(1.11), and let u be any other AB-entropy solution to (1.1) with initial datum $u_0 \in \mathcal{I}_T^{[AB]}(\omega^T)$. Then, there holds*

$$\mathcal{C}(u^*, x) \cap \mathcal{C}(u, x) \neq \emptyset \qquad \forall\ x \in \mathbb{R}\,. \tag{4.21}$$

Proof. To fix the ideas, we will assume that, letting L, R be the quantities defined in (3.13), there holds $\mathsf{L} = 0$, $\mathsf{R} \in \,]0, T\cdot f_r'(B)[\,$, and that ω^T fulfills the conditions (i)'-(ii)' of [3, Theorem 4.7] for a non critical connection (A, B), which in particular require

$$\omega^T(x-) \geq \omega^T(x+) \qquad \forall\ x \neq 0, \tag{4.22}$$

$$\omega^T(x) \geq B \qquad\qquad \forall\ x \in \,]0, \mathsf{R}[\,. \tag{4.23}$$

The cases where ω^T belongs to other classes of reachable profiles described in [3, Theorems 4.2, 4.7, 4.9] can be analyzed with entirely similar arguments. Throughout the proof we let $u_l(t), u_r(t)$, and $u_l^*(t), u_r^*(t)$, denote the one-sided traces of $u(t, \cdot)$ and $u^*(t, \cdot)$, respectively, at $x = 0$.

1. Relying on the fact that any sequence $\{\zeta_n\}_n$ of AB-gic (for u^* and u) admits a subsequence uniformly convergent to some $\zeta \in \mathrm{Lip}([0, T]\ ;\mathbb{R})$ (see point **3.** of the proof of Proposition 4.1), and since the map $x \to \mathcal{C}(u^*, x) \cap \mathcal{C}(u, x)$ has closed graph by Proposition 4.1-(ii), it will be sufficient to show that $\mathcal{C}(u^*, x) \cap \mathcal{C}(u, x) \neq \emptyset$ holds for all point x of continuity for ω^T. Moreover, for every point $x \in \,]-\infty, 0[\, \cup \,]\mathsf{R}, +\infty[\,$ of continuity for ω^T, there exists a unique AB-gic for u^* and u that reaches the point x at time T, which is a classical genuine characteristic ϑ_x for u^* and u (since it never crosses the interface $x = 0$, but at most at $t = 0$, by definition (3.13)). Thus we have $\mathcal{C}(u^*, x) \cap \mathcal{C}(u, x) = \{\vartheta_x\}$ for all point $x \in \,]-\infty, 0[\, \cup \,]\mathsf{R}, +\infty[\,$ of continuity for ω^T. As a consequence, in order to establish (4.21) it will be sufficient to show

$$\mathcal{C}(u^*, x) \cap \mathcal{C}(u, x) \neq \emptyset \qquad \text{for all } x \in \,]0, \mathsf{R}[\, \text{ of continuity for } \omega^T. \tag{4.24}$$

To this end, given any $x \in \,]0, \mathsf{R}[\,$ of continuity for ω^T, we consider the AB-gic $\zeta \in \mathcal{C}(u, x)$ defined in (4.7), with $\overline{\tau}$ as in (4.3) and

$$\tau(x) \doteq T - \frac{x}{f_r'(\omega^T(x))}, \tag{4.25}$$

in place of $\tau_-(x)$. We will show that ζ also belongs to $\mathcal{C}(u^*, x)$. Note that by definition of R at (3.13) we have $\tau(x) > 0$.

2. We determine here explicitly the AB-entropy solution u^* defined by (1.10)-(1.11) when ω^T satisfies the conditions (i)'-(ii)' of [3, Theorem 4.7] for a non critical connection, with $\mathsf{L}=0$, $\mathsf{R}\in\,]0, T\cdot f_r'(B)[\,$. These conditions require in particular that

$$\omega^T(0-)\geq \pi(\omega^T(0+)), \tag{4.26}$$

$$\omega^T(x)\geq B, \qquad \forall\, x\in\,]0,\mathsf{R}[\,, \tag{4.27}$$

$$\omega^T(\mathsf{R}+)\leq \boldsymbol{u}\,, \tag{4.28}$$

where

$$\pi(u)\doteq \big(f_{l\,|[\theta_l,+\infty)}\big)^{-1}\circ f_r(u), \qquad u\in\mathbb{R}, \tag{4.29}$$

and $\boldsymbol{u}\doteq\boldsymbol{u}[\mathsf{R},B,f_r]$ is the quantity defined in [3, § 3.1] that satisfies

$$B>\boldsymbol{u}>\overline{B}, \qquad\qquad f_r'(\boldsymbol{u})<\mathsf{R}/T, \tag{4.30}$$

(with $\overline{B}$ defined as in (2.7)). Because of condition (4.27), according with the analysis in [3, § 5.4] the solution u^* contains a shock curve starting at the interface $x=0$ and reaching the point R at time T, which is parametrized by a map $\gamma:[\boldsymbol{\tau},T]\to[0,\infty[\,$, with the properties that $\gamma(\boldsymbol{\tau})=0$, $\gamma(T)=\mathsf{R}$, where $\boldsymbol{\tau}\doteq\boldsymbol{\tau}[\mathsf{R},B,f_r]$ is a quantity defined as in [3, § 3.4]. The curve $t\to(\gamma(t),t)$ is the location of a shock for the conservation law $u_t+f_r(u)_x=0$, which connects the left state B with the right states $(f_r')^{-1}\big(\big(\gamma(t)-\mathsf{R}+T\cdot f_r'(\boldsymbol{u})\big)/t\big)$, $t\in[\boldsymbol{\tau},T]$. On the right of $\gamma(t)$ there is a rarefaction wave, connecting the left state $\overline{B}$ with the right state $\boldsymbol{u}$, and centered at the point $(\mathsf{R}-T\cdot f_r'(\boldsymbol{u}),0)$. Moreover, there holds

$$\boldsymbol{\tau}=\frac{T\cdot f_r'(\boldsymbol{u})-\mathsf{R}}{f_r'(\overline{B})}. \tag{4.31}$$

Following the procedure described in [3, § 5.4], in order to define u^* we introduce some notations for the polygonal lines along which u^* takes constant values in each region $\{x<0\}$, $\{x>0\}$ (that correspond to AB-gic for u^*). We define

$$\begin{aligned}
&\vartheta_{0,-}(t)\doteq(t-T)\cdot f_l'(\omega^T(0-)),\\
&\vartheta_{0,+}(t)\doteq(t-T)\cdot f_l'\big(\pi(\omega^T(0+))\big),\\
&\vartheta_{\mathrm{R},-}(t)\doteq\begin{cases}\mathsf{R}-(T-t)\cdot f_r'\big(\omega^T(\mathsf{R}-),& \text{if}\quad \tau_-(\mathsf{R})\leq t\leq T,\\ \big(t-\tau_-(\mathsf{R})\big)\cdot f_l'\circ\pi(\omega^T(\mathsf{R}-)),& \text{if}\quad 0\leq t\leq\tau_-(\mathsf{R}),\end{cases}\\
&\vartheta_{\mathrm{R},+}(t)\doteq\mathsf{R}-(T-t)\cdot f_r'\big(\omega^T(\mathsf{R}+)\big),
\end{aligned}\tag{4.32}$$

and, for every $y\in\,]-\infty,0\,[\,\cup\,]\mathsf{R},+\infty[\,$, we define

$$\vartheta_{y,\pm}(t)\doteq\begin{cases}y-(T-t)\cdot f_l'\big(\omega^T(y\pm)\big),& \text{if}\ \ y<0,\quad 0\leq t\leq T,\\ y-(T-t)\cdot f_r'\big(\omega^T(y\pm)\big),& \text{if}\ \ 0<y<\mathsf{R},\quad \tau_\pm(y)\leq t\leq T,\\ \big(t-\tau_\pm(y)\big)\cdot f_l'\circ\pi(\omega^T(y\pm)),& \text{if}\ \ 0<y<\mathsf{R},\quad 0\leq t<\tau_\pm(y),\\ y-(T-t)\cdot f_r'\big(\omega^T(y\pm)\big),& \text{if}\ \ y>\mathsf{R},\quad 0\leq t\leq T,\end{cases}\tag{4.33}$$

where

$$\tau_\pm(y)\doteq T-\frac{y}{f_r'(\omega^T(y\pm))}, \qquad y>0. \tag{4.34}$$

Moreover, letting $\{y_n\}_n$ denote the (at most) countably many discontinuity points of ω^T in the intervals $]-\infty,0]$, $]\mathsf{R},+\infty[$, we set

$$
\begin{aligned}
\mathcal{I}_0^n &=]x_n^-, x_n^+[\,, \quad x_n^\pm = \vartheta_{y_n,\pm}(0), \quad y_n \in]-\infty,0]\,,\\
\mathcal{I}_{\mathsf{R}}^n &=]x_n^-, x_n^+[\,, \quad x_n^\pm = \vartheta_{y_n,\pm}(0), \quad y_n \in]\mathsf{R},+\infty[\,,
\end{aligned}
$$

(here we consider the possibility of a jump of ω^T in $x=0$ when $\omega^T(0-) > \pi(\omega^T(0+))$). The intervals $\mathcal{I}_0^n$, $\mathcal{I}_{\mathsf{R}}^n$, consist of the starting points of compression waves in u^* that generate a shock at (y_n,T).

Next, we introduce the polygonal lines connecting two points $(z,0)$, (y,T) (that correspond to compression fronts for u^* generating a shock at the point (y,T)) defined by

$$
\eta_{y,z} \doteq \begin{cases}
y-(T-t)\cdot\frac{(y-z)}{T}, & \text{if} \quad y\in]-\infty,0]\cup]\mathrm{R},+\infty[\,, \quad 0\le t\le T,\\
y-(T-t)\cdot f_r'(u_{y,z}) & \text{if} \quad 0<y<\mathsf{R}, \quad T-y/f_r'(u_{yz})\le t\le T,\\
\big(t-T+y/f_r'(u_{yz})\big)\cdot f_l'\circ\pi(u_{y,z}) & \text{if} \quad 0<y<\mathsf{R}, \quad 0\le t< T-y/f_r'(u_{yz}),
\end{cases}
\tag{4.35}
$$

where $u_{y,z}$ is the unique constant $u \ge (f_r')^{-1}(y/T)$ satisfying

$$
\Big(\frac{y}{f_r'(u)} - T\Big)\cdot f_l'\circ\pi(u) = z
$$

(see [3, § 5.4.1]). Finally, we set

$$
\begin{aligned}
r_-(t) &\doteq \mathsf{R} - T\cdot f_r'(\boldsymbol{u}) + t\cdot f_r'(\overline{B}), \qquad t\in[0,\boldsymbol{\tau}],\\
r_+(t) &\doteq \mathsf{R} - (T-t)\cdot f_r'(\boldsymbol{u}), \qquad t\in[0,T].
\end{aligned}
$$

