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VARIATIONAL RESULTS ON CAPILLARY ISOPERIMETRIC PROBLEMS AND LIQUID DROP MODELS

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Abstract

The main topic of this thesis concerns some recent developments in Calculus of Variations and Geometric Measure Theory that have been obtained in [MP21; MP24; PP24; Pas25].
After the introductory Chapter 1, where we present the main results, motivations and history on the topics considered in this thesis, in Chapter 2 we collect the preliminaries needed for the presentation.
Chapter 3 is dedicated to the proof of some quantitative isoperimetric inequalities for the classical capillarity problem in a Euclidean halfspace. The results have been obtained in a joint work with M. Pozzetta and are based on a novel combination of a quantitative ABP method with a selection-type method, after a symmetrization procedure.
In Chapter 4 we establish some existence and nonexistence results for the volume-constrained minimization problem of an energy functional given by the sum of a capillarity perimeter, a nonlocal interaction term and a gravitational type energy. The strategy stems from an application of the quantitative isoperimetric inequalities for the capillarity problem in a half-space.
Chapter 5 is devoted to study differentiability and integrability properties of weak solutions to some nonlinear elliptic systems with growth coefficients in BMO. The results have been obtained in collaboration with G. Moscariello. Moreover we derive some local Calderón–Zygmund estimates, which are relevant to provide upper bounds for the Hausdorff dimension of the singular set of minima of general variational integrals.

Notation and symbols

- $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$.
- $|\cdot|$ denotes both the Lebesgue measure in $\mathbb{R}^n$ and the modulus of a vector in $\mathbb{R}^n$, depending on the context.
- A_p, with $p \geq 1$, denotes the class of Muckenhoupt with exponent p.
- $\mathcal{B}(X)$ denotes the σ-algebra of Borel subsets of a topological space X.
- $B_r(x)$ denotes the open ball of center x and radius r in $\mathbb{R}^n$.
- $B_r := B_r(0) \subset \mathbb{R}^n$ for $r > 0$, $B := B_1$.
- $B^\lambda = \{x \in B : \langle x, e_n \rangle > \lambda\}$.
- $B^\lambda(v) := \frac{v^{\frac{1}{n}}}{|B^\lambda|^{\frac{1}{n}}}(B^\lambda - \lambda e_n)$, for any $v > 0$.
- $B^\lambda(v, x) := B^\lambda(v) + x$, for any $x \in \{x_n = 0\}$. In particular $B^\lambda(v) = B^\lambda(v, 0)$.
- $BMO(\Omega)$ denotes the space of functions with bounded mean oscillation in $\Omega \subset \mathbb{R}^n$.
- $c(\cdot), C(\cdot)$ denote strictly positive constants, that may change from line to line.
- $C^{k,\alpha}(X)$ denotes the space of real functions continuously derivable in the topological space X up to the order $k \in \mathbb{N}$, with locally α-Hölder continuous derivatives in X.
- $C_c(X)$ denotes the space of real continuous functions with compact support on X.
- $C_0(X)$ denotes the closure, in the sup norm, of $C_c(X)$.
- $\operatorname{diam} Y$ denotes the diameter of a set Y in a metric space.
- $d_{\mathcal{H}}$ denotes Hausdorff distance in $\mathbb{R}^n$.
- $\dim_{\mathcal{H}}(Y)$ denotes the Hausdorff dimension of a set Y.
- $\mathscr{D}_K$ denotes the distance to L^∞ space of a function K in a weak-L^p space.
- $\Delta_{s,h} f$ denotes the difference quotient of a function f with respect to s-th axis and increment h.
- $\partial^e E$ denotes the essential boundary of a Lebesgue measurable set E.
- $\partial^* E$ denotes the reduced boundary of a set of locally finite perimeter E.
- $E \Delta F$ denotes the symmetric difference between two sets E and F.
- $\mathbf{G}_k$ denotes the set of unoriented k-dimensional subspaces of $\mathbb{R}^n$.
- $H := \{x_n \leq 0\}$.
- $\mathcal{H}^d$ denotes d-dimensional Hausdorff measure in $\mathbb{R}^n$, for $d \geq 0$.
- id denotes identity/inclusion map between given sets.

- $\mathcal{L}_n$ denotes the σ-algebra of Lebesgue measurable sets in $\mathbb{R}^n$.
- $\mathcal{L}^n$ denote the Lebesgue measure in $\mathbb{R}^n$.
- L^p denotes the real valued p-integrable functions with respect to the Lebesgue measure on $\mathbb{R}^n$.
- $L^{p,\infty}$ denotes the Marcinkiewicz class with exponent $p > 1$, i.e. the weak-L^p space.
- L^p_w denotes the real valued p-integrable functions with respect to the measure $w\mathcal{L}^n$ on $\mathbb{R}^n$, where w is a weight.
- $[\mathcal{M}(X)]^m$ denotes the space of the finite $\mathbb{R}^m$-valued Radon measures on a locally compact and separable metric space X.
- $\left[\mathcal{M}_{loc}(X)\right]^m$ denotes the space of the $\mathbb{R}^m$-valued Radon measures on a locally compact and separable metric space X.
- $M^*_{Q_0}(f)$ denotes the Restricted Maximal Function Operator relative to a cube Q_0 for a function $f \in L^1(Q_0)$
- $M^*_{w,Q_0}(f)$ denotes the weighted Restricted Maximal Function Operator relative to a cube Q_0 for a function $f \in L^1(w, Q_0)$
- $P_\lambda(B^\lambda) := P_\lambda(B^\lambda(|B^\lambda|, x)) = P(B, \{x_n > \lambda\}) - \lambda\mathcal{H}^{n-1}(B \cap \{x_n = \lambda\})$.
- $Q_r := [-r, r]^n \subset \mathbb{R}^n$, for any $r > 0$.
- ν^E denotes the generalized outer unit normal to a set of locally finite perimeter E.
- $\mathbb{N}$ denotes the set of natural numbers.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{R}^n$ denotes the Euclidean n-dimensional space.
- $\overline{\mathbb{R}}$ denotes the extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$.
- $r_\lambda := \min\left\{\sqrt{1-\lambda^2}, 1-\lambda\right\}$, $R_\lambda := \max\left\{\sqrt{1-\lambda^2}, 1-\lambda\right\}$, for $\lambda \in (-1, 1)$.
- $\tau_{s,h}$ denotes the finite difference operator with respect to s-th axis and increment h.
- u_Ω denotes the mean value of a function $u \in L^1(\Omega)$.
- $V \subset\subset U$ denotes that the closure of the set V is compact and it is contained in U.
- $W^{k,p}(\Omega)$ denotes the Sobolev space on an open set $\Omega \subset \mathbb{R}^n$, for $k \in \mathbb{N}$, $k \geq 1$, and $p \in [1, +\infty]$.
- $W^{s,p}(\Omega)$ denotes the fractional Sobolev space on an open set $\Omega \subset \mathbb{R}^n$, for $s \in \mathbb{R}$, $s > 0$, and $p \in [1, +\infty)$.
- ω_n denotes the volume of the n-dimensional unit ball in $\mathbb{R}^n$.

Contents

Chapter 1

Introduction

The main topics of this thesis concern recent developments in the stability of minimizers for the classical capillarity problem in a half-space, as well as existence issues under the presence of nonlocal interaction and gravity. The results have been obtained in collaboration with Marco Pozzetta [Pas25; PP24]. Other results concerning regularity theory and obtained during the PhD studies are described in the introduction and in the last chapter of the thesis. These last results have been achieved in collaboration with Gioconda Moscariello [MP21; MP24].
The aim of this chapter is to review the motivations of the considered topics and to present the results, pointing out briefly the new ideas necessary to prove them.

1.1 Isoperimetry for the classical capillarity problem

The classical isoperimetric problem in the Euclidean space $\mathbb{R}^n$, for $n \geq 2$, aims at minimizing the $(n-1)$-dimensional area of boundaries of sets having fixed finite volume. More precisely, given $v > 0$, one aims to characterize minimizers to the problem

$$\inf \{ P(E) \ : \ E \subset \mathbb{R}^n, |E| = v \} , \tag{1.1.1}$$

where $P(E)$ denotes the perimeter of E and $|E|$ the Lebesgue measure of E. It is well-known that balls (uniquely) minimize (1.1.1), cf. [De 58] or [Mag12, Chapter 14], and this is encoded in the classical isoperimetric inequality

$$P(E) \geq n\omega_n^{\frac{1}{n}} |E|^{\frac{n-1}{n}}, \tag{1.1.2}$$

where ω_n denotes the measure of the unit ball in $\mathbb{R}^n$. To prove a *quantitative* version of (1.1.2) means to estimate the distance of a competitor from the set of minimizers in terms of the energy deficit of the competitor with respect to the infimum of the problem. The first quantitative isoperimetric inequality for (1.1.1) with sharp exponents was proved in [FMP08], and it reads

$$\alpha(E)^2 \leq C(n) D(E), \tag{1.1.3}$$

where $\alpha(E)$ and $D(E)$ are respectively the Fraenkel asymmetry and the isoperimetric deficit of E, i.e.,

$$\alpha(E) := \inf \left\{ \frac{E \Delta B(|E|, x)}{|E|} \ : \ x \in \mathbb{R}^n \right\} \qquad D(E) := \frac{P(E) - P(B(|E|))}{P(B(|E|))},$$

where $B(v, x)$ denotes the ball in $\mathbb{R}^n$ with volume v centered at x, for $v > 0$ and $x \in \mathbb{R}^n$, and $B(v) := B(v, 0)$. The underlying idea of the proof is to reduce the problem, by means of suitable geometric constructions, to the case of suitable axially symmetric sets and then to apply an induction argument over the dimension n. The inequality (1.1.3) improves the previous non-sharp inequality proved in [Hal92], after [HHW91; Fug89]. In [FMP10] a sharp quantitative version of the anisotropic isoperimetric inequality is established. The proof is based on a quantitative study of certain transportation maps, through bounds that can be derived from Gromov's proof of the isoperimetric inequality. In [AFM13; CL12] a selection principle is used to prove sharp quantitative isoperimetric inequalities. In a selection-type argument one argues by contradiction assuming existence of sets contradicting the quantitative isoperimetric inequality. These sets are then replaced by the minimizers of suitable auxiliary penalized minimization problems. Such minimizers are tailored in such a way that the quantitative isoperimetric inequality still fails. At the same time they are shown to be small perturbations of some isoperimetric set, contradicting the inequality already proved for sets given by small perturbations of optimal sets. A new direct proof of the classical isoperimetric inequality was given in [Cab00; Cab08] by means of ABP techniques, which were originally employed to derive regularity estimates for second order elliptic equations [GT01, Chapter 9]. We also mention [FI13; FJ14; IN15] for further quantitative isoperimetric inequalities for possibly anisotropic perimeters, [BDS15; CMM19; CES23] for quantitative isoperimetric inequalities on manifolds, and [Cia+11; BDR12; BBJ17; Cin+22; FL23] about weighted quantitative isoperimetric inequalities.

In Chapter 3 we prove quantitative isoperimetric inequalities for the following classical capillarity problem. If E is a measurable set in the half-space $\{x_n > 0\} \subset \mathbb{R}^n$ and $\lambda \in (-1, 1)$, we define the weighted perimeter functional

$$P_\lambda(E) := P(E, \{x_n > 0\}) - \lambda \mathcal{H}^{n-1}(\partial^* E \cap \{x_n = 0\}),$$

where $\mathcal{H}^k$, for $k \geq 0$, denotes the k-dimensional Hausdorff measure in $\mathbb{R}^n$, and $\partial^* E$ denotes the reduced boundary of E (see Chapter 2 for the definitions). Interpreting the perimeter as a measure of the surface tension of a liquid drop, the constant λ basically represent the relative adhesion coefficient between a liquid drop and the solid walls of the container given by $\{x_n > 0\}$.
If $v > 0$, we consider the isoperimetric capillarity problem

$$\inf \left\{ P_\lambda(E) \ : \ E \subset \{x_n > 0\}, |E| = v \right\}. \tag{1.1.4}$$

Minimizers for (1.1.4) are given by suitably truncated balls $B^\lambda(v, x)$, $x \in \{x_n = 0\}$, lying on the boundary of the half-space.

The first variational results regarding capillarity problems go back to works by Giusti, Gonzalez, Massari and Tamanini who established existence, symmetry and regularity results for the isotropic sessile drop problem, where an additional potential energy representing gravity is added to the minimization of P_λ (see [Gon76; Gon77; GT77; GMT80; Giu80; Giu81]; see also [Fin80] where uniqueness results for the symmetric sessile drop were established). We refer to [Fin86] and [Mag12, Chapters 19, 20] for a more complete treatment regarding classical results.
More recently, in [Bae15] the shape of liquid drops and crystals, resting on a horizontal surface and under the influence of gravity, are described in the anisotropic setting. The shape and the fine regularity of volume contrained minimizers of weighted perimeters like P_λ, where the weight on the interface touching the boundary of the container may be nonconstant and where an additional potential term is present, are addressed in [MM16; DM15; CEL24]. In [DM15] the perimeter functional measuring the area of the interface that does not touch the container is also possibly anisotropic. Recently, the isoperimetric problem for the relative perimeter of sets contained in the complement of a convex set had been addressed in [CGR07], where a sharp isoperimetric inequality is established, and in [FM23], where the rigidity of the inequality is addressed in the generality of measurable sets. Extensions of [CGR07] to higher codimension have been considered in [LWW23; Kru17], while the case of capillary energy outside convex cilinders has been considered in [FJM24].

The minimality of sets $B^\lambda(v, x)$ for (1.1.4) comes with an isoperimetric inequality for P_λ

$$P_\lambda(E) \geq c(n, \lambda)|E|^{\frac{n-1}{n}}.$$

In order to prove a quantitative isoperimetric inequality for (1.1.4), we define the corresponding Fraenkel asymmetry and isoperimetric deficit by setting

$$\alpha_\lambda(E) := \inf \left\{ \frac{|E \Delta B^\lambda(v, x)|}{v} \ : \ x \in \{x_n = 0\} \right\}, \qquad D_\lambda(E) := \frac{P_\lambda(E) - P_\lambda(B^\lambda(v))}{P_\lambda(B^\lambda(v))},$$

for any $E \subset \{x_n > 0\}$ with volume $|E| = v$. The infimum defining the asymmetry is, in fact, a minimum.
The first main result is the following

Theorem 1.1.1 ([PP24])**.** *Let $\lambda \in (-1, 1)$ and $n \in \mathbb{N}$ with $n \geq 2$. There exists a constant $c_{\rm iso} = c_{\rm iso}(n, \lambda) > 0$ such that for any measurable set $E \subset \mathbb{R}^n \cap \{x_n > 0\}$ with finite measure there holds*

$$\alpha_\lambda(E)^2 \leq c_{\rm iso} D_\lambda(E). \tag{1.1.5}$$

As for the classical quantitative isoperimetric inequality, perturbing the boundary of an optimal bubble only inside the container $\{x_n > 0\}$, it is possible to check that exponents in (1.1.5) are sharp.
Observing that, roughly speaking, the minimization problem

$$\inf\left\{P_\lambda(E) \ : \ E \subset \{x_n > 0\}, \, |E| = v\right\}$$

is symmetric with respect to the first $n-1$ axes, it is possible to adapt arguments in the spirit of [FMP08] to see that, in order to prove Theorem 1.1.1, it is sufficient to prove (1.1.5) in a class of suitable axially symmetric sets, see Corollary 3.4.12. However, the arguments in [FMP08] require to symmetrize a competitor with respect to a preferred axis depending on the competitor, while in our case it is only possible to symmetrize with respect to the n-th axis. The proof of (1.1.5) in this class of symmetric sets is then achieved here with a new combination of the so-called selection principle [AFM13; CL12] with an Alexandrov–Bakelman–Pucci-type technique in the spirit of [Cin+22].

In the context of these capillarity problems it is also spontaneous to consider a notion of asymmetry for the part of the boundary of a set that touches the plane $\{x_n = 0\}$. For a measurable set $E \subset \{x_n > 0\}$, we define

$$\beta_\lambda(E) := \inf\left\{ \frac{\mathcal{H}^{n-1}\left(\partial^* E \cap \{x_n = 0\} \,\Delta\, \partial^* B^\lambda(|E|, x) \cap \{x_n = 0\}\right)}{\mathcal{H}^{n-1}\left(\partial^* B^\lambda(|E|, x) \cap \{x_n = 0\}\right)} \ : \ x \in \{x_n = 0\} \right\}.$$

The previous quantity measures the asymmetry of the set $\partial^* E \cap \{x_n = 0\}$ with respect to $(n-1)$-dimensional balls in $\{x_n = 0\}$ having volume equal to the one of the trace of the optimal bubble corresponding to the volume of E. We establish the following second quantitative isoperimetric inequality, that provides a quantitative estimate on β_λ.

Theorem 1.1.2 ([PP24])**.** *Let $\lambda \in (-1, 1)$ and $n \in \mathbb{N}$ with $n \geq 2$. There exists a constant $c'_{\rm iso} = c'_{\rm iso}(n, \lambda) > 0$ such that for any measurable set $E \subset \mathbb{R}^n \cap \{x_n > 0\}$ with finite measure there holds*

$$\beta_\lambda(E) \leq c'_{\rm iso} \max\left\{ D_\lambda(E), D_\lambda(E)^{\frac{1}{2n}} \right\}.$$

Theorem 1.1.2 follows by applying again a selection-type argument where now β_λ plays the role of the Fraenkel asymmetry, together with a quantitative inequality that estimates the Hausdorff distance between the relative boundary in $\{x_n > 0\}$ of a suitable competitor E and the relative boundary of some bubble in terms of the Fraenkel asymmetry of E, see Lemma 3.6.4.

1.2 Capillarity in presence of nonlocal repulsion and gravity

The classical liquid drop model for the atomic nucleus in the Euclidean space $\mathbb{R}^n$, for $n \geq 2$, aims to characterize minimizers of the functional

$$P(E) + \int_E \int_E \frac{1}{|y-x|^\alpha} \, \mathrm{d}y \, \mathrm{d}x$$

among sets of given volume, where $0 < \alpha < n$ is a given parameter and $P(E)$ denotes the perimeter of $E \subset \mathbb{R}^n$. There is a clear competition between the two terms in the energy, since the ball at the same time minimizes the perimeter, by the isoperimetric inequality [De 58], [Mag12, Theorem 14.1] and maximizes the second term, by the Riesz rearrangement inequality [Rie30], [LL01, Theorem 3.7]. The physically relevant case is when $\alpha = 1$ and $n = 3$, that is when the second term is the Coulombic energy. This case goes back to Gamow's liquid drop model for atomic nuclei [Gam30], subsequently developed by von Weizsäcker [Wei35], Bohr [Boh36; BW39], and many other researchers. This model is used to explain various properties of nuclear matter [CPS74; CS62; MS96; PTM90], but it also arises in the Ohta-Kawasaki model for diblock copolymers [OK86] and in many other physical situations, see [CM75; CK93; Gen79; EK93; GDM95; KN86; Mam94; Nag95; NKD94]. For a more specific account on the physical background of this kind of problems, we refer to [Mur02].

In the last decades, the model for general α and n has gained renewed interest in mathematics literature, in order to investigate existence and non-existence of minimizers and the minimality of the ball. In [KM13a; KM14], Knüpfer and Muratov proved that balls are the only minimizers in the small mass regime when $n = 2$ and when $3 \leq n \leq 7$ with $0 < \alpha < n-1$. At the same time they obtained nonexistence results when $n \geq 2$ and $\alpha \in (0, 2)$. See also the alternative proofs [FKN16; LO14; Jul14] in the case $\alpha = 1$ and $n = 3$ and [MZ14] in the case $n = 2$ with α

sufficiently small. Later on, Bonacini and Cristoferi [BC14] proved existence and uniqueness results for every n and $0 < \alpha < n-1$. Finally, Figalli, Fusco, Maggi, Millot and Morini [Fig+15] studied the case $0 < \alpha < n$ for every n, even replacing the perimeter $P(E)$ by the fractional perimeter $P_s(E)$, $0 < s \le 1$. We refer to [CMT17; NO23a] for a review on the topic and to [AFM13; Fra19; FL15; FN21; FNV18; Jul17; Mur10; Nam20; NO23b; Ono22] and references therein for a more complete treatment. A variant of the problem with a constant background has been studied by [ACO09; CS13; CP10; CP11; EFK20; FL19; KMN16]; see also [AFM13; Ala+19; CN17; FNV18; GMS13; GMS14; Mur10; Nam20; Ono22; ST11] for further results on related problems.

In Chapter 4 we investigate, under a volume constraint and among sets contained in a Euclidean half-space, the minimization problem of an energy functional given by the sum of the capillarity perimeter, a nonlocal interaction term and a gravitational potential energy. In particular, if $g : \mathbb{R}^n \setminus \{0\} \to (0,\infty)$, we define the Riesz-type potential

$$\mathscr{R}(E) := \int_E \int_E g(y-x)\,\mathrm{d}y\,\mathrm{d}x,$$

and, given a function $G : (0,\infty) \to (0,\infty)$, we define the gravity-type potential

$$\mathscr{G}(E) := \int_E G(x_n)\,\mathrm{d}x.$$

If $v > 0$ and we denote

$$\mathscr{F}^\lambda(E) := P_\lambda(E) + \mathscr{R}(E) + \mathscr{G}(E),$$

we consider the nonlocal problem

$$\inf\{\mathscr{F}^\lambda(E) \ : \ E \subset \{x_n > 0\},\ |E| = v\}.$$

In the context of minimization of energies

$$P(E) + \int_E \int_E g(y-x)\,\mathrm{d}y\,\mathrm{d}x$$

with general Riesz-type potential g in the Euclidean space $\mathbb{R}^n$, Novaga and Pratelli in [NP21] showed the existence of (generalized) minimizers for radially decreasing g. Later on, Carazzato, Fusco and Pratelli in [CFP23] showed that the ball is the unique minimizer in the small mass regime. Pegon in [Peg21] showed that, if the kernel g decays sufficiently fast at infinity and if the volume is sufficiently large, then minimizers exist and converge to a ball as the volume goes to infinity. Then, Merlet and Pegon [MP22] proved that in the planar case minimizers are actually balls in the large mass regime. In [NO22], Novaga and Onoue obtained existence of minimizers for any volume and convergence to a ball as volume goes to infinity, if the Riesz potential decays sufficiently fast and even if the perimeter $P(E)$ is replaced by the fractional perimeter $P_s(E)$, $0 < s < 1$. We refer to [CN18; MW21; MS19; Rig00] and references therein for a more complete treatment on general nonlocal energies.

The first result is an existence result in the small mass regime, together with a bound on the Fraenkel asymmetry and some qualitative properties of volume constrained minimizers. At the same time, under suitable conditions on the potential energies, existence result extends to all masses.

Theorem 1.2.1 ([Pas25]). *Let g be a $\mathscr{R}$-admissible q-growing function, $q \ge 0$, and let G be a $\mathscr{G}$-admissible function. There exists a mass $\bar{m} = \bar{m}(n,\lambda,g,G,q) > 0$ such that, for every $m \in (0,\bar{m})$, there exists a minimizer of $\mathscr{F}^\lambda$ in the class*

$$\mathcal{A}_m := \{\Omega \subset \mathbb{R}^n \setminus H \text{ measurable } : \ |\Omega| = m\}$$

and it satisfies

$$\alpha_\lambda(E) \le c(n,\lambda,g,G) m^{\frac{1}{2n}}.$$

Moreover, if g is also infinitesimal, minimizers are indecomposable and, if in addition g is symmetric, minimizers are essentially bounded.
Furthermore, if g is also 0-growing, infinitesimal and symmetric and G is coercive, minimizers have no holes, i.e., if E is a minimizer of $\mathscr{F}^\lambda$ in $\mathcal{A}_m$, there is no set $F \subset \mathbb{R}^n \setminus (H \cup E)$ with $|F| > 0$ such that

$$P_\lambda(E) = P_\lambda(E \cup F) + P(F, \mathbb{R}^n \setminus H) + \lambda \mathcal{H}^{n-1}(\partial^* F \cap \partial H).$$

Finally, if g is $\mathscr{R}$-admissible and coercive and G is $\mathscr{G}$-admissible and coercive, there exists a minimizer of $\mathscr{F}^\lambda$ in $\mathcal{A}_m$ for any $m > 0$.

Let us make some comments on the definitions present in Theorem 1.2.1, while referring to Section 4.2 for their precise enunciation. The "admissibility" requirements on the kernels just refer to some necessary integrability conditions. The infinitesimality of g and the coercivity of g and G concern the behavior of these functions as the variable diverge, while the symmetry of g is referred to the symmetry with respect to the origin. The q-growing property is satisfied by rather general nonlocal interaction terms, not only by repulsive ones. Indeed, we point out that classical radial decreasing kernels are 0-growing, but at the same time attractive-repulsive kernels of the type

$$g(x) = |x|^{\beta_1} + \frac{1}{|x|^{\beta_2}}, \qquad \beta_1 > 0, \quad \beta_2 \in (0, n), \tag{1.2.1}$$

are q-growing for any $q \geq \beta_1$, even if they diverge positively as $|x| \to +\infty$, see Definition 4.2.1 and Remark 4.2.3. In particular attractive-repulsive kernels as in (1.2.1) represent a possible choice in the definition of $\mathscr{F}^\lambda$ in Theorem 1.2.1. Minimization problems for attractive-repulsive functionals have been widely studied in the last years. Existence and nonexistence results are addressed in [BCT18; FL18; FL21]. Stability and uniqueness of minimizers have been respectively studied in [BCT24; Lop19]. We refer to [Car23; CP22; CPT23; CDM16] for a more complete treatment on this kind of problems.

For large masses and for suitable choices of repulsive kernels g, the repulsive interaction dominates and the variational problem in Theorem 1.2.1 does not admit a minimizer.

Theorem 1.2.2 ([Pas25]). *Let*

$$g(x) = \frac{1}{|x|^\beta}, \qquad 0 < \beta < n,\ x \in \mathbb{R}^n \setminus \{0\}$$

and let G be $\mathscr{G}$-admissible. For every $\beta \in (0, 2]$, there exists $\tilde{m} > 0$, depending on n, λ, β, G, such that for all $m \geq \tilde{m}$ the minimization problem

$$\inf\{\mathscr{F}^\lambda(E) \ : \ E \subset \mathbb{R}^n \setminus H, |E| = m\}$$

has no minimizers.

Therefore, for a general repulsive kernel g, existence may fail for masses large enough, since minimizers tend to split in two or more components which then move apart one from the other in order to decrease the nonlocal energy. To capture this phenomenon, it is convenient to introduce a generalized energy defined as

$$\tilde{\mathscr{F}}^\lambda(E) := \inf_{h \in \mathbb{N}} \tilde{\mathscr{F}}^\lambda_h(E),$$

where

$$\tilde{\mathscr{F}}^\lambda_h(E) := \inf\left\{ \sum_{i=1}^h \mathscr{F}^\lambda(E^i) \ : \ E = \bigcup_{i=1}^h E^i, E^i \cap E^j = \emptyset \quad \text{for } 1 \leq i \neq j \leq h \right\}.$$

Note that in this functional the interaction between different components is not evaluated, which corresponds to consider them "at infinite distance" one from the other.
By considering $\tilde{\mathscr{F}}^\lambda$ instead of $\mathscr{F}^\lambda$, we can prove the following generalized existence result.

Theorem 1.2.3 ([Pas25]). *Let g be a $\mathscr{R}$-admissible q-growing function, $q \geq 0$, and let G be a $\mathscr{G}$-admissible function. For every $m > 0$ there exists a minimizer of $\tilde{\mathscr{F}}^\lambda$ in the class*

$$\mathcal{A}_m = \{\Omega \subset \mathbb{R}^n \setminus H \text{ measurable } : \ |\Omega| = m\}.$$

More precisely, there exist a set $E \in \mathcal{A}$ and a subdivision $E = \cup_{j=1}^h E^j$, with pairwise disjoint sets E^j, such that

$$\tilde{\mathscr{F}}^\lambda(E) = \sum_{j=1}^h \mathscr{F}^\lambda(E^j) = \inf\left\{\tilde{\mathscr{F}}^\lambda(\Omega) \ : \ \Omega \in \mathcal{A}\right\}.$$

Moreover, for every $1 \leq j \leq h$, the set E^j is a minimizer of both the standard and the generalized energy for its volume, i.e.

$$\tilde{\mathscr{F}}^\lambda(E^j) = \mathscr{F}^\lambda(E^j) = \min\left\{\tilde{\mathscr{F}}^\lambda(\Omega) \ : \ \Omega \subset \mathbb{R}^n \setminus H,\ |\Omega| = |E^j|\right\}.$$

We remark that in previous theorems we heavily use the quantitative isoperimetric inequality (1.1.5) for the capillarity problem. Moreover, classical and inspiring arguments as in [FN21; KM14; NP21] must be modified to take into account the presence of the gravitational energy and since the vertical direction must be treated separately.

1.3 Further results: Regularity theory for systems with discontinuous coefficients

In Chapter 5 we consider nonlinear elliptic systems of the type

$$\operatorname{div} A(x, Du(x)) = \operatorname{div} F(x) \tag{1.3.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n > 2$, and with $u : \Omega \to \mathbb{R}^N$, $N > 1$. We suppose that the vector field $A : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ is a Carathéodory function, i.e.

- $x \to A(x,\xi)$ is measurable for all $\xi \in \mathbb{R}^{N\times n}$,
- $\xi \to A(x,\xi)$ is continuous for a.e. $x \in \Omega$.

Furthermore, we assume that there exist a function $b(x) \geq \lambda_0 > 0$, belonging to the space BMO, and a function $K(x)$, belonging to the Marcinkiewicz space $L^{n,\infty}(\Omega)$, such that $F \in W^{1,2}_{loc}(b, \Omega; \mathbb{R}^{N\times n})$ and, for a.e. $x, y \in \Omega$,

$$|A(x,\xi) - A(x,\eta)| \leq kb(x)|\xi - \eta| \,(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}, \tag{1.3.2}$$

$$\frac{1}{k} b(x)|\xi - \eta|^2 \,(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \leq \langle A(x,\xi) - A(x,\eta), \xi - \eta \rangle\,, \tag{1.3.3}$$

$$|A(x,\eta) - A(y,\eta)| \leq |x - y| \; [K(x) + K(y)] \; (\mu^2 + |\eta|^2)^{\frac{p-1}{2}} \tag{1.3.4}$$

$$A(x, 0) = 0 \tag{1.3.5}$$

$$|b(x) - b(y)| \leq |x - y| \,[K(x) + K(y)], \tag{1.3.6}$$

where k is a positive constant, $\mu \in (0,1]$, $p \geq 2$, ξ and η are arbitrary elements of $\mathbb{R}^{N\times n}$. In the account of the typical functions of BMO and $L^{n,\infty}$ respectively, the functions

$$b(x) = \frac{e^{-|x|}}{\Lambda} - \Lambda \log|x|$$
$$K(x) = \frac{e^{-|x|}}{\Lambda} + \Lambda \frac{1}{|x|},$$

defined for a positive Λ with $x \in B(0,1) = \{y \in \mathbb{R}^n : 0 < |y| < 1\}$, satisfy assumption (1.3.6).
A vector field u in the Sobolev space $W^{1,r}_{loc}(b, \Omega; \mathbb{R}^N)$, $r > \frac{2n}{n+2}$, is a local solution of (1.3.1) if it verifies

$$\int_{\operatorname{supp}\varphi} \langle A(x, Du(x)), D\varphi(x) \rangle \,\mathrm{d}x = \int_{\operatorname{supp}\varphi} \langle F(x), D\varphi(x) \rangle \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega, \mathbb{R}^N).$$

Our first goal is to study regularity properties of local solutions to (1.3.1) for r close to p. The existence of second derivatives is not clear due to the degeneracy of the problem; anyway, although the first derivatives of the solutions may not be differentiable, the higher differentiability of solutions holds in the sense that the nonlinear expressions $V_\mu(Du) := (\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du$ of their gradients, with $\mu \in (0,1]$, are weakly differentiable. Therefore, the main result is the following:

Theorem 1.3.1 ([MP24]). *Let Ω be a regular domain, $A(x,\xi)$ a mapping verifying assumptions (1.3.2), (1.3.3), (1.3.4) and (1.3.5), and $F \in W^{1,2}_{loc}(b, \Omega; \mathbb{R}^{N\times n})$, with $b(x)$ as in (1.3.6). There exist $0 < \varepsilon_1 < \frac{1}{2}$, depending on k, n, λ_0, p and the BMO - norm of $b(x)$, and $\alpha_1 > 0$, depending on p, n, λ_0, μ and k, such that, if $u \in W^{1,p-\varepsilon}_{loc}(b, \Omega; \mathbb{R}^N)$, with $0 \leq \varepsilon < \varepsilon_1$, is a local solution of (1.3.1) and*

$$\mathscr{D}_K := \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_1,$$

then $D(V_\mu(Du)) \in L^2_{loc}(b, \Omega)$ and the following estimate holds:

$$\int_{B_R} |D(V_\mu(Du))|^2 b \,\mathrm{d}x \leq c \int_{B_{2R}} \left(\left(1 + \frac{1}{R^2}\right)(\mu^2 + |Du|^2)^{\frac{p}{2}} + (\mu^2 + |DF|^2)\right) b \,\mathrm{d}x,$$

for every ball $B_{2R} \subset\subset \Omega$ and for a constant c depending on p, k, λ_0, n, μ and $\mathscr{D}_K$.

The novelty of Theorem 1.3.1 is to consider nonlinear systems with growth coefficients in BMO and not uniformly continuous in the spatial variable, whose feature is that they are allowed to be very irregular. Moreover we deal with local solutions u to (1.3.1) lying in $W^{1,r}$ with $r \leq p$. In this case the energy functional

$$\int_{\Omega} \langle A(x, Du(x)), Du(x) \rangle \, \mathrm{d}x$$

could not be bounded. We refer to such a solution as a *very weak solution* as stated by Iwaniec and Sbordone in [IS94]. We explicitly remark that, thanks to the embedding theorem, our results apply if the growth coefficients lie in $W^{1,n}$. Theorem 1.3.1 extends analogous results in [MP21], which deals with linear systems of the type

$$\operatorname{div} A(x)Du(x) = \operatorname{div} F(x) + g.$$

In the linear case, the regularity results for systems with continuous coefficients can be considered classical. The first remarkable contribution is due to Agmon, Douglis and Nirenberg [ADN59; ADN64]. Later regularity results of Schauder type in the class of Hölderian functions are proved by Campanato [Cam65] and Morrey [Mor54]. See also [CC81]. A full discussion can be found in [GGM13; GM18].

The study of the second order regularity of solutions to linear equations with discontinuous coefficients goes back to C. Miranda who, in [Mir53; Mir60], considered equations with coefficients in the Sobolev class $W^{1,n}$. Then a significant improvement has been given in [AT85; CFL91; CFL93]. Subsequently, a complete regularity theory for equations in nondivergence form was developed by assuming coefficients in the vanishing mean oscillation space VMO (see e.g. [CFL93; Chi94]). More recently, in connection with the regularity of minimizers of functionals of the Calculus of Variations [AF89], the study of higher differentiability for solutions to problems of the type

$$\operatorname{div} A(x, Du) = \operatorname{div} F(x) \tag{1.3.7}$$

had a remarkable development. In particular, estimates of the type

$$\int_{B_R} |D^2u|^2 \, \mathrm{d}x \leq c \int_{B_{2R}} \left(\left(1 + \frac{1}{R^2} \right) |Du|^p + |DF|^2 \right) \mathrm{d}x$$

are important elements to prove partial regularity properties of solutions to nonlinear elliptic systems with Uhlenbeck structure. Linear equations having coefficients in BMO with small norm have been addressed in [GMR09]. In [Str01] Stroffolini studied the Dirichlet problem for very weak solutions to a linear system with coefficients in BMO. More recently, a smallness condition on

$$\mathscr{D}_{(\cdot)} := \operatorname{dist}_{L^{n,\infty}}(\cdot, L^{\infty})$$

has been considered in [GM18] to study the L^p - regularity of a linear Dirichlet problem. In [DK11] linear systems with coefficients having in some directions locally small mean oscillation have been studied. The case of a nonlinear system with $b(x) \in L^{\infty}(\Omega)$ has been considered in [GM23]. For a more complete treatment about smallness conditions on $\mathscr{D}_{(\cdot)}$, we refer to [Far+23; Far+21]. In these papers a similar bound turned out to be necessary in proving the existence of solutions to noncoercive PDEs having singularities in the coefficients of lower order terms. We also mention the similar conditions in [Boc15; GMZ15; GMZ18]. Optimal second order regularity properties of solutions to nonlinear p-Laplacian systems are given in [CM19], when the datum in the right hand side of (1.3.7) is not in divergence form. We refer also to [KM13b; KM10; Min06] and reference therein. We point out that for local solutions of homogeneous systems

$$\operatorname{div} A(x, Du) = 0,$$

Theorem 1.3.1 also applies in the degenerate case, i.e. $\mu = 0$, with constants independent of μ. As a consequence, in section we establish certain local Calderón and Zygmund type estimates without assuming any differentiability condition on the datum. More precisely, for $G \in L^p_{loc}(b, \Omega; \mathbb{R}^{N\times n})$ we consider the problem

$$\operatorname{div} A(x, Du(x)) = \operatorname{div} |G|^{p-2}G \quad \text{in} \quad \Omega. \tag{1.3.8}$$

Then we prove the following result:

Theorem 1.3.2 ([MP24])**.** *Let Ω be a regular domain and $A(x,\xi)$ a mapping verifying assumptions* (1.3.2), (1.3.3), (1.3.4) *and* (1.3.5)*, with $b(x)$ as in* (1.3.6)*. There exists $\alpha_2 > 0$, depending on p, n, λ_0 and k, such that, if $u \in W^{1,p}_{loc}(b,\Omega;\mathbb{R}^N)$ is a local solution of* (1.3.8) *and*

$$\mathscr{D}_K := \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_2,$$

then

$$G \in L^q_{loc}(b,\Omega;\mathbb{R}^{N\times n}) \quad \Longrightarrow \quad Du \in L^q_{loc}(b,\Omega;\mathbb{R}^{N\times n})$$

for any $q \in (p,s)$, where $s := \frac{np}{n-1} + \delta$ for a suitable $\delta > 0$, depending on p, k, λ_0, n, $\mathscr{D}_K$ and the BMO-norm of b. Moreover, for every cube $Q_{2R} \subset\subset \Omega$ and $\mu \in [0,1]$, we have

$$\left(\fint_{Q_R} \left(\mu^2 + |Du|^2\right)^{\frac{q}{2}} b\,\mathrm{d}x\right)^{\frac{1}{q}} \le c\left(\fint_{Q_{2R}} \left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} b\,\mathrm{d}x\right)^{\frac{1}{p}} + \\ + c\left(\fint_{Q_{2R}} (\mu^2 + |G|^2)^{\frac{q}{2}} b\,\mathrm{d}x\right)^{\frac{1}{q}},$$

where c depends on p, $s-q$, k, λ_0, n, $\mathscr{D}_K$ and the BMO - norm of b and is independent of μ.

Calderòn–Zygmund type estimates in the case of the p-Laplacian equation with $p > 2$ were established in the fundamental paper by T. Iwaniec [Iwa83]. Let us remark that such kind of estimate is relevant to provide upper bounds for the Hausdorff dimension of the singular set of minima of general variational integrals [KM10; KM06; Min03]. Additionally, the a priori knowledge of higher integrability of the gradient allows to implement better schemes in the numerical treatment of problems modeled by energies like $\int_\Omega \langle A(x, Du(x)), Du(x)\rangle\,\mathrm{d}x$, as e.g. electrorheological fluids. Subsequently Iwaniec's results were generalized to systems by DiBenedetto and Manfredi [DM93]. Regarding equations of the type

$$\operatorname{div}\left((A(x)\,\nabla u \cdot \nabla u)^{\frac{p-2}{2}}\, A(x)\,\nabla u\right) = \operatorname{div}|G|^{p-2}G, \tag{1.3.9}$$

with $A(x) : \Omega \to \mathbb{R}^{n\times n}$ symmetric, local and global estimates for the gradient of a solution were considered by Iwaniec and Sbordone [IS01; IS98] and by Kinnunen and Zhou [KZ99] when the coefficients of $A(x)$ are bounded and in VMO. The condition about $A(x)$ in VMO is relaxed to a small BMO condition in [BW04] and [BWZ07]. Recently local and global estimates for degenerate equations of the type (1.3.9) are given in weighted spaces in [Bal+23] and [Bal+22b] assuming a smallness condition for the BMO norm of $\log A(x)$ depending on the exponent q. This result is not strictly comparable with results in [MP21; MP24], where the exponent depends on the BMO norm and on the bound on the distance $\mathscr{D}_{(\cdot)}$. Moreover, as mentioned earlier, we require no condition of smallness of the norm.

New main estimates for the development of nonlinear Calderòn–Zygmund theory for equations and systems are due to Mingione, starting from pioneering papers [Min07a; Min07b]. Regarding systems, weaker results are available unless in the case of the p-Laplacian system (see [DM93; Uhl77]). Indeed, some bounds on exponent q is necessary according to the example exhibited in [ŠY02]. If the vector field $A(\xi)$ is sufficiently regular, then CZ-estimates survive for $q \in \left(p, \frac{np}{n-2}\right)$ (see [DKM07; Min17] and references therein). A significant extension of CZ-theory to non-uniformly elliptic operators shaped on the $p(x)$-Laplacian [AM05; CKP11] and to the double-phase problems [CM15] were also established, following the fundamental paper [Mar89].

Chapter 2

Preliminaries

In this chapter we recall the main definitions and tools we will need, with the aim of fixing the notation and making the exposition self-contained, at least as far as the results are concerned.

2.1 Preliminaries in measure theory

In this section we present the basic notions of measure theory which are exlpoited in the text, while we refer to [AFP00; Bre95; Fed69; Mag12; Rud87] for a complete treatment on the topic

Abstract measure theory

Definition 2.1.1 (σ-algebras and measure spaces)**.** Let X be a nonempty set and let $\mathcal{E}$ be a collection of subsets of X.

- We say that $\mathcal{E}$ is a *σ-algebra* if $\emptyset \in \mathcal{E}$, for any sequence $\{E_h\} \subset \mathcal{E}$ its union $\bigcup_h E_h$ belongs to $\mathcal{E}$ and $X \setminus F \in \mathcal{E}$ whenever $F \in \mathcal{E}$.
- For any collection $\mathcal{G}$ of subsets of X, the σ-algebra generated by $\mathcal{G}$ is the smallest σ-algebra containing $\mathcal{G}$. If (X, τ) is a topological space, we denote by $\mathcal{B}(X)$ the σ-algebra of Borel subsets of X, i.e., the σ-algebra generated by the open subsets of X.
- If $\mathcal{E}$ is a σ-algebra in X, we call the pair $(X, \mathcal{E})$ a *measure space*.

Since the intersection of any family of σ-algebras is a σ-algebra, the definition of generated σ-algebra is well posed.

Definition 2.1.2 (Positive measures)**.** Let $(X, \mathcal{E})$ be a measure space and $\mu : \mathcal{E} \to [0, \infty]$.

- We say that μ is a *positive measure* if $\mu(\emptyset) = 0$ and μ is σ-additive on $\mathcal{E}$, i.e. for any sequence $\{E_h\}$ of pairwise disjoint elements of $\mathcal{E}$
$$\mu\left(\bigcup_{h=0}^{\infty} E_h\right) = \sum_{h=0}^{\infty} \mu(E_h).$$
We say that μ is *finite* if $\mu(X) < \infty$.
- We say that a positive measure μ on X is *σ-finite* if X is the union of an increasing sequence of sets with finite measure.

A positive measure μ such that $\mu(X) = 1$ is also called a *probability measure*.
Beside positive measures, it is also possible to define real- and vector-valued measures. Note that positive measures are not a particular case of real measures, since real measures, according to the following definition, must be finite.

Definition 2.1.3 (Real and vector measures)**.** Let $(X, \mathcal{E})$ is a measure space and let $m \in \mathbb{N}$, $m \geq 1$.

- We say that $\mu : \mathcal{E} \to \mathbb{R}^m$ is a *measure* if $\mu(\emptyset) = 0$ and for any sequence $\{E_h\}$ of pairwise disjoint elements of $\mathcal{E}$
$$\mu\left(\bigcup_{h=0}^{\infty} E_h\right) = \sum_{h=0}^{\infty} \mu(E_h). \tag{2.1.1}$$
If $m = 1$ we say that μ is a *real measure*, if $m > 1$ we say that μ is a *vector measure*.

- If μ is a measure, we define its *total variation* $|\mu|$ for every $E \in \mathcal{E}$ as follows:
$$|\mu|(E) := \sup\left\{\sum_{h=0}^{\infty} |\mu(E_h)| \ : \ E_h \in \mathcal{E} \text{ pairwise disjoint}, \ E = \bigcup_{h=0}^{\infty} E_h\right\}.$$

Note that the absolute convergence of the series in (2.1.1) is a requirement on the set function μ: in fact, the sum of the series cannot depend on the order of its terms, as the union does not. Moreover, by [AFP00, Theorem 1.6] $|\mu|$ is a positive finite measure.

Definition 2.1.4 (μ-negligible sets)**.** Let μ be a positive measure on the measure space $(X, \mathcal{E})$.

- We say that $N \subset X$ is *μ-negligible* if there exists $E \in \mathcal{E}$ such that $N \subset E$ and $\mu(E) = 0$.
- We say that a property $P(x)$ depending on the point $x \in X$ *holds* μ-a.e. *in* X if the set where P fails is a μ-negligible set.
- Let $\mathcal{E}_\mu$ be the collection of all the subsets of X of the form $F = E \cup N$, with $E \in \mathcal{E}$ and N μ-negligible; then $\mathcal{E}_\mu$ is a σ-algebra which is called the *μ-completion of* $\mathcal{E}$, and we say that $E \subset X$ is *μ-measurable* if $E \in \mathcal{E}_\mu$. The measure μ extends to $\mathcal{E}_\mu$ by setting, for F as above, $\mu(F) = \mu(E)$.

If μ is a real or vector measure, we call the completion of $\mathcal{E}$ with respect to the total variation $|\mu|$ of μ the μ-completion $\mathcal{E}_\mu$ of $\mathcal{E}$. Then, the measure μ can be extended to $\mathcal{E}_\mu$ as above. Unless otherwise indicated, from now on each measure μ is tacitly extended to the completion $\mathcal{E}_\mu$.

Definition 2.1.5 (Measurable functions)**.** Let $(X, \mathcal{E})$ be a measure space and (Y, d) a metric space.

- A function $f : X \to Y$ is said to be *$\mathcal{E}$-measurable* if $f^{-1}(A) \in \mathcal{E}$ for every open set $A \subset Y$.
- If μ is a positive measure on $(X, \mathcal{E})$ the function f is said to be *μ-measurable* if it is $\mathcal{E}_\mu$-measurable.

In particular, if f is $\mathcal{E}$-measurable then $f^{-1}(B) \in \mathcal{E}$ for every $B \in \mathcal{B}(Y)$.

Definition 2.1.6 (Integrals)**.** Let $(X, \mathcal{E})$ be a measure space.

- For $E \subset X$ we define the *characteristic function of E*, denoted by χ_E, by
$$\chi_E(x) := \begin{cases} 1 & \text{if} \quad x \in E \\ 0 & \text{if} \quad x \notin E. \end{cases}$$
We say that $f : X \to \mathbb{R}$ is a *simple function* if the image of f is finite.

- Let μ be a positive measure on $(X, \mathcal{E})$; the *integral* of a simple μ-measurable function $u : X \to [0, \infty)$ is defined by
$$\int_X u \, \mathrm{d}\mu := \sum_{z \in \mathrm{im}(u)} z\mu(u^{-1}(z)),$$
where we adopt the convention that whenever $z = 0$ and $\mu(u^{-1}(z)) = \infty$ the product $z\mu(u^{-1}(z))$ is set equal to zero. The definition is extended to any μ-measurable function $u : X \to [0, \infty]$ by setting:
$$\int_X u \, \mathrm{d}\mu := \sup\left\{\int_X v \, \mathrm{d}\mu \ : \ v \ \mu\text{−measurable, simple}, \ v \le u\right\}.$$

We say that a μ-measurable map $u : X \to \overline{\mathbb{R}}$ is *μ-integrable* if either

$$\int_X u^+ \, d\mu < \infty \qquad \text{or} \qquad \int_X u^- \, d\mu < \infty.$$

If u is μ-integrable, we set

$$\int_X u \, d\mu := \int_X u^+ \, d\mu - \int_X u^- \, d\mu.$$

- Let μ be a measure on $(X, \mathcal{E})$ and $u : X \to \overline{\mathbb{R}}$ a $|\mu|$-measurable function; we say that u is *μ-integrable* if u is $|\mu|$-integrable and, if μ is real, we set

$$\int_X u \, d\mu := \int_X u \, d\mu^+ - \int_X u \, d\mu^-.$$

If μ is an $\mathbb{R}^m$-valued vector measure then we set

$$\int_X u \, d\mu := \left(\int_X u \, d\mu_1, \dots, \int_X u \, d\mu_m \right).$$

Note that an immediate consequence of the above definition is the inequality

$$\left| \int_X u \, d\mu \right| \le \int_X |u| \, d|\mu|,$$

which holds for every extended real or vector valued function u with finite integral and for every positive, real or vector measure μ. More generally, let us recall

Theorem 2.1.7 (Jensen inequality, [AFP00, Lemma 1.15])**.** *Let $\Phi : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function, μ a probability measure on $(X, \mathcal{E})$ and $u : X \to \mathbb{R}^k$ a μ-summable function; then*

$$\Phi \left(\int_X u \, d\mu \right) \le \int_X \Phi(u) \, d\mu.$$

When E is a μ-measurable set the integral of a function u on E is defined by

$$\int_E u \, d\mu := \int_X u \chi_E \, d\mu,$$

provided that the right-hand side makes sense. Note also that, if u has finite integral, for any $\varepsilon > 0$ there is a measurable set A with finite measure such that $\int_{X \setminus A} |u| \, d|\mu| < \varepsilon$.

Definition 2.1.8 (L^p spaces)**.** Let $(X, \mathcal{E})$ be a measure space, μ a positive measure on it and $u : X \to \overline{\mathbb{R}}$ a μ-measurable function. We set

$$\|u\|_{L^p} := \left(\int_X |u|^p \, d\mu \right)^{\frac{1}{p}}$$

if $1 \le p < \infty$, and

$$\|u\|_{L^\infty} := \inf \{ C \in [0, \infty] \ : \ |u(x)| \le C \quad \text{for } \mu\text{–a.e. } x \in X \}$$

We say that $u \in L^p(X, \mu)$ if $\|u\|_{L^p} < \infty$. The set $L^p(X, \mu)$ is a real vector space and $\| \cdot \|_{L^p}$ is a semi-norm.

When dealing with measure-theoretic or functional-analytic properties of functions and L^p spaces, it is often convenient to consider functions that agree a.e. as identical, thinking of the elements of L^p spaces as equivalence classes; in particular, this makes $\| \cdot \|_{L^p}$ a norm. However we shall not consider functions agreeing a.e. to be identical if we are concerned with fine properties of the single function.
For $1 < p < \infty$ the Banach space $L^p = L^p(X, \mu)$ is uniformly convex (hence reflexive) and its dual is $L^{p'}$, with $p' = \frac{p}{p-1}$; if μ is σ-finite, the dual of L^1 is L^∞. Accordingly, the weak convergence of sequences is defined:

Definition 2.1.9 (Convergence in L^p spaces)**.** Given f, $\{f_h\} \in L^p$, we say that $f_h \to f$ weakly if

$$\int_X f_h g \, d\mu \to \int_X f g \, d\mu$$

forr any $g \in L^{p'}$ if $1 \leq p < \infty$, and that $f_h \to f$ weakly* in L^∞ if $p = \infty$ and

$$\int_X f_h g \, d\mu \to \int_X f g \, d\mu$$

for any $g \in L^1$.

We state here some relevant theorems concerning convergence of integrals.

Theorem 2.1.10 (Fatou's lemma, [AFP00, Theorem 1.20])**.** *Let $u_h : X \to \overline{\mathbb{R}}$ be a μ-measurable function and $g \in L^1(X, \mu)$. Then*

$$\int_X \liminf_{h\to\infty} u_h \, d\mu \leq \liminf_{h\to\infty} \int_X u_h \, d\mu$$

if $u_h \geq g$ for any $h \in \mathbb{N}$ and

$$\int_X \limsup_{h\to\infty} u_h \, d\mu \geq \limsup_{h\to\infty} \int_X u_h \, d\mu$$

if $u_h \leq g$ for any $h \in \mathbb{N}$.

Theorem 2.1.11 (Dominated convergence theorem, [AFP00, Theorem 1.21])**.** *Let u, $u_h : X \to \overline{\mathbb{R}}$ be μ-measurable functions, and assume that $u_h(x) \to u(x)$ for μ-a.e. $x \in X$ as $h \to \infty$. If*

$$\int_X \sup_h |u_h| \, d\mu < \infty$$

then

$$\lim_{h\to\infty} \int_X u_h \, d\mu = \int_X u \, d\mu.$$

Now we introduce the notion of Borel and Radon measures.

Definition 2.1.12 (Borel and Radon measures)**.** Let X be a locally compact and separable metric space, $\mathcal{B}(X)$ its Borel σ-algebra, and consider the measure space $(X, \mathcal{B}(X))$.

- A positive measure on $(X, \mathcal{B}(X))$ is called a *Borel measure*. If a Borel measure is finite on the compact sets, it is called *positive Radon measure*.
- A (real or vector) set function defined on the relatively compact Borel subsets of X that is a measure on $(K, \mathcal{B}(K))$ for every compact set $K \subset X$ is called a (*real* or *vector*) *Radon measure* on X. If $\mu : \mathcal{B}(X) \to \mathbb{R}^m$ is a measure then we say that is a *finite Radon measure*. We denote by $[\mathcal{M}_{loc}(X)]^m$ (resp. $[\mathcal{M}(X)]^m$) the space of the $\mathbb{R}^m$-valued Radon (resp. finite $\mathbb{R}^m$-valued Radon) measures on X.

Note that if μ is a Radon measure and $\sup\{|\mu|(K) : K \subset X \text{ compact}\} < \infty$ then it can be extended to the whole of $\mathcal{B}(X)$ and the resulting set function, which we still denote μ, is a finite Radon measure.

Definition 2.1.13 (Borel functions)**.** Let X, Y be metric spaces, and let $f : X \to Y$. We say that f is a *Borel function* if $f^{-1}(A) \in \mathcal{B}(X)$ for every open set $A \subset Y$.

We now present the definition of outer measures in metric spaces, which embodies an additivity condition on separated sets.

Definition 2.1.14 (Outer measures)**.** Let X be a metic space and μ a function defined on all the subset of X with values in $[0, \infty]$; we say that μ is an outer measure if $\mu(\emptyset) = 0$, the following subadditivity condition holds

$$E \subset \bigcup_{h=0}^{\infty} E_h \implies \mu(E) \leq \sum_{h=0}^{\infty} \mu(E_h)$$

for any E, $\{E_h\} \subset X$, and moreover the following additivity condition holds:

$$\text{dist}(E, F) > 0 \implies \mu(E \cup F) = \mu(E) + \mu(F)$$

for any E, $F \subset X$.

Theorem 2.1.15 (Carathéodory criterion, [AFP00, Theorem 1.49])**.** *Let μ be an outer measure on the metric space X; then μ is σ-additive on $\mathcal{B}(X)$, hence the restriction of μ to the Borel sets of X is a positive measure.*

Example 2.1.16 (Lebesgue measure). Let $\mathring{Q}_r(x) = \{y \in \mathbb{R}^n \ : \ \max_i |x_i - y_i| < r\}$ be the open cube with side $2r$ centered at x and set

$$\mu(E) := \inf \left\{ \sum_{h=0}^{\infty} (2r_h)^n \ : \ E \subset \bigcup_{h=0}^{\infty} \mathring{Q}_{r_h}(x_h) \right\}$$

for any $E \subset \mathbb{R}^n$. Then μ is an outer measure, that we call *Lebesgue outer measure* and we denote by $\mathcal{L}^n$. Since it is finite on compact sets, according to Carathéodory criterion 2.1.15 its restriction to $\mathcal{B}(\mathbb{R}^n)$ is a Radon measure. We say that $E \subset \mathbb{R}^n$ is *Lebesgue measurable* if E belongs to the completion $\mathcal{B}_{\mathcal{L}^n}(\mathbb{R}^n)$. The σ-algebra of Lebesgue measurable sets is denoted by $\mathcal{L}_n$ and we write $|E|$ for $\mathcal{L}^n(E)$ for any $E \subset \mathbb{R}^n$.

Definition 2.1.17 (Restriction)**.** Let μ be a positive, real or vector measure on the measure space $(X, \mathcal{E})$. If $E \in \mathcal{E}$ we set $\mu \llcorner E(F) := \mu(E \cap F)$ for every $F \in \mathcal{E}$.

Note that the restriction of μ to E can be also defined as $\mu \llcorner E = \chi_E \mu$; moreover, if μ is a Borel (resp. Radon) measure and E is a Borel set, then the measure $\mu \llcorner E$ is a Borel (resp. Radon) measure, too.
Given a measure space $(X, \mathcal{E})$ and a measure on it, we see how it can be carried on another set Y through a function $f : X \to Y$.

Definition 2.1.18 (Push-forward)**.** Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be measure spaces, and let $f : X \to Y$ be such that $f^{-1}(F) \in \mathcal{E}$ whenever $F \in \mathcal{F}$. For any positive, real, or vector measure μ on $(X, \mathcal{E})$ we define a measure $f_\#$ in $(Y, \mathcal{F})$ by

$$f_\#\mu(F) := \mu\left(f^{-1}(F)\right) \qquad \forall F \in \mathcal{F}.$$

From the previous definition the corresponding change of variable formula for integrals follows immediately: if u is a (real- or vector-valued) function on Y summable with respect to $f_\#\mu$, then $u \circ f$ is summable with respect to μ and we have the equality

$$\int_Y u \,\mathrm{d}(f_\#\mu) = \int_X u \circ f \,\mathrm{d}\mu.$$

We now consider two measure spaces and see the resulting structure on their Cartesian product.

Definition 2.1.19 (Product σ-algebra)**.** Let $(X_1, \mathcal{E}_1)$ and $(X_2, \mathcal{E}_2)$ be measure spaces. The *product σ-algebra* of $\mathcal{E}_1$ and $\mathcal{E}_2$, denoted by $\mathcal{E}_1 \times \mathcal{E}_2$, is the σ-algebra generated in $X_1 \times X_2$ by

$$\mathcal{G} = \{E_1 \times E_2 \ : \ E_1 \in \mathcal{E}, \ E_2 \in \mathcal{E}_2\}.$$

Let $E \in \mathcal{E}_1 \times \mathcal{E}_2$; then for every $x \in X_1$ the section $E_x = \{y \in X_2 \ : \ (x, y) \in E\}$ belongs to $\mathcal{E}_2$, and for every $y \in X_2$ the section $E_y = \{x \in X_1 \ : \ (x, y) \in E\}$ belongs to $\mathcal{E}_1$.

Theorem 2.1.20 (Fubini, [AFP00, Theorem 1.74])**.** *Let $(X_1, \mathcal{E}_1)$, $(X_2, \mathcal{E}_2)$ be measure spaces and μ_1, μ_2 be positive σ-finite measures in X_1, X_2 respectively. Then, there is a unique positive σ-finite measure μ on $(X_1 \times X_2, \mathcal{E}_1 \times \mathcal{E}_2)$ such that*

$$\mu(E_1 \times E_2) = \mu_1(E_1) \cdot \mu_2(E_2) \qquad \forall E_1 \in \mathcal{E}_1, \ \forall E_2 \in \mathcal{E}_2.$$

Furthermore, for any μ-measurable function $u : X_1 \times X_2 \to [0, \infty]$ we have that

$$x \mapsto \int_{X_2} u(x, y) \,\mathrm{d}\mu_2(y) \qquad \text{and} \qquad y \mapsto \int_{X_1} u(x, y) \,\mathrm{d}\mu_1(x)$$

are respectively μ_1-measurable and μ_2-measurable and

$$\int_{X_1 \times X_2} u \,\mathrm{d}\mu = \int_{X_1} \left(\int_{X_2} u(x, y) \,\mathrm{d}\mu_2(y) \right) \mathrm{d}\mu_1(x) = \int_{X_2} \left(\int_{X_1} u(x, y) \,\mathrm{d}\mu_1(x) \right) \mathrm{d}\mu_2(y).$$

Once the product measure μ has been introduced on $X_1 \times X_2$, μ-measurability refers to $(\mathcal{E}_1 \times \mathcal{E}_2)_\mu$, the completion of $\mathcal{E}_1 \times \mathcal{E}_2$ with respect to μ. Moreover, if X_1, X_2 and Y are metric spaces and $f : X_1 \times X_2 \to Y$ is a Borel function, then all its sections are Borel.

Weak* convergence

We fix a measure space $(X, \mathcal{E})$. Now we introduce a notion of convergence for Radon measures.

Definition 2.1.21 (Weak* convergence of measures)**.** Let $\mu \in [\mathcal{M}_{loc}(X)]^m$ and let $\{\mu_h\} \subset \mathcal{M}_{loc}(X)]^m$; we say that $\{\mu_h\}$ *locally weakly* converges to* μ if

$$\lim_{h\to\infty} \int_X u \, \mathrm{d}\mu_h = \int_X u \, \mathrm{d}\mu$$

for every $u \in C_c(X)$; if μ and the μ_h are finite, we say that $\{\mu_h\}$ *weakly* converges to* μ if

$$\lim_{h\to\infty} \int_X u \, \mathrm{d}\mu_h = \int_X u \, \mathrm{d}\mu$$

for every $u \in C_0(X)$.

The weak* convergence of a sequence $\{\mu_h\}$ of finite Radon measures is equivalent to the local weak* convergence together with the condition $\sup_h |\mu_h|(X) < \infty$.
Now we state some properties regarding the lower semicontinuity and the continuity under weak* convergence of the functionals

$$\int_\Omega f\left(x, \frac{\mu}{|\mu|}(x)\right) \mathrm{d}|\mu|(x)$$

depending on vector valued Radon measures μ in an open subset Ω of $\mathbb{R}^n$, first studied by Y. G. Reshetnyak in [Res68].

Theorem 2.1.22 (Reshetnyak lower semicontinuity, [AFP00, Theorem 2.38])**.** *Let Ω be an open set of $\mathbb{R}^n$ and μ, μ_h be $\mathbb{R}^m$-valued finite Radon measures in Ω; if $\mu_h \to \mu$ weakly* in Ω then*

$$\int_\Omega f\left(x, \frac{\mu}{|\mu|}(x)\right) \mathrm{d}|\mu|(x) \le \liminf_{h\to\infty} \int_\Omega f\left(x, \frac{\mu_h}{|\mu_h|}(x)\right) \mathrm{d}|\mu_h|(x)$$

for every lower semicontinuous function $f : \Omega\times\mathbb{R}^m \to [0, \infty]$, positively 1-homogeneous and convex in the second variable.

Theorem 2.1.23 (Reshetnyak continuity, [AFP00, Theorem 2.39])**.** *Let Ω, μ_h, μ as in Theorem 2.1.22; if $|\mu_h|(\Omega) \to |\mu|(\Omega)$ then*

$$\lim_{h\to\infty} \int_\Omega f\left(x, \frac{\mu_h}{|\mu_h|}(x)\right) \mathrm{d}|\mu_h|(x) = \int_\Omega f\left(x, \frac{\mu}{|\mu|}(x)\right) \mathrm{d}|\mu|(x)$$

for every continuous and bounded function $f : \Omega \times \mathbf{S}^{m-1} \to \mathbb{R}$.

Disintegration

In this section we define a generalized notion of product of measures, where one of the factors is allowed to vary from a point to another, and we state a disintegration theorem, which allows us to decompose a measure on a product space as a generalized product of this kind.

Definition 2.1.24 (Measurable measure-valued maps)**.** Let $E \subset \mathbb{R}^n$, $F \subset \mathbb{R}^m$ be open sets, μ a positive Radon measure on E, and $x \mapsto \nu_x$ a function which assigns to each $x \in E$ a $\mathbb{R}^m$-valued finite Radon measure ν_x on F. We say that this map is μ-measurable if $x \mapsto \nu_x(B)$ is μ-measurable for any $B \in \mathcal{B}(F)$.

Proposition 2.1.25 ([AFP00, Proposition 2.26])**.** *Let E, F, μ and ν_x be as in Definition 2.1.24. If $x \mapsto \nu_x(A)$ is μ-measurable for any open set $A \subset F$, then $x \mapsto \nu_x$ is μ-measurable. Moreover, $x \mapsto \int_F g(x, y)\, \mathrm{d}\nu_x(y)$ is μ-measurable for any bounded $\mathcal{B}_\mu(E) \times \mathcal{B}(F)$-measurable function $g : E \times F \to \mathbb{R}$.*

As a consequence of Proposition 2.1.25 we have the implication

$$\nu_x \quad \mu - \text{measurable} \qquad \Longrightarrow \qquad |\nu_x| \quad \mu - \text{measurable}.$$

Measurable measure-valued functions give rise to the following notion of integral of measures, which generalizes the product of two measures.

Definition 2.1.26 (Generalized product)**.** Let E, F, μ and ν_x be as in Definition 2.1.24 and assume that

$$\int_{E'} |\nu_x|(F)\,\mathrm{d}\mu(x) < \infty \qquad \forall E' \subset\subset E \text{ open.}$$

We denote by $\mu \otimes \nu_x$ the $\mathbb{R}^m$-valued Radon measure on $E \times F$ defined by

$$\mu \otimes \nu_x(B) := \int_E \left(\int_F \chi_B(x,y)\,\mathrm{d}\nu_x(y) \right) \mathrm{d}\mu(x) \qquad \forall B \in \mathcal{B}(K \times F).$$

where $K \subset E$ is any compact set.

The measure $\mu \otimes \nu_x$ is well defined, thanks to Proposition 2.1.25. Note that the integration formula

$$\int_{E\times F} f(x,y)\,\mathrm{d}(\mu \otimes \nu_x)(x,y) = \int_E \left(\int_F f(x,y)\,\mathrm{d}\nu_x(y) \right) \mathrm{d}\mu(x)$$

holds for every bounded Borel function $f : E \times F \to \mathbb{R}$ with $\operatorname{supp} f \subset E' \times F$, with $E' \subset\subset E$, due to the fact that any bounded Borel function can be uniformly approximated by a sequence of simple functions.
The following theorem shows that under suitable conditions a measure ν on the product $E \times F$ can be written as $\mu \otimes \nu_x$, where μ is the push-forward of $|\nu|$ under the projection on E. This decomposition is known as *disintegration* of ν, or *layerwise decomposition*.

Theorem 2.1.27 (Disintegration, [AFP00, Theorem 2.28])**.** *Let $m \geq 1$, $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$ open sets, ν an $\mathbb{R}^m$-valued Radon measure on $E \times F$, $\pi : E \times F \to E$ the projection on the first factor and $\mu = \pi_\# |\nu|$. Let us assume that μ is a Radon measure, i.e. that $|\nu|(K \times F) < \infty$ for any compact set $K \subset E$. Then there exist $\mathbb{R}^m$-valued finite Radon measures ν_x in F such that $x \mapsto \nu_x$ is μ-measurable,*

$$|\nu_x|(F) = 1 \qquad \mu - \text{a.e.in } E$$

and

$$f(x,\cdot) \in L^1(F, |\nu_x|) \qquad \text{for} \qquad \mu - \text{a.e. } x \in E$$

$$x \mapsto \int_F f(x,y)\,\mathrm{d}\nu_x(y) \in L^1(E,\mu) \tag{2.1.2}$$

$$\int_{E\times F} f(x,y)\,\mathrm{d}\nu(x,y) = \int_E \left(\int_F f(x,y)\,\mathrm{d}\nu_x(y) \right) \mathrm{d}\mu(x) \tag{2.1.3}$$

for any $f \in L^1(E \times F, |\nu|)$. Moreover, if ν'_x is any other μ-measurable map satisfying (2.1.2), (2.1.3) for every bounded Borel function with compact support and such that $\nu'_x(F) \in L^1_{loc}(E,\mu)$, then $\nu_x = \nu'_x$ for μ-a.e. $x \in E$.

Hausdorff measures

If $k \in [0,\infty)$, in the following we denote by ω_k the constant $\frac{\pi^{\frac{k}{2}}}{\Gamma\left(1+\frac{k}{2}\right)}$, where $\Gamma(t) := \int_0^\infty s^{t-1}e^{-s}\,\mathrm{d}s$ is the Euler Γ function. In particular, this constant coincides with the Lebesgue measure of the unit ball of $\mathbb{R}^k$ if $k \geq 1$ is an integer.

Definition 2.1.28 (Hausdorff measures)**.** Let $k \in [0,\infty)$ and $E \subset \mathbb{R}^n$. The *k-dimensional Hausdorff measure* of E is given by

$$\mathcal{H}^k(E) := \lim_{\delta\downarrow 0} \mathcal{H}^k_\delta(E)$$

where, for $0 < \delta \leq \infty$, $\mathcal{H}^k_\delta(E)$ is defined by

$$\mathcal{H}^k_\delta(E) := \frac{\omega_k}{2^k} \inf \left\{ \sum_{i\in I} [\operatorname{diam}(E_i)]^k \ : \ \operatorname{diam}(E_i) < \delta, \quad E \subset \bigcup_{i\in I} E_i \right\}$$

for finite or countable covers $\{E_i\}_{i\in I}$, with the convention $\operatorname{diam}(\emptyset) = 0$.

Since $\delta \mapsto \mathcal{H}^k_\delta(E)$ is decreasing in $(0,\infty]$ the limit defining $\mathcal{H}^k(E)$ exists, finite or infinite. It is also worth noticing that the measure $\mathcal{H}^0$ corresponds to the counting measure.

Proposition 2.1.29 (Properties of Hausdorff measures, [AFP00, Proposition 2.49])**.** *The measures $\mathcal{H}^k$ are outer measures in $\mathbb{R}^n$ and, in particular, they are σ-additive on $\mathcal{B}(\mathbb{R}^n)$. Moreover $\mathcal{H}^k(\cdot)$ is invariant by translations and is positively k-homogeneous. In addition, $\mathcal{H}^k(\cdot)$ is identically zero if $k > n$, while if $k' > k'' \geq 0$ then*

$$\mathcal{H}^{k'}(E) > 0 \implies \mathcal{H}^{k''}(E) = \infty.$$

Finally if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function then

$$\mathcal{H}^k(f(E)) \leq [\mathrm{Lip}(f)]^k \mathcal{H}^k(E) \qquad \forall E \subset \mathbb{R}^n.$$

Note that for $k < n$ the Borel measure $\mathcal{H}^k$ is not even σ-finite in $\mathbb{R}^n$. The theory of integration outlined in Definition 2.1.6, in which no σ-finiteness assumption was made, can be use to integrate with respect to $\mathcal{H}^k$.

Definition 2.1.30 (Hausdorff dimension)**.** The *Hausdorff dimension* of $E \subset \mathbb{R}^n$ is given by

$$\dim_{\mathcal{H}}(E) := \inf \left\{ k \geq 0 \ : \ \mathcal{H}^k(E) = 0 \right\}.$$

By Proposition 2.1.29 $\mathcal{H}^k(E) = \infty$ if $k < \dim_{\mathcal{H}}(E)$ and $\mathcal{H}^k(E) = 0$ if $k > \dim_{\mathcal{H}}(E)$. If $k = \dim_{\mathcal{H}}(E)$ nothing can be said, in general.

Rectifiable sets

In this section we introduce a mild regularity property of $\mathcal{H}^k$-measurable sets.

Definition 2.1.31 (Rectifiable sets)**.** Let $E \subset \mathbb{R}^n$ be an $\mathcal{H}^k$-measurable set, with $k \in [0,n]$ integer. We say that E is *countably k-rectifiable* if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$E \subset \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^k).$$

We say that E is *countably $\mathcal{H}^k$-rectifiable* if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$\mathcal{H}^k\left(E \setminus \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^k) \right) = 0.$$

Finally, we say that E is *$\mathcal{H}^k$-rectifiable* if E is countably $\mathcal{H}^k$-rectifiable and $\mathcal{H}^k(E) < \infty$.

For $k = 0$ countably k-rectifiable and countably $\mathcal{H}^k$-rectifiable sets correspond to finite or countable sets, while $\mathcal{H}^k$-rectifiable sets correspond to finite sets. An immediate consequence of Proposition 2.1.29 is the fact that rectifiable sets are stable under Lipschitz transformations.
Given a Radon measure μ in an open set $\Omega \subset \mathbb{R}^n$, we define the rescaled measures around $x \in \Omega$

$$\mu_{x,\varrho}(B) = \mu(x + \varrho B) \qquad B \in \mathcal{B}(\mathbb{R}^n), \quad B \subset \frac{\Omega - x}{\varrho}.$$

Definition 2.1.32 (Approximate tangent space to a measure)**.** Let μ be an $\mathbb{R}^m$-valued Radon measure in an open set $\Omega \subset \mathbb{R}^n$ and $x \in \Omega$. We say that μ has *approximate tangent space $\pi \in \mathbf{G}_k$ with multiplicity $\vartheta \in \mathbb{R}^m$* at x, and we write

$$\mathrm{Tan}^k(\mu, x) = \vartheta \mathcal{H}^k \llcorner \pi$$

if $\varrho^{-k}\mu_{x,\varrho}$ locally weakly* converge to $\vartheta\mathcal{H}^k \llcorner \pi$ in $\mathbb{R}^n$ as $\varrho \downarrow 0$.

It is possible to define an approximate tangent space $\mathrm{Tan}^k(E,x)$ to countably $\mathcal{H}^k$-rectifiable sets as follows.

Definition 2.1.33 (Approximate tangent space to a set)**.** Let $E \subset \mathbb{R}^n$ be a countably $\mathcal{H}^k$-rectifiable set and let $\{E_i\}$ be a partition of $\mathcal{H}^k$-almost all of E into $\mathcal{H}^k$-rectifiable sets; we define $\mathrm{Tan}^k(E,x)$ to be the *approximate tangent space* to $\mathcal{H}^k \llcorner E_i$ *at* x for any $x \in E_i$ where the latter is defined.

In the following proposition we compare our definition of approximate tangent space with a parametric one which is often useful in applications.

Proposition 2.1.34 ([AFP00, Proposition 2.88]). *Let $\varphi : \mathbb{R}^k \to \mathbb{R}^n$ be a one-to-one Lipschitz function and let $D \subset \mathbb{R}^k$ be a $\mathcal{L}^k$-measurable set. Then $E = \varphi(D)$ satisfies*

$$\mathrm{Tan}^k(E, x) = \mathrm{d}\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k) \qquad \mathrm{for} \mathcal{H}^k - \text{a.e. } x \in E.$$

Definition 2.1.35 (Tangential differential of Lipschitz functions). Let E be a countably $\mathcal{H}^k$-rectifiable set in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ a Lipschitz function. We say that f is *tangentially differentiable at* $x \in E$ if the restriction of f to the affine space $x+ \mathrm{Tan}^k(E, x)$ is differentiable at x. The tangential differential is denoted by $\mathrm{d}^E f_x$ and is a linear map between the spaces $\mathrm{Tan}^k(E, x)$ and $\mathbb{R}^m$.

Clearly, if f is differentiable at $x \in E$, then $\mathrm{d}^E f_x$ is the restriction of the differential $\mathrm{d}f_x$ to $\mathrm{Tan}^k(E, x)$, provided that the approximate tangent space exists. Definition 2.1.35 is motivated by the following natural extension of Rademacher's differentiability theorem.

Theorem 2.1.36 (Tangential differentiability, [AFP00, Theorem 2.90]). *With the notation of Definition 2.1.35, $d^E f_x$ exists for $\mathcal{H}^k$-a.e. $x \in E$.*

Area and coarea formulas

Area formula shows how the k-dimensional Hausdorff measure of sets $B = f(E)$ parametrized by a Lipschitz map $f : \mathbb{R}^k \to \mathbb{R}^n$ can be computed.

Definition 2.1.37 (k-dimensional Jacobian). Let V, W be Hilbert spaces with $\dim(V) = k \leq n = \dim(W)$ and let $L : v \to W$ be a linear map. The *k-dimensional Jacobian* is defined by

$$\mathbf{J}_k L := \sqrt{\det(L^* \circ L)}$$

where $L^* : W^* \to V^*$ is the transpose of L.

Note that $\mathbf{J}_k L = 0$ if and only if the rank of L is strictly less than k. Given a matrix representation L_{ij} of L with respect to orthonormal bases of V and W, it follows directly from the definition that

$$\mathbf{J}_k L = \sqrt{\det(C)} \qquad \text{with} \qquad C_{jl} := \sum_{i=1}^{N} L_{ij} L_{il}.$$

Theorem 2.1.38 (Area formula, [AFP00, Theorem 2.91]). *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz function and $E \subset \mathbb{R}^m$ a countably $\mathcal{H}^k$-rectifiable set. Then, the multiplicity function $\mathcal{H}^0\left(E \cap f^{-1}(y)\right)$ is $\mathcal{H}^k$-measurable in $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} \mathcal{H}^0\left(E \cap f^{-1}(y)\right) \mathrm{d}\mathcal{H}^k(y) = \int_E \mathbf{J}_k \mathrm{d}^E f_x \, \mathrm{d}\mathcal{H}^k(x).$$

The set $f(E)$ is $\mathcal{H}^k$-measurable, being the support of the multiplicity function. If f is one-to-one on E we obtain

$$\mathcal{H}^k(f(E)) = \int_E \mathbf{J}_k \mathrm{d}^E f_x \, \mathrm{d}x.$$

Finally, representing any Borel function $g : E \to [0, \infty]$ as a series of characteristic function one immediately obtains the general change of variables formula

$$\int_{\mathbb{R}^n} \sum_{x \in E \cap f^{-1}(y)} g(x) \, \mathrm{d}\mathcal{H}^k(y) = \int_E g(x) \mathbf{J}_k \mathrm{d}^E f_x \, \mathrm{d}x.$$

Given a Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}^k$ and an n-dimensional domain $E \subset \mathbb{R}^m$ with $n \geq k$, in many applications it is useful to reduce an integral on E to a double integral, where the first integral is computed on the level set $E \cap \{f = t\}$ with respect to $\mathcal{H}^{n-k}$ and the result is integrated in t with respect to $\mathcal{L}^k$. If $m = n$ and f is an orthogonal projection, the level sets of f are $(n-k)$-planes and this procedure corresponds to Fubini's Theorem 2.1.20. Coarea formula is the natural extension of Fubini's theorem to the above mentioned more general setting, first proved by H. Federer in [Fed59].

Definition 2.1.39 (k-dimensional coarea factor)**.** Let V, W be Hilbert spaces with $\dim(V) = n \geq k = \dim(W)$ and let $L : V \to W$ be a linear map. The *k-dimensional coarea factor* $\mathbf{C}_k L$ is given by

$$\mathbf{C}_k L := \sqrt{\det(L \circ L^*)}$$

where $L^* : W^* \to V^*$ is the transpose of L.