Then, the function u^* defined by (1.10)-(1.11) is given by

$$
u^*(x,t) = \begin{cases}
\omega^T(y\pm), & \text{if} \quad x=\vartheta_{y,\pm}(t) \quad \text{for some} \quad y\in]-\infty,0[\,\cup]\mathsf{R},+\infty[\,,\\
\omega^T(y\pm), & \text{if} \quad x=\vartheta_{y,\pm}(t)>0 \quad \text{for some} \quad y\in]0,\mathsf{R}[\,,\\
\pi(\omega^T(y\pm)), & \text{if} \quad x=\vartheta_{y,\pm}(t)<0 \quad \text{for some} \quad y\in]0,\mathsf{R}[\,,\\
(f_r')^{-1}\big(\frac{y_n-z}{T}\big), & \text{if} \quad x=\eta_{y_n,z}(t) \quad \text{for some} \quad z\in\mathcal{I}_{\mathsf{R}}^n,\\
(f_l')^{-1}\big(\frac{y_n-z}{T}\big), & \text{if} \quad x=\eta_{y_n,z}(t) \quad \text{for some} \quad z\in\mathcal{I}_0^n,\\
B & \text{if} \begin{cases}\vartheta_{\mathrm{R},-}(t)\le x<\gamma(t), & t\in[\tau_-(\mathsf{R}),T],\\ 0<x<\gamma(t), & t\in[\boldsymbol{\tau},\tau_-(\mathsf{R})],\end{cases}\\
\overline{A} & \text{if} \quad \vartheta_{\mathrm{R},-}(t)\le x<0, \quad t\in[0,\tau_-(\mathsf{R})],\\
\overline{B} & \text{if} \quad 0<x\le r_-(t), \quad t\in[0,\boldsymbol{\tau}],\\
(f_r')^{-1}\Big(\dfrac{x-\mathsf{R}+T\cdot f_r'(\boldsymbol{u})}{t}\Big) & \text{if} \begin{cases}\gamma(t)<x<r_+(t), & t\in[\boldsymbol{\tau},T],\\ r_-(t)<x<r_+(t), & t\in[0,\boldsymbol{\tau}],\end{cases}\\
(f_r')^{-1}\big(\frac{\mathsf{R}-x}{T-t}\big) & \text{if} \quad r_+(t)\le x\le\vartheta_{\mathrm{R},+}(t), \quad t\in[0,T[\,.
\end{cases}
\tag{4.36}
$$

Observe that the left and right traces of u^* satisfy

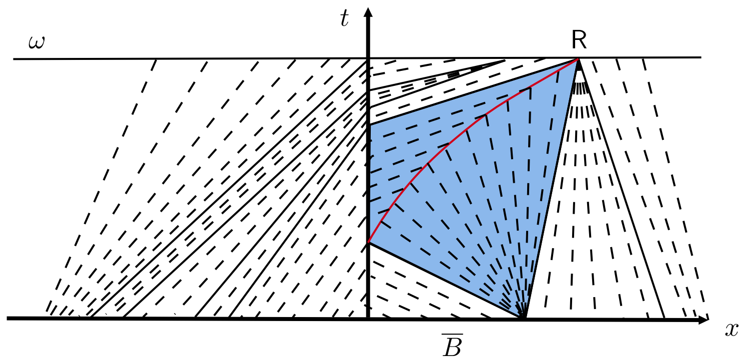

FIGURE 10. The solution u^*

$$\begin{aligned}
&u_l^*(t) \geq \overline{A}, && u_r^*(t) \geq B, && \forall\, t \in]\tau_-(\mathsf{R}), T],\\
&u_l^*(t) = \overline{A}, && u_r^*(t) = B, && \forall\, t \in]\boldsymbol{\tau}, \tau_-(\mathsf{R})],\\
&u_l^*(t) = \overline{A}, && u_r^*(t) = \overline{B}, && \forall\, t \in]0, \boldsymbol{\tau}].
\end{aligned} \tag{4.37}$$

Moreover, since x is a point of continuity for ω^T, it follows that the restriction of ζ to $]\tau(x), T]$ is a (classical) genuine characteristic both for u and u^* with slope $f_r'(\omega^T(x)) > 0$ so that, recalling (2.5), there holds

$$u_r(\tau(x)) = \omega^T(x) = u_r^*(\tau(x)) > \theta_r, \qquad f_l(u_l(\tau(x))) = f_r(u_r^*(\tau(x))). \tag{4.38}$$

Now, we will distinguish two cases according with the position of $\overline{\tau}$ with respect to the time $\tau_-(\mathsf{R})$ defined as in (4.34). Note that by definition of R at (3.13) we have $\tau_-(\mathsf{R}) \geq 0$.

3. Assume that $\overline{\tau} \geq \tau_-(\mathsf{R})$, and suppose first that $\tau(x) = \overline{\tau} > \tau_-(\mathsf{R})$. Note that, since $\overline{\tau} > 0$ is an element of the set E in (4.2), and because of (4.37), (4.38), we have $u_l(\tau(x)) = u_l^*(\tau(x)) > \theta_l$. Therefore, also the restriction of ζ to $[0, \tau(x)[$ is a (classical) genuine characteristic both for u and u^*. Hence, when $\tau(x) = \overline{\tau} > \tau_-(\mathsf{R})$, the map ζ in (4.7) is an AB-gic also for u^*, proving that $\zeta \in \mathcal{C}(u^*, x) \cap \mathcal{C}(u, x)$.

Next, consider the subcase $\tau(x) > \overline{\tau} \geq \tau_-(\mathsf{R})$, and observe that by (4.5), (4.6), (4.38), we have

$$u_l(t) = A, \quad u_r(t) = B \qquad \forall\, t \in]\,\overline{\tau}, \tau(x)], \qquad u_r^*(\tau(x)) = B\,. \tag{4.39}$$

Moreover, we claim that

$$\begin{aligned}
\omega^T(z) &= B, && \forall\, z \in [x, \overline{x}[\,, \quad \overline{x} \doteq (T - \overline{\tau}) \cdot f_r'(B),\\
u_r^*(t) &= B, && \forall\, t \in]\,\overline{\tau}, \tau(x)]\,.
\end{aligned} \tag{4.40}$$

Note that the first equality in (4.40) implies the second one by tracing the backward (genuine) characteristics for u^* at time T, from points $z \in [x, \overline{x}[$. In order to prove the first equality in (4.40), we trace the minimal backward characteristic $\vartheta_{z,-}$ for the solution u, at time T, from points $z \in [x, \overline{x}]$. Observe that $\vartheta_{z,-}$ impacts the interface $x = 0$ at time $\tau_-(z) \doteq T - z/f_r'(\omega^T(z-))$. Moreover, since $\omega^T(\mathsf{R}-) \geq B$ because of (4.23), we deduce from $\overline{\tau} \geq \tau_-(\mathsf{R})$ that

$$\overline{x} \leq \mathsf{R}. \tag{4.41}$$

Furthermore, we have

$$\tau_-(z) \geq T - \frac{z}{f_r'(B)} \geq \overline{\tau} \qquad \forall\, z \in [x, \overline{x}], \tag{4.42}$$

since $\omega^T(z-) \geq B$ by virtue of (4.23). On the other hand, by (4.25) we know that the (genuine) characteristic ϑ_x for u, starting at time T from the point x, reaches the interface $x = 0$ at time $\tau(x)$. Since ϑ_x, $\vartheta_{z,-}$ are (classical) genuine characteristics that cannot cross in the domain $\{x > 0\}$, it follows that

$$\tau(x) \geq \tau_-(z) \qquad \forall\, z \in [x,\, \overline{x}]\,. \tag{4.43}$$

Combining together (4.42), (4.43), we deduce that, for every $z \in [x,\, \overline{x}]$, the minimal backward characteristics $\vartheta_{z,-}$ reaches the interface $x = 0$ at time $\tau_-(z) \in [\,\overline{\tau}, \tau(x)]$. Hence, because of (4.39), we find that for all $z \in [x,\, \overline{x}]$ there holds $\omega^T(z-) = u_r(\tau_-(z)) = B$, and this yields the first equality in (4.40), concluding the proof of claim (4.40).

Relying on (4.40) we will show now that ζ satisfies the condition of an AB-gic also for u^* on the interval $[0, \tau(x)]$. To this end, observe that (4.37) (4.40) together imply

$$u_l^*(t) = \overline{A}, \qquad \forall\, t \in \,]\,\overline{\tau}, \tau(x)]\,. \tag{4.44}$$

Hence, because of (4.40), (4.44), ζ satisfies the condition (ii) of Definition 3.1 of an AB-gic for u^* on the interval $]\overline{\tau}, \tau(x)]$. Moreover, let $\vartheta^*_{t_n}$ denote the (classical genuine) backward characteristic for u^*, on the region $\{x < 0\}$, starting at time $t_n \in \,]\,\overline{\tau}, \tau(x)]$ from $x = 0$, for a sequence $t_n \downarrow \overline{\tau}$. Note that, because of (4.44), all $\vartheta^*_{t_n}$ have slope $f_l'(\,\overline{A})$. Thus $\{\vartheta^*_{t_n}\}_n$ converges uniformly to a function $\vartheta^* : [0, \overline{\tau}] \to \mathbb{R}$ that is as well a (classical) genuine characteristic for u^* with slope $f_l'(\,\overline{A})$ and such that $\vartheta^*(\overline{\tau}) = 0$. This in turn implies that

$$u_l^*(\overline{\tau}) = \overline{A}\,. \tag{4.45}$$

Next, we will prove that

$$u_l(\overline{\tau}) = \overline{A}\,. \tag{4.46}$$

To this end observe that (4.22), (4.23), (4.40), (4.41) together imply $\omega^T(\overline{x}-) = B = \omega^T(\overline{x}+)$. This means that the characteristic $\vartheta_{\overline{x}}$ starting at time T from $\overline{x}$, and reaching $x = 0$ at time $\overline{\tau}$, is a (classical) genuine characteristic for u (on the semiplane $\{x > 0\}$), and hence we deduce that

$$u_r(\overline{\tau}) = \omega^T(\overline{x}) = B. \tag{4.47}$$

Recalling (4.4) and the definition (4.2) of the set E, we derive from (4.47) and from condition (2) of Definition 2.1 that

$$u_l(\overline{\tau}) > \theta_l. \tag{4.48}$$

In turn, condition (4.48), together with (4.39), implies (4.46) by a blow-up argument as in [3, § 5.2.6]. Namely, we can consider the blow ups of u at the point $(0, \overline{\tau}\,)$:

$$u_n(x,t) \doteq u\big(x/n,\, \overline{\tau} + (t - \overline{\tau})/n\big) \qquad x \in \mathbb{R},\ t \geq 0\,, \quad n \in \mathbb{N}, \tag{4.49}$$

and observe that, because of (4.39), the left and right traces of $u_n(\cdot, t)$ at $x = 0$ satisfy

$$(u_{n,l}(t), u_{n,r}(t)) = (A, B) \qquad \forall\, t \in \big]\overline{\tau},\, \overline{\tau} + n\big(\tau(x) - \overline{\tau}\big)\big[\,. \tag{4.50}$$

When $n \to \infty$, up to a subsequence, the blow ups $u_n(\cdot, t)$ converge in $\mathbf{L^1_{loc}}$ to a limiting AB entropy solution $v(\cdot, t)$, for all $t > 0$, and there holds

$$v(x, \overline{\tau}\,) = \begin{cases} u_l(\overline{\tau}), & \text{if } x < 0, \\ u_r(\overline{\tau}), & \text{if } x > 0, \end{cases} \tag{4.51}$$

$$v(0-, t) \in \{A, \overline{A}\,\}, \qquad\qquad v(0+, t) \in \{B, \overline{B}\,\}, \qquad \forall\, t > \overline{\tau}\,. \tag{4.52}$$

Then, by a direct inspection we find that, if an AB entropy solution of a Riemann problem for (1.1), with initial datum (4.51) at time $\overline{\tau}$, enjoys the properties (4.48), (4.52), it follows that the left initial datum at time $\overline{\tau}$ must be

$$v(x,\overline{\tau}) = u_l(\overline{\tau}) = \overline{A}, \qquad \forall\ x < 0,$$

thus proving (4.46).