By Definition 2.1.37, $\mathbf{C}_k L$ corresponds to $\mathbf{J}_k L^*$. As a consequence, $\mathbf{C}_k L > 0$ if and only if $\operatorname{rank}(L) = k$.

Theorem 2.1.40 (Coarea formula, [AFP00, Theorem 2.93])**.** *Let $f : \mathbb{R}^m \to \mathbb{R}^k$ be a Lipschitz function and let E be a countably $\mathcal{H}^n$-rectifiable subset of $\mathbb{R}^m$. Then the function $t \mapsto \mathcal{H}^{n-k}(E \cap f^{-1}(t))$ is $\mathcal{L}^k$-measurable in $\mathbb{R}^k$, $E \cap f^{-1}(t)$ is countably $\mathcal{H}^{n-k}$-rectifiable for $\mathcal{L}^k$-a.e. $t \in \mathbb{R}^k$ and*

$$\int_E \mathbf{C}_k \mathrm{d}^E f_x \,\mathrm{d}\mathcal{H}^n(x) = \int_{\mathbb{R}^k} \mathcal{H}^{n-k}\left(E \cap f^{-1}(t)\right) \mathrm{d}t.$$

We can obtain the more general formula

$$\int_E g(x) \mathbf{C}_k \mathrm{d}^E f_x \,\mathrm{d}\mathcal{H}^n(x) = \int_{\mathbb{R}^k} \left(\int_{E \cap \{f=t\}} g(y) \,\mathrm{d}\mathcal{H}^{n-k}(y) \right) \mathrm{d}t \tag{2.1.4}$$

for any Borel function $g : \mathbb{R}^m \to [0, \infty]$. In the particular case $k = 1$ and $m = n$, (2.1.4) becomes

$$\int_E g(x) \,|\nabla f(x)| \,\mathrm{d}x = \int_{-\infty}^{\infty} \left(\int_{E \cap \{f=t\}} g(y) \,\mathrm{d}\mathcal{H}^{n-1}(y) \right) \mathrm{d}t.$$

2.2 Preliminaries in functional analysis

Sobolev spaces

The aim of this section is to give a brief introduction to the theory of weak derivatives and Sobolev spaces. We refer to the treatises [Ada75; LM68; Mal82; Maz85; Zie89] for a detailed presentation of these topics.

Definition 2.2.1 (Weak derivatives)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $i \in \{1, \ldots, n\}$, $u \in L^1_{loc}(\Omega)$; if there is $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x = -\int_\Omega \varphi g \,\mathrm{d}x \qquad \forall \varphi \in C^\infty_c(\Omega)$$

then we say that u has weak i-th derivative given by g. The i-th weak derivative, if exists, is unique and is denoted by $\nabla_i u$ or $\frac{\partial u}{\partial x_i}$.

The weak derivatives coincide with the classical ones if $u \in C^1(\Omega)$.

Definition 2.2.2 (Sobolev spaces)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set, and $1 \leq p \leq \infty$; we say that $u \in W^{1,p}(\Omega)$ if $u \in L^p(\Omega)$ and has weak derivatives in $L^p(\Omega)$ for every $i = 1, \ldots, n$. For any $u \in W^{1,p}(\Omega)$ we set

$$\nabla u := (\nabla_1 u, \ldots, \nabla_n u).$$

We recall that $W^{1,p}(\Omega)$ becomes a Banach space (Hilbert for $p = 2$) when endowed with the norm $\|\cdot\|_{W^{1,p}(\Omega)}$ defined by

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \sum_{i=1}^n \|\nabla_i u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

if $1 \leq p < \infty$; for $p = \infty$ the norm is given by

$$\|u\|_{W^{1,\infty}(\Omega)} = \|u\|_{L^\infty(\Omega)} + \sum_{i=1}^n \|\nabla_i u\|_{L^\infty(\Omega)}.$$

The space $W^{1,p}(\Omega)$ is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$.

Definition 2.2.3. We denote by $W_0^{1,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

Definition 2.2.4 (Weak convergence in $W^{1,p}$)**.** Let $\Omega \subset \mathbb{R}^n$, $1 \le p \le \infty$ and $u, u_h \in W^{1,p}(\Omega)$; then, we say that $u_h \to u$ *weakly in* $W^{1,p}(\Omega)$ (*weakly** if $p = \infty$) if ∇u_h weakly converges in $L^p(\Omega)$ (weakly* if $p = \infty$) to ∇u and $u_h \to u$ strongly in $L^p(\Omega)$.

Higher order weak derivatives $\nabla^\alpha u$ (with α multiindex) can be introduced, giving rise to the space $W^{k,p}$. If $u \in L^1_{loc}(\Omega)$ we say that $g \in L^1_{loc}(\Omega)$ is the α-th weak derivative of u if

$$\int_\Omega u \nabla^\alpha \varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g\varphi \,\mathrm{d}x \qquad \forall \varphi \in C_c^\infty(\Omega).$$

Given an integer $k > 1$ and $1 \le p \le \infty$ the Sobolev space $W^{k,p}(\Omega)$ is thus defined as the set of functions $u \in L^p(\Omega)$ such that all weak derivatives $\nabla^\alpha u$ belong to $L^p(\Omega)$ for any $|\alpha| \le k$. It can be endowed with a norm, setting for $1 \le p < \infty$

$$\|u\|_{W^{k,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \sum_{i=1}^k \sum_{|\alpha|=i} \|\nabla^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

and for $p = \infty$

$$\|u\|_{W^{k,\infty}(\Omega)} = \|u\|_{L^\infty(\Omega)} + \sum_{i=1}^k \sum_{|\alpha|=i} \|\nabla^\alpha u\|_{L^\infty(\Omega)}.$$

If $0 < s < 1$ and $1 \le p < \infty$, we define

$$W^{s,p}(\Omega) := \left\{ u \in L^p \;:\; \frac{|u(x) - u(y)|}{|x-y|^{s+\frac{n}{p}}} \in L^p(\Omega \times \Omega) \right\},$$

i.e. an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{p}}.$$

If $s > 1$, we write $s = m + \sigma$, with $m = \lfloor s \rfloor$, and we define

$$W^{s,p}(\Omega) := \{ u \in W^{m,p} \;:\; D^\alpha u \in W^{\sigma,p}(\Omega) \qquad \forall \alpha \text{ with } |\alpha| = m \}.$$

BMO spaces

In this section we define BMO spaces, introduced by John and Nirenberg [Joh61; JN61] in connection with problems arising from elasticity theory.

Definition 2.2.5 ([BN95], [JN61])**.** Let Ω be a cube or the entire space $\mathbb{R}^n$. The $BMO(\Omega)$ space consists of all functions b which are integrable on every cube $Q \subset \Omega$ with sides parallel to those of Ω and satisfy:

$$\|b\|_* = \sup_Q \left\{ \frac{1}{|Q|} \int_Q |b - b_Q| \,\mathrm{d}x \right\} < \infty,$$

where $b_Q = \frac{1}{|Q|} \int_Q b(y)\, dy$ and $|Q|$ denotes the Lebesgue measure of Q.

It is clear that the functional $\|\cdot\|_*$ does not define a norm since it vanishes on constant functions. However BMO becomes a Banach space provided we identify functions which differ almost everywhere from a constant.
Bounded functions clearly belong to BMO. On the other hand, BMO contains suitable unbounded functions and it is contained in L^p_{loc} spaces [JN61]. The standard example of unbounded BMO function is

$$f(x) = \log|x|, \qquad x \in B_1(0) \setminus 0.$$

We also recall the following property.

Theorem 2.2.6 ([BN95])**.** *For any cube $Q \subset \mathbb{R}^n$ the following inclusion holds with continuous embedding:*

$$W^{1,n}(Q) \hookrightarrow BMO(Q).$$

Muckenhoupt weights

In this section we define Muckenhoupt weights, introduced in '70s in [FM74; Muc74; Str79]. See also the treatises [GR85; Tor86].

Definition 2.2.7 ([GR85]). Given a weight w, i.e. a nonnegative function locally integrable in $\mathbb{R}^n$, we say that w belongs to the A_p class of Muckenhoupt, with $1 < p < \infty$, if

$$A_p(w) := \sup_Q \left(\fint_Q w \right) \left(\fint_Q w^{-\frac{1}{p-1}} \right)^{p-1} < \infty$$

where the supremum is taken all over cubes Q of $\mathbb{R}^n$. We say w belongs to the A_1 class of Muckenhoupt if

$$A_1(w) := \sup_Q \left(\fint_Q w \right) \left(\operatorname*{ess\,sup}_Q \left(w^{-1} \right) \right) < \infty,$$

where the supremum is taken all over cubes Q of $\mathbb{R}^n$. The number $A_p(w)$ is called the A_p constant of w.

Note that, if $1 \le p < q$, then $A_p \subset A_q$. In fact, if $p > 1$, by Hölder's inequality, we have

$$\begin{aligned} \left(\fint_Q w \right) \left(\fint_Q w^{-\frac{1}{q-1}} \right)^{q-1} &\le \left(\fint_Q w \right) \frac{\left(\int_Q 1 \right)^{q-p} \left(\int_Q w^{-\frac{1}{p-1}} \right)^{p-1}}{|Q|^{q-1}} = \\ &= \left(\fint_Q w \right) \left(\fint_Q w^{-\frac{1}{p-1}} \right)^{p-1} \le A_p(w). \end{aligned}$$

If $p = 1$ then

$$\begin{aligned} \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{q-1}} \right)^{q-1} &\le \operatorname*{ess\,sup}_{x \in Q} \left(w^{-1}(x) \right) = \operatorname*{ess\,sup}_{x \in Q} \left(w^{-1}(x) \right) \left(\fint_Q w \right) \left(\fint_Q w \right)^{-1} \le \\ &\le A_1(w) \left(\frac{|Q|}{w(Q)} \right), \end{aligned}$$

where we have set, for every measurable set $E \subset \mathbb{R}^n$,

$$w(E) := \int_E w \, \mathrm{d}x. \tag{2.2.1}$$

Note that, if w is a Muckenhoupt weight, the measure defined in (2.2.1) is doubling (see [Tor86, Chapter IX, Theorem 2.1]).

Definition 2.2.8 ([Str01]). Let $k(x) : \mathbb{R}^n \to \mathbb{R}^n$. We will call k a Calderon–Zygmund kernel (CZ kernel) if k satisfies the following properties

- $k(x) \in C^\infty(\mathbb{R}^n \setminus \{0\})$,
- $k(x)$ is homogeneous of degree $-n$, i.e. $k(tx) = t^{-n} k(x)$ for any $t > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$,
- $\int_\Sigma k(x) \, d\sigma_x = 0$ where Σ is the unit sphere of $\mathbb{R}^n$.

Given such a kernel, one can define a bounded operator in L^p, $1 < p < \infty$, called Calderon–Zygmund singular operator, as follows

$$Kf(x) = P.V.(k \star f)(x) := P.V. \int_{\mathbb{R}^n} k(x-y) f(y) \, dy.$$

Given a measurable subset E of $\mathbb{R}^n$, we will denote by $L^p(w, E; \mathbb{R}^N)$, $1 < p < \infty$, the Banach space of all measurable functions f defined on E for which

$$\|f\|_{L^p_w(E)} = \left(\int_E |f(x)|^p w(x) \, \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

It is well known that the singular integral operators are bounded on weighted L^p spaces for weights belonging to the A_p class. A theorem due to Buckley explicitly gives the dependence of the $L^p(w, \mathbb{R}^n)$ - norm of a singular integral operator on the A_p constant of w.

Theorem 2.2.9 ([Buc93]). *Let w be an A_p weight and K a singular integral operator. Then, for every $f \in L^p(w, E; \mathbb{R}^N)$, there exists a constant $c = c(n, p)$ such that*

$$\|Kf\|^p_{L^p_w(E)} \leq c A_p(w)^{p+p'} \|f\|^p_{L^p_w(E)}$$

where $p' = \frac{p}{p-1}$.

Since both A_p condition and the definition of BMO deal with the averaging of functions it is natural to consider the connections between these two classes. Among a lot of results in this direction, we point out the following

Lemma 2.2.10 ([JN93]). *Let $b(x)$ be a function such that b, $\frac{1}{b}$ both belong to $BMO(\mathbb{R}^n)$. Then*

$$b \in \bigcap_{p>1} A_p$$

and

$$A_p(b) \leq c + c\|b\|_*$$

where c is a constant depending only on p.

We can state the following weighted versions of Imbedding Theorem and Sobolev – Poincaré inequality:

Theorem 2.2.11 ([FKS82]). *Given $1 < p < \infty$ and $w \in A_p$, there exist constants c, depending on n, p and the A_p constant of w, and $\zeta > 0$, depending on n and p, such that for all balls B_R, all $u \in C^\infty_0(B_R)$ and all numbers k satisfying $1 \leq k \leq \frac{n}{n-1} + \zeta$,*

$$\left(\frac{1}{w(B_R)} \int_{B_R} |u|^{kp} w \,\mathrm{d}x \right)^{\frac{1}{kp}} \leq cR \left(\frac{1}{w(B_R)} \int_{B_R} |\nabla u|^p w \,\mathrm{d}x \right)^{\frac{1}{p}}.$$

Theorem 2.2.12 ([FKS82]). *Let $1 < p < \infty$ and $w \in A_p$. Then there are constants c, depending on n, p and the A_p constant of w, and $\zeta > 0$, depending on n and p, such that for all Lipschitz continuous functions u defined on $\overline{B_R}$ and for all $1 \leq k \leq \frac{n}{n-1} + \zeta$,*

$$\left(\frac{1}{w(B_R)} \int_{B_R} \left| u(x) - A_{B_R} \right|^{kp} w \,\mathrm{d}x \right)^{\frac{1}{kp}} \leq cR \left(\frac{1}{w(B_R)} \int_{B_R} |\nabla u|^p \, w \,\mathrm{d}x \right)^{\frac{1}{p}},$$

where either $A_{B_R} = \frac{1}{w(B_R)} \int_{B_R} u(x)w(x) \,\mathrm{d}x$ or $A_{B_R} = \frac{1}{|B_R|} \int_{B_R} u(x) \,\mathrm{d}x$.

Hodge decomposition

We shall now briefly discuss the Hodge decomposition of vector fields; for a more complete treatment see [IS92] and [IS94]. For a given vector field $L = (l^1, \ldots, l^n) \in L^p(\mathbb{R}^n; \mathbb{R}^n)$, $1 < p < \infty$, the Poisson equation $\Delta u = \operatorname{div} L$ can be solved by using the Riesz transforms in $\mathbb{R}^n$, $\mathcal{R} = (R_1, \ldots, R_n)$,

$$\nabla u = -(\mathcal{R} \otimes \mathcal{R})(L) =: \mathscr{K}(L).$$

Here the tensor product operator $\mathscr{K} = -\mathcal{R} \otimes \mathcal{R} = -[R_{ij}]$ is the $n \times n$ matrix of the second order Riesz transforms $R_{ij} = R_i \circ R_j$, $i, j = 1, \ldots, n$. Notice that the range of the operator

$$\mathscr{H} := \mathrm{Id} - \mathscr{K} : L^p(\mathbb{R}^n; \mathbb{R}^n) \to L^p(\mathbb{R}^n; \mathbb{R}^n)$$

consists of the divergence free vector fields. We then arrive at the familiar Hodge decomposition of L

$$L = \nabla u + H, \quad \operatorname{div} H = 0.$$

Hence, L^p-estimates for Riesz transform yield an uniform estimate

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} + \|H\|_{L^p(\mathbb{R}^n)} \leq c(p)\|L\|_{L^p(\mathbb{R}^n)}.$$

For this result and other dimension free estimates see [IM90].
Let $\Omega \subset \mathbb{R}^n$ be a domain and $G = G(x, y)$ the Green's function. For $h \in C_0^\infty(\Omega)$ the integral

$$u(x) = \int_\Omega G(x, y)h(y)\, dy$$

defines a solution of the Poisson equation $\Delta u = h$ with u vanishing on the boundary of Ω. If h has a divergence form, say $h = \operatorname{div} L$ with $L = (l^1, \dots, l^n) \in C_0^\infty(\Omega; \mathbb{R}^n)$, then integration by parts yields

$$u(x) = -\int_\Omega \nabla_y G(x, y)L(y)\, dy.$$

Hence the gradient of u is expressed by a singular integral

$$\nabla u(x) = -\int_\Omega \nabla_x \nabla_y G(x, y)L(y)\, dy =: (\mathscr{K}_\Omega L)(x).$$

By Theorem 2.2.9 and Lemma 2.2.10, if $b \in BMO$ and $\frac{1}{b} \in BMO$, we have

$$\|\mathscr{K}_\Omega L\|^p_{L^p_b} \le c(1 + \|b\|_*)^{p+p'} \|L\|^p_{L^p_b}$$

If $1 < p < \infty$, let $\mathcal{D}^p(b, \Omega; \mathbb{R}^n)$ denote the closure of the range of the gradient operator $\nabla : C_0^\infty(\Omega) \to L^p(b, \Omega; \mathbb{R}^n)$, i.e.

$$\mathcal{D}^p(b, \Omega; \mathbb{R}^n) := \overline{\left\{\nabla v \ : \ v \in C_0^\infty(\Omega)\right\}}^{L^p_b}.$$

If Ω is smooth, then $\mathscr{K}_\Omega$ extends continuously to all $L^p(b, \Omega; \mathbb{R}^n)$ spaces. Consequently the formula $\nabla u = \mathscr{K}_\Omega L$ extends to all $L \in L^p(b, \Omega; \mathbb{R}^n)$ giving a solution with $\nabla u \in \mathcal{D}^p(b, \Omega; \mathbb{R}^n)$, $1 < p < \infty$.

Definition 2.2.13 ([IS94]). A domain $\Omega \subset \mathbb{R}^n$ will be called regular if the operator $\mathscr{K}_\Omega$ acts boundedly in all $L^p(b, \Omega; \mathbb{R}^n)$-spaces, for $1 < p < \infty$.

For Ω a regular domain we introduce, as before, the operator

$$\mathscr{H}_\Omega := \mathrm{Id} - \mathscr{K}_\Omega \ : \ L^p(b, \Omega; \mathbb{R}^n) \to L^p(b, \Omega; \mathbb{R}^n).$$

Obviously, the range of $\mathscr{H}_\Omega$ consists of the divergence free vector fields on Ω. We have the Hodge decomposition of $L \in L^p(b, \Omega; \mathbb{R}^n)$,

$$L = \nabla u + H, \quad \operatorname{div} H = 0, \quad \nabla u \in \mathcal{D}^p(b, \Omega; \mathbb{R}^n).$$

We deduce the following stability property in our decomposition

Lemma 2.2.14 ([CMP02]). *Let Ω be a regular domain of $\mathbb{R}^n$ and consider $b \in BMO$ such that $\frac{1}{b} \in BMO$. If $u \in W_0^{1,r-\varepsilon}(b, \Omega; \mathbb{R}^N)$, $1 < r < \infty$, $-1 < 2\varepsilon < r - 1$, there exist $\varphi \in W_0^{1,\frac{r-\varepsilon}{1-\varepsilon}}(b, \Omega; \mathbb{R}^N)$ and a divergence free vector field $H \in L^{\frac{r-\varepsilon}{1-\varepsilon}}(b, \Omega; \mathbb{R}^{N\times n})$ such that*

$$|Du|^{-\varepsilon} Du = D\varphi + H.$$

Moreover

$$\|D\varphi\|_{L_b^{\frac{r-\varepsilon}{1-\varepsilon}}(\Omega)} \le c(n, r)(1 + \|b\|_*)^{\gamma(r)} \|Du\|^{1-\varepsilon}_{L_b^{r-\varepsilon}(\Omega)}$$

$$\|H\|_{L_b^{\frac{r-\varepsilon}{1-\varepsilon}}(\Omega)} \le c(n, r)(1 + \|b\|_*)^{\gamma(r)} |\varepsilon| \, \|Du\|^{1-\varepsilon}_{L_b^{r-\varepsilon}(\Omega)}$$

where $\gamma(r)$ is an exponent depending only on r.

Lorentz spaces

Let Ω be a bounded domain in $\mathbb{R}^n$. Given $1 < p, q < \infty$, the Lorentz space $L^{p,q}(\Omega)$ consists of all measurable functions g defined on Ω for which the quantity

$$||g||^q_{L^{p,q}} = p \int_0^\infty |\Omega_t(g)|^{\frac{q}{p}} t^{q-1}\, dt$$

is finite, where $\Omega_t(g) = \{x \in \Omega : |g(x)| > t\}$ and $|\Omega_t|$ is the Lebesgue measure of Ω_t. Note that $|| \cdot ||_{L^{p,q}}$ is equivalent to a norm and $L^{p,q}$ becomes a Banach space when endowed with it. For $p = q$, the Lorentz space $L^{p,p}$ reduces to the standard Lebesgue space L^p. For $q = \infty$, the class $L^{p,\infty}$ consists of all measurable functions g defined on Ω such that

$$||g||^p_{L^{p,\infty}} = \sup_{t>0} t^p |\Omega_t(g)| < \infty$$

and it coincides with the Marcinkiewicz class, weak - L^p. For Lorentz spaces the following inclusions hold

$$L^r(\Omega) \subset L^{p,q}(\Omega) \subset L^{p,r}(\Omega) \subset L^{p,\infty}(\Omega) \subset L^q(\Omega),$$

whenever $1 \leq q < p < r \leq \infty$
Fundamental to us will be the Sobolev embedding theorem in Lorentz spaces.

Theorem 2.2.15 ([Alv77])**.** *Let us assume that $1 < p < n$, $1 \leq q \leq p$, then any function $u \in W_0^{1,1}(\Omega)$ such that $|\nabla u| \in L^{p,q}(\Omega)$ actually belongs to $L^{p^*,q}(\Omega)$ and*

$$||u||_{L^{p^*,q}} \leq S_{n,p} ||\nabla u||_{L^{p,q}}.$$

Here $p^* = \frac{np}{n-p}$ and $S_{n,p} = \omega_n^{-\frac{1}{n}} \frac{p}{n-p}$, where ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

We define the distance of a given $f \in L^{p,\infty}$ to L^∞ as

$$\text{dist}_{L^{p,\infty}}(f, L^\infty) = \inf_{g \in L^\infty} ||f - g||_{L^{p,\infty}}.$$

To find a formula for the distance, we consider the truncation operator. For $k > 0$ and $y \in \mathbb{R}$, we set

$$T_k(y) = \min\{k, \max\{-k, y\}\}.$$

Then

$$\text{dist}_{L^{p,\infty}}(f, L^\infty) = \lim_{k \to \infty} ||f - T_k f||_{L^{p,\infty}}.$$

Indeed, $\forall g \in L^\infty$, $\forall k \geq ||g||_{L^\infty}$, we have for almost every $x \in \Omega$,

$$|f(x) - g(x)| \geq |f(x) - T_k f(x)|.$$

Let Ω be the unit ball of $\mathbb{R}^n$. The function

$$f(x) = \frac{1}{|x|}$$

belongs to $L^{n,\infty}$. Setting $\omega_n = |\Omega|$, we have

$$||f - T_k f||_{L^{p,\infty}} = \omega_n^{1/n}$$

and it does not depend on k. For more details, see [GGM13].
We recall the following relevant properties.

Lemma 2.2.16 ([BBC75])**.** *If $E \in L^{p,\infty}(\mathbb{R}^n)$, $1 < p < \infty$, and $f \in L^1(\mathbb{R}^n)$, then $E \star f \in L^{p,\infty}(\mathbb{R}^n)$ and*

$$||E \star f||_{L^{p,\infty}} \leq ||E||_{L^{p,\infty}} ||f||_{L^1}.$$

Theorem 2.2.17 (Hölder's inequality in Lorentz spaces, [ONe65])**.** *If $0 < p_1, p_2, p < \infty$ and $0 < q_1, q_2, q \leq \infty$ obey $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ then*

$$||fg||_{L^{p,q}} \leq ||f||_{L^{p_1,q_1}} ||g||_{L^{p_2,q_2}}$$

whenever the right - hand side norms are finite.

Maximal Operators

Let $Q_0 \subset \mathbb{R}^n$ be a cube. We shall consider, in the following, the Restricted Maximal Function Operator relative to Q_0. This is defined as

$$M^*_{Q_0}(f)(x) := \sup_{\substack{Q\subset Q_0\\ x\in Q}} \fint_Q |f(y)|\,dy, \qquad x \in Q_0,$$

whenever $f \in L^1(Q_0)$, where Q denotes any cube contained in Q_0 with sides parallel to those of Q_0, as long as $x \in Q$. We recall the following weak type (p,p) estimate for $M^*_{Q_0}$, valid for any $p \in [1,\infty)$:

$$\left|\left\{x \in Q_0 \,:\, M^*_{Q_0}(f)(x) \geq t\right\}\right| \leq \frac{c(n,p)}{t^p}\int_{Q_0} |f(y)|^p\,dy \qquad t > 0 \tag{2.2.2}$$

which is valid for any $f \in L^p(Q_0)$. For this and related issues we refer to [Ste93].
If w is a weight and $Q_0 \subset \mathbb{R}^n$ is a cube, we define the weighted Restricted Maximal Function Operator relative to Q_0 as

$$M^*_{w,Q_0}(f)(x) := \sup_{\substack{Q\subset Q_0\\ x\in Q}} \frac{\int_Q |f(y)|w(y)\,dy}{w(Q)}, \qquad x \in Q_0,$$

whenever $f \in L^1(w, Q_0)$, where Q denotes any cube contained in Q_0 with sides parallel to those of Q_0, as long as $x \in Q$. We have the following weighted generalization of (2.2.2):

Theorem 2.2.18 ([Tor86])**.** *Suppose $w \in A_p$, $1 < p < \infty$. Then M^*_{w,Q_0} maps $L^p(w, Q_0)$ into weak-$L^p(w, Q_0)$, with norm independent in A_p.*

Difference quotients

Definition 2.2.19 ([Giu03])**.** Let $f(x)$ be a function defined in an open set $\Omega \subset \mathbb{R}^n$, and let h be a real number. We shall call a difference quotient of f with respect to x_s the function

$$\Delta_{s,h} f(x) = \frac{f(x+he_s) - f(x)}{h} \equiv \frac{\tau_{s,h} f(x)}{h},$$

where e_s denotes the direction of the x_s axis and $\tau_{s,h}$ is the finite difference operator.

When no confusion can arise, we shall omit the index s, and we shall write simply Δ_h instead of $\Delta_{s,h}$.
The function $\Delta_{s,h} f$ is defined in the set

$$\Delta_{s,h}\Omega := \{x \in \Omega \,:\, x + he_s \in \Omega\},$$

and hence in the set

$$\Omega_{|h|} = \{x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) > |h|\}.$$

The following properties of the difference quotients are immediate:

- If $f \in W^{1,p}(\Omega)$, then $\Delta_h f \in W^{1,p}(\Omega_{|h|})$, and
$$D_i(\Delta_h f) = \Delta_h(D_i f).$$
- If at least one of the functions f or g has support contained in $\Omega_{|h|}$, then
$$\int_\Omega f\Delta_h g\,\mathrm{d}x = -\int_\Omega g\Delta_{-h} f\,\mathrm{d}x. \tag{2.2.3}$$
- We have
$$\Delta_h(fg)(x) = f(x+he_s)\Delta_h g(x) + g(x)\Delta_h f(x).$$

Remark 2.2.20. It follows immediately from (2.2.3) that the derivatives $D_s g$ of a Lipschitz - continuous function g, which exist almost everywhere as limits of the difference quotient $\Delta_{s,h} g$, coincide with its weak derivatives. Indeed, if f is a test function, we can pass to the limit in (2.2.3), getting

$$\int f D_s g \, dx = -\int g D_s f \, dx.$$

In other words, we have $\mathrm{Lip}(\Omega) = W^{1,\infty}(\Omega)$.

Lemma 2.2.21 ([Giu03]). *There exists a constant* $c(n)$ *such that, if* $v \in W^{1,p}(\Omega)$, $\Sigma \subset\subset \Omega$, $1 < p < \infty$, $s \in \{1, \ldots, n\}$ *and* $|h| < h_0 = \frac{1}{10\sqrt{n}} \mathrm{dist}(\Sigma, \partial\Omega)$, *then*

$$||\Delta_{s,h} v||_{L^p(\Sigma)} \le c||D_s v||_{L^p(\Omega)}.$$

Moreover, if $0 < \varrho < R$, $|h| < R - \varrho$,

$$\int_{B_\varrho} |v(x + he_s)|^p \, dx \le c(n, p) \int_{B_R} |v(x)|^p \, dx.$$

2.3 Sets of finite perimeter

The modern notion of sets of finite perimeter is due to Caccioppoli [Cac52] and De Giorgi [De 54; De 55; De 58; De 61]. See also [Fed58; Fed69]. The starting point of the theory of sets of finite perimeter is a generalization of the Gauss-Green theorem based on the notion of vector-valued Radon measure.

Functions of bounded variation

The idea of function of bounded variation developed along different streams, both in analytical and in a geometrical vein. From the point of view of the classical analysis, BV functions were singled out as those for which a control on the oscillation is possible, suitable to ensure the convergence of the Fourier series. The geometric counterpart is that rectifiable curves are precisely those parametrized by BV functions. Functions of bounded variation in $\mathbb{R}$ have been introduced by C. Jordan in 1881 [Jor81] in connection with Dirichlet's test for the convergence of Fourier series. Later developments on BV functions are due to Vitali [Vit05], Levi [Lev06], Lebesgue [Leb10], Fichera [Fic54], De Giorgi [De 54; De 55], Federer [Fed69], Vol'pert [Vol67; HV85]. We refer to [AFP00] for a complete treatment on the topic.

Definition 2.3.1. Let $u \in L^1(\Omega)$, Ω being an open set in $\mathbb{R}^n$; we say that u is a *function of bounded variation in* Ω if the distributional derivative of u is representable by a finite Radon measure in Ω, i.e. if

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, dx = -\int_\Omega \varphi \, dD_i u \qquad \forall \varphi \in C_c^\infty(\Omega), \quad i = 1, \ldots, n \tag{2.3.1}$$

for some $\mathbb{R}^n$-valued measure $Du = (D_1 u, \ldots, D_n u)$ in Ω. The vector space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$.

A smoothing argument shows that the integration by parts formulae (2.3.1) are still true for any $\varphi \in C_c^1(\Omega)$, or even Lipschitz functions φ with compact support in Ω. These formulae can be summarized in a single one by writing

$$\int_\Omega u \, \mathrm{div} \, \varphi \, dx = -\sum_{i=1}^n \int_\Omega \varphi_i \, dD_i u \qquad \forall \varphi \in \left[C_c^1(\Omega)\right]^n.$$

We use the same notation also for functions $u \in [BV(\Omega)]^m$; in this case Du is an $m \times n$ matrix of measures $D_i u^\alpha$ in Ω satisfying

$$\int_\Omega u^\alpha \frac{\partial \varphi}{\partial x_i} \, dx = -\int_\Omega \varphi \, dD_i u^\alpha \qquad \forall \varphi \in C_c^1(\Omega), \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, m$$

or, equivalently,

$$\sum_{\alpha=1}^{m}\int_{\Omega} u^\alpha \operatorname{div}\varphi^\alpha \,\mathrm{d}x = -\sum_{\alpha=1}^{m}\sum_{i=1}^{n}\int_{\Omega}\varphi_i^\alpha \,\mathrm{d}D_i u^\alpha \qquad \forall \varphi \in [C_c^1(\Omega)]^{mn}.$$

The Sobolev space $W^{1,1}(\Omega)$ is contained in $BV(\Omega)$; indeed, for any $u \in W^{1,1}(\Omega)$ the distributional derivative is given by $\nabla u \mathcal{L}^n$. This inclusion is strict: there exist functions $u \in BV(\Omega)$ such that Du is singular with respect to $\mathcal{L}^n$. For instance, the distributional derivative of the Heaviside function $\chi_{(0,\infty)}$ is the Dirac measure δ_0.

Theorem 2.3.2 ([AFP00, Theorem 3.44]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with Lipschitz boundary and let $u_\Omega := ⨍_\Omega u(x)\,\mathrm{d}x$. Then*

$$\int_\Omega |u - u_\Omega|\,\mathrm{d}x \le c\,|Du|(\Omega) \qquad \forall u \in BV(\Omega)$$

for some real constant c depending only on Ω

Remark 2.3.3. We remark that Theorem 2.3.2 holds for a wider class of domains, i.e. if Ω is a bounded connected *extension domain*, see [AFP00, Definition 3.20, Theorem 3.44]. Extension domains are useful when one needs to extend a function $u \in [BV(\Omega)]^m$ to a function $\tilde{u} \in [BV(\mathbb{R}^n)]^m$, in order to deduce global statements in Ω from local ones in $\mathbb{R}^n$. However any open set Ω with compact Lipschitz boundary is an extension domain [AFP00, Proposition 3.21].

Theorem 2.3.4 (Boundary trace theorem, [AFP00, Theorem 3.87]). *Let $\Omega \subset \mathbb{R}^n$ be an open set with bounded Lipschitz boundary and $u \in [BV(\Omega)]^m$. Then, for $\mathcal{H}^{n-1}$-almost every $x \in \partial\Omega$ there exists $u^\Omega(x) \in \mathbb{R}^m$ such that*

$$\lim_{\varrho \downarrow 0} \varrho^{-n} \int_{\Omega \cap B_\varrho(x)} |u(y) - u^\Omega(x)|\,\mathrm{d}y = 0.$$

Moreover, $\|u^\Omega\|_{L^1(\partial\Omega)} \le c\|u\|_{BV}$ for some constant c depending only on Ω, the extension $\tilde{u}$ of u to 0 out of Ω belongs to $u \in [BV(\mathbb{R}^n)]^m$ and, viewing Du as a measure on the whole of $\mathbb{R}^n$ and concentrated on Ω, $D\tilde{u}$ is given by

$$D\tilde{u} = Du + \left(u^\Omega \otimes \nu_\Omega\right)\mathcal{H}^{n-1} \llcorner \partial\Omega.$$

E. Gagliardo proved in [Gag57] that any function $u \in [L^1(\partial\Omega, \mathcal{H}^{n-1} \llcorner \partial\Omega)]^m$ is the trace of a suitable function in $[W^{1,1}(\Omega)]^m$, and this proves that the trace operator in Theorem 2.3.4 is onto.
Now we introduce the so-called *weak* convergence* and *strict convergence* in $BV(\Omega)$.

Definition 2.3.5 (Weak* convergence). Let $u, u_h \in [BV(\Omega)]^m$. We say that $\{u_h\}$ *weakly* converges in* $[BV(\Omega)]^m$ *to* u if $\{u_h\}$ converges to u in $[L^1(\Omega)]^m$ and $\{Du_h\}$ weakly* converges to Du in Ω, i.e.

$$\lim_{h\to\infty}\int_\Omega \varphi\,\mathrm{d}Du_h = \int_\Omega \varphi\,\mathrm{d}Du \qquad \forall \varphi \in C_0(\Omega).$$

Definition 2.3.6 (Strict convergence). Let $u, u_h \in [BV(\Omega)]^m$. We say that $\{u_h\}$ *strictly converges in* $[BV(\Omega)]^m$ *to* u if $\{u_h\}$ converges to u in $[L^1(\Omega)]^m$ and the variations $|Du_h|(\Omega)$ converge to $|Du|(\Omega)$ as $h \to \infty$.

The trace operator is not continuous if $[BV(\Omega)]^m$ is endowed with the topology of weak* convergence: for instance the sequence $(1 \wedge ht)$ weakly* converges in $BV(0,1)$ to 1 as $h \to \infty$ but the traces at 0 do not converge to 1. We have continuity, however, under strict convergence.

Theorem 2.3.7 (Continuity of the trace operator, [AFP00, Theorem 3.88]). *Let Ω be an open subset of $\mathbb{R}^n$ with bounded Lipschitz boundary. Then, the trace operator $u \mapsto u^\Omega$ is continuous between $[BV(\Omega)]^m$, endowed with the topology induced by strict convergence, and $[L^1(\partial\Omega, \mathcal{H}^{n-1} \llcorner \partial\Omega)]^m$.*

Perimeter

We recall basic definitions and properties regarding sets of finite perimeter, referring to [AFP00; Mag12] for a complete treatment on the subject.
Let E be a Lebesgue measurable set in $\mathbb{R}^n$. We say that E is a *set of locally finite perimeter* in $\mathbb{R}^n$ if for every compact set $K \subset \mathbb{R}^n$ we have

$$\sup \left\{ \int_E \operatorname{div} T(x)\, \mathrm{d}x \ : \ T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n), \ \operatorname{spt} T \subset K, \ \sup_{\mathbb{R}^n} |T| \leq 1 \right\} < \infty.$$

If this quantity is bounded independently of K, then we say that E is a *set of finite perimeter* in $\mathbb{R}^n$.

Proposition 2.3.8 ([Mag12, Proposition 12.1]). *If E is a Lebesgue measurable set in $\mathbb{R}^n$, then E is a set of locally finite perimeter if and only if there exists a $\mathbb{R}^n$-valued Radon measure μ_E on $\mathbb{R}^n$ such that*

$$\int_E \operatorname{div} T = \int_{\mathbb{R}^n} T \cdot \mathrm{d}\mu_E, \qquad \forall T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n). \tag{2.3.2}$$

Moreover, E is a set of finite perimeter if and only if $|\mu_E|(\mathbb{R}^n) < \infty$.

Of course (2.3.2) is equivalent to

$$\int_E \nabla \varphi = \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu_E, \qquad \forall \varphi \in C_c^1(E). \tag{2.3.3}$$

We call μ_E the *Gauss–Green measure* of E, and define the *relative perimeter* of E in $F \subset \mathbb{R}^n$, and the *perimeter* of E, as

$$P(E, F) = |\mu_E|(F), \qquad P(E) = |\mu_E|(\mathbb{R}^n).$$

By the fundamental lemma of the Calculus of Variations, μ_E is uniquely determined as a Radon measure on $\mathbb{R}^n$.
We note that a Lebesgue measurable set $E \subset \mathbb{R}^n$ is a set of locally finite perimeter if and only if the distributional gradient $D\chi_E$ of $\chi_E \in L^1_{loc}(\mathbb{R}^n)$ can be represented as the integration with respect to the $\mathbb{R}^n$-valued Radon measure $-\mu_E$. Therefore we speak of *distributional perimeter* and we refer to (2.3.3) as the *distributional Gauss-Green theorem.*

Example 2.3.9. By the Gauss–Green theorem, if $E \subset \mathbb{R}^n$ is an open set with C^1 boundary and ν^E is the outer unit normal to E, then $\nu^E \mathcal{H}^{n-1} \llcorner \partial E$ is a $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^n$ such that (2.3.3) holds true. Hence E is a set of locally finite perimeter, with Gauss–Green measure $\mu_E = \nu^E \mathcal{H}^{n-1} \llcorner \partial E$, $P(E) = \mathcal{H}^{n-1}(\partial E)$ and $P(E, F) = \mathcal{H}^{n-1}(F \cap \partial E)$ for every $F \subset \mathbb{R}^n$.

Theorem 2.3.10 ([Mag12, Theorem 12.26]). *If $R > 0$ and $\{E_h\}_{h \in \mathbb{N}}$ are sets of finite perimeter in $\mathbb{R}^n$, with*

$$\sup_{h \in \mathbb{N}} P(E_h) < \infty, \tag{2.3.4}$$

$$E_h \subset B_R, \qquad \forall h \in \mathbb{N},$$

then there exist E of finite perimeter in $\mathbb{R}^n$ and $h(k) \to \infty$ as $k \to \infty$, with

$$E_{h(k)} \to E, \qquad \mu_{E_{h(x)}} \overset{*}{\rightharpoonup} \mu_E, \qquad E \subset B_R.$$

We cannot conclude the compactness of sets from the perimeter bound (2.3.4) only. For example, if $\{x_h\} \subset \mathbb{R}^n$ is such that $|x_h| \to \infty$, then the sequence $E_h = B_1(x_h)$ satisfies $P(E_h) = n\omega_n$ for every $h \in \mathbb{N}$, while $|E \Delta E_h| \to 2\omega_n$ as $h \to \infty$ for every Lebesgue measurable set E with $|E| = \omega_n$. Thus, $\{E_h\}$ does not admit any converging subsequence. It is clear, however, that $\{E_h\}$ locally converges to the empty set, so that compactness with respect to the local convergence still holds.
The following isoperimetric-type inequality is related to the relative isoperimetric problem

$$\inf \left\{ P(E, B_r(x)) \ : \ E \subset B_r(x), \ |E| = v \right\}.$$

For this reason, it is usually called the *relative isoperimetric inequality on a ball.*

Proposition 2.3.11 (Local perimeter bound on volume, [Mag12, Proposition 12.37]). *If $n \geq 2$, $t \in (0,1)$, $x \in \mathbb{R}^n$ and $r > 0$, then there exists a positive constant $c(n,t)$ such that*

$$P(E, B_r(x)) \geq c(n,t)\, |E \cap B_r(x)|^{\frac{n-1}{n}},$$

for every set of locally finite perimeter E such that $|E \cap B_r(x)| \leq t|B_r(x)|$.

We recall the classical *Euclidean isoperimetric inequality*.

Theorem 2.3.12 ([De 58], [Mag12, Theorem 14.1]). *If E is a Lebesgue measurable set in $\mathbb{R}^n$ with $|E| < \infty$, then*

$$P(E) \geq n\omega_n^{\frac{1}{n}} |E|^{\frac{n-1}{n}}.$$

Equality holds if and only if $|E \Delta B_r(x)| = 0$ for some $x \in \mathbb{R}^n$, $r > 0$.

The following approximation theorem shows that the sets of finite perimeter in $\mathbb{R}^n$ can be approximated in measure by open sets with smooth boundaries in an optimal way, i.e. also getting convergence of perimeters to perimeters.

Theorem 2.3.13 (Density of smooth sets, [AFP00, Theorem 3.42]). *Let E be a set of finite perimeter in $\mathbb{R}^n$, $n \geq 2$. Then, there exists a sequence $\{E_h\}$ of open sets with smooth boundaries converging in measure to E and such that*

$$\lim_{h\to\infty} P(E_h, \mathbb{R}^n) = P(E, \mathbb{R}^n).$$

We define the *reduced boundary* $\partial^* E$ of a set of locally finite perimeter E in $\mathbb{R}^n$ as

$$\partial^* E := \left\{ x \in \operatorname{spt}|D\chi_E| \ : \ \exists\, \nu^E(x) := -\lim_{r\to 0} \frac{D\chi_E(B_r(x))}{|D\chi_E(B_r(x))|} \text{ with } |\nu^E(x)| = 1 \right\}.$$

The vector ν^E is called the *generalized outer unit normal to E*. By Example 2.3.9, if E is an open set with C^1 boundary, then $\partial^* E = \partial E$ and the generalized outer unit normal coincides with classical notion of outer unit normal.

Theorem 2.3.14 (De Giorgi's structure theorem, [Mag12, Theorem 15.9]). *If E is a set of locally finite perimeter in $\mathbb{R}^n$, then the Gauss-Green measure μ_E of E satisfies*

$$\mu_E = \nu^E \mathcal{H}^{n-1} \llcorner \partial^* E, \qquad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E,$$

and the generalized Gauss-Green formula holds true:

$$\int_E \nabla\varphi = \int_{\partial^* E} \varphi \nu^E \, \mathrm{d}\mathcal{H}^{n-1}, \qquad \forall \varphi \in C_c^1(\mathbb{R}^n).$$

Introducing the sets of density $t \in [0,1]$ points for E defined by

$$E^{(t)} := \left\{ x \in \mathbb{R}^n \ : \ \lim_{r\to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = t \right\},$$

we define the *essential boundary* $\partial^e E$ of a Lebesgue measurable set $E \subset \mathbb{R}^n$

$$\partial^e E = \mathbb{R}^n \setminus \left(E^{(0)} \cup E^{(1)}\right).$$

Theorem 2.3.15 (Federer's theorem, [Mag12, Theorem 16.2]). *If E is a set of locally finite perimeter in $\mathbb{R}^n$, then $\partial^* E \subset E^{(1/2)} \subset \partial^e E$, with*

$$\mathcal{H}^{n-1}\left(\partial^e E \setminus \partial^* E\right) = 0.$$

Now we state some properties of sets of finite perimeter under the action of diffeomorphisms.

Proposition 2.3.16 (Diffeomorphic images of sets of finite perimeter, [Mag12, Proposition 17.1]). *If E is a set of locally finite perimeter in $\mathbb{R}^n$ and f is a diffeomorphism of $\mathbb{R}^n$ with $g = f^{-1}$, then $f(E)$ is a set of locally finite perimeter in $\mathbb{R}^n$ with*

$$\mathcal{H}^{n-1}(f(\partial^* E)\Delta\partial^* f(E)) = 0,$$

$$\int_{\partial^* f(E)} \varphi\nu^{f(E)}\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\partial^* E} (\varphi\circ f)\mathbf{J}f(\nabla g\circ f)^*\nu^E\,\mathrm{d}\mathcal{H}^{n-1},$$

for every $\varphi \in C_c(\mathbb{R}^n)$. In particular, for every Borel set $F \subset \mathbb{R}^n$,

$$\mathcal{H}^{n-1}(F\cap\partial^* f(E)) = \int_{\varphi(F)\cap\partial^* E} \mathbf{J}f\left|(\nabla g\circ f)^*\nu^E\right|\,\mathrm{d}\mathcal{H}^{n-1}.$$

A *one parameter family of diffeomorphisms of* $\mathbb{R}^n$ is a smooth function

$$(x,t)\in\mathbb{R}^n\times(-\varepsilon,\varepsilon)\mapsto f(t,x) = f_t(x)\in\mathbb{R}^n, \qquad \varepsilon>0,$$

such that, for each fixed $|t|<\varepsilon$, $f_t : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism of $\mathbb{R}^n$. Given an open set A in $\mathbb{R}^n$, we say that $\{f_t\}_{|t|<\varepsilon}$ is a *local variation in A* if it defines a one parameter family of diffeomorphisms such that

$$f_0(x) = x, \qquad \forall x\in\mathbb{R}^n,$$

$$\{x\in\mathbb{R}^n \ : \ f_t(x)\neq x\} \subset\subset A, \qquad \forall |t|<\varepsilon.$$

Lemma 2.3.17 ([Mag12, Lemma 17.9]). *If E is a set of locally finite perimeter in $\mathbb{R}^n$, A is open and $\{f_t\}_{|t|<\varepsilon}$ is a local variation in A, then there exist positive constants C and $\varepsilon_0 < \varepsilon$ such that, if K is a compact set with $\{x\neq f_t(x)\}\subset K\subset A$, then*

$$|f_t(E)\Delta E| \le C|t|P(E,K).$$

Lemma 2.3.18 (Volume-fixing variations, [Mag12, Lemma 17.21]). *If E is a set of finite perimeter and A is an open set such that $\mathcal{H}^{n-1}(A\cap\partial^* E)>0$, then there exist $\sigma_0 = \sigma_0(E,A)>0$ and $C = C(E,A)<\infty$ such that for every $\sigma\in(-\sigma_0,\sigma_0)$ we can find a set of finite perimeter F with $F\Delta E\subset\subset A$ and*

$$|F| = |E|+\sigma, \qquad |P(F,A)-P(E,A)|\le C|\sigma|.$$

Finally we recall the following

Proposition 2.3.19 ([Mag12, Proposition 12.20]). *If E is a set of locally finite perimeter in $\mathbb{R}^n$, then*

$$(\mu_E)_\varepsilon = -\nabla(1_E\star\varrho_\varepsilon)\mathcal{L}^n, \qquad \forall\varepsilon>0,$$

$$-\nabla(1_E\star\varrho_\varepsilon)\mathcal{L}^n \overset{*}{\rightharpoonup} \mu_E, \qquad |\nabla(1_E\star\varrho_\varepsilon)\mathcal{L}^n| \overset{*}{\rightharpoonup} |\mu_E|$$

as $\varepsilon\to 0^+$. If, conversely, E is a Lebesgue measurable set in $\mathbb{R}^n$ such that

$$\limsup_{\varepsilon\to 0^+}\int_K |\nabla(1_E\star\varrho_\varepsilon)(x)|\,\mathrm{d}x<\infty,$$

for every compact set K, then E is of locally finite perimeter.

Slicing

Let us introduce the following notation: when we need to decompose $\mathbb{R}^n$ as the Cartesian product $\mathbb{R}^k\times\mathbb{R}^{n-k}$, we denote by $\mathbf{p} : \mathbb{R}^n\to\mathbb{R}^k\times\{0\} = \mathbb{R}^k$ and $\mathbf{q} : \mathbb{R}^n\to\{0\}\times\mathbb{R}^{n-k} = \mathbb{R}^{n-k}$ the horizontal and vertical projections, so that $x = (\mathbf{p}x,\mathbf{q}x)$, $x\in\mathbb{R}^n$. If we write $\mathbb{R}^n = \mathbb{R}^{n-1}\times\mathbb{R}$, we denote by E_t the horizontal slice of $E\subset\mathbb{R}^n$ with $t\in\mathbb{R}$

$$E_t = \left\{z\in\mathbb{R}^{n-1} \ : \ (z,t)\in E\right\}, \qquad t\in\mathbb{R}.$$

By the coarea formula for rectifiable sets 2.1.40 we can study the slicing by hyperplanes of a set of finite perimeter.

Theorem 2.3.20 (Slicing boundaries by hyperplanes, [Mag12, Theorem 18.11], [FMP08, Theorem 6.1]). *If E is a set of locally finite perimeter in $\mathbb{R}^n$ then, for a.e. $t \in \mathbb{R}$, the horizontal section E_t of E is a set of locally finite perimeter in $\mathbb{R}^{n-1}$, with*

$$\mathcal{H}^{n-2}(\partial^* E_t \Delta (\partial^* E)_t) = 0,$$

$$\mathbf{p}\nu^E(z,t) \neq 0, \qquad \text{for } \mathcal{H}^{n-2} - \text{a.e. } z \in (\partial^* E)_t,$$

and

$$\mu_{E_t} = \frac{\mathbf{p}\nu^E(\cdot,t)}{|\mathbf{p}\nu^E(\cdot,t)|}\mathcal{H}^{n-2} \llcorner (\partial^* E)_t.$$

Moreover, if E has finite Lebesgue measure and

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* E \ : \ \nu^E(x) = \pm e_n\right\}\right) = 0,$$

then $v_E(t) = \mathcal{H}^{n-1}(E_t)$, $t \in \mathbb{R}$ is such that $v_E \in W^{1,1}_{loc}(\mathbb{R})$, with

$$v_E'(t) = -\int_{(\partial^* E)_t} \frac{\mathbf{q}\nu^E(z,t)}{|\mathbf{p}\nu^E(z,t)|}\, \mathrm{d}\mathcal{H}^{n-2}(z), \qquad \text{for a.e. } t \in \mathbb{R}.$$

2.4 Capillarity functional

Definitions and main results

In this section we prove some preparatory results on the capillarity functional. From now on, H denotes the closed half-space $H := \{x_n \le 0\}$. If E is a measurable set in the half-space $\{x_n > 0\} \subset \mathbb{R}^n$ and $\lambda \in (-1,1)$, we define the weighted perimeter functional

$$P_\lambda(E) := P(E, \{x_n > 0\}) - \lambda \mathcal{H}^{n-1}(\partial^* E \cap \{x_n = 0\}).$$

Interpreting the perimeter as a measure of the surface tension of a liquid drop, the constant λ basically represent the relative adhesion coefficient between a liquid drop and the solid walls of the container given by $\{x_n > 0\}$.
If $v > 0$, we consider the isoperimetric capillarity problem

$$\inf\left\{P_\lambda(E) \ : \ E \subset \{x_n > 0\}, |E| = v\right\}. \tag{2.4.1}$$

Minimizers for (2.4.1), below called isoperimetric sets for (2.4.1), are given by suitably truncated balls lying on the boundary of the half-space. More precisely, if $B^\lambda = \{x \in B_1(0) \subset \mathbb{R}^n : \langle x, e_n\rangle > \lambda\}$, and for $v > 0$ we set

$$B^\lambda(v) := \frac{v^{\frac{1}{n}}}{|B^\lambda|^{\frac{1}{n}}}(B^\lambda - \lambda e_n)$$

then minimizers for (2.4.1) are sets of the form

$$B^\lambda(v,x) := B^\lambda(v) + x, \tag{2.4.2}$$

for $x \in \{x_n = 0\}$. Explicitly

Theorem 2.4.1 (Liquid drops in the absence of gravity, [Mag12, Theorem 19.21]). *For every $\lambda \in (-1,1)$ and $m > 0$, there exists a unique $\sigma(\lambda, m)$ with the following property: a set of finite perimeter $E \subset \mathbb{R}^n \setminus H$ with $|E| = m$ is a minimizer in the variational problem (2.4.1) if and only if, up to horizontal translation, is equivalent to the set*

$$B^\lambda(m) = B_r(s\, e_n) \cap (\mathbb{R}^n \setminus H),$$

where $s \in \mathbb{R}$ and $r > 0$ are uniquely determined by the constraints

$$|B^\lambda(m)| = m, \qquad P(B^\lambda(m), \partial H) = \sigma.$$

Moreover,

$$\left\langle \nu^{B^\lambda(m)}, e_n \right\rangle = \lambda \qquad \text{on the boundary of the hypersurface} \quad \partial B^\lambda(m) \cap (\mathbb{R}^n \setminus H).$$

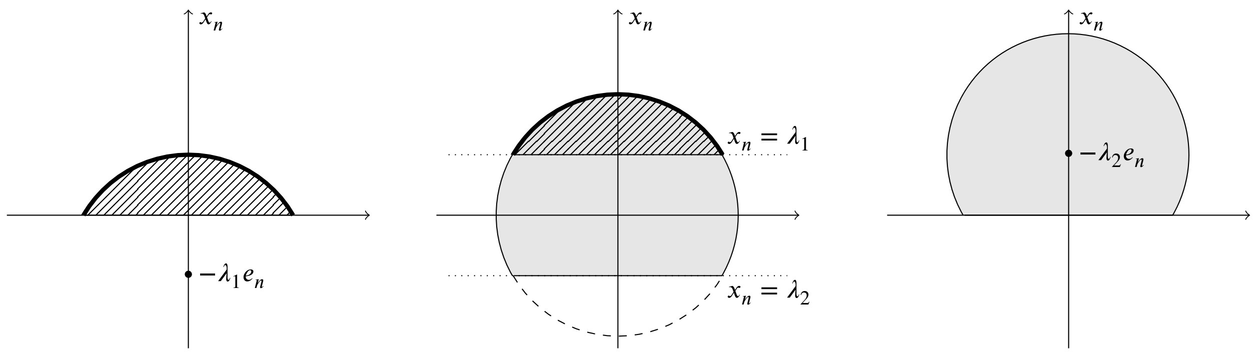

Figure 2.1: Left: $B^{\lambda_1}(|B^{\lambda_1}|)$ for some $\lambda_1 > 0$. Middle: Diagonally striped set $= B^{\lambda_1}$, Gray set $= B^{\lambda_2}$, for some $\lambda_1 > 0, \lambda_2 < 0$. Right: $B^{\lambda_2}(|B^{\lambda_2}|)$ for some $\lambda_2 < 0$.

We remark that the problem is trivial if $\lambda \geq 1$, and that it reduces to the Euclidean isoperimetric problem if $\lambda \leq -1$. The minimality of sets $B^\lambda(v, x)$ for (2.4.1) comes with an isoperimetric inequality for P_λ

$$P_\lambda(E) \geq n|B^\lambda|^{\frac{1}{n}}|E|^{\frac{n-1}{n}},$$

see Theorem 3.2.3 below.

Remark 2.4.2. Let $E \subset \mathbb{R}^n \setminus H$ be a measurable set. We observe that

$$P_\lambda(E) = \int_{\partial^* E\setminus H} 1 - \lambda \left\langle e_n, \nu^E \right\rangle \,\mathrm{d}\mathcal{H}^{n-1},$$

where ν^E is the generalized outer normal to E. In particular, since $|\lambda| < 1$, we have that $P_\lambda(E) \geq 0$. The previous identity follows from the divergence theorem, indeed

$$0 = \int_E \operatorname{div} e_n \,\mathrm{d}x = -\mathcal{H}^{n-1}(\partial^* E \cap \partial H) + \int_{\partial^* E\setminus H} \left\langle e_n, \nu^E \right\rangle \,\mathrm{d}\mathcal{H}^{n-1}.$$

We now aim at proving an approximation result for sets of finite perimeter contained in $\mathbb{R}^n \setminus H$ by sequences of sets having smooth boundary relative in $\mathbb{R}^n \setminus H$. We need the following lemma first.

Lemma 2.4.3. *Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter with finite measure. For any $t \geq 0$ define the diffeomorphism $\varphi_t : \{x_n \geq 0\} \to \{x_n \geq 0\}$ given by $\varphi_t(x', x_n) := (x', x_n(1 + te^{-|x'|^2}))$, where we wrote $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ for any $x \in \{x_n \geq 0\} \subset \mathbb{R}^n$. Then*

1. *for at most countably many $t \in [0, +\infty)$ there holds*

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^*(\varphi_t(E)) \setminus H \ : \ \nu^{\varphi_t(E)}(x) = \pm e_n\right\}\right) > 0;$$

2. *for at most countably many $\nu \in \mathbb{S}^{n-1}$ there holds*

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* E \ : \ \nu^E(x) = \pm\nu\right\}\right) > 0.$$

Proof. We begin by proving (1). Define $f_t : \mathbb{R}^{n-1} \to \mathbb{R}$ by $f_t(x') := 1 + te^{-|x'|^2}$, and let

$$A_t := \left\{(x', x_n) \in \partial^* E \cap \mathbb{R}^n \setminus H \text{ with } x' \in \mathbb{R}^{n-1} \setminus \{0\} \ : \ \nu^E(x', x_n) \text{ is proportional to } (x_n \nabla f_t(x'), f_t(x'))\right\}.$$

We claim that, if $s, r > 0$, with $s \neq r$, then $A_r \cap A_s = \emptyset$. Indeed, if $A_r \cap A_s \neq \emptyset$, there exist $(\bar{x}', \bar{x}_n) \in \partial^* E \cap \mathbb{R}^n \setminus H$ and $\bar{\alpha} \in \mathbb{R} \setminus \{0\}$ such that

$$(\bar{x}_n \nabla f_r(\bar{x}'), f_r(\bar{x}')) = \bar{\alpha}(\bar{x}_n \nabla f_s(\bar{x}'), f_s(\bar{x}')).$$

Since $x_n > 0$ and $x' \neq 0$, then $\bar{x}_n \nabla f_r(\bar{x}') = \bar{\alpha}\bar{x}_n \nabla f_s(\bar{x}')$ implies $r = \bar{\alpha}\, s$. Thus $f_r(\bar{x}') = \bar{\alpha} f_s(\bar{x}')$ implies $1 = \bar{\alpha}$, which in turn implies $r = s$, contradiction.
For $k \in \mathbb{N}_{\geq 1}$ let us define

$$M_k := \left\{ t > 0 \ : \ \mathcal{H}^{n-1}(A_t) > \frac{1}{k} \right\}.$$

We want to prove that M_k is finite. Let $t_1, \dots, t_l \in M_k$, with $t_i \neq t_j$ if $i \neq j$. Since $A_{t_i} \neq A_{t_j} = \emptyset$, then

$$P(E) \geq \sum_{i=1}^{l} \mathcal{H}^{n-1}(A_{t_i}) \geq \frac{1}{k} l,$$

and then $\# M_k \leq k P(E)$. Therefore the set

$$\left\{ t > 0 \ : \ \mathcal{H}^{n-1}(A_t) > 0 \right\} = \bigcup_k M_k,$$

is countable.
Since the differential of φ_t is represented by the matrix

$$\mathrm{d}\varphi_t(x', x_n) = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & 0 \\ x_n \nabla f_t(x') & f_t(x') \end{pmatrix},$$

requiring that $\nu^{\varphi_t(E)}(\varphi_t(x)) = \pm e_n$ for some $x = (x', x_n)$ means that

$$\nu^E(x', x_n) \text{ is proportional to } (x_n \nabla f_t(x'), f_t(x')),$$

see Proposition 2.3.16 and Proposition 2.1.34. Therefore for any $t \notin \cup_k M_k$ there holds

$$\mathcal{H}^{n-1}\left(\left\{ x \in \partial^*(\varphi_t(E)) \setminus H \ : \ \nu^{\varphi_t(E)}(x) = \pm e_n \right\} \right) = 0.$$

The proof of (2) is analogous. For $k \in \mathbb{N}_{\geq 1}$ let

$$N_k := \left\{ \nu \in \mathbb{S}^{n-1} \ : \ \mathcal{H}^{n-1}\left(\left\{ x \in \partial^* E \ : \ \nu^E = \nu \right\} \right) > \frac{1}{k} \right\}.$$

As above one gets that $\cup_k N_k$ is at most countable, and (2) follows. □

We are now ready to prove

Lemma 2.4.4 (Approximation with regular sets). *Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter with finite measure. Then there exists a sequence of sets $E_i \subset \mathbb{R}^n \setminus H$ such that*

1. *E_i is a bounded set such that $\partial E_i \setminus \partial H$ is a smooth hypersurface (possibly with smooth boundary) such that either $\partial E_i \cap \partial H = \emptyset$ or $\partial E_i \setminus \partial H$ intersects ∂H orthogonally;*
2. *$E_i \to E$ in L^1, $P(E_i, \mathbb{R}^n \setminus H) \to P(E, \mathbb{R}^n \setminus H)$, $\mathcal{H}^{n-1}(\partial E_i \cap \partial H) \to \mathcal{H}^{n-1}(\partial^* E \cap \partial H)$, as $i \to +\infty$. In particular, $P(E_i) \to P(E)$ and $P_\lambda(E_i) \to P_\lambda(E)$ as $i \to +\infty$, for any $\lambda \in (-1, 1)$;*
3. *$\mathcal{H}^{n-1}(\{x \in \partial^* E_i \ : \ \nu^{E_i}(x) = \pm e_j\}) = 0$ for any $j = 1, \dots, n-1$;*
4. *$\mathcal{H}^{n-1}(\{x \in \partial^* E_i \setminus H \ : \ \nu^{E_i}(x) = \pm e_n\}) = 0$.*

Proof. Since $P(E), |E| < +\infty$, by a diagonal argument we can assume without loss of generality that E is bounded.
Step 1. We first construct a sequence $F_i \subset \mathbb{R}^n \setminus H$ such that 1. and 2. hold with F_i in place of E_i. Let us denote by F the union of E with the reflection of E with respect to the hyperplane $\{x_n = 0\}$. There exists a sequence of smooth sets $\tilde{F}_i \subset \mathbb{R}^n$, symmetric with respect to $\{x_n = 0\}$, such that they converge to F in $L^1(\mathbb{R}^n)$ and $P(\tilde{F}_i) \to P(F)$ (see Theorem 2.3.13). The fact that $\tilde{F}_i$ is symmetric with respect to $\{x_n = 0\}$ follows as $\tilde{F}_i$ can be obtained as superlevel set of a convolution of χ_F with a symmetric mollifier. In particular, if $\partial \tilde{F}_i \cap \partial H \neq \emptyset$, then $\partial \tilde{F}_i$ intersects H orthogonally, and thus $\partial \tilde{F}_i \cap \partial H$ is a smooth $(n-2)$-dimensional manifold. Let $F_i := \tilde{F}_i \setminus H$. Then $F_i \to E$ in L^1 and

$$P(F_i, \mathbb{R}^n \setminus H) = \frac{1}{2} P(\tilde{F}_i) \to \frac{1}{2} P(F) = P(E, \mathbb{R}^n \setminus H).$$

Let us define the function

$$f : \mathbb{R}^n \to [0, +\infty)$$
$$f(v) = |v| - \lambda \langle e_n, v\rangle .$$

Note that f is continuous; moreover $f(tv) = tf(v)$ for any $t \ge 0$ and f is convex. Let us set

$$\mu_i := \nu^{F_i}\mathcal{H}^{n-1} \llcorner (\partial^* F_i \cap (\mathbb{R}^n \setminus H))$$
$$\mu := \nu^{E}\mathcal{H}^{n-1} \llcorner (\partial^* E \cap (\mathbb{R}^n \setminus H)).$$

Since $\nu^{F_i}\mathcal{H}^{n-1}\llcorner\partial^* F_i \to \nu^E\mathcal{H}^{n-1}\llcorner\partial^* E$ weakly* in $\mathbb{R}^n$, then $\mu_i \to \mu$ weakly* in $\mathbb{R}^n\setminus H$. Since we already know that $|\mu_i|(\mathbb{R}^n \setminus H) \to |\mu|(\mathbb{R}^n \setminus H)$, by Reshetnyak continuity theorem 2.1.23 we deduce $P_\lambda(F_i) = \int f(\mu_i/|\mu_i|)\,\mathrm{d}|\mu_i| \to \int f(\mu/|\mu|)\,\mathrm{d}|\mu| = P_\lambda(E)$.
Step 2. Let us consider the flow $\varphi_t : \mathbb{R}^n \setminus H \to \mathbb{R}^n \setminus H$ such that

$$\varphi_t(x', x_n) := (x', x_n(1 + te^{-|x'|^2})),$$

for $t \ge 0$. By Lemma 2.4.3 for a.e. t we have that $\varphi_t(F_i)$ satisfies (4). Moreover, for any $t \ge 0$ there exists a sequence of rotations $\mathcal{R}_{j,t} : (x', x_n) \mapsto (R_{j,t}(x'), x_n)$ along the n-th axis converging to the identity such that

$$\mathcal{H}^{n-1}(\{x \in \partial^*(\mathcal{R}_{j,t}(\varphi_t(F_i)) : \nu^{\mathcal{R}_{j,t}(\varphi_t(F_i))}(x) = \pm e_l\}) = 0$$

for all $l = 1, \dots, n-1$. By a diagonal argument, since $\varphi_t(F_i)$ maintains orthogonal intersection with ∂H, one extract the desired sequence E_i. □

Corollary 2.4.5. *Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter with finite measure. Then*

$$P_\lambda(E) \ge \frac{1-\lambda}{2} P(E).$$

Proof. Let $\{E_i\}_i$ be the sequence of smooth sets in $\mathbb{R}^n\setminus H$ given by Lemma 2.4.4. Since the orthogonal projection on ∂H is a 1-Lipschitz and surjective map from $\partial E_i \cap (\mathbb{R}^n \setminus H)$ onto $\partial E_i \cap \partial H$, then by Proposition 2.1.29

$$P(E_i, \mathbb{R}^n \setminus H) \ge \mathcal{H}^{n-1}(\partial^* E_i \cap \partial H).$$

Since

$$P_\lambda(E_i) = \frac{1+\lambda}{2}(P(E_i, \mathbb{R}^n \setminus H) - \mathcal{H}^{n-1}(\partial^* E_i \cap \partial H)) + \frac{1-\lambda}{2} P(E_i), \tag{2.4.3}$$

the claim follows by passing to the limit $i \to \infty$. □

2.5 Regularity of (Λ, r_0)-minimizers

We recall definitions and basic properties of local (Λ, r_0)-minimizers of the perimeter. A detailed account on the theory of (Λ, r_0)-minimizers can be found in [Mag12].

Definition 2.5.1. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $E \subset \mathbb{R}^n$ be a set of finite perimeter. We say that E is a local (Λ, r_0)-minimizer of the perimeter in Ω, with $\Lambda, r_0 > 0$, if

$$P(E, B_r(x)) \le P(F, B_r(x)) + \Lambda|E\Delta F|,$$

whenever $E\Delta F \subset\subset B_r(x) \subset\subset \Omega$ and $r \le r_0$.

It is well-known that local (Λ, r_0)-minimizers have bounded mean curvature in a generalized sense. We provide a proof of this fact in the following lemma.

Lemma 2.5.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $E \subset \mathbb{R}^n$ be a local (Λ, r_0)-minimizer of the perimeter in Ω. Then there exists $H \in L^\infty(P(E, \cdot), \mathbb{R}^n)$ such that $\|H\|_{L^\infty} \le \Lambda$ and*

$$\int_{\partial^* E} \mathrm{div}_T X = -\int_{\partial^* E} \langle X, H\rangle \qquad \forall X \in C^1_c(\Omega, \mathbb{R}^n),$$

where $\mathrm{div}_T X$ is the tangential divergence of X along the ($\mathcal{H}^{n-1}$-a.e. defined) tangent space of $\partial^* E$. We shall refer to H as to the (generalized) mean curvature *of E.*

Proof. Let $X \in C^1_c(B_r(x))$, with $B_r(x) \subset\subset \Omega$ and $r \le r_0$. Let $\{g_t\}_{t<|\eta|}$ be the flow of the vector field X, and define $F_t = g_t(E)$. Then $E\Delta F_t \subset\subset B_r(x)$. Let $f_t = g_t^{-1} = g_{-t}$. If $u \in C^1(\Omega)$ then

$$\begin{aligned}
\int_{B_r(x)} |u(f_t(y)) - u(y)|\,\mathrm{d}y &= \int_{B_r(x)} \left| \int_0^t \partial_s[u(f_s(y))]\,\mathrm{d}s \right| \mathrm{d}y \\
&\le \int_{B_r(x)} \int_0^t |\nabla u(f_s(y))|\; |X(f_s(y))|\,\mathrm{d}s\,\mathrm{d}y \\
&= \int_0^t \int_{B_r(x)} |\nabla u(f_s(y))|\; |X(f_s(y))| \frac{Jf_s(y)}{Jf_s(y)}\,\mathrm{d}y\,\mathrm{d}s \\
&\le (1+o(1)) \int_0^t \int_{B_r(x)} |\nabla u(f_s(y))|\; |X(f_s(y))|\; Jf_s(y)\,\mathrm{d}y\,\mathrm{d}s \\
&= (1+o(1)) \int_0^t \int_{B_r(x)} |\nabla u(z)|\; |X(z)|\,\mathrm{d}z\,\mathrm{d}s \\
&= (1+o(1))t \int_{B_r(x)} |\nabla u(z)|\; |X(z)|\,\mathrm{d}z,
\end{aligned} \tag{2.5.1}$$

where $o(1) \to 0$ as $t \to 0$.
Setting $u = u_\varepsilon = \chi_E \star \varrho_\varepsilon$, then $|\nabla u_\varepsilon|\mathcal{L}^n \to P(E,\cdot)$ as $\varepsilon \to 0$ by Proposition 2.3.19. Also

$$\int_{B_r(x)} \left| u_\varepsilon(f_t(y)) - \chi_{f_{-t}(E)}(y) \right| \frac{Jf_t}{Jf_t}\,\mathrm{d}y \le 2 \int_{B_r(x)} |u_\varepsilon(z) - \chi_E(z)|\,\mathrm{d}z.$$

Hence setting $u = u_\varepsilon$ in (2.5.1) and letting $\varepsilon \to 0$ implies

$$|E\Delta F_t| = \int_{B_r(x)} |\chi_{F_t} - \chi_E| \le (1+o(1))t \int_{B_r(x)} |X|\,\mathrm{d}P(E,\cdot).$$

By (Λ, r_0)-minimality we deduce

$$P(E, B_r(x)) - P(F_t, B_r(x)) \le (1+o(1))\Lambda t \int_{B_r(x)} |X|\,\mathrm{d}P(E,\cdot).$$

Dividing by $t > 0$ and letting $t \to 0^+$ we get

$$-\int_{\partial^* E} \mathrm{div}_T X\,\mathrm{d}P(E,\cdot) \le \Lambda \int_{\partial^* E} |X|\,\mathrm{d}P(E,\cdot).$$

Up to changing X with $-X$ we obtain

$$\left| \int_{\partial^* E} \mathrm{div}_T X\,\mathrm{d}P(E,\cdot) \right| \le \Lambda \int_{\partial^* E} |X|\,\mathrm{d}P(E,\cdot),$$

that implies the existence of the generalized mean curvature $H \in L^\infty(P(E,\cdot) \llcorner B_r(x))$ for E in $B_r(x)$ with $\|H\|_{L^\infty} \le \Lambda$. Since $B_r(x)$ was arbitrary in Ω, by a partition of unity argument the claim follows. □

Let us further recall the following fundamental regularity properties of local (Λ, r_0)-minimizers.

Theorem 2.5.3 ([Tam84], [Mag12, Theorem 26.3, Theorem 26.6])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $E \subset \Omega$ be a local (Λ, r_0)-minimizer in Ω. Then the set $E^{(1)}$ of points of density 1 for E is an open representative for E. Moreover, representing E with $E^{(1)}$, we have that $\partial^* E \cap \Omega$ is a $C^{1,\frac{1}{2}}$ manifold and $\mathcal{H}^d(\partial E \cap \Omega \setminus \partial^* E) = 0$ for any $d > n-8$.*
Let $E_i \subset \Omega$ be a sequence of local (Λ, r_0)-minimizers in Ω that converges to E in $L^1(\Omega)$. If $\partial E \cap B_r(x)$ is of class C^2, for some $B_r(x) \subset \Omega$, then $\partial E_i \cap B_{r/2}(x)$ is of class $C^{1,\frac{1}{2}}$ for large i and converges to $\partial E \cap B_{r/2}(x)$ in $C^{1,\alpha}$ for any $\alpha \in (0, 1/2)$.

Thanks to Theorem 2.5.3, we will always identify a local (Λ, r_0)-minimizer E with the open set $E^{(1)}$.

2.6 Schwarz symmetrization and axially symmetric hypersurfaces

We introduce an important class of axially symmetric hypersurfaces.

Definition 2.6.1. Let $E \subset \mathbb{R}^n$ be a Borel set. Then its Schwarz symmetrization (with respect to the n-th axis) is the set

$$E^* := \{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R} \ : \ \omega_{n-1}|x'|^{n-1} < \mathcal{H}^{n-1}(E \cap \{x_n = t\}), \ t \in \mathbb{R}\}.$$

A Borel set $E \subset \mathbb{R}^n$ is said to be Schwarz-symmetric if it coincides with its Schwarz symmetrization, up to negligible sets. The profile function of a Schwarz-symmetric set E is the a.e. defined map $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) := \left(\frac{\mathcal{H}^{n-1}(E \cap \{x_n = t\})}{\omega_{n-1}} \right)^{\frac{1}{n-1}}.$$

Remark 2.6.2. Let $E \subset \{x_n > 0\}$ be a set of finite perimeter. Let E be Schwarz-symmetric with profile function f. Assume that $\mathcal{H}^{n-1}(\{x \in \partial^* E \ : \ \nu^E(x) = \pm e_n\} = 0$.
Then f is differentiable almost everywhere by Theorem 2.3.20 with derivative

$$f'(t) = -\left(\frac{1}{\omega_{n-1} \left(\mathcal{H}^{n-1}(E \cap \{x_n = t\})\right)^{n-2}} \right)^{\frac{1}{n-1}} \int_{\partial^* E \cap \{x_n = t\}} \frac{\left\langle \nu^E(z,t), e_n \right\rangle}{\left|\nu^E(z,t) - \langle \nu^E(z,t), e_n\rangle \, e_n\right|} \, \mathrm{d}\mathcal{H}^{n-2}(z),$$

at a.e. $t \in \mathbb{R}$ such that $f(t) > 0$. The generalized outer normal to E can be written as

$$\nu^E(f(t)e, t) = \frac{1}{\sqrt{1 + (f')^2}}(e, -f'),$$

for any $e = (e_1, \dots, e_{n-1}) \in \mathbb{R}^{n-1}$ with $|e| = 1$ and for a.e. $t \in \mathbb{R}$ such that $f(t) > 0$.
Moreover by area formula 2.1.38 we have

$$\begin{aligned} P_\lambda(E) &= \int_{\partial^* E \setminus H} 1 - \lambda \left\langle e_n, \nu^E \right\rangle \, \mathrm{d}\mathcal{H}^{n-1} \\ &= \int_0^{+\infty} \left((n-1)\omega_{n-1} \sqrt{1 + (f')^2} f^{n-2} - \lambda \int_{\partial^* E \cap \{x_n = t\}} \left\langle e_n, \nu^E \right\rangle \sqrt{1 + (f')^2} \, \mathrm{d}\mathcal{H}^{n-2} \right) \mathrm{d}t \\ &= (n-1)\omega_{n-1} \int_0^{+\infty} \left(\sqrt{1 + (f')^2} f^{n-2} + \lambda f' f^{n-2} \right) \mathrm{d}t. \end{aligned}$$

We recall the following properties of Schwarz rearrangement.

Proposition 2.6.3 (Contractivity of Schwarz rearrangement, [Mag12, Exercise 19.14])**.** *If E and F are Lebesgue measurable sets, then $|E^* \Delta F^*| \le |E \Delta F|$.*

Proof. Let us recall the definition of horizontal slice of a Lebesgue measurable set E as

$$E_t := \left\{ z \in \mathbb{R}^{n-1} \ : \ (z, t) \in E \right\}.$$

Note that by Fubini's Theorem 2.1.20

$$|E \Delta F| = \int_{\mathbb{R}} \mathcal{L}^{n-1} \left(E_t \Delta F_t \right) \mathrm{d}t \ge \int_{\mathbb{R}} \left| \mathcal{L}^{n-1}(E_t) - \mathcal{L}^{n-1}(F_t) \right| = |E^* \Delta F^*|.$$

□

Theorem 2.6.4 (Schwarz inequality, [Mag12, Theorem 19.11])**.** *If E is a set of finite perimeter in $\mathbb{R}^n$ with $|E| < \infty$, then E^* is a set of finite perimeter in $\mathbb{R}^n$ and*

$$P(E) \ge P(E^*). \tag{2.6.1}$$

If equality holds in (2.6.1), then, for a.e. $t \in \mathbb{R}$, E_t is $\mathcal{H}^{n-1}$-equivalent to an $(n-1)$-dimensional ball, and $\mathbf{q}\nu^E$ is $\mathcal{H}^{n-2}$-a.e. constant on $\partial^* E_t$.

Proposition 2.6.5 ([Mag12, Proposition 19.17])**.** *If $E \subset \mathbb{R}^n \setminus H$ is of locally finite perimeter, then $P(E, \partial H) = P(E^*, \partial H)$.*

We recall a formula for the mean curvature of axially symmetric hypersurfaces of class $W^{2,p}$.

Lemma 2.6.6. *Let $a < b$. Let α, $\beta : (a,b) \to (0,\infty)$ be $W^{2,p}$ functions, with $p \in (1,\infty]$, parametrizing the curve $\gamma : (a,b) \to \operatorname{span}\{e_1, e_n\} \subset \mathbb{R}^n$ given by $\gamma(t) = (\alpha(t), 0, \dots, 0, \beta(t))$, and assume that $|\gamma'(t)| = 1$ and that $\inf_{(a,b)} \alpha > 0$. Let S be the axially symmetric hypersurface around the n-th axis parametrized by*

$$\varphi_S : \mathbb{S}^{n-2} \times (a,b) \to \mathbb{R}^n$$
$$\varphi_S(\vartheta, t) = (\alpha(t)\vartheta, \beta(t)).$$

Then the vector

$$H = \left(\left\langle k_\gamma, \nu \right\rangle - (n-2)\frac{\beta'}{\alpha} \right) \left(-\beta'\vartheta, \alpha' \right),$$

for every ϑ and a.e. t, where k_γ is the curvature of γ and $\nu(t) = (-\beta', 0, \dots, 0, \alpha')$, is the (generalized) mean curvature of S. More precisely

$$\int_S \operatorname{div}_T X = -\int_S \langle X, H \rangle,$$

for any $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{spt} X \cap \partial S = \emptyset$, where $\operatorname{div}_T X$ is the tangential divergence of X along S.

Proof. Let α_i, $\beta_i : (a,b) \to (0,+\infty)$ be smooth functions converging in $W^{2,p}$ to α, β respectively ([Eva94, Section 5.3, Theorem 2]). Up to reparametrization, we can also assume $(\alpha_i')^2 + (\beta_i')^2 = 1$. Then the tangent vector to the curve

$$\gamma_i(t) := (\alpha_i(t), 0, \dots, 0, \beta_i(t))$$

is

$$\tau_i := (\alpha_i', 0, \dots, 0, \beta_i')$$

and the curvature is

$$k_{\gamma_i} = (\alpha_i'', 0, \dots, 0, \beta_i'').$$

Also, fix the normal vector

$$\nu_i := (-\beta_i', 0, \dots, 0, \alpha_i').$$

The corresponding choice of unit normal on the hypersurface S_i parametrized by φ_{S_i} given by the rotation of γ_i is

$$N_i(\vartheta, t) = (-\beta_i'\vartheta, \alpha_i'),$$

for $\vartheta \in \mathbb{S}^{n-2}$. The second fundamental form of S_i in direction N_i is given by

$$\mathrm{II}_{N_i}(\partial_\mu, \partial_\eta) = -\left\langle \partial_\mu, \partial_\eta N_i \right\rangle,$$

where $\partial_\mu, \partial_\eta$ denote coordinate vector fields on $\mathbb{S}^{n-2} \times (a,b)$.
If $\mu, \eta \neq t$, then

$$\mathrm{II}_{N_i}(\partial_\mu, \partial_\eta) = -\left\langle \partial_\mu, \partial_\eta(-\beta_i'\vartheta, \alpha_i') \right\rangle = \beta_i' \left\langle (\alpha_i \partial_\mu \vartheta, 0), (\partial_\eta \vartheta, 0) \right\rangle = -\alpha_i \beta_i' \, \mathrm{II}^{\mathbb{S}^{n-2}}_\vartheta(\partial_\mu, \partial_\eta),$$

where $\mathrm{II}^{\mathbb{S}^{n-2}}_\vartheta$ denotes second fundamental form of $\mathbb{S}^{n-2}$ with respect to the inner normal given by ϑ.
If $\mu \neq t$ and $\eta = t$, then

$$\mathrm{II}_{N_i}(\partial_\mu, \partial_t) = -\left\langle \partial_\mu, \partial_t(-\beta_i'\vartheta, \alpha_i') \right\rangle = -\left\langle \partial_\mu \varphi_S, (-\beta_i''\vartheta, \alpha_i'') \right\rangle = 0,$$

hence by symmetry $\mathrm{II}_{N_i}(\partial_t, \partial_\mu) = 0$ for $\mu \neq t$ as well.
If $\mu = \eta = t$ then

$$\mathrm{II}_{N_i}(\partial_t, \partial_t) = -\left\langle \partial_t, (-\beta_i''\vartheta, \alpha_i'') \right\rangle = -\left\langle \partial_t \varphi_{S_i}, (-\beta_i''\vartheta, \alpha_i'') \right\rangle = -\left\langle (\alpha_i'\vartheta, \beta_i'), (-\beta_i''\vartheta, \alpha_i'') \right\rangle = \alpha_i'\beta_i'' - \beta_i'\alpha_i''.$$

In local coordinates, the metric on S_i is given by

$$g_{lk} = \left\langle \partial_l \varphi_{S_i}, \partial_k \varphi_{S_i} \right\rangle,$$

then $g_{tt} = 1$ and $g_{hj} = \alpha_i^2 (g_{\mathbb{S}^{n-2}})_{hj}$ for $h, j \neq t$. Hence the mean curvature of S_i is determined by

$$\begin{aligned}
\langle H_i, N_i \rangle &= g^{\mu\eta} \mathrm{II}_N(\partial_\mu, \partial_\eta) \\
&= g^{tt} \mathrm{II}_{N_i}(\partial_t, \partial_t) + g^{ht} \mathrm{II}_{N_i}(\partial_h, \partial_t) + g^{hj} \mathrm{II}_{N_i}(\partial_h, \partial_j) \\
&= (\alpha_i' \beta_i'' - \beta_i' \alpha_i'') - \alpha_i^{-2} g^{hj}_{\mathbb{S}^{n-2}} \alpha_i \beta_i' \mathrm{II}^{\mathbb{S}^{n-2}}_{\vartheta}(\partial_h, \partial_j) \\
&= \left\langle k_{\gamma_i}, \nu_i \right\rangle - (n-2) \frac{\beta_i'}{\alpha_i},
\end{aligned}$$

where indices h, j are understood to be different from t. Since φ_{S_i} converges in C^1-sense to φ_S and $\langle H_i, N_i \rangle\, N_i$ $\mathcal{H}^{n-1} \llcorner S_i$ converges in duality with bounded continuous vector fields, then the identity defining the generalized mean curvature passes to the limit on S, and the claimed formula also follows. □

Chapter 3

Quantitative isoperimetry for the classical capillarity problem

3.1 Main results

In order to prove a quantitative isoperimetric inequality for

$$\inf\left\{P_\lambda(E)\ :\ E\subset\{x_n>0\},\ |E|=v\right\}, \tag{3.1.1}$$

we define the corresponding Fraenkel asymmetry and isoperimetric deficit by setting

$$\alpha_\lambda(E):=\inf\left\{\frac{|E\Delta B^\lambda(v,x)|}{v}\ :\ x\in\{x_n=0\}\right\},\qquad D_\lambda(E):=\frac{P_\lambda(E)-P_\lambda(B^\lambda(v))}{P_\lambda(B^\lambda(v))},$$

for any $E\subset\{x_n>0\}$ with volume $|E|=v$. The infimum defining the asymmetry is, in fact, a minimum. The first main result of the chapter is the following

Theorem 3.1.1 ([PP24]). *Let $\lambda\in(-1,1)$ and $n\in\mathbb{N}$ with $n\geq 2$. There exists a constant $c_{\rm iso}=c_{\rm iso}(n,\lambda)>0$ such that for any measurable set $E\subset\mathbb{R}^n\cap\{x_n>0\}$ with finite measure there holds*

$$\alpha_\lambda(E)^2\leq c_{\rm iso}D_\lambda(E). \tag{3.1.2}$$

As for the classical quantitative isoperimetric inequality [FMP08], perturbing the boundary of an optimal bubble only inside the container $\{x_n>0\}$, it is possible to check that exponents in (3.1.2) are sharp.

In the context of these capillarity problems it is also spontaneous to consider a notion of asymmetry for the part of the boundary of a set that touches the half-plane $\{x_n=0\}$. For a measurable set $E\subset\{x_n>0\}$, we define

$$\beta_\lambda(E):=\inf\left\{\frac{\mathcal{H}^{n-1}\left(\partial^*E\cap\{x_n=0\}\ \Delta\ \partial^*B^\lambda(|E|,x)\cap\{x_n=0\}\right)}{\mathcal{H}^{n-1}\left(\partial^*B^\lambda(|E|,x)\cap\{x_n=0\}\right)}\ :\ x\in\{x_n=0\}\right\}.$$

The previous quantity measures the asymmetry of the set $\partial^*E\cap\{x_n=0\}$ with respect to $(n-1)$-dimensional balls in $\{x_n=0\}$ having volume equal to the one of the trace of the optimal bubble corresponding to the volume of E. We establish the following second quantitative isoperimetric inequality, that provides a quantitative estimate on β_λ.

Theorem 3.1.2 ([PP24]). *Let $\lambda\in(-1,1)$ and $n\in\mathbb{N}$ with $n\geq 2$. There exists a constant $c'_{\rm iso}=c'_{\rm iso}(n,\lambda)>0$ such that for any measurable set $E\subset\mathbb{R}^n\cap\{x_n>0\}$ with finite measure there holds*

$$\beta_\lambda(E)\leq c'_{\rm iso}\max\left\{D_\lambda(E),D_\lambda(E)^{\frac{1}{2n}}\right\}. \tag{3.1.3}$$

Strategy of the proof and comments. Observing that, roughly speaking, the minimization problem

$$\inf\left\{P_\lambda(E)\ :\ E\subset\{x_n>0\},\ |E|=v\right\}$$

is symmetric with respect to the first $n-1$ axes, it is possible to adapt arguments in the spirit of [FMP08] to see that, in order to prove Theorem 3.1.1, it is sufficient to prove (3.1.2) in the class of Schwarz-symmetric sets, see Corollary 3.4.12. However, we point out that it seems not possible to push the strategy of [FMP08] to the very end to prove Theorem 3.1.1. Indeed the arguments in [FMP08] require to Schwarz-symmetrize a competitor with respect to a preferred axis depending on the competitor, while in our case it is only possible to symmetrize with respect to the n-th axis. One finds an analogous obstruction also in a possible adaptation of the proof via symmetrization revised in [Mag08] (see also [Fus15]); in [Mag08] the quantitative isoperimetric inequality for Schwarz-symmetric sets is eventually obtained performing a quantitative version of Gromov's proof [MS86] of the isoperimetric inequality, but again after having symmetrized a competitor with respect to a convenient axis.

The proof of (3.1.2) in the class of Schwarz-symmetric sets is achieved here with a new combination of the so-called selection principle [AFM13; CL12] with an Alexandrov–Bakelman–Pucci-type technique in the spirit of [Cin+22].

In the recent [Cin+22], the authors prove sharp quantitative isoperimetric inequalities for a class of isoperimetric problems in cones where volume and perimeter are weighted in terms of a function satisfying suitable homogeneity and concavity properties. The proof in [Cin+22] stems from the fact that the isoperimetric inequality for the corresponding problem was proved in [CRS16] by a so-called ABP argument. The methods that go under the name of ABP techniques were originally employed to derive regularity estimates for second order elliptic equations [GT01, Chapter 9] and they were applied to give a new direct proof of the classical isoperimetric inequality in [Cab00; Cab08] (see [Cab17] for a detailed account on the method). More precisely, for the classical isoperimetric problem, if $E \subset \mathbb{R}^n$ is a smooth connected open set, one would consider a solution u to

$$\begin{cases} \Delta u = \frac{P(E)}{|E|} & \text{on } E, \\ \partial_\nu u = 1 & \text{on } \partial E, \end{cases} \tag{3.1.4}$$

where $\partial_\nu u$ denotes outward normal derivative. It is immediate to check that $\nabla u(E') \supset B_1(0)$ where $E' := \{x \in E \ : \ \nabla^2 u(x) \geq 0\}$, hence the area formula together with the arithmetic-geometric mean inequality readily imply the Euclidean isoperimetric inequality, indeed

$$\omega_n = |B_1(0)| \leq \int_{E'} \det \nabla^2 u \leq \int_E \left(\frac{\Delta u}{n}\right)^n = \frac{P(E)^n}{n^n |E|^{n-1}}.$$

Now the rough idea is that a control on the energy deficit should control the "asymmetry" of the solution u with respect to the solution corresponding to the optimal shape $B_1(0)$, that is the radially symmetric parabola $|x|^2/2$. In fact, this is achieved in [Cin+22] by controlling the asymmetry of a *coupling* function which is defined as a suitable convex envelope of u. Adapting arguments from [FMP10], in [Cin+22] the authors then show that it is possible to employ trace-type theorems to estimate the asymmetry of a competitor set in terms of the asymmetry of the coupling function, which is in turn estimated by the energy deficit.

In Theorem 3.2.3 below we will give an ABP proof of the isoperimetric inequality for the problem (3.1.1) by analyzing an elliptic problem analogous to (3.1.4), see (3.2.4). We are then in position to consider a coupling function as done in [Cin+22] and we can quantitatively estimate its asymmetry, which will be achieved in Proposition 3.5.5. Moreover, Schwarz-symmetric sets that are sufficiently small C^1-perturbations (in the sense of Definition 3.5.11) of an optimal bubble (2.4.2) readily verify the needed trace-type inequalities that relate asymmetry of the competitor with the asymmetry of the coupling. Hence this establishes the quantitative inequality (3.1.2) for Schwarz-symmetric C^1-perturbations of optimal bubbles, see Corollary 3.5.12. Observe that, in our setting, isoperimetric sets are just Lipschitz-regular and a set E is C^1-close to an optimal bubble if just the relative boundary $\partial E \cap \{x_n > 0\}$ is close to the relative boundary of an optimal bubble as C^1-hypersurfaces with boundary.

Once (3.1.2) is proved for C^1-perturbations of optimal bubbles (Corollary 3.5.12), we want apply a selection-type argument in the spirit of [AFM13; CL12] in the class of Schwarz-symmetric sets in order to extend the validity of the quantitative inequality to the whole class of Schwarz-symmetric sets. In this way we also avoid the implementation of further technical results that in [FMP10; Cin+22] allow to reduce to just consider sets that enjoy the required trace-type inequalities.

Roughly speaking, in a selection-type argument one argues by contradiction assuming existence of sets contradicting the quantitative isoperimetric inequality and one uses such sets to define an auxiliary minimization problem, cf. (3.5.19). Minimizers to the previous problem still contradict the quantitative isoperimetric inequality, but at the same time they are shown to be small C^1-perturbations of some isoperimetric set, contradicting the inequality

already proved for sets given by small perturbations of optimal ones.
In our case, we will prove that minimizers E to the auxiliary minimization problem are C^1-perturbations of optimal bubbles up to the boundary of the half-space $\{x_n > 0\}$ as a consequence of the classical interior regularity of (Λ, r_0)-minimizers of the perimeter (Definition 2.5.1), see [Tam84] and [Mag12, Chapter 26], together with a simple variational argument that allows us to propagate the regularity up to the boundary of the half-space $\{x_n > 0\}$. This essentially follows from the fact that a Schwarz-symmetric local (Λ, r_0)-minimizer E in $\{x_n > 0\}$ is locally of class C^1 and has bounded mean curvature (in a generalized sense, see Lemma 2.5.2); hence a uniform bound on the whole second fundamental form on a portion of boundary $\partial E \cap \{0 < x_n < \varepsilon\}$ follows just by showing that the set $\partial E \cap \{0 < x_n < \varepsilon\}$ is far from the axis of revolution of E (see Lemma 2.6.6), and the latter holds in the proof of Theorem 3.1.1 by an energy estimate holding for minimizers to the auxiliary minimization problems.

We stress that in our case it is not clear how to apply a Fuglede-type argument [CL12; Fug89] to prove the quantitative inequality for sets given by small C^1-perturbations of optimal ones. Indeed, the classical Fuglede's method relies on the precise knowledge of the eigenvalues of the Laplace–Beltrami operator, which is not available for the operator on spherical caps corresponding to optimal bubbles (2.4.2) for generic $\lambda \in (-1, 1)$. Moreover, observe that in our case it is not possible to globally parametrize C^1-close boundaries one on the other as normal graphs in general, introducing a further nontrivial technical difficulty in the implementation of a Fuglede-type argument.

Once Theorem 3.1.1 is proved, for the proof of Theorem 3.1.2 we argue as follows. First we establish a quantitative inequality that estimates the Hausdorff distance between the relative boundary in $\{x_n > 0\}$ of a competitor E and the relative boundary of some bubble in terms of the Fraenkel asymmetry of E, under the assumption that E is a so-called (K, r_0)-quasiminimal set, see Definition 3.6.2 and Lemma 3.6.4. This is achieved since quasiminimal sets enjoy uniform density estimates at boundary points, see Theorem 3.6.3. Exploiting Theorem 3.1.1, the previous quantitative inequality yields an inequality of the form (3.1.3) in the class of quasiminimal sets. Eventually, Theorem 3.1.2 follows by applying again a selection-type argument where now β_λ plays the role of the Fraenkel asymmetry.

From now on and for the rest of the chapter it is assumed that $\lambda \in (-1, 1)$ and $n \in \mathbb{N}$ with $n \geq 2$ are fixed.

3.2 Two proofs of the capillarity isoperimetric inequality

We first give a proof of the isoperimetric inequality for the capillarity functional P_λ exploiting an ABP method. Since the bubbles $B^\lambda(v)$ are Schwarz-symmetric, we can exploit Remark 2.6.2 to compute their energy.

Lemma 3.2.1. *There holds $P_\lambda(B^\lambda(v)) = n|B^\lambda|^{\frac{1}{n}} v^{\frac{n-1}{n}}$, for any $v \geq 0$ and $\lambda \in (-1, 1)$.*

First proof of Lemma 3.2.1. By scale invariance, it is sufficient to prove that $P_\lambda(B^\lambda) := P(B^\lambda, \{x_n > \lambda\}) - \lambda \mathcal{H}^{n-1}(\partial B^\lambda \cap \{x_n = \lambda\})$ is equal to $n|B^\lambda|$.
If $\lambda = 0$, we have $P_0(B^0) = \frac{1}{2}P(B) = \frac{1}{2}n|B| = n|B^0|$. So if we prove that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(P_\lambda(B^\lambda) - n|B^\lambda|\right) = 0, \tag{3.2.1}$$

for any $\lambda \in (-1, 1)$, the claim follows. Let $\varphi(t) := (1 - t^2)^{\frac{1}{2}}$, for $t \in [-1, 1]$, be the profile function of the standard unit ball in $\mathbb{R}^n$.

- By coarea formula 2.1.40, the volume of B^λ equals

$$|B^\lambda| = \int_\lambda^1 \omega_{n-1}\varphi^{n-1}\,\mathrm{d}t.$$

- By Remark 2.6.2 we get

$$\begin{aligned} P_\lambda(B^\lambda) &= \int_\lambda^1 \left((n-1)\omega_{n-1}\sqrt{1 + (\varphi')^2}\varphi^{n-2} + \lambda(n-1)\omega_{n-1}\varphi'\varphi^{n-2}\right)\,\mathrm{d}t \\ &= \int_\lambda^1 (n-1)\omega_{n-1}(1 - \lambda t)\varphi^{n-3}\,\mathrm{d}t. \end{aligned}$$

- Since $\varphi' = -t/\varphi$, the derivative of $P_\lambda(B^\lambda)$ equals:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}P_\lambda(B^\lambda) &= \int_\lambda^1 (n-1)\omega_{n-1}(-t)\varphi^{n-3}\,\mathrm{d}t - (n-1)\omega_{n-1}(1-\lambda^2)\varphi^{n-3}(\lambda)\\ &= (n-1)\omega_{n-1}\int_\lambda^1 \varphi'\varphi^{n-2}\,\mathrm{d}t - (n-1)\omega_{n-1}\varphi^{n-1}(\lambda)\\ &= -\omega_{n-1}\varphi^{n-1}(\lambda) - (n-1)\omega_{n-1}\varphi^{n-1}(\lambda)\\ &= -n\omega_{n-1}\varphi^{n-1}(\lambda).\end{aligned}$$

- The derivative of the volume $|B^\lambda|$ equals

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}|B^\lambda| = -\omega_{n-1}\varphi^{n-1}(\lambda).$$

Putting together the above computations we have $\frac{\mathrm{d}}{\mathrm{d}\lambda}P_\lambda(B^\lambda) = n\frac{\mathrm{d}}{\mathrm{d}\lambda}|B^\lambda|$, which is (3.2.1). □

Now we provide a second proof of Lemma 3.2.1.

Second proof of Lemma 3.2.1. By scale invariance, it is sufficient to prove that $P(B^\lambda, \{x_n > \lambda\}) - \lambda\mathcal{H}^{n-1}(\partial B^\lambda \cap \{x_n = \lambda\})$ is equal to $n|B^\lambda|$. Indeed, let $u(x) = \frac{1}{2}|x|^2$. Then

$$n|B^\lambda| = \int_{B^\lambda}\Delta u = P(B^\lambda, \{x_n > \lambda\}) + \int_{\partial B^\lambda\cap\{x_n=\lambda\}}\langle -e_n, x\rangle = P(B^\lambda, \{x_n > \lambda\}) - \lambda\mathcal{H}^{n-1}(\partial B^\lambda \cap \{x_n = \lambda\}).$$

□

Remark 3.2.2. Let $E \subset \mathbb{R}^n \setminus H$ be a connected open set such that $\partial E \setminus H$ is a smooth hypersurface with boundary that intersects ∂H orthogonally. Then the Neumann problem

$$\begin{cases}\Delta u = \frac{P_\lambda(E)}{|E|} & \text{in} \quad E,\\ \frac{\partial u}{\partial \nu} = 1 & \text{on} \quad \partial E\setminus\partial H,\\ \frac{\partial u}{\partial \nu} = -\lambda & \text{on} \quad \partial E\cap\partial H,\end{cases} \tag{3.2.2}$$

has a solution $u \in C^1(\overline{E}) \cap C^\infty(E)$.
Indeed, existence of a weak solution of (3.2.2) follows by classical arguments exploiting the Riesz representation theorem. By [Nit11, Proposition 3.6] there exists $\gamma > 0$ such that every weak solution is in $C^{0,\gamma}(E)$. Hence we can apply [Lie88, Theorem 1] to the equivalent problem

$$\begin{cases}\Delta u - u = \frac{P_\lambda(E)}{|E|} - u =: f & \text{in} \quad E,\\ \frac{\partial u}{\partial \nu} = 1 & \text{on} \quad \partial E\setminus\partial H,\\ \frac{\partial u}{\partial \nu} = -\lambda & \text{on} \quad \partial E\cap\partial H\end{cases}$$

getting that a weak solution is in fact $C^1(\overline{E})$.

Theorem 3.2.3 (Isoperimetric inequality for P_λ)**.** *Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter with $|E| \in (0, +\infty)$. Then*

$$\frac{P_\lambda(E)}{|E|^{\frac{n-1}{n}}} \geq \frac{P_\lambda(B^\lambda)}{|B^\lambda|^{\frac{n-1}{n}}} = n|B^\lambda|^{\frac{1}{n}}. \tag{3.2.3}$$

Moreover, equality occurs in (3.2.3) if and only if $E = B^\lambda(|E|)$ up to a translation and up to negligible sets.

We just give a proof of the inequality (3.2.3) here, referring to the proof of Theorem 2.4.1 in [Mag12] for an alternative proof comprising the characterization of minimizers.

First proof of Theorem 3.2.3. By the standard isoperimetric inequality, we can assume that $\mathcal{H}^{n-1}(\partial^* E \cap \partial H) > 0$. By Lemma 2.4.4, we can further assume that E is a bounded set such that $\partial E \setminus \partial H$ is smooth and intersects ∂H orthogonally.
Let us further assume for the moment that E is connected. Let u be the solution of the Neumann problem

$$\begin{cases} \Delta u = \frac{P_\lambda(E)}{|E|} & \text{in} \quad E, \\ \frac{\partial u}{\partial \nu} = 1 & \text{on} \quad \partial E \setminus \partial H, \\ \frac{\partial u}{\partial \nu} = -\lambda & \text{on} \quad \partial E \cap \partial H, \end{cases} \tag{3.2.4}$$

where $\partial u / \partial \nu$ denotes the outer normal derivative of u on ∂E. Observe that such a solution exists and $u \in C^1(\overline{E}) \cap C^\infty(E)$ (see Remark 3.2.2). We consider the "lower contact set" of u defined by

$$\Gamma_u := \left\{ x \in E \,:\, u(y) \geq u(x) + \langle \nabla u(x), y - x \rangle \ \text{ for all } y \in \overline{E} \right\}.$$

We claim that

$$B^\lambda \subset \nabla u(\Gamma_u). \tag{3.2.5}$$

To show (3.2.5), take any $p \in B^\lambda$. Let $x \in \overline{E}$ be a point such that

$$\min_{y \in \overline{E}} \{ u(y) - \langle p, y \rangle \} = u(x) - \langle p, x \rangle .$$

If $x \in \partial E \setminus \partial H$ then the exterior normal derivative of $u(y) - \langle p, y\rangle$ at x would be nonpositive and hence $(\partial u/\partial \nu)(x) \leq |p| < 1$, a contradiction with (3.2.4). Similarly, if $x \in \partial E \cap \partial H$ then $(\partial u/\partial \nu)(x) \leq \langle p, -e_n \rangle < -\lambda$, a contradiction with (3.2.4). It follows that $x \in E$ and, therefore, that x is an interior minimum of the function $u(y) - \langle p, y \rangle$ over $\overline{E}$. In particular $p = \nabla u(x)$ and $x \in \Gamma_u$, hence Claim (3.2.5) is now proved.
From (3.2.5), since $u \in C^\infty(E)$ and $\Gamma_u \subset E$, we can apply the area formula on ∇u to deduce

$$|B^\lambda| \leq |\nabla u(\Gamma_u)| = \int_{\nabla u(\Gamma_u)} \mathrm{d}p \leq \int_{\Gamma_u} |\det \nabla^2 u(x)| \, \mathrm{d}x.$$

Since points $x \in \Gamma_u$ are interior minima for $y \mapsto u(y) - \langle \nabla u(x), y \rangle$, then $\nabla^2 u(x)$ is positively semi-definite. Hence by the arithmetic-geometric mean inequality

$$|\det \nabla^2 u| = \det \nabla^2 u \leq \left(\frac{\Delta u}{n} \right)^n \qquad \text{in } \Gamma_u.$$

Hence

$$|B^\lambda| \leq \int_{\Gamma_u} \det \nabla^2 u \, \mathrm{d}x \leq \int_{\Gamma_u} \left(\frac{\Delta u}{n} \right)^n \mathrm{d}x \leq \int_E \left(\frac{\Delta u}{n} \right)^n \mathrm{d}x,$$

since $\Delta u \equiv P_\lambda(E)/|E|$. Plugging in the value of Δu, the claimed inequality follows.
It remains to consider the case when E is not connected, hence when E is a disjoint union of finitely many bounded sets E_i, for $i = 1, \dots, k$, such that $\partial E_i \setminus \partial H$ is smooth and intersects ∂H orthogonally. We can apply the isoperimetric inequality that we just proved for P_λ on each component E_i. Summing the inequalities and exploiting the subadditivity of $t \mapsto t^{\frac{n-1}{n}}$, the final inequality follows. □

Now we present another version of the proof of the inequality in Theorem 3.2.3, inspired by Gromov's proof of the isoperimetric inequality [CNV04; FMP10; Mag08; MS86].

Second proof of Theorem 3.2.3. By scale invariance, we can assume $|E| = |B^\lambda|$. At the same time, by the subadditivity of $t \mapsto t^{\frac{n-1}{n}}$ it is sufficient to prove Theorem 3.2.3 when E is connected. Finally, by Theorem 2.6.4 and Proposition 2.6.5 we can assume that E is Schwarz symmetric and by density we can assume that E is smooth. The proof is based on a parametrization of B^λ in terms of E via the function

$$\tau \,:\, [0, +\infty) \to [\lambda, \infty)$$

defined by

$$|E \cap \{x_n < t\}| = |B^\lambda \cap \{x_n < \tau(t)\}|.$$

If we define

$$v(t) := \mathcal{H}^{n-1}(E \cap \{x_n = t\})$$
$$w(s) := \mathcal{H}^{n-1}(B^\lambda \cap \{x_n = s\}),$$

by construction

$$\int_0^t v(s)\,\mathrm{d}s = \int_0^{\tau(t)} w(s)\,\mathrm{d}s.$$

Then τ is smooth with $\tau' > 0$ on the set $\{t > 0 \,:\, v(t) > 0\}$, where τ' is given by

$$\tau'(t) = \frac{v(t)}{w(\tau(t))}.$$

Let us consider the deformation $T \,:\, E \to B^\lambda$ defined by

$$T(x) := \sum_{i=1}^{n-1} \left(\frac{w(\tau(x_n))}{v(x_n)} \right)^{\frac{1}{n-1}} x_i e_i + \tau(x_n) e_n = \sum_{i=1}^{n-1} \tau'(x_n)^{-\frac{1}{n-1}} x_i e_i + \tau(x_n) e_n.$$

Note that

$$T(E \cap \{x_n = t\}) = B^\lambda \cap \{x_n = \tau(t)\}$$

for every $t > 0$, therefore $T(E) = B^\lambda$. For any $x \in E$, the differential $\mathrm{d}T$ of T is represented by the matrix

$$\mathrm{d}T(x) = \sum_{i=1}^{n-1} \frac{1}{\tau'(x_n)^{\frac{1}{n-1}}} e_i \otimes e_i + \left(\sum_{i=1}^{n-1} \frac{\partial \left(\tau'(x_n)^{\frac{1}{1-n}} \right)}{\partial x_n} x_i e_i \right) \otimes e_n + \tau'(x_n) e_n \otimes e_n,$$

where $\{e_i\}$ is an orthonormal basis of $\{x_n = 0\}$. Taking the trace of $\mathrm{d}T$ we deduce that

$$\operatorname{div} T(x) = (n-1) \frac{1}{\tau'(x_n)^{\frac{1}{n-1}}} + \tau'(x_n),$$

thus by Young's inequality we estimate

$$\frac{\operatorname{div} T(x)}{n} = \frac{n-1}{n} \left(\frac{1}{\tau'(x_n)^{\frac{1}{n}}} \right)^{\frac{n}{n-1}} + \frac{1}{n} \left(\tau'(x_n)^{\frac{1}{n}} \right)^n \geq 1.$$

If for $\varepsilon > 0$ we define $E_\varepsilon := E \cap \{\varepsilon < x_n\}$, by divergence theorem

$$\begin{aligned} P_\lambda(E) &= \lim_{\varepsilon \to 0} P(E_\varepsilon, \{x_n > \varepsilon\}) - \lambda \mathcal{H}^{n-1}(E_\varepsilon \cap \{x_n = \varepsilon\}) \\ &\geq \lim_{\varepsilon \to 0} P(E_\varepsilon, \{x_n > \varepsilon\}) + \int_{\partial^* E_\varepsilon \cap \{x_n = \varepsilon\}} \left\langle T, \nu^{E_\varepsilon} \right\rangle \mathrm{d}\mathcal{H}^{n-1} \\ &\geq \lim_{\varepsilon \to 0} \int_{\partial^* E_\varepsilon} \left\langle T, \nu^{E_\varepsilon} \right\rangle \mathrm{d}\mathcal{H}^{n-1} = \lim_{\varepsilon \to 0} \int_{E_\varepsilon} \operatorname{div} T \,\mathrm{d}\mathcal{H}^n \geq \lim_{\varepsilon \to 0} n|E_\varepsilon| = n|E| = n|B^\lambda|, \end{aligned}$$

and we conclude the proof. □

3.3 Asymmetry and deficit

We recall the definition of the Fraenkel asymmetry with respect to optimal bubbles $B^\lambda(v, x)$ and the deficit corresponding to the functional P_λ, proving some preliminary properties on these quantities.

Definition 3.3.1 (Fraenkel asymmetry)**.** Let $E \subset \mathbb{R}^n \setminus H$ be a Borel set with measure $|E| = v \in (0, +\infty)$. We define

$$\alpha_\lambda(E) := \inf \left\{ \frac{|E \Delta B^\lambda(v, x)|}{v} \,:\, x \in \{x_n = 0\} \right\}.$$

It is readily checked that the Fraenkel asymmetry of E is a minimum.

Definition 3.3.2 (Isoperimetric deficit)**.** Let $E \subset \mathbb{R}^n \setminus H$ be a Borel set with measure $|E| = v \in (0, +\infty)$. We define

$$D_\lambda(E) := \frac{P_\lambda(E) - P_\lambda(B^\lambda(v))}{P_\lambda(B^\lambda(v))}.$$

The following lemma states that, if the isoperimetric deficit of a competitor is sufficiently small, then the competitor touches the hyperplane $\{x_n = 0\}$.

Lemma 3.3.3. *There exists $\bar{c} = \bar{c}(n, \lambda) > 0$ such that if a Borel set $E \subset \mathbb{R}^n \setminus H$ satisfies $D_\lambda(E) < \bar{c}$ then $P(E, \partial H) > 0$.*

Proof. If E is a Borel set such that $D_\lambda(E) < \bar{c}$ and $P(E, \partial H) = 0$, then the standard isoperimetric inequality together with Lemma 3.2.1 imply

$$n\omega_n^{\frac{1}{n}}|E|^{\frac{n-1}{n}} \le P(E) = P_\lambda(E) < (1+\bar{c})P_\lambda(B^\lambda(|E|)) = n|B^\lambda|^{\frac{1}{n}}(1+\bar{c})|E|^{\frac{n-1}{n}}.$$

Since $|E| > 0$ for $\bar{c}$ small enough, we get a contradiction if $\bar{c}$ is sufficiently small. □

The following lemmas prove continuity properties of deficit, asymmetry and P_λ under convergence of sets.

Lemma 3.3.4. *If $\{E_i\}_{i\in\mathbb{N}}$ and E are sets of finite perimeter in $\mathbb{R}^n \setminus H$ with finite measure such that*

$$E_i \xrightarrow{L^1_{\rm loc}} E.$$

Then

$$\liminf_{i\to+\infty} P_\lambda(E_i) \ge P_\lambda(E), \qquad \liminf_{i\to+\infty} D_\lambda(E_i) \ge D_\lambda(E).$$

Proof. Let us define the function

$$f : \mathbb{R}^n \to [0, +\infty)$$
$$f(v) = |v| - \lambda \langle e_n, v\rangle .$$

Note that f is continuous; moreover $f(tv) = tf(v)$ for any $t \ge 0$ and f is convex. Let us set

$$\mu_i := \nu^{E_i}\mathcal{H}^{n-1} \llcorner (\partial^* E_i \cap (\mathbb{R}^n \setminus H))$$
$$\mu := \nu^{E}\mathcal{H}^{n-1} \llcorner (\partial^* E \cap (\mathbb{R}^n \setminus H)).$$

Since $\nu^{E_i}\mathcal{H}^{n-1}\llcorner\partial^* E_i \to \nu^{E}\mathcal{H}^{n-1}\llcorner\partial^* E$ weakly* in $\mathbb{R}^n$, then $\mu_i \to \mu$ weakly* in $\mathbb{R}^n \setminus H$. Recalling Remark 2.4.2, the Reshetnyak lower semicontinuity theorem 2.1.22 guarantees the lower semicontinuity of P_λ. Therefore, the lower semicontinuity of the deficit also follows by lower semicontinuity of P_λ. □

Lemma 3.3.5. *If $\{E_i\}_{i\in\mathbb{N}}$ and E are sets of finite perimeter in $\mathbb{R}^n \setminus H$ with finite measure such that $|E| > 0$ and*

$$E_i \xrightarrow{L^1} E.$$

Then

$$\lim_{i\to+\infty} \alpha_\lambda(E_i) = \alpha_\lambda(E)$$

Proof. Let $v := |E|$ and $v_i := |E_i|$, then $v_i \to v$.
If $\alpha_\lambda(E) = 2$, there exists $y > 0$ such that $|E \cap \{x \in \mathbb{R}^n \setminus H \ : \ \langle x, e_n\rangle < y\}| = 0$. Let y_0 be the least upper bound of such y's; then every spherical cap $B^\lambda(v, x)$, with $x \in \partial H$, is contained in $\{x \in \mathbb{R}^n \setminus H \ : \ \langle x, e_n\rangle < y_0\}$. By L^1-convergence of $\{E_i\}_i$, for every $z \in \partial H$ and i sufficiently large we have

$$\begin{aligned}\sup_{z\in\partial H} |E_i \cap B^\lambda(v_i, z)| = \sup_{z\in\partial H} & |E_i \cap B^\lambda(v_i, z) \cap \{x \in \mathbb{R}^n \setminus H \ : \ \langle x, e_n\rangle < y_0\}| + \\ & + |E_i \cap B^\lambda(v_i, z) \cap \{x \in \mathbb{R}^n \setminus H \ : \ \langle x, e_n\rangle > y_0\}| \\ \le |E_i \cap \{x \in \mathbb{R}^n \setminus H \ : \ \langle x, e_n\rangle < y_0\}| & + |B^\lambda(v_i, 0) \cap \{x \in \mathbb{R}^n \setminus H \ : \ \langle x, e_n\rangle > y_0\}| \xrightarrow[i]{} 0,\end{aligned}$$

hence $\alpha_\lambda(E_i) \to 2$.
Suppose then that $\alpha_\lambda(E) < 2$, let $\alpha_\lambda(E)$ be attained by some $B^\lambda(v, x_0)$. Then

$$\begin{aligned}\alpha_\lambda(E) = \frac{|E \Delta B^\lambda(v, x_0)|}{v} &= \lim_{i \to +\infty} \left(\frac{|E_i \Delta B^\lambda(v, x_0)|}{v_i} \frac{v_i}{v} \right) \\ &\geq \limsup_{i \to +\infty} \frac{|E_i \Delta B^\lambda(v_i, x_0)| - |B^\lambda(v_i, x_0) \Delta B^\lambda(v, x_0)|}{v_i} \\ &\geq \limsup_{i \to +\infty} \left(\alpha_\lambda(E_i) - \frac{|B^\lambda(v_i, x_0) \Delta B^\lambda(v, x_0)|}{v_i} \right) = \limsup_{i \to +\infty} \alpha_\lambda(E_i).\end{aligned}$$

On the other hand, let $\alpha_\lambda(E_i)$ be attained by $B_i^\lambda(v_i, x_i)$. We claim that $\{x_i\}_i$ is bounded. If $\{x_i\}_i \subset \partial H$ were unbounded, then there would be a subsequence $\{x_{i_j}\}_j$ such that $|x_{i_j}| \to +\infty$. From the hypothesis we have that, for sufficiently large j, $|B^\lambda(v_{i_j}, x_0) \cap E_{i_j}| > |B^\lambda(v_{i_j}, x_{i_j}) \cap E_{i_j}| \underset{j}{\to} 0$, in contradiction with the definition of asymmetry. Therefore $\{x_i\}_i$ is bounded. Let $\{x_{i_k}\}_k$ be a subsequence such that $\lim_{k \to +\infty} \alpha_\lambda(E_{i_k}) = \liminf_{i \to +\infty} \alpha_\lambda(E_i)$. By the boundedness of $\{x_i\}_i$ there is a subsequence $\{x_{i_{k_l}}\}_l$ of x_{i_k} such that $x_{i_{k_l}} \to x \in \partial H$. Then

$$\begin{aligned}\alpha_\lambda(E) \leq \frac{|E \Delta B^\lambda(v, x)|}{v} &\leq \lim_{i \to +\infty} \frac{|E_i \Delta B^\lambda(v, x)| + |E_i \Delta E|}{v_i} = \lim_{i \to +\infty} \frac{|E_i \Delta B^\lambda(v, x)|}{v_i} \\ &\leq \liminf_{l \to +\infty} \frac{|E_{i_{k_l}} \Delta B^\lambda(v_{i_{k_l}}, x_{i_{k_l}})| + |B^\lambda(v_{i_{k_l}}, x_{i_{k_l}}) \Delta B^\lambda(v, x)|}{v_{i_{k_l}}} \\ &= \liminf_{l \to +\infty} \alpha_\lambda(E_{i_{k_l}}) = \lim_{k \to +\infty} \alpha_\lambda(E_{i_k}) = \liminf_{i \to +\infty} \alpha_\lambda(E_i),\end{aligned}$$

which gives the needed reversed inequality. □

Corollary 3.3.6. *If $\{E_i\}_{i \in \mathbb{N}}$ are sets of finite perimeter in $\mathbb{R}^n \setminus H$ such that $|E_i \setminus K| = 0$ for any i and for some compact set $K \subset \mathbb{R}^n$, and if*

$$\sup_{i \in \mathbb{N}} |E_i| + P_\lambda(E_i) < \infty,$$

then there exists E of finite perimeter in $\mathbb{R}^n \setminus H$ and $i_k \to \infty$ as $k \to \infty$ such that

$$E_{i_k} \xrightarrow{L^1} E \qquad \liminf_k P_\lambda(E_{i_k}) \geq P_\lambda(E).$$

Proof. By Corollary 2.4.5 we have that $P_\lambda(E_i) \geq \frac{1-\lambda}{2} P(E_i)$. Then $\sup_{i \in \mathbb{N}} P(E_i) < \infty$. Hence by classical precompactness of sets of finite perimeter 2.3.10 and recalling Lemma 3.3.4 the claim follows. □

Now we give another proof of the lower semicontinuity of P_λ when the convergence of $\{E_i\}$ is in L^1, inspired by [Mag12, Proposition 19.1, Proposition 19.3].

Lemma 3.3.7. *If $\{E_i\}_{i \in \mathbb{N}}$ and E are sets of finite perimeter in $\mathbb{R}^n \setminus H$ with finite measure such that*

$$E_i \xrightarrow{L^1} E.$$

Then

$$\liminf_{i \to +\infty} P_\lambda(E_i) \geq P_\lambda(E).$$

Proof. First let us assume that $\lambda \leq 0$. We can suppose that $\liminf P_\lambda(E_i)$ is finite. By Corollary 2.4.5 we find that $\sup_{i \in \mathbb{N}} P(E_i)$ is finite. The weak convergence of the perimeter measures implies

$$\liminf_{i \to +\infty} P(E_i) \geq P(E), \qquad \liminf_{i \to +\infty} P(E_i, \mathbb{R}^n \setminus H) \geq P(E, \mathbb{R}^n \setminus H).$$

The identity

$$P_\lambda(E) = (1 + \lambda) P(E, \mathbb{R}^n \setminus H) - \lambda P(E)$$

and the nonnegativity of $1 + \lambda$ and $-\lambda$ guarantee the lower semicontinuity.
Let us assume $\lambda > 0$. Let us consider the function

$$T_\delta : \mathbb{R}^n \setminus H \to \mathbb{R}^n$$
$$T_\delta := \chi(x_n)\, e_n,$$

where $\chi(x_n) : [0, +\infty) \to [0, +\infty)$ is a cut-off function such that $\chi(t) = 0$ if $t \geq \delta$ and $\chi(0) = 1$. Note that

$$\langle T_\delta(x_1, \dots, x_{n-1}, 0), e_n \rangle = 1 \qquad \text{on} \quad \partial H$$

$$|T_\delta| \leq 1 \qquad \text{on} \quad \mathbb{R}^n \setminus H$$

$$T_\delta(x_1, \dots, x_n) = 0 \qquad \text{if} \quad x_n > \delta.$$

If $F \subset \mathbb{R}^n \setminus H$ is a set of finite perimeter, by the divergence theorem

$$\int_F \operatorname{div} T_\delta = \int_{\partial^* F \cap (\mathbb{R}^n \setminus H)} \left\langle T_\delta, \nu^F \right\rangle \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\partial^* F \cap \partial H} \left\langle T_\delta, \nu^{\mathbb{R}^n \setminus H} \right\rangle \, \mathrm{d}\mathcal{H}^{n-1}.$$

We deduce

$$P(F, \partial H) \leq P(F, \{(x_1, \dots, x_n) \in \mathbb{R}^n \setminus H \ : \ x_n < \delta\}) + c(\delta)|F|, \tag{3.3.1}$$

with $c(\delta) = \sup_{\mathbb{R}^n \setminus H} |\nabla T_\delta|$. If E_i, E are taken as in the statement, we apply (3.3.1) to $F = E_i \setminus E$, in order to have

$$\begin{aligned} |P(E_i, \partial H) - P(E, \partial H)| &\leq \mathcal{H}^{n-1}(\partial H \cap (\partial^* E_i \Delta \partial^* E)) \\ &= P(E_i \Delta E, \partial H) \\ &\leq P(E_i \Delta E, \{(x_1, \dots, x_n) \in \mathbb{R}^n \setminus H \ : \ x_n < \delta\}) + c(\delta)|E_i \Delta E| \\ &\leq P(E_i, \{(x_1, \dots, x_n) \in \mathbb{R}^n \setminus H \ : \ x_n < \delta\}) \\ &\quad + P(E, \{(x_1, \dots, x_n) \in \mathbb{R}^n \setminus H \ : \ x_n < \delta\}) + c(\delta)|E_i \Delta E|. \end{aligned}$$

Then

$$\begin{aligned} P_\lambda(E_i) - P_\lambda(E) &\geq P(E_i, \mathbb{R}^n \setminus H) - P(E, \mathbb{R}^n \setminus H) - |P(E_i, \partial H) - P(E, \partial H)| \\ &\geq P(E_i, \{(x_1, \dots, x_n) \in \mathbb{R}^n \setminus H \ : \ x_n > \delta\}) - P(E, \mathbb{R}^n \setminus H) \\ &\quad - P(E, \{(x_1, \dots, x_n) \in \mathbb{R}^n \setminus H \ : \ x_n < \delta\}) - c(\delta)|E_i \Delta E|. \end{aligned}$$

By the lower semicontinuity of the perimeter we have

$$\begin{aligned} \liminf_{i \to +\infty} P_\lambda(E_i) \geq P_\lambda(E) &+ P(E, \{(x_1, \dots, x_n) \in \mathbb{R}^n \setminus H \ : \ x_n > \delta\}) - P(E, \mathbb{R}^n \setminus H) \\ &- P(E, \{(x_1, \dots, x_n) \in \mathbb{R}^n \setminus H \ : \ x_n < \delta\}), \end{aligned}$$

and the right-hand side converges to $P_\lambda(E)$ as $\delta \to 0^+$. □

3.4 Reduction to bounded symmetric sets

In the following arguments, in order to prove Theorem 3.1.1, we will repeatedly reduce ourselves to consider sets E having isoperimetric deficit smaller than some chosen constant. This reduction is always possible.
Indeed, let $\delta > 0$ be some positive constant; if E is a set of finite perimeter such that $D_\lambda(E) \geq \delta$, since $\alpha_\lambda(E) \leq 2$, we immediately get

$$\alpha_\lambda^2(E) \leq \frac{4}{\delta}\delta \leq \frac{4}{\delta} D_\lambda(E).$$

Therefore, if Theorem 3.1.1 is proved on sets with deficit $\leq \delta$, then it is proved for any set.
Hence,

within this section we will assume that $D_\lambda(E) < \bar{c}$ for any competitor E involved, whit $\bar{c}$ given by Lemma 3.3.3.

In particular $P(E, \partial H) > 0$.

Reduction to bounded sets

In this section we prove that, in order to prove Theorem 3.1.1, it is sufficient to prove the quantitative isoperimetric inequality (3.1.2) among suitably uniformly bounded sets.
From now on, we shall denote $Q_l := [-l, l]^n \subset \mathbb{R}^n$. We start by proving an estimate on the area of horizontal slices of a set in terms of its deficit.

Lemma 3.4.1. *Let $E \subset \mathbb{R}^n \setminus H$ be a bounded set of finite perimeter such that $\partial E \cap \mathbb{R}^n \setminus H$ is a smooth hypersurface (possibly with smooth boundary) with $|E| = |B^\lambda|$ and such that $\mathcal{H}^{n-1}(\{x \in \partial^* E \setminus H : \nu^E(x) = \pm e_n\}) = 0$. Then*

$$\mathcal{H}^{n-1}(E \cap \{x_n = t\}) \geq \frac{1}{2} P_\lambda(B^\lambda)\left(\left(\frac{\omega_n}{|B^\lambda|}\right)^{\frac{1}{n}}\left(1 - \frac{|E \cap \{x_n < t\}|}{|B^\lambda|}\right)^{\frac{n-1}{n}} - 1 - D_\lambda(E)\right), \tag{3.4.1}$$

for every $t > 0$. In particular

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial H) \geq \frac{1}{2} P_\lambda(B^\lambda)\left(\left(\frac{\omega_n}{|B^\lambda|}\right)^{\frac{1}{n}} - 1 - D_\lambda(E)\right). \tag{3.4.2}$$

Moreover, if $F \subset \mathbb{R}^n \setminus H$ is a set of finite perimeter with $|F| = |B^\lambda|$, then (3.4.1) holds with F in place of E for almost every $t > 0$, and (3.4.2) holds with F in place of E.

The estimates given by Lemma 3.4.1 are clearly nontrivial only when the deficit is sufficiently small. On the other hand, if the deficit $D_\lambda(E)$ is sufficiently small, since $\omega_n/|B^\lambda| > 1$, (3.4.1) and (3.4.2) essentially yield a quantitative version of Lemma 3.3.3, nontrivial also for slices $\mathcal{H}^{n-1}(E \cap \{x_n = t\})$ with $t > 0$ as long as $|E \cap \{x_n < t\}|$ is small.

Proof of Lemma 3.4.1. Let $v_E(t) := \mathcal{H}^{n-1}(E \cap \{x_n = t\})$ for any $t > 0$, and let $g(t) := |E \cap \{x_n < t\}|/|B^\lambda|$. By the standard isoperimetric inequality we have that

$$P(E, \{x_n > t\}) + v_E(t) = P(E \cap \{x_n > t\}) \geq n\omega_n^{\frac{1}{n}}|E \cap \{x_n > t\}|^{\frac{n-1}{n}} = P_\lambda(B^\lambda)\left(\frac{\omega_n}{|B^\lambda|}\right)^{\frac{1}{n}}(1 - g(t))^{\frac{n-1}{n}}, \tag{3.4.3}$$

for any $t > 0$. Moreover, for any $t > 0$, we observe that for any $x' \in \partial^* E \cap \partial H$, the halfline $[0, t] \ni x_n \mapsto (x', x_n)$ either intersects $\partial^* E \cap \{0 < x_n \leq t\}$ or it intersects $E \cap \{x_n = t\}$. Therefore

$$P(E, \{0 < x_n \leq t\}) + v_E(t) \geq \mathcal{H}^{n-1}(\partial^* E \cap \partial H), \tag{3.4.4}$$

for any $t > 0$. Hence we conclude that

$$\begin{aligned} P_\lambda(B^\lambda)(1 + D_\lambda(E)) = P_\lambda(E) &= P(E, \{x_n > t\}) + P(E, \{0 < x_n \leq t\}) - \lambda\mathcal{H}^{n-1}(\partial^* E \cap \partial H) \\ &\overset{(3.4.4)}{\geq} P(E, \{x_n > t\}) - v_E(t) \\ &\overset{(3.4.3)}{\geq} P_\lambda(B^\lambda)\left(\frac{\omega_n}{|B^\lambda|}\right)^{\frac{1}{n}}(1 - g(t))^{\frac{n-1}{n}} - 2v_E(t), \end{aligned}$$

for any $t > 0$, which yields (3.4.1). By Theorem 2.3.20, the function v_E belongs to $W^{1,1}(0, +\infty)$, thus (3.4.2) follows by letting $t \to 0^+$ in (3.4.1).
Now if $F \subset \mathbb{R}^n \setminus H$ is as in the assumptions, let E_i be given by Lemma 2.4.4 applied to F, and let $\tilde{E}_i := (|B^\lambda|/|E_i|)^{\frac{1}{n}} E_i$. Hence the inequality (3.4.2) and the right hand side of (3.4.1) applied with $E = \tilde{E}_i$ pass to the limit as $i \to \infty$. Moreover

$$|\tilde{E}_i \Delta E| = \int_0^{+\infty} \mathcal{H}^{n-1}(\tilde{E}_i \Delta E \cap \{x_n = t\})\, \mathrm{d}t \geq \int_0^{+\infty} \left|\mathcal{H}^{n-1}(\tilde{E}_i \cap \{x_n = t\}) - \mathcal{H}^{n-1}(E \cap \{x_n = t\})\right| \mathrm{d}t.$$

Since $|\tilde{E}_i \Delta E| \to 0$, the left hand side of (3.4.1) passes to the limit as well as $i \to \infty$, for a.e. $t > 0$. □

We are ready to prove the claimed reduction to bounded sets. The proof follows the line of [FMP08, Lemma 5.1], essentially truncating a competitor with coordinate slabs having estimated width. To give a bound for the truncation in the n-th direction will need to modify the argument in [FMP08, Lemma 5.1] and we will exploit Lemma 3.4.1.

Lemma 3.4.2 (Reduction to bounded sets)**.** *There exist $l = l(n, \lambda) > 0$ and $C_1 = C_1(n, \lambda) > 0$ such that, if $E \subset \mathbb{R}^n \setminus H$ is a set of finite perimeter with $|E| \in (0, +\infty)$, then there exists a set of finite perimeter $E' \subset Q_l \cap (\mathbb{R}^n \setminus H)$ such that $|E'| = |B^\lambda|$ and*

$$\alpha_\lambda(E) \leq \alpha_\lambda(E') + C_1 D_\lambda(E), \qquad D_\lambda(E') \leq C_1 D_\lambda(E). \tag{3.4.5}$$

Proof. By scale-invariance of the asymmetry and of the deficit, it is sufficient to prove the claim assuming also $|E| = |B^\lambda|$. First of all we observe that we may prove the claim assuming that $\partial E \cap \mathbb{R}^n \setminus H$ is smooth and

$$\mathcal{H}^{n-1}(\{x \in \partial^* E \cap \mathbb{R}^n \setminus H : \nu^E(x) = \pm e_i\}) = 0 \tag{3.4.6}$$

for all $i = 1, \dots, n$. Indeed, if E is a generic set of finite perimeter, then by Lemma 2.4.4 there exists a sequence of smooth sets $\{E_i\}_{i\in\mathbb{N}}$ converging to E such that (3.4.6) holds. If we know that the claim holds for E_i, we get the existence of $E_i' \subset Q_l \cap (\mathbb{R}^n \setminus H)$ such that (3.4.5) holds with E, E' replaced by E_i, E_i'. Hence we can apply Corollary 3.3.6 on the sequence E_i', and by Lemma 3.3.4 and Lemma 3.3.5 the inequalities (3.4.5) pass to the limit. Without loss of generality, we can further assume that

$$D_\lambda(E) < (2^{1/n} - 1)/4. \tag{3.4.7}$$

Let us consider the axis x_1 first. Thanks to (3.4.6), by Theorem 2.3.20 (see also [FMP08, Theorem 6.1]) we deduce that

$$v_E(t) := \mathcal{H}^{n-1}(\{x' \in \mathbb{R}^{n-1} : (t, x') \in E\}) \qquad \text{for } t \in \mathbb{R}$$

belongs to $W^{1,1}(\mathbb{R})$, hence we may assume that v_E is continuous. Setting

$$E_t^- := \{x \in E : x_1 < t\}$$

and

$$P_\lambda(E, \{x_1 < t\}) := P(E, \{x \in \mathbb{R}^n \setminus H : x_1 < t\}) - \lambda \mathcal{H}^{n-1}(\{x \in \partial^* E \cap \partial H : x_1 < t\})$$

for all $t \in \mathbb{R}$, by smoothness of E we have that

$$P_\lambda(E_t^-) = P_\lambda(E, \{x_1 < t\}) + v_E(t), \qquad P_\lambda(E \setminus E_t^-) = P_\lambda(E, \{x_1 > t\}) + v_E(t), \tag{3.4.8}$$

where $P_\lambda(E, \{x_1 > t\})$ is defined analogously. Let us now define the function $g : \mathbb{R} \to [0, +\infty)$ given by

$$g(t) := \frac{|E_t^-|}{|B^\lambda|}.$$

Hence g is a nondecreasing C^1 function with $g'(t) = v_E(t)/|B^\lambda|$. Let $-\infty \leq a < b \leq +\infty$ be such that $\{t : 0 < g(t) < 1\} = (a, b)$. If $t \in (a, b)$, then by (3.2.3) we have

$$P_\lambda(E_t^-) \geq g(t)^{\frac{n-1}{n}} P_\lambda(B^\lambda).$$

Similarly,

$$P_\lambda(E \setminus E_t^-) \geq (1 - g(t))^{\frac{n-1}{n}} P_\lambda(B^\lambda).$$

Therefore, from (3.4.6) and (3.4.8) we get that

$$P_\lambda(E) + 2v_E(t) \geq P_\lambda(B^\lambda) \left(g(t)^{\frac{n-1}{n}} + (1 - g(t))^{\frac{n-1}{n}} \right)$$

for all $t \in (a, b)$. Since by definition we have $P_\lambda(E) = P_\lambda(B^\lambda)(1 + D_\lambda(E))$, we obtain

$$v_E(t) \geq \frac{1}{2} P_\lambda(B^\lambda) \left(g(t)^{\frac{n-1}{n}} + (1 - g(t))^{\frac{n-1}{n}} - 1 - D_\lambda(E) \right). \tag{3.4.9}$$

Let us now define the concave function

$$\psi : [0,1] \to [0,+\infty) \qquad\qquad \psi(t) := t^{\frac{n-1}{n}} + (1-t)^{\frac{n-1}{n}} - 1.$$

Note that $\psi(0) = \psi(1) = 0$ and ψ achieves its maximum at $\psi(1/2) = 2^{1/n} - 1$, hence by concavity

$$\psi(t) = \psi\left(2t\frac{1}{2} + 0\right) \geq 2t\psi(1/2) + 0 = 2(2^{1/n} - 1)t \qquad \forall t \in \left[0, \frac{1}{2}\right]. \tag{3.4.10}$$

Recall that by (3.4.7) there holds $2D_\lambda(E) < \psi(1/2)$. Let $a < t_1 < t_2 < b$ be such that $g(t_1) = 1 - g(t_2)$ and $\psi(g(t_1)) = \psi(g(t_2)) = 2D_\lambda(E)$. Then

$$\psi(g(t)) \geq 2D_\lambda(E) \qquad \forall t \in (t_1, t_2) \tag{3.4.11}$$

and, by (3.4.10),

$$g(t_1) = 1 - g(t_2) \leq \frac{D_\lambda(E)}{2^{1/n} - 1}. \tag{3.4.12}$$

For any $t_1 \leq t \leq t_2$ we have

$$\begin{aligned} v_E(t) &\overset{(3.4.9)}{\geq} \frac{1}{2}P_\lambda(B^\lambda)(\psi(g(t)) - D_\lambda(E)) = \frac{1}{4}P_\lambda(B^\lambda)\psi(g(t)) + \frac{1}{4}P_\lambda(B^\lambda)(\psi(g(t)) - 2D_\lambda(E)) \\ &\overset{(3.4.11)}{\geq} \frac{n|B^\lambda|}{4}\psi(g(t)). \end{aligned} \tag{3.4.13}$$

Since $v_E(t) = |B^\lambda| g'(t)$, we have

$$t_2 - t_1 \overset{(3.4.13)}{\leq} \frac{4}{n}\int_{t_1}^{t_2} \frac{g'(t)}{\psi(g(t))}\,\mathrm{d}t = \frac{4}{n}\int_{g(t_1)}^{g(t_2)} \frac{1}{\psi(s)}\,\mathrm{d}s \leq \frac{4}{n}\int_0^1 \frac{1}{\psi(s)}\,\mathrm{d}s =: \alpha, \tag{3.4.14}$$

for some $\alpha = \alpha(n) > 0$.
Let

$$\tau_1 = \max\left\{t \in (a, t_1] : v_E(t) \leq \frac{n|B^\lambda|D_\lambda(E)}{2}\right\},$$
$$\tau_2 = \min\left\{t \in [t_2, b) : v_E(t) \leq \frac{n|B^\lambda|D_\lambda(E)}{2}\right\}.$$

Note that τ_1 and τ_2 are well defined since v_E is continuous and $v_E(t) \to 0$ as $t \to a$ or $t \to b$; moreover, by (3.4.11) and (3.4.13), $v_E(\tau_1) = v_E(\tau_2) = \frac{n|B^\lambda|D_\lambda(E)}{2}$. Moreover, from (3.4.12) and by definition of τ_1, we have

$$t_1 - \tau_1 \leq \frac{2}{n|B^\lambda|D_\lambda(E)}\int_{\tau_1}^{t_1} v_E(t)\,\mathrm{d}t = \frac{2}{nD_\lambda(E)}\int_{\tau_1}^{t_1} g'(t)\,\mathrm{d}t \leq \frac{2g(t_1)}{nD_\lambda(E)} \leq \frac{2}{n(2^{1/n} - 1)},$$

and an analogous estimate holds for $\tau_2 - t_2$.
We consider the truncation $\tilde{E} := E \cap \{x : \tau_1 < x_1 < \tau_2\}$. From the above estimate and (3.4.14), we have that $\tau_2 - \tau_1 < \beta$ for some $\beta = \beta(n) > 0$. Moreover by (3.4.8) and (3.4.12), by the definition of τ_1, τ_2, and since $P(E, \{x_1 < \tau_1, x_1 > \tau_2\}) \geq \lambda\mathcal{H}^{n-1}(\partial^* E \cap \partial H \cap \{x_1 < \tau_1, x_1 > \tau_2\})$ (see the proof of Corollary 2.4.5) we can estimate

$$|\tilde{E}| \geq \left|B^\lambda\right|\left(1 - 2\frac{D_\lambda(E)}{2^{1/n} - 1}\right), \qquad\qquad P_\lambda(\tilde{E}) \leq P_\lambda(E) + n|B^\lambda|D_\lambda(E). \tag{3.4.15}$$

We finally define

$$\sigma := \left(\frac{|B^\lambda|}{|\tilde{E}|}\right)^{1/n}, \qquad\qquad E' := \sigma\tilde{E}.$$

Clearly, $|E'| = |B^\lambda|$ and by (3.4.15) we get that E' is contained in a strip $\{\tau_1' < x_1 < \tau_2'\}$, with $\tau_2' - \tau_1' \leq \sigma(\tau_2 - \tau_1) \leq l'$, where $l' = l'(n, \lambda) > 0$. Let us now show that E' satisfies (3.4.5) for a suitable constant $C_1 = C_1(n, \lambda) > 0$ that may change from line to line. To this aim, since we are assuming $D_\lambda(E)$ small by (3.4.7), from (3.4.15) we get that $1 \le \sigma \le 1 + C_0 D_\lambda(E)$, with $C_0 = C_0(n)$. Thus, from (3.4.15) and (3.4.7), we get

$$\begin{aligned} P_\lambda(E') &= \sigma^{n-1} P_\lambda(\tilde{E}) \le \sigma^{n-1}(P_\lambda(E) + n|B^\lambda| D_\lambda(E)) \\ &= \sigma^{n-1} P_\lambda(B^\lambda)(1 + 2D_\lambda(E)) \le P_\lambda(B^\lambda)(1 + C_1 D_\lambda(E)). \end{aligned}$$

Hence, the second inequality in (3.4.5) follows. To prove the first inequality, let us denote by $B^\lambda(|B^\lambda|, p)$, with $p \in \partial H$, a spherical cap such that $\alpha_\lambda(E') = \frac{|E' \Delta B^\lambda(|B^\lambda|,p)|}{|B^\lambda|}$. From the first inequality in (3.4.15), recalling that $|E| = |B^\lambda|$, we then get

$$\begin{aligned} \alpha_\lambda(E) &\le \frac{|E \Delta B^\lambda(|B^\lambda|, p/\sigma)|}{|B^\lambda|} \\ &\le \frac{|E \Delta \tilde{E}|}{|B^\lambda|} + \frac{|\tilde{E} \Delta B^\lambda(|B^\lambda|/\sigma^n, p/\sigma)|}{|B^\lambda|} + \frac{|B^\lambda(|B^\lambda|/\sigma^n, p/\sigma) \Delta B^\lambda(|B^\lambda|, p/\sigma)|}{|B^\lambda|} \\ &= \frac{|E \setminus \tilde{E}|}{|B^\lambda|} + \frac{\alpha_\lambda(E')}{\sigma^n} + \frac{|B^\lambda(|B^\lambda|) \setminus B^\lambda(|B^\lambda|/\sigma^n)|}{|B^\lambda|} \\ &\overset{(3.4.15)}{\le} C_1 D_\lambda(E) + \alpha_\lambda(E') + C_1(\sigma - 1) \\ &\le \alpha_\lambda(E') + C_1 D_\lambda(E). \end{aligned}$$

Thus the set E' satisfies (3.4.5) and points in E' have first coordinate contained in an interval of length bounded by l'.

Starting from E', we can repeat the same construction finitely many times with respect to the axes $x_2, \dots, x_{n-1}$, thus getting a new set, still denoted by E', satisfying (3.4.5).

It remains to adapt the construction with respect to the coordinate axis x_n. In this case we eventually aim at truncating the set E' in some controlled slab of the form $\{0 < x_n < \bar{\tau}_2\}$. Define

$$\bar{v}(t) := \mathcal{H}^{n-1}(\{x' \in \mathbb{R}^{n-1} : (x', t) \in E'\}), \qquad \bar{E}_t^- := \{x \in E' : x_n < t\}, \qquad \bar{g}(t) := \frac{|\bar{E}_t^-|}{|B^\lambda|},$$

for $t > 0$. It is readily checked that, arguing as above, one estimates

$$\bar{v}(t) \ge \frac{1}{2} P_\lambda(B^\lambda) \left(\psi(\bar{g}(t)) - D_\lambda(E)\right), \tag{3.4.16}$$

which is analogous to (3.4.9), for any t such that $\bar{g}(t) \in (0, 1)$. Similarly as before, we define $0 < \bar{t}_1 < \bar{t}_2$ such that $\bar{g}(\bar{t}_1) = 1 - \bar{g}(\bar{t}_2)$ and $\psi(\bar{g}(\bar{t}_1)) = \psi(\bar{g}(\bar{t}_2)) = 2D_\lambda(E')$. Therefore, using (3.4.16) and the concavity of ψ, arguing as before one estimates

$$\bar{v}(t) \ge \frac{n|B^\lambda|}{4} \psi(\bar{g}(t)) \qquad \forall\, t \in [\bar{t}_1, \bar{t}_2], \tag{3.4.17}$$

which is analogous to (3.4.13).

Let

$$A := \frac{1}{2} P_\lambda(B^\lambda) \left(\left(\frac{\omega_n}{|B^\lambda|} \right)^{\frac{1}{n}} - 1 \right) > 0.$$

We claim that there exists $\bar{\varepsilon} = \bar{\varepsilon}(n, \lambda) > 0$ such that if $D_\lambda(E) < \bar{\varepsilon}$ then

$$\bar{v}(t) \ge \frac{A}{2} \qquad \text{for a.e. } t \in (0, \bar{t}_1). \tag{3.4.18}$$

Indeed, since $D_\lambda(E') \le C_1 D_\lambda(E)$ and $\psi(\bar{g}(\bar{t}_1)) = 2D_\lambda(E')$, then for any $\omega > 0$ there is $\bar{\varepsilon} = \bar{\varepsilon}(n, \lambda) > 0$ such that $\bar{g}(\bar{t}_1) < \omega$ whenever $D_\lambda(E) < \bar{\varepsilon}$. Applying Lemma 3.4.1 with $F = E'$, for almost every $t \in (0, \bar{t}_1)$ we find

$$\begin{aligned} \bar{v}(t) &\ge \frac{1}{2} P_\lambda(B^\lambda) \left(\left(\frac{\omega_n}{|B^\lambda|} \right)^{\frac{1}{n}} (1 - \bar{g}(t))^{\frac{n-1}{n}} - 1 - D_\lambda(E') \right) \\ &\ge \frac{1}{2} P_\lambda(B^\lambda) \left(\left(\frac{\omega_n}{|B^\lambda|} \right)^{\frac{1}{n}} (1 - \bar{g}(\bar{t}_1))^{\frac{n-1}{n}} - 1 - D_\lambda(E') \right) \\ &\ge \frac{1}{2} P_\lambda(B^\lambda) \left(\left(\frac{\omega_n}{|B^\lambda|} \right)^{\frac{1}{n}} (1 - \omega)^{\frac{n-1}{n}} - 1 - C_1 \bar{\varepsilon} \right) \ge \frac{A}{2}, \end{aligned}$$

provided $\bar{\varepsilon}$ is small enough.
Therefore, assuming without loss of generality that $D_\lambda(E) < \bar{\varepsilon}$, since $\bar{g}'(t) = \bar{v}(t)/|B^\lambda|$ we estimate

$$\bar{t}_2 \overset{(3.4.17)}{\leq} \bar{t}_1 + \frac{4}{n}\int_{\bar{t}_1}^{\bar{t}_2} \frac{\bar{g}'(t)}{\psi(\bar{g}(t))}\,\mathrm{d}t \overset{(3.4.18)}{\leq} \frac{2}{A}\int_0^{\bar{t}_1} \bar{v}(t)\,\mathrm{d}t + \frac{4}{n}\int_0^1 \frac{1}{\psi} \leq \frac{2|B^\lambda|}{A} + \frac{4}{n}\int_0^1 \frac{1}{\psi} =: \alpha'(n,\lambda).$$

The rest of the construction follows analogously as above by defining

$$\bar{\tau}_2 := \min\left\{ t \geq \bar{t}_2 \ : \ \bar{g}(t) < 1,\ \bar{v}(t) \leq \frac{n|B^\lambda|D_\lambda(E')}{2} \right\},$$

estimating $\bar{\tau}_2 - \bar{t}_2 \leq \beta'(n,\lambda)$, hence finally taking the set

$$\left(\frac{|B^\lambda|}{|E' \cap \{x_n < \bar{\tau}_2\}|}\right)^{\frac{1}{n}} (E' \cap \{x_n < \bar{\tau}_2\}). \tag{3.4.19}$$

Up to translation along ∂H, the set defined in (3.4.19) yields the final one satisfying the claim of the lemma. □

Corollary 3.4.3 (Non-quantitative stability)**.** *For any $\bar{\varepsilon} > 0$ there exists $\bar{\delta} = \bar{\delta}(n,\lambda,\bar{\varepsilon}) > 0$ such that if $E \subset \mathbb{R}^n \setminus H$ is a Borel set such that $D_\lambda(E) \leq \bar{\delta}$, then $\alpha_\lambda(E) \leq \bar{\varepsilon}$.*

Proof. By scale-invariance of the asymmetry and of the deficit, it is sufficient to prove the claim assuming also $|E| = |B^\lambda|$. We argue by contradiction. Suppose there exist a number $\bar{\varepsilon} > 0$ and a sequence of sets $\{E_i\}_i$, with $E_i \subset \mathbb{R}^n \setminus H$ and $|E_i| = |B^\lambda|$, such that $D_\lambda(E_i) < \frac{1}{i}$ and $\alpha_\lambda(E_i) > \bar{\varepsilon}$ for all $i \in \mathbb{N}$. Let us consider the sequence of sets $\{E_i'\}_i$, with $E_i' \subset Q_l \cap (\mathbb{R}^n \setminus H)$ and $|E_i'| = |B^\lambda|$, given by Lemma 3.4.2. Moreover Lemma 3.4.2 assures that $\alpha_\lambda(E_i') > \bar{\varepsilon}/2$ for large i, and $D_\lambda(E_i') \to 0$. Since each set E_i' is contained in the same Q_l, by Corollary 3.3.6 we can assume, up to a subsequence, that $E_i' \xrightarrow{L^1} E'$ for some set E' of finite perimeter with $|E'| = |B^\lambda|$. By the lower semicontinuity of the perimeters we get $P_\lambda(E') \leq P_\lambda(B^\lambda)$, hence $E' = B^\lambda(|B^\lambda|, x)$ for some $x \in \partial H \cap Q_l$ by uniqueness of minimizers. The convergence of E_i' to E' implies that $|E_i' \Delta E'| \to 0$, against the assumption $\alpha_\lambda(E_i') > \frac{\bar{\varepsilon}}{2}$. □

Corollary 3.4.4. *There exist $A_\lambda, T_\lambda, \eta > 0$ depending on n, λ such that for any set of finite perimeter $E \subset \mathbb{R}^n \setminus H$ with $\big||E| - |B^\lambda|\big| \leq \eta$ and $D_\lambda(E) \leq \eta$ there holds*

$$\mathcal{H}^{n-1}(E \cap \{x_n = t\}) \geq A_\lambda,$$

for almost every $t \in (0, T_\lambda)$.

Proof. Let us prove the inequality assuming $|E| = |B^\lambda|$ first. Fix $T_\lambda' > 0$ such that

$$\left(\frac{\omega_n}{|B^\lambda|}\right)^{\frac{1}{n}} \left(1 - \frac{|B^\lambda(|B^\lambda|, 0) \cap \{x_n < T_\lambda'\}|}{|B^\lambda|}\right)^{\frac{n-1}{n}} \geq 1 + a,$$

for some $a > 0$. By Corollary 3.4.3, for any $\omega > 0$ there is $\eta > 0$ such that if $D_\lambda(E) < \eta$ then $|E \cap \{x_n < T_\lambda'\}| \leq |B^\lambda(|B^\lambda|, 0) \cap \{x_n < T_\lambda'\}| + \omega$. For such a set E, Lemma 3.4.1 implies that for almost every $t \in (0, T_\lambda')$ there holds

$$\begin{aligned}
\mathcal{H}^{n-1}(E \cap \{x_n = t\}) &\geq \frac{1}{2}P_\lambda(B^\lambda)\left(\left(\frac{\omega_n}{|B^\lambda|}\right)^{\frac{1}{n}}\left(1 - \frac{|E \cap \{x_n < t\}|}{|B^\lambda|}\right)^{\frac{n-1}{n}} - 1 - D_\lambda(E)\right) \\
&\geq \frac{1}{2}P_\lambda(B^\lambda)\left(\left(\frac{\omega_n}{|B^\lambda|}\right)^{\frac{1}{n}}\left(1 - \frac{|B^\lambda(|B^\lambda|, 0) \cap \{x_n < T_\lambda'\}| + \omega}{|B^\lambda|}\right)^{\frac{n-1}{n}} - 1 - \eta\right).
\end{aligned}$$

Hence for sufficiently small $\eta > 0$ the right hand side in the previous estimate is bounded below by some constant $A_\lambda'(n,\lambda) > 0$.
For a generic set E such that $\big||E| - |B^\lambda|\big| \leq \eta$ and $D_\lambda(E) \leq \eta$, the set $E' = \left(|B^\lambda|^{\frac{1}{n}}/|E|^{\frac{1}{n}}\right) E$ has measure equal to $|B^\lambda|$ and deficit $D_\lambda(E') \leq \eta$. Up to decreasing $\eta > 0$, applying the first part of the proof to E', the desired estimate holds on E for $T_\lambda = T_\lambda'/2$ and $A_\lambda = A_\lambda'/2$. □

Reduction to $(n-1)$-symmetric sets

In this section we prove that, in order to prove Theorem 3.1.1, it is sufficient to further reduce to show (3.1.2) among $(n-1)$-symmetric sets, i.e., sets which are symmetric with respect to reflection across $n-1$ orthogonal hyperplanes, each one orthogonal to $\{x_n = 0\}$. The results are analogous to [Mag08, Section 6].

Lemma 3.4.5. *Let $E \subset \mathbb{R}^n \setminus H$ be a Borel set with finite measure, symmetric with respect to $k \in \{1, \dots, n-1\}$ orthogonal half-hyperplanes $H_j = \left\{x \in \mathbb{R}^n \setminus H : \left\langle x, v_j \right\rangle = 0\right\}$ for $1 \le j \le k$, where $|v_j| = 1$ and $\left\langle v_j, e_n \right\rangle = 0$ for any $1 \le j \le k$. Then*

$$\min_{x \in \partial H} |E \Delta B^\lambda(|E|, x)| \le \min_{y \in \partial H \cap \bigcap_{j=1}^k H_j} |E \Delta B^\lambda(|E|, y)| \le 3 \min_{x \in \partial H} |E \Delta B^\lambda(|E|, x)|. \tag{3.4.20}$$

Proof. We can suppose for simplicity that $\forall j \in \{1, \dots, k\}$ we have $v_j = e_j$. If $x^0 = (x^0_1, \dots, x^0_{n-1}, 0)$ is such that $\alpha_\lambda(E)$ is achieved by $B^\lambda(|E|, x^0)$, then $\alpha_\lambda(E)$ is achieved also by $B^\lambda(|E|, \bar{x}^0)$, where $\bar{x}^0 = (-x^0_1, \dots, -x^0_k, x^0_{k+1}, \dots, x^0_{n-1}, 0)$. Since $|B^\lambda(|E|, x^0) \Delta B^\lambda(|E|)| \le |B^\lambda(|E|, x^0) \Delta B^\lambda(|E|, \bar{x}^0)|$ we have

$$\begin{aligned}
|E \Delta B^\lambda(|E|)| &\le |E \Delta B^\lambda\left(|E|, x^0\right)| + |B^\lambda\left(|E|, x^0\right) \Delta B^\lambda(|E|)| \\
&\le |E \Delta B^\lambda\left(|E|, x^0\right)| + |B^\lambda\left(|E|, x^0\right) \Delta B^\lambda\left(|E|, \bar{x}^0\right)| \\
&\le |E \Delta B^\lambda\left(|E|, x^0\right)| + |B^\lambda\left(|E|, x^0\right) \Delta E| + |E \Delta B^\lambda\left(|E|, \bar{x}^0\right)| \\
&= 3 |E \Delta B^\lambda\left(|E|, x^0\right)|.
\end{aligned}$$

□

Given a Borel set $E \subset \mathbb{R}^n \setminus H$ with finite measure and a unit vector v with $\langle v, e_n \rangle = 0$, we denote by $H^+_v = \{x \in \mathbb{R}^n : \langle x, v \rangle > t\}$ an open half-space orthogonal to v where $t \in \mathbb{R}$ is chosen in such a way that

$$|E \cap H^+_v| = \frac{|E|}{2}.$$

We also denote by $r_v : \mathbb{R}^n \setminus H \to \mathbb{R}^n \setminus H$ the reflection with respect to $H_v := \partial H^+_v$, and by $H^-_v := r_v(H^+_v)$ the open half-space complementary to H^+_v. Finally we write $E^\pm_v := E \cap H^\pm_v$.
Observe that

$$D_\lambda(E^\pm_v \cup r_v(E^\pm_v)) \le 2 D_\lambda(E). \tag{3.4.21}$$

Indeed

$$\begin{aligned}
P_\lambda(E^\pm_v \cup r_v(E^\pm_v)) - P_\lambda(B^\lambda(|E|)) &\le 2P_\lambda(E) - P_\lambda(E^\mp_v \cup r_v(E^\mp_v)) - P_\lambda(B^\lambda(|E|)) \\
&= 2(P_\lambda(E) - P_\lambda(B^\lambda(|E|))) + P_\lambda(B^\lambda(|E|)) - P_\lambda(E^\mp_v \cup r_v(E^\mp_v)) \\
&\le 2(P_\lambda(E) - P_\lambda(B^\lambda(|E|))),
\end{aligned}$$

where in the last inequality we used the isoperimetric inequality of Theorem 3.2.3.