The two conditions equalities (4.45), (4.46) and the definition (4.7) imply that the restriction of ζ to $[0,\overline{\tau}[$ is a (classical) genuine characteristic both for u and u^* with slope $f_l'(\overline{A}) > 0$. Therefore we can conclude that ζ satisfies the condition of an AB-gic also for u^* on the interval $[0,\tau(x)]$, and hence on the whole interval $[0,T]$ by the analysis in the point **2**. This completes the proof that $\zeta \in \mathcal{C}(u^*,x)\cap\mathcal{C}(u,x)$ when $\overline{\tau} \geq \tau_-(\mathsf{R})$.

4. Assume that $\overline{\tau} < \tau_-(\mathsf{R})$, with $\tau_-(\mathsf{R})$ as in (4.34). If we suppose that (4.48) holds, since (4.39) is still verified we can deduce as above that (4.46) holds as well, and then we conclude that $\zeta \in \mathcal{C}(u^*,x)\cap\mathcal{C}(u,x)$ with the same arguments of point **3**.

Therefore, let us assume that $u_l(\overline{\tau}) \leq \theta_l$ and that $\zeta(t) > 0$ for all $t \in [0,\overline{\tau}[$. Observe that because of (4.4), and by definition (4.2) of the set E, we have

$$u_r(\overline{\tau}) < \theta_r. \tag{4.53}$$

Then, relying on (4.39), we deduce with the same blow up argument of above that

$$u_r(\overline{\tau}) = \overline{B}\,. \tag{4.54}$$

On the other hand, if we show that

$$\overline{\tau} \leq \boldsymbol{\tau}, \tag{4.55}$$

it would follow from (4.37) that

$$u_r^*(\overline{\tau}) = \overline{B}\,. \tag{4.56}$$

The two conditions (4.54), (4.56) and the definition (4.7) imply that the restriction of ζ to $[0,\overline{\tau}[$ is a (classical) genuine characteristic both for u and u^* with slope $f_r'(\overline{B}) < 0$. Therefore, if (4.55) holds, we can conclude that ζ satisfies the condition of an AB-gic also for u^* on the interval $[0,\tau(x)]$, and hence on the whole interval $[0,T]$ by the analysis in the point **2**. Hence, in order to completes the proof that $\zeta \in \mathcal{C}(u^*,x)\cap\mathcal{C}(u,x)$ when $\overline{\tau} < \tau_-(\mathsf{R})$, it remains to establish (4.55).

By contradiction, assume that $\overline{\tau} > \boldsymbol{\tau}$. Define the curve

$$\xi(t) \doteq \inf\Big\{R > 0 \;\Big|\; x - t\, f_r'(u(x,t)) \geq 0 \quad \forall\ R > 0\Big\} \qquad t \in [\overline{\tau},T].$$

Notice that, because of (4.54), we have $\xi(\overline{\tau}) = 0$, while the definition (3.13) yields $\xi(T) = \mathsf{R}$. But now using a comparison argument between $\xi(t)$ and the map $\gamma(t)$ defining the shock curve of u^* at point **2**, we obtain as in [3, §5.2.3] that $\xi(t) < \gamma(t)$ for all $t \in [\overline{\tau},T]$. Thus, we find in particular that $\xi(T) < \gamma(T) = \mathsf{R}$, which gives a contradiction. This concludes the proof of the proposition.

□

The next Lemma shows that the initial positions of the AB-gics of the AB-entropy solution u^* defined in (1.11) provide a partition of $\mathbb{R}$.

Lemma 4.3. *Given $\omega^T \in \mathcal{A}^{[AB]}(T)$, let u^* be the AB-entropy solution defined by (1.10)-(1.11). Then, there holds*

$$\mathbb{R} = \bigcup_{x\in\mathbb{R}} \mathcal{C}_0(u^*, x). \tag{4.57}$$

Proof. By the analysis in [3, §5.4.3] we deduce that $\mathbb{R}$ can be partitioned as the union of sets containing points of three types:

- starting points of compression fronts (possibly refracted by the interface $x = 0$) which meet together generating a shock at time T;
- starting points of classical genuine characteristics or of polygonal lines made of two segments consisting of classical genuine characteristics in each semiplane $\{x < 0\}$, $\{x > 0\}$, which reach at time T a point of continuity of ω^T.
- starting points y of polygonal lines $\xi : [0, T] \to \mathbb{R}$ with $\xi(0) = y$, composed of three segments of the form

$$\xi(t) = \begin{cases} y + t\, f'(u(t, \xi(t)), \xi(t)), & \text{if } 0 \le t \le t_1, \\ 0, & \text{if } t_1 \le t \le t_2, \\ (t - t_2) f'(u(t, \xi(t)), \xi(t)) & \text{if } t_2 \le t \le T \end{cases}$$

  where

$$f(u(t, 0\pm)) = \gamma \qquad \forall t \in (t_1, t_2).$$

  Notice that these polygonal lines may belong to the near-interface regions $\Delta_{\mathsf{L}}, \Gamma_{\mathsf{R}}$ defined in ([3], §5.4.4).

In all cases they are starting points of segments or of polygonal lines which are AB-gics for u^*, and the result follows. □

We introduce now a functional that, for any given function $v(x, t)$, measures the total amount of flux of the vector field $\big(f(x, v(x, t)),\, v(x, t)\big)$ passing through a curve $t \mapsto (\alpha(t), t)$, from each side of the curve.

Definition 4.4. Given a function $v \in \mathbf{L}^\infty(\mathbb{R}\times[0, T]\,;\,\mathbb{R})$ that admits one-sided limits $v(x\pm, t)$ at every point $(t, x) \in]0, T] \times \mathbb{R}$, and $\alpha \in \mathrm{Lip}([0, T]\,;\,\mathbb{R})$, we define

$$\mathcal{F}_t(\alpha\pm, v) \doteq \int_t^T \Big\{ f\big(\alpha(t)\pm, v(\alpha(t)\pm, t)\big) - \dot\alpha(t)\, v(\alpha(t)\pm, t) \Big\}\, \mathrm{d}t, \qquad t \in [0, T], \tag{4.58}$$

where $f(x, u)$ is the flux (1.3). We also set

$$\mathcal{F}(\alpha\pm, v) \doteq \mathcal{F}_0(\alpha\pm, v). \tag{4.59}$$

Remark 4.5. Notice that if u is an AB-entropy solution of (1.1), since u is in particular a distributional solution of (1.1) on $\mathbb{R}\times\,]0, +\infty[$, it follows that for any curve $\alpha \in \mathrm{Lip}([0, T]\,;\,\mathbb{R})$, the Rankine-Hugoniot conditions yield, for a.e. $t \in [0, T]$, the equality

$$f\big(\alpha(t)-, u(\alpha(t)-, t)\big) - \dot\alpha(t)\, u(\alpha(t)-, t) = f\big(\alpha(t)+, u(\alpha(t)+, t)\big) - \dot\alpha(t)\, u(\alpha(t)+, t). \tag{4.60}$$

Therefore, in this case we have $\mathcal{F}(\alpha+, u) = \mathcal{F}(\alpha-, u)$. Hence, since there is no ambiguity, whenever u is an AB-entropy solution of (1.1), we will simply write

$$\mathcal{F}_t(\alpha, u) \doteq \mathcal{F}_t(\alpha+, u) \equiv \mathcal{F}_t(\alpha-, u) \quad \forall\ t, \qquad \mathcal{F}(\alpha, u) \doteq \mathcal{F}(\alpha+, u) \equiv \mathcal{F}(\alpha-, u).$$

Lemma 4.6. *Let $u, u^* \in \mathbf{L}^\infty(\mathbb{R}\times[0,T]\,;\mathbb{R})$ be AB-entropy solutions to (1.1), and let $\zeta \in \mathcal{C}(u^*,x)$, $x\in\mathbb{R}$. Then, there holds*

$$\mathcal{F}_t(\zeta,u) \geq \mathcal{F}_t(\zeta,u^*), \quad \forall\, t\in[0,T]. \tag{4.61}$$

Moreover, one has

$$\mathcal{F}(\zeta,u) = \mathcal{F}(\zeta,u^*) \quad\iff\quad \zeta\in\mathcal{C}(u^*,x)\cap\mathcal{C}(u,x). \tag{4.62}$$

Proof. Let $0\leq\tau_1\leq\tau_2\leq T$, be the partition of $[0,T]$ for $\zeta\in\mathcal{C}(u^*,x)$ given by Remark 3.2, so that there holds

$$\zeta(t)=0, \qquad f_l(u_l^*(t)) = f_r(u_r^*(t)) = \gamma \qquad \forall\, t\in[\tau_1,\,\tau_2].$$

To fix the ideas we assume that

$$\zeta(t)<0 \qquad \forall\, t\in[0,\tau_1[\,, \qquad\qquad \zeta(t)>0 \qquad \forall\, t\in\,]\tau_2,\,T]. \tag{4.63}$$

The cases where $\zeta(t)>0$ for all $t\in[0,\tau_1[\,$, or $\zeta(t)<0$ for all $t\in\,]\tau_2,\,T]$ are entirely similar. Then, setting

$$\begin{aligned}
g(t) &\doteq \begin{cases} f_r(u(\zeta(t)+,t)) - \dot\zeta(t)\,u(\zeta(t)+,t) & \text{if} \quad t\in\,]\tau_2,\,T],\\ f_r(u_r(t)) & \text{if} \quad t\in[\tau_1,\tau_2],\\ f_l(u(\zeta(t)+,t)) - \dot\zeta(t)\,u(\zeta(t)+,t) & \text{if} \quad t\in[0,\tau_1[\,,\end{cases}\\
g^*(t) &\doteq \begin{cases} f_r(u^*(\zeta(t),t)) - \dot\zeta(t)\,u^*(\zeta(t),t) & \text{if} \quad t\in\,]\tau_2,\,T],\\ \gamma & \text{if} \quad t\in[\tau_1,\tau_2],\\ f_l(u^*(\zeta(t),t)) - \dot\zeta(t)\,u^*(\zeta(t),t) & \text{if} \quad t\in[0,\tau_1[\,,\end{cases}
\end{aligned} \tag{4.64}$$

we write

$$\mathcal{F}_t(\zeta,u) = \int_t^T g(t)\,\mathrm{d}t, \qquad\qquad \mathcal{F}_t(\zeta,u^*) = \int_t^T g^*(t)\,\mathrm{d}t, \qquad \forall\, t\in[0,T]. \tag{4.65}$$