Lemma 3.4.6. *There exist $\bar{C}_2, \bar{\delta}_2 > 0$ depending on n, λ such that, if $E \subset \mathbb{R}^n \setminus H$ is a Borel set with finite measure such that $D_\lambda(E) \le \bar{\delta}_2$, and if v_1 and v_2 are two orthogonal vectors, with $\langle v_i, e_n \rangle = 0$, such that H_{v_1} and H_{v_2} divide E in four parts of equal measure, then there exist $i \in \{1, 2\}$ and $s \in \{+, -\}$ such that, setting $E' = E^s_{v_i} \cup r_{v_i}(E^s_{v_i})$, there holds*

$$\alpha_\lambda(E) \le \bar{C}_2 \alpha(E'). \tag{3.4.22}$$

Proof. By scale-invariance of Fraenkel asymmetry, it is sufficient to prove the claim assuming also $|E| = |B^\lambda|$. If $i \in \{1, 2\}$ and $s \in \{+, -\}$, let $E'^s_{v_i}$ denote the sets obtained by reflecting $E^s_{v_i}$ along H_{v_i} and let $B^{\lambda, s}_i = B^\lambda(|B^\lambda|, x^s_i)$ be four spherical caps such that

$$|E'^s_{v_i} \Delta B^{\lambda, s}_i| = \min_{x \in H_{v_i} \cap \partial H} |E'^s_{v_i} \Delta B^\lambda(|B^\lambda|, x)|.$$

For $i = 1, 2$, by the triangular inequality we have

$$\begin{aligned}
\min_{x \in \partial H} |E \Delta B^\lambda(|B^\lambda|, x)| &\le |E \Delta B^{\lambda,+}_i| \\
&= |(E \Delta B^{\lambda,+}_i) \cap H^+_{v_i}| + |(E \Delta B^{\lambda,+}_i) \cap H^-_{v_i}| \\
&\le |(E \Delta B^{\lambda,+}_i) \cap H^+_{v_i}| + |(E \Delta B^{\lambda,-}_i) \cap H^-_{v_i}| + |(B^{\lambda,+}_i \Delta B^{\lambda,-}_i) \cap H^-_{v_i}| \\
&= \frac{1}{2}|E'^+_{v_i} \Delta B^{\lambda,+}_i| + \frac{1}{2}|E'^-_{v_i} \Delta B^{\lambda,-}_i| + \frac{1}{2}|B^{\lambda,+}_i \Delta B^{\lambda,-}_i|.
\end{aligned} \tag{3.4.23}$$

Once we show that if $D_\lambda(E)$ is sufficiently small then there exists $c_{n,\lambda} > 0$ such that at least one the following

$$\begin{aligned} |B_1^{\lambda,+}\Delta B_1^{\lambda,-}| &\le 2c_{n,\lambda}\Big(|E_{\nu_1}'^{+}\Delta B_1^{\lambda,+}| + |E_{\nu_1}'^{-}\Delta B_1^{\lambda,-}|\Big) \\ |B_2^{\lambda,+}\Delta B_2^{\lambda,-}| &\le 2c_{n,\lambda}\Big(|E_{\nu_2}'^{+}\Delta B_2^{\lambda,+}| + |E_{\nu_2}'^{-}\Delta B_2^{\lambda,-}|\Big) \end{aligned} \tag{3.4.24}$$

holds, then we soon conclude the proof. Indeed, assume for example that the first inequality in (3.4.24) holds. Then, from (3.4.20) and (3.4.23) with $i = 1$, we get

$$\begin{aligned} \min_{x\in\partial H} |E\Delta B^\lambda(|B^\lambda|,x)| &\overset{(3.4.23)}{\le} (c_{n,\lambda}+1/2)\Big(|E_{\nu_1}'^{+}\Delta B_1^{\lambda,+}| + |E_{\nu_1}'^{-}\Delta B_1^{\lambda,-}|\Big) \\ &\overset{(3.4.20)}{\le} 3(c_{n,\lambda}+1/2)\Big(\min_{x\in\partial H}|E_{\nu_1}'^{+}\Delta B^\lambda(|B^\lambda|,x)| + \min_{x\in\partial H}|E_{\nu_1}'^{-}\Delta B^\lambda(|B^\lambda|,x)|\Big), \end{aligned}$$

thus proving (3.4.22) with $\bar{C}_2 = 6(c_{n,\lambda}+1/2)$ and E' equal to $E_{\nu_1}'^{+}$ or $E_{\nu_1}'^{-}$.
Observe that, given $\tilde{\varepsilon} > 0$, Corollary 3.4.3, (3.4.20) and (3.4.21) imply that there exists $\bar{\delta}_2(n,\lambda) > 0$ such that if $D_\lambda(E) < \bar{\delta}_2$ then

$$\max\left\{\alpha_\lambda(E), \frac{|E_{\nu_i}'^{\pm}\Delta B_i^{\lambda,\pm}|}{|B^\lambda|} \ :\ i = 1,2\right\} < \tilde{\varepsilon}. \tag{3.4.25}$$

Thanks to (3.4.25), we can show that the caps $B_i^{\lambda,\pm}$ get closer and closer to the optimal ones for E, as $\bar{\delta}_2$ decreases. Indeed, let us assume by contradiction that there exists $\eta > 0$ such that for every $j \in \mathbb{N}$ there exist E_j, with $|E_j| = |B^\lambda|$, $D_\lambda(E_j) < \frac{1}{j}$, with $B_j^\lambda := B^\lambda(|B^\lambda|, x_j)$ realizing the asymmetry of E_j, but for $i \in \{1,2\}$ and $s \in \{+,-\}$, if $B_{i,j}^{\lambda,s} := B^\lambda(|B^\lambda|, x_{i,j}^s)$ is such that

$$|E'^{s}_{j,\nu_i^j}\Delta B_{i,j}^{\lambda,s}| = \min_{x\in H_{\nu_i^j}\cap\partial H} |E'^{s}_{j,\nu_i^j}\Delta B^\lambda(|B^\lambda|,x)|,$$

where E'^{s}_{j,ν_i^j} is given by reflections of truncations of E_j along orthogonal subspaces $H_{\nu_1^j}, H_{\nu_2^j}$, then $|x_{i,j}^s - x_j| > \eta$ for some $i \in \{1,2\}$, $s \in \{+,-\}$ and any j. Without loss of generality we can assume that that $i = 1$ and $s = +$.
Let us translate every set in the above contradiction assumption by $-x_j$. Without relabeling the objects involved, up to subsequences, we have that $E_j \to B_0^\lambda := B^\lambda(|B^\lambda|, 0)$ in L^1. We can show that $E'^{+}_{j,\nu_1^j} \to B_0^\lambda$ as well. Indeed, up to a rotation we can further assume that

$$H_{\nu_1^j} = \left\{(a_j, 0, \dots, 0) + e_1^\perp\right\}, \qquad \nu_1^j = e_1, \qquad \forall j,$$

with $a_j \in \mathbb{R}$. By the definition of $H_{\nu_1^j}$, we have

$$\frac{|B^\lambda|}{2} = \frac{|E_j|}{2} = |E_j \cap \{x_1 > a_j\}|.$$

Then $\{a_j\}$ is bounded and, up to subsequence, converges to $a_\infty = 0$ because, if $a_\infty \neq 0$, then

$$E_j \cap \{x_1 > a_j\} \to B_0^\lambda \cap \{x_1 > a_\infty\}$$

with $\left|B_0^\lambda \cap \{x_1 > a_\infty\}\right| \neq \frac{|B^\lambda|}{2}$. In particular $E'^{+}_{j,\nu_1^j} \to B_0^\lambda$. Finally by (3.4.25)

$$\tilde{\varepsilon}|B^\lambda| \ge \lim_j \left|E'^{+}_{j,\nu_1^j}\Delta B_{1,j}^{\lambda,+}\right| \ge \min_{x\in\partial H, |x|\ge\eta}\left\{|B^\lambda(|B^\lambda|,x)\Delta B_0^\lambda\right\} =: C(\eta) > 0.$$

But for j sufficiently large, since $D_\lambda(E_j) \to 0$, by (3.4.25) we can choose $\tilde{\varepsilon}$ such that $\tilde{\varepsilon}|B^\lambda| < C(\eta)/2$, getting a contradiction.

We observe that for $\tilde{\varepsilon} > 0$ sufficiently small, that is for $\bar{\delta}_2 > 0$ sufficiently small, there exists $c_{n,\lambda} > 0$ such that for all possible choices of $s, t \in \{+, -\}$ there holds

$$|(B_1^{\lambda,s} \Delta B_2^{\lambda,t}) \cap (H_{\nu_1}^s \cap H_{\nu_2}^t)| > \frac{|B_1^{\lambda,s} \Delta B_2^{\lambda,t}|}{c_{n,\lambda}}. \tag{3.4.26}$$

We only sketch the argument for (3.4.26). Letting $Q := (H_{\nu_1}^s \cap H_{\nu_2}^t)$ and $B_1(h) := B^\lambda(|B^\lambda|, h\, x_1^s)$, $B_2(h) := B^\lambda(|B^\lambda|, h\, x_2^t)$ for $h \in [0,1]$, one can compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}h}\Big|(B_1(h)\Delta B_2(h)) \cap Q\Big| &= \int_{\partial B_1(h) \cap B_2(h) \cap Q} \left\langle \nu^{B_1(h)}, x_2^t - x_1^s \right\rangle \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\partial B_2(h) \cap B_1(h) \cap Q} \left\langle \nu^{B_2(h)}, x_1^s - x_2^t \right\rangle \, \mathrm{d}\mathcal{H}^{n-1} \\
&= \left(\int_{\partial B_1(h) \cap B_2(h) \cap Q} \left\langle \nu^{B_1(h)}, \frac{x_2^t - x_1^s}{|x_1^s - x_2^t|} \right\rangle + \int_{\partial B_2(h) \cap B_1(h) \cap Q} \left\langle \nu^{B_2(h)}, \frac{x_1^s - x_2^t}{|x_1^s - x_2^t|} \right\rangle \right) |x_1^s - x_2^t| \\
&= \int_{\partial(B_1(h) \cap B_2(h)) \cap Q} \left\langle \nu^{B_1(h) \cap B_2(h)}, \upsilon_{12}^{s,t} \right\rangle \, \mathrm{d}\mathcal{H}^{n-1} \, |x_1^s - x_2^t| \\
&\geq c|x_1^s - x_2^t|,
\end{aligned}$$

where $\upsilon_{12}^{s,t}$ is obviously defined, provided $\tilde{\varepsilon}$ is small enough, for some $c = c(n, \lambda) > 0$ that will change from line to line. The last estimate follows since $\left\langle \nu^{B_1(h) \cap B_2(h)}, \upsilon_{12}^{s,t} \right\rangle \geq 0$ pointwise and, for $\tilde{\varepsilon}$ small, centers x_1^s, x_2^t are so close that $\left\langle \nu^{B_1(h) \cap B_2(h)}, \upsilon_{12}^{s,t} \right\rangle$ can be estimated from below by a positive constant on a set of $\mathcal{H}^{n-1}$-measure uniformly bounded from below away from zero. On the other hand one can estimate

$$|(B_1^{\lambda,s} \Delta B_2^{\lambda,t})| \leq c|x_1^s - x_2^t|.$$

Hence

$$|(B_1^{\lambda,s} \Delta B_2^{\lambda,t}) \cap (H_{\nu_1}^s \cap H_{\nu_2}^t)| = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}h}\Big|(B_1(h)\Delta B_2(h)) \cap Q\Big| \, \mathrm{d}h \geq c|x_1^s - x_2^t| \geq c|(B_1^{\lambda,s} \Delta B_2^{\lambda,t})|,$$

and (3.4.26) follows.
Letting

$$S_1 = (B_1^{\lambda,+} \cap H_{\nu_1}^+) \cup (B_1^{\lambda,-} \cap H_{\nu_1}^-), \quad S_2 = (B_2^{\lambda,+} \cap H_{\nu_2}^+) \cup (B_2^{\lambda,-} \cap H_{\nu_2}^-),$$

we deduce

$$|S_1 \Delta S_2| \geq |(S_1 \Delta S_2) \cap (H_{\nu_1}^s \cap H_{\nu_2}^t)| = |(B_1^{\lambda,s} \Delta B_2^{\lambda,t}) \cap (H_{\nu_1}^s \cap H_{\nu_2}^t)| > \frac{|B_1^{\lambda,s} \Delta B_2^{\lambda,t}|}{c_{n,\lambda}}.$$

In particular we have

$$\begin{aligned}
|B_1^{\lambda,+} \Delta B_1^{\lambda,-}| &\leq |B_1^{\lambda,+} \Delta B_2^{\lambda,+}| + |B_2^{\lambda,+} \Delta B_1^{\lambda,-}| < 2c_{n,\lambda}|S_1 \Delta S_2|, \\
|B_2^{\lambda,+} \Delta B_2^{\lambda,-}| &\leq |B_2^{\lambda,+} \Delta B_1^{\lambda,+}| + |B_1^{\lambda,+} \Delta B_2^{\lambda,-}| < 2c_{n,\lambda}|S_1 \Delta S_2|.
\end{aligned} \tag{3.4.27}$$

If by contradiction (3.4.24) were false, then

$$|E_{\nu_1}'^{+} \Delta B_1^{\lambda,+}| + |E_{\nu_1}'^{-} \Delta B_1^{\lambda,-}| < \frac{|B_1^{\lambda,+} \Delta B_1^{\lambda,-}|}{2c_{n,\lambda}} \quad \text{and} \quad |E_{\nu_2}'^{+} \Delta B_2^{\lambda,+}| + |E_{\nu_2}'^{-} \Delta B_2^{\lambda,-}| < \frac{|B_2^{\lambda,+} \Delta B_2^{\lambda,-}|}{2c_{n,\lambda}}. \tag{3.4.28}$$

Hence

$$\begin{aligned}
|S_1 \Delta S_2| \leq |S_1 \Delta E| + |E \Delta S_2| &= \frac{1}{2} \sum_{i=1}^{2} \left(|E_{\nu_i}'^{+} \Delta B_i^{\lambda,+}| + |E_{\nu_i}'^{-} \Delta B_i^{\lambda,-}| \right) \\
&\overset{(3.4.28)}{<} \frac{1}{4c_{n,\lambda}} \sum_{i=1}^{2} |B_i^{\lambda,+} \Delta B_i^{\lambda,-}| \overset{(3.4.27)}{\leq} |S_1 \Delta S_2|,
\end{aligned}$$

getting a contradiction. □

Lemma 3.4.7 (Reduction to $(n-1)$-symmetric sets)**.** *There exist $C_2, \delta_2 > 0$ depending on n, λ such that, if E is a Borel set with $E \subset \mathbb{R}^n \setminus H$, $E \subset Q_l$, $|E| = |B^\lambda|$ and $D_\lambda(E) \le \delta_2$, there exists a Borel set $F \subset \mathbb{R}^n \setminus H$, $F \subset Q_{2l}$, $|F| = |B^\lambda|$, symmetric with respect to $n-1$ orthogonal half-hyperplanes (each orthogonal to ∂H) and such that*

$$\alpha_\lambda(E) \le C_2 \alpha_\lambda(F), \qquad D_\lambda(F) \le 2^{n-1} D_\lambda(E).$$

Proof. Let us define $\delta_2 := \bar{\delta}_2 2^{-(n-2)}$, where $\bar{\delta}_2$ is the constant appearing in Lemma 3.4.6. We can apply Lemma 3.4.6 $n-2$ times to different pairs of orthogonal vectors in $\{e_1, \dots, e_{n-2}\}$ normal to corresponding pairs of affine hyperplanes splitting the measure of E in two halves. Therefore, also recalling (3.4.21), we find an $(n-2)$-symmetric set E' such that $|E'| = |B^\lambda|$ and

$$\alpha_\lambda(E) \le \bar{C}_2^{n-2} \alpha_\lambda(E'), \quad D_\lambda(E') \le 2^{n-2} D_\lambda(E).$$

To perform the last symmetrization, let us consider a half-hyperplane H_{n-1} orthogonal to e_{n-1} and dividing E' into two parts of equal measure. For simplicity let us assume that $H_{n-1} = \{x_{n-1} = 0\} \setminus H$. We denote by $E^{'+}$ (resp. $E^{'-}$) the set obtained by the union of $E' \cap \{x_{n-1} > 0\}$ (resp. $E' \cap \{x_{n-1} < 0\}$) with its reflection along H_{n-1}. By (3.4.21) we have

$$D_\lambda(E^{'\pm}) \le 2 D_\lambda(E') \le 2^{n-1} D_\lambda(E).$$

Regarding the asymmetry of $E^{'\pm}$ note that since E' is symmetric with respect to the first $n-2$ coordinate hyperplanes, $E^{'+}$ and $E^{'-}$ are $(n-1)$-symmetric. By Lemma 3.4.5 we get

$$\begin{aligned}
|B^\lambda| \alpha_\lambda(E') &\le |E' \Delta B^\lambda(|B^\lambda|)| \\
&= |(E' \Delta B^\lambda(|B^\lambda|)) \cap \{x_{n-1} > 0\}| + |(E' \Delta B^\lambda(|B^\lambda|)) \cap \{x_{n-1} < 0\}| \\
&= \frac{1}{2}\Big(|E^{'+} \Delta B^\lambda(|B^\lambda|)| + |E^{'-} \Delta B^\lambda(|B^\lambda|)| \Big) \\
&\le \frac{3|B^\lambda|}{2} \Big(\alpha_\lambda(E^{'+}) + \alpha_\lambda(E^{'-}) \Big).
\end{aligned}$$

Therefore at least one of the sets $E^{'+}, E^{'-}$ has asymmetry greater than $\frac{1}{3}\alpha_\lambda(E')$ and, denoting by F this set, we have

$$\begin{aligned}
D_\lambda(F) &\le 2 D_\lambda(E') \le 2^{n-1} D_\lambda(E) \\
\alpha_\lambda(E) &\le \bar{C}_2^{n-2} \alpha_\lambda(E') \le 3 \bar{C}_2^{n-2} \alpha_\lambda(F).
\end{aligned}$$

Finally, the inclusion $F \subset Q_{2l}$ follows since F was obtained by performing reflections of $E \subset Q_l$ along affine hyperplanes of the form $\{x_j = a_j\}$ for $j = 1, \dots, n-1$ with $a_j \in (-l, l)$. ☐

Reduction to Schwarz-symmetric sets

In this section we observe that, in order to prove Theorem 3.1.1, it is sufficient to further reduce to show (3.1.2) just among Schwarz-symmetric sets. The proof is analogous to [Fus15, Proposition 4.9].

Lemma 3.4.8 (Reduction to Schwarz-symmetric sets)**.** *There exists $C_3 = C_3(n, \lambda) > 0$ such that the following holds. Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter with $|E| = |B^\lambda|$. Suppose that E is symmetric with respect to the coordinate hyerplanes $\{x_1 = 0\}, \dots, \{x_{n-1} = 0\}$ and that*

$$D_\lambda(E) < 1, \qquad E \subset Q_{2l},$$

where $l = l(n, \lambda)$ is as in in Lemma 3.4.2. Then

$$|E \Delta E^*| \le C_3 \sqrt{D_\lambda(E)} \qquad \text{and} \qquad D_\lambda(E^*) \le D_\lambda(E), \tag{3.4.29}$$

where E^* denotes the Schwarz symmetrization of E with respect to the n-th axis.

Proof. The second inequality in (3.4.29) follows from the fact $|E^*| = |E|$ and $P_\lambda(E^*) \le P_\lambda(E)$. Exploiting Lemma 2.4.4, we may assume

$$\mathcal{H}^{n-1}(\{x \in \partial^* E \setminus H \ : \ \nu^E(x) = \pm e_n\}) = 0,$$

and thus that $v_E \in W^{1,1}(\mathbb{R})$. Indeed, if E_i is given by Lemma 2.4.4 and the claim is proved for E_i, by the contractivity of Schwartz rearrangement 2.6.3, for every $\tilde{\varepsilon} > 0$ and with i sufficiently large, we get

$$\begin{aligned}|E\Delta E^*| &\le |E\Delta E_i| + |E_i\Delta E_i^*| + |E_i^*\Delta E^*| \\ &\le 2|E\Delta E_i| + |E_i\Delta E_i^*| \\ &\le 2\tilde{\varepsilon} + C_3\sqrt{D_\lambda(E_i)},\end{aligned}$$

and the last term tends to $C_3\sqrt{D_\lambda(E)}$ as $i \to \infty$.
For $\mathcal{H}^1$-a.e. $t \in (0,\infty)$ denote

$$\begin{aligned}v_E(t) &:= \mathcal{H}^{n-1}(\{x' \in \mathbb{R}^{n-1} \ : \ (x',t) \in E\}), \\ p_E(t) &:= \mathcal{H}^{n-2}(\partial^*\{x' \in \mathbb{R}^{n-1} \ : \ (x',t) \in E\}),\end{aligned}$$

and employ analogous notation for E^*. Since $|\partial^* E \cap \partial H| = |\partial^* E^* \cap \partial H|$, we get

$$P_\lambda(E) - P_\lambda(B^\lambda) \ge P_\lambda(E) - P_\lambda(E^*) = P(E, \mathbb{R}^n \setminus H) - P(E^*, \mathbb{R}^n \setminus H).$$

It is therefore possible to reproduce the computation in [Fus15, Proposition 4.9] verbatim up to [Fus15, Eq. (4.29)] to estimate the right hand side in the last equation from below. We include the computation for the convenience of the reader. [Mag12, Eq. (19.30)] states that

$$P(E) \ge \int_0^\infty \sqrt{p_E(t)^2 + v_E'(t)^2}\,\mathrm{d}t,$$

therefore by Theorem 2.6.4 one has

$$\begin{aligned}P_\lambda(E) - P_\lambda(B^\lambda) &\ge P(E,\mathbb{R}^n\setminus H) - P(E^*,\mathbb{R}^n\setminus H) \ge \int_0^\infty \left(\sqrt{p_E^2 + v_E'^2} - \sqrt{p_{E^*}^2 + v_E'^2}\right)\mathrm{d}t \\ &= \int_0^\infty \frac{p_E^2 - p_{E^*}^2}{\sqrt{p_E^2 + v_E'^2} + \sqrt{p_{E^*}^2 + v_E'^2}}\,\mathrm{d}t \ge \left(\int_0^\infty \sqrt{p_E^2 - p_{E^*}^2}\,\mathrm{d}t\right)^2 \frac{1}{\int_0^\infty \sqrt{p_E^2 + v_E'^2} + \sqrt{p_{E^*}^2 + v_E'^2}\,\mathrm{d}t} \\ &\ge \left(\int_0^\infty \sqrt{p_E^2 - p_{E^*}^2}\,\mathrm{d}t\right)^2 \frac{1}{P(E,\mathbb{R}^n\setminus H) + P(E^*,\mathbb{R}^n\setminus H)} \\ &\ge c(\lambda)\left(\int_0^\infty \sqrt{p_E^2 - p_{E^*}^2}\,\mathrm{d}t\right)^2 \frac{1}{P_\lambda(E) + P_\lambda(E^*)},\end{aligned}$$

where in the last inequality we used Corollary 2.4.5. Since $D_\lambda(E) < 1$ and $p_E \ge p_{E^*}$, we have $P_\lambda(E^*) \le P_\lambda(E) \le 2P_\lambda(B^\lambda)$ and

$$\begin{aligned}\sqrt{D_\lambda(E)} &\ge c\int_0^\infty \sqrt{p_E^2 - p_{E^*}^2}\,\mathrm{d}t = c\int_0^\infty \sqrt{p_E + p_{E^*}}\sqrt{p_{E^*}}\sqrt{\frac{p_E - p_{E^*}}{p_{E^*}}}\,\mathrm{d}t \\ &\ge \sqrt{2}c\int_0^\infty p_{E^*}\sqrt{\frac{p_E - p_{E^*}}{p_{E^*}}}\,\mathrm{d}t,\end{aligned} \tag{3.4.30}$$

for some constant $c = c(n,\lambda) > 0$ changing from line to line. Note that $(E^*)_t$ is a $(n-1)$-dimensional ball with the same $\mathcal{H}^{n-1}$ measure of E_t. Then the quantity

$$\frac{p_E(t) - p_{E^*}(t)}{p_{E^*}(t)}$$

is the classical isoperimetric deficit in $\mathbb{R}^{n-1}$ of E_t with respect to the standard perimeter. By the quantitative isoperimetric inequality in $\mathbb{R}^{n-1}$ [FMP08], the fact that E_t is $n-1$ symmetric with $(E^*)_t$ centered at the center of symmetry of E_t and Lemma 3.4.5, we have

$$\frac{\mathcal{H}^{n-1}(E_t\Delta E_t^*)}{\mathcal{H}^{n-1}((E^*)_t)} \le c(n)\sqrt{\frac{p_E(t) - p_{E^*}(t)}{p_{E^*}(t)}}.$$

By (3.4.30) and the inclusion $E \subset Q_{2l}$ we conclude

$$\sqrt{D_\lambda(E)} \ge c\int_0^\infty \frac{p_{E^*}(t)}{\mathcal{H}^{n-1}((E^*)_t)}\mathcal{H}^{n-1}(E_t\Delta E_t^*)\,\mathrm{d}t \ge \frac{c}{l}\int_0^\infty \mathcal{H}^{n-1}(E_t\Delta E_t^*)\,\mathrm{d}t = \frac{c}{l}|E\Delta E^*|.$$

□

Reduction to indecomposable sets

Definition 3.4.9. A set of finite perimeter E with finite measure is said to be indecomposable if, whenever there exist two sets of finite perimeter E_1, E_2 such that $|E\Delta(E_1 \cup E_2)| = 0$ with $|E_1 \cap E_2| = 0$ and $P(E) = P(E_1)+P(E_2)$, then

$$\min\{|E_1|, |E_2|\} = 0.$$

The next result is analogous to [Mag08, Theorem 4.4].

Lemma 3.4.10. *There exist two $\delta_4 = \delta_4(n) > 0$ and $C_4 = C_4(n, \lambda) > 0$ such that if E is a set of finite measure with $D_\lambda(E) \le \delta_4$, then there exists an indecomposable set F such that $|F| > |E|/2$ and*

$$\alpha_\lambda(E) \le \alpha_\lambda(F) + C_4 D_\lambda(E), \qquad D_\lambda(F) \le C_4 D_\lambda(E). \tag{3.4.31}$$

Proof. By scale-invariance of the asymmetry and of the deficit, it is sufficient to prove the claim assuming also $|E| = |B^\lambda|$. Without loss of generality, let us assume that E is not indecomposable. By [Amb+01, Theorem 1], there exist at most countably many disjoint indecomposable sets E_h, for $h \in I \subset \mathbb{N}$, such that $|E_h| > 0$, $|E_h \setminus E| = 0$, $|E \setminus \cup_h E_h| = 0$ and $P(E) = \sum_h P(E_h)$. In particular $\mathcal{H}^{n-1} \llcorner \partial^* E = \sum_h \mathcal{H}^{n-1} \llcorner \partial^* E_h$ as measures, thus $P(E, \mathbb{R}^n \setminus H) = \sum_h P(E_h, \mathbb{R}^n \setminus H)$ and $\mathcal{H}^{n-1}(\partial^* E \cap \partial H) = \sum_h \mathcal{H}^{n-1}(\partial^* E_h \cap \partial H)$. Rewriting P_λ as in (2.4.3), we conclude that and $P_\lambda(E) = \sum_{h\in I} P_\lambda(E_h)$ holds. From the isoperimetric inequality in Theorem 3.2.3 we get

$$D_\lambda(E) = \frac{1}{n|B^\lambda|}\left(\sum_{h\in I} P_\lambda(E_h) - P_\lambda(B^\lambda)\right) \ge \frac{1}{n|B^\lambda|}\left(n|B^\lambda|^{\frac{1}{n}} \sum_{h\in I} |E_h|^{\frac{n-1}{n}} - n|B^\lambda|\right) = \sum_{h\in I} a_h^{\frac{n-1}{n}} - 1$$

with $a_h := |E_h|/|B^\lambda|$. Note that $\sum_{h\in I} a_h = 1$ as $|E| = |B^\lambda|$. Assume for simplicity that the a_h are ordered decreasingly.
If $a_1 > 1/2$, from (3.4.10) we get

$$D_\lambda(E) \ge a_1^{\frac{n-1}{n}} + (1-a_1)^{\frac{n-1}{n}} - 1 \ge 2\left(2^{\frac{1}{n}} - 1\right)(1-a_1), \tag{3.4.32}$$

hence $|E \setminus E_1| \le C_4 D_\lambda(E)$, with $C_4 = C_4(n, \lambda) > 0$ that may change from line to line. Setting $F = E_1$, since $t \mapsto t^{\frac{n-1}{n}}$ is Lipschitz on $[|B^\lambda|/2, |B^\lambda|]$, we find

$$\begin{aligned} P_\lambda(F) - P_\lambda(B^\lambda(|F|)) &\le P_\lambda(E) - P_\lambda(B^\lambda) + P_\lambda(B^\lambda) - P_\lambda(B^\lambda(|F|)) = P_\lambda(E) - P_\lambda(B^\lambda) + n|B^\lambda|^{\frac{1}{n}}(|E|^{\frac{n-1}{n}} - |F|^{\frac{n-1}{n}}) \\ &\le P_\lambda(E) - P_\lambda(B^\lambda) + C_4|E \setminus E_1| \le P_\lambda(E) - P_\lambda(B^\lambda) + C_4 D_\lambda(E). \end{aligned}$$

Since

$$P_\lambda(B^\lambda(|F|)) \ge n|B^\lambda|^{\frac{1}{n}}|F|^{\frac{n-1}{n}} > \frac{n}{2^{\frac{n-1}{n}}}|B^\lambda| = \frac{1}{2^{\frac{n-1}{n}}} P_\lambda(B^\lambda),$$

we get

$$D_\lambda(F) \le C_4 D_\lambda(E).$$

Similarly, denoting by $B^\lambda(|F|, x_0)$ an optimal spherical cap realizing $\alpha_\lambda(F)$ we have

$$\begin{aligned} |B^\lambda|\alpha_\lambda(E) &\le |E\Delta B^\lambda(|B^\lambda|, x_0)| \\ &\le |E\Delta F| + |F\Delta B^\lambda(|F|, x_0)| + |B^\lambda(|F|, x_0)\Delta B^\lambda(|B^\lambda|, x_0)| \\ &= |F\Delta B^\lambda(|F|, x_0)| + 2|E \setminus F| \\ &\le |F\Delta B^\lambda(|F|, x_0)| + 2C_4 D_\lambda(E), \end{aligned}$$

thus completing the proof of (3.4.31).
Therefore, it remains to show that if δ_4 is sufficiently small then $a_1 > 1/2$. By contradiction assume that $a_1 \le 1/2$ and let $N \ge 2$ be the greatest integer such that $\sum_{h<N} a_h \le 1/2$. Then, by (3.4.10) and arguing as in (3.4.32), we

get

$$D_\lambda(E) \geq \left(\sum_{h<N} a_h\right)^{\frac{n-1}{n}} + \left(\sum_{h\geq N} a_h\right)^{\frac{n-1}{n}} \geq 2\left(2^{\frac{1}{n}}-1\right)\sum_{h<N} a_h$$
$$D_\lambda(E) \geq \left(\sum_{h\leq N} a_h\right)^{\frac{n-1}{n}} + \left(\sum_{h> N} a_h\right)^{\frac{n-1}{n}} \geq 2\left(2^{\frac{1}{n}}-1\right)\sum_{h>N} a_h.$$

Adding these two inequalities we get

$$D_\lambda(E) \geq \left(2^{\frac{1}{n}}-1\right)\sum_{h\neq N} a_h = \left(2^{\frac{1}{n}}-1\right)(1-a_N) \geq \left(2^{\frac{1}{n}}-1\right)(1-a_1) > \frac{2^{\frac{1}{n}}-1}{2},$$

which is impossible for $\delta_4 < \frac{2}{2^{\frac{1}{n}}-1}$. □

Remark 3.4.11. If E as in Lemma 3.4.10 is also symmetric with respect to the first $n-1$ variables, then the indecomposable component F given by Lemma 3.4.10 is $n-1$-symmetric as well.
Indeed, consider for instance the hyperplane $H_1 = \{x_1 = 0\}$ and assume by contradiction that there exists $A \subset F^+ := F \cap \{x_1 > 0\}$ with $|A| > 0$ such that its reflection $r_{H_1}(A)$ with respect to H_1 satisfies $|r_{H_1}(A) \cap F| = \emptyset$, where $r_{H_1} : \{x_1 > 0\} \to \{x_1 < 0\}$. Decomposing $F^+ = \cup_i F_i$ in maximal indecomposable components (see [Amb+01, Theorem 1]) we can assume without loss of generality that $A \subset F_1$ (containments understood up to Lebesgue-null sets). If $P(F_1, H_1) > 0$, then [Amb+01, Proposition 5(i)] implies that $F_1 \cup r_{H_1}(F_1)$ is indecomposable. Also $F_1 \cup r_{H_1}(F_1) \subset E$ by symmetry of E, hence $F_1 \cup r_{H_1}(F_1)$ is contained in one indecomposable component of E; since $|(F_1 \cup r_{H_1}(F_1)) \cap F| > 0$, then $F_1 \cup r_{H_1}(F_1) \subset F$, which implies $r_{H_1}(A) \subset F$, against the contradiction assumption. Therefore there must hold that $P(F_1, H_1) = 0$, hence $F_1 = F$ since F is indecomposable. Thus $F \subset \{x_1 > 0\}$, but then $|E| \geq |F| + |r_{H_1}(F)| > |E|/2 + |E|/2 = |E|$, that gives a final contradiction.

Corollary 3.4.12. *There exist $\delta_5, \tilde{l} > 0$ depending on n, λ such that for every $0 < \delta \leq \delta_5$, if the quantitative isoperimetric inequality (3.1.2) holds (with some constant depending on n, λ) for every Schwarz-symmetric indecomposable set F, with $|F| = |B^\lambda|$, contained in a cube $Q_{\tilde{l}}$ and with $D_\lambda(F) < \delta$, then (3.1.2) holds for any measurable set of finite measure (up to changing the multiplicative constant, depending on n, λ only).*

3.5 First quantitative isoperimetric inequality

Coupling

We start by recalling the general definition and properties of the restricted envelopes introduced in [Cin+22].

Definition 3.5.1 ([Cin+22, Definition 3.1])**.** Let $K \subset \mathbb{R}^n$ be a compact convex set, $E \subset \mathbb{R}^n$ be a bounded open set, and $u \in C^0(\overline{E}) \cap C^2(E)$. The K-envelope of u is the function $\bar{u}^K : \mathbb{R}^n \to \mathbb{R}$ given by

$$\bar{u}^K(x) := \sup\{a + \langle \xi, x\rangle \ : \ \xi \in K, \quad a + \langle \xi, y\rangle \leq u(y) \quad \forall y \in \overline{E}\}.$$

The desired coupling corresponding to a competitor E will be essentially the K-envelope of the solution to the elliptic problem (3.2.4), for K equal to the closure of the optimal bubble B^λ.

Definition 3.5.2 ([Cin+22, Definition 3.9])**.** Let $K \subset \mathbb{R}^n$ be a compact convex set, $E \subset \mathbb{R}^n$ be a bounded open set and $u \in C^0(\overline{E}) \cap C^2(E)$. Given $x \in \mathbb{R}^n$ and $\xi \in K$, we define

$$S_\xi := \arg\min_{x \in \overline{E}} \{u(x) - \langle \xi, x\rangle\}$$

and

$$H(x, \xi, K) := \left\{ \sum_{i=1}^m \lambda_i \nabla^2 u(s_i) \ : \ \begin{array}{l} 1 \leq m \leq n+1 \\ \lambda_i \geq 0, \ \sum \lambda_i = 1 \\ s_i \in S_\xi \cap E \\ x - \sum \lambda_i s_i \in N(\xi, K) \end{array} \right\},$$

where $N(\xi, K)$ is the normal cone of K at ξ, defined as

$$N(\xi, K) := \{v \in \mathbb{R}^n \ : \ \langle v, \xi' - \xi\rangle \leq 0 \text{ for all } \xi' \in K\}.$$

We will need the following

Proposition 3.5.3 ([Cin+22, Proposition 3.10]). *Let $K \subset \mathbb{R}^n$ be a compact convex set, $E \subset \mathbb{R}^n$ be a bounded open set, and $u \in C^0(\overline{E}) \cap C^2(E)$. Assume that for any $\xi \in K$ it holds $S_\xi \subset E$, then $\bar{u}^K : \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,1}$ convex function such that*

$$\nabla \bar{u}^K(\mathbb{R}^n) = \nabla \bar{u}^K(E) = K.$$

Moreover, for any $x \in \mathbb{R}^n$ and any $H_x \in H(x, \nabla \bar{u}^K(x), K) \neq \emptyset$, it holds $\nabla^2 \bar{u}^K(x) \le H_x$.

Lemma 3.5.4 ([Cin+22, Lemma A.1]). *Let $\{\lambda_i\}_{i=1,\dots,m}$ be positive real numbers with $s := \lambda_1 + \cdots + \lambda_m \ge 1$ and let $\{x_i\}_{i=1,\dots,m}$ be nonnegative real numbers. If $\sum \lambda_i x_i \le cs$ for some $c > 0$, then it holds*

$$\sum_{i=1}^m \lambda_i (x_i - c)^2 \le \frac{8}{3} \frac{c^{2-s} s^3}{\min_{i=1,\dots,m} \lambda_i^2} \left(c^s - x_1^{\lambda_1} \cdots x_m^{\lambda_m} \right).$$

We will need to associate a coupling only to Lipschitz-regular connected competitors, having C^1 relative boundary in $\mathbb{R}^n \setminus H$. This is established in the next result, whose proof is analogous to the one in [Cin+22].

Proposition 3.5.5. *There exists $\hat{C} = \hat{C}(n, \lambda) > 0$ such that the following holds. Let $E \subset \mathbb{R}^n \setminus H$ be a connected bounded open set. Suppose that E has Lipschitz boundary and that $\partial E \cap \{x_n \ge 0\}$ is a hypersurface of class C^1 with boundary.*
Then there exists a 1-Lipschitz convex function $\Psi : \mathbb{R}^n \to \mathbb{R}$ of class $C^{1,1}$ such that $\Delta \Psi \le \frac{P_\lambda(E)}{|E|}$ and such that $\nabla \Psi(\mathbb{R}^n) = \nabla \Psi(E) = B^\lambda$ up to negligible sets. Moreover, if $|E| = |B^\lambda|$ and $D_\lambda(E) \le 1$, then

$$\int_E |\nabla^2 \Psi - \mathrm{id}| \, \mathrm{d}x \le \hat{C} \sqrt{D_\lambda(E)} \tag{3.5.1}$$

$$\int_{\partial^* E \cap (\mathbb{R}^n \setminus H)} (1 - |\nabla \Psi|) \, \mathrm{d}\mathcal{H}^{n-1} \le \hat{C} D_\lambda(E). \tag{3.5.2}$$

Proof. Let us assume first that $\partial E \setminus \partial H$ is a smooth hypersurface with smooth boundary intersecting ∂H orthogonally. Let $u : E \to \mathbb{R}$ be a solution of (3.2.4) and $(K_i)_{i \in \mathbb{N}}$ a sequence of compact convex sets such that $K_i \subset\subset \mathring{K}_{i+1}$ and $\cup_{i \in \mathbb{N}} K_i = B^\lambda$. We showed in the proof of Theorem 3.2.3 that for any $\xi \in B^\lambda$ the minimum of $u(x) - \langle \xi, x \rangle$ cannot be achieved on the boundary of E. Moreover, at any point $x \in E$ such that $\nabla^2 u(x) \ge 0$, it holds

$$0 \le \nabla^2 u(x) \le \Delta u(x) \, \mathrm{id} = \frac{P_\lambda(E)}{|E|} \mathrm{id}.$$

Therefore, recalling Remark 3.2.2, by Proposition 3.5.3 we get that $\bar{u}^{K_i}$ is a sequence of 1-Lipschitz functions, being suprema of 1-Lipschitz functions, that is uniformly bounded in $C^{1,1}$ on compact sets; hence up to subsequence it converges to a limit function Ψ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. Since $\nabla \bar{u}^{K_i}(\mathbb{R}^n) = \nabla \bar{u}^{K_i}(E) = K_i$, then $\nabla \Psi(\mathbb{R}^n) = \nabla \Psi(\overline{E}) = \overline{B^\lambda}$, Ψ is a convex function of class $C^{1,1}$, and writing the inequality $\Delta \bar{u}^{K_i} \le \frac{P_\lambda(E)}{|E|}$ in the sense of distributions, one checks that it readily passes to the limit as $i \to +\infty$ for the function Ψ.
To prove that $|\nabla \Psi(E) \Delta B^\lambda| = 0$, let $Z \subset E$ be a compact set and notice that

$$|\nabla \bar{u}^{K_i}(E \setminus Z)| \le c(n, |E|, P_\lambda(E)) |E \setminus Z|,$$

$$|\nabla \bar{u}^{K_i}(Z)| = |\nabla \bar{u}^{K_i}(E \setminus (E \setminus Z))| \ge |\nabla \bar{u}^{K_i}(E)| - |\nabla \bar{u}^{K_i}(E \setminus Z)| = |K_i| - c|E \setminus Z|.$$

Passing to the limit we find

$$|\nabla \Psi(Z)| \ge \left| \limsup_i \nabla \bar{u}^{K_i}(Z) \right| \ge \limsup_i |\nabla \bar{u}^{K_i}(Z)| \ge |B^\lambda| - c|E \setminus Z|,$$

hence letting $Z \nearrow E$, we get that $\nabla \Psi(\mathbb{R}^n) = \nabla \Psi(E) = B^\lambda$ up to negligible sets.
Suppose now that E is a generic connected set as in the assumptions. If $n \ge 3$ we can apply the above argument to a sequence of sets E_i approximating E given by Lemma 2.4.4, suitably modified connecting possibly disconnected components with thin tubes vanishing in the limit. If $n = 2$, then $\partial E \setminus \partial H$ is a union of C^1 curves, which thus can

be approximated by smooth ones touching ∂H orthogonally preserving the connectedness of the set. Applying the first part of the proof on the approximating sequence E_i we get a corresponding sequence of functions Ψ_i uniformly bounded in $C^{1,1}$ on compact sets, hence converging in C^1_{loc} up to subsequence to a convex function Ψ of class $C^{1,1}$ with $\Delta\Psi \le P_\lambda(E)/|E|$. Also, since $\nabla\Psi_i(\mathbb{R}^n) = \overline{B^\lambda}$, then $\nabla\Psi(\mathbb{R}^n) \subset \overline{B^\lambda}$ by C^1_{loc}-convergence. Moreover, since E has Lipschitz boundary, there exists a sequence of compact sets $Z_j \subset E$ such that $Z_j \subset E_i$ for any $i \ge i_j$ and such that $Z_j \nearrow E$. Hence one can repeat the above argument with Ψ_i, E, Z_j in place of $\bar{u}^{K_i}, E, Z$, respectively, to deduce that

$$|\nabla\Psi_i(Z_j)| \ge |B^\lambda| - c(n, |E_i|, P_\lambda(E_i))|E_i \setminus Z_j| \ge |B^\lambda| - c(n, |E|, P_\lambda(E))|E_i \setminus Z_j|.$$

Letting $i \to \infty$ first, and then $j \to \infty$, we get that $\nabla\Psi(\mathbb{R}^n) = \nabla\Psi(E) = B^\lambda$ up to negligible sets.
We now prove (3.5.1) and (3.5.2). The symbol $\hat{C}$ shall denote a positive constant depending on n, λ changing from line to line. By the area formula, the arithmetic-geometric mean inequality and the properties of Ψ we get

$$\begin{aligned} |B^\lambda| = |\nabla\Psi(E)| &\le \int_E \det(\nabla^2\Psi)\,\mathrm{d}\mathcal{H}^n \le \int_E \left(\frac{\Delta\Psi}{n}\right)^n \mathrm{d}x \le \int_E \left(\frac{P_\lambda(E)}{n|E|}\right)^n \mathrm{d}x = \int_E \left(\frac{P_\lambda(E)}{n|B^\lambda|}\right)^n \mathrm{d}x \\ &= \left(\frac{P_\lambda(E)}{P_\lambda(B^\lambda)}\right)^n |B^\lambda| = (1 + D_\lambda(E))^n |B^\lambda|. \end{aligned} \tag{3.5.3}$$

Hence

$$\int_E \left(\left(\frac{P_\lambda(E)}{n|E|}\right)^n - \det(\nabla^2\Psi)\right) \mathrm{d}x \le |B^\lambda|(1 + D_\lambda(E))^n - |B^\lambda| \le \hat{C} D_\lambda(E). \tag{3.5.4}$$

By Lemma 3.5.4 applied with $m = n$, $\lambda_1 = \ldots = \lambda_n = 1$, $(x_1, \ldots, x_n)$ equal to the eigenvalues of $\nabla^2\Psi$ and $c = \frac{P_\lambda(E)}{n|E|} = \frac{P_\lambda(E)}{P_\lambda(B^\lambda)} \ge 1$, we obtain

$$\begin{aligned} |\nabla^2\Psi - \mathrm{id}|^2 &\le 2|\nabla^2\Psi - c\,\mathrm{id}|^2 + 2|(c-1)\mathrm{id}|^2 \le \hat{C}\left(\left(\left(\frac{P_\lambda(E)}{n|E|}\right)^n - \det(\nabla^2\Psi)\right) + 2n\left(\frac{P_\lambda(E) - n|E|}{n|E|}\right)^2\right) \\ &\le \hat{C}\left(\left(\frac{P_\lambda(E)}{n|E|}\right)^n - \det(\nabla^2\Psi) + D_\lambda(E)^2\right). \end{aligned} \tag{3.5.5}$$

Therefore by (3.5.4) and (3.5.5) we get

$$\int_E |\nabla^2\Psi - \mathrm{id}|^2\,\mathrm{d}x \le \hat{C} D_\lambda(E),$$

which implies (3.5.1).
Arguing as in (3.5.3), by the divergence theorem and using that $\langle \nabla\Psi, -e_n\rangle \le -\lambda$ since $\nabla\Psi(\mathbb{R}^n) \subset \overline{B^\lambda}$, we get

$$\begin{aligned} |B^\lambda| &\le \int_E \left(\frac{\Delta\Psi}{n}\right)^n \mathrm{d}x \le \frac{P_\lambda(E)^{n-1}}{n^n|E|^{n-1}} \int_E \Delta\Psi\,\mathrm{d}x = \frac{P_\lambda(E)^{n-1}}{n^n|E|^{n-1}} \int_{\partial^* E} \left\langle \nabla\Psi, \nu^E \right\rangle \mathrm{d}\mathcal{H}^{n-1} \\ &\le \frac{P_\lambda(E)^{n-1}}{n^n|E|^{n-1}} \left(\int_{\partial^* E\cap(\mathbb{R}^n\setminus H)} \left\langle \nabla\Psi, \nu^E \right\rangle \mathrm{d}\mathcal{H}^{n-1} + \int_{\partial^* E\cap\partial H} \langle \nabla\Psi, -e_n\rangle\,\mathrm{d}\mathcal{H}^{n-1}\right) \\ &\le \frac{P_\lambda(E)^{n-1}}{n^n|E|^{n-1}} \left(\int_{\partial^* E\cap(\mathbb{R}^n\setminus H)} \left\langle \nabla\Psi, \nu^E \right\rangle - 1\,\mathrm{d}\mathcal{H}^{n-1} + P_\lambda(E)\right) \\ &\le \frac{P_\lambda(E)^n}{n^n|E|^{n-1}} - \frac{P_\lambda(E)^{n-1}}{n^n|E|^{n-1}} \int_{\partial^* E\cap(\mathbb{R}^n\setminus H)} (1 - |\nabla\Psi|)\,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

Rearranging terms, since $D_\lambda(E) \le 1$ and $|E| = |B^\lambda|$, we obtain

$$\int_{\partial^* E\cap(\mathbb{R}^n\setminus H)} (1 - |\nabla\Psi|)\,\mathrm{d}\mathcal{H}^{n-1} \le \frac{P_\lambda(E)^n - P_\lambda(B^\lambda)^n}{P_\lambda(E)^{n-1}} \le \hat{C} D_\lambda(E).$$

□

We now want to translate the quantitative estimates obtained on the coupling in Proposition 3.5.5 into quantitative estimates on the asymmetry of a competitor. We will need some technical results first.
The next lemma is analogous to [Cin+22, Lemma 6.2], but with a varying range of parameters that here must depend on λ.

Lemma 3.5.6. *Let $E \subset [0,\infty)$ be a 1-dimensional set of locally finite perimeter with $|E| < \infty$ and set*

$$r_\lambda := \min\left\{\sqrt{1-\lambda^2}, 1-\lambda\right\}, \qquad R_\lambda := \max\left\{\sqrt{1-\lambda^2}, 1-\lambda\right\}.$$

There exists $\hat{c} = \hat{c}(n,\lambda) > 0$ such that for any $\frac{7}{8}r_\lambda \le l \le \frac{9}{8}R_\lambda$ there holds

$$\int_{E\Delta[0,l]} t^{n-1}\,\mathrm{d}t \le \hat{c}\left(\int_{\left[0,\frac{r_\lambda}{2}\right]\setminus E} t^{n-1}\,\mathrm{d}t + \int_{\partial^* E} t^{n-1}|l-t|\,\mathrm{d}\mathcal{H}^0\right).$$

Proof. It holds $E\Delta[0,l] = ((l,\infty)\cap E)\cup([0,l]\setminus E)$. We claim that

$$\max\left\{\int_{[l,\infty)\cap E} t^{n-1}\,\mathrm{d}t, \int_{\left[\frac{r_\lambda}{2},l\right]\setminus E} t^{n-1}\,\mathrm{d}t\right\} \le \hat{c}(n,\lambda)\left(\int_{\left[0,\frac{r_\lambda}{2}\right]\setminus E} t^{n-1}\,\mathrm{d}t + \int_{\partial^* E} t^{n-1}|l-t|\,\mathrm{d}\mathcal{H}^0(t)\right). \tag{3.5.6}$$

We will estimate the two terms on the left-hand side separately.
Without loss of generality, suppose $[l,\infty)\cap E \neq \emptyset$. Since $|E| < \infty$, then $\partial^* E\cap[l,\infty)$ is nonempty and we can assume it has finite supremum $\bar{t}$ (otherwise the right hand side in (3.5.6) equals $+\infty$). In particular the right-hand side in (3.5.6) is finite. It holds

$$\int_{[l,\infty)\cap E} t^{n-1}\,\mathrm{d}t \le \int_l^{\bar{t}} t^{n-1}\,\mathrm{d}t \le \bar{t}^{n-1}|\bar{t}-l| \le \int_{\partial^* E} t^{n-1}|l-t|\,\mathrm{d}\mathcal{H}^0(t),$$

and the first term in the left-hand side of (3.5.6) is bounded as wished.
Let us now consider $\int_{\left[\frac{r_\lambda}{2},l\right]\setminus E} t^{n-1}\,\mathrm{d}t$. Its value is a priori bounded by $\left(\frac{9}{8}R_\lambda\right)^n$. If $\partial^* E\cap\left[\frac{r_\lambda}{4},\frac{3}{4}r_\lambda\right]\neq\emptyset$ and τ is one of its elements, then

$$\int_{\partial^* E} t^{n-1}|l-t|\,\mathrm{d}\mathcal{H}^0(t) \ge \tau^n|l-\tau| \ge \left(\frac{r_\lambda}{4}\right)^n\frac{1}{8}r_\lambda \ge \hat{c}(n,\lambda)\left(\frac{9}{8}R_\lambda\right)^{n-1}\left|\frac{9}{8}R_\lambda - \frac{r_\lambda}{2}\right| \ge \hat{c}(n,\lambda)\int_{\left[\frac{r_\lambda}{2},l\right]\setminus E} t^{n-1}\,\mathrm{d}t.$$

So from now on we can assume that $\partial^* E\cap\left[\frac{r_\lambda}{4},\frac{3}{4}r_\lambda\right]=\emptyset$. If $\left[\frac{r_\lambda}{4},\frac{3}{4}r_\lambda\right]\setminus E\neq\emptyset$, then $E\cap\left[\frac{r_\lambda}{4},\frac{3}{4}r_\lambda\right]=\emptyset$ and

$$\begin{aligned}\int_{\left[0,\frac{r_\lambda}{2}\right]\setminus E} t^{n-1}\,\mathrm{d}t + \int_{\partial^* E} t^{n-1}|l-t|\,\mathrm{d}\mathcal{H}^0(t) &\ge \int_{\left[\frac{r_\lambda}{4},\frac{r_\lambda}{2}\right]} t^{n-1}\,\mathrm{d}t = \hat{c}(n,\lambda) \ge \hat{c}(n,\lambda)\left(\frac{9}{8}R_\lambda\right)^{n-1}\left|\frac{9}{8}R_\lambda-\frac{r_\lambda}{2}\right| \\ &\ge \hat{c}(n,\lambda)\int_{\left[\frac{r_\lambda}{2},l\right]\setminus E} t^{n-1}\,\mathrm{d}t.\end{aligned}$$

So we can further assume $\left[\frac{r_\lambda}{4},\frac{3}{4}r_\lambda\right]\subset E$. If $\partial^* E\cap\left[\frac{r_\lambda}{4},l\right]=\emptyset$, then $\left[\frac{r_\lambda}{2},l\right]\subset E$ and there is nothing to prove. Finally, if $\partial^* E\cap\left[\frac{r_\lambda}{4},l\right]\neq\emptyset$, let us denote by $\underline{t}$ the infimum of $\partial^* E\cap\left[\frac{r_\lambda}{4},l\right]$. Then

$$\begin{aligned}\int_{\left[\frac{r_\lambda}{2},l\right]\setminus E} t^{n-1}\,\mathrm{d}t &\le \int_{\underline{t}}^l t^{n-1}\,\mathrm{d}t \le l^{n-1}|l-\underline{t}| = \underline{t}^{n-1}|l-\underline{t}|\left(\frac{l}{\underline{t}}\right)^{n-1} \le \hat{c}(n,\lambda)\underline{t}^{n-1}|l-\underline{t}| \\ &\le \hat{c}(n,\lambda)\int_{\partial^* E} t^{n-1}|l-t|\,\mathrm{d}\mathcal{H}^0(t).\end{aligned}$$

This concludes the proof of the claim (3.5.6). Hence

$$\begin{aligned}\int_{E\Delta[0,l]} t^{n-1}\,\mathrm{d}t &\le \max\left\{\int_{[l,\infty)\cap E} t^{n-1}\,\mathrm{d}t, \int_{\left[\frac{r_\lambda}{2},l\right]\setminus E} t^{n-1}\,\mathrm{d}t\right\} + \int_{\left[0,\frac{r_\lambda}{2}\right]\setminus E} t^{n-1}\,\mathrm{d}t \\ &\overset{(3.5.6)}{\le} \hat{c}(n,\lambda)\left(\int_{\left[0,\frac{r_\lambda}{2}\right]\setminus E} t^{n-1}\,\mathrm{d}t + \int_{\partial^* E} t^{n-1}|l-t|\,\mathrm{d}\mathcal{H}^0(t)\right) + \int_{\left[0,\frac{r_\lambda}{2}\right]\setminus E} t^{n-1}\,\mathrm{d}t,\end{aligned}$$

and the proof follows. $\square$

We shall also need the following standard technical result stating that a Vol'pert property holds for the intersections of a set of finite perimeter with rays from the origin, cf. [Vol67]. The proof follows, for example, by adapting the proof of [Fus04, Theorem 3.21] working in polar coordinates rather than in Cartesian coordinates.

Lemma 3.5.7. *Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter with $|E| < +\infty$. If $\vartheta \in \mathbb{S}^{n-1} \cap (\mathbb{R}^n \setminus H)$, we define*

$$E_\vartheta := \{t \geq 0 \ : \ t\vartheta \in E\}.$$

Then, for $\mathcal{H}^{n-1}$-almost every $\vartheta \in \mathbb{S}^{n-1} \cap (\mathbb{R}^n \setminus H)$, E_ϑ is a 1-dimensional set of locally finite perimeter such that

$$\partial^* E_\vartheta \cap \{t > 0\} = \{t > 0 \ : \ t\vartheta \in \partial^* E\}.$$

Moreover, if $\eta \in L^1(\partial^* E)$ is nonnegative, we have

$$\int_{\partial^* E \setminus H} \eta \, \mathrm{d}\mathcal{H}^{n-1} \geq \int_{\mathbb{S}^{n-1} \setminus H} \left(\int_{\partial^* E_\vartheta} t^{n-1} \eta(t\vartheta) \, \mathrm{d}\mathcal{H}^0(t) \right) \mathrm{d}\mathcal{H}^{n-1}(\vartheta).$$

Combining Lemma 3.5.6 with Lemma 3.5.7 we get the following result that estimates the symmetric difference of a competitor with a bubble that is just close to a standard bubble $B^\lambda(v, x)$. The result is analogous to [Cin+22, Proposition 6.1].

Lemma 3.5.8. *There exist $\varepsilon, \tilde{c} > 0$ depending on n, λ such that the following holds. Let $E \subset \mathbb{R}^n \setminus H$ be a bounded set of finite perimeter. If $\left| E \cap B_{\frac{r_\lambda}{2}}(0) \right| \geq \frac{1}{2} \left| B_{\frac{r_\lambda}{2}}(0) \setminus H \right|$, then*

$$|E \Delta (B_1(x_0) \setminus H)| \leq \tilde{c} \int_{\partial^* E \setminus H} ||x - x_0| - 1| \, \mathrm{d}\mathcal{H}^{n-1}(x),$$

for any $x_0 \in \mathbb{R}^n$ such that $|x_0 - (0, \dots, 0, -\lambda)| < \varepsilon$.

Proof. If ε is sufficiently small, depending only on n, λ, then for any $\vartheta \in \mathbb{S}^{n-1} \cap (\mathbb{R}^n \setminus H)$ the set $\{t > 0 \ : \ |t\vartheta - x_0| < 1\}$ is an open segment $(0, t(\vartheta))$, with $t(\vartheta)$ close to the number

$$T_\vartheta \in \left[\min\left\{ \sqrt{1-\lambda^2}, 1-\lambda \right\}, \max\left\{ \sqrt{1-\lambda^2}, 1-\lambda \right\} \right] =: [r_\lambda, R_\lambda]$$

such that $|T_\vartheta \vartheta + \lambda e_n| = 1$. In particular, if ε is sufficiently small, then $\frac{9}{8} R_\lambda \geq t(\vartheta) \geq \frac{7}{8} r_\lambda$ for any $\vartheta \in \mathbb{S}^{n-1} \cap (\mathbb{R}^n \setminus H)$. As before, for any $\vartheta \in \mathbb{S}^{n-1} \setminus H$ let

$$E_\vartheta := \{t \geq 0 \ : \ t\vartheta \in E\}.$$

By coarea formula we get

$$|E \Delta B_1(x_0) \cap (\mathbb{R}^n \setminus H)| = \int_{\mathbb{S}^{n-1} \setminus H} \left(\int_{E_\vartheta \Delta [0, t(\vartheta)]} t^{n-1} \, \mathrm{d}t \right) \mathrm{d}\mathcal{H}^{n-1}(\vartheta).$$

By Lemma 3.5.6 we obtain

$$|E \Delta B_1(x_0) \setminus H| \leq c(n, \lambda) \int_{\mathbb{S}^{n-1} \setminus H} \left(\int_{\left[0, \frac{r_\lambda}{2}\right] \setminus E_\vartheta} t^{n-1} \, \mathrm{d}t + \int_{\partial^* E_\vartheta} t^{n-1} |t(\vartheta) - t| \, \mathrm{d}\mathcal{H}^0 \right) \mathrm{d}\mathcal{H}^{n-1}(\vartheta). \tag{3.5.7}$$

For every $t > 0$, we claim that

$$||t\vartheta - x_0| - 1| \geq c(n, \lambda) |t - t(\vartheta)|. \tag{3.5.8}$$

Note that there exists $\delta = \delta(n, \lambda, \varepsilon) \in (0, r_\lambda/8)$ such that for $t \in [t(\vartheta) - \delta, t(\vartheta) + \delta]$ there holds

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} |t\vartheta - x_0| \right| = \left| \langle (t\vartheta - x_0) / \left| t\vartheta - x_0 \right|, \vartheta \rangle \right| \geq c(n, \lambda, \varepsilon) > 0.$$

Then, for $t \in [t(\vartheta) - \delta, t(\vartheta) + \delta]$,

$$|t(\vartheta) - t| \leq c(n, \lambda) ||t\vartheta - x_0| - 1|,$$

and in this case the claim follows. Regarding the remaining cases, note that

$$\frac{||t\vartheta - x_0| - 1|}{|t(\vartheta) - t|} \to 1 \qquad \text{as } |t| \to +\infty$$

and the claim follows for $t \geq R = R(n, \lambda, \varepsilon) > 0$ big enough. Finally, if $0 < t < R$ and $t \notin [t(\vartheta) - \delta, t(\vartheta) + \delta]$, then

$$\big||t\vartheta - x_0| - 1\big| \geq c(n, \lambda, \delta) > 0$$

$$|t(\vartheta) - t| \leq c(n, \lambda, R)$$

hence the claim follows as well.
Therefore

$$\int_{\mathbb{S}^{n-1}\setminus H} \int_{\partial^* E_\vartheta} t^{n-1}|t - t(\vartheta)| \, d\mathcal{H}^0(t) \, d\mathcal{H}^{n-1}(\vartheta) \overset{(3.5.8)}{\leq} c(n, \lambda) \int_{\mathbb{S}^{n-1}\setminus H} \int_{\partial^* E_\vartheta} t^{n-1}||t\vartheta - x_0| - 1| \, d\mathcal{H}^0(t) \, d\mathcal{H}^{n-1}(\vartheta). \tag{3.5.9}$$

By Lemma 3.5.7 we deduce

$$\int_{\partial^* E\setminus H} ||x - x_0| - 1| \, d\mathcal{H}^{n-1}(x) \geq \int_{\mathbb{S}^{n-1}\setminus H} \left(\int_{\partial^* E_\vartheta} t^{n-1}||t\vartheta - x_0| - 1| \, d\mathcal{H}^0(t) \right) d\mathcal{H}^{n-1}(\vartheta). \tag{3.5.10}$$

Since $||x - x_0| - 1| \geq c(n, \lambda) > 0$ in $B_{\frac{r_\lambda}{2}}(0)$, by coarea formula and relative isoperimetric inequality we get

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}\setminus H} \left(\int_{\left[0, \frac{r_\lambda}{2}\right]\setminus E_\vartheta} t^{n-1} \, dt \right) d\mathcal{H}^{n-1}(\vartheta) &= \left| \left(B_{\frac{r_\lambda}{2}}(0) \setminus H \right) \setminus E \right| = \left| \left(B_{\frac{r_\lambda}{2}}(0) \setminus H \right) \setminus E \right|^{\frac{n-1}{n}} \left| \left(B_{\frac{r_\lambda}{2}}(0) \setminus H \right) \setminus E \right|^{\frac{1}{n}} \\
&\leq c(n, \lambda) \int_{\partial^* E \cap \left(B_{\frac{r_\lambda}{2}}(0)\setminus H \right)} d\mathcal{H}^{n-1} \\
&\leq c(n, \lambda) \int_{\partial^* E\setminus H} ||x - x_0| - 1| \, d\mathcal{H}^{n-1}(x).
\end{aligned} \tag{3.5.11}$$

Putting together (3.5.7), (3.5.9), (3.5.10) and (3.5.11), the proof follows. □

We can finally show that if a suitably regular Schwarz-symmetric set satisfies a trace inequality, then the quantitative estimates in Proposition 3.5.5 imply a quantitative isoperimetric inequality.

Proposition 3.5.9. *There exists $\delta_6 = \delta_6(n, \lambda) > 0$ such that for any $c_T > 0$ there exists $\gamma = \gamma(n, \lambda, c_T) > 0$ such that the following holds. Let $E \subset \mathbb{R}^n \setminus H$ be a bounded connected open set with $|E| = |B^\lambda|$. Suppose that E has Lipschitz boundary and that $\partial E \cap \{x_n \geq 0\}$ is a hypersurface of class C^1 with boundary. Assume that E is Schwarz-symmetric with respect to the n-th axis and that there exists a constant c_T such that for every function $f \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ there is a constant $c \in \mathbb{R}$ such that the following holds*

$$\int_E d|Df|(x) \geq c_T \int_{\partial^* E\cap(\mathbb{R}^n\setminus H)} \mathrm{tr}_E(|f - c|) \, d\mathcal{H}^{n-1}(x). \tag{3.5.12}$$

If $D_\lambda(E) < \delta_6$, then

$$\alpha_\lambda^2(E) \leq \gamma D_\lambda(E).$$

Proof. Let Ψ be given by Proposition 3.5.5. By (3.5.12) and (3.5.1) we get

$$\int_{\partial^* E\cap(\mathbb{R}^n\setminus H)} |(\nabla\Psi - x) + x_0| \, d\mathcal{H}^{n-1}(x) \leq c(n, \lambda, c_T)\sqrt{D_\lambda(E)},$$

where $x_0 = (x_0^1, \dots, x_0^n)$ is the vector whose i-th component is the constant c of (3.5.12) corresponding to the i-th component of $\nabla\Psi - x$. Therefore

$$\begin{aligned}
\int_{\partial^* E\cap(\mathbb{R}^n\setminus H)} \big||x - x_0| - 1\big| \, d\mathcal{H}^{n-1}(x) &\leq \int_{\partial^* E\cap(\mathbb{R}^n\setminus H)} |\nabla\Psi - (x - x_0)| + |1 - |\nabla\Psi|| \, d\mathcal{H}^{n-1}(x) \\
&\overset{(3.5.2)}{\leq} c(n, \lambda, c_T)\sqrt{D_\lambda(E)}.
\end{aligned} \tag{3.5.13}$$

We observe that if $\varepsilon = \varepsilon(n, \lambda)$ is given by Lemma 3.5.8, then for δ_6 small enough depending on ε, we ensure that $|x_0 - (0, \dots, 0, -\lambda)| < \varepsilon$. Indeed, if for every $i \in \mathbb{N}$ there were E_i satisfying the hypotheses of Proposition 3.5.9 such that $D_\lambda(E) < \frac{1}{i}$ with corresponding $x_{0,i}$ verifying $|x_{0,i} - (0, \dots, 0, -\lambda)| \geq \varepsilon$, passing to limit in (3.5.13) we would get a contradiction with the fact that E_i converges to $B^\lambda(|B^\lambda|)$.
Hence we can apply Lemma 3.5.8. Since E is Schwarz-symmetric, we get

$$\begin{aligned} |E \Delta B_1(0, \dots, 0, x_0^n) \cap (\mathbb{R}^n \setminus H)| &\leq |E \Delta B_1(x_0) \cap (\mathbb{R}^n \setminus H)| \\ &\leq c(n, \lambda, c_T)\sqrt{D_\lambda(E)}. \end{aligned} \tag{3.5.14}$$

Arguing as above, up to taking a smaller δ_6, we can assume that $|x_0^n + \lambda| \leq |x_0 - (0, \dots, 0, -\lambda)|$ is so small that

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} |B_1(0, \dots, 0, t) \setminus H| \right| \geq \frac{1}{2}\omega_{n-1}(1 - \lambda^2)^{\frac{n-1}{2}},$$

for any $t \in [-|x_0^n + \lambda|, |x_0^n + \lambda|]$. Hence

$$\begin{aligned} c(n, \lambda, c_T)\sqrt{D_\lambda(E)} &\overset{(3.5.14)}{\geq} \Big| |B_1(0, \dots, 0, x_0^n) \setminus H| - |E| \Big| \\ &= \Big| |B_1(0, \dots, 0, x_0^n) \setminus H| - |B_1(0, \dots, 0, -\lambda) \setminus H| \Big| \\ &\geq \frac{1}{2}\omega_{n-1}(1 - \lambda^2)^{\frac{n-1}{2}} |x_0^n + \lambda|, \end{aligned}$$

which implies

$$|x_0^n + \lambda| \leq c(n, \lambda, c_T)\sqrt{D_\lambda(E)}, \tag{3.5.15}$$

for a suitable constant. Therefore

$$\begin{aligned} \Big| \big(B_1(0, \dots, 0, x_0^n) \setminus H\big) \,\Delta\, \big(B_1(0, \dots, 0, -\lambda) \setminus H\big) \Big| &\leq c(n, \lambda)|x_0^n + \lambda| \\ &\overset{(3.5.15)}{\leq} c(n, \lambda, c_T)\sqrt{D_\lambda(E)}, \end{aligned} \tag{3.5.16}$$

where in the first inequality we used that $t \mapsto \Big| \big(B_1(0, \dots, 0, t) \setminus H\big) \,\Delta\, \big(B_1(0, \dots, 0, -\lambda) \setminus H\big) \Big|$ is Lipschitz for some Lipschitz constant $c(n, \lambda) > 0$.
Finally

$$\begin{aligned} &|E \Delta B_1(0, \dots, 0, x_0^n) \cap (\mathbb{R}^n \setminus H)| \\ &\quad \geq |B^\lambda(|B^\lambda|) \Delta E| - \Big| \big(B_1(0, \dots, 0, x_0^n) \setminus H\big) \,\Delta\, \big(B_1(0, \dots, 0, -\lambda) \setminus H\big) \Big| \\ &\quad \overset{(3.5.16)}{\geq} \alpha_\lambda(E) - c(n, \lambda, c_T)\sqrt{D_\lambda(E)}. \end{aligned}$$

□

In the next lemma we observe that optimal bubbles do satisfy trace inequalities.

Lemma 3.5.10. *There exists $\bar{c} = \bar{c}(n, \lambda) > 0$ such that for every function $f \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ there is a constant $c \in \mathbb{R}$ such that the following holds*

$$\int_{B^\lambda(|B^\lambda|)} \mathrm{d}|Df|(x) \geq \bar{c} \int_{\partial B^\lambda(|B^\lambda|) \setminus H} \mathrm{tr}_{B^\lambda(|B^\lambda|)}(|f - c|) \,\mathrm{d}\mathcal{H}^{n-1}(x). \tag{3.5.17}$$

Proof. We may assume $f \in \mathrm{Lip}_c(\mathbb{R}^n)$. If we set $f_+ := f|_{\{x_n \geq 0\}}$, we define

$$\bar{f}(x) := \begin{cases} f_+(x) & \text{if} \quad x \in \{x_n \geq 0\}, \\ f_+(-x) & \text{if} \quad x \in \{x_n < 0\}. \end{cases}$$

Hence $\bar{f}$ is Lipschitz. Let B be the union of $B^\lambda(|B^\lambda|)$ with its reflection across $\{x_n = 0\}$. By the Poincaré inequality 2.3.2, there exist $c = c(n, \lambda) > 0$ and $C = C(n, \lambda, \bar{f}) > 0$ such that

$$\int_B |\bar{f} - C| \,\mathrm{d}x \leq c(n, \lambda) \int_B |\nabla \bar{f}| \,\mathrm{d}x.$$

By the boundary trace theorem 2.3.4 applied to the function $\bar{f} - C$, we get

$$2\int_{\partial B^\lambda(|B^\lambda|)\cap\mathbb{R}^n\setminus H} |f - C|\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\partial B} |\bar{f} - C|\,\mathrm{d}\mathcal{H}^{n-1} \le c(n,\lambda)\left(\int_B |\bar{f} - C|\,\mathrm{d}x + \int_B |\nabla(\bar{f} - C)|\,\mathrm{d}x\right)$$
$$= c(n,\lambda)\left(\int_B |\bar{f} - C|\,\mathrm{d}x + \int_B |\nabla \bar{f}|\,\mathrm{d}x\right) \le c(n,\lambda)\int_B |\nabla \bar{f}|\,\mathrm{d}x$$
$$= c(n,\lambda)\int_{B^\lambda(|B^\lambda|)} |\nabla f|\,\mathrm{d}x.$$

□

We now introduce a notion of C^1-distance from $B^\lambda(|B^\lambda|)$ for sets in the half-space $\mathbb{R}^n \setminus H$, and we deduce that Schwarz-symmetric sets sufficiently close in C^1 to $B^\lambda(|B^\lambda|)$ enjoy a quantitative isoperimetric inequality.

Definition 3.5.11. Let $\varphi_\lambda : \partial B_1 \setminus H \to \partial B^\lambda(|B^\lambda|) \setminus H$ be such that $\partial B^\lambda(|B^\lambda|) \setminus H = \{\varphi_\lambda(x)\, x \;:\; x \in \partial B_1 \setminus H\}$. Let $E \subset \mathbb{R}^n \setminus H$ be a bounded open set. Suppose that E has Lipschitz boundary and that $\partial E \cap \{x_n \ge 0\}$ is a hypersurface of class C^1 with boundary. Assume that E is Schwarz-symmetric. Suppose that there exists a C^1 functions

$$\varphi : \partial B_1 \setminus H \to \partial E \setminus H$$

whose graph parametrizes the boundary of E in $\mathbb{R}^n \setminus H$, that is

$$\partial E \setminus H = \{\varphi(x)\, x \;:\; x \in \partial B_1 \setminus H\}.$$

We define the C^1 distance of E to $B^\lambda(|B^\lambda|)$ by $\mathrm{d}_{C^1}(E, B^\lambda(|B^\lambda|)) := \|\varphi - \varphi_\lambda\|_{C^1(\partial B_1 \cap \mathbb{R}^n \setminus H)}$.
A sequence of sets E_j as above is said to converge to $B^\lambda(|B^\lambda|)$ in C^1 if $\mathrm{d}_{C^1}(E_j, B^\lambda(|B^\lambda|)) \to 0$ as $j \to +\infty$.

Corollary 3.5.12. *There exist $\hat{\varepsilon}, \hat{\gamma} > 0$ depending only on n, λ such that the following holds. Let $E \subset \mathbb{R}^n \setminus H$ be as in Definition 3.5.11. If $\mathrm{d}_{C^1}(E, B^\lambda(|B^\lambda|)) \le \hat{\varepsilon}$, then*

$$\alpha_\lambda^2(E) \le \hat{\gamma}(n,\lambda) D_\lambda(E).$$

Proof. Let φ, φ_λ be as in Definition 3.5.11. If $\chi : [0,+\infty) \to [0,1]$ is a smooth cut-off function such that $\chi(t) = 0$ for $t < \frac{1}{4}\min\{\sqrt{1-\lambda^2}, 1-\lambda\}$ and such that $\chi(t) = 1$ for $t > \frac{1}{2}\min\{\sqrt{1-\lambda^2}, 1-\lambda\}$, we define the diffeomorphism

$$\psi : \mathbb{R}^n \setminus H \to \mathbb{R}^n \setminus H \qquad \psi(x) = \left(1 - \chi(|x|) + \chi(|x|)\frac{\varphi\left(\frac{x}{|x|}\right)}{\varphi_\lambda\left(\frac{x}{|x|}\right)}\right)x.$$

Note that

$$\|\psi - \mathrm{id}\|_{C^1} \le c\hat{\varepsilon},$$
$$\psi(\partial B^\lambda(|B^\lambda|) \setminus H) = \partial E \setminus H,$$

for some $c = c(n, \lambda, \chi)$, if $\mathrm{d}_{C^1}(E, B^\lambda(|B^\lambda|)) \le \hat{\varepsilon} < 1$.
Let $g \in \mathrm{Lip}_c(\mathbb{R}^n)$ and define $f := g \circ \psi$. If c is the constant in (3.5.17) corresponding to f, then by area formula and (3.5.17) we get

$$\int_{\partial^* E \cap \setminus H} |g - c|\,\mathrm{d}\mathcal{H}^{n-1} \le C(n,\lambda)\int_{\partial B^\lambda \setminus H} |f - c|\,\mathrm{d}\mathcal{H}^{n-1} \le C(n,\lambda)\int_{B^\lambda} |\nabla f|\,\mathrm{d}x$$
$$\le C(n,\lambda)\int_E |\nabla g|\,\mathrm{d}x.$$

Therefore, if $\hat{\varepsilon}$ is small enough, we can apply Proposition 3.5.9 with c_T therein depending on n, λ only, and we get

$$\alpha_\lambda(E) \le \hat{\gamma}(n,\lambda) D_\lambda(E).$$

□

Proof of the first quantitative isoperimetric inequality

We are ready to prove the main quantitative isoperimetric inequality. Let us recall the following immediate result, completely analogous to [Fus15, Lemma 5.3].

Lemma 3.5.13. *The standard bubble $B^\lambda(|B^\lambda|)$ is the unique solution, up to translations along ∂H, of*

$$\min\left\{ P_\lambda(F) + \Lambda \left| |F| - |B^\lambda| \right| \ : \ F \subset \mathbb{R}^n \setminus H \right\},$$

for any $\Lambda > n$.

Proof. By Theorem 3.2.3 we may restrict ourselves to consider rescalings of $B^\lambda(|B^\lambda|)$. Hence we just need to minimize the function

$$[0,+\infty) \ni r \mapsto P_\lambda(rB^\lambda(|B^\lambda|)) + \Lambda \left| |rB^\lambda(|B^\lambda|)| - |B^\lambda(|B^\lambda|)| \right| = |B^\lambda(|B^\lambda|)| \left(nr^{n-1} + \Lambda |r^n - 1| \right).$$

To show that $r = 1$ minimize the above function, it is sufficient to consider $r \in [0,1)$. Since $\Lambda > n$, for $r \in [0,1)$ we have

$$nr^{n-1} + \Lambda |r^n - 1| = nr^{n-1} + \Lambda(1 - r^n) > n(r^{n-1} + 1 - r^n) > n,$$

which proves the claim. □

Proof of Theorem 3.1.1. Let $\hat\gamma(n,\lambda)$ be the constant given by Corollary 3.5.12 and let $\delta_4, \tilde l$ be given by Corollary 3.4.12. By Corollary 3.4.12 it is sufficient to prove that there exists $\delta \in (0,\delta_4)$ such that, if E is a Schwarz-symmetric set contained in $Q_{\tilde l}$ such that $|E| = |B^\lambda|$ and $D_\lambda(E) < \delta$, then $\alpha_\lambda(E) \le 2\hat\gamma(n,\lambda)\sqrt{D_\lambda(E)}$.
We argue by contradiction. Let $\{E_j\}_j$ be a sequence of Schwarz-symmetric sets contained in $Q_{\tilde l}$ such that $|E_j| = |B^\lambda|$, with $P_\lambda(E_j) \to P_\lambda(B^\lambda)$ and

$$\alpha_\lambda(E_j) > 2\hat\gamma(n,\lambda)\sqrt{D_\lambda(E_j)}. \tag{3.5.18}$$

For every j we consider a minimizer F_j of the problem

$$\min\{ P_\lambda(F) + |\alpha_\lambda(F) - \alpha_\lambda(E_j)| + \Lambda ||F| - |E_j|| \ : \ F \text{ Schwarz-symmetric contained in } Q_{\tilde l} \}, \tag{3.5.19}$$

for $\Lambda > 0$ to be chosen large. Up to subsequence, F_j converges in L^1 to a minimizer of $F \mapsto P_\lambda(F) + \alpha_\lambda(F) + \Lambda \left| |F| - |B^\lambda| \right|$, hence, taking $\Lambda > n$, we have that F_j converges to $B^\lambda(|B^\lambda|)$ by Lemma 3.5.13. Also, by comparison with with E_j, we have that $P_\lambda(F_j) \to P_\lambda(B^\lambda)$.
We prove that F_j is a local (Λ_1, r_0)-minimizer in $\mathbb{R}^n \setminus H$, for some $\Lambda_1, r_0 > 0$ and j large. Let us consider a ball $B_r(x) \subset\subset \mathbb{R}^n \setminus H$, with $r < \min\{r_0, d(x,\partial H)\}$, and a set G such that $F_j \Delta G \subset\subset B_r(x)$. Denoting by $(\cdot)^*$ the Schwarz symmetrization with respect to the n-th axis and by $Z := G \cap Q_{\tilde l}$, we have

$$\begin{aligned} P(F_j, \mathbb{R}^n \setminus H) &\le P(Z^*, \mathbb{R}^n \setminus H) + |\alpha_\lambda(Z^*) - \alpha_\lambda(E_j)| - |\alpha_\lambda(F_j) - \alpha_\lambda(E_j)| + \Lambda[||Z| - |E_j|| - ||F_j| - |E_j||] \\ &\le P(Z, \mathbb{R}^n \setminus H) + |\alpha_\lambda(Z^*) - \alpha_\lambda(F_j)| + \Lambda |Z \Delta F_j| \\ &\le P(G, \mathbb{R}^n \setminus H) + |\alpha_\lambda(Z^*) - \alpha_\lambda(F_j)| + \Lambda |G \Delta F_j|. \end{aligned}$$

For r_0 small enough and j sufficiently large we have that $|G| \ge |Z| \ge c(n,\lambda) > 0$. Assume for instance that $\alpha_\lambda(Z^*) \ge \alpha_\lambda(F_j)$ (the opposite case being symmetric), then

$$\begin{aligned} \alpha_\lambda(Z^*) - \alpha_\lambda(F_j) &\le |Z|^{-1} \left(|Z^* \Delta F_j| + |F_j \Delta B^\lambda(|F_j|)| + \left| |Z| - |F_j| \right| \right) - |F_j|^{-1} |F_j \Delta B^\lambda(|F_j|)| \\ &\le c(n,\lambda)|Z \Delta F_j| + c(n,\lambda) \left(|F_j| - |Z| \right) + |Z \Delta F_j| \\ &\le c(n,\lambda)|G \Delta F_j|. \end{aligned}$$

Arguing analogously in case $\alpha_\lambda(Z^*) < \alpha_\lambda(F_j)$, we deduce

$$P(F_j, \mathbb{R}^n \setminus H) \le P(G, \mathbb{R}^n \setminus H) + \Lambda_1 |G \Delta F_j|,$$

for some $\Lambda_1 = (\Lambda, n, \lambda)$.

By Theorem 2.5.3 we know that $\partial^* F_j \cap \{x_n > 0\}$ is a $C^{1,\frac{1}{2}}$ manifold and $\partial F_j \cap \{x_n > 0\} \setminus \partial^* F_j$ has Hausdorff dimension $\leq n-8$. Since F_j is Schwarz-symmetric, if there exists a point $(r\vartheta, t) \in \partial F_j \cap \{x_n > 0\} \setminus \partial^* F_j$ for some $r, t > 0, \vartheta \in \mathbb{S}^{n-2}$, then $(r\vartheta', t) \in \partial F_j \cap \{x_n > 0\} \setminus \partial^* F_j$ for any $\vartheta' \in \mathbb{S}^{n-2}$. Hence $\partial F_j \cap \{x_n > 0\} \setminus \partial^* F_j \subset \{te_n : t > 0\}$. However by Theorem 2.5.3 for every $\varepsilon > 0$ the set $\partial F_j \cap \{x_n \geq \varepsilon\}$ converges to $\partial B^\lambda(|B^\lambda|) \cap \{x_n \geq \varepsilon\}$ in $C^{1,\alpha}$ for any $0 < \alpha < \frac{1}{2}$. Also, for j large we can apply Corollary 3.4.4 which implies that $\mathcal{H}^{n-1}(F_j \cap \{x_n = t\}) \geq A_\lambda$ for a.e. $t \in (0, T_\lambda)$, for some $A_\lambda, T_\lambda > 0$ depending on n, λ. Then points te_n for $t \in (0, T_\lambda/2)$ are points of density 1 for F_j, hence they belong to the interior of F_j. Therefore, for j large enough, $\partial F_j \cap \{x_n > 0\} \setminus \partial^* F_j$ must be empty and $\partial F_j \cap \{x_n > 0\}$ is an axially symmetric hypersurface of class $C^{1,\frac{1}{2}}$.

By the minimality of the F_j, (3.5.18) and Lemma 3.5.13 we observe that

$$
\begin{aligned}
P_\lambda(F_j) + \Lambda \left| |F_j| - |B^\lambda| \right| + \left| \alpha_\lambda(F_j) - \alpha_\lambda(E_j) \right| &\leq P_\lambda(E_j) \\
\leq P_\lambda(B^\lambda(|B^\lambda|)) + \frac{P_\lambda(B^\lambda(|B^\lambda|))}{4\hat{\gamma}^2(n,\lambda)} \alpha_\lambda^2(E_j) &\leq P_\lambda(F_j) + \Lambda \left| |F_j| - |B^\lambda| \right| + \frac{P_\lambda(B^\lambda(|B^\lambda|))}{4\hat{\gamma}^2(n,\lambda)} \alpha_\lambda^2(E_j).
\end{aligned}
\tag{3.5.20}
$$

Therefore, we have that

$$
|\alpha_\lambda(F_j) - \alpha_\lambda(E_j)| \leq \frac{P_\lambda(B^\lambda(|B^\lambda|))}{4\hat{\gamma}^2(n,\lambda)} \alpha_\lambda^2(E_j).
$$

Since $\alpha_\lambda(E_j) \to 0$ we get that

$$
\frac{\alpha_\lambda(F_j)}{\alpha_\lambda(E_j)} \to 1.
$$

Let $\{\hat{\lambda}_j\} \subset (0, \infty)$ such that, setting $\tilde{F}_j := \hat{\lambda}_j F_j$, then $|\tilde{F}_j| = |B^\lambda|$. Clearly $\hat{\lambda}_j \to 1$ since $|F_j| \to |B^\lambda|$. Since $P_\lambda(F_j) \to P_\lambda(B^\lambda(|B^\lambda|))$ and $\Lambda > n$, for j sufficiently large we have $P_\lambda(F_j) < \Lambda|F_j|$ and

$$
\left| P_\lambda(\tilde{F}_j) - P_\lambda(F_j) \right| = P_\lambda(F_j) \left| \hat{\lambda}_j^{n-1} - 1 \right| \leq P_\lambda(F_j) \left| \hat{\lambda}_j^n - 1 \right| \leq \Lambda \left| \hat{\lambda}_j^n - 1 \right| \, |F_j| = \Lambda \left| |\tilde{F}_j| - |F_j| \right|.
$$

Hence, by definition of $\hat{\lambda}_j$ and by (3.5.20) we get

$$
\begin{aligned}
P_\lambda(\tilde{F}_j) \leq P_\lambda(F_j) + \Lambda \left| |\tilde{F}_j| - |F_j| \right| &= P_\lambda(F_j) + \Lambda \left| |F_j| - |B^\lambda| \right| \\
&\overset{(3.5.20)}{\leq} P_\lambda(B^\lambda(|B^\lambda|)) + \frac{P_\lambda(B^\lambda(|B^\lambda|))}{4\hat{\gamma}^2(n,\lambda)} \alpha_\lambda^2(E_j).
\end{aligned}
\tag{3.5.21}
$$

Since $\alpha_\lambda(F_j)/\alpha_\lambda(E_j) \to 1$ as $j \to \infty$ we have $\alpha_\lambda(E_j)^2 < 2\alpha_\lambda(\tilde{F}_j)^2$ for j sufficiently large. Hence from (3.5.21) we finally obtain

$$
\alpha_\lambda(\tilde{F}_j) > \sqrt{2}\hat{\gamma}(n,\lambda)\sqrt{D_\lambda(\tilde{F}_j)}. \tag{3.5.22}
$$

For $t > 0$ let

$$
\varphi^-_{\tilde{F}_j}(t) := \begin{cases} \min_{x \in \partial \tilde{F}_j \cap \{x_n = t\}} \left\{ |x - te_n| \right\} & \text{if} \quad \partial \tilde{F}_j \cap \{x_n = t\} \neq \emptyset, \\ 0 & \text{if} \quad \partial \tilde{F}_j \cap \{x_n = t\} = \emptyset. \end{cases}
$$

be the function measuring the distance of $\partial \tilde{F}_j \cap \{x_n = t\}$ from the n-th axis, set to zero in case $\partial \tilde{F}_j \cap \{x_n = t\} = \emptyset$. For j large we can apply Corollary 3.4.4 again to deduce that there exists $T_\lambda, A_\lambda > 0$ such that $\mathcal{H}^{n-1}(\tilde{F}_j \cap \{x_n = t\}) \geq A_\lambda$ for almost every $t \in (0, T_\lambda)$. Since $\tilde{F}_j$ is Schwarz-symmetric and its relative boundary in $\{x_n > 0\}$ is C^1 regular, then we can write that $\varphi^-_{\tilde{F}_j}(t) \geq A'_\lambda > 0$ for j large and for any $t \in (0, T_\lambda)$.

Recalling that F_j is a local (Λ_1, r_0)-minimizer, by Lemma 2.5.2 its boundary has generalized mean curvature bounded by Λ_1 for any j. Since $\tilde{F}_j = \hat{\lambda}_j F_j$ with $\hat{\lambda}_j \to 1$, then $\partial \tilde{F}_j \cap (\mathbb{R}^n \setminus H)$ is a hypersurface of class $C^{1,\frac{1}{2}}$ with generalized mean curvature $H_{\partial \tilde{F}_j}$ bounded by $2\Lambda_1$ for any j. Observe that if we locally parametrize $\partial \tilde{F}_j \cap (\mathbb{R}^n \setminus H)$ with the graph of a function Φ_j, then Φ_j weakly solves the mean curvature equation

$$
\operatorname{div}\left(\frac{\nabla \Phi_j}{\sqrt{1 + |\nabla \Phi_j|^2}} \right) = \left\langle H_{\partial \tilde{F}_j}, N_{\Phi_j} \right\rangle,
$$

where N_{Φ_j} is the unit normal corresponding to Φ_j and $H_{\partial \tilde{F}_j}$ is evaluated along the graph of Φ_j. Since $H_{\partial \tilde{F}_j}$ is bounded, we get that Φ_j is of class $W^{2,p}$ for every $p < \infty$ (see [GT01]).

Fix $p_0 \in \partial \tilde{F}_j \cap \{x_1 > 0, 0 < x_n \leq T_\lambda\} \cap \operatorname{span}\{e_1, e_n\}$. Since $\partial \tilde{F}_j \cap (\mathbb{R}^n \setminus H)$ is $C^{1,\frac{1}{2}}$, there exists a curve $\gamma_j = (\alpha_j, 0, \dots, 0, \beta_j) : (a,b) \to \operatorname{span}\{e_1, e_n\} \setminus H$ such that the map $\mathbb{S}^{n-2} \times (a,b) \ni (\vartheta, t) \mapsto (\alpha_j(t)\vartheta, \beta_j(t))$ parametrizes $\partial \tilde{F}_j$ in a neighborhood of p_0. We claim that $\alpha_j, \beta_j \in W^{2,p}$, up to reparametrization.
Indeed, we can also parametrize $\partial \tilde{F}_j$ in a neighborhood U of p_0 as the graph of a function Φ_j with domain contained in some affine hyperplane of the form $p_0 + V$, and without loss of generality we can assume that either $V = \{x_1 = 0\}$ or $V = \{x_n = 0\}$. If $n = 2$, then the claimed regularity immediately follows from the regularity of Φ_j. Then assume $n \geq 3$, and suppose for example that $V = \{x_1 = 0\}$. The image of the curve γ_j in U can be parametrized as the graph of a function $t \mapsto (f(t), 0, \dots, 0, t)$. Writing as $(x', x_n) \in p_0 + V$ the variable for Φ_j, the fact that the distance from the n-th axis is constant on the intersection of $\partial \tilde{F}_j$ with any horizontal hyperplane yields the identity

$$\left(\Phi_j(x', x_n) + \operatorname{dist}_{x_n}(p_0)\right)^2 + |x'|^2 = f(x_n)^2,$$

where $\operatorname{dist}_{x_n}(p_0)$ denotes distance of p_0 from the n-th axis. Since Φ_j is of class C^1 and $W^{2,p}$ and $\operatorname{dist}_{x_n}(p_0) > 0$ because $\varphi^-_{\tilde{F}_j} \geq A'_\lambda > 0$, inverting the above identity we find that f is of class $W^{2,p}$, hence so is γ_j, up to reparametrization. In case $V = \{x_n = 0\}$, the observation follows analogously relating Φ_j with a parametrization for γ_j.
We further observe that, for $\alpha_j, \beta_j : (a,b) \to (0, \infty)$ as above, since α_j, β_j are of class $W^{2,p}_{\mathrm{loc}}$, up to reparametrization by arclength we can apply Lemma 2.6.6 to get that

$$H_{\partial \tilde{F}_j}\Big|_{(\alpha_j(t)\vartheta, \beta_j(t))} = \left(\left\langle k_{\gamma_j}, \nu \right\rangle - (n-2)\frac{\beta'_j}{\alpha_j} \right)(-\beta'_j \vartheta, \alpha'_j),$$

in the notation of Lemma 2.6.6. Recalling that $\varphi^-_{\tilde{F}_j} \geq A'_\lambda > 0$ on $(0, T_\lambda)$, we have that $|\alpha_j| \geq A'_\lambda$ and thus

$$|k_{\gamma_j}| \leq 2\Lambda_1 + \frac{n-2}{A'_\lambda}. \tag{3.5.23}$$

Observe that the upper bound in (3.5.23) is independent of j and of the initially chosen point p_0.
Fix now $q_0 \in \partial \tilde{F}_j \cap \{x_1 > 0, x_n = T_\lambda\} \cap \operatorname{span}\{e_1, e_n\}$, let $\gamma^0_j : [0, l_0) \to \operatorname{span}\{e_1, e_n\}$ be part of a curve defined as before, parametrized by arclength, such that $\left\langle \gamma^0_j(t), e_n \right\rangle \leq T_\lambda$ for any t. If $\lim_{t \to l_0^-} \gamma^0_j(t) \notin \partial H$, the curve can be extended to a longer one, parametrized by arclength, by joining γ^0_j with a curve defined as before for the choice $p_0 = \lim_{t \to l_0^-} \gamma^0_j(t)$. Hence we can consider $\sigma_j : [0, L_j) \to \operatorname{span}\{e_1, e_n\}$ the maximal extension of γ^0_j parametrized by arclength that parametrizes $\tilde{F}_j \cap \{x_1 > 0, 0 < x_n \leq T_\lambda\} \cap \operatorname{span}\{e_1, e_n\}$. Since the perimeter $P(\tilde{F}_j, \mathbb{R}^n \setminus H)$ is uniformly bounded, then $\sup_j L_j < +\infty$. Obviously $\lim_{t \to L_j} \sigma_j(t) \in \partial H$, for otherwise the curve could be further extended. By construction, the uniform bound in (3.5.23) holds pointwise for the curvature of σ_j. Therefore σ_j can be extended to a curve $\gamma_j : [0, L_j] \to \operatorname{span}\{e_1, e_n\}$ such that

$$\|\gamma_j\|_{C^{1,1}([0,L_j])} \leq C, \tag{3.5.24}$$

with C independent of j, depending only on $n, \lambda, \tilde{l}$ and the upper bound on the curvature given by (3.5.23).
Up to a subsequence, since $\tilde{F}_j \to B^\lambda(|B^\lambda|)$ and we already know that for every $\varepsilon > 0$ the set $\partial \tilde{F}_j \cap \{x_n \geq \varepsilon\}$ converges to $\partial B^\lambda(|B^\lambda|) \cap \{x_n \geq \varepsilon\}$ in $C^{1,\alpha}$ for any $0 < \alpha < \frac{1}{2}$, the bound (3.5.24) implies that $\tilde{F}_j$ converges in C^1 sense to $B^\lambda(|B^\lambda|)$ in the sense of Definition 3.5.11. Hence, by Corollary 3.5.12, for j sufficiently large there holds

$$\alpha^2_\lambda(\tilde{F}_j) \leq \hat{\gamma}(n, \lambda) D_\lambda(\tilde{F}_j),$$

in contradiction with (3.5.22). $\square$

Remark 3.5.14. In the proof of it is possible to directly prove that the boundary of $\tilde{F}_j$ remains uniformly far away from the n-th axis in a small slab $\{0 < x_n \leq \varepsilon_0\}$ without appealing to Corollary 3.4.4. We present here an alternative proof of this fact.

Let

$$h_0 := (1-\lambda)\frac{9}{10}, \qquad \varepsilon_0 := (1-\lambda)\frac{1}{10}.$$

Let φ be the profile function parametrizing $\partial B^\lambda(|B^\lambda|) \setminus H$. For $t > 0$, we define

$$\varphi^-_{\tilde{F}_j}(t) := \begin{cases} \min_{x\in\partial\tilde{F}_j\cap\{x_n=t\}} \{|x-te_n|\} & \text{if} \quad \partial\tilde{F}_j \cap \{x_n = t\} \neq \emptyset, \\ 0 & \text{if} \quad \partial\tilde{F}_j \cap \{x_n = t\} = \emptyset. \end{cases}$$

$$\varphi^+_{\tilde{F}_j}(t) := \begin{cases} \max_{x\in\partial\tilde{F}_j\cap\{x_n=t\}} \{|x-te_n|\} & \text{if} \quad \partial\tilde{F}_j \cap \{x_n = t\} \neq \emptyset, \\ 0 & \text{if} \quad \partial\tilde{F}_j \cap \{x_n = t\} = \emptyset. \end{cases}$$

Note that $\varphi^-_{\tilde{F}_j}(t)$ is lower semicontinuous. Indeed, let $t_k \to t_\infty \in (0,\infty)$. If there exists $t_{k_l} \to t_\infty$ such that $\varphi^-_{\tilde{F}_j}(t_{k_l}) = 0$ then, by regularity of $\partial\tilde{F}_j$, we infer

$$\liminf_{k\to\infty} \varphi^-_{\tilde{F}_j}(t_k) = \lim_{l\to\infty} \varphi^-_{\tilde{F}_j}(t_{k_l}) = 0 = \varphi^-_{\tilde{F}_j}(t_\infty).$$

Otherwise, for k sufficiently large let $\varphi^-_{\tilde{F}_j}(t_k)$ be achieved in x_k. Then

$$\begin{aligned} \liminf_{k\to\infty} \varphi^-_{\tilde{F}_j}(t_k) &= \liminf_{k\to\infty} \min_{x\in\partial\tilde{F}_j\cap\{x_n=t_k\}} \{|x-t_ke_n|\} \\ &= \liminf_{k\to\infty} \{|x_k - t_ke_n|\} \\ &\geq \varphi^-_{\tilde{F}_j}(t_\infty). \end{aligned}$$

By the convergence of $\partial\tilde{F}_j$ to $\partial B^\lambda(|B^\lambda|)$ in C^1-sense in $\{x_n \geq \varepsilon_0\}$, for any $\underline{\sigma} > 0$ and for sufficiently large $j \in \mathbb{N}$ we have

$$\left|\varphi^z_{\tilde{F}_j}(\varepsilon_0) - \varphi(\varepsilon_0)\right| < \frac{\underline{\sigma}}{1000}, \tag{3.5.25}$$

with $z \in \{+,-\}$. Let us define the neighborhood of $\partial B^\lambda(|B^\lambda|)$ in $\{x_n > 0\}$ given by

$$U_{\underline{\sigma}} := \left\{ x = (x', x_n) \in \mathbb{R}^n \setminus H \ : \ ||x'| - \varphi(x_n)| < \frac{\underline{\sigma}}{1000} \right\}$$

and let

$$\tilde{s}_j := \inf \left\{ 0 < s \leq \varepsilon_0 \ : \ \varphi^-_{\tilde{F}_j}(t) > \frac{1}{1000}\sqrt{1-\lambda^2} \quad \forall t \in [s, \varepsilon_0] \right\}.$$

We claim that, for sufficiently large j, $\tilde{s}_j = 0$, which coincides to prove that $\partial\tilde{F}_j \cap \{x_n = t\}$ is nonempty and uniformly far away from the n-th axis, for $t \in (0, \varepsilon_0]$.
By contradiction let us assume that $\tilde{s}_j$ is strictly positive for any large j. Note that the lower semicontinuity of $\varphi^-_{\tilde{F}_j}$ implies that $\varphi^-_{\tilde{F}_j}(\tilde{s}_j) \leq \frac{1}{1000}\sqrt{1-\lambda^2}$. This follows by taking a sequence $\hat{s}_k \nearrow \tilde{s}_j$, with

$$\varphi^-_{\tilde{F}_j}(\hat{s}_k) \leq \frac{1}{1000}\sqrt{1-\lambda^2}.$$

We claim that there exists $\widetilde{c}(n, \lambda, \underline{\sigma})$ such that

$$\mathcal{H}^{n-1}((\partial\tilde{F}_j \setminus H) \setminus U_{\underline{\sigma}} \cap \{\tilde{s}_j < x_n < \varepsilon_0\}) > \widetilde{c}(n, \lambda, \underline{\sigma}) > 0. \tag{3.5.26}$$

Indeed, if there exists $\widetilde{c}_1 > 0$ such that

$$\liminf_{j\to\infty} \varphi^+_{\tilde{F}_j}(\tilde{s}_j) > \liminf_{j\to\infty} \varphi^-_{\tilde{F}_j}(\tilde{s}_j) + \widetilde{c}_1,$$

then

$$\mathcal{H}^{n-1}((\partial\tilde{F}_j \setminus H) \setminus U_{\underline{\sigma}} \cap \{\tilde{s}_j < x_n < \varepsilon\}) \geq \omega_{n-1}(\varphi^+_{\tilde{F}_j}(\tilde{s}_j)^{n-1} - \varphi^-_{\tilde{F}_j}(\tilde{s}_j)^{n-1}) \geq \widetilde{c}_1 > 0.$$

Otherwise $\lim_{j\to\infty}\varphi^+_{\tilde F_j}(\tilde s_j)=\lim_{j\to\infty}\varphi^-_{\tilde F_j}(\tilde s_j)$ up to subsequence; hence let $A_{\tilde s_j}$ be the annulus in the plane $\{x_n=\tilde s_j\}$ with center in the origin and radii $\frac{1}{1000}\sqrt{1-\lambda^2}$ and $\min\{\varphi(0),\varphi(\varepsilon_0)\}-\frac{1}{1000}\underline{\sigma}$. If $p\in A_{\tilde s_j}$ and $p\notin \bar F_j$, by (3.5.25) and the regularity of $\partial\tilde F_j$, there exists $\underline{t}>0$ such that $p+\underline{t}e_n\in\partial\tilde F_j$. In particular, the projection of $(\partial\tilde F_j\setminus H)\setminus U_{\underline{\sigma}}\cap\{\tilde s_j<x_n<\varepsilon_0\}$ over the plane $\{x_n=\tilde s_j\}$ is surjective on $A_{\tilde s_j}$. Therefore

$$\begin{aligned}\mathcal{H}^{n-1}((\partial\tilde F_j\setminus H)\setminus U_{\underline{\sigma}}\cap\{\tilde s_j<x_n<\varepsilon_0\})&\geq\mathcal{H}^{n-1}\left(\pi_{\partial H}\left((\partial\tilde F_j\setminus H)\setminus U_{\underline{\sigma}}\cap\{\tilde s_j<x_n<\varepsilon_0\}\right)\right)\\&\geq\omega_{n-1}\left[\left(\min\left\{\varphi(0),\varphi(\varepsilon_0)\right\}-\frac{\underline{\sigma}}{1000}\right)^{n-1}-\left(\frac{1}{1000}\right)^{n-1}\left(1-\lambda^2\right)^{\frac{n-1}{2}}\right]>0\end{aligned}$$

and we have proved (3.5.26).
However (3.5.26) would imply that $P_\lambda(\tilde F_j)\not\to P_\lambda(B^\lambda)$. Indeed, since

$$P_\lambda(\tilde F_j)=\int_{(\partial\tilde F_j\setminus H)\cap U_{\underline{\sigma}}}1-\lambda\left\langle e_n,\nu^{\tilde F_j}\right\rangle\,\mathrm{d}\mathcal{H}^{n-1}+\int_{(\partial\tilde F_j\setminus H)\setminus U_{\underline{\sigma}}}1-\lambda\left\langle e_n,\nu^{\tilde F_j}\right\rangle\,\mathrm{d}\mathcal{H}^{n-1},$$

the first integral in the right-hand side tends to $P_\lambda(B^\lambda)$ by Reshetnyak lower semicontinuity theorem 2.1.22 while the second integral is greater than a strictly positive constant because of (3.5.26).

We can further prove that $\varphi^z_{\tilde F_j}$, with $z\in\{+,-\}$, defined in the proof of Theorem 3.1.1, converges uniformly to the profile function φ of $B^\lambda(|B^\lambda|)$ in $(0,h_0]$.
Indeed, let us assume by contradiction that there exists $\underline{\sigma}_0>0$ such that

$$\max_{z\in\{+,-\}}\sup_{(0,h_0]}\left|\varphi^z_{\tilde F_j}(t)-\varphi(t)\right|>\underline{\sigma}_0$$

for any j large. We can assume that

$$\max_{z\in\{+,-\}}\sup_{(0,\varepsilon_0]}\left|\varphi^z_{\tilde F_j}(t)-\varphi(t)\right|>\underline{\sigma}_0$$

for sufficiently large j. Let $\{t_j\}_{j\in\mathbb{N}}\subset(0,\varepsilon_0]$ and $z\in\{+,-\}$ be such that

$$\left|\varphi^z_{\tilde F_j}(t_j)-\varphi(t_j)\right|>\underline{\sigma}_0.$$

Then there exists $t_\infty\in[0,\varepsilon_0]$ such that, up to a subsequence, $t_j\to t_\infty$. For $U_{\underline{\sigma}_0}$ defined analogously as above, since

$$P_\lambda(\tilde F_j)\to P_\lambda(|B^\lambda|)$$

and

$$P_\lambda(\tilde F_j)=\int_{(\partial\tilde F_j\setminus H)\cap U_{\underline{\sigma}_0}}1-\lambda\left\langle e_n,\nu^{\tilde F_j}\right\rangle\,\mathrm{d}\mathcal{H}^{n-1}+\int_{(\partial\tilde F_j\setminus H)\setminus U_{\underline{\sigma}_0}}1-\lambda\left\langle e_n,\nu^{\tilde F_j}\right\rangle\,\mathrm{d}\mathcal{H}^{n-1},$$

then

$$\lim_{j\to\infty}\int_{(\partial\tilde F_j\setminus H)\setminus U_{\underline{\sigma}_0}}1-\lambda\left\langle e_n,\nu^{\tilde F_j}\right\rangle\,\mathrm{d}\mathcal{H}^{n-1}=0.$$

For j large enough $\varphi^z_{\tilde F_j}(\varepsilon_0)\in U_{\underline{\sigma}_0}$, with $z\in\{+,-\}$. Moreover

$$\int_{(\partial\tilde F_j\setminus H)\setminus U_{\underline{\sigma}_0}}1-\lambda\left\langle e_n,\nu^{\tilde F_j}\right\rangle\,\mathrm{d}\mathcal{H}^{n-1}\geq(1-\lambda)\mathcal{H}^{n-1}\left(\left(\partial\tilde F_j\setminus H\right)\setminus U_{\underline{\sigma}_0}\cap\left\{t_j<x_n<\varepsilon_0\right\}\right).$$

Let us define

$$s_j:=\inf\left\{0<s\leq\varepsilon_0\ :\ |\varphi^+_{\tilde F_j}(t)-\varphi(t)|<\frac{\underline{\sigma}}{1000},|\varphi^-_{\tilde F_j}(t)-\varphi(t)|<\frac{\underline{\sigma}}{1000}\ \forall t\in[s,\varepsilon_0]\right\}.$$

Observe that $t_j < s_j$. There exists $s_\infty \in [0, \varepsilon_0]$ such that, up to a subsequence, $s_j \to s_\infty$. If $s_\infty = t_\infty$ then

$$\begin{aligned}\mathcal{H}^{n-1}\left(\left(\partial\tilde{F}_j \setminus H\right) \setminus U_{\underline{\sigma}_0} \cap \left\{t_j < x_n < \varepsilon_0\right\}\right) &\geq \mathcal{H}^{n-1}\left(\pi_{\partial H}\left(\left(\partial\tilde{F}_j \setminus H\right) \setminus U_{\underline{\sigma}_0} \cap \left\{t_j < x_n < \varepsilon_0\right\}\right)\right) \\ &> \omega_{n-1}\left[\underline{\sigma}_0^{n-1} - \left(\frac{\underline{\sigma}_0}{1000}\right)^{n-1}\right]\left(1-\lambda^2\right)^{\frac{n-1}{2}} > 0.\end{aligned}$$

If $s_\infty > t_\infty$ then

$$\begin{aligned}\mathcal{H}^{n-1}\left(\left(\partial\tilde{F}_j \setminus H\right) \setminus U_{\underline{\sigma}_0} \cap \left\{t_j < x_n < \varepsilon_0\right\}\right) &\geq \int_{t_j}^{s_j} (n-1)\omega_{n-1}\sqrt{1+\left((\varphi^z_{\tilde{F}_j})'\right)^2}\,(\varphi^z_{\tilde{F}_j})^{n-2}\,\mathrm{d}x \\ &\geq (n-1)\omega_{n-1}\int_{t_j}^{s_j} (\varphi^z_{\tilde{F}_j})^{n-2}\,\mathrm{d}x \\ &\geq (n-1)\omega_{n-1}\int_{\frac{t_\infty+s_j}{2}}^{s_j} (\varphi^z_{\tilde{F}_j})^{n-2}\,\mathrm{d}x \geq c > 0.\end{aligned}$$

In both cases, as before we would get that $P_\lambda(\tilde{F}_j) \not\to P_\lambda(B^\lambda)$.

3.6 Second quantitative isoperimetric inequality

We will need the following technical lemma, proving that if the energy $P_\lambda(E_i)$ a sequence of sets E_i converges to the energy of the limit, then the sequence strictly converges in the sense of BV functions. The proof essentially follows by analyzing the equality case in the Reshetnyak lower semicontinuity theorem 2.1.22.

Lemma 3.6.1. *Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of sets of finite perimeter in $\mathbb{R}^n \setminus H$ such that $E_i \to E$ in L^1, for some set of finite perimeter E with $|E| < +\infty$. If $P_\lambda(E_i) \to P_\lambda(E)$, then*

$$\lim_i P(E_i, \mathbb{R}^n \setminus H) = P(E, \mathbb{R}^n \setminus H), \qquad \lim_i \mathcal{H}^{n-1}(\partial^* E_i \cap \partial H) = \mathcal{H}^{n-1}(\partial^* E \cap \partial H).$$

Proof. Let $f(v) := |v| - \lambda\langle e_n, v\rangle$, for any $v \in \mathbb{R}^n$, and let $\nu_i := |D\chi_{E_i}| \otimes \delta_{\nu^{E_i}}$ be a measure on $\mathbb{R}^n \setminus H \times \mathbb{S}^{n-1}$. Since $|\nu_i|(\mathbb{R}^n \setminus H \times \mathbb{S}^{n-1}) \leq P(E_i) \leq 2P_\lambda(E_i)/(1-\lambda)$ by Corollary 2.4.5, up to subsequence, ν_i weakly* converges to a finite measure ν. Up to subsequence, also $|D\chi_{E_i}|$ weakly* converges to a finite measure μ on $\mathbb{R}^n \setminus H$. Denoting by $\pi : \mathbb{R}^n \setminus H \times \mathbb{S}^{n-1} \to \mathbb{R}^n \setminus H$ the natural projection, we have $\pi_\sharp \nu_i = |D\chi_{E_i}| \to \mu = \pi_\sharp \nu$. Moreover, $\mu \geq |D\chi_E|$ by lower semicontinuity. By the disintegration theorem 2.1.27, we can write $\nu = \mu \otimes \nu_x$, for a μ-measurable map $\mathbb{R}^n \setminus H \ni x \mapsto \nu_x$, where ν_x is a probability measure on $\mathbb{S}^{n-1}$. Analogously to [AFP00, Eq. (2.30)], we observe that

$$\int_{\mathbb{S}^{n-1}} v\,\mathrm{d}\nu_x(v) = \nu^E(x)\frac{|D\chi_E|}{\mu}(x), \tag{3.6.1}$$

at μ-a.e. $x \in \mathbb{R}^n \setminus H$. Indeed, for any continuous function g with $\mathrm{spt}(g) \subset\subset \mathbb{R}^n \setminus H$ we find

$$\begin{aligned}\int_{\mathbb{R}^n\setminus H} g(x)\int_{\mathbb{S}^{n-1}} v\,\mathrm{d}\nu_x(v)\,\mathrm{d}\mu(x) &= \int_{\mathbb{R}^n\setminus H\times\mathbb{S}^{n-1}} g(x)\,v\,\mathrm{d}\nu(x,v) = \lim_i \int_{\mathbb{R}^n\setminus H\times\mathbb{S}^{n-1}} g(x)\,v\,\mathrm{d}\nu_i(x,v) \\ &= -\lim_i \int_{\mathbb{R}^n\setminus H} g(x)\,\mathrm{d}D\chi_{E_i}(x) = -\int_{\mathbb{R}^n\setminus H} g(x)\,\mathrm{d}D\chi_E(x) = \int_{\mathbb{R}^n\setminus H} g(x)\nu^E(x)\frac{|D\chi_E|}{\mu}(x)\,\mathrm{d}\mu(x).\end{aligned}$$

Since f is nonnegative, convex and continuous, by Remark 2.4.2 we find

$$\begin{aligned}\lim_i P_\lambda(E_i) &= \lim_i \int_{\mathbb{R}^n\setminus H} f(\nu^{E_i})\,\mathrm{d}|D\chi_{E_i}| = \lim_i \int_{\mathbb{R}^n\setminus H\times\mathbb{S}^{n-1}} f(v)\,\mathrm{d}\nu_i(x,v) \geq \int_{\mathbb{R}^n\setminus H\times\mathbb{S}^{n-1}} f(v)\,\mathrm{d}\nu(x,v) \\ &= \int_{\mathbb{R}^n\setminus H}\int_{\mathbb{S}^{n-1}} f(v)\,\mathrm{d}\nu_x(v)\,\mathrm{d}\mu(x) \geq \int_{\mathbb{R}^n\setminus H} f\left(\int_{\mathbb{S}^{n-1}} v\,\mathrm{d}\nu_x(v)\right)\mathrm{d}\mu(x) \\ &\overset{(3.6.1)}{=} \int_{\mathbb{R}^n\setminus H} f\left(\nu^E(x)\frac{|D\chi_E|}{\mu}(x)\right)\mathrm{d}\mu(x) = \int_{\mathbb{R}^n\setminus H} f(\nu^E(x))\,\mathrm{d}|D\chi_E|(x) = P_\lambda(E),\end{aligned} \tag{3.6.2}$$

where in the second inequality we applied Jensen inequality 2.1.7, and where the last equality follows since f is positively 1-homogeneous. Since $\lim_i P_\lambda(E_i) = P_\lambda(E)$ by assumption and since f is not affine, equality in Jensen inequality 2.1.7 implies that the identity map $\mathbb{S}^{n-1} \ni v \mapsto v$ is constant ν_x-a.e., for μ-a.e. $x \in \mathbb{R}^n \setminus H$. This means that $\nu_x = \delta_{v_x}$ for some $v_x \in \mathbb{S}^{n-1}$ for μ-a.e. $x \in \mathbb{R}^n \setminus H$. Hence (3.6.1) implies

$$v_x = \nu^E(x)\frac{|D\chi_E|}{\mu}(x),$$

at μ-a.e. $x \in \mathbb{R}^n \setminus H$, and since $|v_x| = |\nu^E(x)| = 1$, then $|D\chi_E|/\mu(x) = 1$ at μ-a.e. $x \in \mathbb{R}^n \setminus H$, and $v_x = \nu^E(x)$ μ-almost everywhere. Inserting in (3.6.2) we deduce

$$\int_{\mathbb{R}^n\setminus H} f\left(\nu^E(x)\right) \mathrm{d}\mu(x) = \int_{\mathbb{R}^n\setminus H} f\left(\int_{\mathbb{S}^{n-1}} v \,\mathrm{d}\nu_x(v)\right) \mathrm{d}\mu(x) = \int_{\mathbb{R}^n\setminus H} f(\nu^E(x))\,\mathrm{d}|D\chi_E|(x).$$

Since $f(\nu^E(x)) > 0$ and $\mu \ge |D\chi_E|$, we deduce that $\mu = |D\chi_E|$, and then $|D\chi_{E_i}|$ weakly* converges to $|D\chi_E|$. We can now fix an increasing sequence of Lipschitz bounded open sets $\Omega_j \subset\subset \mathbb{R}^n \setminus H$ such that $\cup_j \Omega_j = \mathbb{R}^n \setminus H$ and $P(E_i, \partial\Omega_j) = P(E, \partial\Omega_j) = 0$ for every i, j. Hence $\lim_i P(E_i, \Omega_j) = P(E, \Omega_j)$ for any j. Moreover

$$P(E_i, \mathbb{R}^n \setminus (H \cup \Omega_j)) \le \frac{1}{1-|\lambda|}\left(P_\lambda(E_i) - \int_{\Omega_j} f(\nu^{E_i})\,\mathrm{d}|D\chi_{E_i}|\right),$$

for any i, j. Applying Reshetnyak continuity theorem 2.1.23 on Ω_j we get

$$\limsup_i P(E_i, \mathbb{R}^n \setminus (H \cup \Omega_j)) \le \frac{1}{1-|\lambda|}\int_{\mathbb{R}^n\setminus(H\cup\Omega_j)} f(\nu^E)\,\mathrm{d}|D\chi_E| \le \frac{1+|\lambda|}{1-|\lambda|} P(E, \mathbb{R}^n \setminus (H \cup \Omega_j)),$$

for any j. Therefore

$$\limsup_i P(E_i, \mathbb{R}^n \setminus H) \le P(E, \Omega_j) + \frac{1+|\lambda|}{1-|\lambda|} P(E, \mathbb{R}^n \setminus (H \cup \Omega_j)),$$

for any j. Letting $j \to \infty$, the proof follows. □

We will also exploit the concept of (K, r_0)-quasiminimal set.

Definition 3.6.2. Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter with finite measure, and let $K \ge 1, r_0 > 0$. We say that E is a (K, r_0)-quasiminimal set (relatively in $\mathbb{R}^n \setminus H$) if

$$P(E, \mathbb{R}^n \setminus H) \le K P(F, \mathbb{R}^n \setminus H),$$

for any $F \subset \mathbb{R}^n \setminus H$ such that $E \Delta F \subset\subset B_r(x)$, for some ball $B_r(x) \subset \mathbb{R}^n$ with $r \le r_0$ and $x \in \{x_n \ge 0\}$.

Quasiminimal sets have well-known topological regularity properties following from uniform density estimates at boundary points. We recall these facts in the following statement. The proof follows, for example, by repeatedly applying [Kin+13, Theorem 4.2] with $X = \{x_n \ge 0\}$ in domains $\Omega = X \cap B_{r_0}(x)$ for $x \in X$, in the notation of [Kin+13, Theorem 4.2]. Observe that in [Kin+13], the perimeter functional coincides with the relative perimeter in $\mathbb{R}^n \setminus H$, hence the definition of quasiminimal set in [Kin+13, Definition 3.1] coincides with our Definition 3.6.2. Alternatively, the proof follows by adapting the proof of [Mag12, Theorem 21.11] working with (K, r_0)-quasiminimal sets instead of (Λ, r_0)-minimizers.

Theorem 3.6.3. *Let $E \subset \mathbb{R}^n \setminus H$ be a (K, r_0)-quasiminimal set, for some $K \ge 1, r_0 > 0$. Then there exist $m = m(n, K, r_0) \in (0, 1)$ and $r_0' = r_0'(n, K, r_0) \in (0, r_0]$ such that*

$$m \le \frac{|E \cap B_r(x)|}{|B_r(x) \setminus H|} \le 1 - m \qquad \forall x \in \overline{\partial E \setminus H},\ \forall r \in (0, r_0'].$$

In particular the set $E^{(1)}$ of points of density 1 for E is an open representative for E.

We will identify a (K, r_0)-quasiminimal set with its open representative $E^{(1)}$. In order to prove Theorem 1.1.2 we need two preparatory lemmas.

Lemma 3.6.4. *For any $K \geq 1, r_0 > 0$ there exist $\delta_7, C_6, C_7 > 0$ depending on n, λ, K, r_0 such that the following holds. If $E \subset \mathbb{R}^n \setminus H$ is a bounded (K, r_0)-quasiminimal set with $|E| = |B^\lambda|$ and $D_\lambda(E) \leq \delta_7$, then*

$$d_{\mathcal{H}}\left(\overline{\partial E \setminus H}, \overline{\partial B^\lambda(|B^\lambda|, x) \setminus H}\right) \leq C_6 \alpha_\lambda(E)^{\frac{1}{n}}, \tag{3.6.3}$$

where $B^\lambda(|B^\lambda|, x)$ is a bubble realizing the asymmetry of E. Moreover

$$\beta_\lambda(E) \leq C_7 D_\lambda(E)^{\frac{1}{2n}}. \tag{3.6.4}$$

Proof. Up to translation, we can assume that $x = 0$. Also, letting m, r_0' be given by Theorem 3.6.3, up to decreasing r_0 we can assume that $r_0 = r_0'$. Let $p \in \overline{\partial E \setminus H}$ be such that

$$d_0 := \operatorname{dist}\left(p, \overline{\partial B^\lambda(|B^\lambda|) \setminus H}\right) = \max\left\{\operatorname{dist}\left(y, \overline{\partial B^\lambda(|B^\lambda|) \setminus H}\right) \ : \ y \in \overline{\partial E \setminus H}\right\}.$$

Hence $B_{d_0}(p) \cap \overline{\partial B^\lambda(|B^\lambda|) \setminus H} = \emptyset$. Then either $B_{d_0}(p) \setminus H \subset B^\lambda(|B^\lambda|)$ or $B_{d_0}(p) \setminus H \subset \mathbb{R}^n \setminus (H \cup B^\lambda(|B^\lambda|))$. In the first case Theorem 3.6.3 implies

$$m|B_r(x) \setminus H| \leq |B_r(x) \setminus (H \cup E)| \leq |B^\lambda(|B^\lambda|) \setminus E| = \frac{1}{2}\alpha_\lambda(E) \qquad \forall r \in (0, \min\{d_0, r_0\}),$$

while in the second case Theorem 3.6.3 implies

$$m|B_r(x) \setminus H| \leq |B_r(x) \cap E| \leq |E \setminus B^\lambda(|B^\lambda|)| = \frac{1}{2}\alpha_\lambda(E) \qquad \forall r \in (0, \min\{d_0, r_0\}).$$

Since $|B_R(x) \setminus H| \geq Cr^n$, then $\min\{d_0, r_0\}^n \leq C\alpha_\lambda(E)$, for $C = C(n, \lambda, K, r_0)$. So by Corollary 3.4.3, choosing δ_7 small enough we have that $\alpha_\lambda(E)$ is so small that $\min\{d_0, r_0\} = d_0$ and then

$$d_0^n \leq C\alpha_\lambda(E).$$

Since density estimates as those in Theorem 3.6.3 hold for $B^\lambda(|B^\lambda|)$, repeating the above argument exchanging the roles of E and $B^\lambda(|B^\lambda|)$, (3.6.3) follows.
From (3.6.3), we deduce that

$$\overline{\partial E \setminus H} \subset \left\{y \in \{x_n \geq 0\} \ : \ \operatorname{dist}\left(y, \overline{\partial B^\lambda(|B^\lambda|) \setminus H}\right) \leq C_6 \alpha_\lambda(E)^{\frac{1}{n}}\right\}.$$

Hence

$$\begin{aligned}
&\mathcal{H}^{n-1}\left(\partial^* E \Delta \partial B^\lambda(|B^\lambda|) \cap \partial H\right) \\
&\quad \leq \mathcal{H}^{n-1}\left(\left\{(x', 0) \in \mathbb{R}^n \ : \ (1-\lambda^2)^{\frac{1}{2}} - C_6\alpha_\lambda(E)^{\frac{1}{n}} \leq |x'| \leq (1-\lambda^2)^{\frac{1}{2}} + C_6\alpha_\lambda(E)^{\frac{1}{n}}\right\}\right) \\
&\quad \leq C\alpha_\lambda(E)^{\frac{1}{n}} \leq CD_\lambda(E)^{\frac{1}{2n}},
\end{aligned}$$

for some $C = C(n, \lambda, K, r_0)$, where we used Theorem 3.1.1 in the last inequality. Hence (3.6.4) follows. □

Lemma 3.6.5. *There exists $\delta_8, C_8 > 0$ depending on n, λ such that for any measurable set $E \subset \mathbb{R}^n \setminus H$ with $|E| = |B^\lambda|$ and $D_\lambda(E) \leq \delta_8$ there holds*

$$\beta_\lambda(E) \leq C_8 D_\lambda(E)^{\frac{1}{2n}}.$$

Proof. Fix $\Lambda > n$. Let $Q \subset \mathbb{R}^n$ be a large cube whose interior contains the closure of $B^\lambda(|B^\lambda|)$, and let $F \subset Q \setminus H$ be such that $|B^\lambda|/2 \leq |F| \leq 2|B^\lambda|$. Let $G \subset \mathbb{R}^n \setminus H$ be such that $G \Delta F \subset\subset B_{r_0}(x)$, for $x \in \{x_n \geq 0\}$ and $r_0 \in (0, 1)$ to be chosen small. Let $Z := G \cap Q$. Observe that

$$\begin{aligned}
||Z| - |F|| \leq |Z \Delta F| \leq |G \Delta F|^{\frac{1}{n}} |G \Delta F|^{\frac{n-1}{n}} &\leq \omega_n^{\frac{1}{n}} r \left(|G|^{\frac{n-1}{n}} + |F|^{\frac{n-1}{n}}\right) \\
&\leq \overline{C}(n) r_0 \left(P(G, \mathbb{R}^n \setminus H) + P(F, \mathbb{R}^n \setminus H)\right),
\end{aligned} \tag{3.6.5}$$

where in the last inequality we used the relative isoperimetric inequality in a half-space (see [CGR07] for the sharp inequality). Let $y, z \in \partial H$ be such that

$$\beta_\lambda(F) = \frac{\mathcal{H}^{n-1}\left(\partial^* F \Delta \partial B^\lambda(|F|, y) \cap \partial H\right)}{\mathcal{H}^{n-1}\left(\partial B^\lambda(|F|, y) \cap \partial H\right)}, \qquad \beta_\lambda(Z) = \frac{\mathcal{H}^{n-1}\left(\partial^* Z \Delta \partial B^\lambda(|Z|, z) \cap \partial H\right)}{\mathcal{H}^{n-1}\left(\partial B^\lambda(|Z|, z) \cap \partial H\right)},$$

Observe that y, z exist since $F, Z \subset Q$. Suppose, for instance, that $\beta_\lambda(Z) \geq \beta_\lambda(F)$. Then

$$\begin{aligned}
\beta_\lambda(Z) - \beta_\lambda(F) &\leq \frac{\mathcal{H}^{n-1}\left(\partial^* Z \Delta \partial B^\lambda(|Z|, y) \cap \partial H\right)}{\mathcal{H}^{n-1}\left(\partial B^\lambda(|Z|, y) \cap \partial H\right)} - \frac{\mathcal{H}^{n-1}\left(\partial^* F \Delta \partial B^\lambda(|F|, y) \cap \partial H\right)}{\mathcal{H}^{n-1}\left(\partial B^\lambda(|F|, y) \cap \partial H\right)} \\
&\leq \frac{\mathcal{H}^{n-1}(\partial^* Z \Delta \partial^* F \cap \partial H) + \mathcal{H}^{n-1}\left(\partial^* F \Delta \partial B^\lambda(|F|, y) \cap \partial H\right) + \mathcal{H}^{n-1}\left(\partial B^\lambda(|F|, y) \Delta \partial B^\lambda(|Z|, y) \cap \partial H\right)}{\mathcal{H}^{n-1}\left(\partial B^\lambda(|Z|, y) \cap \partial H\right)} + \\
&\quad - \frac{\mathcal{H}^{n-1}\left(\partial^* F \Delta \partial B^\lambda(|F|, y) \cap \partial H\right)}{\mathcal{H}^{n-1}\left(\partial B^\lambda(|F|, y) \cap \partial H\right)} \\
&= \frac{\mathcal{H}^{n-1}(\partial^* Z \Delta \partial^* F \cap \partial H)}{\mathcal{H}^{n-1}\left(\partial B^\lambda(|Z|, y) \cap \partial H\right)} + \frac{\left|\mathcal{H}^{n-1}\left(\partial B^\lambda(|F|, y) \cap \partial H\right) - \mathcal{H}^{n-1}\left(\partial B^\lambda(|Z|, y) \cap \partial H\right)\right|}{\mathcal{H}^{n-1}\left(\partial B^\lambda(|Z|, y) \cap \partial H\right)} + \\
&\quad + \mathcal{H}^{n-1}\left(\partial^* F \Delta \partial B^\lambda(|F|, y) \cap \partial H\right) \left(\frac{1}{\mathcal{H}^{n-1}\left(\partial B^\lambda(|Z|, y) \cap \partial H\right)} - \frac{1}{\mathcal{H}^{n-1}\left(\partial B^\lambda(|F|, y) \cap \partial H\right)} \right)
\end{aligned} \tag{3.6.6}$$

By the trace inequality in Theorem 2.3.4 we estimate

$$\begin{aligned}
\mathcal{H}^{n-1}\left(\partial^* Z \Delta \partial^* F \cap \partial H\right) &\leq C(n)\left(|Z \Delta F| + P(Z, \mathbb{R}^n \setminus H) + P(F, \mathbb{R}^n \setminus H)\right) \\
&\leq C(n)\left(|G \Delta F| + P(G, \mathbb{R}^n \setminus H) + P(F, \mathbb{R}^n \setminus H)\right) \\
&\leq C(n)\left(P(G, \mathbb{R}^n \setminus H) + P(F, \mathbb{R}^n \setminus H)\right),
\end{aligned}$$

where the last inequality follows as in (3.6.5), and C denotes a constant depending on suitable parameters that changes from line to line. For r_0 small, depending only on n, λ, we can ensure that

$$\mathcal{H}^{n-1}\left(\partial B^\lambda(|Z|, y) \cap \partial H\right) \geq C(n, \lambda) > 0.$$

Finally

$$\begin{aligned}
\left|\mathcal{H}^{n-1}\left(\partial B^\lambda(|F|, y) \cap \partial H\right) - \mathcal{H}^{n-1}\left(\partial B^\lambda(|Z|, y) \cap \partial H\right)\right| &\leq L\,\big||Z| - |F|\big| \\
&\overset{(3.6.5)}{\leq} C(n, \lambda)\left(P(G, \mathbb{R}^n \setminus H) + P(F, \mathbb{R}^n \setminus H)\right),
\end{aligned}$$

for a suitable Lipschitz constant $L = L(n, \lambda)$. Therefore (3.6.6) becomes

$$\beta_\lambda(Z) - \beta_\lambda(F) \leq C(n, \lambda)\left(P(G, \mathbb{R}^n \setminus H) + P(F, \mathbb{R}^n \setminus H)\right).$$

In case $\beta_\lambda(Z) < \beta_\lambda(F)$, the very same argument leads to an analogous estimate. Hence

$$\left|\beta_\lambda(Z) - \beta_\lambda(F)\right| \leq \widetilde{C}(n, \lambda)\left(P(G, \mathbb{R}^n \setminus H) + P(F, \mathbb{R}^n \setminus H)\right). \tag{3.6.7}$$

Up to taking a smaller r_0, we fix $r_0 = r_0(n, \lambda, \Lambda) \in (0, 1)$ and $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda) \in (0, 1)$ such that $1 - |\lambda| - \varepsilon_0 \widetilde{C}(n, \lambda) - \Lambda \overline{C}(n) r_0 > 0$, and we define

$$K := \frac{1 + |\lambda| + \varepsilon_0 \widetilde{C}(n, \lambda) + \Lambda \overline{C}(n) r_0}{1 - |\lambda| - \varepsilon_0 \widetilde{C}(n, \lambda) - \Lambda \overline{C}(n) r_0} > 1.$$

Let δ_7, C_7 be given by Lemma 3.6.4 corresponding to the parameters $K, r_0/2$. We want to prove that if δ_7 is sufficiently small, then

$$\beta_\lambda(E) \leq 2 C_7 D_\lambda(E)^{\frac{1}{2n}}.$$

We argue by contradiction assuming that there exist sets $E_j \subset \mathbb{R}^n \setminus H$ with $|E_j| = |B^\lambda|$ and $D_\lambda(E_j) \le 1/j$ such that

$$\beta_\lambda(E_j) > 2C_7 D_\lambda(E_j)^{\frac{1}{2n}},$$

for any j. Up to translation, $E_j \to B^\lambda(|B^\lambda|, 0)$ and $P_\lambda(E_j) \to P_\lambda(B^\lambda)$. Since the trace operator is continuous with respect to strict convergence of BV functions by Theorem 2.3.7, by Lemma 3.6.1 we deduce that $\beta_\lambda(E_j) \to 0$.
Let F_j be a minimizer of the problem

$$\min\left\{ P_\lambda(E) + \varepsilon_0|\beta_\lambda(E) - \beta_\lambda(E_j)| + \Lambda\left||E| - |E_j|\right| \ : \ E \subset Q \right\}. \tag{3.6.8}$$

By Corollary 3.3.6, up to subsequence F_j converges to a limit set F in L^1. If by contradiction $\beta_j(F_j) \not\to 0$, by Lemma 3.5.13 for large j we would have that

$$P_\lambda(B^\lambda(|B^\lambda|)) + \varepsilon_0\beta_\lambda(E_j) < P_\lambda(F_j) + \varepsilon_0|\beta_\lambda(F_j) - \beta_\lambda(E_j)| + \Lambda\left||F_j| - |E_j|\right|,$$

contradicting the minimality of F_j. Hence $\beta_\lambda(F_j) \to 0$. It follows that, up to translation, F_j converges to $B^\lambda(|B^\lambda|)$ in L^1. Comparing with E_j, we also see that $P_\lambda(F_j) \to P_\lambda(B^\lambda)$.
We want to show that F_j is (K, r_0)-quasiminimal for j large. Indeed, $|B^\lambda|/2 \le |F_j| \le 2|B^\lambda|$ for j large. Hence we can apply (3.6.5) and (3.6.7) with $F = F_j$. Letting $G \subset \mathbb{R}^n \setminus H$ such that $G\Delta F \subset\subset B_{r_0}(x)$, for $x \in \{x_n \ge 0\}$, denoting $Z := G \cap Q$, by minimality of F_j for (3.6.8) we find

$$\begin{aligned}(1-|\lambda|)P(F_j, \mathbb{R}^n \setminus H) &\le (1+|\lambda|)P(Z, \mathbb{R}^n \setminus H) + \varepsilon_0\left|\beta_\lambda(Z) - \beta_\lambda(F_j)\right| + \Lambda\left||Z| - |F_j|\right| \\ &\le (1+|\lambda|)P(G, \mathbb{R}^n \setminus H) + \left(\varepsilon_0\widetilde{C}(n,\lambda) + \Lambda\overline{C}(n)r_0\right)\left(P(G, \mathbb{R}^n \setminus H) + P(F_j, \mathbb{R}^n \setminus H)\right),\end{aligned}$$

proving that F_j is (K, r_0)-quasiminimal.
By minimality of F_j, we have

$$\begin{aligned}&P_\lambda(F_j) + \Lambda\left||F_j| - |B^\lambda|\right| + \varepsilon_0\left|\beta_\lambda(F_j) - \beta_\lambda(E_j)\right| \le P_\lambda(E_j) \\ &\le P_\lambda(B^\lambda) + \frac{P_\lambda(B^\lambda)}{(2C_7)^{2n}}\beta_\lambda^{2n}(E_j) \le P_\lambda(F_j) + \Lambda\left||F_j| - |B^\lambda|\right| + \frac{P_\lambda(B^\lambda)}{(2C_7)^{2n}}\beta_\lambda^{2n}(E_j)\end{aligned} \tag{3.6.9}$$

Therefore

$$|\beta_\lambda(F_j) - \beta_\lambda(E_j)| \le \frac{P_\lambda(B^\lambda)}{\varepsilon_0(2C_7)^{2n}}\beta_\lambda^{2n}(E_j),$$

and then

$$\frac{\beta_\lambda(F_j)}{\beta_\lambda(E_j)} \to 1.$$

Next we select $\{\hat\lambda_j\} \subset (0, \infty)$ such that, setting $\tilde F_j := \hat\lambda_j F_j$, then $|\tilde F_j| = |B^\lambda|$. Clearly $\hat\lambda_j \to 1$ since $|F_j| \to |B^\lambda|$. Since $P_\lambda(F_j) \to P_\lambda(B^\lambda))$ and $\Lambda > n$, for j sufficiently large we have $P_\lambda(F_j) < \Lambda|F_j|$ and

$$\left|P_\lambda(\tilde F_j) - P_\lambda(F_j)\right| = P_\lambda(F_j)\left|\hat\lambda_j^{n-1} - 1\right| \le P_\lambda(F_j)\left|\hat\lambda_j^{n} - 1\right| \le \Lambda\left|\hat\lambda_j^{n} - 1\right| |F_j| = \Lambda\left||\tilde F_j| - |F_j|\right|.$$

Hence, by definition of $\hat\lambda_j$ and by (3.6.9) we get

$$P_\lambda(\tilde F_j) \le P_\lambda(F_j) + \Lambda\left||\tilde F_j| - |F_j|\right| = P_\lambda(F_j) + \Lambda\left||F_j| - |B^\lambda|\right| \overset{(3.6.9)}{\le} P_\lambda(B^\lambda) + \frac{P_\lambda(B^\lambda)}{(2C_7)^{2n}}\beta_\lambda^{2n}(E_j). \tag{3.6.10}$$

Since $\beta_\lambda(F_j)/\beta_\lambda(E_j) \to 1$ as $j \to \infty$ and β_λ is scale invariant, we have $\beta_\lambda(E_j)^{2n} < 2\beta_\lambda(\tilde F_j)^{2n}$ for j sufficiently large. Hence from (3.6.10) we obtain

$$\beta_\lambda(\tilde F_j)^{2n} \ge 2^{2n-1}C_7^{2n}D_\lambda(\tilde F_j),$$

that is $\beta_\lambda(\tilde F_j) \ge 2^{1-\frac{1}{2n}}C_7 D_\lambda(\tilde F_j)^{\frac{1}{2n}}$. On the other hand, for j large, $\tilde F_j$ is $(K, \hat\lambda_j r_0)$-quasiminimal. As $\hat\lambda_j \to 1$, then $\tilde F_j$ is $(K, r_0/2)$-quasiminimal for j large. Moreover $D_\lambda(\tilde F_j) \to 0$. By the choice of C_7 above, Lemma 3.6.4 implies that

$$\beta_\lambda(\tilde F_j) \le C_7 D_\lambda(\tilde F_j)^{\frac{1}{2n}},$$

giving a contradiction. □

Proof of Theorem 3.1.2. By Lemma 3.6.5 it follows that for any $A > 0$ there exists $C_A > 0$ such that for any set $E \subset \mathbb{R}^n \setminus H$ with $|E| = |B^\lambda|$ and $\mathcal{H}^{n-1}(\partial^* E \cap \partial H) \leq A$ there holds

$$\beta_\lambda(E) \leq C_A D_\lambda(E)^{\frac{1}{2n}}. \tag{3.6.11}$$

Indeed, if $D_\lambda(E) \leq \delta_8$, for δ_8 as in Lemma 3.6.5, then (3.6.11) follows with $C_A = C_8$. Otherwise we just have

$$\beta_\lambda(E) \leq C(n,\lambda)\left(\mathcal{H}^{n-1}(\partial^* E \cap \partial H) + \mathcal{H}^{n-1}(\partial^* B^\lambda(|B^\lambda|,0) \cap \partial H)\right) \leq C(n,\lambda,A)\frac{\delta_8^{\frac{1}{2n}}}{\delta_8^{\frac{1}{2n}}} \leq C(n,\lambda,A)D_\lambda^{\frac{1}{2n}}.$$

Next we observe that, letting C_λ such that $P_\lambda(B^\lambda) \leq C_\lambda \mathcal{H}^{n-1}(\partial B^\lambda(|B^\lambda|) \cap \partial H)$, then for any set $E \subset \mathbb{R}^n \setminus H$ with $|E| = |B^\lambda|$ and $\mathcal{H}^{n-1}(\partial^* E \cap \partial H) \geq \frac{2C_\lambda}{1-\lambda}\mathcal{H}^{n-1}(\partial B^\lambda(|B^\lambda|) \cap \partial H)$ there holds

$$\beta_\lambda(E) \leq C_9 D_\lambda(E), \tag{3.6.12}$$

for a constant $C_9 = C_9(n,\lambda) > 0$.
Indeed

$$P_\lambda(E) - P_\lambda(B^\lambda) \geq (1-\lambda)\mathcal{H}^{n-1}(\partial^* E \cap \partial H) - C_\lambda \mathcal{H}^{n-1}(\partial B^\lambda(|B^\lambda|) \cap \partial H) \geq \frac{1-\lambda}{2}\mathcal{H}^{n-1}(\partial^* E \cap \partial H),$$

and

$$\beta_\lambda(E) \leq C(n,\lambda)\left(\mathcal{H}^{n-1}(\partial^* E \cap \partial H) + \mathcal{H}^{n-1}(\partial^* B^\lambda(|B^\lambda|,0) \cap \partial H)\right) \leq C(n,\lambda,C_\lambda)\mathcal{H}^{n-1}(\partial^* E \cap \partial H).$$

Setting now $A := \frac{2C_\lambda}{1-\lambda}\mathcal{H}^{n-1}(\partial B^\lambda(|B^\lambda|) \cap \partial H)$ in (3.6.11), taking into account (3.6.12) we conclude that for any set $E \subset \mathbb{R}^n \setminus H$ with $|E| = |B^\lambda|$ there holds

$$\beta_\lambda(E) \leq \max\{C_A, C_9\} \max\left\{D_\lambda(E), D_\lambda(E)^{\frac{1}{2n}}\right\}.$$

□

Chapter 4

Existence and nonexistence results for the capillarity problem with nonlocal repulsion and gravity

4.1 Main results

If $g : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$, we define the Riesz-type potential

$$\mathscr{R}(E) := \int_E \int_E g(y - x)\, \mathrm{d}y\, \mathrm{d}x.$$

Given a function $G : (0, \infty) \to (0, \infty)$, we define the gravity-type potential

$$\mathscr{G}(E) := \int_E G(x_n)\, \mathrm{d}x.$$

If $v > 0$ and we denote

$$\mathscr{F}^\lambda(E) := P_\lambda(E) + \mathscr{R}(E) + \mathscr{G}(E),$$

we consider the nonlocal problem

$$\inf\{\mathscr{F}^\lambda(E) \ : \ E \subset \{x_n > 0\},\ |E| = v\}. \tag{4.1.1}$$

The first result in this chapter is an existence result in the capillarity context and in the small mass regime, together with qualitative properties of volume constrained minimizers.

Theorem 4.1.1 ([Pas25]). *Let g be a $\mathscr{R}$-admissible q-growing function, $q \geq 0$, and let G be a $\mathscr{G}$-admissible function. There exists a mass $\bar{m} = \bar{m}(n, \lambda, g, G, q) > 0$ such that, for every $m \in (0, \bar{m})$, there exists a minimizer of $\mathscr{F}^\lambda$ in the class*

$$\mathcal{A}_m := \{\Omega \subset \mathbb{R}^n \setminus H \text{ measurable} \ : \ |\Omega| = m\}$$

and it satisfies

$$\alpha_\lambda(E) \leq c(n, \lambda, g, G) m^{\frac{1}{2n}}.$$

Moreover, if g is also infinitesimal, minimizers are indecomposable and, if in addition g is symmetric, minimizers are essentially bounded.
Furthermore, if g is also 0-growing, infinitesimal and symmetric and G is coercive, minimizers have no holes, i.e., if E is a minimizer of $\mathscr{F}^\lambda$ in $\mathcal{A}_m$, there is no set $F \subset \mathbb{R}^n \setminus (H \cup E)$ with $|F| > 0$ such that

$$P_\lambda(E) = P_\lambda(E \cup F) + P(F, \mathbb{R}^n \setminus H) + \lambda \mathcal{H}^{n-1}(\partial^* F \cap \partial H).$$

Finally, if g is $\mathscr{R}$-admissible and coercive and G is $\mathscr{G}$-admissible and coercive, there exists a minimizer of $\mathscr{F}^\lambda$ in $\mathcal{A}_m$ for any $m > 0$.

We remark that, by a symmetry argument, analyzing the Euler-Lagrange equation of problem (4.1.1), it is possible to verify that the sets $B^\lambda(m,x)$ are not volume constrained minimizers of $\mathscr{F}^\lambda$; actually, the isoperimetric bubbles $B^\lambda(m,x)$ are not even volume constrained critical points of $\mathscr{F}^\lambda$. It is left as a future project to study quantitative properties of minimizers to (4.1.1), such as the proximity of minimizers from bubbles $B^\lambda(m,x)$ in terms of the smallness of the mass.

For large masses and for suitable choices of g, the repulsive interaction dominates and the variational problem in Theorem 4.1.1 does not admit a minimizer.

Theorem 4.1.2 ([Pas25]). *Let*

$$g(x) = \frac{1}{|x|^\beta}, \qquad 0 < \beta < n,\ x \in \mathbb{R}^n \setminus \{0\}$$

and let G be $\mathscr{G}$-admissible. For every $\beta \in (0,2]$, there exists $\tilde{m} > 0$, depending on n, λ, β, G, such that for all $m \geq \tilde{m}$ the minimization problem

$$\inf\{\mathscr{F}^\lambda(E) \ : \ E \subset \mathbb{R}^n \setminus H, |E| = m\}$$

has no minimizers.

Therefore, for a general g, existence may fail for masses large enough, since minimizers tend to split in two or more components which then move apart one from the other in order to decrease the nonlocal energy. To capture this phenomenon, it is convenient to introduce a generalized energy defined as

$$\tilde{\mathscr{F}}^\lambda(E) := \inf_{h\in\mathbb{N}} \tilde{\mathscr{F}}^\lambda_h(E),$$

where

$$\tilde{\mathscr{F}}^\lambda_h(E) := \inf\left\{ \sum_{i=1}^h \mathscr{F}^\lambda(E^i) \ : \ E = \bigcup_{i=1}^h E^i, E^i \cap E^j = \emptyset \quad \text{for } 1 \leq i \neq j \leq h \right\}.$$

Note that in this functional the interaction between different components is not evaluated, which corresponds to consider them "at infinite distance" one from the other. By considering $\tilde{\mathscr{F}}^\lambda$ instead of $\mathscr{F}^\lambda$, we can prove the following generalized existence result.

Theorem 4.1.3 ([Pas25]). *Let g be a $\mathscr{R}$-admissible q-growing function, $q \geq 0$, and let G be a $\mathscr{G}$-admissible function. For every $m > 0$ there exists a minimizer of $\tilde{\mathscr{F}}^\lambda$ in the class*

$$\mathcal{A}_m = \{\Omega \subset \mathbb{R}^n \setminus H \text{ measurable } : \ |\Omega| = m\}.$$

More precisely, there exist a set $E \in \mathcal{A}$ and a subdivision $E = \cup_{j=1}^h E^j$, with pairwise disjoint sets E^j, such that

$$\tilde{\mathscr{F}}^\lambda(E) = \sum_{j=1}^h \mathscr{F}^\lambda(E^j) = \inf\left\{\tilde{\mathscr{F}}^\lambda(\Omega) \ : \ \Omega \in \mathcal{A}\right\}.$$

Moreover, for every $1 \leq j \leq h$, the set E^j is a minimizer of both the standard and the generalized energy for its volume, i.e.

$$\tilde{\mathscr{F}}^\lambda(E^j) = \mathscr{F}^\lambda(E^j) = \min\left\{\tilde{\mathscr{F}}^\lambda(\Omega) \ : \ \Omega \subset \mathbb{R}^n \setminus H,\ |\Omega| = |E^j|\right\}.$$

Remark 4.1.4. We note that, if g is infinitesimal, then

$$\inf\left\{\tilde{\mathscr{F}}^\lambda(\Omega) \ : \ \Omega \in \mathcal{A}_m\right\} = \inf\left\{\mathscr{F}^\lambda(\Omega) \ : \ \Omega \in \mathcal{A}_m\right\}.$$

The proof can be easily adapted by [NP21, Lemma 3.4], with attention given to translating the components of a partition without changing the n-th component. In this case, every minimizer of $\mathscr{F}^\lambda$ is also a minimizer of $\tilde{\mathscr{F}}^\lambda$.

Strategy of the proof and comments The proof of Theorem 4.1.1 is divided into several steps. In the spirit of [KM14], the existence of minimizers in the small mass regime follows by the direct method of the calculus of variations, see Theorem 4.3.1, once we show that for sufficiently small mass every minimizing sequence of the energy may be replaced by another minimizing sequence where all sets have uniformly bounded diameter, see Lemma 4.3.5. We remark that we heavily use the quantitative isoperimetric inequality for the capillarity problem proved in [PP24] and described in Chapter 3, which estimates the Fraenkel asymmetry of a competitor with respect to the optimal sets in terms of the energy deficit. As pointed out in Chapter 3, it is unclear at the moment how to apply stronger isoperimetric inequalities of Fuglede-type [CL12; Fug89] for nearly spherical sets in the present capillarity framework; instead, stronger isoperimetric inequalities have been used as fundamental tools, for example, in [AFM13; BC14; CFP23]. In fact, the classical Fuglede's method relies on the precise knowledge of the eigenvalues of the Laplace-Beltrami operator, which is not available for P_λ on optimal sets for generic $\lambda \in (-1, 1)$. Moreover, in our case it is in general not possible to globally parametrize C^1-close boundaries one on the other as normal graphs.

The boundedness result in Theorem 4.1.1 follows once we show that minimizers enjoy uniform density estimates at boundary points. In order to do so, we prove that, under suitable conditions on the Riesz potential, minimizers are (K, r_0)-quasiminimal sets for all masses, see Definition 3.6.2 and Lemma 4.3.13. Indeed quasiminimal sets have well-known topological regularity properties (Theorem 3.6.3), which easily guarantee boundedness, see Theorem 4.3.8. Note that the lack of symmetry of the problem, due to the presence of gravitational potential and the fact that ambient space is a half-space, forces us to deal with the vertical direction in a separate way, see Lemma 4.3.12.

The absence of holes is based on the combination of some techniques from [KM14] and [NP21]. We firstly prove some density estimates which improve, under suitable hypotheses on g, the analogous estimates for quasiminimal sets, by providing bounds independent of the minimizer, see Lemma 4.4.2. In fact, this allows to prove the boundedness in the vertical direction with a bound independent of the minimizer, see Lemma 4.4.8, and to obtain absence of holes arguing by contradiction, see Theorem 4.4.7.

The proof of Theorem 4.1.2 is based on the combination of some techniques from [FN21] and [KM14] and exploits some estimates on the diameter and the nonlocal potential energy of minimizers. We remark that the range of the exponent β in Theorem 4.1.2 is the same as the analogous nonexistence results in the classical setting [CNT22; FKN16; FN21; KM14; LO14].

The proof of Theorem 4.1.3 is inspired by [NP21] and exploits the isoperimetric inequality for the capillarity functional P_λ. In our case the argument must be modified to take into account the presence of the gravitational energy and, as before, estimates in the vertical direction must be treated separately. We remark that also the possible choices for the kernels g in our Theorem 4.1.3 allow for more freedom than those considered in [NP21].

From now on and for the rest of the chapter we assume that $\lambda \in (-1, 1)$ and $n \in \mathbb{N}$ with $n \geq 2$ are fixed.

4.2 Definitions

We provide some definitions for the kernels of Riesz-type potential $\mathscr{R}$ and gravity-type potential $\mathscr{G}$. In particular, in the following Definition 4.2.1 and Definition 4.2.4 we will impose pointwise requirements on the functions g, G, i.e., it is understood that we fix pointwise defined representatives for the functions g, G.

Definition 4.2.1. A function $g : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ is $\mathscr{R}$-admissible if $g \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$ and $\mathscr{R}(B_1) < \infty$. A $\mathscr{R}$-admissible function $g : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ is q-growing, for some $q \in [0, \infty)$, if for every $x \in \mathbb{R}^n \setminus \{0\}$ and every $\alpha > 1$ it holds

$$g(\alpha x) \leq \alpha^q g(x).$$

A $\mathscr{R}$-admissible function $g : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ is infinitesimal if

$$\lim_{|x| \to +\infty} g(x) = 0.$$

A $\mathscr{R}$-admissible function $g : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ is symmetric if

$$g(-x) = g(x) \qquad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

A $\mathscr{R}$-admissible function $g : \mathbb{R}^n \setminus \{0\} \to (0,\infty)$ is coercive if

$$g(x) \to +\infty \qquad \text{as } |x| \to +\infty.$$

Given two measurable sets $L, M \subset \mathbb{R}^n \setminus H$ we let

$$\mathscr{R}(L,M) := \int_L \int_M g(y-x)\,\mathrm{d}y\,\mathrm{d}x.$$

Remark 4.2.2. The functions $\frac{1}{|x|^\beta}$, for $\beta \in (0,n)$, are $\mathscr{R}$-admissible, 0-growing, infinitesimal and symmetric.

Remark 4.2.3. The attractive-repulsive kernels $|x|^{\beta_1} + \frac{1}{|x|^{\beta_2}}$, for $\beta_1 > 0$ and $\beta_2 \in (0,n)$, are $\mathscr{R}$-admissible β_1-growing symmetric functions. At the same time they diverge positively as $|x| \to +\infty$.

Definition 4.2.4. A function $G : (0,\infty) \to (0,\infty)$ is $\mathscr{G}$-admissible if $G \in L^1_{loc}(0,\infty)$,

$$\sup_{t\in(0,2)} G(t) < \infty, \tag{4.2.1}$$

and

$$G(\alpha t) \leq \alpha^n G(t), \qquad \forall \alpha > 1,\, t > 0. \tag{4.2.2}$$

A $\mathscr{G}$-admissible function $G : (0,\infty) \to (0,\infty)$ is coercive if

$$G(t) \to +\infty \qquad \text{as } t \to +\infty.$$

Remark 4.2.5. The identity function $G(t) = t$ on $(0,+\infty)$ is a $\mathscr{G}$-admissible function.

Remark 4.2.6. Conditions (4.2.1) and (4.2.2) easily imply

$$G(t) = G(t\cdot 1) \leq t^n G(1) \leq c(G)\,t^n, \qquad \forall t > 1.$$

4.3 Existence of minimizers for small masses

Existence

The goal of this Section is to prove the following

Theorem 4.3.1. *Let g be a $\mathscr{R}$-admissible q-growing function and let G be a $\mathscr{G}$-admissible function. There exists a mass $\bar{m} = \bar{m}(n,\lambda,g,G,q) > 0$ such that, for all $m \in (0,\bar{m})$, there exists a minimizer of $\mathscr{F}^\lambda$ in the class*

$$\mathcal{A}_m := \{\Omega \subset \mathbb{R}^n \setminus H \text{ measurable } : |\Omega| = m\}.$$

We begin by proving some preparatory lemmas, which estimate the energy of some competitors.

Lemma 4.3.2. *Let g be $\mathscr{R}$-admissible and G be $\mathscr{G}$-admissible. There exists a constant $c = c(n,\lambda,g,G)$ such that*

$$\mathscr{R}(B^\lambda(m)) \leq c\,m, \qquad \mathscr{G}(B^\lambda(m)) \leq c\,m$$

for every $0 < m \leq |B^\lambda|$.

Proof. Let us denote by $\bar{Q}_l \subset \mathbb{R}^n \setminus H$, with $l > 0$, the cube $[-l,l]\times\cdots\times[-l,l]\times[0,2l]$. For any $N \in \mathbb{N}$ the cube $\bar{Q}_1$ is the essential union of $(2N)^n$ disjoint isometric cubes $Q^i_{\frac{1}{2N}}$ of side $1/N$. If $\bar{Q}_{\frac{1}{2N}} \subset \bar{Q}_1$ is the cube $\left[-\frac{1}{2N},\frac{1}{2N}\right]\times\cdots\times\left[-\frac{1}{2N},\frac{1}{2N}\right]\times\left[0,\frac{1}{N}\right]$, evidently

$$\mathscr{R}(\bar{Q}_1) \geq \sum_{i=1}^{(2N)^n} \mathscr{R}(Q^i_{\frac{1}{2N}}) = (2N)^n \mathscr{R}(\bar{Q}_{\frac{1}{2N}}).$$

Moreover

$$2^n \sup_{(0,2)} G = |\bar{Q}_1| \sup_{(0,2)} G = (2N)^n \frac{|\bar{Q}_1|}{(2N)^n} \sup_{(0,2)} G = (2N)^n \int_{\bar{Q}_{\frac{1}{2N}}} \sup_{(0,2)} G \,\mathrm{d}x \geq (2N)^n \mathscr{G}(\bar{Q}_{\frac{1}{2N}}).$$

For any $0 < r \leq 1$ we denote by N the integer part of $\frac{1}{2r}$, so that $(2r)^{-1} \leq 2N \leq r^{-1}$. The above estimates, together with $4rN \geq 1$, imply that

$$\mathscr{R}(\bar{Q}_r) \leq (4r)^n N^n \mathscr{R}(\bar{Q}_{\frac{1}{2N}}) \leq 2^n \mathscr{R}(\bar{Q}_1) r^n$$

and

$$\mathscr{G}(\bar{Q}_r) \leq (4r)^n N^n \mathscr{G}(\bar{Q}_{\frac{1}{2N}}) \leq 4^n \left(\sup_{(0,2)} G \right) r^n.$$

If $r = \frac{m^{\frac{1}{n}}}{|B^\lambda|^{\frac{1}{n}}} \leq 1$, since $B^\lambda(|B^\lambda|) \subset \bar{Q}_1$, we get $B^\lambda(m) \subset \bar{Q}_r$ and we conclude that

$$\mathscr{R}(B^\lambda(m)) \leq \mathscr{R}(\bar{Q}_r) \leq cr^n \leq cm$$

and

$$\mathscr{G}(B^\lambda(m)) \leq \mathscr{G}(\bar{Q}_r) \leq cr^n \leq cm.$$

□

Corollary 4.3.3. *Let g be $\mathscr{R}$-admissible and infinitesimal and let G be $\mathscr{G}$-admissible. For every $m \geq 1$ there exists $E \subset \mathbb{R}^n \setminus H$ with $|E| = m$ such that $\mathscr{F}^\lambda(E) \leq cm$ for some c depending on n, λ, g and G.*

Proof. Let us consider the set E given by a collection of $N \geq 1$ spherical caps $\left\{B^\lambda(v, x_i)\right\}_{1 \leq i \leq N}$ of equal volume v and with centers located at $x_i = iRe_1$, $i = 1, \dots, N$, with R large enough so that $B^\lambda(v, x_i)$ are pairwise disjoint. We choose the number N as the smallest integer for which the volume of each spherical cap does not exceed $\min\left\{1, |B^\lambda|\right\}$. In particular $Nv = m$ and $N = \left\lceil \frac{m}{\min\{1,|B^\lambda|\}} \right\rceil$. Note that, by [PP24, Lemma 3.1], since $v \leq 1 \leq m = |E|$,

$$\begin{aligned} P_\lambda(E) = P_\lambda\left(\cup_{i=1}^N B^\lambda(v, x_i)\right) = \sum_{i=1}^N P_\lambda(B^\lambda(v, x_i)) = c(n, \lambda) N v^{\frac{n-1}{n}} &\leq c(n, \lambda) \left(\frac{m}{\min\{1, |B^\lambda|\}} + 1 \right) v^{\frac{n-1}{n}} \\ &\leq c(n, \lambda)\,(m\, v^{\frac{n-1}{n}} + v^{\frac{n-1}{n}}) \leq c(n, \lambda)\,(m\, 1^{\frac{n-1}{n}} + m\, 1^{\frac{n-1}{n}}) = c(n, \lambda) m. \end{aligned}$$

Moreover, let R be so large that $g(x-y) < \frac{1}{N}$ for every $x \in B^\lambda(v, x_j)$, $y \in B^\lambda(v, x_k)$ with $j \neq k$. Then, by Lemma 4.3.2, since $v \leq 1 \leq m = |E|$,

$$\begin{aligned} \mathscr{R}(E) &= \int_{\bigcup_{i=1}^N B^\lambda(v,x_i)} \int_{\bigcup_{i=1}^N B^\lambda(v,x_i)} g(y-x) \,\mathrm{d}y \,\mathrm{d}x \\ &= \sum_{i=1}^N \mathscr{R}(B^\lambda(v, x_i)) + \sum_{i=1}^N \int_{B^\lambda(v,x_i)} \int_{B^\lambda(v,x_1) \cup \cdots \cup \widehat{B^\lambda(v,x_i)} \cup \cdots \cup B^\lambda(v,x_N)} g(y-x) \,\mathrm{d}y \,\mathrm{d}x \\ &\leq c(n, \lambda, g, G) N v + N \frac{1}{N} v(m - v) \leq c(n, \lambda, g, G) m + (m - v) \leq c(n, \lambda, g, G) m, \end{aligned}$$

where the $B^\lambda(v, x_1) \cup \cdots \cup \widehat{B^\lambda(v, x_i)} \cup \cdots \cup B^\lambda(v, x_N)$ denotes union over all the bubbles except for $B^\lambda(v, x_i)$. Finally, by Lemma 4.3.2

$$\begin{aligned} \mathscr{F}^\lambda(E) &= P_\lambda(E) + \mathscr{R}(E) + \mathscr{G}(E) \\ &\leq c(n, \lambda)|E| + c(n, \lambda, g, G)|E| + \sum_{i=1}^N \mathscr{G}(B^\lambda(v, x_i)) \\ &\leq c(n, \lambda)|E| + c(n, \lambda, g, G)|E| + c(n, \lambda, g, G) N v = c(n, \lambda, g, G) m. \end{aligned}$$

□

Lemma 4.3.4. *Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter. Let g be $\mathscr{R}$-admissible and q-growing and let G be $\mathscr{G}$-admissible. If $\alpha > 1$, then*

$$\mathscr{F}^\lambda(\alpha E) \leq \alpha^{2n+q} \mathscr{F}^\lambda(E).$$

Proof. Note that, if $E \subset \mathbb{R}^n \setminus H$, then $\alpha E \subset \mathbb{R}^n \setminus H$. Since $\alpha > 1$, by the positivity of P_λ we get

$$P_\lambda(\alpha E) = \alpha^{n-1} P_\lambda(E) \leq \alpha^{2n+q} P_\lambda(E).$$

Since g is q-growing, we have

$$\begin{aligned}
\mathscr{R}(\alpha E) &= \int\int_{(\alpha E)^2} g(y-x)\,\mathrm{d}y\,\mathrm{d}x \\
&= \alpha^{2n} \int\int_{E\times E} g(\alpha(y-x))\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \alpha^{2n+q} \int\int_{E\times E} g(y-x)\,\mathrm{d}y\,\mathrm{d}x = \alpha^{2n+q}\mathscr{R}(E).
\end{aligned}$$

Finally, by (4.2.2) we get

$$\int_{\alpha E} G(x_n)\,\mathrm{d}x = \alpha^n \int_E G(\alpha x_n)\,\mathrm{d}x \leq \alpha^{2n+q} \int_E G(x_n)\,\mathrm{d}x.$$

□

The following lemma allows to suitably localize minimizing sequences with sufficiently small volume.