Note that, because of (4.60), in the definition of g we may equivalently take $u(\zeta(t)-,t)$ instead of $u(\zeta(t)+,t)$, while in the definition of g^* we take u^* continuous at $(\zeta(t),t)$ since ζ is a classical genuine characteristic for u^* when $t\in\,]0,\tau_1[\,\cup\,]\tau_2,T[\,$. Observe that, since u is an AB-entropy solutions to (1.1), by the interface condition (2.5) we have

$$f_r(u_r(t))\geq\gamma \qquad \forall\, t\in[\tau_1,\tau_2]. \tag{4.66}$$

On the other hand, because of the convexity of f_r there holds

$$f_r(v) - f_r'(w)\,v \geq f_r(w) - f_r'(w)\,w, \quad \forall\, v,w\in\mathbb{R}. \tag{4.67}$$

Moreover, since ζ is an AB-gic for u^*, and because of (4.63), note that the restriction of ζ to $[0,\tau_1[$ is a classical genuine characteristic for u^* as solution of $u_t+f_l(u)_x=0$, and the restriction of ζ to $]\tau_2,T]$ is a classical genuine characteristic for u^* as solution of $u_t+f_r(u)_x=0$. Hence, it follows that

$$\dot\zeta(t) = \begin{cases} f_l'(u^*(\zeta(t),t)) & \text{if} \quad t\in\,]0,\tau_1[\,,\\ 0 & \text{if} \quad t\in\,]\tau_1,\tau_2[\,,\\ f_r'(u^*(\zeta(t),t)) & \text{if} \quad t\in\,]\tau_2,T[\,.\end{cases} \tag{4.68}$$

Thus, (4.67)-(4.68) together imply

$$\begin{aligned} f_l(u(\zeta(t)+,t)) - \dot\zeta(t)\, u(\zeta(t)+,t) &\geq f_l(u^*(\zeta(t),t)) - \dot\zeta(t)\, u^*(\zeta(t),t) \qquad \forall\ t \in]0,\tau_1[\,, \\ f_r(u(\zeta(t)+,t)) - \dot\zeta(t)\, u(\zeta(t)+,t) &\geq f_r(u^*(\zeta(t),t)) - \dot\zeta(t)\, u^*(\zeta(t),t) \qquad \forall\ t \in]\tau_2,T[\,. \end{aligned} \tag{4.69}$$

Therefore, from (4.66), (4.69) we deduce that

$$g(t) \geq g^*(t) \qquad \forall\ t \in [0,T], \tag{4.70}$$

which, because of (4.65), yields (4.61).

Concerning (4.62), if $\zeta \in \mathcal{C}(u^*,x)\cap\mathcal{C}(u,x)$, then the inequality (4.61) is verified also when u and u^* switch their places, so that we have $\mathcal{F}_t(u,\zeta) \geq \mathcal{F}_t(u^*,\zeta)$ for all $t \in [0,T]$, thus proving

$$\zeta \in \mathcal{C}(u^*,x)\cap\mathcal{C}(u,x) \quad\Longrightarrow\quad \mathcal{F}(u,\zeta) = \mathcal{F}(u^*,\zeta). \tag{4.71}$$

Next, given $\zeta \in \mathcal{C}(u^*,x)$, assume that

$$\mathcal{F}(\zeta,u) = \mathcal{F}(\zeta,u^*). \tag{4.72}$$

Since, by the above analysis we have (4.66), it follows from (4.72) that $g(t) = g^*(t)$ for a.e. $t \in [0,T]$. Because of (4.64), (4.68), this in particular implies that, for a.e. $t \in [0,T]$, there holds

$$f_r(u_r(t)) = \gamma \qquad \text{if} \quad t \in [\tau_1,\tau_2]\,, \tag{4.73}$$

and

$$\begin{aligned} & f_l(u(\zeta(t)+,t)) - f_l'(u^*(\zeta(t),t))\, u(\zeta(t)+,t) = \\ & \qquad = f_l(u^*(\zeta(t),t)) - f_l'(u^*(\zeta(t),t))\, u^*(\zeta(t),t) \qquad \text{if} \quad t \in]0,\tau_1[\,, \\ & f_r(u(\zeta(t)+,t)) - f_r'(u^*(\zeta(t),t))\, u(\zeta(t)+,t) = \\ & \qquad = f_r(u^*(\zeta(t),t)) - f_r'(u^*(\zeta(t),t))\, u^*(\zeta(t),t) \qquad \text{if} \quad t \in]\tau_2,T[\,. \end{aligned} \tag{4.74}$$

Since f_l, f_r are strictly convex functions, we deduce from (4.74) that $u(\zeta(t)+,t) = u^*(\zeta(t),t)$ for a.e. $t \in]0,\tau_1[\cup]\tau_2,T[$. If we repeat the same analysis taking $u(\zeta(t)-,t)$ instead of $u(\zeta(t)+,t)$ in the definition (4.64) of g, we find that also $u(\zeta(t)-,t) = u^*(\zeta(t),t)$ for a.e. $t \in]0,\tau_1[\cup]\tau_2,T[$. This shows that the restriction of ζ to $[0,\tau_1[$ and to $]\tau_2,T]$ is a classical genuine characteristic for u as well, as solution of $u_t + f_l(u)_x = 0$, and of $u_t + f_r(u)_x = 0$, respectively. Hence, by Remark 3.2 we deduce that ζ is an AB-igc also for u, which means that $\zeta \in \mathcal{C}(u,x)$, completing the proof of

$$\mathcal{F}(\zeta,u) = \mathcal{F}(\zeta,u^*) \quad\Longrightarrow\quad \zeta \in \mathcal{C}(u^*,x)\cap\mathcal{C}(u,x), \tag{4.75}$$

and thus concluding the proof of the Lemma. □

5. Proof of Theorem 3.4

In this section we provide a proof of the initial data identification Theorem 3.4. To this end we first state a technical Lemma that we are going to use repeatedly in the proof of Theorem 3.4.

Lemma 5.1. *Let u be an AB-entropy solution to (1.1), (1.3), and let $\alpha,\beta : [\tau,T] \to \mathbb{R}$, $\tau < T$, be two Lipschitz continuous maps such that $\alpha(t) \leq \beta(t)$ for all $t \in [\tau,T]$. Then it holds*

$$\int_{\alpha(T)}^{\beta(T)} u(x,T)\,\mathrm{d}x - \int_{\alpha(\tau)}^{\beta(\tau)} u(x,\tau)\,\mathrm{d}x \ = \mathcal{F}_\tau(\alpha-,u) - \mathcal{F}_\tau(\beta+,u)\,. \tag{5.1}$$

Proof. Observe that, by property (1) of Definition 2.2, u is a weak distributional solution to (1.1), (1.3). Moreover, by Remark 2.3, $u(t,\cdot)$ is a function of locally bounded variation on $\{x<0\}$, $\{x>0\}$, and it admits left and right strong traces at $x=0$, for all $t>0$. Thus, we can recover the equality (5.1) recalling definition (4.58), applying the divergence theorem to the vector field $(f(x,u),u)$ on each domain $\Delta\cap\{x<\rho\}$, $\Delta\cap\{x>\rho\}$, with $\Delta\doteq\{(x,t)\mid\alpha(t)\le x\le\beta(t),\ t\in[t_0,T]\}$, and then taking the limit as $\rho\to 0$. □

Proof of Theorem 3.4. Given $\omega^T\in\mathcal{A}^{[AB]}(T)$, let u^* be the AB-entropy solution defined by (1.10)-(1.11).

1. We will show that if $u_0\in\mathcal{I}_T^{[AB]}(\omega^T)$, then for every point $\overline{x}\in\mathbb{R}$ there exists $\overline{y}\in\mathcal{C}_0(u^*,\overline{x})$ such that there hold (3.8). The proof of (3.9) is entirely similar. By Proposition 4.2, choose $\zeta_{\overline{x}}\in\mathcal{C}(u^*,\overline{x})\cap\mathcal{C}(u,\overline{x})$, with $u\doteq\mathcal{S}^{[AB]+}u_0(\cdot)$, and set $\overline{y}\doteq\zeta_{\overline{x}}(0)$. Then, consider any $y<\min\mathcal{C}_0(u^*,\overline{x})$. By Lemma 4.3, and because of Proposition 4.1-(iv), there will be some $x<\bar{x}$, and some $\zeta_x\in\mathcal{C}(u^*,x)$, such that $y=\zeta_x(0)$. Hence, applying Lemma 4.6, we deduce that

$$\mathcal{F}(u,\zeta_x)\ge\mathcal{F}(u^*,\zeta_x). \tag{5.2}$$

Moreover, since $\zeta_{\bar{x}}\in\mathcal{C}(u^*,\overline{x})\cap\mathcal{C}(u,\overline{x})$, by the second part of Lemma 4.6 we have

$$\mathcal{F}(u^*,\zeta_{\bar{x}})=\mathcal{F}(u,\zeta_{\bar{x}}). \tag{5.3}$$

On the other hand, applying Lemma 5.1 to the solution u^* with $\alpha=\zeta_x,\beta=\zeta_{\overline{x}}$, $\tau=0$, and recalling Remark 4.5, one obtains

$$\int_x^{\bar{x}}\omega^T(\xi)\,\mathrm{d}\xi-\int_y^{\bar{y}}u_0^*(\xi)\,\mathrm{d}\xi=\mathcal{F}(u^*,\zeta_x)-\mathcal{F}(u^*,\zeta_{\bar{x}}). \tag{5.4}$$

With the same arguments, applying Lemma 5.1 to the solution u, and relying on (5.2), (5.3), we find

$$\int_x^{\overline{x}}\omega^T(\xi)\,\mathrm{d}\xi-\int_y^{\overline{y}}u_0(\xi)\,\mathrm{d}\xi=\mathcal{F}(u,\zeta_x)-\mathcal{F}(u,\zeta_{\overline{x}})\ge\mathcal{F}(u^*,\zeta_x)-\mathcal{F}(u^*,\zeta_{\overline{x}}). \tag{5.5}$$

Combining (5.4), (5.5) we deduce

$$-\int_y^{\overline{y}}u_0(\xi)\,\mathrm{d}\xi\ge-\int_y^{\overline{y}}u_0^*(\xi)\,\mathrm{d}\xi,$$

which yields (3.8).