Lemma 4.3.5. *Let g be $\mathscr{R}$-admissible and q-growing and let G be $\mathscr{G}$-admissible. There exists $\bar{m} > 0$, depending on n, λ, g, G and q, such that, for every $m \in (0, \bar{m})$ and every set of finite perimeter $F \subset \mathbb{R}^n \setminus H$ with $|F| = m$, there exists a set of finite perimeter L with*

$$\mathscr{F}^\lambda(L) \leq \mathscr{F}^\lambda(F) \qquad \text{and} \qquad L \subset \bar{Q}_1 := [-1,1]\times\cdots\times[-1,1]\times[0,2]. \tag{4.3.1}$$

Proof. Throughout the proof we will assume that $\bar{m} < \frac{|B^\lambda|}{4^n}$. If $\mathscr{F}^\lambda(B^\lambda(m)) \leq \mathscr{F}^\lambda(F)$, then the assertion of the lemma is proved by choosing $L = B^\lambda(m)$. Then we can assume, by Lemma 4.3.2, that

$$\mathscr{F}^\lambda(F) < \mathscr{F}^\lambda(B^\lambda(m)) \leq c(n,\lambda,g,G) \max\left\{m, m^{\frac{n-1}{n}}\right\}. \tag{4.3.2}$$

By Lemma 4.3.2

$$\begin{aligned}
D_\lambda(F) &= \frac{c(n,\lambda)}{m^{\frac{n-1}{n}}}(P_\lambda(F) - P_\lambda(B^\lambda(m))) \\
&\leq \frac{c(n,\lambda)}{m^{\frac{n-1}{n}}}([\mathscr{R}(B^\lambda(m)) - \mathscr{R}(F)] + [\mathscr{G}(B^\lambda(m)) - \mathscr{G}(F)]) \\
&\leq \frac{c(n,\lambda)}{m^{\frac{n-1}{n}}}(\mathscr{R}(B^\lambda(m)) + \mathscr{G}(B^\lambda(m))) \\
&\leq c(n,\lambda,g,G)\left(\frac{m}{|B^\lambda|}\right)^{\frac{1}{n}}.
\end{aligned} \tag{4.3.3}$$

By the quantitative isoperimetric inequality 3.1.1

$$\alpha_\lambda(F) \leq c(n,\lambda)\sqrt{D_\lambda(F)} \leq c(n,\lambda,g,G)\left(\frac{m}{|B^\lambda|}\right)^{\frac{1}{2n}} \tag{4.3.4}$$

and, after a suitable translation,

$$|B^\lambda(m)\Delta F| \leq c(n,\lambda,g,G)\left(\frac{m}{|B^\lambda|}\right)^{1+\frac{1}{2n}}.$$

Since $|F| = m = |B^\lambda(m)|$ we also have

$$|B^\lambda(m)\Delta F| = 2|F \setminus B^\lambda(m)|$$

and

$$|F \setminus B^\lambda(m)| \le c(n,\lambda,g,G)\left(\frac{m}{|B^\lambda|}\right)^{1+\frac{1}{2n}}. \tag{4.3.5}$$

For any $\varrho > 0$ let $F_1 = F \cap B_\varrho(0)$ and $F_2 = F \setminus B_\varrho(0)$. Note that for every $\varepsilon > 0$ there exists $\bar m$ sufficiently small such that, if $\varrho \ge \frac{m^{\frac{1}{n}}}{|B^\lambda|^{\frac{1}{n}}} R_\lambda =: \varrho_m$, with $m < \bar m$, then

$$|F_2| \le \varepsilon |F_1|. \tag{4.3.6}$$

Indeed, since $B^\lambda(|B^\lambda|) \subset B_{R_\lambda}(0)$, we get $B^\lambda(m) \subset B_\varrho(0)$. Moreover, by (4.3.5) and for sufficiently small $\bar m$ we estimate

$$\begin{aligned}
|F_1| = |F \cap B_\varrho(0)| &\ge |F \cap B^\lambda(m)| \\
&= |F| - |F \setminus B^\lambda(m)| \\
&\ge |B^\lambda|\left(\frac{m}{|B^\lambda|}\right) - c(n,\lambda,g,G)\left(\frac{m}{|B^\lambda|}\right)^{1+\frac{1}{2n}} \\
&\ge \frac{|B^\lambda|}{2}\left(\frac{m}{|B^\lambda|}\right).
\end{aligned}$$

and

$$|F \setminus B_\varrho(0)| \le |F \setminus B^\lambda(m)| \le c(n,\lambda,g,G)\left(\frac{m}{|B^\lambda|}\right)^{1+\frac{1}{2n}} \le c(n,\lambda,g,G)\left(\frac{m}{|B^\lambda|}\right)^{\frac{1}{2n}} |F_1| \le \varepsilon |F_1|.$$

Let us define the monotonically decreasing function $U(\varrho) = |F \setminus B_\varrho(0)|$.
We now distinguish two cases. Let us firstly prove (4.3.1) when we assume that

$$\Sigma := P_\lambda(F_1) + P_\lambda(F_2) - P_\lambda(F) > \frac{1}{2}\mathscr{F}^\lambda(F_2) \qquad \forall \varrho \in \left(\varrho_m, \frac{R_\lambda}{2}\right) \tag{4.3.7}$$

By (4.3.5) we have

$$U(\varrho_m) = |F \setminus B^\lambda(m)| \le c(n,\lambda,g,G)\, m^{1+\frac{1}{2n}} \le c(n,\lambda,g,G)\varrho_m^{n+\frac{1}{2}}$$

Furthermore, by Theorem 3.2.3 and (4.3.7) we have

$$-2\frac{\mathrm{d}U(\varrho)}{\mathrm{d}\varrho} = \Sigma > \frac{1}{2}\mathscr{F}^\lambda(F_2) \ge \frac{1}{2}P_\lambda(F \setminus B_\varrho(0)) \ge c(n,\lambda)U^{\frac{n-1}{n}}(\varrho).$$

In particular we have

$$\begin{cases}
\frac{\mathrm{d}U(\varrho)}{\mathrm{d}\varrho} \le -c(n,\lambda)U^{\frac{n-1}{n}}(\varrho) & \text{for a.e. } \varrho \in \left(\varrho_m, \frac{R_\lambda}{2}\right) \\
U(\varrho_m) \le c(n,\lambda,g,G)\varrho_m^{n+\frac{1}{2}}.
\end{cases}$$

By ODE comparison we deduce that, if $\bar m < 1$,

$$\begin{aligned}
U(\varrho)^{\frac{1}{n}} &\le U(\varrho_m)^{\frac{1}{n}} - c(n,\lambda)(\varrho - \varrho_m) \\
&\le c(n,\lambda,g,G)\,\varrho_m^{1+\frac{1}{2n}} - c(n,\lambda)\varrho + c(n,\lambda)\varrho_m \\
&= c(n,\lambda,g,G)\, m^{\frac{1}{n}+\frac{1}{2n^2}} - c(n,\lambda)\,\varrho + c(n,\lambda)\, m^{\frac{1}{n}} \\
&\le c(n,\lambda,g,G)\, m^{\frac{1}{n}} - c(n,\lambda)\,\varrho + c(n,\lambda)\, m^{\frac{1}{n}} \\
&= c(n,\lambda,g,G)\, m^{\frac{1}{n}} - c(n,\lambda)\,\varrho.
\end{aligned}$$

For $\bar{m}$ sufficiently small, it follows that $U(\varrho) = 0$ for $\varrho \geq \frac{R_\lambda}{2}$, and we obtain (4.3.1) with $L = F$.
Let us prove (4.3.1) assuming that

$$\Sigma \leq \frac{1}{2}\mathscr{F}^\lambda(F_2) \tag{4.3.8}$$

holds for some $\varrho_0 \in \left(\varrho_m, \frac{R_\lambda}{2}\right)$. Let $m_1 := |F_1|$, $m_2 := |F_2|$ and $\gamma := \frac{m_2}{m_1} \leq \varepsilon$, with ε that will be chosen suitably small later. Let us also denote $\tilde{F} = l\, F_1$, with $l := (1+\gamma)^{\frac{1}{n}}$. In particular $|\tilde{F}| = m$ and, if ε is sufficiently small,

$$\tilde{F} = (1+\gamma)^{\frac{1}{n}} F_1 = (1+\gamma)^{\frac{1}{n}} \left(F \cap B_{\varrho_0}(0)\right) \subset B_{\varrho_0 \sqrt[n]{1+\gamma}}(0) \subset B_{R_\lambda}(0) \subset \bar{Q}_1.$$

By Lemma 4.3.4

$$\begin{aligned} \mathscr{F}^\lambda(\tilde{F}) = \mathscr{F}^\lambda(lF_1) &\leq l^{2n+q}\mathscr{F}^\lambda(F_1) \\ &= \mathscr{F}^\lambda(F_1) + \left(l^{2n+q} - 1\right)\mathscr{F}^\lambda(F_1). \end{aligned} \tag{4.3.9}$$

Choosing $\varepsilon \leq 1$, we have $1 \leq l \leq 2^{\frac{1}{n}}$, and by Taylor's formula we obtain $l^{2n+q} - 1 = (1+\gamma)^{2+q/n} - 1 \leq \gamma K$ for some $K > 0$ independent of γ and for ε sufficiently small. By (4.3.9) we arrive at

$$\mathscr{F}^\lambda(\tilde{F}) - \mathscr{F}^\lambda(F_1) \leq \gamma K \mathscr{F}^\lambda(F_1).$$

By the definition of Σ and since $\mathscr{R}(F_1) + \mathscr{R}(F_2) \leq \mathscr{R}(F)$

$$\begin{aligned} \mathscr{F}^\lambda(\tilde{F}) - \mathscr{F}^\lambda(F) &\leq \mathscr{R}(F_1) + \mathscr{G}(F_1) + \mathscr{R}(F_2) + \mathscr{G}(F_2) - \mathscr{R}(F) - \mathscr{G}(F) + \Sigma - \mathscr{F}^\lambda(F_2) + \gamma K \mathscr{F}^\lambda(F_1) \\ &\leq -\frac{1}{2}\mathscr{F}^\lambda(F_2) + \gamma K \mathscr{F}^\lambda(F_1). \end{aligned} \tag{4.3.10}$$

By positivity of $\mathscr{R}$ and $\mathscr{G}$ and the isoperimetric inequality Theorem 3.2.3, we have $\mathscr{F}^\lambda(F_2) > P_\lambda(F_2) \geq c(n,\lambda) m_2^{\frac{n-1}{n}}$.
By (4.3.8) we obtain

$$\begin{aligned} \mathscr{F}^\lambda(F) - \mathscr{F}^\lambda(F_1) &= P_\lambda(F) + \mathscr{R}(F) + \mathscr{G}(F) - P_\lambda(F_1) - \mathscr{R}(F_1) - \mathscr{G}(F_1) - P_\lambda(F_2) + P_\lambda(F_2) \\ &\geq -\frac{1}{2}\mathscr{F}^\lambda(F_2) + \mathscr{R}(F) + \mathscr{G}(F) - \mathscr{R}(F_1) - \mathscr{G}(F_1) + P_\lambda(F_2) \\ &= -\frac{1}{2}P_\lambda(F_2) - \frac{1}{2}\mathscr{R}(F_2) - \frac{1}{2}\mathscr{G}(F_2) + \mathscr{R}(F) + \mathscr{G}(F) - \mathscr{R}(F_1) - \mathscr{G}(F_1) + P_\lambda(F_2) \geq 0, \end{aligned} \tag{4.3.11}$$

that is $\mathscr{F}^\lambda(F_1) \leq \mathscr{F}^\lambda(F)$. By (4.3.2), since $\gamma m \leq 2m_2$ and $\gamma \leq \varepsilon$, (4.3.10) turns into

$$\begin{aligned} \mathscr{F}^\lambda(\tilde{F}) - \mathscr{F}^\lambda(F) &\leq -c(n,\lambda) m_2^{\frac{n-1}{n}} + \gamma K \mathscr{F}^\lambda(F) \\ &\leq -c(n,\lambda) m_2^{\frac{n-1}{n}} + C(n,\lambda,g,G,q) \max\left\{ m_2, \varepsilon^{\frac{1}{n}} m_2^{\frac{n-1}{n}} \right\}. \end{aligned}$$

Since $m_2 \leq c(n,\lambda)\varepsilon$ by (4.3.6), for ε sufficiently small (4.3.1) follows with $L = \tilde{F}$. □

Remark 4.3.6. Arguing exactly as in (4.3.2), (4.3.3) and (4.3.4), we easily deduce that if E is a minimizer of $\mathscr{F}^\lambda$ in $\mathcal{A}_m$ with m sufficiently small then

$$\alpha_\lambda(E) \leq c(n,\lambda,g,G) m^{\frac{1}{2n}}.$$

Now we are ready to prove Theorem 4.3.1.

Proof of Theorem 4.3.1. By Lemma 4.3.5 there exists a minimizing sequence with uniformly bounded sets. The lower semicontinuity of P_λ [PP24, Lemma 3.7] and the continuity of $\mathscr{R}$ and $\mathscr{G}$ under strong L^1 convergence (which holds by dominated convergence theorem and uniformly boundedness of the minimizing sequence) allow to conclude the proof. □

The following proposition states that, if the nonlocal kernel g and the gravitational term G are coercive, we have existence of minimizers for *all* values of m.

Proposition 4.3.7. *Let g be a $\mathcal{R}$-admissible coercive function and let G be a $\mathcal{G}$-admissible coercive function. Then, for every $m > 0$, there exists a minimizer of $\mathcal{F}^\lambda$ in the class*

$$\mathcal{A}_m := \{\Omega \subset \mathbb{R}^n \setminus H \text{ measurable } : |\Omega| = m\}.$$

Proof. Let us consider a minimizing sequence $\{E_i\}_{i\in\mathbb{N}}$ of $\mathcal{F}^\lambda$ in $\mathcal{A}_m$. In particular

$$\sup_{i\in\mathbb{N}} \mathcal{F}^\lambda(E_i) = C < +\infty.$$

By [PP24, Corollary 2.5] it holds

$$P(E_i) \leq \frac{2}{1-\lambda} P_\lambda(E_i) \leq \frac{2}{1-\lambda}\mathcal{F}^\lambda(E_i) \leq C, \qquad \forall i \in \mathbb{N}.$$

By [ANP22, Lemma 2.10] and [LRV22, Corollary 3.25] we get the existence of a constant $\bar{c} = \bar{c}\big(m, n, \sup_i P(E_i)\big) > 0$ such that for every $i \in \mathbb{N}$ there exists $x_i \in \{x_n > 0\}$ with

$$\left|E_i \cap B_1(x_i)\right| \geq \bar{c}.$$

By the coercivity of G we have that $\left(x_i\right)_n$ is uniformly bounded. Indeed, if by contradiction $\left(x_i\right)_n \to +\infty$, then

$$C \geq \int_{E_i} G(x_n)\,\mathrm{d}x \geq \int_{E_i\cap B_1(x_i)} G(x_n)\,\mathrm{d}x \geq \bar{c}\left(\inf_{B_1(x_i)} G(x_n)\right) \to +\infty.$$

By the lower semicontinuity of $\mathcal{F}^\lambda$ under strong L^1 convergence, the existence of a minimizer follows if we show that for every $\varepsilon > 0$ and $i \in \mathbb{N}$ there exists $R > 0$ with

$$\sup_{i\in\mathbb{N}} \left|E_i \setminus B_R(x_i)\right| < \varepsilon. \tag{4.3.12}$$

In order to prove (4.3.12), note that

$$\begin{aligned} C \geq \mathcal{R}(E_i) &\geq \int_{E_i}\int_{E_i\setminus B_R(x)} g(y-x)\,\mathrm{d}y\,\mathrm{d}x \geq \left(\inf_{|x|>R} g(x)\right)\int_{E_i} \left|E_i \setminus B_R(x)\right|\,\mathrm{d}x \\ &\geq \left(\inf_{|x|>R} g(x)\right)\int_{E_i\cap B_1(x_i)} \left|E_i \setminus B_R(x)\right|\,\mathrm{d}x \geq \bar{c}\left(\inf_{|x|>R} g(x)\right)\left|E_i \setminus B_{R+1}(x_i)\right| \end{aligned}$$

and

$$\left|E_i \setminus B_{R+1}(x_i)\right| \leq \frac{C}{\bar{c}}\left(\inf_{|x|>R} g(x)\right)^{-1} \to 0 \qquad \text{as} \quad R \to +\infty.$$

□

Boundedness and indecomposability of minimizers

In this section we will prove two qualitative properties of volume constrained minimizers of $\mathcal{F}^\lambda$, namely boundedness and indecomposability. We begin with the following

Theorem 4.3.8. *Let g be $\mathcal{R}$-admissible, infinitesimal and symmetric and let G be $\mathcal{G}$-admissible. Let $E \subset \mathbb{R}^n \setminus H$ be a minimizer of $\mathcal{F}^\lambda$ with $|E| = m$, $m > 0$. Then E is essentially bounded.*

Remark 4.3.9. We remark that Theorem 4.3.8 proves boundedness of minimizers without requiring growing properties of the Riesz-type kernel, but only infinitesimality and symmetry.

Before giving the proof, we recall the definition and some properties of the so-called (K, r_0)-quasiminimal sets.

Definition 4.3.10. Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter with finite measure, and let $K \geq 1$, $r_0 > 0$. We say that E is a (K, r_0)-quasiminimal set (relatively in $\mathbb{R}^n \setminus H$) if

$$P(E, \mathbb{R}^n \setminus H) \leq K P(F, \mathbb{R}^n \setminus H),$$

for any $F \subset \mathbb{R}^n \setminus H$ such that $E\Delta F \subset\subset B_r(x)$, for some ball $B_r(x) \subset \mathbb{R}^n$ with $r \leq r_0$ and $x \in \{x_n \geq 0\}$.

Theorem 4.3.11. *Let $E \subset \mathbb{R}^n \setminus H$ be a (K, r_0)-quasiminimal set, for some $K \geq 1$, $r_0 > 0$. Then there exist $c = c(n, K, r_0) \in \left(0, \frac{1}{2}\right]$ and $r_0' = r_0'(n, K, r_0) \in (0, r_0]$ such that*

$$c \leq \frac{|E \cap B_r(x)|}{|B_r(x) \setminus H|} \leq 1 - c \qquad \forall x \in \overline{\partial E \setminus H},\ \forall r \in (0, r_0'].$$

In particular the set $E^{(1)}$ of points of density 1 for E is an open representative for E.

The proof of Theorem 4.3.11 follows, for instance, by repeatedly applying [Kin+13, Theorem 4.2] with $X = \{x_n \geq 0\}$ in domains $\Omega = X \cap B_{r_0}(x)$ for $x \in X$, in the notation of [Kin+13, Theorem 4.2]. Observe also that in [Kin+13], the perimeter functional coincides with the relative perimeter in $\mathbb{R}^n \setminus H$, hence the definition of quasiminimal set in [Kin+13, Definition 3.1] coincides with Definition 4.3.10. Alternatively, Theorem 4.3.11 follows also by adapting the classical argument in the proof of [Mag12, Theorem 21.11] working with (K, r_0)-quasiminimal sets instead of (Λ, r_0)-minimizers.

The aim of the following lemmas is to prove that minimizers of $\mathcal{F}^\lambda$ are (K, r_0)-quasiminimal sets, in order to apply Theorem 4.3.11.

Lemma 4.3.12. *Let g be $\mathscr{R}$-admissible, infinitesimal and symmetric and let G be $\mathscr{G}$-admissible. Let $E \subset \mathbb{R}^n \setminus H$ be a minimizer of $\mathcal{F}^\lambda$ with $|E| = m$, $m > 0$. Then there exists $\bar{x}_n > 0$, depending on n, g, G, E, such that*

$$|E \cap \{x_n > \bar{x}_n\}| = 0. \tag{4.3.13}$$

Proof. Let us define, for every $t > 0$,

$$E_t := E \cap \{x_n \leq t\}, \qquad V(t) := \big|E \cap \{x_n > t\}\big|.$$

Fix $x_0 \in \partial^* E$ such that $x_0 \in \partial^* E \cap \{0 < x_n < t\}$ and $r_0 > 0$ such that $B_{r_0}(x_0) \subset\subset \{0 < x_n < t\}$ for any t large enough. By [Mag12, Lemma 17.21] there exist $\sigma_0, c_0 \in (0, \infty)$, depending on E, x_0 and r_0, such that for every $\sigma \in (-\sigma_0, \sigma_0)$ we can find a set of finite perimeter F, given by a suitable local variation of E, such that

$$F \Delta E \subset\subset B_{r_0}(x_0) \qquad |F| = |E| + \sigma, \qquad |P(F, B_{r_0}(x_0)) - P(E, B_{r_0}(x_0))| \leq c_0|\sigma|. \tag{4.3.14}$$

Now consider $t_0 = t_0(E) > 0$ large enough such that $V(t_0) < \sigma_0$, and set $\sigma = V(t)$ for $t > t_0$. Then there exists $\tilde{F}$ such that (4.3.14) holds. Define also $\tilde{E}_t := \tilde{F} \cap \{0 < x_n \leq t\}$, so that

$$|\tilde{E}_t| = |\tilde{F}| - |\tilde{F} \cap \{x_n > t\}| = |\tilde{F}| - V(t) = |E| + \sigma - \sigma = |E|.$$

Moreover by [Mag12, Lemma 17.9, Lemma 17.21] and properties of local variations, we get

$$\begin{aligned} &|\tilde{E}_t \Delta E_t| = |\tilde{F} \Delta E| \leq c(E)\, ||\tilde{F}| - |E|| = c(E) V(t), \\ &|P(\tilde{E}_t, B_{r_0}(x_0)) - P(E, B_{r_0}(x_0))| \leq c(E)\, V(t). \end{aligned}$$

By the minimality of E

$$P_\lambda(E) + \mathscr{R}(E) + \mathscr{G}(E) \leq P_\lambda(\tilde{E}_t) + \mathscr{R}(\tilde{E}_t) + \mathscr{G}(\tilde{E}_t).$$

Since $\mathcal{H}^{n-1}(\partial^* E \cap \partial H) = \mathcal{H}^{n-1}(\partial^* \tilde{E}_t \cap \partial H)$, we get

$$\begin{aligned} P(E, \mathbb{R}^n \setminus H) + \mathscr{R}(E) + \mathscr{G}(E) &\leq P(\tilde{E}_t, \mathbb{R}^n \setminus H) + \mathscr{R}(\tilde{E}_t) + \mathscr{G}(\tilde{E}_t) \\ &\leq P(E_t, \mathbb{R}^n \setminus (H \cup B_{r_0}(x_0))) + P(E, B_{r_0}(x_0)) + c(E)\, V(t) + \mathscr{R}(\tilde{E}_t) + \mathscr{G}(\tilde{E}_t) \\ &= P(E, \{x_n < t\}) + |V'(t)| + \mathscr{R}(\tilde{E}_t) + \mathscr{G}(\tilde{E}_t) + c(E) V(t) \\ &= P(E, \mathbb{R}^n \setminus H) - P(E, \{x_n > t\}) + |V'(t)| + \mathscr{R}(\tilde{E}_t) + \mathscr{G}(\tilde{E}_t) + c(E)\, V(t). \end{aligned}$$

Then

$$P(E, \{x_n > t\}) \leq |V'(t)| + \mathscr{R}(\tilde{E}_t) - \mathscr{R}(E) + \mathscr{G}(\tilde{E}_t) - \mathscr{G}(E) + c(E)\, V(t)$$

By Fubini theorem and symmetry of g

$$
\begin{aligned}
\mathscr{R}(\tilde{E}_t) - \mathscr{R}(E) &= \int_{\tilde{E}_t\setminus E}\int_{\tilde{E}_t} g(y-x)\,\mathrm{d}y\,\mathrm{d}x + \int_{\tilde{E}_t\cap E}\int_{\tilde{E}_t\setminus E} g(y-x)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad - \int_{E\setminus\tilde{E}_t}\int_{E} g(y-x)\,\mathrm{d}y\,\mathrm{d}x - \int_{E\cap\tilde{E}_t}\int_{E\setminus\tilde{E}_t} g(y-x)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\tilde{E}_t\setminus E}\int_{\tilde{E}_t} g(y-x)\,\mathrm{d}y\,\mathrm{d}x + \int_{\tilde{E}_t\setminus E}\int_{\tilde{E}_t\cap E} g(x-y)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad - \int_{E\setminus\tilde{E}_t}\int_{E} g(y-x)\,\mathrm{d}y\,\mathrm{d}x - \int_{E\setminus\tilde{E}_t}\int_{E\cap\tilde{E}_t} g(x-y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\tilde{E}_t\setminus E}\int_{\tilde{E}_t} g(y-x)\,\mathrm{d}y\,\mathrm{d}x + \int_{\tilde{E}_t\setminus E}\int_{\tilde{E}_t\cap E} g(y-x)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad - \int_{E\setminus\tilde{E}_t}\int_{E} g(y-x)\,\mathrm{d}y\,\mathrm{d}x - \int_{E\setminus\tilde{E}_t}\int_{E\cap\tilde{E}_t} g(y-x)\,\mathrm{d}y\,\mathrm{d}x \\
&\le \int_{\tilde{E}_t\Delta E}\left(\int_{\tilde{E}_t} g(y-x)\,\mathrm{d}y + \int_{E} g(y-x)\,\mathrm{d}y\right)\,\mathrm{d}x
\end{aligned}
$$

Since g is infinitesimal there exists $R_g > 0$ such that

$$
g(x) < 1 \qquad \forall x \;:\; |x| > R_g.
$$

Then

$$
\begin{aligned}
\mathscr{R}(\tilde{E}_t) - \mathscr{R}(E) &\le \int_{\tilde{E}_t\Delta E}\left(2\int_{B_{R_g}(0)} g(z)\,\mathrm{d}z + \left|\tilde{E}_t \setminus B_{R_g(0)}\right| + \left|E\setminus B_{R_g(0)}\right|\right)\,\mathrm{d}x \\
&\le 2\int_{\tilde{E}_t\Delta E}\left(\int_{B_{R_g}(0)} g(z)\,\mathrm{d}z + |E|\right)\,\mathrm{d}x \\
&\le c(g,m)(|\tilde{E}_t\Delta E_t| + |E_t\Delta E|) \le c(g,E)V(t).
\end{aligned}
$$

By Remark 4.2.6 $\left(\sup_{B_{r_0}(x_0)} G\right) < \infty$ and

$$
\mathscr{G}(\tilde{E}_t) - \mathscr{G}(E) = \int_{\tilde{E}_t\setminus E} G\,\mathrm{d}x - \int_{E\setminus\tilde{E}_t} G\,\mathrm{d}x \le \int_{\tilde{E}_t\setminus E} G\,\mathrm{d}x \le \left(\sup_{B_{r_0}(x_0)} G\right)\,|\tilde{E}_t\setminus E|.
$$

Therefore, for almost every t sufficiently large,

$$
P(E,\{x_n > t\}) \le |V'(t)| + c(n,g,G,E)\,V(t). \tag{4.3.15}
$$

Finally, if $c_{iso} = c_{iso}(n)$ is the constant in the classical isoperimetric inequality and t is large enough, (4.3.15) yields

$$
\begin{aligned}
c_{iso}V(t)^{\frac{n-1}{n}} = c_{iso}|E\setminus E_t|^{\frac{n-1}{n}} &\le P(E\setminus E_t) = \mathcal{H}^{n-1}(\partial^* E_t \cap \{x_n = t\}) + P(E,\{x_n > t\}) \\
&\le 2|V'(t)| + c(n,g,G,E)\,V(t) < 2|V'(t)| + \frac{c_{iso}}{2}V(t)^{\frac{n-1}{n}}
\end{aligned}
$$

and

$$
-V'(t) \ge cV(t)^{\frac{n-1}{n}}.
$$

Therefore ODE comparison implies that $V(t)$ vanishes at some $t = \bar{x}_n < +\infty$. □

Lemma 4.3.13. *Let g be $\mathscr{R}$-admissible, infinitesimal and symmetric and let G be $\mathscr{G}$-admissible. Let $E \subset \mathbb{R}^n \setminus H$ be a minimizer of $\mathscr{F}^\lambda$ with $|E| = m$, $m > 0$. Then E is a (K, r_0)-quasiminimal set, for suitable $K \ge 1$ and $r_0 > 0$, depending on n, λ, g, G, E.*

Proof. Let us consider $x_1, x_2 \in \partial^* E \setminus H$ and $t_0 > 0$ such that we have $B_{t_0}(x_1) \cap B_{t_0}(x_2) = \emptyset$ and $B_{t_0}(x_1) \cup B_{t_0}(x_2) \subset\subset \mathbb{R}^n \setminus H$. By applying [Mag12, Lemma 17.21] we find two positive constants σ_0 and c_0, depending on E, such that, given $|\sigma| < \sigma_0$, there exist two sets of finite perimeter F_1 and F_2 with

$$E\Delta F_k \subset\subset B_{t_0}(x_k), \qquad |F_k| = |E| + \sigma, \qquad \left|P(E, B_{t_0}(x_k)) - P(F_k, B_{t_0}(x_k))\right| \le c_0|\sigma|, \quad k \in \{1,2\}. \tag{4.3.16}$$

Let $r_0 = r_0(n, \lambda, g, G, E) > 0$ to be determined later. At the moment assume that

$$r_0 < \min\left\{\frac{t_0}{2}, \frac{\sigma_0^{\frac{1}{n}}}{\omega_n}, \frac{|x_1 - x_2| - 2t_0}{2}\right\}.$$

In particular, if a ball of radius r_0 intersects $B_{t_0}(x_1)$ (resp. $B_{t_0}(x_2)$), then it is disjoint from $B_{t_0}(x_2)$ (resp. from $B_{t_0}(x_1)$). Let F be such that $E\Delta F \subset\subset B_r(x) \cap (\mathbb{R}^n \setminus H)$, where $r < r_0$. Then, by the definition of r_0,

$$||E| - |F|| \le |E\Delta F| \le \omega_n r^n < \omega_n r_0^n \le \sigma_0$$

and we can compensate for the volume deficit $||E| - |F||$ between E and F by modifying F inside either $B_{t_0}(x_1)$ or $B_{t_0}(x_2)$. Precisely, by the definition of r_0, we may assume without loss of generality that $B_r(x)$ does not intersect $B_{t_0}(x_1)$, set $\sigma = |E| - |F|$, and consider F_1 verifying (4.3.16), so that

$$E\Delta F_1 \subset\subset B_{t_0}(x_1), \qquad E\Delta F \subset\subset B_r(x) \cap (\mathbb{R}^n \setminus H) \subset\subset \mathbb{R}^n \setminus \left(H \cup \overline{B_{t_0}(x_1)}\right). \tag{4.3.17}$$

By (4.3.16) $\sigma = |F_1| - |E|$ and, if we define

$$\tilde{F} = (F \cap B_r(x)) \cup (F_1 \cap B_{t_0}(x_1)) \cup (E \setminus (B_r(x) \cup B_{t_0}(x_1))),$$

then $|\tilde{F}| = |E|$ and $\tilde{F}\Delta E \subset\subset \{x_n > 0\}$. By the minimality of E

$$(1 - |\lambda|)P(E, \mathbb{R}^n \setminus H) \le P_\lambda(E) \le P_\lambda(\tilde{F}) + \mathscr{R}(\tilde{F}) - \mathscr{R}(E) + \mathscr{G}(\tilde{F}) - \mathscr{G}(E). \tag{4.3.18}$$

By (4.3.16) and (4.3.17) we get

$$\begin{aligned} P_\lambda(\tilde{F}) &\le (1 + |\lambda|)P(\tilde{F}, \mathbb{R}^n \setminus H) \\ &\le (1 + |\lambda|)\left[P(\tilde{F}, \mathbb{R}^n \setminus (H \cup \overline{B_{t_0}(x_1)})) + P(\tilde{F}, B_{t_0}(x_1)) + P(\tilde{F}, \partial B_{t_0}(x_1))\right] \\ &= (1 + |\lambda|)\left[P(F, \mathbb{R}^n \setminus (H \cup \overline{B_{t_0}(x_1)})) + P(F_1, B_{t_0}(x_1)) + P(F, \partial B_{t_0}(x_1))\right] \\ &\le (1 + |\lambda|)\left[P(F, \mathbb{R}^n \setminus (H \cup B_{t_0}(x_1))) + P(E, B_{t_0}(x_1)) + c_0(E)|\sigma|\right] \\ &\le (1 + |\lambda|)\, P(F, \mathbb{R}^n \setminus H) + c_0(\lambda, E)|F\Delta E|. \end{aligned} \tag{4.3.19}$$

As in the proof of Lemma 4.3.12 one estimates

$$\mathscr{R}(\tilde{F}) - \mathscr{R}(E) \le c(g, E)\,|F\Delta E|, \tag{4.3.20}$$

and by Remark 4.2.6, (4.3.13) and if $t_0 < 1$

$$\begin{aligned} \mathscr{G}(\tilde{F}) - \mathscr{G}(E) &\le \left(\sup_{\tilde{F}\setminus E} G\right)|\tilde{F} \setminus E| \le \left(\sup_{\tilde{F}\setminus E} G\right)\left(|\tilde{F}\Delta F| + |F\Delta E|\right) \\ &\le c(G, E)\,(\bar{x}_n + 1)^n|F\Delta E| = c(n, g, G, E)\,|F\Delta E|. \end{aligned} \tag{4.3.21}$$

However, by the relative isoperimetric inequality [CGR07; FM23] and (4.3.17)

$$\begin{aligned} |F\Delta E| = |F\Delta E|^{\frac{1}{n}}|F\Delta E|^{\frac{n-1}{n}} &\le c(n)|F\Delta E|^{\frac{1}{n}}P(F\Delta E, \mathbb{R}^n \setminus H) \\ &\le c(n)|F\Delta E|^{\frac{1}{n}}(P(F, \mathbb{R}^n \setminus H) + P(E, \mathbb{R}^n \setminus H)) \\ &\le c(n)\,r_0\,(P(F, \mathbb{R}^n \setminus H) + P(E, \mathbb{R}^n \setminus H)). \end{aligned} \tag{4.3.22}$$

Putting together (4.3.18)-(4.3.22) we obtain

$$\left((1 - |\lambda|) - c(n, \lambda, g, G, E)\,r_0\right) P(E, \mathbb{R}^n \setminus H) \le \left((1 + |\lambda|) + c(n, \lambda, g, G, E)\,r_0\right) P(F, \mathbb{R}^n \setminus H)$$

If r_0 is sufficiently small, we conclude the proof. □

By Theorem 4.3.11 and Lemma 4.3.13, from now on we can identify any minimizer E of $\mathscr{F}^\lambda$, with $|E| = m$, $m > 0$, g $\mathscr{R}$-admissible infinitesimal symmetric function and G $\mathscr{G}$-admissible function, with the open set $E^{(1)}$ of points of density 1 for E.

Now we are ready to prove Theorem 4.3.8.

Proof of Theorem 4.3.8. By Lemma 4.3.13 and Theorem 4.3.11 there exist $r > 0$ and $c > 0$ such that for every $x \in \overline{\partial E \setminus H}$ we have $|E \cap B_r(x)| \geq cr^n$. If E were not bounded, one would easily get $|E| = \infty$. □

Now we prove indecomposability of minimizers.

Theorem 4.3.14. *Let g be $\mathscr{R}$-admissible and infinitesimal and let G be $\mathscr{G}$-admissible. Let $E \subset \mathbb{R}^n \setminus H$ be a minimizer of $\mathscr{F}^\lambda$ with $|E| = m$, $m > 0$. Then E is indecomposable.*

Proof. We argue by contradiction. Assume that there exist two sets of finite perimeter E_1 and E_2 such that $|E_1 \cap E_2| = 0$, $E = E_1 \cup E_2$ and $P(E) = P(E_1) + P(E_2)$. If $R > 0$ is sufficiently large, letting $e_1 = (1, 0, \dots, 0)$ and defining $E_R := E_1 \cup (E_2 + e_1 R)$, we have $|E_R| = m$, $P_\lambda(E_R) = P_\lambda(E)$ and $\mathscr{G}(E_R) = \mathscr{G}(E)$. At the same time, the nonlocal energy decreases, precisely

$$\liminf_{R\to\infty} \left(\int_{E_1} \int_{E_2+e_1R} g(y-x)\,\mathrm{d}y\,\mathrm{d}x + \int_{E_2+e_1R} \int_{E_1} g(y-x)\,\mathrm{d}y\,\mathrm{d}x \right) = 0,$$

and

$$\begin{aligned}
\liminf_{R\to\infty} \mathscr{F}^\lambda(E_R) &= P_\lambda(E) + \mathscr{R}(E_1) + \mathscr{R}(E_2) + \mathscr{G}(E) \\
&< P_\lambda(E) + \mathscr{R}(E_1) + \mathscr{R}(E_2) + \int_{E_1}\int_{E_2} g(y-x)\,\mathrm{d}y\,\mathrm{d}x + \int_{E_2}\int_{E_1} g(y-x)\,\mathrm{d}y\,\mathrm{d}x + \mathscr{G}(E) \\
&= \mathscr{F}^\lambda(E).
\end{aligned}$$

Therefore, if R is sufficiently large, we obtain $\mathscr{F}^\lambda(E_R) < \mathscr{F}^\lambda(E)$, in contradiction with the minimizing property of E. □

4.4 Nonexistence of minimizers for large masses

The goal of this Section is to prove Theorem 4.1.2. We begin by proving some preparatory lemmas. Let us start with a non-optimality criterion.

Lemma 4.4.1. *Let g be $\mathscr{R}$-admissible, q-growing and infinitesimal and let G be $\mathscr{G}$-admissible. There exists $\varepsilon > 0$, depending on n, λ, g, G and q, such that the following holds. Let $F \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter and assume there exist two sets of finite perimeter F_1, $F_2 \subset F$ such that $|F_1|, |F_2| > 0$, $|F_1 \cap F_2| = 0$, $|F \setminus (F_1 \cup F_2)| = 0$ and*

$$\Sigma := P_\lambda(F_1) + P_\lambda(F_2) - P_\lambda(F) \leq \frac{1}{2}\mathscr{F}^\lambda(F_2). \tag{4.4.1}$$

Then, if

$$|F_2| \leq \varepsilon \min\{1, |F_1|\}, \tag{4.4.2}$$

there exists a set $G \subset \mathbb{R}^n \setminus H$ with $|G| = |F|$ and $\mathscr{F}^\lambda(G) < \mathscr{F}^\lambda(F)$.

Proof. Let us denote $m := |F|$, $m_1 := |F_1|$, $m_2 := |F_2|$ and $\gamma := \frac{m_2}{m_1} \leq \varepsilon$. Let us define the sets $\tilde{F}$ and $\hat{F}$ in the following way: $\tilde{F}$ is given by $\tilde{F} = l\,F_1$, with $l := \sqrt[n]{1+\gamma}$, so that $|\tilde{F}| = |F|$, and $\hat{F}$ is given by a collection of $N \geq 1$ spherical caps $\left\{B^\lambda(v, x_i)\right\}_{1\leq i\leq N}$ of equal volume v and with centers located at $x_i = iRe_1$, $i = 1, \dots, N$, with R large enough so that the $B^\lambda(v, x_i)$, $1 \leq i \leq N$ are pairwise disjoint. The number N is the smallest integer for which the volume of each spherical cap does not exceed $\min\left\{1, |B^\lambda|\right\}$. Hence $Nv = m$ and $N = \left\lceil \frac{m}{\min\{1,|B^\lambda|\}} \right\rceil$. If there exists $R > 0$ such that, for the corresponding $\hat{F}$, one has $\mathscr{F}^\lambda(\hat{F}) < \mathscr{F}^\lambda(F)$, then the proof is concluded with $G = \hat{F}$. So we can assume that for any $R > 0$ there holds $\mathscr{F}^\lambda(\hat{F}) \geq \mathscr{F}^\lambda(F)$. Hence for R large enough we claim that

$$\mathscr{F}^\lambda(F) \leq \mathscr{F}^\lambda(\hat{F}) \leq c(n, \lambda, g, G) \max\left\{m, m^{\frac{n-1}{n}}\right\}. \tag{4.4.3}$$

Indeed, if $m \geq 1$, estimate (4.4.3) follows by the same computations done in the proof of Corollary 4.3.3. If instead $m < 1$, by [PP24, Lemma 3.1] we find

$$P_\lambda(E) = P_\lambda\left(\cup_{i=1}^N B^\lambda(v, x_i)\right) = \sum_{i=1}^N P_\lambda(B^\lambda(v, x_i)) = c(n,\lambda) N v^{\frac{n-1}{n}} \leq c(n,\lambda)\left(\frac{m}{\min\{1, |B^\lambda|\}} + 1\right) v^{\frac{n-1}{n}}$$
$$\leq c(n,\lambda)(m v^{\frac{n-1}{n}} + v^{\frac{n-1}{n}}) \leq c(n,\lambda)(m + m^{\frac{n-1}{n}}) = c(n,\lambda) m^{\frac{n-1}{n}}.$$

Arguing as in Corollary 4.3.3, if R is so large that $g(x-y) < \frac{1}{N}$ for every $x \in B^\lambda(v, x_j)$, $y \in B^\lambda(v, x_k)$ with $j \neq k$, then

$$\mathscr{R}(E) \leq c(n,\lambda,g,G) m \leq c(n,\lambda,g,G) m^{\frac{n-1}{n}}$$

and

$$\mathscr{F}^\lambda(E) = P_\lambda(E) + \mathscr{R}(E) + \mathscr{G}(E) \leq c(n,\lambda)|E|^{\frac{n-1}{n}} + c(n,\lambda,g,G)|E|^{\frac{n-1}{n}} + \sum_{i=1}^N \mathscr{G}(B^\lambda(v, x_i)) \leq c(n,\lambda,g,G) m^{\frac{n-1}{n}},$$

therefore (4.4.3) holds.
We want to show that if ε sufficiently small, then $\mathscr{F}^\lambda(\tilde{F}) < \mathscr{F}^\lambda(F)$, implying the claim with $G = \tilde{F}$. By Lemma 4.3.4

$$\mathscr{F}^\lambda(\tilde{F}) = \mathscr{F}^\lambda(l F_1) \leq l^{2n+q} \mathscr{F}^\lambda(F_1) = \mathscr{F}^\lambda(F_1) + (l^{2n+q} - 1)\mathscr{F}^\lambda(F_1). \tag{4.4.4}$$

Choosing $\varepsilon \leq 1$, we have $1 \leq l \leq 2^{\frac{1}{n}}$, and by Taylor's formula we obtain $l^{2n+q} - 1 = (1+\gamma)^{2+q/n} - 1 \leq \gamma K$ for some $K > 0$ independent of γ, for ε sufficiently small. By (4.4.4) we arrive at

$$\mathscr{F}^\lambda(\tilde{F}) - \mathscr{F}^\lambda(F_1) \leq \gamma K \mathscr{F}^\lambda(F_1).$$

By the definition of Σ and since $\mathscr{R}(F_1) + \mathscr{R}(F_2) \leq \mathscr{R}(F)$

$$\begin{aligned}\mathscr{F}^\lambda(\tilde{F}) - \mathscr{F}^\lambda(F) &\leq \mathscr{R}(F_1) + \mathscr{G}(F_1) + \mathscr{R}(F_2) + \mathscr{G}(F_2) - \mathscr{R}(F) - \mathscr{G}(F) + \Sigma - \mathscr{F}^\lambda(F_2) + \gamma K \mathscr{F}^\lambda(F_1) \\ &\leq -\frac{1}{2}\mathscr{F}^\lambda(F_2) + \gamma K \mathscr{F}^\lambda(F_1).\end{aligned} \tag{4.4.5}$$

By positivity of $\mathscr{R}$ and $\mathscr{G}$ and the isoperimetric inequality, we have $\mathscr{F}^\lambda(F_2) > P_\lambda(F_2) \geq c(n,\lambda) m_2^{\frac{n-1}{n}}$. As in (4.3.11) we obtain $\mathscr{F}^\lambda(F_1) \leq \mathscr{F}^\lambda(F)$. By (4.4.3), since $\gamma m \leq 2m_2$ and $\gamma \leq \varepsilon$, (4.4.5) turns into

$$\mathscr{F}^\lambda(\tilde{F}) - \mathscr{F}^\lambda(F) \leq -c(n,\lambda) m_2^{\frac{n-1}{n}} + \gamma K \mathscr{F}^\lambda(F) \leq -c(n,\lambda) m_2^{\frac{n-1}{n}} + C(n,\lambda,g,G,q) \max\left\{ m_2, \varepsilon^{\frac{1}{n}} m_2^{\frac{n-1}{n}} \right\}.$$

Since $m_2 \leq \varepsilon$ by (4.4.2), for ε sufficiently small the assertion of the lemma holds with $G = \tilde{F}$. □

Next lemma is an improvement of the standard density estimate for quasiminimizers.

Lemma 4.4.2. *Let g be $\mathscr{R}$-admissible, q-growing and infinitesimal, and let G be $\mathscr{G}$-admissible. Then there exists $c = c(n,\lambda,g,G,q) > 0$ such that the following holds. Let $E \subset \mathbb{R}^n \setminus H$ be a minimizer of $\mathscr{F}^\lambda$ with $|E| = m$, $m > 0$. Then for almost every $x \in E$ there holds*

$$|E \cap B_1(x)| \geq c \min\{1, m\}.$$

Proof. For $r > 0$ and $x \in E$, let $F_1^r := E \setminus B_r(x)$ and $F_2^r := E \cap B_r(x)$. Note that $|F_1^r| + |F_2^r| = m$ and $|F_2^r| \leq \omega_n r^n$. Then there exists $C > 0$, depending on n, λ, g, G and q, such that (4.4.2) holds for all $r \leq r_1 := C \min\left\{1, \sqrt[n]{m}\right\}$. Note that we can choose $C \leq 1$. Since E is a minimizer, Lemma 4.4.1 implies that (4.4.1) cannot be satisfied for any $r \leq r_1$. Equivalently, recalling also [PP24, Corollary 2.5], for all $r \leq r_1$ we have

$$\Sigma^r := P_\lambda(F_1^r) + P_\lambda(F_2^r) - P_\lambda(E) > \frac{1}{2}\mathscr{F}^\lambda(F_2^r) > \frac{1}{2}P_\lambda(F_2^r) \geq \frac{1-\lambda}{4}P(F_2^r). \tag{4.4.6}$$

At the same time, for almost every r we have

$$\Sigma^r = 2\mathcal{H}^{n-1}(E^{(1)} \cap \partial B_r(x)).$$

By (4.4.6), for a constant $c(n,\lambda) \in \left(0, \frac{1}{2}\right)$ there holds

$$2\mathcal{H}^{n-1}(E^{(1)} \cap \partial B_r(x)) > c(n,\lambda)\,(\mathcal{H}^{n-1}(\partial^* E \cap B_r(x)) + \mathcal{H}^{n-1}(E^{(1)} \cap \partial B_r(x))), \tag{4.4.7}$$

for almost every r. Let us now distinguish two cases. If there exists $r_2 \in \left(\frac{r_1}{2}, r_1\right)$ such that $|E \cap B_{r_2}| \geq \frac{1}{2}\omega_n r_2^n$, then by the choice of r_1 we get

$$|E \cap B_1| \geq |E \cap B_{r_2}| \geq c(n) \left(\frac{r_1}{2}\right)^n = c(n,\lambda,g,G,q)\min\{1,m\},$$

and the proof is concluded.

Let us assume that $|E \cap B_r| < \frac{1}{2}\omega_n r^n$ for all $r \in \left(\frac{r_1}{2}, r_1\right)$. Then we rearrange terms in (4.4.7) and apply the relative isoperimetric inequality [Mag12, Proposition 12.37] to the right-hand side to obtain

$$\mathcal{H}^{n-1}(E^{(1)} \cap \partial B_r(x)) \geq c(n,\lambda)|E \cap B_r(x)|^{\frac{n-1}{n}}.$$

Let us denote $U(r) := |E \cap B_r(x)|$. Then $\frac{\mathrm{d}U(r)}{\mathrm{d}r} = \mathcal{H}^{n-1}(E^{(1)} \cap \partial B_r(x))$ for all $r \in \left(\frac{r_1}{2}, r_1\right)$ and

$$\frac{\mathrm{d}U(r)}{\mathrm{d}r} \geq c(n,\lambda)U^{\frac{n-1}{n}}(r) \qquad \forall r \in \left(\frac{r_1}{2}, r_1\right).$$

For $\mathcal{H}^n$-a.e. $x \in E$, we have $U(r) > 0$ for all $r > 0$ and ODE comparison in $r \in \left(\frac{r_1}{2}, r_1\right)$ implies that

$$U^{1/n}(r) \geq U^{1/n}\left(\frac{r_1}{2}\right) + c(n,\lambda)\left(r - \frac{r_1}{2}\right) \geq c(n,\lambda)\left(r - \frac{r_1}{2}\right) \qquad \forall r \in \left(\frac{r_1}{2}, r_1\right).$$

Then the lemma follows as

$$c(n,\lambda,g,G)\min\{1,m\} = c(n,\lambda)r_1^n \leq U(r_1) = |E \cap B_{r_1}(x)| = \left|E \cap B_{C\min\left\{1,\sqrt[n]{m}\right\}}(x)\right| \leq |E \cap B_1(x)|.$$

□

Remark 4.4.3. We remark that the density estimate in Lemma 4.4.2 is more precise than the one provided in Section 4.3. Indeed, in Lemma 4.3.13 K and r_0 depend on the minimizer, and consequently c in Theorem 4.3.11 also inherits this dependence. At the same time, Lemma 4.4.2 requires g to be q-growing, which is not required in Lemma 4.3.13.

The following lemma will imply Theorem 4.1.2 for $\beta \in (0,1)$.

Lemma 4.4.4. *Let*

$$g(x) = \frac{1}{|x|^\beta}, \qquad 0 < \beta < n,\ x \in \mathbb{R}^n \setminus \{0\},$$

and let G be $\mathcal{G}$-admissible. Let E be a minimizer of $\mathcal{F}^\lambda$ with $|E| = m$ and $m \geq 1$. Then

$$cm^{\frac{1}{\beta}} \leq \operatorname{diam} E \leq Cm, \tag{4.4.8}$$

for some C, $c > 0$ depending only on n, β, λ, G.

Proof. By Theorem 4.3.8 and Theorem 4.3.14 we know that E is essentially bounded and indecomposable. In particular $d := \operatorname{diam} E < \infty$. By Corollary 4.3.3 we get the existence of $c(n,\lambda,\beta,G) > 0$ such that

$$\frac{m^2}{d^\beta} \leq \int_E \int_E \frac{1}{|x-y|^\beta}\,\mathrm{d}y\,\mathrm{d}x = \mathcal{R}(E) \leq \mathcal{F}^\lambda(E) \leq cm,$$

which implies the first bound in (4.4.8).
In order to prove the upper bound in (4.4.8), we may clearly assume that $\frac{d}{\sqrt{2}} > 3$. Recalling that we identify E with the bounded open set $E^{(1)}$, we let $x^{(1)}, x^{(2)} \in \bar{E}$ such that

$$|x^{(1)} - x^{(2)}| = d.$$

Up to a rotation with respect to an axis orthogonal to $\{x_n = 0\}$, we can write

$$x^{(2)} - x^{(1)} = \left\langle x^{(2)} - x^{(1)}, e_1 \right\rangle e_1 + \left\langle x^{(2)} - x^{(1)}, e_n \right\rangle e_n.$$

In particular,

$$\max \left\{ \left| \left\langle x^{(2)} - x^{(1)}, e_1 \right\rangle \right|, \left| \left\langle x^{(2)} - x^{(1)}, e_n \right\rangle \right| \right\} \geq \frac{d}{\sqrt{2}}.$$

Assume for simplicity that

$$\left| \left\langle x^{(2)} - x^{(1)}, e_n \right\rangle \right| \geq \frac{d}{\sqrt{2}},$$

the remaining case being analogous. Up to relabeling, assume also that

$$\left\langle x^{(2)}, e_n \right\rangle > \left\langle x^{(1)}, e_n \right\rangle$$

Let N be the largest integer smaller than $\frac{d}{3\sqrt{2}}$, i.e. $N := \left\lfloor \frac{d}{3\sqrt{2}} \right\rfloor$. Since E is indecomposable, for every $j = 1, \dots, N$ there holds

$$\left| E \cap \left\{ 3j - 1 + \left\langle x^{(1)}, e_n \right\rangle < x_n < 3j + \left\langle x^{(1)}, e_n \right\rangle \right\} \right| > 0.$$

For every $j = 1, \dots, N$, let

$$x_j \in E \cap \left\{ 3j - 1 + \left\langle x^{(1)}, e_n \right\rangle < x_n < 3j + \left\langle x^{(1)}, e_n \right\rangle \right\}.$$

The balls $B_1(x_j)$, $j = 1, \dots, N$, are pairwise disjoint and, for a suitable choice of x_j, we can apply Lemma 4.4.2 to get

$$m = |E| \geq \sum_{j=1}^{N} |B_1(x_j) \cap E| \geq c(n, \lambda, \beta, G)\, N \geq c(n, \lambda, \beta, G)\, d.$$

□

The following lemma will imply Theorem 4.1.2 for $\beta = 1$.

Lemma 4.4.5. *Let*

$$g(x) = \frac{1}{|x|^{\beta}}, \qquad \beta \in (0, n),\ x \in \mathbb{R}^n \setminus \{0\}$$

and let G be $\mathscr{G}$-admissible. Let $E \subset \mathbb{R}^n \setminus H$ be a minimizer for $\mathscr{F}^{\lambda}$ with $|E| = m$, $m > 0$. Then

$$\int_E \int_E \frac{1}{|x - y|^{\beta - 1}}\, \mathrm{d}y\, \mathrm{d}x \leq c(n)\, m.$$

Proof. Let $\nu \in \mathbb{S}^{n-1} \setminus \{\pm e_n\}$ and $t \in \mathbb{R}$. Denote

$$E^+_{\nu,t} := E \cap \{\langle \nu, x \rangle > t\}$$
$$E^-_{\nu,t} := E \cap \{\langle \nu, x \rangle < t\}.$$

Let $\nu_1 := \frac{\nu_{hor}}{|\nu_{hor}|}$, where ν_{hor} is the orthogonal projection of ν on $\{x_n = 0\}$. For any $\varrho \geq 0$, the set

$$E^+_{\nu,t} \cup (E^-_{\nu,t} - \varrho \nu_1)$$

has measure m and, by minimality of E,

$$\mathscr{F}^{\lambda}(E^+_{\nu,t} \cup (E^-_{\nu,t} - \varrho \nu_1)) \geq \mathscr{F}^{\lambda}(E). \tag{4.4.9}$$

For any $\varrho > 0$ and for a.e. $t \in \mathbb{R}$

$$P_\lambda(E^+_{\nu,t} \cup (E^-_{\nu,t} - \varrho \nu_1)) = P_\lambda(E^+_{\nu,t}) + P_\lambda(E^-_{\nu,t}) \leq P_\lambda(E) + 2\mathcal{H}^{n-1}(E \cap \{\langle \nu, x\rangle = t\}).$$

For any $\varrho \geq 0$ we have

$$\begin{aligned}\int_{E^+_{\nu,t}\cup(E^-_{\nu,t}-\varrho\nu_1)} \int_{E^+_{\nu,t}\cup(E^-_{\nu,t}-\varrho\nu_1)} \frac{1}{|x-y|^\beta}\,\mathrm{d}y\,\mathrm{d}x &= \int_{E^+_{\nu,t}}\int_{E^+_{\nu,t}} \frac{1}{|x-y|^\beta}\,\mathrm{d}y\,\mathrm{d}x + \int_{E^-_{\nu,t}}\int_{E^-_{\nu,t}} \frac{1}{|x-y|^\beta}\,\mathrm{d}y\,\mathrm{d}x \\ &\quad + 2\int_{E^+_{\nu,t}}\int_{E^-_{\nu,t}} \frac{1}{|x-y+\varrho\nu_1|^\beta}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Moreover

$$\int_{E^+_{\nu,t}}\int_{E^-_{\nu,t}} \frac{1}{|x-y+\varrho\nu_1|^\beta}\,\mathrm{d}y\,\mathrm{d}x \to 0$$

as $\varrho \to \infty$. Hence, by (4.4.9), letting $\varrho \to \infty$, we get

$$\begin{aligned}&P_\lambda(E) + 2\mathcal{H}^{n-1}(E \cap \{\langle \nu, x\rangle = t\}) + \int_{E^+_{\nu,t}}\int_{E^+_{\nu,t}} \frac{1}{|x-y|^\beta}\,\mathrm{d}y\,\mathrm{d}x + \int_{E^-_{\nu,t}}\int_{E^-_{\nu,t}} \frac{1}{|x-y|^\beta}\,\mathrm{d}y\,\mathrm{d}x + \mathcal{G}(E^+_{\nu,t}) + \mathcal{G}(E^-_{\nu,t}) \\ &\geq P_\lambda(E) + \int_{E^+_{\nu,t}}\int_{E^+_{\nu,t}} \frac{1}{|x-y|^\beta}\,\mathrm{d}y\,\mathrm{d}x + \int_{E^-_{\nu,t}}\int_{E^-_{\nu,t}} \frac{1}{|x-y|^\beta}\,\mathrm{d}y\,\mathrm{d}x + 2\int_{E^+_{\nu,t}}\int_{E^-_{\nu,t}} \frac{1}{|x-y|^\beta}\,\mathrm{d}y\,\mathrm{d}x + \mathcal{G}(E).\end{aligned}$$

Then

$$\begin{aligned}\mathcal{H}^{n-1}(E \cap \{\langle \nu, x\rangle\} = t) &\geq \int_{E^+_{\nu,t}}\int_{E^-_{\nu,t}} \frac{1}{|x-y|^\beta}\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_E\int_E \chi_{\{\langle \nu,\cdot\rangle < t\}}(y)\chi_{\{\langle \nu,\cdot\rangle > t\}}(x)\frac{1}{|x-y|^\beta}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Integrating the last inequality with respect to $t \in \mathbb{R}$, by Fubini's theorem we get

$$\begin{aligned}m &\geq \int_E\int_E\int_{-\infty}^{+\infty} \chi_{\{\langle \nu,\cdot\rangle < t\}}(y)\chi_{\{\langle \nu,\cdot\rangle > t\}}(x)\,\mathrm{d}t\frac{1}{|x-y|^\beta}\,\mathrm{d}y\,\mathrm{d}x = \int_E\int_E\int_{-\infty}^{+\infty} \chi_{(\langle \nu,y\rangle,\langle \nu,x\rangle)}(t)\,\mathrm{d}t\frac{1}{|x-y|^\beta}\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_E\int_E \frac{\langle \nu, x-y\rangle_+}{|x-y|^\beta}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Further integrating over $\mathbb{S}^{n-1} \setminus \{\pm e_n\}$, since

$$\int_{\mathbb{S}^{n-1}} \langle \nu, x-y\rangle_+\,\mathrm{d}\nu = c(n)|x-y|,$$

by symmetry of $\mathbb{S}^{n-1}$, we conclude that

$$|\mathbb{S}^{n-1}|m \geq \int_E\int_E\int_{\mathbb{S}^{n-1}} \langle \nu, x-y\rangle_+\,\mathrm{d}\nu\frac{1}{|x-y|^\beta}\,\mathrm{d}y\,\mathrm{d}x = c(n)\int_E\int_E \frac{1}{|x-y|^{\beta-1}}\,\mathrm{d}y\,\mathrm{d}x.$$

□

The following lemma will imply Theorem 4.1.2 for $\beta \in (1, 2]$.

Lemma 4.4.6. *Let*

$$g(x) = \frac{1}{|x|^\beta}, \qquad 0 < \beta < n,\ x \in \mathbb{R}^n \setminus \{0\}$$

and let G be $\mathcal{G}$-admissible. Let $E \subset \mathbb{R}^n \setminus H$ be a minimizer for $\mathcal{F}^\lambda$ with $|E| = m$, $m > \omega_n$. Then, for $1 \leq r \leq \frac{\operatorname{diam} E}{2}$,

$$|E \cap B_r(x)| \geq c(n, \lambda, \beta, G)\, r \qquad \text{for a.e. } x \in E.$$

Proof. Let $N := \left\lfloor \frac{r-1}{3} \right\rfloor$ and $x \in E$. If $r < 4$, by Lemma 4.4.2

$$|E \cap B_r(x)| \geq |E \cap B_1(x)| \geq c(n,\lambda,\beta,G) = \frac{c(n,\lambda,\beta,G)}{4}\, 4 \geq c(n,\lambda,\beta,G)\, r.$$

So we can assume that $r \geq 4$, in particular $N \geq 1$. Since E is indecomposable by Theorem 4.3.14, for every $i = 0, \dots, N-1$ there holds

$$|E \cap (B_{3i+3}(x) \setminus B_{3i+2}(x))| > 0.$$

For every $i = 0, \dots, N-1$, let

$$y_i \in E \cap (B_{3i+3}(x) \setminus B_{3i+2}(x)).$$

The balls $B_1(y_i)$, $i = 0, \dots N-1$, are pairwise disjoint and, for a suitable choice of y_i, by Lemma 4.4.2 there exists $c(n,\lambda,\beta,G)$ such that $|E \cap B_1(y_i)| \geq c$ for $i = 0, \dots, N-1$. Finally

$$|E \cap B_r(x)| \geq \sum_{i=0}^{N-1} |E \cap B_1(y_i)| + |E \cap B_1(x)| \geq (N+1)c \geq c\frac{r-1}{3} \geq c\, r.$$

□

Now we are ready to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. Lemma 4.4.4 and Lemma 4.4.5 easily imply Theorem 4.1.2 for $\beta \in (0,1]$ and mass m sufficiently large. Then it remains to consider $\beta \in (1,2]$. Let $E \subset \mathbb{R}^n \setminus H$ be a volume constrained minimizer for $\mathcal{F}^\lambda$ with $|E| = m$. By Lemma 4.4.4, for m large enough, we can assume that $\operatorname{diam} E > 4$. We observe first that

$$\frac{1}{r^{\beta-1}} |\{(x,y) \in E \times E \ : \ |x-y| < r\}| = \frac{1}{r^{\beta-1}} \int_E |E \cap B_r(x)|\, \mathrm{d}x \leq \frac{\omega_n r^n}{r^{\beta-1}} |E| \xrightarrow[r\to 0]{} 0. \tag{4.4.10}$$

Applying the coarea formula on $\mathbb{R}^n \times \mathbb{R}^n$ for the Lipschitz function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $f(x,y) := |x-y|$, observing that $|\nabla f| = \sqrt{2}$ and that

$$\frac{\mathrm{d}}{\mathrm{d}r} |\{(x,y) \in E \times E \ : \ |x-y| < r\}| = \frac{1}{\sqrt{2}} \int_{\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n \ : \ |x-y|=r\}} \chi_{E\times E}(x,y)\, \mathrm{d}\mathcal{H}^{2n-1}(x,y) \quad \text{for a.e. } r > 0,$$

and integrating by parts, we estimate

$$\begin{aligned}
\int_E \int_E \frac{1}{|x-y|^{\beta-1}}\, \mathrm{d}y\, \mathrm{d}x &= \frac{1}{\sqrt{2}} \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\chi_{E\times E}(x,y)}{|x-y|^{\beta-1}} |\nabla f|\, \mathrm{d}x\, \mathrm{d}y \\
&= \frac{1}{\sqrt{2}} \int_0^{+\infty} \int_{\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n \ : \ |x-y|=r\}} \frac{\chi_{E\times E}(x,y)}{r^{\beta-1}}\, \mathrm{d}\mathcal{H}^{2n-1}(x,y)\, \mathrm{d}r \\
&= \lim_{\varepsilon\to 0^+} \int_\varepsilon^{+\infty} \frac{1}{r^{\beta-1}} \frac{\mathrm{d}}{\mathrm{d}r} |\{(x,y) \in E\times E \ : \ |x-y| < r\}|\, \mathrm{d}r \\
&= \lim_{\varepsilon\to 0^+} -\frac{1}{\varepsilon^{\beta-1}} |\{(x,y) \in E\times E \ : \ |x-y| < \varepsilon\}| + \\
&\qquad - \int_\varepsilon^{+\infty} (1-\beta)\frac{1}{r^\beta} |\{(x,y) \in E\times E \ : \ |x-y| < r\}|\, \mathrm{d}r.
\end{aligned}$$

Exploiting (4.4.10) and Lemma 4.4.6 we deduce

$$\begin{aligned}
\int_E \int_E \frac{1}{|x-y|^{\beta-1}}\, \mathrm{d}y\, \mathrm{d}x &= (\beta-1) \int_0^{+\infty} |\{(x,y) \in E\times E \ : \ |x-y| < r\}| \frac{\mathrm{d}r}{r^\beta} \\
&= (\beta-1) \int_0^{+\infty} \int_E \frac{|E \cap B_r(x)|}{r^\beta}\, \mathrm{d}x\, \mathrm{d}r \\
&\geq (\beta-1) \int_1^{\frac{\operatorname{diam} E}{2}} \int_E \frac{|E \cap B_r(x)|}{r^\beta}\, \mathrm{d}x\, \mathrm{d}r \\
&\geq c \int_1^{\frac{\operatorname{diam} E}{2}} \frac{|E|}{r^{\beta-1}}\, \mathrm{d}r.
\end{aligned}$$

The final right-hand side in the previous chain of inequalities is bounded from below by $c\ |E|\ (\operatorname{diam} E)^{2-\beta}$ if $\beta < 2$, and by $c\ |E|\ \log(\operatorname{diam} E)$ if $\beta = 2$. Combining this bounds with Lemma 4.4.5, we get a contradiction for $|E|$ large enough. □

Absence of holes in minimizers

As a corollary of the estimates proved in the last section, we prove here a further qualitative property of volume constrained minimizers of $\mathscr{F}^\lambda$. The next theorem essentially tells that volume constrained minimizers of $\mathscr{F}^\lambda$ do not have "interior holes".

Theorem 4.4.7. *Let g be $\mathscr{R}$-admissible, 0-growing, infinitesimal and symmetric and let G be $\mathscr{G}$-admissible and coercive. There exists $\bar{m} > 0$, depending on n, λ, g and G, such that, for all $m \in (0, \bar{m})$, every minimizer E of $\mathscr{F}^\lambda$ with $|E| = m$ has the following property. There is no set $F \subset \mathbb{R}^n \setminus (H \cup E)$ with $|F| > 0$ such that*

$$P_\lambda(E) = P_\lambda(E \cup F) + P(F, \mathbb{R}^n \setminus H) + \lambda \mathcal{H}^{n-1}(\partial^* F \cap \partial H). \tag{4.4.11}$$

We begin with a preparatory lemma.

Lemma 4.4.8. *Let g be $\mathscr{R}$-admissible, q-growing, infinitesimal and symmetric and let G be $\mathscr{G}$-admissible and coercive. There exist $\bar{m} > 0$ and $\bar{T} > 0$, depending on n, λ, g, G, q such that, for all $m \in (0, \bar{m})$, every volume constrained minimizer E of $\mathscr{F}^\lambda$ with $|E| = m$ satisfies*

$$|E \cap \{x_n > \bar{T}\}| = 0.$$

Proof. By Lemma 4.3.12 there exists $\bar{T}_E < \infty$, depending on n, λ, g, G and E with

$$\bar{T}_E := \sup\{t\ :\ |E \cap \{x_n > t\}| > 0\}.$$

Let $x_E \in E$ such that

$$(x_E)_n \geq \frac{1}{2}\bar{T}_E.$$

By Lemma 4.4.2 there exists $c = c(n, \lambda, g, G, q) > 0$ such that, if $\bar{m} < 1$, then

$$|E \cap B_1(x_E)| \geq c\, m.$$

Therefore

$$\mathscr{F}^\lambda(E) \geq P_\lambda(E) + \int_{E \cap B_1(x_E)} G \geq P_\lambda(B^\lambda(m)) + cm \inf_{((x_E)_n - 1, (x_E)_n + 1)} G. \tag{4.4.12}$$

On the other hand, by Lemma 4.3.2, if $\bar{m} \leq |B^\lambda|$ we have

$$\mathscr{F}^\lambda(E) \leq \mathscr{F}^\lambda(B^\lambda(m)) \leq P_\lambda(B^\lambda(m)) + c(n, \lambda, g, G)m. \tag{4.4.13}$$

Putting together (4.4.12) and (4.4.13) we obtain

$$\inf_{((x_E)_n - 1, (x_E)_n + 1)} G \leq c(n, \lambda, g, G, q).$$

Since G is coercive, then $(x_E)_n$, and in particular also $\bar{T}_E$, is bounded by a constant independent of E, and we conclude the proof. □

Remark 4.4.9. We remark that Lemma 4.4.8 is a stronger result than Lemma 4.3.12. Indeed, the bound in Lemma 4.4.8 does not depend on the minimizer. At the same time, Lemma 4.3.12 does not require that g is q-growing and G is coercive.

Now we are ready to prove Theorem 4.4.7.

Proof of Theorem 4.4.7. If $\bar{m}$ is sufficiently small, Lemma 4.4.8 guarantees that there exists $\bar{T} > 0$, depending on n, λ, g, G, such that

$$|E \cap \{x_n > \bar{T}\}| = 0.$$

Assume that there exists a set $F \subset \mathbb{R}^n \setminus (H \cup E)$ with $v := |F| > 0$ and such that (4.4.11) holds. We aim to find a contradiction if $\bar{m}$ is chosen suitably small. Let $\bar{m} \leq |B^\lambda|$. By the minimality of E, the isoperimetric inequality Theorem 3.2.3, the relative isoperimetric inequality outside convex sets [CGR07; FM23] and since $P_\lambda(B^\lambda(m)) = n|B^\lambda|^{\frac{1}{n}} m^{\frac{n-1}{n}}$ [PP24, Lemma 3.1], we find

$$\begin{aligned}
n|B^\lambda|^{\frac{1}{n}} m^{\frac{n-1}{n}} + \mathscr{R}(B^\lambda(m)) + \mathscr{G}(B^\lambda(m)) = \mathscr{F}^\lambda(B^\lambda(m)) \geq \mathscr{F}^\lambda(E) &\geq P_\lambda(E) \\
&= P_\lambda(E \cup F) + P(F, \mathbb{R}^n \setminus H) + \lambda \mathcal{H}^{n-1}(\partial^* F \cap \partial H) \\
&\geq P_\lambda(E \cup F) + (1 - |\lambda|) P(F, \mathbb{R}^n \setminus H) \\
&\geq n|B^\lambda|^{\frac{1}{n}} (m + v)^{\frac{n-1}{n}} + (1 - |\lambda|)\, n \left(\frac{\omega_n}{2}\right)^{\frac{1}{n}} v^{\frac{n-1}{n}} \\
&\geq n|B^\lambda|^{\frac{1}{n}} m^{\frac{n-1}{n}} + (1 - |\lambda|)\, n \left(\frac{\omega_n}{2}\right)^{\frac{1}{n}} v^{\frac{n-1}{n}}
\end{aligned}$$

which gives, by Lemma 4.3.2,

$$v \leq \left(\frac{\mathscr{R}(B^\lambda(m)) + \mathscr{G}(B^\lambda(m))}{(1 - |\lambda|)\, n \sqrt[n]{\frac{\omega_n}{2}}} \right)^{\frac{n}{n-1}} \leq c(n, \lambda, g, G) \left(\frac{m}{(1 - |\lambda|)\, n \sqrt[n]{\frac{\omega_n}{2}}} \right)^{\frac{n}{n-1}}. \tag{4.4.14}$$

Since $\bar{m} < c(n, \lambda, g, G)$, also v is bounded by a suitable $\bar{v}(n, \lambda, g, G)$. By [NP21, Lemma 3.5] there exists a continuous and increasing function $\varphi : (0, \infty) \to (0, \infty)$, with $\varphi(0) = 0$, such that for every two sets $F_1, F_2 \subset \mathbb{R}^n \setminus H$ one has

$$\mathscr{R}(F_1, F_2) \leq |F_1|\, \varphi(|F_2|).$$

Then

$$\mathscr{R}(E \cup F) - \mathscr{R}(E) = \mathscr{R}(F, F) + 2\mathscr{R}(F, E) \leq v\, \varphi(\bar{v}) + 2v\, \varphi(\bar{m}) \leq c(n, \lambda, g, G) v.$$

Let us prove that also F is essentially contained in $\{0 < x_n \leq \bar{T}\}$. To this end, assume by contradiction that

$$\mathcal{H}^{n-1}\left(\partial^* F \setminus \left(H \cup \partial^* E\right)\right) > 0. \tag{4.4.15}$$

Let us denote

$$\begin{aligned}
\Sigma_E &:= \mathcal{H}^{n-1}\left(\partial^* E \setminus \left(H \cup \partial^* F\right)\right) \\
\Sigma &:= \mathcal{H}^{n-1}\left(\left(\partial^* E \cap \partial^* F\right) \setminus H\right) \\
\Sigma_F &:= \mathcal{H}^{n-1}\left(\partial^* F \setminus \left(H \cup \partial^* E\right)\right) \\
\Theta_E &:= \mathcal{H}^{n-1}\left(\partial^* E \cap \partial H\right) \\
\Theta_F &:= \mathcal{H}^{n-1}\left(\partial^* F \cap \partial H\right).
\end{aligned}$$

By (4.4.11) we obtain

$$\begin{aligned}
\Sigma_E + \Sigma - \lambda \Theta_E &= P_\lambda(E) \\
&= P_\lambda(E \cup F) + P(F, \mathbb{R}^n \setminus H) + \lambda \Theta_F \\
&= \Sigma_E + \Sigma_F - \lambda \Theta_E - \lambda \Theta_F + \Sigma + \Sigma_F + \lambda \Theta_F \\
&= \Sigma_E + 2\Sigma_F + \Sigma - \lambda \Theta_E.
\end{aligned}$$

In particular, we get $\Sigma_F = 0$, contradicting (4.4.15). Therefore $F \subset \{0 < x_n \leq \bar{T}\}$ and we also deduce

$$\mathscr{G}(E \cup F) - \mathscr{G}(E) = \mathscr{G}(F) \leq \int_F \sup_{(0, \bar{T})} G \, \mathrm{d}x = c(n, \lambda, g, G)\, v. \tag{4.4.16}$$

By (4.4.11), we obtain

$$
\begin{aligned}
\mathcal{F}^\lambda(E\cup F) &= P_\lambda(E\cup F)+\mathcal{R}(E\cup F)+\mathcal{G}(E\cup F)\\
&= P_\lambda(E)-P(F,\mathbb{R}^n\setminus H)-\lambda\mathcal{H}^{n-1}(\partial^* F\cap\partial H)+\mathcal{R}(E\cup F)+\mathcal{G}(E\cup F)\\
&\le P_\lambda(E)-(1-|\lambda|)P(F,\mathbb{R}^n\setminus H)+\mathcal{R}(E\cup F)+\mathcal{G}(E\cup F)\\
&\le P_\lambda(E)-(1-|\lambda|)\,n\left(\frac{\omega_n}{2}\right)^{\frac{1}{n}} v^{\frac{n-1}{n}}+\mathcal{R}(E)+c\,v+\mathcal{G}(E)+c\,v\\
&= \mathcal{F}^\lambda(E)-(1-|\lambda|)\,n\left(\frac{\omega_n}{2}\right)^{\frac{1}{n}} v^{\frac{n-1}{n}}+c\,v<\mathcal{F}^\lambda(E),
\end{aligned}
\tag{4.4.17}
$$

where the last inequality holds if v is sufficiently small, hence by (4.4.14) as soon as $\bar{m}$ is sufficiently small. Let $t\in(0,\infty)$ be such that, if $D=\{x\in E\cup F, x_n<t\}$, then $|D|=m$. Clearly $\mathcal{F}^\lambda(D)\le\mathcal{F}^\lambda(E\cup F)$, hence (4.4.17) implies $\mathcal{F}^\lambda(D)<\mathcal{F}^\lambda(E)$, contradicting the minimality of E. ☐

Remark 4.4.10. Note that if the function G were globally bounded, Theorem 4.4.7 could be easily extended to minimizers of $\mathcal{F}^\lambda$ in the class

$$
\mathcal{A}_m=\{\Omega\subset\mathbb{R}^n\setminus H \text{ measurable } : |\Omega|=m\}
$$

for every mass $m>0$.
Indeed, in the proof of Theorem 4.4.7 we exploited Lemma 4.4.8 just to get the estimate (4.4.16), which is trivial in case G were assumed to be globally bounded.

Now we are ready to complete the proof of Theorem 4.1.1.

Proof of Theorem 4.1.1. Theorem 4.1.1 follows by Theorem 4.3.1, Remark 4.3.6, Proposition 4.3.7, Theorem 4.3.8, Theorem 4.3.14 and Theorem 4.4.7. ☐

4.5 Generalized minimizers

Let us give the following definition.

Definition 4.5.1. If $E\subset\mathbb{R}^n\setminus H$ is a measurable set, g is a $\mathcal{R}$-admissible function, G is a $\mathcal{R}$-admissible function and $\varepsilon_1,\varepsilon_2>0$, we define the functional

$$
\begin{aligned}
\mathcal{F}^\lambda_\varepsilon(E) &:= P_\lambda(E)+\varepsilon_1\mathcal{R}(E)+\varepsilon_2\mathcal{G}(E)\\
&= P_\lambda(E)+\varepsilon_1\int_E\int_E g(y-x)\,\mathrm{d}y\,\mathrm{d}x+\varepsilon_2\int_E G(x_n)\,\mathrm{d}x.
\end{aligned}
$$

Remark 4.5.2. We remark that minimizing the functional $\mathcal{F}^\lambda$ in the small mass regime is equivalent to minimizing the functional $\mathcal{F}^\lambda_\varepsilon$ for $\varepsilon_1,\varepsilon_2$ small and among sets of a fixed volume. Indeed, let for instance $|E|=|B^\lambda|$, if $m>0$ and $\bar\varepsilon:=\frac{m^{\frac{1}{n}}}{|B^\lambda|^{\frac{1}{n}}}$, then $\tilde{E}:=\bar\varepsilon E$ has volume m and by scaling we have

$$
\mathcal{F}^\lambda(\tilde{E})=\bar\varepsilon^{n-1}\left(P_\lambda(E)+\bar\varepsilon^{n+1}\int_E\int_E g(\bar\varepsilon(y-x))\,\mathrm{d}y\,\mathrm{d}x+\bar\varepsilon\int_E G(\bar\varepsilon x_n)\,\mathrm{d}x\right).
$$

In particular we deduce that

- for every $\mathcal{R}$-admissible function g_1, $\mathcal{G}$-admissible function G_1 and $m>0$, there exist $\bar\varepsilon>0$, a $\mathcal{R}$-admissible function g_2 and a $\mathcal{G}$-admissible function G_2 such that, if

  $$
  \mathcal{F}^\lambda(E)=P_\lambda(E)+\int_E\int_E g_1(y-x)\,\mathrm{d}y\,\mathrm{d}x+\int_E G_1(x_n)\,\mathrm{d}x
  $$

  and

  $$
  \mathcal{F}^\lambda_{\bar\varepsilon}(E)=P_\lambda(E)+\bar\varepsilon^{n+1}\int_E\int_E g_2(y-x)\,\mathrm{d}y\,\mathrm{d}x+\bar\varepsilon\int_E G_2(x_n)\,\mathrm{d}x,
  $$

  then $\inf_{|E|=m}\mathcal{F}^\lambda(E)$ is proportional to $\inf_{|E|=|B^\lambda|}\mathcal{F}^\lambda_{\bar\varepsilon}(E)$ and the variational problems are equivalent.

- for every $\mathscr{R}$-admissible function g_2, $\mathscr{G}$-admissible function G_2, $\varepsilon_1, \varepsilon_2 > 0$ and $m > 0$, there exist a $\mathscr{R}$-admissible function g_1 and a $\mathscr{G}$-admissible function G_1 such that, if

$$\mathscr{F}^\lambda(E) = P_\lambda(E) + \int_E \int_E g_1(y-x)\,\mathrm{d}y\,\mathrm{d}x + \int_E G_1(x_n)\,\mathrm{d}x$$

and

$$\mathscr{F}_\varepsilon^\lambda(E) = P_\lambda(E) + \varepsilon_1 \int_E \int_E g_2(y-x)\,\mathrm{d}y\,\mathrm{d}x + \varepsilon_2 \int_E G_2(x_n)\,\mathrm{d}x,$$

then $\inf_{|E|=|B^\lambda|} \mathscr{F}_\varepsilon^\lambda(E)$ is proportional to $\inf_{|E|=m} \mathscr{F}^\lambda(E)$ and the variational problems are equivalent.

From now on for the rest of the section, we assume that $\varepsilon_1, \varepsilon_2 > 0$ and g, G as in Definition 4.5.1 are given. Hence we also define the generalized energy corresponding to $\mathscr{F}_\varepsilon^\lambda$ as

$$\tilde{\mathscr{F}}_\varepsilon^\lambda(E) := \inf_{h\in\mathbb{N}} \tilde{\mathscr{F}}_{\varepsilon,h}^\lambda(E),$$

where

$$\tilde{\mathscr{F}}_{\varepsilon,h}^\lambda(E) := \inf\left\{ \sum_{i=1}^h \mathscr{F}_\varepsilon^\lambda(E^i) \;:\; E = \bigcup_{i=1}^h E^i, E^i \cap E^j = \emptyset \quad \text{for } 1 \le i \ne j \le h \right\}.$$

The goal of this Section is to prove the following version of Theorem 4.1.3, suitably modified for the functional $\mathscr{F}_\varepsilon^\lambda$.

Theorem 4.5.3. *Let g be $\mathscr{R}$-admissible and q-growing and let G be $\mathscr{G}$-admissible. For every $\varepsilon_1, \varepsilon_2 > 0$ there exists a minimizer of $\tilde{\mathscr{F}}_\varepsilon^\lambda$ in the class*

$$\mathcal{A} := \left\{\Omega \subset \mathbb{R}^n \setminus H \text{ measurable } : \; |\Omega| = |B^\lambda|\right\}.$$

More precisely, there exist a set $E \in \mathcal{A}$ and a subdivision $E = \cup_{j=1}^h E^j$, with pairwise disjoint sets E^j, such that

$$\tilde{\mathscr{F}}_\varepsilon^\lambda(E) = \sum_{j=1}^h \mathscr{F}_\varepsilon^\lambda(E^j) = \inf\left\{\tilde{\mathscr{F}}_\varepsilon^\lambda(\Omega) \;:\; \Omega \in \mathcal{A}\right\}.$$

Moreover, for every $1 \le j \le h$, the set E^j is a minimizer of both the standard and the generalized energy for its volume, i.e.

$$\tilde{\mathscr{F}}_\varepsilon^\lambda(E^j) = \mathscr{F}_\varepsilon^\lambda(E^j) = \min\left\{\tilde{\mathscr{F}}_\varepsilon^\lambda(\Omega) \;:\; \Omega \subset \mathbb{R}^n \setminus H,\, |\Omega| = |E^j|\right\}. \tag{4.5.1}$$

Note that an analogous version of Lemma 4.3.4 holds.

Lemma 4.5.4. *Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter. Let g be $\mathscr{R}$-admissible and q-growing and let G be $\mathscr{G}$-admissible. If $\alpha > 1$, then*

$$\mathscr{F}_\varepsilon^\lambda(\alpha E) \le \alpha^{2n+q} \mathscr{F}_\varepsilon^\lambda(E).$$

We begin by proving some preparatory lemmas. The next geometric lemma allows to modify an excessively long and thin set decreasing its energy.

Lemma 4.5.5. *Let g be $\mathscr{R}$-admissible and G be $\mathscr{G}$-admissible. For every $\bar{m} > 0$ there exists $L(n, \lambda, \bar{m}) > 0$ such that the following holds. Let $E \subset \mathbb{R}^n \setminus H$, and let $a < b$ be two numbers with $b > a + 2L$ and such that*

$$\left|\left\{x \in E \;:\; a \le x_1 \le b\right\}\right| < \bar{m}.$$

Then there exist two numbers $a^+ \in [a, a+L]$ and $b^- \in [b-L, b]$ such that, denoting $E^- = E \setminus ([a^+, b^-] \times \mathbb{R}^{n-2} \times (0,\infty))$ and $m = |E| - |E^-| < \bar{m}$, one has

$$\mathscr{F}_\varepsilon^\lambda(E^-) \le \mathscr{F}_\varepsilon^\lambda(E) - \frac{1}{2} n |B^\lambda|^{\frac{1}{n}} m^{\frac{n-1}{n}}. \tag{4.5.2}$$

Proof. It is sufficient to prove the claim for bounded sets E such that $\partial E \setminus \partial H$ is a smooth hypersurface with $\mathcal{H}^{n-1}(\{x \in \partial E \setminus H \ : \ \nu^E(x) = \pm e_j\}) = 0$ for any $j = 1, \dots, n$. Indeed, if E is a generic set of finite perimeter satisfying the hypotheses of Lemma 4.5.5, let $E_i \xrightarrow{L^1} E$ be the sequence of sets given by Lemma 2.4.4. For i sufficiently large,

$$\left|\left\{x \in E_i \ : \ a \le x_1 \le b\right\}\right| < \bar{m}$$

holds. Then there exist $a_i^+ \in [a, a+L]$ and $b_i^- \in [b-L, b]$ such that, if we set $E_i^- = E_i \setminus \left([a_i^+, b_i^-] \times (0,\infty)\right)$ and $m_i = |E_i| - |E_i^-|$, we obtain

$$\mathcal{F}_\varepsilon^\lambda(E_i^-) \le \mathcal{F}_\varepsilon^\lambda(E_i) - \frac{1}{2} n |B^\lambda|^{\frac{1}{n}} m_i^{\frac{n-1}{n}}.$$

Up to subsequence, a_i^+ and b_i^- converge to certain $a^+ \in [a, a+L]$ and $b^- \in [b-L, b]$ respectively. By Lemma 2.4.4 $P_\lambda(E_i) \to P_\lambda(E)$. By the lower semicontinuity of P_λ 3.3.4, the continuity of $\mathcal{R}$ and $\mathcal{G}$ under strong L^1 convergence and the properties of $\{E_i\}$, if $E^- = E \setminus \left([a^+, b^-] \times (0,\infty)\right)$ then

$$\begin{aligned}
\mathcal{F}_\varepsilon^\lambda(E^-) &= P_\lambda(E^-) + \varepsilon_1 \mathcal{R}(E^-) + \varepsilon_2 \mathcal{G}(E^-) \\
&\le \liminf_i (P_\lambda(E_i^-) + \varepsilon_1 \mathcal{R}(E_i^-) + \varepsilon_2 \mathcal{G}(E_i^-)) \\
&\le \liminf_i \left(P_\lambda(E_i) + \varepsilon_1 \mathcal{R}(E_i) + \varepsilon_2 \mathcal{G}(E_i) - \frac{1}{2} n |B^\lambda|^{\frac{1}{n}} m_i^{\frac{n-1}{n}} \right) \\
&= \mathcal{F}_\varepsilon^\lambda(E) - \frac{1}{2} n |B^\lambda|^{\frac{1}{n}} m^{\frac{n-1}{n}}.
\end{aligned}$$

So let us fix $\bar{m}$ and consider a and b as in the claim, with L to be determined later. For $t \in \mathbb{R}$ let

$$\sigma(t) := \mathcal{H}^{n-1}\left(E \cap \left\{x \in \mathbb{R}^n \ : \ x_1 = t\right\}\right).$$

If $c := \frac{a+b}{2}$, let

$$\varphi(t) := \int_t^c \sigma(s)\, \mathrm{d}s.$$

We note that there exists $a^+ \in [a, a+L] \subset [a, c)$ such that, if

$$m_1 := \left|\left\{x \in E \ : \ a^+ < x_1 < c\right\}\right|,$$

then

$$\sigma(a^+) \le \frac{1}{8} n |B^\lambda|^{\frac{1}{n}} m_1^{\frac{n-1}{n}}. \tag{4.5.3}$$

Indeed, assume by contradiction that for every $t \in (a, a+L)$ it holds

$$-\varphi'(t) = \sigma(t) > \frac{1}{8} n |B^\lambda|^{\frac{1}{n}} \varphi(t)^{\frac{n-1}{n}}.$$

Then $\varphi|_{(a,a+L)}$ is a positive decreasing function satisfying

$$\begin{cases} \varphi(a) \le \bar{m}, \\ |\varphi'(t)| > \frac{1}{8} n |B^\lambda|^{\frac{1}{n}} \varphi(t)^{\frac{n-1}{n}}. \end{cases}$$

By standard ODE comparison there exists a constant $d > 0$ depending only on n, λ and $\bar{m}$ such that, if $a+d < a+L$, then $\varphi(t) \to 0$ as $t \to (a+d)^-$. Hence L could be chosen so big that $a+d < a+L < c$, and then $\varphi(t) = 0$ for any $t \in (a+d, c)$. It follows that there exists a^+ such that (4.5.3) holds. Similarly, up to choosing a larger L, we have the existence of $b^- \in [b-L, b] \subset (c, b]$ such that, if

$$m_2 := \left|\left\{x \in E \ : \ c < x_1 < b^-\right\}\right|,$$

then

$$\sigma(b^-) \le \frac{1}{8} n |B^\lambda|^{\frac{1}{n}} m_2^{\frac{n-1}{n}}.$$

Let $E^- := E \setminus \left(\left[a^+, b^-\right] \times (0,\infty)\right)$ and $F := E \setminus E^-$. Then

$$|F| = m = m_1 + m_2,$$

and, by isoperimetric inequality 3.2.3, there holds

$$P_\lambda(F) \geq n|B^\lambda|^{\frac{1}{n}} m^{\frac{n-1}{n}}.$$

Hence

$$\begin{aligned}
P_\lambda(E^-) &\leq P_\lambda(E) - P_\lambda(F) + 2(\sigma(a^+) + \sigma(b^-)) \\
&\leq P_\lambda(E) - P_\lambda(F) + \frac{1}{4} n|B^\lambda|^{\frac{1}{n}} \left(m_1^{\frac{n-1}{n}} + m_2^{\frac{n-1}{n}}\right) \\
&\leq P_\lambda(E) - P_\lambda(F) + \frac{1}{2} n|B^\lambda|^{\frac{1}{n}} (m_1 + m_2)^{\frac{n-1}{n}} \\
&\leq P_\lambda(E) - \frac{1}{2} n|B^\lambda|^{\frac{1}{n}} (m_1 + m_2)^{\frac{n-1}{n}}.
\end{aligned}$$

Since $E^- \subset E$, then $\mathscr{R}(E^-) \leq \mathscr{R}(E)$ and $\mathscr{G}(E^-) \leq \mathscr{G}(E)$, and we deduce (4.5.2). □

The following variant of Lemma 4.5.5 concerns the case of the vertical direction when we modify a part lying on the hyperplane $\{x_n = 0\}$.

Lemma 4.5.6. *Let g be $\mathscr{R}$-admissible and G be $\mathscr{G}$-admissible. For every $\bar{m} \in \mathbb{R}$ there exists $L(n, \lambda, \bar{m}) > 0$ such that the following holds. Let $E \subset \mathbb{R}^n \setminus H$, and let b be a number with $b > L$ and such that*

$$\left|\left\{x \in E \ : \ 0 < x_n \leq b\right\}\right| < \bar{m}.$$

There exists then $b^- \in [b - L, b]$ such that, denoting $E^- = E \setminus \left(\mathbb{R}^{n-1} \times [0, b^-]\right)$ and $m = |E| - |E^-| \leq \bar{m}$, one has

$$\mathscr{F}_\varepsilon^\lambda(E^-) \leq \mathscr{F}_\varepsilon^\lambda(E) - \frac{1}{2} n|B^\lambda|^{\frac{1}{n}} m^{\frac{n-1}{n}}. \tag{4.5.4}$$

Proof. As in the proof of Lemma 4.5.5, we may assume that $\partial E \setminus \partial H$ is a smooth hypersurface with $\mathcal{H}^{n-1}(\{x \in \partial E \setminus H \ : \ \nu^{E_i}(x) = \pm e_j\}) = 0$ for any $j = 1, \dots n$. Let us fix $\bar{m}$ and consider b as in the claim, with L to be determined later. For almost every $t \in \mathbb{R}$, let

$$\sigma(t) := \mathcal{H}^{n-1}\left(E \cap \left\{x \in \mathbb{R}^n \ : \ x_1 = t\right\}\right).$$

Let

$$\varphi(t) := \int_0^t \sigma(s)\, ds.$$

As in the proof of Lemma 4.5.5, we note that there exists $b^- \in [b - L, b] \subset (0, b]$ such that, if

$$m := \left|\left\{x \in E \ : \ 0 < x_n < b^-\right\}\right|,$$

then

$$\sigma(b^-) \leq \frac{1}{4} n|B^\lambda|^{\frac{1}{n}} m^{\frac{n-1}{n}}.$$

If $E^- := E \setminus \left(\mathbb{R}^{n-1} \times [0, b^-]\right)$ and $F := E \setminus E^-$, then

$$|F| = m$$

and, by isoperimetric inequality 3.2.3,

$$P_\lambda(F) \geq n|B^\lambda|^{\frac{1}{n}} m^{\frac{n-1}{n}}.$$

We establish that

$$\begin{aligned}
P_\lambda(E^-) &\leq P_\lambda(E) - P_\lambda(F) + 2\sigma(b^-) \\
&\leq P_\lambda(E) - P_\lambda(F) + \frac{1}{2} n|B^\lambda|^{\frac{1}{n}} m^{\frac{n-1}{n}} \\
&\leq P_\lambda(E) - \frac{1}{2} n|B^\lambda|^{\frac{1}{n}} m^{\frac{n-1}{n}}.
\end{aligned}$$

Since $E^- \subset E$, then $\mathscr{R}(E^-) \leq \mathscr{R}(E)$, $\mathscr{G}(E^-) \leq \mathscr{G}(E)$ and we deduce (4.5.4). □

We now prove a uniform boundedness result.

Lemma 4.5.7. *Let g be $\mathscr{R}$-admissible and q-growing and let G be $\mathscr{G}$-admissible. Let $\varepsilon_1, \varepsilon_2 > 0$. For every $m \in (0,\infty)$ there exist $R > 0$ and $\bar{h} \in \mathbb{N}$, depending on n, λ, m, ε_1, ε_2, g, G and q, such that*

$$\inf\left\{\mathscr{F}_\varepsilon^\lambda(\Omega) \;:\; \Omega \subset \mathbb{R}^n \setminus H,\, |\Omega| = m\right\} \geq \inf\left\{\tilde{\mathscr{F}}_{\varepsilon,\bar{h}}^{\lambda,R}(\Omega) \;:\; \Omega \subset \mathbb{R}^n \setminus H,\, |\Omega| = m\right\},$$

where

$$\tilde{\mathscr{F}}_{\varepsilon,\bar{h}}^{\lambda,R}(\Omega) := \inf\left\{\sum_{i=1}^{\bar{h}} \mathscr{F}_\varepsilon^\lambda(\Omega^i) \;:\; \Omega = \cup_{i=1}^{\bar{h}}\Omega^i,\, \Omega^i \cap \Omega^j = \emptyset,\, \operatorname{diam}\Omega^i \leq R \quad \forall 1 \leq i \neq j \leq \bar{h}\right\}.$$

Proof. Let $M(n,\lambda,m,\varepsilon_1,\varepsilon_2,g,G,q) \in \mathbb{N}$ be a natural number to be determined later and let us denote $\bar{m} = m/M$. Let $E \subset R^n \setminus H$ be a bounded set with $|E| = m$ and

$$\mathscr{F}_\varepsilon^\lambda(E) \leq \inf\left\{\mathscr{F}_\varepsilon^\lambda(\Omega) \;:\; \Omega \in \mathcal{A}\right\} + \frac{n|B^\lambda|^{\frac{1}{n}}}{3}\left(\frac{m}{M^2}\right)^{\frac{n-1}{n}}. \tag{4.5.5}$$

This is possible since the infimum is reached by a sequence of bounded sets. Let $t_0 < t_1 < \cdots < t_{M-1} < t_M$ be real numbers such that

$$\left|E \cap \left((t_i, t_{i+1}) \times \mathbb{R}^{n-2} \times (0,\infty)\right)\right| = \bar{m},$$

for every $0 \leq i \leq M-1$ and let $L(n,\lambda,\bar{m})$ be given by Lemma 4.5.5. For every $0 \leq i \leq M-1$ let us define the interval I_i in the following way. If $t_{i+1} - t_i \leq 2L$ we set $I_i = \emptyset$, otherwise we apply Lemma 4.5.5 with $a = t_i$ and $b = t_{i+1}$ and we set $I_i = [a^+, b^-]$. If $m_i = |E \cap (I_i \times \mathbb{R}^{n-2} \times (0,\infty))|$, then

$$m_i \leq \frac{m}{M^2}. \tag{4.5.6}$$

Indeed, if $I_i = \emptyset$, then (4.5.6) is clearly true. If $I_i \neq \emptyset$, we set

$$E' = \alpha\left(E \setminus \left(I_i \times \mathbb{R}^{n-2} \times (0,\infty)\right)\right),$$

with $\alpha = \left(\frac{m}{m-m_i}\right)^{\frac{1}{n}}$. Note that $\frac{m_i}{m} \leq \frac{1}{M}$ by construction. By Lemma 4.5.4 and (4.5.2) we have

$$\begin{aligned}
\mathscr{F}_\varepsilon^\lambda(E') &\leq \left(\frac{m}{m-m_i}\right)^{2+\frac{q}{n}} \mathscr{F}_\varepsilon^\lambda\left(E \setminus (I_i \times \mathbb{R}^{n-2} \times (0,\infty))\right) \\
&\leq \left(\frac{1}{1-\frac{m_i}{m}}\right)^{2+\frac{q}{n}} \left(\mathscr{F}_\varepsilon^\lambda(E) - \frac{1}{2} n|B^\lambda|^{\frac{1}{n}} m_i^{\frac{n-1}{n}}\right).
\end{aligned}$$

Moreover, if M is large enough,

$$\begin{aligned}
\mathscr{F}_\varepsilon^\lambda(E') &\leq \left(1 + \left(3 + \frac{q}{n}\right)\frac{m_i}{m}\right)\left(\mathscr{F}_\varepsilon^\lambda(E) - \frac{1}{2} n|B^\lambda|^{\frac{1}{n}} m_i^{\frac{n-1}{n}}\right) \\
&\leq \mathscr{F}_\varepsilon^\lambda(E) - \frac{1}{3} n|B^\lambda|^{\frac{1}{n}} m_i^{\frac{n-1}{n}}.
\end{aligned} \tag{4.5.7}$$

Estimates (4.5.5) and (4.5.7) imply that

$$\mathscr{F}_\varepsilon^\lambda(E') \leq \inf\left\{\mathscr{F}_\varepsilon^\lambda(\Omega) \;:\; \Omega \in \mathcal{A}\right\} + \frac{n|B^\lambda|^{\frac{1}{n}}}{3}\left(\frac{m}{M^2}\right)^{\frac{n-1}{n}} - \frac{n|B^\lambda|^{\frac{1}{n}}}{3} m_i^{\frac{n-1}{n}},$$

and, since $|E'| = m$, (4.5.6) holds.
Let

$$\tilde{E} = E \setminus \left(\bigcup_{i=0}^{M-1} I_i \times \mathbb{R}^{n-2} \times (0,\infty)\right)$$

and $\mu = \sum_{i=0}^{M-1} m_i$, so that $|\tilde{E}| = m - \mu$. By Lemma 4.5.5 and the subadditivity of power function with exponent less than 1 we get

$$\mathcal{F}_\varepsilon^\lambda(\tilde{E}) \le \mathcal{F}_\varepsilon^\lambda(E) - \frac{1}{2} n|B^\lambda|^{\frac{1}{n}} \sum_{i=0}^{M-1} m_i^{\frac{n-1}{n}} \le \mathcal{F}_\varepsilon^\lambda(E) - \frac{1}{2} n|B^\lambda|^{\frac{1}{n}} \mu^{\frac{n-1}{n}}.$$

The set $F := \left(\frac{m}{m-\mu}\right)^{\frac{1}{n}} \tilde{E}$ has volume m. We can use Lemma 4.5.4 to obtain

$$\mathcal{F}_\varepsilon^\lambda(F) \le \left(\frac{m}{m-\mu}\right)^{2+\frac{q}{n}} \mathcal{F}_\varepsilon(\tilde{E}) \le \left(\frac{m}{m-\mu}\right)^{2+\frac{q}{n}} \left(\mathcal{F}_\varepsilon^\lambda(E) - \frac{1}{2} n|B^\lambda|^{\frac{1}{n}} \mu^{\frac{n-1}{n}}\right).$$

If μ is small enough, which happens as soon as M is large enough thanks to (4.5.6), we deduce

$$\begin{aligned} \mathcal{F}_\varepsilon^\lambda(F) &\le \left(\frac{m}{m-\mu}\right)^{2+\frac{q}{n}} \left(\mathcal{F}_\varepsilon^\lambda(E) - \frac{1}{2} n|B^\lambda|^{\frac{1}{n}} \mu^{\frac{n-1}{n}}\right) \\ &\le \mathcal{F}_\varepsilon^\lambda(E) + C\mu\left(\inf\left\{\mathcal{F}_\varepsilon^\lambda(\Omega) \ : \ \Omega \in \mathcal{A}\right\} + c\left(\frac{m}{M^2}\right)^{\frac{n-1}{n}}\right) + (1 + C\mu)\left(-\frac{1}{2} n|B^\lambda|^{\frac{1}{n}} \mu^{\frac{n-1}{n}}\right) \\ &\le \mathcal{F}_\varepsilon^\lambda(E). \end{aligned}$$

Note that $\tilde{E}$ is the union of at most $M + 1$ sets, each contained in a slab having width at most equal to $2L$ by Lemma 4.5.5. In particular, F is the union of at most $M + 1$ parts and each of them has horizontal width at most equal $3L$.
If we repeat the arguments in the remaining directions, with care to apply also Lemma 4.5.6 in the n-th direction, we get the boundedness of the pieces in all the n directions. Then there exist $R \in (0, \infty)$, $\bar{h} \in \mathbb{N}$ and $G \subset \mathbb{R}^n$ such that $|G| = m$, $G = \cup_{i=1}^{\bar{h}} G_i$, $G_i \cap G_j = \emptyset$, $\operatorname{diam} G_i \le R$ and $\mathcal{F}_\varepsilon^\lambda(G) \le \mathcal{F}_\varepsilon^\lambda(E)$. Finally

$$\mathcal{F}_\varepsilon^\lambda(G) \ge \sum_{i=1}^{\bar{h}} \mathcal{F}_\varepsilon^\lambda(G_i) \ge \tilde{\mathcal{F}}_{\varepsilon,\bar{h}}^{\lambda,R}(G).$$

□

Now we are ready to prove Theorem 4.5.3.

Proof of Theorem 4.5.3. We begin by proving the existence of $h' \in \mathbb{N}$ and of a sequence $\{G_i\}_{i\in\mathbb{N}} \subset \mathcal{A}$ such that

$$\inf\left\{\mathcal{F}_\varepsilon^\lambda(\Omega) \ : \ \Omega \in \mathcal{A}\right\} = \lim_{i\to\infty} \tilde{\mathcal{F}}_{\varepsilon,h'}^\lambda(G_i). \tag{4.5.8}$$

Let $h'(n, \lambda, \varepsilon_1, \varepsilon_2, g, G, q)$ be an integer to be determined later and consider a sequence $\{E_i\}_{i\in\mathbb{N}} \subset \mathcal{A}$ such that

$$K := \inf\left\{\tilde{\mathcal{F}}_\varepsilon^\lambda(\Omega) \ : \ \Omega \in \mathcal{A}\right\} = \lim_{i\to\infty} \tilde{\mathcal{F}}_\varepsilon^\lambda(E_i). \tag{4.5.9}$$

For every $i \in \mathbb{N}$ let $h(i) \in \mathbb{N}$ such that there exists a subdivision $E_i = E_i^1 \cup E_i^2 \cup \cdots \cup E_i^{h(i)}$ with

$$\tilde{\mathcal{F}}_\varepsilon^\lambda(E_i) > \left(1 - \frac{1}{i+1}\right) \sum_{j=1}^{h(i)} \mathcal{F}_\varepsilon^\lambda(E_i^j). \tag{4.5.10}$$

Without loss of generality, we can assume $h(i) \to \infty$, so that $h(i) > h'$ for i large enough. Let us fix a generic $i \in \mathbb{N}$. For simplicity of notation, let us denote $h = h(i)$ and $m_j = |E_i^j|$ for every $1 \le j \le h$. Let us also assume, without loss of generality, that m_j is decreasing with respect to j. By (4.5.10) we get

$$\tilde{\mathcal{F}}_\varepsilon^\lambda(E_i) \ge \frac{1}{2} \sum_{j=1}^{h} P_\lambda(E_i^j) \ge \frac{1}{2} \sum_{j=1}^{h} n|B^\lambda|^{\frac{1}{n}} m_j^{\frac{n-1}{n}} \ge \frac{1}{2\sqrt[n]{m_1}} \sum_{j=1}^{h} n|B^\lambda|^{\frac{1}{n}} m_j = \frac{1}{2\sqrt[n]{m_1}} n|B^\lambda|^{\frac{n+1}{n}}.$$

If i is large enough, by (4.5.9) we deduce that

$$m_1 \geq \left(\frac{n|B^\lambda|^{\frac{n+1}{n}}}{4K} \right)^n .$$

For every such i, we define

$$G_i = \alpha \bigcup_{j=1}^{h'} E_i^j ,$$

with

$$\alpha = \left(\frac{|B^\lambda|}{|B^\lambda| - \sum_{j>h'} m_j} \right)^{\frac{1}{n}} \leq 1 + c_1 \sum_{j=h'+1}^{h} m_j ,$$

where c_1 is a constant depending on n, λ and K (that is on n, λ, ε_1, ε_2, g and G). Note that also G_i belongs to $\mathcal{A}$. By Lemma 4.5.4 we deduce

$$\begin{aligned} \tilde{\mathscr{F}}^\lambda_{\varepsilon,h'}(G_i) &\leq \sum_{j=1}^{h'} \mathscr{F}^\lambda_\varepsilon(\alpha E_i^j) \leq \alpha^{2n+q} \sum_{j=1}^{h'} \mathscr{F}_\varepsilon(E_i^j) \\ &\leq \left(1 + c_2(n, \lambda, \varepsilon_1, \varepsilon_2, g, G, q) \sum_{j=h'+1}^{h} m_j \right) \sum_{j=1}^{h'} \mathscr{F}^\lambda_\varepsilon(E_i^j). \end{aligned} \tag{4.5.11}$$

By (4.5.10) we get

$$\begin{aligned} \tilde{\mathscr{F}}^\lambda_\varepsilon(E_i) &> \left(1 - \frac{1}{i+1}\right) \sum_{j=1}^{h} \mathscr{F}^\lambda_\varepsilon(E_i^j) \\ &\geq \left(1 - \frac{1}{i+1}\right) \sum_{j=1}^{h'} \mathscr{F}^\lambda_\varepsilon(E_i^j) + \frac{1}{2} \sum_{j=h'+1}^{h} P_\lambda(E_i^j) \\ &\geq \left(1 - \frac{1}{i+1}\right) \sum_{j=1}^{h'} \mathscr{F}^\lambda_\varepsilon(E_i^j) + \frac{1}{2} n |B^\lambda|^{\frac{1}{n}} \sum_{j=h'+1}^{h} m_j^{\frac{n-1}{n}} . \end{aligned}$$

If i is large enough, by (4.5.9) and (4.5.10)

$$\sum_{j=1}^{h'} \mathscr{F}^\lambda_\varepsilon(E_i^j) \leq 2K .$$

By (4.5.11)

$$\tilde{\mathscr{F}}^\lambda_{\varepsilon,h'}(G_i) - \tilde{\mathscr{F}}^\lambda_{\varepsilon,h}(E_i) \leq 2K \left(c_2(n, \lambda, \varepsilon_1, \varepsilon_2, g, G, q) \sum_{j=h'+1}^{h} m_j + \frac{1}{i+1} \right) - \frac{1}{2} n |B^\lambda|^{\frac{1}{n}} \sum_{j=h'+1}^{h} m_j^{\frac{n-1}{n}} . \tag{4.5.12}$$

Now we can define $h' \in \mathbb{N}$ so that

$$h' \geq \left(\frac{4Kc_2(n, \lambda, \varepsilon_1, \varepsilon_2, g, G, q)}{n} \right)^n .$$

Since $m_j \leq \frac{|B^\lambda|}{h'}$ for $j > h'$, if i is large enough we get from (4.5.12)

$$\tilde{\mathscr{F}}^\lambda_{\varepsilon,h'}(G_i) \leq \tilde{\mathscr{F}}^\lambda_{\varepsilon,h}(E_i) + \frac{2K}{i+1} .$$

By (4.5.9) we finally deduce that $\{G_i\}_{i \in \mathbb{N}} \subset \mathcal{A}$ satisfies (4.5.8).

Let us now show that for every $m > 0$ there exist $\bar{h} \in \mathbb{N}$, a bounded set E with $|E| = m$ and a subdivision $E = \bigcup_{k=1}^{\bar{h}} E^k$ such that

$$\tilde{\mathscr{F}}_\varepsilon^\lambda(E) \leq \sum_{k=1}^{\bar{h}} \mathscr{F}_\varepsilon^\lambda(E^k) \leq \inf\left\{\mathscr{F}_\varepsilon^\lambda(\Omega) \;:\; \Omega \in \mathbb{R}^n \setminus H, |\Omega| = m\right\}. \tag{4.5.13}$$

Let R and $\bar{h}$ be as in Lemma 4.5.7. By Lemma 4.5.7 there is a sequence of sets $\{\Omega_i\}_{i\in\mathbb{N}}$ of volume m such that

$$\inf\left\{\mathscr{F}_\varepsilon^\lambda(\Omega) \;:\; \Omega \subset \mathbb{R}^n \setminus H, |\Omega| = m\right\} \geq \lim_{i\to+\infty} \tilde{\mathscr{F}}_{\varepsilon,\bar{h}}^{\lambda,R}(\Omega_i), \tag{4.5.14}$$

where $\tilde{\mathscr{F}}_{\varepsilon,\bar{h}}^{\lambda,R}$ is defined in Lemma 4.5.7. For every $i \in \mathbb{N}$ there exists a partition $\Omega_i = \Omega_i^1 \cup \Omega_i^2 \cup \cdots \cup \Omega_i^{\bar{h}}$ with $\operatorname{diam}(\Omega_i^j) \leq R$ and

$$\sum_{j=1}^{\bar{h}} \mathscr{F}_\varepsilon^\lambda(\Omega_i^j) \leq \tilde{\mathscr{F}}_{\varepsilon,\bar{h}}^{\lambda,R}(\Omega_i) + \frac{1}{i}. \tag{4.5.15}$$

Up to a subsequence there exist $m_j \in (0,\infty)$, with $1 \leq j \leq \bar{h}$, such that

$$m_j = \lim_{i\to\infty} |\Omega_i^j| \qquad \forall 1 \leq j \leq \bar{h}, \qquad\qquad m = \sum_{j=1}^{\bar{h}} m_j.$$

Let us fix $1 \leq k \leq \bar{h}$ and consider the sets $\{\Omega_i^k\}_{i\in\mathbb{N}}$. Since their diameters are uniformly bounded by R, up to translations we can assume that all the Ω_i^k are pairwise disjoint and contained in a fixed ball with radius R. Therefore the characteristic functions $f_i = \chi_{\Omega_i^k}$ have uniformly bounded supports and are bounded in BV. Up to a subsequence, we can assume that f_i is weakly* convergent in BV, and in particular strongly convergent in L^1, to a certain function f. Then f is the characteristic function of a bounded set E^k with volume m_k. By the lower-semicontinuity of the perimeter under weak* BV-convergence and the continuity of $\mathscr{R}$ and $\mathscr{G}$ under strong L^1 convergence, we obtain that

$$\mathscr{F}_\varepsilon^\lambda(E^k) \leq \liminf_{i\to\infty} \mathscr{F}_\varepsilon^\lambda(\Omega_i^k). \tag{4.5.16}$$

Up to a translation we can assume that the sets E^k are pairwise disjoint. In particular the set $E = \cup_{k=1}^{\bar{h}} E^k$ is bounded with $|E| = m$. By (4.5.14), (4.5.15) and (4.5.16) we get

$$\begin{aligned}\tilde{\mathscr{F}}_\varepsilon^\lambda(E) &\leq \sum_{k=1}^{\bar{h}} \mathscr{F}_\varepsilon^\lambda(E^k) \leq \sum_{k=1}^{\bar{h}} \liminf_{i\to\infty} \mathscr{F}_\varepsilon^\lambda(\Omega_i^k) \leq \liminf_{i\to\infty} \sum_{k=1}^{\bar{h}} \mathscr{F}_\varepsilon^\lambda(\Omega_i^k)\\ &\leq \liminf_{i\to\infty} \tilde{\mathscr{F}}_{\varepsilon,\bar{h}}^{\lambda,R}(\Omega_i) \leq \inf\left\{\mathscr{F}_\varepsilon^\lambda(\Omega) \;:\; \Omega \subset \mathbb{R}^n \setminus H,\ |\Omega| = m\right\},\end{aligned}$$

so (4.5.13) is proved.

We can now conclude the proof of the theorem. Let $\{G_i\}_{i\in\mathbb{N}}$ as in (4.5.8) and let us consider a subdivision $G_i = G_i^1 \cup G_i^2 \cup \cdots \cup G_i^{h'}$ such that

$$\inf\left\{\tilde{\mathscr{F}}_\varepsilon^\lambda(\Omega) \;:\; \Omega \in \mathcal{A}\right\} = \lim_{i\to\infty} \sum_{j=1}^{h'} \mathscr{F}_\varepsilon^\lambda(G_i^j). \tag{4.5.17}$$

Up to a subsequence there exist $\mu_j > 0$, $1 \leq j \leq h'$, such that

$$\mu_i = \lim_{i\to\infty} |G_i^j| \qquad \forall 1 \leq j \leq h',\ |B^\lambda| = \sum_{j=1}^{h'} \mu_j.$$

Let

$$K_j := \inf\left\{\mathscr{F}_\varepsilon^\lambda(\Omega) \;:\; \Omega \subset \mathbb{R}^n \setminus H, |\Omega| = \mu_j\right\}.$$

By (4.5.17)

$$\inf\left\{\tilde{\mathscr{F}}_\varepsilon^\lambda(\Omega) \;:\; \Omega \in \mathcal{A}\right\} = \sum_{j=1}^{h'} K_j. \tag{4.5.18}$$

By (4.5.13) for every $1 \le j \le h'$ there exist $\bar h(j) \in \mathbb{N}$, a bounded set $E_j \subset \mathbb{R}^n \setminus H$ with $|E_j| = \mu_j$ and a subdivision in pairwise disjoint sets $E_j = \bigcup_{k=1}^{\bar h(j)} E_{j,k}$ such that

$$\tilde{\mathscr{F}}_\varepsilon^\lambda(E_j) \le \sum_{k=1}^{\bar h(j)} \mathscr{F}_\varepsilon^\lambda(E_{j,k}) \le K_j. \tag{4.5.19}$$

Since the sets E_j are bounded, up to translations we can assume that the set $E = \cup_{j=1}^{h'} E_j$ has volume $|B^\lambda|$. Therefore E is the disjoint union of all the sets $E_{j,k}$ with $1 \le j \le h'$ and $1 \le k \le \bar h(j)$. Let us denote these sets as E^l with $1 \le l \le h$ and $h = \sum_{j=1}^{h'} \bar h(j)$. By (4.5.18) and (4.5.19) we deduce that

$$\tilde{\mathscr{F}}_\varepsilon^\lambda(E) \le \sum_{l=1}^{h} \mathscr{F}_\varepsilon^\lambda(E^l) \le \sum_{j=1}^{h'} K_j = \inf \left\{ \tilde{\mathscr{F}}_\varepsilon^\lambda(\Omega) \ : \ \Omega \in \mathcal{A} \right\},$$

that is E is a minimizer of $\tilde{\mathscr{F}}_\varepsilon^\lambda$ and the subdivision $E = \bigcup_{l=1}^{h} E^l$ is optimal.
The proof of (4.5.1) for a given $1 \le \bar j \le h$ easily follows as in the proof of [NP21, Proposition 1.2]. □

Now we are ready to prove Theorem 4.1.3.

Proof of Theorem 4.1.3. With the notation of Theorem 4.1.3, if $\Omega \subset \mathbb{R}^n \setminus H$ is a measurable set with $|\Omega| = m$, let

$$\mathscr{F}^\lambda(\Omega) = P_\lambda(\Omega) + \int_\Omega \int_\Omega g(y-x)\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega G(x_n)\,\mathrm{d}x$$

and

$$\tilde{\mathscr{F}}^\lambda(\Omega) := \inf_{h \in \mathbb{N}} \tilde{\mathscr{F}}_h^\lambda(\Omega),$$

where

$$\tilde{\mathscr{F}}_h^\lambda(\Omega) := \inf \left\{ \sum_{i=1}^{h} \mathscr{F}^\lambda(\Omega^i) \ : \ \Omega = \bigcup_{i=1}^{h} \Omega^i, \Omega^i \cap \Omega^j = \emptyset \quad \text{for } 1 \le i \ne j \le h \right\}.$$

If $F \subset \mathbb{R}^n \setminus H$ has measure $|B^\lambda|$ and $\bar\varepsilon = \frac{m^{\frac{1}{n}}}{|B^\lambda|^{\frac{1}{n}}}$, the set $\tilde F := \bar\varepsilon F$ has volume m and by Remark 4.5.2 there exist $\tilde g$ $\mathscr{R}$-admissible and $\tilde G$ $\mathscr{G}$-admissible such that

$$\mathscr{F}^\lambda(\tilde F) = \bar\varepsilon^{n-1} \left(P_\lambda(F) + \bar\varepsilon^{n+1} \int_F \int_F \tilde g(y-x)\,\mathrm{d}y\,\mathrm{d}x + \bar\varepsilon \int_F \tilde G(x_n)\,\mathrm{d}x \right) =: \bar\varepsilon^{n-1} \mathscr{F}_{\bar\varepsilon}^\lambda(F).$$

Note that $\tilde{\mathscr{F}}^\lambda(\tilde F) = \bar\varepsilon^{n-1} \tilde{\mathscr{F}}_{\bar\varepsilon}^\lambda(F)$ and that, if g is q-growing, also $\tilde g$ is q-growing (see Remark 4.5.2). By Theorem 4.5.3 there exists $E \subset \mathbb{R}^n \setminus H$ with $|E| = |B^\lambda|$ which minimizes $\tilde{\mathscr{F}}_{\bar\varepsilon}^\lambda$. Then the set $\tilde E := \bar\varepsilon E$ minimizes $\tilde{\mathscr{F}}^\lambda$ among sets with volume m and Theorem 4.1.3 easily follows. □

Chapter 5

Regularity results for some nonlinear elliptic systems with discontinuous coefficients

5.1 Main results

We consider nonlinear elliptic systems of the type

$$\operatorname{div} A(x, Du(x)) = \operatorname{div} F(x) \tag{5.1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n > 2$, and with $u : \Omega \to \mathbb{R}^N$, $N > 1$. We suppose that the vector field $A : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ is a Carathéodory function, i.e.

- $x \to A(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^{N\times n}$,
- $\xi \to A(x, \xi)$ is continuous for a.e. $x \in \Omega$.

Furthermore, we assume that there exist a function $b(x) \geq \lambda_0 > 0$, belonging to the space BMO, and a function $K(x)$, belonging to the Marcinkiewicz space $L^{n,\infty}(\Omega)$, such that $F \in W^{1,2}_{loc}(b, \Omega; \mathbb{R}^{N\times n})$ and, for a.e. $x, y \in \Omega$,

$$|A(x, \xi) - A(x, \eta)| \leq k b(x)|\xi - \eta| \,(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}, \tag{5.1.2}$$

$$\frac{1}{k} b(x)|\xi - \eta|^2 \,(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \leq \langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle\,, \tag{5.1.3}$$

$$|A(x, \eta) - A(y, \eta)| \leq |x - y| \; [K(x) + K(y)] \; (\mu^2 + |\eta|^2)^{\frac{p-1}{2}} \tag{5.1.4}$$

$$A(x, 0) = 0 \tag{5.1.5}$$

$$|b(x) - b(y)| \leq |x - y| \,[K(x) + K(y)], \tag{5.1.6}$$

where k is a positive constant, $\mu \in (0, 1]$, $p \geq 2$, ξ and η are arbitrary elements of $\mathbb{R}^{N\times n}$. Note that, by virtue of a characterization of the Sobolev functions due to Hajlasz [Haj96], the conditions (5.1.4) and (5.1.6) describe a weak form of continuity with respect to the x-variable since the function K may blow up at some points.
In the account of the typical functions of BMO and $L^{n,\infty}$ respectively, the functions

$$b(x) = \frac{e^{-|x|}}{\Lambda} - \Lambda \log |x|$$
$$K(x) = \frac{e^{-|x|}}{\Lambda} + \Lambda \frac{1}{|x|},$$

defined for a positive Λ with $x \in B(0, 1) = \{y \in \mathbb{R}^n : 0 < |y| < 1\}$, satisfy assumption (5.1.6).
A vector field u in the Sobolev space $W^{1,r}_{loc}(b, \Omega; \mathbb{R}^N)$, $r > \frac{2n}{n+2}$, is a local solution of (5.1.1) if it verifies

$$\int_{\operatorname{supp}\varphi} \langle A(x, Du(x)), D\varphi(x)\rangle \,\mathrm{d}x = \int_{\operatorname{supp}\varphi} \langle F(x), D\varphi(x)\rangle \,\mathrm{d}x \qquad \forall \varphi \in C^\infty_0(\Omega, \mathbb{R}^N). \tag{5.1.7}$$

In this paper our first goal is to study regularity properties of local solutions to (5.1.1) for r close to p. The existence of second derivatives is not clear due to the degeneracy of the problem; anyway, although the first derivatives of the solutions may not be differentiable, the higher differentiability of solutions holds in the sense that the nonlinear expressions $V_\mu(Du) := (\mu^2+|Du|^2)^{\frac{p-2}{4}} Du$ of their gradients, with $\mu \in (0,1]$, are weakly differentiable. Therefore, the main result will be the following:

Theorem 5.1.1 ([MP24]). *Let Ω be a regular domain, $A(x,\xi)$ a mapping verifying assumptions (5.1.2), (5.1.3), (5.1.4) and (5.1.5), and $F \in W^{1,2}_{loc}(b,\Omega;\mathbb{R}^{N\times n})$, with $b(x)$ as in (5.1.6). There exist $0<\varepsilon_1<\frac{1}{2}$, depending on k, n, λ_0, p and the BMO - norm of $b(x)$, and $\alpha_1>0$, depending on p, n, λ_0, μ and k, such that, if $u \in W^{1,p-\varepsilon}_{loc}(b,\Omega;\mathbb{R}^N)$, with $0 \le \varepsilon < \varepsilon_1$, is a local solution of (5.1.1) and*

$$\mathscr{D}_K := \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_1,$$

then $D(V_\mu(Du)) \in L^2_{loc}(b,\Omega)$ and the following estimate holds:

$$\int_{B_R} |D(V_\mu(Du))|^2 b \,\mathrm{d}x \le c\int_{B_{2R}} \left(\left(1+\frac{1}{R^2}\right)(\mu^2+|Du|^2)^{\frac{p}{2}} + (\mu^2+|DF|^2)\right) b\,\mathrm{d}x,$$

for every ball $B_{2R} \subset\subset \Omega$ and for a constant c depending on p, k, λ_0, n, μ and $\mathscr{D}_K$.

We point out that for local solutions of homogeneous systems

$$\operatorname{div} A(x,Du) = 0,$$

Theorem 5.1.1 also applies in the degenerate case, i.e. $\mu = 0$, with constants independent of μ (see Proposition 5.4.1). As a consequence, we establish certain local Calderón and Zygmund type estimates without assuming any differentiability condition on the datum. More precisely, for $G \in L^p_{loc}(b,\Omega;\mathbb{R}^{N\times n})$ we consider the problem

$$\operatorname{div} A(x,Du(x)) = \operatorname{div} |G|^{p-2}G \quad \text{in} \quad \Omega. \tag{5.1.8}$$

Then we prove the following result:

Theorem 5.1.2 ([MP24]). *Let Ω be a regular domain and $A(x,\xi)$ a mapping verifying assumptions (5.1.2), (5.1.3), (5.1.4) and (5.1.5), with $b(x)$ as in (5.1.6). There exists $\alpha_2>0$, depending on p, n, λ_0 and k, such that, if $u \in W^{1,p}_{loc}(b,\Omega;\mathbb{R}^N)$ is a local solution of (5.1.8) and*

$$\mathscr{D}_K := \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_2,$$

then

$$G \in L^q_{loc}(b,\Omega;\mathbb{R}^{N\times n}) \quad \Longrightarrow \quad Du \in L^q_{loc}(b,\Omega;\mathbb{R}^{N\times n})$$

for any $q \in (p,s)$, where $s := \frac{np}{n-1}+\delta$ for a suitable $\delta>0$, depending on p, k, λ_0, n, $\mathscr{D}_K$ and the BMO-norm of b. Moreover, for every cube $Q_{2R} \subset\subset \Omega$ and $\mu \in [0,1]$, we have

$$\left(\fint_{Q_R} (\mu^2+|Du|^2)^{\frac{q}{2}}\, b\,\mathrm{d}x\right)^{\frac{1}{q}} \le c\left(\fint_{Q_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}}\, b\,\mathrm{d}x\right)^{\frac{1}{p}} + c\left(\fint_{Q_{2R}} (\mu^2+|G|^2)^{\frac{q}{2}} b\,\mathrm{d}x\right)^{\frac{1}{q}},$$

where c depends on p, $s-q$, k, λ_0, n, $\mathscr{D}_K$ and the BMO - norm of b and is independent of μ.

We point out that through all the paper we consider $p \ge 2$. As known, in the subquadratic case the assumptions and the results change, according to the properties of the p-Laplacian operator and the degeneracy of the problem. In the case of systems with right-hand side affected by weak integrability properties, the existence of solutions to boundary value problems obtained as the limit of smooth solutions to approximating problems is only known under the assumption that $p > 2-\frac{1}{n}$ [DHM97]. Theorem 5.1.1 extends for homogeneous regular systems [AF89; Bec07; CM19; DKM07]. For regular systems also CZ type estimates are available when $p > 2-\frac{1}{n}$ [Min17]. For nonhomogeneous p-Laplacian systems Theorem 5.1.1 holds when $p > \frac{3}{2}$ for a datum not in divergence form and lying in L^2 [CM19]. An improvement of the range of p is given in [Bal+22a]. Some techniques presented here are suitable to be extended, but since they are already delicate, at this stage we prefer to confine ourselves to the superquadratic case in order to highlight the main ideas and novelties.

Strategy of the proof and comments. In Theorem 5.1.1 we deal with very weak solutions and so, a priori, we cannot use in (5.1.7) test functions proportional to a solution u. Then, in Section 5.2, we first achieve a higher integrability result. Following [IS92] and [IS94], via a weighted version of Hodge decomposition [CMP02] and connectedness arguments, we construct suitable test functions and in Lemma 5.2.1 we prove a reverse Hölder inequality for Du. The statement of this lemma does not require assumptions (5.1.4) and (5.1.6) and extends a result proved in [GLS96]. For another approach to treat very weak solutions see [Lew93]. Once acquired the higher integrability of Du, in Section 5.3 we prove an a priori estimate by using the classic difference quotient method (for details see for example [AF89], [GM86] and [Giu03]). Finally, in Section 5.4, the proof of Theorem 5.1.1 follows by constructing appropriate approximating boundary value problems, whose solvability is known and for which the a priori estimate applies.
In order to prove Theorem 5.1.2 in Section 5.5, the main difficulty is the interplay of the nonlinearity and the presence of a weight which does not allow us to follow the scheme of classical papers [Iwa83; KM10; KM06; Min03], based on comparing a solution w to the initial problem with the solution to homogeneous systems with frozen coefficients, i.e. $\operatorname{div} A(x_0, Dw) = 0$. In order to deal with such a peculiarity, we first compare a local solution to (5.1.8) with the solution to a related homogeneous Dirichlet problem for which higher integrability follows from Theorem 5.1.1. So, as in [KM06], we shall rely on a technique introduced by Caffarelli and Peral [Caf89], [CP98], and based on Calderón and Zygmund type covering arguments and iteration of level sets, combined with a clever use of Harmonic Analysis tools such as weighted versions of Maximal function inequalities. Finally, in Section 5.6 we present global versions of Theorem 5.1.1 and Theorem 5.1.2. We study the Dirichlet problem with zero boundary condition on a regular C^2 domain. Since mollifiers and quasiconformal homeomorphisms preserve the BMO norm [Ast83] and the distance $\mathscr{D}_K$ [BBC75] respectively, the proof of these results follows in a standard way (see Theorem 5.6.2 and Theorem 5.6.4). When Ω is not regular the problem is more delicate [CM19; DKM07; KM10].

5.2 Higher integrability

In this section we prove a higher integrability result useful to our aims. We consider the nonlinear elliptic system

$$\operatorname{div} A(x, Du(x)) = \operatorname{div} |G|^{p-2}G \quad \text{in} \quad \Omega. \tag{5.2.1}$$

We begin with a reverse Hölder inequality.

Lemma 5.2.1. *Let Ω be a regular domain, $A(x,\xi)$ a mapping verifying assumptions (5.1.2), (5.1.3) and (5.1.5), $\mu \in [0,1]$, and let $G \in L^{p+\delta}_{loc}(b,\Omega;\mathbb{R}^{N\times n})$, for $\delta \geq 0$. Then there exist positive constants ε_1 and c, depending on n, λ_0, $\|b\|_*$, k, p, with $0 < \varepsilon_1 < \frac{1}{2}$, such that if $u \in W^{1,p-\varepsilon}_{loc}$ verifies (5.2.1) and $-\min\{\varepsilon_1, \delta\} < \varepsilon \leq \varepsilon_1$, then*

$$\begin{aligned} \frac{1}{b(B_R)} \int_{B_R} \left(\mu^2 + |Du|^2 + |u|^2\right)^{\frac{p-\varepsilon}{2}} b \,\mathrm{d}x \leq & c \left(\frac{1}{b(B_{2R})} \int_{B_{2R}} (\mu^2 + |Du|^2 + |u|^2)^{\frac{\sigma}{2}} b \,\mathrm{d}x \right)^{\frac{p-\varepsilon}{\sigma}} \\ & + \frac{c}{b(B_{2R})} \int_{B_{2R}} \left(\mu^2 + |G|^2\right)^{\frac{p-\varepsilon}{2}} b \,\mathrm{d}x \end{aligned} \tag{5.2.2}$$

for every σ with $\max\left\{1, \frac{(n-1)(p-\varepsilon)}{n}\right\} \leq \sigma < p - \varepsilon$ and for every pair of concentric balls $B_R \subset B_{2R} \subset\subset \Omega$ with $R < 1$.

We recall the following

Lemma 5.2.2 ([Gia83])**.** *For $R_0 < R_1$, consider a bounded function $f : [R_0, R_1] \to [0, \infty)$ with*

$$f(s) \leq \vartheta f(t) + \frac{A}{(s-t)^\delta} + B \quad \textit{for all } R_0 < s < t < R_1,$$

where A, B and δ denote non - negative constants and $\vartheta \in (0,1)$. Then we have

$$f(R_0) \leq c(\delta, \vartheta) \left(\frac{A}{(R_1 - R_0)^\delta} + B \right),$$

where $c(\delta, \vartheta)$ is increasing with respect to δ.

By means of an analogous proof of [GM79, Proposition 5.1] we get

Lemma 5.2.3. *Let $f \in L^r(w,\Omega)$ and $g \in L^s(w,\Omega)$ be non-negative functions, where $1 < r < s$, Ω is an open set, w is a weight. If the following*

$$\frac{\int_{B_R} f^r w\, \mathrm{d}x}{w(B_R)} \leq c\left(\left(\frac{\int_{B_{2R}} f w\, \mathrm{d}x}{w(B_{2R})}\right)^r + \frac{\int_{B_{2R}} g^r w\, \mathrm{d}x}{w(B_{2R})}\right), \qquad c > 1,$$

holds for every pair of concentric balls $B_R \subset B_{2R} \subset\subset \Omega$, then there exists $\varepsilon > 0$ such that $f \in L^{r+\varepsilon}_{loc}(w,\Omega)$.

Now we are ready to prove Lemma 5.2.1.

Proof of Lemma 5.2.1. Fix a ball $B_R(x_0)$ with $R < 1$ such that $B_{2R}(x_0) \subset\subset \Omega$. For $R \leq s < t \leq 2R$, we consider the balls centered at x_0 with radii $R, s, t, 2R$. Let $\xi : \mathbb{R}^n \to \mathbb{R}$ be the usual cut-off function, that is $\xi \in C_0^\infty(B_t)$ with $0 \leq \xi \leq 1$, $\xi = 1$ on B_s and $|\nabla \xi| \leq \frac{1}{t-s}$. Let us assume that $u \in W^{1,p-\varepsilon}_{loc}(b,\Omega;\mathbb{R}^N)$ is a local solution of (5.2.1), with $-1 < 2\varepsilon < p-1$. By Lemma 2.2.14 applied to $\xi(u-\lambda)$, $\lambda \in \mathbb{R}^N$, there exist $\varphi \in W_0^{1,\frac{p-\varepsilon}{1-\varepsilon}}(b,\Omega;\mathbb{R}^N)$ and $H \in L^{\frac{p-\varepsilon}{1-\varepsilon}}(b,\Omega;\mathbb{R}^{N\times n})$ such that

$$\begin{aligned}
&|D[\xi(u-\lambda)]|^{-\varepsilon} D[\xi(u-\lambda)] = D\varphi + H\\
&\|D\varphi\|_{L_b^{\frac{p-\varepsilon}{1-\varepsilon}}(\Omega)} \leq c(n,p)(1+\|b\|_*)^\gamma \|D[\xi(u-\lambda)]\|^{1-\varepsilon}_{L_b^{p-\varepsilon}(\Omega)}\\
&\|H\|_{L_b^{\frac{p-\varepsilon}{1-\varepsilon}}(\Omega)} \leq c(n,p)(1+\|b\|_*)^\gamma |\varepsilon|\, \|D[\xi(u-\lambda)]\|^{1-\varepsilon}_{L_b^{p-\varepsilon}(\Omega)},
\end{aligned}$$

where γ is an exponent depending only on p. We use φ as a test function in (5.1.7); this yields

$$\begin{aligned}
&\int_{B_t} \left\langle |G|^{p-2}G, D\varphi\right\rangle \mathrm{d}x = \int_{B_t} \langle A(x,Du), D\varphi\rangle\, \mathrm{d}x =\\
&= \int_{B_t} \langle A(x, D[\xi(u-\lambda)]), D\varphi\rangle\, \mathrm{d}x + \int_{B_t} \langle [A(x,Du) - A(x, D[\xi(u-\lambda)])], D\varphi\rangle\, \mathrm{d}x =\\
&= \int_{B_t} \langle A(x, D[\xi(u-\lambda)]), |D[\xi(u-\lambda)]|^{-\varepsilon} D[\xi(u-\lambda)]\rangle\, \mathrm{d}x +\\
&\quad - \int_{B_t} \langle A(x, D[\xi(u-\lambda)]), H\rangle\, \mathrm{d}x + \int_{B_t} \langle [A(x,Du) - A(x, D[\xi(u-\lambda)])], D\varphi\rangle\, \mathrm{d}x.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{B_t} \langle A(x, D[\xi(u-\lambda)]), |D[\xi(u-\lambda)]|^{-\varepsilon} D[\xi(u-\lambda)]\rangle = \int_{B_t} \langle A(x, D[\xi(u-\lambda)]), H\rangle\\
&- \int_{B_t} \langle [A(x,Du) - A(x, D[\xi(u-\lambda)])], D\varphi\rangle + \int_{B_t} \left\langle |G|^{p-2}G, D\varphi\right\rangle.
\end{aligned}$$

Now we use (5.1.3) and (5.1.5) to obtain

$$\frac{1}{k}\int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\, \mathrm{d}x \leq \int_{B_t} \langle A(x, D[\xi(u-\lambda)]), |D[\xi(u-\lambda)]|^{-\varepsilon} D[\xi(u-\lambda)]\rangle\, \mathrm{d}x. \tag{5.2.3}$$

We then apply (5.1.2), (5.1.5), together with Hölder's and Young's inequalities with exponents $\frac{p-\varepsilon}{p-1}$ and $\frac{p-\varepsilon}{1-\varepsilon}$, to get

$$\begin{aligned}
&\int_{B_t} |\langle A(x, D[\xi(u-\lambda)]), H\rangle|\, \mathrm{d}x \leq k\int_{B_t} (\mu^2 + |D[\xi(u-\lambda)]|^2)^{\frac{p-1}{2}} |H| b\, \mathrm{d}x \leq\\
&\leq c(k,p)\int_{B_t} (\mu^{p-1} + |D[\xi(u-\lambda)]|^{p-1})|H| b\, \mathrm{d}x \leq
\end{aligned} \tag{5.2.4}$$

$$\le c(n,\|b\|_*,k,p)\,|\varepsilon|\Bigg(\left(\left(\int_{B_t}\mu^{p-\varepsilon}b\,\mathrm{d}x\right)^{\frac{p-1}{p-\varepsilon}}\left(\int_{B_t}|D[\xi(u-\lambda)]|^{p-\varepsilon}b\,\mathrm{d}x\right)^{\frac{1-\varepsilon}{p-\varepsilon}}+\right.$$

$$\left.+\int_{B_t}|D[\xi(u-\lambda)]|^{p-\varepsilon}b\,\mathrm{d}x\right)\le$$

$$\le c(n,\|b\|_*,k,p)|\varepsilon|\left(\int_{B_t}\mu^{p-\varepsilon}b\,\mathrm{d}x+\int_{B_t}|D[\xi(u-\lambda)]|^{p-\varepsilon}b\,\mathrm{d}x\right).$$

On the other hand,

$$|Du|\le|Du-D[\xi(u-\lambda)]|+|D[\xi(u-\lambda)]|,$$

thus

$$(\mu^2+|Du|^2+|D[\xi(u-\lambda)]|^2)^{\frac{p-2}{2}}\le c(p)(\mu^{p-2}+|Du-D[\xi(u-\lambda)]|^{p-2}+|D[\xi(u-\lambda)]|^{p-2})$$

and, by (5.1.2), we have

$$\int_{B_t}|\langle A(x,Du)-A(x,D[\xi(u-\lambda)]),D\varphi\rangle|\,\mathrm{d}x\le$$

$$\le c\int_{B_t}|Du-D[\xi(u-\lambda)]|\,|D\varphi|(\mu^{p-2}+|Du-D[\xi(u-\lambda)]|^{p-2}+|D[\xi(u-\lambda)]|^{p-2})b\,\mathrm{d}x$$

$$=c\Bigg(\int_{B_t}\mu^{p-2}|Du-D[\xi(u-\lambda)]|\,|D\varphi|b\,\mathrm{d}x+\int_{B_t}|Du-D[\xi(u-\lambda)]|^{p-1}|D\varphi|b\,\mathrm{d}x+$$

$$+\int_{B_t}|Du-D[\xi(u-\lambda)]|\,|D[\xi(u-\lambda)]|^{p-2}\,|D\varphi|b\,\mathrm{d}x\Bigg),$$

with $c=c(k,p)$. Next, we apply the straightforward equality $Du-D[\xi(u-\lambda)]=(1-\xi)Du-(u-\lambda)\nabla\xi$:

$$\int_{B_t}|\langle A(x,Du)-A(x,D[\xi(u-\lambda)]),D\varphi\rangle|\,\mathrm{d}x\le c(k,p)\Bigg(\int_{B_t}\mu^{p-2}(1-\xi)\,|Du|\,|D\varphi|b\,\mathrm{d}x$$

$$+\int_{B_t}\mu^{p-2}|\nabla\xi|\,|u-\lambda||D\varphi|b\,\mathrm{d}x+\int_{B_t}|(1-\xi)Du|^{p-1}|D\varphi|b\,\mathrm{d}x+$$

$$+\int_{B_t}|(u-\lambda)\nabla\xi|^{p-1}|D\varphi|b\,\mathrm{d}x+\int_{B_t}(1-\xi)|Du|\,|D[\xi(u-\lambda)]|^{p-2}|D\varphi|b\,\mathrm{d}x+$$

$$+\int_{B_t}|\nabla\xi|\,|u-\lambda|\,|D[\xi(u-\lambda)[|^{p-2}|D\varphi|b\,\mathrm{d}x\Bigg)=:\ I+II+III+IV+V+VI.$$

In order to estimate I, if $p>2$, using Hölder's and Young's inequalities with exponents $p-\varepsilon$, $\frac{p-\varepsilon}{p-2}$ and $\frac{p-\varepsilon}{1-\varepsilon}$, and for $\dot{\varepsilon}>0$, we get:

$$\begin{aligned}I&\le\left(\int_{B_t}|(1-\xi)Du|^{p-\varepsilon}b\,\mathrm{d}x\right)^{\frac{1}{p-\varepsilon}}\left(\int_{B_t}\mu^{p-\varepsilon}b\,\mathrm{d}x\right)^{\frac{p-2}{p-\varepsilon}}\left(\int_{B_t}|D\varphi|^{\frac{p-\varepsilon}{1-\varepsilon}}b\,\mathrm{d}x\right)^{\frac{1-\varepsilon}{p-\varepsilon}}\le\\&\le c(n,\|b\|_*,p)\left(\int_{B_t}|(1-\xi)Du|^{p-\varepsilon}b\,\mathrm{d}x\right)^{\frac{1}{p-\varepsilon}}\left(\int_{B_t}\mu^{p-\varepsilon}b\,\mathrm{d}x\right)^{\frac{p-2}{p-\varepsilon}}\cdot\\&\quad\cdot\left(\int_{B_t}|D[\xi(u-\lambda)]|^{p-\varepsilon}b\,\mathrm{d}x\right)^{\frac{1-\varepsilon}{p-\varepsilon}}\le\\&\le c(n,\|b\|_*,p)^{p-\varepsilon}\left(\frac{1}{\dot{\varepsilon}}\right)^{p-\varepsilon}\int_{B_t}|(1-\xi)Du|^{p-\varepsilon}b\,\mathrm{d}x+\int_{B_t}\mu^{p-\varepsilon}b\,\mathrm{d}x+\\&\quad+\dot{\varepsilon}^{\frac{p-\varepsilon}{1-\varepsilon}}\int_{B_t}|D[\xi(u-\lambda)]|^{p-\varepsilon}b\,\mathrm{d}x.\end{aligned}\tag{5.2.5}$$

Now, since c may be assumed greater than 1, $c^{p-\varepsilon}$ is less than c^p. Therefore, we can assume that the constant c is independent of ε. If $p = 2$, since $\mu \leq 1$, we argue as before by applying Hölder's and Young's inequalities with exponents $p-\varepsilon$ and $\frac{p-\varepsilon}{p-\varepsilon-1}$:

$$\begin{aligned} I &\leq \left(\int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\,\mathrm{d}x\right)^{\frac{1}{p-\varepsilon}} \left(\int_{B_t} |D\varphi|^{\frac{p-\varepsilon}{p-\varepsilon-1}} b\,\mathrm{d}x\right)^{\frac{p-\varepsilon-1}{p-\varepsilon}} \leq \\ &\leq c(n,\|b\|_*,p)\left(\int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\,\mathrm{d}x\right)^{\frac{1}{p-\varepsilon}} \left(\int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,\mathrm{d}x\right)^{\frac{1-\varepsilon}{p-\varepsilon}} \leq \\ &\leq c(n,\|b\|_*,p)\left(\frac{1}{\dot{\varepsilon}}\right)^{p-\varepsilon} \int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\,\mathrm{d}x + \dot{\varepsilon}^{\frac{p-\varepsilon}{1-\varepsilon}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,\mathrm{d}x. \end{aligned} \tag{5.2.6}$$

Replacing $|(1-\xi)Du|$ by $|(u-\lambda)\nabla\xi|$, we get the desired estimate for II, if $p > 2$:

$$II \leq c\left(\frac{1}{\dot{\varepsilon}}\right)^{p-\varepsilon} \int_{B_t} |(u-\lambda)\nabla\xi|^{p-\varepsilon} b\,\mathrm{d}x + \int_{B_t} \mu^{p-\varepsilon} b\,\mathrm{d}x + \dot{\varepsilon}^{\frac{p-\varepsilon}{1-\varepsilon}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,\mathrm{d}x \tag{5.2.7}$$

and $p = 2$:

$$II \leq c\left(\frac{1}{\dot{\varepsilon}}\right)^{p-\varepsilon} \int_{B_t} |(u-\lambda)\nabla\xi|^{p-\varepsilon} b\,\mathrm{d}x + \dot{\varepsilon}^{\frac{p-\varepsilon}{1-\varepsilon}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,\mathrm{d}x, \tag{5.2.8}$$

with $c = c(n,\|b\|_*,p)$. Using Hölder's and Young's inequalities with exponents $\frac{p-\varepsilon}{p-1}$ and $\frac{p-\varepsilon}{1-\varepsilon}$, and if $\ddot{\varepsilon} > 0$, we obtain:

$$\begin{aligned} III &\leq c(n,\|b\|_*,p)\left(\int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\,\mathrm{d}x\right)^{\frac{p-1}{p-\varepsilon}} \left(\int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,\mathrm{d}x\right)^{\frac{1-\varepsilon}{p-\varepsilon}} \leq \\ &\leq c(n,\|b\|_*,p)\left(\frac{1}{\ddot{\varepsilon}}\right)^{\frac{p-\varepsilon}{p-1}} \int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\,\mathrm{d}x + \ddot{\varepsilon}^{\frac{p-\varepsilon}{1-\varepsilon}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,\mathrm{d}x. \end{aligned} \tag{5.2.9}$$

In the same way, we estimate IV, namely

$$\begin{aligned} IV &= \int_{B_t} |(u-\lambda)\nabla\xi|^{p-1} |D\varphi| b\,\mathrm{d}x \leq \\ &\leq c(n,\|b\|_*,p)\left(\frac{1}{\ddot{\varepsilon}}\right)^{\frac{p-\varepsilon}{p-1}} \int_{B_t} |(u-\lambda)\nabla\xi|^{p-\varepsilon} b\,\mathrm{d}x + \ddot{\varepsilon}^{\frac{p-\varepsilon}{1-\varepsilon}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,\mathrm{d}x. \end{aligned} \tag{5.2.10}$$

Arguing as for I and II, with $\dddot{\varepsilon} > 0$, we estimate V

$$\begin{aligned} V &= \int_{B_t} |(1-\xi)Du|\, |D[\xi(u-\lambda)]|^{p-2} |D\varphi| b\,\mathrm{d}x \leq \\ &\leq c(n,\|b\|_*,p)\left(\frac{1}{\dddot{\varepsilon}}\right)^{p-\varepsilon} \int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\,\mathrm{d}x + \dddot{\varepsilon}^{\frac{p-\varepsilon}{p-\varepsilon-1}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,\mathrm{d}x \end{aligned} \tag{5.2.11}$$

and VI

$$\begin{aligned} VI &= \int_{B_t} |(u-\lambda)\nabla\xi|\, |D[\xi(u-\lambda)]|^{p-2} |D\varphi| b\,\mathrm{d}x \leq \\ &\leq c(n,\|b\|_*,p)\left(\frac{1}{\dddot{\varepsilon}}\right)^{p-\varepsilon} \int_{B_t} |(u-\lambda)\nabla\xi|^{p-\varepsilon} b\,\mathrm{d}x + \dddot{\varepsilon}^{\frac{p-\varepsilon}{p-\varepsilon-1}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,\mathrm{d}x. \end{aligned} \tag{5.2.12}$$

Finally, by Hölder's and Young's inequalities with exponents $\frac{p-\varepsilon}{p-1}$ and $\frac{p-\varepsilon}{1-\varepsilon}$, supposing $|\varepsilon| \leq \delta$, and if $\ddddot{\varepsilon} > 0$, we have

$$\begin{aligned} &\int_{B_t} \left|\left\langle |G|^{p-2}G, D\varphi\right\rangle\right| \mathrm{d}x \leq \frac{1}{\lambda_0}\int_{B_t} |G|^{p-1}|D\varphi| b\,\mathrm{d}x \leq \frac{1}{\lambda_0}\|G\|^{p-1}_{L_b^{p-\varepsilon}(B_t)} \|D\varphi\|_{L_b^{\frac{p-\varepsilon}{1-\varepsilon}}(B_t)} \\ &\leq c(n,\|b\|_*,p,\lambda_0)\left(\frac{1}{\ddddot{\varepsilon}}\right)^{\frac{p-\varepsilon}{p-1}} \int_{B_t} |G|^{p-\varepsilon} b\,\mathrm{d}x + \ddddot{\varepsilon}^{\frac{p-\varepsilon}{1-\varepsilon}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,\mathrm{d}x. \end{aligned} \tag{5.2.13}$$

Combining estimates (5.2.3) - (5.2.13) yields

$$\begin{aligned}
&\frac{1}{k}\int_{B_t}|D[\xi(u-\lambda)]|^{p-\varepsilon}b\,\mathrm{d}x \le\\
&\le c(n,\|b\|_*,k,p,\lambda_0)\Bigg(\left(|\varepsilon|+(\dot\varepsilon+\ddot\varepsilon+\dddot\varepsilon)^{\frac{p-\varepsilon}{1-\varepsilon}}+\ddddot\varepsilon^{\frac{p-\varepsilon}{p-\varepsilon-1}}\right)\int_{B_t}|D[\xi(u-\lambda)]|^{p-\varepsilon}b\,\mathrm{d}x+\\
&+\left(\left(\frac{1}{\dot\varepsilon}+\frac{1}{\ddot\varepsilon}\right)^{p-\varepsilon}+\left(\frac{1}{\ddot\varepsilon}\right)^{\frac{p-\varepsilon}{p-1}}\right)\left(\int_{B_t}|(1-\xi)Du|^{p-\varepsilon}b\,\mathrm{d}x+\int_{B_t}|(u-\lambda)\nabla\xi|^{p-\varepsilon}b\,\mathrm{d}x\right)+\\
&+(|\varepsilon|+1)\int_{B_t}\mu^{p-\varepsilon}b\,\mathrm{d}x+\left(\frac{1}{\dddot\varepsilon}\right)^{\frac{p-\varepsilon}{1-\varepsilon}}\int_{B_t}|G|^{p-\varepsilon}b\,\mathrm{d}x\Bigg)
\end{aligned}$$

for arbitrary positive numbers $\dot\varepsilon$, $\ddot\varepsilon$, $\dddot\varepsilon$, $\ddddot\varepsilon$. We now choose ε, $\dot\varepsilon$, $\ddot\varepsilon$, $\dddot\varepsilon$ and $\ddddot\varepsilon$ to be such that $c(n,\|b\|_*,k,p,\lambda_0)\cdot\left(|\varepsilon|+(\dot\varepsilon+\ddot\varepsilon+\dddot\varepsilon)^{\frac{p-\varepsilon}{1-\varepsilon}}+\ddddot\varepsilon^{\frac{p-\varepsilon}{p-\varepsilon-1}}\right)<\frac{1}{2k}$. To this effect, we fix $\varepsilon_1>0$ sufficiently small. Accordingly,

$$\begin{aligned}
\int_{B_t}|D[\xi(u-\lambda)]|^{p-\varepsilon}b\,\mathrm{d}x &\le c(n,\|b\|_*,k,p,\lambda_0)\left(\int_{B_t}|(1-\xi)Du|^{p-\varepsilon}b\,\mathrm{d}x+\right.\\
&\left.+\int_{B_t}|(u-\lambda)\nabla\xi|^{p-\varepsilon}b\,\mathrm{d}x+\int_{B_t}\mu^{p-\varepsilon}b\,\mathrm{d}x+\int_{B_t}|G|^{p-\varepsilon}b\,\mathrm{d}x\right),
\end{aligned}$$

for $-\min\{\varepsilon_1,\delta\}<\varepsilon\le\varepsilon_1$ and c independent of ε. The properties of the cut-off function ξ and the previous inequality yield

$$\begin{aligned}
\int_{B_s}|Du|^{p-\varepsilon}b\,\mathrm{d}x\le c\Big(&\int_{B_t\setminus B_s}|Du|^{p-\varepsilon}b\,\mathrm{d}x+\frac{1}{(t-s)^{p-\varepsilon}}\int_{B_t}|u-\lambda|^{p-\varepsilon}b\,\mathrm{d}x+\\
&+\int_{B_t}(\mu^{p-\varepsilon}+|G|^{p-\varepsilon})b\,\mathrm{d}x\Big).
\end{aligned}$$

Adding $c\int_{B_s}|Du|^{p-\varepsilon}b\,\mathrm{d}x$ to both sides yields

$$\begin{aligned}
\int_{B_s}|Du|^{p-\varepsilon}b\,\mathrm{d}x\le\frac{c}{c+1}\int_{B_t}|Du|^{p-\varepsilon}b\,\mathrm{d}x&+\frac{c}{(c+1)(t-s)^{p-\varepsilon}}\int_{B_{2R}}|u-\lambda|^{p-\varepsilon}b\,\mathrm{d}x+\\
&+\frac{c}{c+1}\int_{B_{2R}}(\mu^{p-\varepsilon}+|G|^{p-\varepsilon})b\,\mathrm{d}x.
\end{aligned}$$

With the notation (2.2.1), we set

$$\lambda=u_R:=\frac{\int_{B_{2R}}u(x)b\,\mathrm{d}x}{b(B_{2R})}$$

and we apply Lemma 5.2.2 to get

$$\int_{B_R}|Du|^{p-\varepsilon}b\,\mathrm{d}x\le c\left(R^{-(p-\varepsilon)}\int_{B_{2R}}|u-u_R|^{p-\varepsilon}b\,\mathrm{d}x+\int_{B_{2R}}(\mu^{p-\varepsilon}+|G|^{p-\varepsilon})b\,\mathrm{d}x\right). \tag{5.2.14}$$

We add $\int_{B_R}(|u|^{p-\varepsilon}+\mu^{p-\varepsilon})\,b\,\mathrm{d}x$ to both sides. Since $|u|^{p-\varepsilon}\le c(|u-u_R|^{p-\varepsilon}+|u_R|^{p-\varepsilon})$ and $R<1$, we obtain

$$\begin{aligned}
\int_{B_R}(\mu^{p-\varepsilon}+|Du|^{p-\varepsilon}+|u|^{p-\varepsilon})b\,\mathrm{d}x\le c\Bigg(&R^{-(p-\varepsilon)}\int_{B_{2R}}|u-u_R|^{p-\varepsilon}b\,\mathrm{d}x+\\
&+\int_{B_{2R}}|u_R|^{p-\varepsilon}b\,\mathrm{d}x+\int_{B_{2R}}\left(\mu^2+|G|^2\right)^{\frac{p-\varepsilon}{2}}b\,\mathrm{d}x\Bigg),
\end{aligned} \tag{5.2.15}$$

with $c=c(n,\|b\|_*,k,p,\lambda_0)$. Note that by Jensen's inequality 2.1.7

$$\begin{aligned}
\int_{B_{2R}}|u_R|^{p-\varepsilon}b\,\mathrm{d}x&=b(B_{2R})\left(\frac{\int_{B_{2R}}|u(x)|\,b\,\mathrm{d}x}{b(B_{2R})}\right)^{p-\varepsilon}\le\\
&\le\int_{B_{2R}}|u(x)|^{p-\varepsilon}b\,\mathrm{d}x\le R^{-(p-\varepsilon)}\int_{B_{2R}}|u(x)|^{p-\varepsilon}b\,\mathrm{d}x.
\end{aligned} \tag{5.2.16}$$

Theorem 2.2.11 and Lemma 2.2.10 yield

$$\frac{R^{-(p-\varepsilon)}}{b(B_{2R})}\int_{B_{2R}}|u|^{p-\varepsilon}b\,dx\le c\left(\frac{1}{b(B_{2R})}\int_{B_{2R}}|\nabla u|^{\sigma}b\,dx\right)^{\frac{p-\varepsilon}{\sigma}} \tag{5.2.17}$$

for all σ such that $\max\left\{1,\frac{(n-1)p-\varepsilon}{n}\right\}\le\sigma<p-\varepsilon$ and with $c=c(n,\|b\|_*,k,p,\lambda_0)$. Theorem 2.2.12 and Lemma 2.2.10 yield

$$\frac{R^{-(p-\varepsilon)}}{b(B_{2R})}\int_{B_{2R}}|u-u_R|^{p-\varepsilon}b\,dx\le c\left(\frac{1}{b(B_{2R})}\int_{B_{2R}}|\nabla u|^{\sigma}b\,dx\right)^{\frac{p-\varepsilon}{\sigma}} \tag{5.2.18}$$

for all σ such that $\max\left\{1,\frac{(n-1)p-\varepsilon}{n}\right\}\le\sigma<p-\varepsilon$ and with $c=c(n,\|b\|_*,k,p,\lambda_0)$.
Putting together (5.2.15)- (5.2.18), since the measure $b\,dx$ is doubling and by means of Lemma 2.2.10, we have

$$\begin{aligned}
&\frac{c(n,p,\|b\|_*)}{b(B_R)}\int_{B_R}(\mu^2+|Du|^2+|u|^2)^{\frac{p-\varepsilon}{2}}b\,dx\le\\
&\le\frac{c(n,p,\|b\|_*)}{b(B_R)}\int_{B_R}(\mu^{p-\varepsilon}+|Du|^{p-\varepsilon}+|u|^{p-\varepsilon})b\,dx\le\\
&\le\frac{1}{b(B_{2R})}\int_{B_R}(\mu^{p-\varepsilon}+|Du|^{p-\varepsilon}+|u|^{p-\varepsilon})b\,dx\le\\
&\le\frac{c(n,\|b\|_*,k,p,\lambda_0)}{b(B_{2R})}\Bigg(R^{-(p-\varepsilon)}\int_{B_{2R}}|u-u_R|^{p-\varepsilon}b\,dx+\\
&\qquad+R^{-(p-\varepsilon)}\int_{B_{2R}}|u(x)|^{p-\varepsilon}b\,dx+\int_{B_{2R}}\left(\mu^2+|G|^2\right)^{\frac{p-\varepsilon}{2}}b\,dx\Bigg)\le\\
&\le c(n,\|b\|_*,k,p,\lambda_0)\Bigg[\left(\frac{1}{b(B_{2R})}\int_{B_{2R}}(\mu^{\sigma}+|Du|^{\sigma}+|u|^{\sigma})b\,dx\right)^{\frac{p-\varepsilon}{\sigma}}+\\
&\qquad+\frac{1}{b(B_{2R})}\int_{B_{2R}}\left(\mu^2+|G|^2\right)^{\frac{p-\varepsilon}{2}}b\,dx\Bigg],
\end{aligned} \tag{5.2.19}$$

for all σ such that $\max\left\{1,\frac{(n-1)p-\varepsilon}{n}\right\}\le\sigma<p-\varepsilon$ and we can conclude the proof of the reverse Hölder's inequality (5.2.2). □

Now we are ready to prove the main result of this section.