2. Now we prove that if $u_0\in\mathbf{L}^\infty(\mathbb{R})$ satisfies (3.8), (3.9), then $\mathcal{S}_T^{[AB]+}u_0=\omega^T$. Namely, we are going to prove that under the conditions (3.8), (3.9), there hold

$$\int_{x_1}^{x_2}\big(\omega^T(x)-\mathcal{S}_T^{[AB]+}u_0(x)\big)\,\mathrm{d}x=0,\qquad\forall\ x_1<x_2, \tag{5.6}$$

which clearly implies that $\mathcal{S}_T^{[AB]+}u_0=\omega^T$.

Towards a proof of (5.6) we will first show that

$$\int_{x_1}^{x_2}\big(\omega^T(x)-\mathcal{S}_T^{[AB]+}u_0(x)\big)\,\mathrm{d}x\ge 0,\qquad\forall\ x_1<x_2, \tag{5.7}$$

distinguishing two cases.

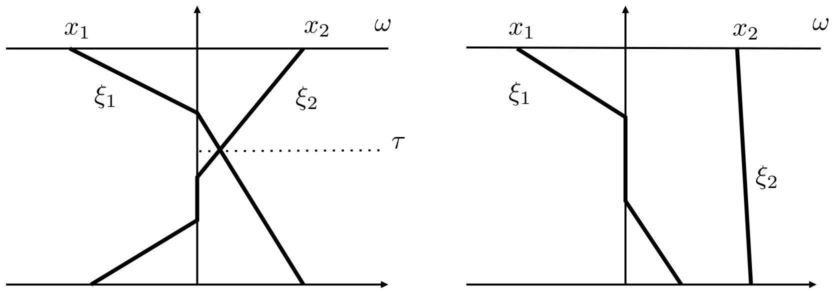

FIGURE 11. CASE 1: $\max \mathcal{C}_0(u, x_1) \geq \min \mathcal{C}_0(u^*, x_2)$ (right); CASE 2: $\max \mathcal{C}_0(u, x_1) < \min \mathcal{C}_0(u^*, x_2)$ (left)

CASE 1. $\max \mathcal{C}_0(u, x_1) \geq \min \mathcal{C}_0(u^*, x_2)$ (see Figure 11, right). Then we can choose $\zeta_1 \in \mathcal{C}(u, x_1)$ and $\zeta_2 \in \mathcal{C}(u^*, x_2)$ such that $\zeta_1(0) \geq \zeta_2(0)$. By continuity there will be a point $\tau \in [0, T[$ such that $\zeta_1(\tau) = \zeta_2(\tau)$, $\zeta_1(t) < \zeta_2(t)$ for all $t \in]\tau, T]$. Applying Lemma 5.1 to the solution u^*, with the curves $\alpha = \zeta_1$, $\beta = \zeta_2$, and using the first part of Lemma 4.6 for ζ_1, we obtain

$$\int_{x_1}^{x_2} \omega^T(x)\,\mathrm{d}x = \mathcal{F}_\tau(u^*, \zeta_1) - \mathcal{F}_\tau(u^*, \zeta_2) \geq \mathcal{F}_\tau(u, \zeta_1) - \mathcal{F}_\tau(u^*, \zeta_2). \tag{5.8}$$

Next, applying again Lemma 5.1 to the solution u, with the curves $\alpha = \zeta_1$, $\beta = \zeta_2$, and then Lemma 4.6 for ζ_2, we obtain

$$\int_{x_1}^{x_2} \mathcal{S}_T^{[AB]+} u_0(x)\,\mathrm{d}x = \mathcal{F}_\tau(u, \zeta_1) - \mathcal{F}_\tau(u, \zeta_2) \leq \mathcal{F}_\tau(u, \zeta_1) - \mathcal{F}_\tau(u^*, \zeta_2). \tag{5.9}$$

Taking the difference of the above two inequalities, we derive (5.7). Note that in this case we are not using the conditions (3.8), (3.9) to establish (5.7).

CASE 2. $\max \mathcal{C}_0(u, x_1) < \min \mathcal{C}_0(u^*, x_2)$ (see Figure 11, left). Choose any $\zeta_1 \in \mathcal{C}_0(u, x_1)$, and set $y \doteq \zeta_1(0)$. Since $y < \min \mathcal{C}_0(u^*, x_2)$, invoking condition (3.8) we find that there exists $\zeta_2 \in \mathcal{C}(u^*, x_2)$ such that, setting $y_2 = \zeta_2(0)$, there holds

$$\int_{y}^{y_2} u_0(x)\,\mathrm{d}x \leq \int_{y}^{y_2} u_0^*(x)\,\mathrm{d}x\,. \tag{5.10}$$

Note that, since $\max \mathcal{C}_0(u, x_1) < \min \mathcal{C}_0(u^*, x_2)$, we have $\zeta_1(t) < \zeta_2(t)$ for all $t \in [0, T]$. Hence, applying Lemma 5.1 to u^*, with the curves $\alpha = \zeta_1$, $\beta = \zeta_2$, and using the first part of Lemma 4.6 for ζ_1, we obtain

$$\begin{aligned} \int_{x_1}^{x_2} \omega^T(x)\,\mathrm{d}x &= \mathcal{F}(u^*, \zeta_1) - \mathcal{F}(u^*, \zeta_2) + \int_{y}^{y_2} u_0^*(x)\,\mathrm{d}x \\ &\geq \mathcal{F}(u, \zeta_1) - \mathcal{F}(u^*, \zeta_2) + \int_{y}^{y_2} u_0^*(x)\,\mathrm{d}x\,. \end{aligned} \tag{5.11}$$

Next, applying Lemma 5.1 to u, with the curves $\alpha = \zeta_1$, $\beta = \zeta_2$, and using the first part of Lemma 4.6 for ζ_2, we obtain

$$\begin{aligned}\int_{x_1}^{x_2} \mathcal{S}_T^{[AB]+} u_0(x)\,\mathrm{d}x &= \mathcal{F}(u,\zeta_1) - \mathcal{F}_\tau(u,\zeta_2) + \int_y^{y_2} u_0(x)\,\mathrm{d}x \\ &\leq \mathcal{F}(u,\zeta_1) - \mathcal{F}(u^*,\zeta_2) + \int_y^{y_2} u_0(x)\,\mathrm{d}x\,.\end{aligned} \tag{5.12}$$

Taking the difference of the above two inequalities, and using (5.10), we derive

$$\int_{x_1}^{x_2} \omega^T(x)\,\mathrm{d}x - \int_{x_1}^{x_2} \mathcal{S}_t^{[AB]+} u_0(x)\,\mathrm{d}x \geq \int_y^{y_2} u_0^*(x)\,\mathrm{d}x - \int_y^{y_2} u_0(x)\,\mathrm{d}x \geq 0 \tag{5.13}$$

which proves (5.7) also in Case 2.

The proof of the opposite inequality of (5.7) is entirely symmetric and is accordingly omitted. Thus the proof of (5.6) is completed, and this concludes the proof of the Theorem. □

Remark 5.2. By the proof of Theorem 3.4 it follows that it is sufficient to assume:

for every point $\overline{x} \in \mathbb{R}$ of continuity of ω^T, there exists $\overline{y} \in \mathcal{C}_0(u^*,\overline{x})$ such that there hold (3.8), (3.9),

to conclude that $u_0 \in \mathcal{I}_T^{[AB]}(\omega^T)$. In fact, in order to show that $\omega^T = \mathcal{S}_T^{[AB]+} u_0$, it is sufficient to prove that (5.6) is verified whenever x_1, x_2 are points of continuity for ω^T, since they are dense in $\mathbb{R}$.

6. Proof of Theorem 3.6

Proof. Given $\omega^T \in \mathcal{A}^{[AB]}(T)$, let u^* be the AB-entropy solution defined by (1.10)-(1.11). We prove the Theorem point by point, in order.

1. Proof of (i). First assume that $|\mathcal{C}_0(u^*,x)| = 1$ for every $x \in \mathbb{R}$. We will show that any initial data $u_0 \in \mathcal{I}_T^{[AB]}(\omega^T)$ satisfies

$$\int_{y_1}^{y_2} u_0(x)\,\mathrm{d}x = \int_{y_1}^{y_2} u_0^*(x)\,\mathrm{d}x, \quad \forall\, y_1 < y_2, \tag{6.1}$$

and this uniquely identifies u_0 as an element of $L^\infty(\mathbb{R})$, thus proving that $\mathcal{I}_T^{[AB]}(\omega^T) = \{u_0^*\}$. Given any two points $y_1 < y_2$, by Lemma 4.3 and because of Proposition 4.1-(iv), there exist $x_1 < x_2$, and $\zeta_i \in \mathcal{C}(u^*,x_i)$, $i = 1,2$, such that $\zeta_i(0) = y_i$, $i = 1,2$. Then, applying (3.8) of Theorem 3.4 we find

$$\int_{y_1}^{y_2} u_0(x)\,\mathrm{d}x \leq \int_{y_1}^{y_2} u_0^*(x)\,\mathrm{d}x\,. \tag{6.2}$$

Next, if we exchange the role of y_1 and y_2, applying this time (3.9) of Theorem 3.4 we find the opposite inequality

$$\int_{y_1}^{y_2} u_0(x)\,\mathrm{d}x \geq \int_{y_1}^{y_2} u_0^*(x)\,\mathrm{d}x\,. \tag{6.3}$$

Combining together the above two inequalities we obtain (6.1).

Conversely, assume that $\mathcal{I}_T^{[AB]}(\omega^T) = \{u_0^*\}$, and by contradiction suppose that there is some $\widetilde{x} \in \mathbb{R}$ such that $|\mathcal{C}_0(u^*,\widetilde{x})| \neq 1$. Using the characterization of Theorem 3.4 we will then show that there exist infinitely many initial data $u_0 \neq u_0^*$ such that $\mathcal{S}_T^{[AB]+} u_0 = \omega^T$. To this end, set

$$\operatorname{conv} \mathcal{C}_0(u^*, \widetilde{x}\,) \doteq [\min \mathcal{C}_0(u^*, \widetilde{x}\,),\, \max \mathcal{C}_0(u^*, \widetilde{x}\,)]\,,$$

and let $\mathbf{L}^\infty(\operatorname{conv} \mathcal{C}_0(u^*, \widetilde{x}\,))$ denote the space of $\mathbf{L}^\infty(\mathbb{R})$ function with essential support in $\operatorname{conv} \mathcal{C}_0(u^*, \widetilde{x}\,))$. Note that $\operatorname{conv} \mathcal{C}_0(u^*, \widetilde{x}\,)$ is a non trivial interval because $|\mathcal{C}_0(u^*, \widetilde{x}\,)| \neq 1$, and hence $\mathbf{L}^\infty(\operatorname{conv} \mathcal{C}_0(u^*, \widetilde{x}\,))$ is an infinite dimensional space. Next, consider the infinite dimensional cone $\mathrm{V}_0 \subset \mathbf{L}^\infty(\operatorname{conv} \mathcal{C}_0(u^*, \widetilde{x}\,))$ consisting of all $v_0 \in \mathbf{L}^\infty(\operatorname{conv} \mathcal{C}_0(u^*, \widetilde{x}\,))$ that satisfy

$$\begin{aligned} &\int_y^{\max \mathcal{C}_0(u^*, \widetilde{x}\,)} v_0(x)\, \mathrm{d}x \leq 0, \qquad \forall\, y \in \operatorname{conv}\, \mathcal{C}_0(u^*, \widetilde{x}\,), \\ &\int_{\min \mathcal{C}_0(u^*, \widetilde{x}\,)}^{y} v_0(x)\, \mathrm{d}x \geq 0, \qquad \forall\, y \in \operatorname{conv}\, \mathcal{C}_0(u^*, \widetilde{x}\,). \end{aligned} \tag{6.4}$$

Note that (6.4) in particular imply

$$\int_{\min \mathcal{C}_0(u^*, \widetilde{x}\,)}^{\max \mathcal{C}_0(u^*, \widetilde{x}\,)} v_0(x)\, \mathrm{d}x = 0\,. \tag{6.5}$$

We will show that

$$V \doteq u_0^* + V_0 \subset \mathcal{I}_T^{[AB]}(\omega^T). \tag{6.6}$$

Relying on Theorem 3.4 this is equivalent to prove that, for any $v_0 \in V_0$, and for every $\overline{x} \in \mathbb{R}$, there exists $\overline{y} \in \mathcal{C}_0(u^*, \overline{x})$ such that (3.8), (3.9) hold for $u_0 \doteq u_0^* + v_0$. We will verify only (3.8), the proof of the other inequality being entirely symmetric.

Then, consider first any $\overline{x} \leq \widetilde{x}$, and choose $\overline{y} = \min \mathcal{C}_0(u^*, \overline{x})$. Observe that, for every $y < \min \mathcal{C}_0(u^*, \overline{x})$, we have $u_0 = u_0^*$ on the interval $[y, \overline{y}]$ since $\overline{x} \leq \widetilde{x}$, together with Proposition 4.1-(iv), implies

$$\overline{y} \leq \min \mathcal{C}_0(u^*, \widetilde{x}),$$

and hence $v_0 = 0$ on $[y, \overline{y}]$, because the essential support of v_0 is contained in $\operatorname{conv} \mathcal{C}_0(u^*, \widetilde{x}\,))$. This implies that, for every $y < \min \mathcal{C}_0(u^*, \overline{x})$, we have

$$\int_y^{\overline{y}} u_0(x)\, \mathrm{d}x = \int_y^{\overline{y}} u_0^*(x)\, \mathrm{d}x\,, \tag{6.7}$$

which proves (3.8) as an equality.

Next, consider any $\overline{x} > \widetilde{x}$, and choose $\overline{y} = \max \mathcal{C}_0(u^*, \overline{x})$. Then, for every $y < \min \mathcal{C}_0(u^*, \overline{x})$, one of the following three cases occurs:

Case 1. If $y \in \,]\max \mathcal{C}_0(u^*, \tilde{x}), \min \mathcal{C}_0(u^*, \bar{x})[\,$, then (3.8) holds again as an equality, because u_0 coincides with u_0^* in the interval $[y, \overline{y}]$ as in the case $\overline{x} \leq \widetilde{x}$ considered above, and thus (6.7) is verified.

Case 2. If $y \in \operatorname{conv}\, \mathcal{C}_0(u^*, \tilde{x})$, then by (6.4) we have

$$\begin{aligned} \int_y^{\overline{y}} u_0(x)\, \mathrm{d}x &= \int_y^{\max \mathcal{C}_0(u^*, \widetilde{x})} u_0(x)\, \mathrm{d}x + \int_{\max \mathcal{C}_0(u^*, \widetilde{x})}^{\overline{y}} u_0^*(x)\, \mathrm{d}x \\ &\leq \int_y^{\overline{y}} u_0^*(x)\, \mathrm{d}x\,, \end{aligned}$$

which proves (3.8).

CASE 3. If $y < \min \mathcal{C}_0(u^*, \tilde{x})$, we obtain (3.8) relying on (6.5), since

$$\begin{aligned}\int_y^{\overline{y}} u_0(x)\,\mathrm{d}x &= \int_y^{\min \mathcal{C}_0(u^*,\widetilde{x})} u_0^*(x)\,\mathrm{d}x + \int_{\min \mathcal{C}_0(u^*,\widetilde{x})}^{\max \mathcal{C}_0(u^*,\widetilde{x})} u_0(x)\,\mathrm{d}x + \int_{\max \mathcal{C}_0(u^*,\widetilde{x})}^{\overline{y}} u_0^*(x)\,\mathrm{d}x \\ &= \int_y^{\overline{y}} u_0^*(x)\,\mathrm{d}x\,.\end{aligned}$$

Thus, for all $u_0 = u_0^* + v_0$, $v_0 \in V_0$, and for every $\overline{x} \in \mathbb{R}$, there exists $\overline{y} \in \mathcal{C}_0(u^*, \overline{x})$ such that (3.8), (3.9) hold. Hence (6.6) is verified, which contradicts the assumption $\mathcal{I}_T^{[AB]}(\omega^T) = \{u_0^*\}$, and thus completes the proof of the first part of property (i).

Finally, observe that if x is a point of discontinuity for ω^T, then one can consider the AB-gics $\vartheta_{x,-}, \vartheta_{x,+} : [0,T] \to \mathbb{R}$ that are the minimal and maximal AB-gics for u^* reaching at time T the point x (e.g. see point **2** of the proof of Proposition 4.2). Since $\vartheta_{x,-}(0) \neq \vartheta_{x,+}(0)$ if $x \neq 0$, and because $\{\vartheta_{x,-}(0), \vartheta_{x,+}(0)\} \subset \mathcal{C}_0(u^*, \tilde{x})$, this implies $|\mathcal{C}_0(u^*, x\,)| \neq 1$, thus proving by contradiction that if $\mathcal{I}_T^{[AB]}(\omega^T)$ is a singleton, then ω^T must be continuous at any point $x \neq 0$. This concludes the proof of property (i).

2. Proof of (ii). To prove that the set $\mathcal{I}_T^{[AB]}(\omega^T) - u_0^*$ is a linear cone, we will show that, for every $u_0 \in \mathcal{I}_T^{[AB]}(\omega^T)$ and $\lambda \geq 0$, it holds $u_0^* + \lambda(u_0 - u_0^*) \in \mathcal{I}_T^{[AB]}(\omega^T)$. To see this, applying Theorem 3.4 it's sufficient to prove that, given any $\overline{x} \in \mathbb{R}$, there exists $\overline{y} \in \mathcal{C}_0(u^*, \overline{x})$ such that (3.8), (3.9) hold with $u_0^* + \lambda(u_0 - u_0^*)$ in place of u_0. Since $u_0 \in \mathcal{I}_T^{[AB]}(\omega^T)$, by Theorem 3.4 we know that there is some $\overline{y} \in \mathcal{C}_0(u^*, \overline{x})$ such that (3.8) holds. Then, for all $y < \min \mathcal{C}_0(u^*, \overline{x})$, one finds

$$\int_y^{\overline{y}} \big(u_0^*(x) + \lambda(u_0(x) - u_0^*(x))\big)\,\mathrm{d}x \leq \int_y^{\overline{y}} \big(u_0^*(x) + \lambda(u_0^*(x) - u_0^*(x))\big)\,\mathrm{d}x = \int_y^{\overline{y}} u_0^*(x)\,\mathrm{d}x\,.$$

This proves that (3.8) is verified with $u_0^* + \lambda(u_0 - u_0^*)$ in place of u_0. The proof that also (3.9) holds, is entirely symmetric.

Next, we prove that u_0^* is an extremal point of $\mathcal{I}_T^{[AB]}(\omega^T)$. Assume by contradiction that there exist $u_{0,i} \in \mathcal{I}_T^{[AB]}(\omega^T)$, $u_{0,i} \neq u_0^*$, $i = 1, 2$, and $\lambda \in]0, 1[$, such that

$$u_0^* = \lambda u_{0,1} + (1 - \lambda) u_{0,2}\,. \tag{6.8}$$

Take any $\overline{x} \in \mathbb{R}$ for which $\mathcal{C}_0(u^*, \overline{x})$ is a singleton (one can choose $\overline{x}$ as a point of continuity for ω^T belonging to the set $]-\infty, \mathsf{L}[\,\cup\,]\mathsf{R}, +\infty[$, with L, R as in (3.13)), and call $\overline{y}$ the unique element of $\mathcal{C}_0(u^*, \overline{x})$. Because of (6.8) it holds

$$\int_{\overline{y}}^{y} u_0^*(x)\,\mathrm{d}x = \lambda \int_{\overline{y}}^{y} u_{0,1}(x)\,\mathrm{d}x + (1 - \lambda) \int_{\overline{y}}^{y} u_{0,2}(x)\,\mathrm{d}x, \qquad \forall\, y \in \mathbb{R}\,. \tag{6.9}$$

Then, since $u_{0,1}, u_{0,2}$ are different from u_0^*, there must be some $y \in \mathbb{R}$ such that one of the following three cases occurs:

CASE 1.

$$\int_{\overline{y}}^{y} u_{0,1}(x)\,\mathrm{d}x \neq \int_{\overline{y}}^{y} u_0^*(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_{\overline{y}}^{y} u_{0,2}(x)\,\mathrm{d}x = \int_{\overline{y}}^{y} u_0^*(x)\,\mathrm{d}x. \tag{6.10}$$

CASE 2.

$$\int_{\overline{y}}^{y} u_{0,1}(x)\,\mathrm{d}x = \int_{\overline{y}}^{y} u_0^*(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_{\overline{y}}^{y} u_{0,2}(x)\,\mathrm{d}x \neq \int_{\overline{y}}^{y} u_0^*(x)\,\mathrm{d}x. \tag{6.11}$$

CASE 3.

$$\int_{\overline{y}}^{y} u_{0,1}(x)\,\mathrm{d}x \neq \int_{\overline{y}}^{y} u_0^*(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_{\overline{y}}^{y} u_{0,2}(x)\,\mathrm{d}x \neq \int_{\overline{y}}^{y} u_0^*(x)\,\mathrm{d}x. \tag{6.12}$$

Assume that CASE 1 holds, with $y \neq \overline{y}$. Then, applying conditions (3.8), (3.9) of Theorem 3.4 to u_0^1, we find that

$$\int_{\overline{y}}^{y} u_{0,1}(x)\,\mathrm{d}x > \int_{\overline{y}}^{y} u_0^*\,\mathrm{d}x. \tag{6.13}$$

But (6.13), together with the equality in (6.10), is in contradiction with (6.9). The analysis of the other two cases is entirely similar, thus it is omitted. This proves that u_0^* is an extremal point of $\mathcal{I}_T^{[AB]}(\omega^T)$ (and of course it is unique since u_0^* is the vertex of the affine cone $\mathcal{I}_T^{[AB]}(\omega^T)$), and thus concludes the proof of property (ii).

3. Proof of (iii). We first show that, if condition (3.16) is verified than the set $\mathcal{I}_T^{[AB]}(\omega^T)$ is convex. Given $u_{0,1}, u_{0,2} \in \mathcal{I}_T^{[AB]}(\omega^T)$, and $\lambda \in]0,1[$, let $\overline{x} \in]\mathsf{L}, \mathsf{R}[$ be a point of continuity for ω^T. By Theorem 3.4, and because $\mathcal{C}_0(u^*, \overline{x})$ is a singleton $\{\overline{y}\}$, we know that there hold

$$\int_{y}^{\overline{y}} u_{0,1}(x)\,\mathrm{d}x \leq \int_{y}^{\overline{y}} u_0^*(x)\,\mathrm{d}x, \qquad \int_{y}^{\overline{y}} u_{0,2}(x)\,\mathrm{d}x \leq \int_{y}^{\overline{y}} u_0^*(x)\,\mathrm{d}x, \quad \forall\, y < \overline{y}. \tag{6.14}$$

Then, using (6.14), we derive

$$\int_{y}^{\overline{y}} \big(\lambda u_{0,1}(x) + (1-\lambda) u_{0,2}(x)\big)\,\mathrm{d}x \leq \int_{y}^{\overline{y}} u^*(x)\,\mathrm{d}x, \qquad \forall\, y < \overline{y}, \quad \forall\, \lambda \in]0,1[\,, \tag{6.15}$$

so that $\lambda u_{0,1} + (1-\lambda)u_{0,2}$ satisfies (3.8) for all $\lambda \in]0,1[$, and $\overline{y} \in \mathcal{C}_0(u^*, \overline{x})$ with $\overline{x} \in]\mathsf{L}, \mathsf{R}[$ of continuity for ω^T. The proof that also (3.9) holds for the same $\overline{y}$ is entirely similar and is accordingly omitted. Next, consider a point $\overline{x} \in]-\infty, \mathsf{L}[\,\cup\,]\mathsf{R}, +\infty[$ of continuity for ω^T. Notice that by definition (3.13) the classical backward characteristics starting from $(\overline{x}, T)$ never cross the interface $x = 0$ at positive times. Therefore the unique AB-gic reaching the point $\overline{x}$ at time $t = T$ is a classical genuine characteristic starting say at $\overline{y}$ at time $t = 0$. Hence $\mathcal{C}_0(u^*, \overline{x}) = \{\overline{y}\}$, and we can proceed as above to show that $\lambda u_{0,1} + (1-\lambda)u_{0,2}$ satisfies (3.8), (3.9) for all $\lambda \in]0,1[$, also when $\overline{y} \in \mathcal{C}_0(u^*, \overline{x})$ with $\overline{x} \in]-\infty, \mathsf{L}[\,\cup\,]\mathsf{R}, +\infty[$ of continuity for ω^T. Then, by Remark 5.2 we can conclude that $\lambda u_{0,1} + (1-\lambda)u_{0,2} \in \mathcal{I}_T^{[AB]}(\omega^T)$, for all $\lambda \in]0,1[$.

Now assume that condition (3.15) is verified. By the above analysis it is clear that in order to prove the convexity of $\mathcal{I}_T^{[AB]}(\omega^T)$ it is sufficient to show that, for any $\overline{x} \in]\mathsf{L}, \mathsf{R}[$ of continuity for ω^T there exists $\overline{y} \in \mathcal{C}_0(u^*, \overline{x})$ such that there holds

$$\int_{y}^{\overline{y}} \big(\lambda u_{0,1}(x) + (1-\lambda) u_{0,2}(x)\big)\,\mathrm{d}x \leq \int_{y}^{\overline{y}} u^*(x)\,\mathrm{d}x, \qquad \forall\, y < \min \mathcal{C}_0(u^*, \overline{x})\,, \quad \forall\, \lambda \in]0,1[\,. \tag{6.16}$$

The problem in this case is the following. Since $\mathcal{C}_0(u^*, \overline{x})$ may not be a singleton, by Theorem 3.4 we know that there will be in general $\overline{y}_i \in \mathcal{C}_0(u^*, \overline{x})$, $i = 1, 2$, $\overline{y}_1 \neq \overline{y}_2$, such that there hold

$$\int_{y}^{\overline{y}_1} u_{0,1}(x)\,\mathrm{d}x \leq \int_{y}^{\overline{y}_1} u_0^*(x)\,\mathrm{d}x, \qquad \int_{y}^{\overline{y}_2} u_{0,2}(x)\,\mathrm{d}x \leq \int_{y}^{\overline{y}_2} u_0^*(x)\,\mathrm{d}x, \quad \forall\, y < \min \mathcal{C}_0(u^*, \overline{x}). \tag{6.17}$$

Here the choice of $\overline{y}_i$ depends on the initial datum $u_{0,i} \in \mathcal{C}_0(u^*, \overline{x})$, $i = 1, 2$. Hence, we cannot rely on (6.17) to derive immediately the existence of $\overline{y} \in \mathcal{C}_0(u^*, \overline{x})$ such that (6.16) holds. However, thanks to the assumption (3.15) we can show that, for every point $\overline{x} \in]\mathsf{L}, \mathsf{R}[$ of continuity for ω^T, there exists $\overline{y} \in \mathcal{C}_0(u^*, \overline{x})$, independent on the initial datum u_0 taken in consideration, such that (3.8), (3.9) are verified. In fact, given any point $\overline{x} \in]\mathsf{L}, \mathsf{R}[$ of continuity for ω^T, let $\{\overline{x}_n\}_n$ be a sequence of points in $\mathcal{X}(\omega^T)$ such that $\overline{x}_n \to \overline{x}$. Letting $\{\overline{y}_n\} = \mathcal{C}_0(u^*, \overline{x}_n)$, we may assume that, up to a subsequence, $\{\overline{y}_n\}_n$ converges to some point $\overline{y} \in \mathbb{R}$. Since $x \mapsto \mathcal{C}_0(u^*, x)$ has closed graph by Proposition 4.1 it follows that $\overline{y} \in \mathcal{C}_0(u^*, \overline{x})$. Hence, applying Theorem 3.4 we find that, for any $y < \min \mathcal{C}_0(u^*, \overline{x})$, and for n sufficiently large, there hold

$$\int_y^{\overline{y}_n} u_{0,1}(x)\, \mathrm{d}x \leq \int_y^{\overline{y}_n} u_0^*(x)\, \mathrm{d}x, \qquad \int_y^{\overline{y}_n} u_{0,2}(x)\, \mathrm{d}x \leq \int_y^{\overline{y}_n} u_0^*(x)\, \mathrm{d}x\,. \tag{6.18}$$

taking the limit as $n \to \infty$ in (6.18) we derive

$$\int_y^{\overline{y}} u_{0,1}(x)\, \mathrm{d}x \leq \int_y^{\overline{y}} u_0^*(x)\, \mathrm{d}x, \qquad \int_y^{\overline{y}} u_{0,2}(x)\, \mathrm{d}x \leq \int_y^{\overline{y}} u_0^*(x)\, \mathrm{d}x, \quad \forall\, y < \min \mathcal{C}_0(u^*, \overline{x})\,,$$

which yields (6.16). This completes the proof of property (iii), and thus concludes the proof of the theorem.

□

6.1. **A nonconvex set of initial data.** We provide here an example of attainable profile $\omega^T \in \mathcal{A}^{[AB]}(T)$ for which the set $\mathcal{I}_T^{[AB]}(\omega^T)$ is not convex. Let $f \doteq f_l = f_r = u^2/2$, and set

$$A \doteq 4L_0, \qquad B \doteq -4L_0, \tag{6.19}$$

for some constant $L_0 < 0$. By definition (2.7) we have

$$\overline{A} = -4L_0, \qquad \overline{B} = 4L_0. \tag{6.20}$$

Then, consider the profile

$$\omega_3(x) = \begin{cases} \overline{A} & x \leq L_0, \\ A & x \in]L_0, 0[\,, \\ p & x > 0, \end{cases} \tag{6.21}$$

with

$$p < 12 L_0\,. \tag{6.22}$$

0bserve that, by definition (3.13) we have $\mathsf{L} = \mathsf{L}[\omega_3, f] = L_0$, and $\mathsf{R} = \mathsf{R}[\omega_3, f] = 0$ since $f'(p) = p < 0$. Moreover, recall that the quantity $\boldsymbol{v} \doteq \boldsymbol{v}[L_0, A, f]$ defined in [3, § 3.2] satisfies

$$\overline{A} > \boldsymbol{v} > A. \tag{6.23}$$

Then, one can readly verify that ω_3 fulfills the conditions (i)-(ii) of [3, Theorem 4.7]. To simplify the analysis we shall consider a time horizon $T = 1$. With this choice, by a direct computation one finds that $\boldsymbol{v} = A - 2\sqrt{A\, L_0} = 0$. Following the same type of procedure of Remark 3.8 we now construct explicitly the AB-entropy solution u^* defined by

$$u_0^* \doteq \mathcal{S}_T^{[AB]-} \omega_3, \qquad u^*(\cdot, t) \doteq \mathcal{S}_t^{[AB]+} u_0^* \qquad \forall\, t \in [0, 1]\,. \tag{6.24}$$

Observe that condition (6.23) ensures the existence in u^* of a shock curve parametrized by a map $\gamma : [\boldsymbol{\sigma}, 1] \to]-\infty, 0]$, with $\boldsymbol{\sigma} = -L_0/\overline{A} = 1/4$, such that $\gamma(\boldsymbol{\sigma}) = 0$, $\gamma(1) = L_0$. The curve $t \to (\gamma(t), t)$, $t \in [1/4, 1]$, is a shock curve for the conservation law $u_t + f(u)_x = 0$,

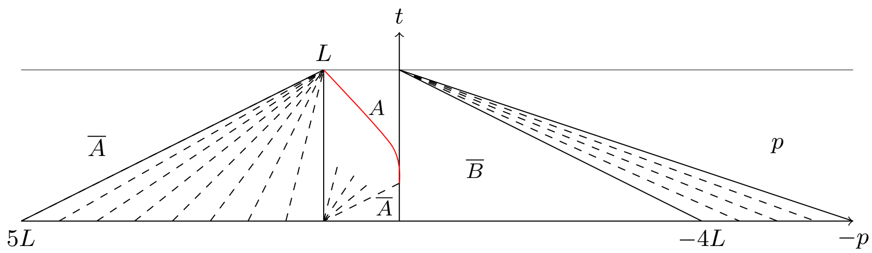

FIGURE 12. The solution produced by the initial datum u_0^*.

which connects the left states $(\gamma(t) - L_0)/t$ with the right state A. On the left of $\gamma(t)$ there is a rarefaction wave connecting the left state 0 with the right state $\overline{A}$, and centered at the point $(L_0, 0)$. Then, u^* is defined by (see Figure 12)

$$u^*(x,t) = \begin{cases} \overline{A} & \text{if} \quad x < L_0 - (1-t)\cdot\overline{A}, \quad t \in [0,1], \\ \dfrac{L_0 - x}{1-t} & \text{if} \quad L_0 - (1-t)\cdot\overline{A} \le x \le L_0, \quad t \in [0,1[\,, \\ \dfrac{x - L_0}{t} & \text{if} \quad \begin{cases} L_0 < x < \gamma(t), & t \in [1/4, 1], \\ L_0 < x < L_0 + t\cdot\overline{A}, & t \in]0, 1/4], \end{cases} \\ A & \text{if} \quad \gamma(t) < x < 0, \quad t \in [1/4, 1], \\ \overline{A} & \text{if} \quad L_0 + t\cdot\overline{A} < x < 0, \quad t \in [0, 1/4], \\ \overline{B} & \text{if} \quad 0 < x < (t-1)\cdot\overline{B}, \quad t \in [0,1], \\ \dfrac{x}{t-1} & \text{if} \quad (t-1)\cdot\overline{B} \le x \le (t-1)\cdot p, \quad t \in [0,1[\,, \\ p & \text{if} \quad x > (t-1)\cdot p, \quad t \in [0,1]\,, \end{cases} \tag{6.25}$$

and the corresponding initial datum is given by

$$u_0^*(x) = \begin{cases} \overline{A} & \text{if} \quad x < 5L_0, \\ L_0 - x & \text{if} \quad 5L_0 < x < L_0, \\ \overline{A} & \text{if} \quad L_0 < x < 0, \\ \overline{B} & \text{if} \quad 0 < x < -4L_0, \\ -x & \text{if} \quad -4L_0 < x < -p, \\ p & \text{if} \quad -p < x. \end{cases} \tag{6.26}$$

Our goal is to find two initial data $u_{0,1}, u_{0,2} \in \mathcal{I}_T^{[AB]}(\omega_3)$ such that for some $\lambda \in]0,1[\,$, we have $\lambda u_{0,1} + (1-\lambda)u_{0,2} \notin \mathcal{I}_T^{[AB]}(\omega_3)$. Then, consider the following two initial data (see Figure 13 and Figure 14):

$$u_{0,1}(x) = \begin{cases} \overline{A} & \text{if} \quad x < L_0, \\ A & \text{if} \quad L_0 < x < 0, \\ B & \text{if} \quad 0 < x < -\lambda(B,p), \\ p & \text{if} \quad -\lambda(B,p) < x, \end{cases} \tag{6.27}$$

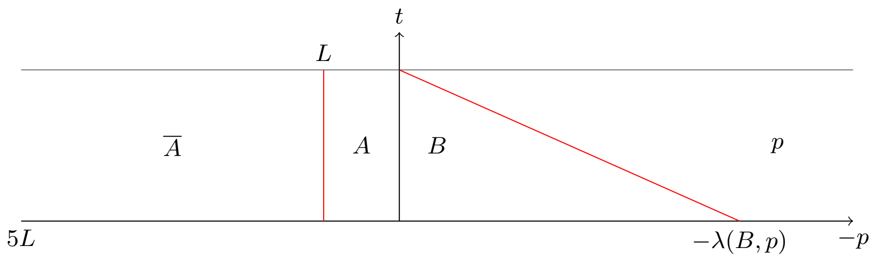

FIGURE 13. The solution produced by the initial datum u_0^1.

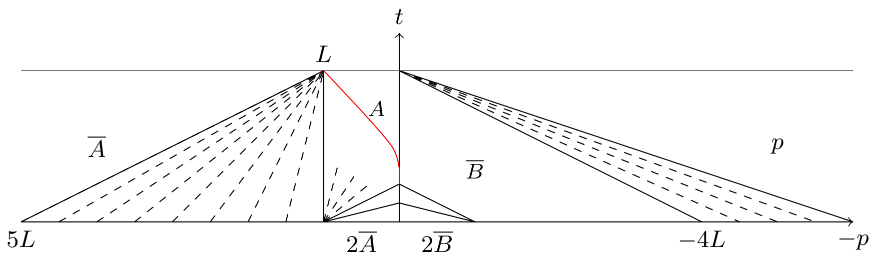

FIGURE 14. The solution produced by the initial datum u_0^2.

where $\lambda(B,p) = (B+p)/2 = (-4L_0+p)/2$ denotes the Rankine-Hugoniot speed of the jump with left state B and right state p,

$$u_{0,2}(x) = \begin{cases} \overline{A} & \text{if} \quad x < 5L_0, \\ L_0 - x & \text{if} \quad 5L_0 < x < L_0, \\ 2\,\overline{A} & \text{if} \quad L_0 < x < 0, \\ 2\,\overline{B} & \text{if} \quad 0 < x < -L_0, \\ \overline{B} & \text{if} \quad -L_0 < x < -4L_0, \\ -x & \text{if} \quad -4L_0 < x < -p, \\ p & \text{if} \quad x > -p. \end{cases} \tag{6.28}$$

With similar arguments as for the construction of u^* above, one can easily see that the AB-entropy solutions to (1.1), (1.3), with initial data $u_{0,1}, u_{0,2}$, reach at time $T=1$ the profile ω_3 in (6.21) (see Figures 13, 14). Hence, we have $u_{0,i} \in \mathcal{I}_T^{[AB]}(\omega_3)$, $i=1,2$. We will now show that

$$u_0^\lambda \doteq \lambda u_{0,1} + (1-\lambda)u_{0,2} \;\notin\; \mathcal{I}_T^{[AB]}(\omega_3) \qquad \forall\, \lambda \in]0,1[\,. \tag{6.29}$$

Toward this end, we will first show that, if $u_0^\lambda \in \mathcal{I}_T^{[AB]}(\omega_3)$ for some $\lambda \in]0,1[$, then there exists $\overline{y} \in [L_0, -3L_0]$ such that there holds

$$\int_{5L_0}^{\overline{y}} u_0^\lambda(x)\,\mathrm{d}x \le \int_{5L_0}^{\overline{y}} u_0^*(x)\,\mathrm{d}x\,. \tag{6.30}$$

In fact, observe first that with the same analysis in Remark 3.8 we deduce that (3.21) is verified also for u^* defined in (6.25). Since here we have $\boldsymbol{v}=0$, $T=1$, one thus finds that there holds

$$\mathcal{C}_0(u^*, x) = [\,L_0,\,(x/A-1)\cdot\overline{B}\,] = [\,L_0,\,x-4\,L_0],\qquad \forall\, x \in]\,L_0, 0[\,. \tag{6.31}$$

Then, considering a sequence of points $\overline{x}_n \downarrow L_0$, and applying Theorem 3.4 with u_0^θ in place of u_0, we deduce that for every n there exists $\overline{y}_n \in \mathcal{C}_0(u^*, \overline{x}_n) = [\,L_0,\, \overline{x}_n - 4\,L_0]$ such that there holds

$$\int_{5L_0}^{\overline{y}_n} u_0^\lambda(x)\,\mathrm{d}x \le \int_{5L_0}^{\overline{y}_n} u_0^*(x)\,\mathrm{d}x\,. \tag{6.32}$$

We may assume that, up to a subsequence, $\{\overline{y}_n\}_n$ converges to some point $\overline{y} \in [L_0, -3L_0]$. Then, taking the limit in (6.32) as $n \to \infty$, we derive that (6.30) holds for such $\overline{y}$.

We will now show that, by definitions of u_0^*, $u_{0,i}$, $i = 1, 2$, in (6.26), (6.27), (6.28), it follows

$$\int_{5L_0}^{\overline{y}} \big(u_0^\lambda(x) - u_0^*(x)\big)\,\mathrm{d}x > 0, \qquad \forall\, \overline{y} \in [L_0, -3L_0]\,, \tag{6.33}$$

which is in contrast with (6.30), thus proving (6.29) by contradiction. We distinguish three cases:

Case 1. $\overline{y} \in [L_0, 0]$. By direct computations we find that

$$\int_{5L_0}^{\overline{y}} u_{0,1}(x)\,\mathrm{d}x = 12L_0^2 + 4L_0\,\overline{y}, \qquad \int_{5L_0}^{\overline{y}} u_{0,2}(x)\,\mathrm{d}x = 16L_0^2 - 8L_0\,\overline{y}, \tag{6.34}$$

and

$$\int_{5L_0}^{\overline{y}} u_0^*(x)\,\mathrm{d}x = 12L_0^2 - 4L_0\,\overline{y}\,. \tag{6.35}$$

Thus, for every $\overline{y} \in [L_0, 0]$, we derive

$$\int_{5L_0}^{\overline{y}} \big(u_0^\lambda(x) - u_0^*(x)\big)\,\mathrm{d}x = 8\,\lambda L_0\,\overline{y} + 8(1-\lambda)L_0(L_0 - \overline{y}) > 0, \qquad \forall\, \lambda \in [0,1]. \tag{6.36}$$

Case 2. $\overline{y} \in [0, -L_0]$. Observe that, because of (6.22), we have $\lambda(B, v) > -4L_0$, which implies that $u_{0,1}(x) = B$ for all $x \in]0, \overline{y}]$. Then, by computations as in previous case, for $\overline{y} \in [0, -L_0]$ we find

$$\int_{5L_0}^{\overline{y}} u_{0,1}(x)\,\mathrm{d}x = 12L_0^2 - 4L_0\,\overline{y}, \qquad \int_{5L_0}^{\overline{y}} u_{0,2}(x)\,\mathrm{d}x = 16L_0^2 + 8L_0\,\overline{y}, \tag{6.37}$$

and

$$\int_{5L_0}^{\overline{y}} u_0^*(x)\,\mathrm{d}x = 12L_0^2 + 4L_0\,\overline{y}\,. \tag{6.38}$$

Thus, for every $\overline{y} \in [0, -L_0]$, we derive

$$\int_{5L_0}^{\overline{y}} \big(u_0^\lambda(x) - u_0^*(x)\big)\,\mathrm{d}x = -8\,\lambda L_0\,\overline{y} + 4(1-\lambda)L_0(L_0 + \overline{y}) > 0, \qquad \forall\, \lambda \in [0,1]. \tag{6.39}$$

Case 3. $\overline{y} \in [-L_0, -3L_0]$. Note that, as in Case 2, we have $u_{0,1}(x) = B$ for all $x \in]0, \overline{y}]$. Then, by computations as in previous cases, for $\overline{y} \in [-L_0, -3L_0]$ we find

$$\int_{5L_0}^{\overline{y}} u_{0,1}(x)\,\mathrm{d}x = 12L_0^2 - 4L_0\,\overline{y}, \qquad \int_{5L_0}^{\overline{y}} u_{0,2}(x)\,\mathrm{d}x = 12L_0^2 + 4L_0\,\overline{y}, \tag{6.40}$$

and

$$\int_{5L_0}^{\overline{y}} u_0^*(x)\,\mathrm{d}x = 12L_0^2 + 4L_0\,\overline{y}\,. \tag{6.41}$$

Hence, for every $\overline{y} \in [-L_0, -3L_0]$, we derive

$$\int_{5L_0}^{\overline{y}} \big(u_0^\lambda(x) - u_0^*(x)\big)\,\mathrm{d}x = -8\,\lambda L_0\,\overline{y} > 0, \qquad \forall\ \lambda \in [0,1]. \tag{6.42}$$

The analysis of all three cases shows that (6.33) is verified, and thus concludes the proof that $\mathcal{I}_T^{[AB]}(\omega_3)$ is not convex since (6.29) holds.

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