Theorem 5.2.4. *Let Ω be a regular domain, $A(x,\xi)$ a mapping verifying assumptions* (5.1.2), (5.1.3) *and* (5.1.5), *$\mu\in[0,1]$, and let $G\in L^{p+\delta}_{loc}(b,\Omega;\mathbb{R}^{N\times n})$, for $\delta\ge0$. Then there exists $0<\varepsilon_1<\frac{1}{2}$, depending on k, n, λ_0, p and the BMO - norm of $b(x)$, such that, if $u\in W^{1,p-\varepsilon}_{loc}(b,\Omega;\mathbb{R}^N)$, with $0\le\varepsilon<\varepsilon_1$, is a local solution of* (5.2.1), *then $u\in W^{1,p-\tilde\varepsilon}_{loc}(b,\Omega,\mathbb{R}^N)$, for any $0\le|\tilde\varepsilon|\le\min\{\delta,\varepsilon_1\}$.*

Proof. Let $u\in W^{1,p-\bar\varepsilon}_{loc}(b,\Omega;\mathbb{R}^N)$ verify the equation (5.1.7), with $0\le\bar\varepsilon\le\varepsilon_1$. For $\Omega'\subset\subset\Omega$, set $A=\{\varepsilon\in[-\min\{\delta,\varepsilon_1\},\varepsilon_1]\,:\,u\in W^{1,p-\varepsilon}_{loc}(b,\Omega';\mathbb{R}^N)\}$. We claim that $A=[-\min\{\delta,\varepsilon_1\},\varepsilon_1]$. In order to see this, we first note that A is not empty, since $\varepsilon_1\in A$. Our goal is to show that A is open and closed in $[-\min\{\delta,\varepsilon_1\},\varepsilon_1]$. First we prove that A is open. Indeed, if $\varepsilon_2\in A$, by the reverse Hölder inequality in Lemma 5.2.1 and by the higher integrability result stated in Lemma 5.2.3, there exists $\varepsilon>0$ such that $\max\{\varepsilon_2-\varepsilon,-\min\{\delta,\varepsilon_1\}\}\in A$. Therefore A is open. Now we prove that A is closed too. Let $\{\varrho_k\}\subset A$ be such that $\varrho_k\to\varrho$. We want to prove that $\varrho\in A$. Obviously, $u\in W^{1,p-\varrho_k}_{loc}(b,\Omega';\mathbb{R}^N)$ and

$$\begin{aligned}
&\frac{1}{b(B_R)}\int_{B_R}\left(\mu^2+|Du|^2+|u|^2\right)^{\frac{p-\varrho_k}{2}}b\,dx\le\\
&\le c\left(\frac{1}{b(B_{2R})}\int_{B_{2R}}(\mu^2+|Du|^2+|u|^2)^{\frac{\sigma_k}{2}}b\,dx\right)^{\frac{p-\varrho_k}{\sigma_k}}+\frac{c}{b(B_{2R})}\int_{B_{2R}}\left(\left(\mu^2+|G|^2\right)^{\frac{p-\varrho_k}{2}}b\,dx\right)
\end{aligned} \tag{5.2.20}$$

with $\sigma_k = \max\{1, (n-1)(p-\varrho_k)/n\}$. Let us observe that $(\mu^2+|Du|^2+|u|^2)^{\frac{p-\varrho_k}{2}} \to (\mu^2+|Du|^2+|u|^2)^{\frac{p-\varrho}{2}}$, almost everywhere in B_{2R}. Therefore, we can apply Fatou's Lemma in order to estimate the left-hand side of (5.2.20). We obtain

$$\frac{1}{b(B_R)}\int_{B_R}(\mu^2+|Du|^2+|u|^2)^{\frac{p-\varrho}{2}}\,b\,\mathrm{d}x \le \liminf_k \frac{1}{b(B_R)}\int_{B_R}(\mu^2+|Du|^2+|u|^2)^{\frac{p-\varrho_k}{2}}\,b\,\mathrm{d}x. \tag{5.2.21}$$

In order to pass to the limit on the right-hand side of (5.2.20), we assume that $\varrho_k > \varrho$ for every k, since otherwise $u \in W^{1,p-\varrho}_{loc}(b,\Omega';\mathbb{R}^N)$ and the conclusion is obvious. With this assumption we have

$$\sigma_k \le \max\{1,(n-1)(p-\varrho)/n\} < p-\varrho,$$

since $1 < 2-\varepsilon_1 \le p-\varrho_k < p-\varrho$. Set $\sigma = \max\{1,(n-1)(p-\varrho)/n\}$; the following inequality holds:

$$(\mu^2+|Du|^2+|u|^2)^{\frac{\sigma_k}{2}} \le 1 + (\mu^2+|Du|^2+|u|^2)^{\frac{\sigma}{2}}. \tag{5.2.22}$$

For k large enough, we have $\sigma < p-\varrho_k < p-\varrho$ and, since $u \in W^{1,\sigma_k}_{loc}(b,\Omega';\mathbb{R}^N)$, the right-hand side of (5.2.22) is in L^1. Therefore, we can apply Lebesgue's Convergence Theorem 2.1.11 in order to pass to the limit on the right-hand side of (5.2.20). Taking also into account that we may assume $|\varrho_k| \le \delta$ for every k and $G \in L^{p+\delta}$, we get, recalling (5.2.21),

$$\frac{1}{b(B_R)}\int_{B_R}\left(\mu^2+|Du|^2+|u|^2\right)^{\frac{p-\varrho}{2}} b\,\mathrm{d}x \le$$
$$\le c\left(\frac{1}{b(B_{2R})}\int_{B_{2R}}(\mu^2+|Du|^2+|u|^2)^{\frac{\sigma}{2}}b\,\mathrm{d}x\right)^{\frac{p-\varrho}{\sigma}} + \frac{c}{b(B_{2R})}\int_{B_{2R}}\left(\left(\mu^2+|G|^2\right)^{\frac{p-\varrho}{2}} b\,\mathrm{d}x\right)$$

Hence $\varrho \in A$, therefore A is closed. □

Remark 5.2.5. By virtue of Theorem 5.2.4, in Section 5.3 and Section 5.4 we can assume $u \in W^{1,p+\varepsilon_1}_{loc}(b,\Omega;\mathbb{R}^N)$, for $\varepsilon_1 > 0$ sufficiently small. In fact, in Theorem 5.1.1 and in a priori estimate 5.3.1 we assume $F \in W^{1,2}_{loc}(b,\Omega;\mathbb{R}^{N\times n})$. Therefore, with the notation $F = |G|^{p-2}G$, we have $|G|^{p-1} \in L^{2^*}_b$, with $2^* := \frac{2n}{n-2}$, i.e. equivalently $|G| \in L_b^{\frac{2np-2n}{n-2}}$. Finally, we note that $\frac{2np-2n}{n-2} > p$ for $p > 2_* := \frac{2n}{n+2}$.

5.3 A priori estimate

In this section we assume the weak differentiability of $V_\mu(Du)$ in order prove an a priori estimate.

Theorem 5.3.1. *Let Ω be a regular domain and $A(x,\xi)$ a mapping verifying assumptions (5.1.2), (5.1.3), (5.1.4) and (5.1.5). If $D(V_\mu(Du)) \in L^2_{loc}(b,\Omega)$, $\mu \in (0,1]$ and $F \in W^{1,2}_{loc}(b,\Omega;\mathbb{R}^{N\times n})$, there exists $\alpha > 0$, depending on p, n, λ_0, μ and k, such that, if*

$$\mathscr{D}_K := \mathrm{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha, \tag{5.3.1}$$

the following estimate holds:

$$\int_{B_R}|D(V_\mu(Du))|^2 b\,\mathrm{d}x \le c\int_{B_{2R}}\left(\left(1+\frac{1}{R^2}\right)(\mu^2+|Du|^2)^{\frac{p}{2}} + (\mu^2+|DF|^2)\right) b\,\mathrm{d}x \tag{5.3.2}$$

for every ball $B_{2R} \subset\subset \Omega$ and for a constant c depending on p, k, λ_0, n, μ and $\mathscr{D}_K$.

We recall the following

Lemma 5.3.2 ([GM86]). *For any $p \ge 2$, we have*

$$c^{-1}(\mu^2+|\eta|^2+|\xi|^2)^{\frac{p-2}{2}}|\eta-\xi|^2 \le |V_\mu(\eta)-V_\mu(\xi)|^2 \le c(\mu^2+|\eta|^2+|\xi|^2)^{\frac{p-2}{2}}|\eta-\xi|^2$$

for any η, $\xi \in \mathbb{R}^k$, $\mu \in [0,1]$ and a constant $c = c(p) > 0$.

Now we are ready to prove Theorem 5.3.1.

Proof of Theorem 5.3.1. For a fixed ball $B_{2R} \subset\subset \Omega$ and radii $R < s < t < 2R$ with R small enough, consider a function $\xi \in C_0^\infty(B_t)$, $0 \le \xi \le 1$, $\xi = 1$ on B_s, $|\nabla \xi| \le \frac{1}{t-s}$ and set $\psi = \xi^2 \tau_h u$ for sufficiently small $h > 0$. Since u is a local solution of (5.1.1), we can choose $\varphi = \tau_{-h}\psi$ as a test function. By virtue of the properties of the difference quotients, we have

$$\int_{B_t} \langle \tau_h A(x, Du), D\psi \rangle \, dx = \int_{B_t} \langle \tau_h F(x), D\psi \rangle \, dx$$

that is

$$\int_{B_t} \left\langle \tau_h A(x, Du), D(\xi^2 \tau_h u) \right\rangle \, dx = \int_{B_t} \left\langle \tau_h F(x), D(\xi^2 \tau_h u) \right\rangle \, dx.$$

It follows that

$$\begin{aligned} \int_{B_t} \xi^2 \langle \tau_h A(x, Du), \tau_h Du \rangle \, dx + 2 \int_{B_t} \xi \langle \tau_h A(x, Du), \nabla \xi \otimes \tau_h u \rangle \, dx = \\ = \int_{B_t} \left\langle \tau_h F(x), D(\xi^2 \tau_h u) \right\rangle \, dx, \end{aligned} \tag{5.3.3}$$

and observing that

$$\begin{aligned} \tau_h A(x, Du) =& [A(x + he_i, Du(x + he_i)) - A(x + he_i, Du(x))] + \\ &+ [A(x + he_i, Du(x)) - A(x, Du(x))] =: \mathcal{A}_h + \mathcal{A}'_h \end{aligned}$$

the equality (5.3.3) can be rewritten as

$$\begin{aligned} &\int_{B_t} \xi^2 \langle \mathcal{A}_h, \tau_h Du \rangle \, dx = - \int_{B_t} \xi^2 \left\langle \mathcal{A}'_h, \tau_h Du \right\rangle \, dx - 2 \int_{B_t} \xi \langle \mathcal{A}_h, \nabla \xi \otimes \tau_h u \rangle \, dx + \\ &- 2 \int_{B_t} \xi \left\langle \mathcal{A}'_h, \nabla \xi \otimes \tau_h u \right\rangle \, dx + \int_{B_t} \left\langle \tau_h F, D(\xi^2 \tau_h u) \right\rangle \, dx =: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

By assumption (5.1.3), we immediately obtain for the left hand side that

$$\int_{B_t} \xi^2 \langle \mathcal{A}_h, \tau_h Du \rangle \, dx \ge \frac{1}{k} \int_{B_t} \xi^2 (\mu^2 + |Du(x)|^2 + |Du(x + he_i)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 b \, dx$$

and hence

$$\frac{1}{k} \int_{B_t} \xi^2 (\mu^2 + |Du(x)|^2 + |Du(x + he_i)|^2)^{\frac{p-2}{2}} \frac{|\tau_h Du|^2}{|h|^2} b \, dx \le \frac{1}{|h|^2} \sum_{i=1}^{4} |I_i|.$$

Now let $K_0 \in L^\infty(\Omega)$. In order to estimate $|I_j|$, $j = 1, \dots, 4$, we introduce the notation

$$\begin{aligned} \mathcal{K}(h) &:= K(x + he_i) + K(x), \\ \mathcal{D}(h) &:= (\mu^2 + |Du(x)|^2 + |Du(x + he_i)|^2)^{\frac{1}{2}}. \end{aligned}$$

By assumption (5.1.4), we immediately have

$$\begin{aligned} |I_1| &\le \int_{B_t} \xi^2 |\mathcal{A}'_h| \, |\tau_h Du| \, dx \le \\ &\le \int_{B_t} \xi^2 |h| \mathcal{K}(h) (\mu^2 + |Du|^2)^{\frac{p-1}{2}} |\tau_h Du| \, dx. \end{aligned}$$

Then, defining

$$\mathcal{K}_0(h) := K_0(x + he_i) + K_0(x),$$

the use of Young's inequality with a constant $\nu \in (0,1)$ that will be chosen later yields

$$
\begin{aligned}
|I_1| \le & \int_{B_t} \xi^2 |h|\, \mathcal{K}(h)\, (\mu^2 + |Du|^2)^{\frac{p-1}{2}}\, |D\tau_h u|\, dx \le \\
\le & \int_{B_t} \xi^2 |h|\, |\mathcal{K}(h) - \mathcal{K}_0(h)|\, (\mu^2 + |Du|^2)^{\frac{p-1}{2}}\, |D\tau_h u|\, dx + \\
& + \int_{B_t} \xi^2 |h|\, \|\mathcal{K}_0(h)\|_{L^\infty}\, (\mu^2 + |Du|^2)^{\frac{p-1}{2}}\, |D\tau_h u|\, dx \le \\
\le & \frac{\nu}{2} \int_{B_t} \xi^2 (\mu^2 + |Du|^2)^{\frac{p-2}{2}}\, |D\tau_h u|^2\, dx + \frac{|h|^2}{2\nu} \int_{B_t} |\mathcal{K}(h) - \mathcal{K}_0(h)|^2 (\xi(\mu^2 + |Du|^2)^{\frac{p}{4}})^2 \\
& + \frac{\nu}{2} \int_{B_t} \xi^2 (\mu^2 + |Du|^2)^{\frac{p-2}{2}} |D\tau_h u|^2\, dx + \frac{|h|^2}{2\nu} \int_{B_t} \xi^2 \|\mathcal{K}_0(h)\|^2_{L^\infty} (\mu^2 + |Du|^2)^{\frac{p}{2}} \\
\le & \nu \int_{B_t} \xi^2 (\mu^2 + |Du|^2)^{\frac{p-2}{2}}\, |D\tau_h u|^2\, dx + \frac{|h|^2}{\nu} \int_{B_t} |K(x) - K_0(x)|^2 (\xi(\mu^2 + |Du|^2)^{\frac{p}{4}})^2 \\
& + \frac{|h|^2}{\nu} \int_{B_t} |K(x + he_i) - K_0(x + he_i)|^2 (\xi(\mu^2 + |Du|^2)^{\frac{p}{4}})^2\, dx + \\
& + \frac{|h|^2}{2\nu} \int_{B_t} \xi^2 \|\mathcal{K}_0(h)\|^2_{L^\infty} (\mu^2 + |Du|^2)^{\frac{p}{2}}\, dx
\end{aligned}
$$

Now note that, thanks to Lemma 5.3.2, the assumption $V_\mu(Du) \in W^{1,2}_{loc}(b, \Omega; \mathbb{R}^{N\times n})$ guarantees that $(\mu^2 + |Du|^2)^{\frac{p}{2}} \in W^{1,2}_{loc}(b, \Omega)$, and therefore in $W^{1,2}(\Omega)$, and in particular, by Sobolev embedding Theorem in Lorentz spaces 2.2.15, that $\xi(\mu^2 + |Du|^2)^{\frac{p}{2}} \in L^{\frac{n}{n-2},1}$. Consequently, by Hölder's inequality in Lorentz spaces 2.2.17, set $2^* := \frac{2n}{n-2}$, we can estimate the second integral in the right hand side of previous inequality as follows

$$
\begin{aligned}
|I_1| \le & \nu \int_{B_t} \xi^2 (\mu^2 + |Du|^2)^{\frac{p-2}{2}} |D\tau_h u|^2\, dx + \frac{2|h|^2}{\nu} \|K(x) - K_0\|^2_{L^{n,\infty}(B_t)} \|\xi(\mu^2 + |Du|^2)^{\frac{p}{4}}\|^2_{L^{2^*,2}(B_t)} \\
& + \frac{|h|^2}{2\nu} \int_{B_t} \xi^2 \|\mathcal{K}_0(h)\|^2_{L^\infty(\Omega)} (\mu^2 + |Du|^2)^{\frac{p}{2}}\, dx.
\end{aligned}
$$

Finally by Sobolev embedding Theorem in Lorentz spaces 2.2.15, and taking into account that $b(x) \ge \lambda_0$ and $(\mu^2 + |Du(x)|^2)^{\frac{p-2}{2}} \le \mathcal{D}(h)^{p-2}$, we have that

$$
\begin{aligned}
|I_1| \le & \frac{\nu}{\lambda_0} \int_{B_t} \xi^2 (\mu^2 + |Du|^2)^{\frac{p-2}{2}} |D\tau_h u|^2 b\, dx + \\
& + 2\frac{|h|^2}{\nu} S^2_{2,n} \|K(x) - K_0\|^2_{L^{n,\infty}(\Omega)} \|D(\xi(\mu^2 + |Du|^2)^{\frac{p}{4}})\|^2_{L^2(B_t)} + \\
& + \frac{|h|^2}{2\nu\lambda_0} \int_{B_t} \xi^2 \|\mathcal{K}_0(h)\|^2_{L^\infty(\Omega)} (\mu^2 + |Du|^2)^{\frac{p}{2}} b\, dx \le \\
\le & \frac{\nu}{\lambda_0} \int_{B_t} \xi^2\, \mathcal{D}(h)^{p-2} |D\tau_h u|^2 b\, dx + \\
& + 2\frac{|h|^2}{\nu} S^2_{2,n} \|K(x) - K_0\|^2_{L^{n,\infty}(\Omega)} \|D(\xi(\mu^2 + |Du|^2)^{\frac{p}{4}})\|^2_{L^2(B_t)} + \\
& + \frac{|h|^2}{2\nu\lambda_0} \int_{B_t} \xi^2 \|\mathcal{K}_0(h)\|^2_{L^\infty(\Omega)} (\mu^2 + |Du|^2)^{\frac{p}{2}} b\, dx.
\end{aligned}
\tag{5.3.4}
$$

Next we estimate I_2. Observe that assumption (5.1.2) yields

$$
\begin{aligned}
|\mathcal{A}_h| \le & k\, b(x + he_i)\, |\tau_h Du|\, \left(\mu^2 + |Du(x)|^2 + |Du(x + he_i)|^2\right)^{\frac{p-2}{2}} = \\
= & k\, b(x + he_i)\, |\tau_h Du|\, \mathcal{D}(h)^{p-2}
\end{aligned}
$$

and hence, by the aid of Young's inequality, we obtain

$$
\begin{aligned}
|I_2| &\le 2\int_{B_t\setminus B_s} \xi\,|\mathcal{A}_h|\,|\nabla\xi|\,|\tau_h u|\,\mathrm{d}x \le \\
&\le 2k\int_{B_t\setminus B_s} \xi\, b(x+he_i)\,|\tau_h Du|\,\mathcal{D}(h)^{p-2}\,|\nabla\xi|\,|\tau_h u|\,\mathrm{d}x \le \\
&\le \nu\int_{B_t\setminus B_s} \xi^2\mathcal{D}(h)^{p-2}|\tau_h Du|^2 b(x+he_i)\,\mathrm{d}x + \\
&\quad + \frac{k^2}{\nu}\int_{B_t\setminus B_s} |\nabla\xi|^2\,\mathcal{D}(h)^{p-2}|\tau_h u|^2 b(x+he_i)\,\mathrm{d}x \le \\
&\le \nu\int_{B_t\setminus B_s} \xi^2\mathcal{D}(h)^{p-2}|\tau_h Du|^2 b(x+he_i)\,\mathrm{d}x + \\
&\quad + \frac{k^2}{\nu\,(t-s)^2}\int_{B_t\setminus B_s} \mathcal{D}(h)^{p-2}|\tau_h u|^2 b(x+he_i)\,\mathrm{d}x.
\end{aligned}
\tag{5.3.5}
$$

For I_3 we proceed as follows. The assumption (5.1.4) yields

$$
|I_3| \le 2\int_{B_t\setminus B_s} \xi|\mathcal{A}'_h|\,|\nabla\xi|\,|\tau_h u|\,\mathrm{d}x \le 2|h|\int_{B_t\setminus B_s} \xi\,\mathcal{K}(h)\,|\nabla\xi|\,(\mu^2+|Du|^2)^{\frac{p-1}{2}}|\tau_h u|.
$$

Arguing similarly as we have done for I_1, we have

$$
\begin{aligned}
|I_3| &\le 2|h|\int_{B_t\setminus B_s} \xi|\mathcal{K}(h)-\mathcal{K}_0(h)|\,|\nabla\xi|(\mu^2+|Du|^2)^{\frac{p-1}{2}}|\tau_h u|\,\mathrm{d}x + \\
&\quad + 2|h|\,\|\mathcal{K}_0(h)\|_{L^\infty(\Omega)}\int_{B_t\setminus B_s} \xi\,|\nabla\xi|(\mu^2+|Du|^2)^{\frac{p-1}{2}}|\tau_h u|\,\mathrm{d}x \le \\
&\le \frac{|h|^2}{\nu}\int_{B_t} \xi^2|\mathcal{K}(h)-\mathcal{K}_0(h)|^2(\mu^2+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x + \frac{2\nu}{(t-s)^2}\int_{B_t}(\mu^2+|Du|^2)^{\frac{p-2}{2}}|\tau_h u|^2 \\
&\quad + \frac{|h|^2}{\nu}\|\mathcal{K}_0(h)\|^2_{L^\infty(\Omega)}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x \le \\
&\le \frac{2|h|^2}{\nu}\int_{B_t}\xi^2|K(x)-K_0(x)|^2(\mu^2+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x + \\
&\quad + \frac{2|h|^2}{\nu}\int_{B_t}\xi^2|K(x+he_i)-K_0(x+he_i)|^2(\mu^2+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x + \\
&\quad + \frac{2\nu}{(t-s)^2}\int_{B_t}(\mu^2+|Du|^2)^{\frac{p-2}{2}}|\tau_h u|^2 + \frac{|h|^2}{\nu}\|\mathcal{K}_0(h)\|^2_{L^\infty(\Omega)}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p}{2}} \\
&\le 4\frac{|h|^2}{\nu}S^2_{2,n}\|K(x)-K_0\|^2_{L^{n,\infty}(\Omega)}\|D(\xi(\mu^2+|Du|^2)^{\frac{p}{4}})\|^2_{L^2(B_t)} + \\
&\quad + \tfrac{2\nu}{\lambda_0(t-s)^2}\int_{B_t}(\mu^2+|Du|^2)^{\frac{p-2}{2}}|\tau_h u|^2 b\,\mathrm{d}x + \tfrac{|h|^2}{\nu\lambda_0}\|K_0\|^2_{L^\infty(\Omega)}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p}{2}} b\,\mathrm{d}x.
\end{aligned}
\tag{5.3.6}
$$

Finally we estimate I_4. By using Young's inequality, together with $b(x)\ge\lambda_0$, we get

$$
\begin{aligned}
|I_4| &= \left|\int_{B_t}\left\langle \tau_h F, D(\xi^2\tau_h u)\right\rangle\,\mathrm{d}x\right| \le \\
&\le \int_{B_t}\xi^2|\tau_h F|\,|D\tau_h u|\,\mathrm{d}x + 2\int_{B_t}\xi\,|\nabla\xi|\,|\tau_h F|\,|\tau_h u|\,\mathrm{d}x \le \\
&\le \frac{\nu}{2}\int_{B_t}\xi^2|D\tau_h u|^2\,\mathrm{d}x + \frac{1}{2\nu}\int_{B_t}\xi^2|\tau_h F|^2\,\mathrm{d}x + \int_{B_t}\xi^2|\tau_h F|^2\,\mathrm{d}x + \int_{B_t}|\nabla\xi|^2|\tau_h u|^2 \\
&\le \frac{\nu}{2\lambda_0}\int_{B_t}\xi^2|D\tau_h u|^2 b\,\mathrm{d}x + \left(\frac{1}{2\nu}+1\right)\int_{B_t}\xi^2|\tau_h F|^2\,\mathrm{d}x + \frac{1}{(t-s)^2}\int_{B_t\setminus B_s}|\tau_h u|^2\,\mathrm{d}x
\end{aligned}
\tag{5.3.7}
$$

$$\leq \frac{\nu}{2\mu^2\lambda_0}\int_{B_t}\xi^2\left(\mu^2+|Du|^2\right)^{\frac{p-2}{2}}|D\tau_h u|^2 b\,\mathrm{d}x+\frac{1}{\lambda_0}\left(\frac{1}{2\nu}+1\right)\int_{B_t}\xi^2|\tau_h F|^2 b\,\mathrm{d}x+$$
$$+\frac{1}{\mu^2\lambda_0(t-s)^2}\int_{B_t\setminus B_s}(\mu^2+|Du|^2)^{\frac{p-2}{2}}|\tau_h u|^2 b\,\mathrm{d}x.$$

Combining estimates (5.3.4) - (5.3.7), we get

$$\frac{1}{k}\int_{B_t}\xi^2\mathcal{D}(h)^{p-2}\left|\frac{\tau_h Du}{h}\right|^2 b\,\mathrm{d}x\leq\frac{\nu}{\lambda_0}\int_{B_t}\xi^2\mathcal{D}(h)^{p-2}\left|\frac{D\tau_h u}{h}\right|^2 b\,\mathrm{d}x+$$
$$+\frac{6}{\nu}S_{2,n}^2\|K(x)-K_0\|^2_{L^{n,\infty}(\Omega)}\|D(\xi(\mu^2+|Du|^2)^{\frac{p}{4}})\|^2_{L^2(B_t)}+$$
$$+\frac{\nu}{2\mu^2\lambda_0}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p-2}{2}}\left|\frac{D\tau_h u}{h}\right|^2 b\,\mathrm{d}x+\nu\int_{B_t\setminus B_s}\xi^2\mathcal{D}(h)^{p-2}\left|\frac{\tau_h Du}{h}\right|^2 b(x+he_i)\,\mathrm{d}x$$
$$+\frac{3\|\mathcal{K}_0(h)\|^2_{L^\infty(\Omega)}}{2\nu\lambda_0}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x+\frac{2\mu^2\nu+1}{\mu^2\lambda_0(t-s)^2}\int_{B_t}(1+|Du|^2)^{\frac{p-2}{2}}\left|\frac{\tau_h u}{h}\right|^2 b\,\mathrm{d}x$$
$$+\frac{k^2}{\nu(t-s)^2}\int_{B_t\setminus B_s}\mathcal{D}(h)^{p-2}\left|\frac{\tau_h u}{h}\right|^2 b(x+he_i)\,\mathrm{d}x+\frac{1}{\lambda_0}\left(\frac{1}{2\nu}+1\right)\int_{B_t}\xi^2\left|\frac{\tau_h F}{h}\right|^2 b\,\mathrm{d}x.$$

Assuming $\nu<\frac{\lambda_0}{k}$ and reabsorbing the first integral in the right hand side by the left hand side, we get

$$\left(\frac{1}{k}-\frac{\nu}{\lambda_0}\right)\int_{B_t}\xi^2\mathcal{D}(h)^{p-2}\left|\frac{\tau_h Du}{h}\right|^2 b\,\mathrm{d}x\leq$$
$$\leq\frac{6}{\nu}S_{2,n}^2\|K(x)-K_0\|^2_{L^{n,\infty}(\Omega)}\|D(\xi(\mu^2+|Du|^2)^{\frac{p}{4}})\|^2_{L^2(B_t)}+$$
$$+\frac{\nu}{2\mu^2\lambda_0}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p-2}{2}}\left|\frac{D\tau_h u}{h}\right|^2 b\,\mathrm{d}x+$$
$$+\nu\int_{B_t\setminus B_s}\xi^2\mathcal{D}(h)^{p-2}\left|\frac{\tau_h Du}{h}\right|^2 b(x+he_i)\,\mathrm{d}x+$$
$$+\frac{3\|\mathcal{K}_0(h)\|^2_{L^\infty(\Omega)}}{2\nu\lambda_0}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x+\frac{2\mu^2\nu+1}{\mu^2\lambda_0(t-s)^2}\int_{B_t}(\mu^2+|Du|^2)^{\frac{p-2}{2}}\left|\frac{\tau_h u}{h}\right|^2 b\,\mathrm{d}x$$
$$+\frac{k^2}{\nu(t-s)^2}\int_{B_t\setminus B_s}\mathcal{D}(h)^{p-2}\left|\frac{\tau_h u}{h}\right|^2 b(x+he_i)\,\mathrm{d}x+\frac{1}{\lambda_0}\left(\frac{1}{2\nu}+1\right)\int_{B_t}\xi^2\left|\frac{\tau_h F}{h}\right|^2 b\,\mathrm{d}x.$$

Let us note that, by the properties of ξ and using the fact that $b(x)\geq\lambda_0$,

$$\|D(\xi(\mu^2+|Du|^2)^{\frac{p}{4}})\|^2_{L^2(B_t)}\leq\frac{1}{\lambda_0}\int_{B_t}|D(\xi(\mu^2+|Du|^2)^{\frac{p}{4}})|^2 b\,\mathrm{d}x\leq$$
$$\leq\frac{2}{\lambda_0}\int_{B_t}|\nabla\xi|^2(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x+\frac{2}{\lambda_0}\int_{B_t}\xi^2|D[(\mu^2+|Du|^2)^{\frac{p}{4}}]|^2 b\,\mathrm{d}x\leq$$
$$\leq\frac{2}{\lambda_0(t-s)^2}\int_{B_t}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x+\frac{2}{\lambda_0}\int_{B_t}\xi^2|D[(\mu^2+|Du|^2)^{\frac{p}{4}}]|^2 b\,\mathrm{d}x,$$

and, by (5.1.6) and since by Lemma 2.2.21 we are legitimate to pass to the limit for $h\to 0$,

$$\int_{B_t\setminus B_s}\lim_{h\to 0}\left(\mathcal{D}(h)^{p-2}\left|\frac{\tau_h u}{h}\right|^2 b(x+he_i)\right)\,\mathrm{d}x=\int_{B_t\setminus B_s}(\mu^2+2|Du|^2)^{\frac{p-2}{2}}|Du|^2 b\,\mathrm{d}x\leq$$
$$\leq\int_{B_t\setminus B_s}(2\mu^2+2|Du|^2)^{\frac{p-2}{2}}(\mu^2+|Du|^2)b\,\mathrm{d}x=2^{\frac{p-2}{2}}\int_{B_t\setminus B_s}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x.$$

Applying now Lemma 5.3.2 and again Lemma 2.2.21, we have

$$
\begin{aligned}
&\frac{1}{c(p)}\left(\frac{1}{k}-\frac{\nu}{\lambda_0}\right)\int_{B_t}\xi^2|D[(\mu^2+|Du|^2)^{\frac{p-2}{4}}Du]|^2 b\,\mathrm{d}x\le\\
&\le\frac{12}{\nu\lambda_0}S_{2,n}^2\|K(x)-K_0\|^2_{L^{n,\infty}(\Omega)}\int_{B_t}\xi^2(D[(\mu^2+|Du|^2)^{\frac{p}{4}}])^2 b\,\mathrm{d}x+\\
&+\frac{\nu}{2\mu^2\lambda_0}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2 b\,\mathrm{d}x+\\
&+c(p)\,\nu\int_{B_t\setminus B_s}\xi^2|D[(\mu^2+|Du|^2)^{\frac{p-2}{4}}Du]|^2 b\,\mathrm{d}x+\\
&+\frac{3\|K_0\|^2_{L^\infty(\Omega)}}{\nu\lambda_0}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x+\frac{c(p,k,\lambda_0,\nu,\mu)}{(t-s)^2}\int_{B_t}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x+\\
&+\frac{12}{\nu\lambda_0(t-s)^2}S_{2,n}^2\|K(x)-K_0\|^2_{L^{n,\infty}(\Omega)}\int_{B_t}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x+\\
&+\frac{1}{\lambda_0}\left(\frac{1}{2\nu}+1\right)\int_{B_t}\xi^2\left|\frac{\tau_h F}{h}\right|^2 b\,\mathrm{d}x.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{1}{c(p)}\left(\frac{1}{k}-\frac{\nu}{\lambda_0}\right)\int_{B_t}\xi^2|D[(\mu^2+|Du|^2)^{\frac{p-2}{4}}Du]|^2 b\,\mathrm{d}x\le\\
&\le\left(\frac{12}{\nu\,\lambda_0}S_{2,n}^2\|K(x)-K_0\|^2_{L^{n,\infty}(\Omega)}+\frac{\nu}{2\mu^2\lambda_0}\right)\int_{B_t}\xi^2|D[(\mu^2+|Du|^2)^{\frac{p-2}{4}}Du]|^2 b\,\mathrm{d}x+\\
&+c(p)\nu\int_{B_t\setminus B_s}\xi^2|D[(\mu^2+|Du|^2)^{\frac{p-2}{4}}Du]|^2 b\,\mathrm{d}x+\\
&+\frac{3\|K_0\|^2_{L^\infty(\Omega)}}{\nu\lambda_0}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x+\frac{c(p,k,\lambda_0,\nu,\mu)}{(t-s)^2}\int_{B_t}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x+\\
&+\frac{12}{\nu\lambda_0(t-s)^2}S_{2,n}^2\|K(x)-K_0\|^2_{L^{n,\infty}(\Omega)}\int_{B_t}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x+\\
&+\frac{1}{\lambda_0}\left(\frac{1}{2\nu}+1\right)\int_{B_t}\xi^2(\mu^2+|DF|^2)b\,\mathrm{d}x.
\end{aligned}
\tag{5.3.8}
$$

Let us fix $\nu := \nu_0$ such that

$$\frac{1}{c(p)}\left(\frac{1}{k}-\frac{\nu_0}{\lambda_0}\right)-\frac{\nu_0}{2\mu^2\lambda_0}>0,$$

for example $\nu_0 := \frac{\mu^2\lambda_0}{k(2\mu^2+c(p))}$. Set $\eta := \frac{\sqrt{\nu_0\lambda_0}}{2\sqrt{3}S_{2,n}}\sqrt{\frac{1}{c(p)}\left(\frac{1}{k}-\frac{\nu_0}{\lambda_0}\right)-\frac{\nu}{2\mu^2\lambda_0}}$, let α be a number such that $0<\alpha<\eta$. If

$$\mathscr{D}_K<\alpha,$$

then we can choose $K_0\in L^\infty(\Omega)$ such that

$$\left(\frac{1}{c(p)}\left(\frac{1}{k}-\frac{\nu_0}{\lambda_0}\right)-\frac{12}{\nu_0\lambda_0}S_{2,n}^2\|K(x)-K_0\|^2_{L^{n,\infty}(\Omega)}-\frac{\nu_0}{2\mu^2\lambda_0}\right)>0.$$

Then, by reabsorbing the first term of the right hand side of (5.3.8) in the left hand side, since $\xi=1$ on B_s and $0\le\xi\le1$, we get

$$
\begin{aligned}
&C\int_{B_s}|D[(\mu^2+|Du|^2)^{\frac{p-2}{4}}Du]|^2 b\,\mathrm{d}x\le c(p,k,\lambda_0,\mu)\int_{B_t\setminus B_s}|D[(\mu^2+|Du|^2)^{\frac{p-2}{4}}Du]|^2 b\,\mathrm{d}x\\
&+c(p,k,\lambda_0,n,\mathscr{D}_K,\mu)\left(\int_{B_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x+\frac{1}{(t-s)^2}\int_{B_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x+\right.\\
&\left.+\int_{B_{2R}}(\mu^2+|DF|^2)b\,\mathrm{d}x\right),
\end{aligned}
$$

where $C = \frac{1}{c(p)}\left(\frac{1}{k} - \frac{\nu_0}{\lambda_0}\right) - \frac{3}{\nu_0\lambda_0} S_{2,n}^2 \alpha^2 - \frac{\nu_0}{2\mu^2\lambda_0}$.
Now we fill the hole, having

$$\begin{aligned}
\int_{B_s} |D[(\mu^2+|Du|^2)^{\frac{p-2}{4}} Du]|^2 b\,\mathrm{d}x \le \tfrac{c(p,k,\lambda_0,\mu)}{C+c(p,k,\lambda_0,\mu)} \int_{B_t} |D[(\mu^2+|Du|^2)^{\frac{p-2}{4}} Du]|^2 b\,\mathrm{d}x+ \\
+ \frac{c(p,k,\lambda_0,n,\mathscr{D}_K,\mu)}{C+c(p,k,\lambda_0,\mu)} \left(\int_{B_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}} b\,\mathrm{d}x + \frac{1}{(t-s)^2} \int_{B_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}} b\,\mathrm{d}x+ \right. \\
\left. + \int_{B_{2R}} (\mu^2+|DF|^2) b\,\mathrm{d}x \right).
\end{aligned}$$

Then by Lemma 5.2.2

$$\int_{B_R} |D(V_\mu(Du))|^2 b\,\mathrm{d}x \le c \int_{B_{2R}} \left(1+\frac{1}{R^2}\right) (\mu^2+|Du|^2)^{\frac{p}{2}} b\,\mathrm{d}x + c\int_{B_{2R}} (\mu^2+|DF|^2) b\,\mathrm{d}x, \tag{5.3.9}$$

where $c = c(p,k,\lambda_0,n,\mathscr{D}_K,\mu)$, and therefore we have the result. □

Remark 5.3.3. Note that, even if we do not provide the exact value of the constant α in (5.3.1), a bound on it is given at the end of the proof of Theorem 5.3.1.

Remark 5.3.4. We point out that the dependence of the constant c in (5.3.9) on $\mathscr{D}_K$ occurs only through the norm of K_0 in L^∞.

Remark 5.3.5. By a careful analysis of the proof, it is evident that the degenerate case, that is for $\mu = 0$, causes further difficulties only when dealing with the integral involving the datum F. More specifically, in the estimate of $|I_4|$, an integral which can blow up appears. Consequently, if $F \equiv 0$, the proof proceeds in the same way even if $\mu = 0$.

5.4 Regularity

In this section we prove Theorem 5.1.1.

Proof of Theorem 5.1.1. Fix a ball $B_{2R} \subset\subset \Omega$, let u be a local solution of the system (5.1.1) and let us consider, for $x \in \Omega$, $\xi \in \mathbb{R}^{N\times n}$ and $j \in \mathbb{N}$ sufficiently large,

$$A_j(x,\xi) := \begin{cases} A(x,\xi) & \text{if} \quad b(x) < j \\ j\frac{A(x,\xi)}{b(x)} & \text{if} \quad b(x) \ge j. \end{cases}$$

Let b_j be the truncated of b at level j, i.e. for $x \in \Omega$ and $j \in \mathbb{N}$ sufficiently large

$$b_j(x) := \begin{cases} b(x) & \text{if} \quad b(x) < j \\ j & \text{if} \quad b(x) \ge j. \end{cases}$$

Since

$$A_j(x,\xi) = \frac{b_j(x)}{b(x)} A(x,\xi),$$

it is easy to check that for a.e. $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^{N\times n}$ we have

$$|A_j(x,\xi) - A_j(x,\eta)| \le k b_j(x) |\xi-\eta| \, (\mu^2+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}, \tag{5.4.1}$$

$$\frac{1}{k} b_j(x) |\xi-\eta|^2 \, (\mu^2+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} \le \left\langle A_j(x,\xi) - A_j(x,\eta), \xi-\eta \right\rangle, \tag{5.4.2}$$

$$A_j(x,0) = 0.$$

For a.e. $x, y \in \Omega$ one easily gets

$$|b_j(x) - b_j(y)| \le |b(x)-b(y)| \le |x-y| \, [K(x)+K(y)].$$

In particular

$$|b_j(x) - b_j(y)| \leq (k+1)\,|x-y|\,[K(x)+K(y)].$$

In order to simplify the proof, we will prove

$$|A_j(x,\eta) - A_j(y,\eta)| \leq (k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}}, \tag{5.4.3}$$

at the end of the paper, see the Appendix.
Let us consider the following Dirichlet problem

$$\begin{cases} \operatorname{div} A_j(x,Dv) = \operatorname{div} F & \text{in} \quad B_{2R} \\ v = u & \text{on} \quad \partial B_{2R}. \end{cases}$$

If we denote by $u_j \in W^{1,p}(B_{2R})$ the solution of this problem, then $D(V_\mu(Du_j)) \in L^2_{loc}(b_j,\Omega)$ (see [GM18]) and, if

$$\mathscr{D}_K < \alpha_1 := \frac{1}{k+1}\alpha,$$

we can use estimate (5.3.2) to obtain

$$\fint_{B_R} |D(V_\mu(Du_j))|^2 b_j \,\mathrm{d}x \leq c \fint_{B_{2R}} \left(\left(1+\tfrac{1}{R^2}\right)(\mu^2+|Du_j|^2)^{\frac{p}{2}} + (\mu^2+|DF|^2)\right) b_j \,\mathrm{d}x. \tag{5.4.4}$$

Let us remark that by Lemma 5.2.1 there exists $\delta > 0$ such that $|Du| \in L^{p+\delta}(b,B_{2R})$. Now we prove the strong convergence of $\{|Du_j|\}_j$ to $|Du|$ in $L^p(b,B_{2R})$. Using $(u-u_j)$ as test function, we easily get, thanks to (5.4.2),

$$\begin{aligned} \int_{B_{2R}} |Du - Du_j|^p b_j(x)\,\mathrm{d}x &\leq c(k) \int_{B_{2R}} \left\langle A_j(x,Du_j) - A_j(x,Du), Du_j - Du \right\rangle \,\mathrm{d}x = \\ &= c(k) \int_{B_{2R}} \left\langle F, Du_j - Du \right\rangle \,\mathrm{d}x - \int_{B_{2R}} \left\langle A_j(x,Du), Du_j - Du \right\rangle \,\mathrm{d}x = \\ &= c(k) \int_{B_{2R}} \left\langle A(x,Du) - A_j(x,Du), Du_j - Du \right\rangle \,\mathrm{d}x = \\ &= c(k) \int_{B_{2R}} \left(1 - \frac{b_j}{b}\right) \left\langle A(x,Du), Du_j - Du \right\rangle \,\mathrm{d}x. \end{aligned}$$

Then from (5.1.2) and (5.1.5) we derive

$$\int_{B_{2R}} |Du - Du_j|^p b_j(x)\,\mathrm{d}x \leq c(k) \int_{B_{2R}} (b-b_j)|Du - Du_j|(\mu^2+|Du|^2)^{\frac{p-1}{2}}\,\mathrm{d}x.$$

Finally by Hölder's inequality, since $b_j(x) \leq b(x)$, we obtain

$$\begin{aligned} \int_{B_{2R}} |Du - Du_j|^p b_j(x)\,\mathrm{d}x \leq c(k,\lambda_0) \left(\int_{B_{2R}} (\mu^2+|Du|^2)^{\frac{p+\delta}{2}} b(x)\,\mathrm{d}x\right)^{\frac{p}{p+\delta}} \cdot \\ \cdot \left(\int_{B_{2R}} b(x)|b-b_j|^{p'\left(\frac{p+\delta}{\delta}\right)}\,\mathrm{d}x\right)^{\frac{\delta}{p+\delta}} \end{aligned} \tag{5.4.5}$$

and the last term goes to zero. Previous relation easily implies the conclusion.
At this point, estimate (5.4.4) and (5.4.5) yield $\|D(V_\mu(Du_j))\|_{L^2_b(B_R)} \leq C$, so that we deduce that, up to a subsequence, $D(V_\mu(Du_j))$ is weakly converging to $D(V_\mu(Du))$ in $L^2(b,B_R)$ and $V_\mu(Du_j)$ is strongly converging in $L^2(b,B_R)$. Therefore, we can pass to the limit in the estimate (5.4.4) having the validity of the desired inequality for the function u. □

In account of Remark 5.3.5, we can state the following

Proposition 5.4.1. *Let Ω be a regular domain and $A(x,\xi)$ a mapping verifying assumptions* (5.1.2)*,* (5.1.3)*,* (5.1.4) *and* (5.1.5)*. There exist $0 < \varepsilon_1 < \frac{1}{2}$, depending on k, n, λ_0, p and the BMO - norm of $b(x)$, and $\alpha_2 > 0$, depending on p, n, λ_0 and k, such that, if $u \in W^{1,p-\varepsilon}_{loc}(b,\Omega;\mathbb{R}^N)$, with $0 \le \varepsilon < \varepsilon_1$, is a local solution of*

$$\operatorname{div} A(x, Du(x)) = 0$$

and

$$\mathscr{D}_K := \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_2,$$

then $D(V_\mu(Du)) \in L^2_{loc}(b,\Omega)$ and the following estimate holds:

$$\int_{B_R} |D(V_\mu(Du))|^2 b\,\mathrm{d}x \le c\left(1+\frac{1}{R^2}\right)\int_{B_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}} b\,\mathrm{d}x,$$

for every ball $B_{2R} \subset\subset \Omega$ and for a constant c depending on p, k, λ_0, n and $\mathscr{D}_K$.

For $2 < p < n$, the following corollaries of fractional higher integrability easily derive from Theorem 5.1.1.

Corollary 5.4.2. *Let Ω be a regular domain, $A(x,\xi)$ a mapping verifying assumptions* (5.1.2)*,* (5.1.3)*,* (5.1.4) *and* (5.1.5)*, and $F \in W^{1,2}_{loc}(b,\Omega;\mathbb{R}^{N\times n})$. There exist $0 < \varepsilon_1 < \frac{1}{2}$, depending on k, n, λ_0, p and the BMO - norm of $b(x)$, and $\alpha_1 > 0$, depending on p, n, λ_0, μ and k, such that, if $u \in W^{1,p-\varepsilon}_{loc}(b,\Omega;\mathbb{R}^N)$, with $0 \le \varepsilon < \varepsilon_1$, is a local solution of* (5.1.1) *and*

$$\mathscr{D}_K := \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_1,$$

then $Du \in W^{\beta,p}_{loc}(b,\Omega;\mathbb{R}^N)$ for every $\beta \in \left(0,\frac{2}{p}\right)$.

Proof. Since we can estimate for every $i \in \{1,\dots,n\}$

$$|\tau_{h,i}Du|^p \le c(n,p)|\tau_{h,i}V_\mu(Du)|^2, \tag{5.4.6}$$

summing up on $i \in \{1,\dots,n\}$ and taking into account the estimate given by Theorem 5.1.1, we get for $\varrho \in (0,R)$ and h sufficiently small

$$\int_{B_\varrho}\sum_{i=1}^n |\tau_{h,i}Du|^p b\,\mathrm{d}x \le$$
$$\le c\cdot\left(|h|^{\frac{2}{p}}\right)^p \int_{B_{2R}}\left(\left(1+\frac{1}{R^2}\right)(\mu^2+|Du|^2)^{\frac{p}{2}} + (\mu^2+|DF|^2)\right) b\,\mathrm{d}x.$$

It follows that Du belongs to the Nikolskii space $\mathcal{H}^{\frac{2}{p},p}$ and hence the conclusion by embedding (see [Ada75], 7.73 and also [Min03]). □

In the next corollary we show that assuming a higher integrability of the function F improves the integrability of the fractional derivatives.

Corollary 5.4.3. *Let Ω be a regular domain, $A(x,\xi)$ a mapping verifying assumptions* (5.1.2)*,* (5.1.3)*,* (5.1.4) *and* (5.1.5)*, and $F \in W^{1,r}_{loc}(b,\Omega;\mathbb{R}^{N\times n})$, for some $r > 2$. There exist $0 < \varepsilon_1 < \frac{1}{2}$, depending on k, n, λ_0, p and the BMO - norm of $b(x)$, and $\alpha_1 > 0$, depending on p, n, λ_0, μ and k, such that, if $u \in W^{1,p-\varepsilon}_{loc}(b,\Omega;\mathbb{R}^N)$, with $0 \le \varepsilon < \varepsilon_1$, is a local solution of* (5.1.1) *and*

$$\mathscr{D}_K := \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_1,$$

then $Du \in W^{\beta,q}_{loc}(b,\Omega;\mathbb{R}^N)$ for some $q > p$ and for every $\beta \in \left(0,\frac{2}{p}\right)$.

Proof. Without loss of generality, we assume $0 < R < 1$. The estimate given by Theorem 5.1.1 and the use of Lemma 5.3.2 yield

$$\frac{1}{b(B_R)}\int_{B_R}|DV_\mu(Du)|^2 b\,\mathrm{d}x \le$$
$$\le c\left(\left(1+\frac{1}{R^2}\right)\frac{1}{b(B_{2R})}\int_{B_{2R}}|V_\mu(Du)-(V(Du))_{B_{2R}}|^2 + (\mu^2+|DF|^2)b\,\mathrm{d}x\right).$$

Hence, applying Sobolev – Poincaré inequality, we have the following reverse Hölder's inequality

$$\frac{1}{B_R}\int_{B_R}|DV_\mu(Du)|^2 b\,dx \leq$$

$$\leq c\left[\left(\frac{1}{b(B_{2R})}\int_{B_{2R}}(|DV_\mu(Du)|^2)^{\frac{n-1}{n}}b\,dx\right)^{\frac{n}{n-1}}+\frac{1}{b(B_{2R})}\int_{B_{2R}}(\mu^2+|DF|^2)b\,dx\right]$$

getting the existence of an exponent $s>2$ such that $|DV_\mu(Du)|\in L^s_{loc}$ and

$$\frac{1}{b\left(B_{\frac{R}{2}}\right)}\int_{B_{\frac{R}{2}}}|DV_\mu(Du)|^s b\,dx \leq$$

$$\leq c\left[\left(\frac{1}{b\left(B_{\frac{R}{2}}\right)}\int_{B_{\frac{R}{2}}}|DV_\mu(Du)|^2 b\,dx\right)^{\frac{s}{2}}+\frac{1}{b(B_{2R})}\int_{B_{2R}}(\mu^2+|DF|^2)^{\frac{s}{2}}b\,dx\right].$$

Then, using the pointwise inequality in (5.4.6), we easily obtain that

$$\frac{\|\tau_h Du\|_{L_b^{\frac{ps}{2}}}}{|h|^2}\leq c\|DV_\mu(Du)\|^{\frac{2}{p}}_{L^2_b}$$

which allows to conclude that Du belongs to the Nikolskii space $\mathcal{H}^{\frac{2}{p},\frac{ps}{2}}$ and hence, setting $q:=\frac{ps}{2}$, by embedding $Du\in W^{\beta,q}_{loc}(b,\Omega;\mathbb{R}^N)$ for every $\beta\in\left(0,\frac{2}{p}\right)$. □

5.5 Calderón–Zygmund estimates

In this section we prove Theorem 5.1.2. Here the cubes considered will always have sides parallel to the coordinate axes.

We recall a few basic facts concerning the interior regularity of solutions of elliptic systems in divergence form of the type

$$\operatorname{div} A(x,Dw)=0 \qquad \text{in} \quad 3Q\subset\Omega,$$

where Q is a generic cube. The same proof of Theorem 5.1.1 works for balls of the type B_{2R}, B_{3R} and, by a covering argument by means of countable disjoint balls, we have the estimate also over cubes instead of balls. Thanks to Theorem 5.1.1, Remark 5.3.5 and Sobolev embedding Theorem 2.2.12, arguing as in (5.2.19) and if $\mathscr{D}_K<\alpha_2$, with α_2 as in Proposition 5.4.1, we get the following reverse Hölder inequality:

$$\left(\frac{1}{b(2Q)}\int_{2Q}\left(\mu^2+|Dw|^2\right)^{\frac{r}{2}}b\,dx\right)^{\frac{1}{r}}\leq c\left(\frac{1}{b(3Q)}\int_{3Q}\left(\mu^2+|Dw|^2\right)^{\frac{p}{2}}b\,dx\right)^{\frac{1}{p}},$$

with $c=c(p,k,\lambda_0,n,\|b\|_*,\mathscr{D}_K)$ and $r=\frac{np}{n-1}$. Then, by Gehring's Lemma, there exists an exponent

$$s:=r+\delta, \tag{5.5.1}$$

with $\delta=\delta(p,k,\lambda_0,n,\|b\|_*,\mathscr{D}_K)$, such that

$$\left(\frac{1}{b(2Q)}\int_{2Q}(\mu^2+|Dw|^2)^{\frac{s}{2}}b\,dx\right)^{\frac{1}{s}}\leq c\left(\frac{1}{b(3Q)}\int_{3Q}(\mu^2+|Dw|^2)^{\frac{p}{2}}b\,dx\right)^{\frac{1}{p}}$$

with $c=c(p,k,\lambda_0,n,\|b\|_*,\mathscr{D}_K)$. Then

$$c\left(\int_{3Q}(\mu^2+|Dw|^2)^{\frac{p}{2}}b\,dx\right)^{\frac{1}{p}}\geq\frac{[b(3Q)]^{\frac{1}{p}}}{[b(2Q)]^{\frac{1}{s}}}\left(\int_{2Q}(\mu^2+|Dw|^2)^{\frac{s}{2}}b\,dx\right)^{\frac{1}{s}}\geq$$

$$\geq[b(2Q)]^{\frac{1}{p}-\frac{1}{s}}\left(\int_{2Q}(\mu^2+|Dw|^2)^{\frac{s}{2}}b\,dx\right)^{\frac{1}{s}}\geq$$

$$\geq\lambda_0|2Q|^{\frac{1}{p}-\frac{1}{s}}\left(\int_{2Q}(\mu^2+|Dw|^2)^{\frac{s}{2}}b\,dx\right)^{\frac{1}{s}}\geq\lambda_0c'(n)\frac{|3Q|^{\frac{1}{p}}}{|2Q|^{\frac{1}{s}}}\left(\int_{2Q}(\mu^2+|Dw|^2)^{\frac{s}{2}}b\,dx\right)^{\frac{1}{s}}.$$

Equivalently we have

$$\left(\fint_{2Q}\left(\mu^2+|Dw|^2\right)^{\frac{s}{2}} b\,\mathrm{d}x\right)^{\frac{1}{s}} \le c\left(\fint_{3Q}\left(\mu^2+|Dw|^2\right)^{\frac{p}{2}} b\,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{5.5.2}$$

with $c = c(p,k,\lambda_0,n,\|b\|_*,\mathscr{D}_K)$. We wish to emphasize the fact that the above constants and exponents are independent of the number $\mu \in [0,1]$.
We recall some useful lemmas.

Lemma 5.5.1 ([KM06])**.** *Let $p \in [2,\infty)$ and $\mu \in [0,1]$, then*

$$(\mu^2+|z|^2)^{\frac{p-2}{2}}|z|\,|\zeta| \le \varepsilon(\mu^2+|z|^2)^{\frac{p-2}{2}}|z|^2+\varepsilon^{1-p}(\mu^2+|\zeta|^2)^{\frac{p-2}{2}}|\zeta|^2$$

for every z, $\zeta \in \mathbb{R}^{N\times n}$ and $\varepsilon \in (0,1]$.

Lemma 5.5.2 ([CP98])**.** *Let $Q_0 \subset \mathbb{R}^n$ be a cube and $\mathcal{D}(Q_0)$ be the class of all dyadic cubes obtained from Q_0. Let $a \in (0,1)$. Assume that $X \subset Y \subset Q_0$ are measurable sets satisfying the following conditions:*

- $|X| < a|Q_0|$
- *if $Q \in \mathcal{D}(Q_0)$ then $|X \cap Q| > a|Q| \implies \tilde{Q} \subset Y$,*

where $\tilde{Q}$ denotes the predecessor of Q. Then

$$|X| < a|Y|.$$

The following Lemma is fundamental in order to prove Theorem 5.1.2.

Lemma 5.5.3. *Let $u \in W^{1,p}(b,Q_{2R};\mathbb{R}^N)$ be a solution to (5.1.8). Let $B > 1$; there exists a number $\varepsilon = \varepsilon(p,k,\lambda_0,n,\mathscr{D}_K,\|b\|_*,B)$ such that the following is true:*
If $\lambda > 0$ and $Q \subset Q_R$ is a dyadic subcube of Q_R such that

$$\begin{aligned}\Bigg|Q \cap \Bigg\{ x \in Q_R \,:\, M_b^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x) > \frac{AB\lambda}{\lambda_0},\\ M^*\left[\left(\mu^2+|G|^2\right)^{\frac{p}{2}}\right](x) < \varepsilon\lambda\Bigg\}\Bigg| > B^{-\frac{s}{p}}|Q|,\end{aligned} \tag{5.5.3}$$

then its predecessor $\tilde{Q}$ satisfies

$$\tilde{Q} \subset \left\{x \in Q_R \,:\, M^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}} b\right](x) > \lambda\right\}. \tag{5.5.4}$$

Here $M_{(b)}^* \equiv M_{(b,)Q_{2R}}^*$ denotes the (weighted) restricted Maximal Function relative to Q_{2R}, i.e. if $f_1 \in L^1(Q_{2R})$, $f_2 \in L^1(b,Q_{2R})$ and $x \in Q_{2R}$

$$M^*(f_1)(x) := \sup_{\substack{Q\subset Q_{2R}\\ x\in Q}} \fint_Q |f_1(y)|\,dy, \qquad M_b^*(f_2)(x) := \sup_{\substack{Q\subset Q_{2R}\\ x\in Q}} \frac{\int_Q |f_2(y)| b(y)\,dy}{b(Q)}.$$

Moreover here s is the number defined in (5.5.1), and $A = A(p,k,\lambda_0,n,\mathscr{D}_K,\|b\|_*) > 1$ is an absolute constant. All the constants and quantities are uniform with respect to $\mu \in [0,1]$.

Proof. We prove this lemma by contradiction. The constants A and ε will be chosen toward the end while all the arguments will be worked out for a general $\mu \in [0,1]$. Suppose (5.5.4) is not satisfied although (5.5.3) holds. Then there exists $\tilde{x} \in \tilde{Q}$ such that

$$\lambda_0 M_b^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](\tilde{x}) \le M^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}} b\right](\tilde{x}) \le \lambda. \tag{5.5.5}$$

Since $\tilde{Q} \subset 3Q$ because $\tilde{Q}$ is the predecessor of Q, we have

$$\fint_{3Q} \left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} b \, dx \le \lambda.$$

Note that $3Q \subset Q_{2R}$. Moreover by (5.5.3) we can find $\bar{x} \in Q$ such that

$$M^* \left[\left(\mu^2 + |G|^2\right)^{\frac{p}{2}}\right](\bar{x}) \le \varepsilon\lambda$$

and therefore

$$\fint_{3Q} (\mu^2 + |G|^2)^{\frac{p}{2}} \, dx \le \varepsilon\lambda. \tag{5.5.6}$$

Now we define $w \in W^{1,p}(b, 3Q; \mathbb{R}^N)$ as the unique solution of the following Dirichlet problem

$$\begin{cases} \operatorname{div} A(x, Dw) = 0 & \text{in} \quad 3Q \\ w - u \in W_0^{1,p}(3Q, \mathbb{R}^N). \end{cases} \tag{5.5.7}$$

The existence and the uniqueness of such a solution follows from Minty-Browder Theorem. Let us first derive the following estimate

$$\int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p}{2}} b \, dx \le c \int_{3Q} \left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} b \, dx \tag{5.5.8}$$

where $c = c(k, p)$. By using $w - u$ as test function in (5.5.7) we get

$$\int_{3Q} \langle A(x, Dw), Dw \rangle \, dx = \int_{3Q} \langle A(x, Dw), Du \rangle \, dx.$$

By (5.1.2), (5.1.3) and (5.1.5) we get

$$\int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p-2}{2}} |Dw|^2 b \, dx \le k^2 \int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p-2}{2}} |Dw| \, |Du| b \, dx$$

Using the Young's type inequality in Lemma 5.5.1, which holds uniformly in $\mu \in [0, 1]$, with $\varepsilon < \min\left\{\frac{1}{2}, \frac{1}{2k^2}\right\}$, we obtain

$$\int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p-2}{2}} |Dw|^2 b \, dx \le c(k, p) \int_{3Q} \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{2}} |Du|^2 b \, dx.$$

Now using again Young's inequality, from previous relation we have

$$\begin{aligned} &\int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p}{2}} b \, dx = \int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p-2}{2}} (\mu^2 + |Dw|^2) b \, dx \le \\ &\le c \int_{3Q} \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{2}} |Du|^2 b \, dx + \frac{1}{2} \int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p}{2}} b \, dx + c \int_{3Q} \mu^p b \, dx \\ &\le \frac{1}{2} \int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p}{2}} b \, dx + c \int_{3Q} \left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} b \, dx, \end{aligned}$$

with $c = c(k, p)$ independent of μ. Then estimate (5.5.8) follows.
Now by (5.5.2) we find that

$$\begin{aligned} \left(\fint_{2Q} \left(\mu^2 + |Dw|^2\right)^{\frac{s}{2}} b \, dx\right)^{\frac{1}{s}} &\le c \left(\fint_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p}{2}} b \, dx\right)^{\frac{1}{p}} \le \\ &\le c \left(\fint_{3Q} \left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} b \, dx\right)^{\frac{1}{p}} \le c\lambda^{\frac{1}{p}}, \end{aligned} \tag{5.5.9}$$

where $c = c(p, k, \lambda_0, n, \|b\|_*, \mathscr{D}_K)$. Notice that since $p \geq 2$, by (5.1.3)

$$\begin{aligned}
&\fint_{3Q} |Du - Dw|^p b \,\mathrm{d}x \leq c(p,k) \fint_{3Q} \langle A(x, Du) - A(x, Dw), Du - Dw \rangle \,\mathrm{d}x = \\
&= c(p,k) \fint_{3Q} \langle A(x, Du), Du - Dw \rangle \,\mathrm{d}x = c(p,k) \fint_{3Q} \left\langle |G|^{p-2} G, Du - Dw \right\rangle \,\mathrm{d}x \leq \\
&\leq c(p,k,\lambda_0) \left(\fint_{3Q} (\mu^2 + |G|^2)^{\frac{p}{2}} \,\mathrm{d}x \right) + \frac{\lambda_0}{2} \fint_{3Q} |Du - Dw|^p \,\mathrm{d}x \leq \\
&\leq c(p,k,\lambda_0) \left(\fint_{3Q} (\mu^2 + |G|^2)^{\frac{p}{2}} \,\mathrm{d}x \right) + \frac{1}{2} \fint_{3Q} |Du - Dw|^p b \,\mathrm{d}x.
\end{aligned}$$

Then from (5.5.6)

$$\fint_{3Q} |Du - Dw|^p b \,\mathrm{d}x \leq c\varepsilon\lambda \tag{5.5.10}$$

where $c = c(p, k, \lambda_0)$. In the following we shall denote by $M^{**}_{(b)}$ the (weighted) Restricted Maximal operator relative to the cube $2Q$, while $M^*_{(b)}$ keeps on denoting the (weighted) Restricted Maximal Operator relative to the cube Q_{2R}. Now, with the notation (2.2.1),

$$\begin{aligned}
&\left| \left\{ x \in Q \,:\, M^{**}_b \left[\left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} \right](x) > \frac{AB\lambda}{\lambda_0} \right\} \right| \leq \\
&\leq \left| \left\{ x \in Q \,:\, M^{**}_b \left[\left(\mu^2 + |Dw|^2\right)^{\frac{p}{2}} \right](x) > \frac{AB\lambda}{\lambda_0 8^p} \right\} \right| + \\
&\qquad + \left| \left\{ x \in Q \,:\, M^{**}_b \left[|Du - Dw|^p \right](x) > \frac{AB\lambda}{\lambda_0 8^p} \right\} \right| \leq \\
&\leq \frac{1}{\lambda_0} b \left(\left\{ x \in Q \,:\, M^{**}_b \left[\left(\mu^2 + |Dw|^2\right)^{\frac{p}{2}} \right](x) > \frac{AB\lambda}{\lambda_0 8^p} \right\} \right) + \\
&\qquad + \left| \left\{ x \in Q \,:\, M^{**} \left[|Du - Dw|^p \, b \right](x) > \frac{AB\lambda}{\lambda_0^2 8^p} \right\} \right| \leq \\
&\leq \frac{c(s, p, \lambda_0, \|b\|_*)}{(AB\lambda)^{\frac{s}{p}}} \int_{2Q} \left(\mu^2 + |Dw|^2\right)^{\frac{s}{2}} b \,\mathrm{d}x + \frac{c(n, p, \lambda_0)}{AB\lambda} \int_{2Q} |Du - Dw|^p b \,\mathrm{d}x,
\end{aligned} \tag{5.5.11}$$

where in the last inequality we used (2.2.2), Theorem 2.2.18 and Lemma 2.2.10. From (5.5.1) and (5.5.9) we get

$$\begin{aligned}
\frac{c(s, p, \lambda_0, \|b\|_*)}{(AB\lambda)^{\frac{s}{p}}} \int_{2Q} \left(\mu^2 + |Dw|^2\right)^{\frac{s}{2}} b \,\mathrm{d}x &\leq \tilde{c}(p, k, \lambda_0, n, \|b\|_*, \mathscr{D}_K) \frac{|Q|}{(AB)^{\frac{s}{p}}} \leq \\
&\leq \frac{1}{100^{n+2} B^{\frac{s}{p}}} |Q|,
\end{aligned} \tag{5.5.12}$$

where the last inequality is true provided we choose, for instance, $A := \left(100^{n+2}(\tilde{c} + 1)\right)^{\frac{p}{s}}$; this fixes the constant A and yields the absolute dependence mentioned in the statement. From (5.5.11) thanks to (5.5.10) and (5.5.12) we obtain

$$\left| \left\{ x \in Q \,:\, M^{**}_b \left[\left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} \right](x) > \frac{AB\lambda}{\lambda_0} \right\} \right| \leq \frac{|Q|}{100^{n+2} B^{\frac{s}{p}}} + c\frac{\varepsilon}{AB} |Q|, \tag{5.5.13}$$

with $c = c(n, p, k, \lambda_0)$. Now we can choose ε such that

$$c(n, p, k, \lambda_0) \frac{\varepsilon}{A} \leq \frac{1}{8 B^{\frac{s}{p} - 1}}$$

and so from (5.5.13)

$$\left| \left\{ x \in Q \,:\, M^{**}_b \left[\left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} \right](x) > \frac{AB\lambda}{\lambda_0} \right\} \right| \leq \frac{|Q|}{8^n B^{\frac{s}{p}}}. \tag{5.5.14}$$

To conclude we remark that (5.5.5) implies

$$M_b^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x)\le \max\left\{M_b^{**}\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x), 100^n\frac{\lambda}{\lambda_0}\right\} \tag{5.5.15}$$

for every $x\in Q$. Indeed, let $x_0\in Q$ and let $C\subset Q_{2R}$ be a cube such that $x_0\in C$. In the case when $C\subset 2Q$, by the definition of M^{**}, we trivially have

$$\frac{\int_C(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x}{b(C)}\le M_b^{**}\left[(\mu^2+|Du|^2)^{\frac{p}{2}}\right](x_0).$$

In the case when $C\not\subset 2Q$, we must have $2^n|C|\ge|Q|$, then $\tilde{Q}\subset 10C$ and in particular $\tilde{x}\in 10C$; at this point we further distinguish two cases. If $20C\subset Q_{2R}$ then, using (5.5.5), we obtain

$$\begin{aligned}\frac{\int_C(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x}{b(C)}&\le\frac{1}{\lambda_0}\fint_C(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x\le\frac{20^n}{\lambda_0}\fint_{20C}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x\\&\le\frac{20^n}{\lambda_0}M^*\left[(\mu^2+|Du|^2)^{\frac{p}{2}}b\right](\tilde{x})\le\frac{100^n}{\lambda_0}\lambda.\end{aligned}$$

If finally $20C\not\subset Q_{2R}$ then $Q_{2R}\subset 70C$ and therefore

$$\begin{aligned}\frac{\int_C(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x}{b(C)}&\le\frac{1}{\lambda_0}\fint_C(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x\le\frac{70^n}{\lambda_0}\fint_{20C}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x\\&\le\frac{70^n}{\lambda_0}M^*\left[(\mu^2+|Du|^2)^{\frac{p}{2}}b\right](\tilde{x})\le\frac{100^n}{\lambda_0}\lambda,\end{aligned}$$

so that (5.5.15) is completely proved. Since $AB>A>100^n$, by using (5.5.14) we get

$$\left|\left\{x\in Q\,:\,M_b^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x)>\frac{AB\lambda}{\lambda_0}\right\}\right|\le\frac{|Q|}{2B^{\frac{s}{p}}},$$

which is a contradiction to (5.5.3). The proof is complete. □

When applying Lemma 5.5.3 we shall need to fix the constant B, depending on the choice of an integrability exponent $q\in(p,s)$. With q being fixed, we do this in the following canonical way:

$$\frac{1}{B^{\frac{s-q}{p}}}=\frac{1}{2A^{\frac{q}{p}}}, \tag{5.5.16}$$

where the constant $A\equiv A(p,k,\lambda_0,n,\|b\|_*,\mathscr{D}_K)$ is the absolute constant appearing in Lemma 5.5.3. This fixes in turn $B\equiv B(p,k,\lambda_0,n,s-q,\|b\|_*,\mathscr{D}_K)$. Note that $B\nearrow\infty$ when $q\nearrow s$; consequently, in Lemma 5.5.3 $\varepsilon\searrow 0$ when $q\nearrow s$. Once the choice of B has been made, this canonically fixes the choice of ε, with the following absolute dependence

$$\varepsilon_0\equiv\varepsilon\equiv\varepsilon(p,k,\lambda_0,n,s-q,\|b\|_*,\mathscr{D}_K)>0. \tag{5.5.17}$$

Proof of Theorem 5.1.2. Following the notation of Lemma 5.5.3, let us set

$$\mu_1(t):=\left|\left\{x\in Q_R\,:\,M_b^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x)>t\right\}\right|,$$

$$\mu_2(t):=\left|\left\{x\in Q_R\,:\,M^*\left[\left(\mu^2+|G|^2\right)^{\frac{p}{2}}\right](x)>t\right\}\right|.$$

Then, with $B>1$ as in (5.5.16), we take

$$\tilde{\lambda}:=\frac{10^n}{\lambda_0}cB^{\frac{s}{p}}\fint_{Q_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,\mathrm{d}x,$$

where $c \equiv c(n)$ is the constant appearing in the weak type inequality (2.2.2) when $p \equiv 1$; note that $\tilde{\lambda}$ is positive. Therefore,

$$\begin{aligned} \mu_1(\tilde{\lambda}) &\leq \left|\left\{x \in Q_R \,:\, M^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}} b\right](x) > \tilde{\lambda}\lambda_0\right\}\right| \leq \\ &\leq \frac{c}{\tilde{\lambda}\lambda_0}\int_{Q_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}} b \,\mathrm{d}x < \frac{|Q_R|}{2B^{\frac{s}{p}}}, \end{aligned} \tag{5.5.18}$$

and consequently, since $AB > 1$,

$$\mu_1((AB)^h\tilde{\lambda}) < \frac{|Q_R|}{2B^{\frac{s}{p}}} \qquad \forall h \in \mathbb{N}, \tag{5.5.19}$$

where A is the constant appearing in Lemma 5.5.3. Next, we recall that the constant B has been chosen according to (5.5.16). With such a choice of B, and in view of (5.5.18) - (5.5.19), we can combine Lemma 5.5.3 and Lemma 5.5.2 at the levels $\lambda \equiv (AB)^h\tilde{\lambda}$, $h \in \mathbb{N}$. To this end, note that

$$\begin{aligned} &\left\{x \in Q_R \,:\, M_b^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x) > (AB)^h\tilde{\lambda}\right\} \subset \\ &\qquad \subset \left\{x \in Q_R \,:\, M^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}} b\right](x) > \tilde{\lambda}\lambda_0\right\} \end{aligned}$$

Therefore, an elementary induction argument leads to

$$\mu_1((AB)^{h+1}\tilde{\lambda}) \leq B^{-\frac{s}{p}(h+1)}\mu_1(\tilde{\lambda}) + \sum_{i=0}^{h} B^{-\frac{s}{p}(h-i)}\mu_2((AB)^i\varepsilon_0\tilde{\lambda})$$

for every $h \in \mathbb{N}$; the number ε_0 is defined in (5.5.17). Summing up over h, we have, for every $M \in \mathbb{N}$

$$\begin{aligned} \sum_{h=0}^{M}(AB)^{\frac{q}{p}(h+1)}\mu_1((AB)^{h+1}\tilde{\lambda}) &\leq \left(\sum_{h=0}^{M}\left[B^{-\frac{s}{p}}(AB)^{\frac{q}{p}}\right]^{h+1}\right)\mu_1(\tilde{\lambda}) + \\ &\quad + \sum_{h=0}^{M}\sum_{i=0}^{h}(AB)^{\frac{q}{p}(h+1)}B^{-\frac{s}{p}(h-i)}\mu_2((AB)^i\varepsilon_0\tilde{\lambda}). \end{aligned} \tag{5.5.20}$$

As for the first sum in the right-hand side, we notice that (5.5.16) leads to

$$\sum_{h=0}^{\infty}\left[B^{-\frac{s}{p}}(AB)^{\frac{q}{p}}\right]^{h+1} \leq 1.$$

Concerning the second sum appearing in the right-hand side of (5.5.20), we have

$$\begin{aligned} &\sum_{h=0}^{M}\sum_{i=0}^{h}(AB)^{\frac{q}{p}(h+1)}B^{-\frac{s}{p}(h-i)}\mu_2((AB)^i\varepsilon_0\tilde{\lambda}) = \\ &\qquad = (AB)^{\frac{q}{p}}\sum_{i=0}^{M}(AB)^{\frac{q}{p}i}\mu_2((AB)^i\varepsilon_0\tilde{\lambda})\sum_{h=0}^{M-i}\left[B^{-\frac{s}{p}}(AB)^{\frac{q}{p}}\right]^h \leq \\ &\qquad \leq 2(AB)^{\frac{q}{p}}\sum_{k=0}^{M}(AB)^{\frac{q}{p}k}\mu_2((AB)^k\varepsilon_0\tilde{\lambda}). \end{aligned}$$

Combining the previous estimates with (5.5.20) we finally obtain

$$\sum_{k=1}^{\infty}(AB)^{\frac{q}{p}k}\mu_1((AB)^k\tilde{\lambda}) \leq \mu_1(\tilde{\lambda}) + 2(AB)^{\frac{q}{p}}\sum_{k=0}^{\infty}(AB)^{\frac{q}{p}k}\mu_2((AB)^k\varepsilon_0\tilde{\lambda}).$$

Now we will do a straight, readable estimate, although it is justified only if read backwards: when the series in (5.5.22) will be shown to converge, we will have proved that the power q of the maximal function is integrable, which implies that also the first integral we are about to write is finite. We observe that

$$\begin{aligned}\int_{Q_R} (\mu^2+|Du|^2)^{\frac{q}{2}} b\,\mathrm{d}x &\le \int_{Q_R} \left| M_b^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x)\right|^{\frac{q}{p}} \mathrm{d}x = \\ &= \int_0^\infty \frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_1(\lambda)\,d\lambda = \\ &= \int_0^{\tilde\lambda} \frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_1(\lambda)\,d\lambda + \int_{\tilde\lambda}^\infty \frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_1(\lambda)\,d\lambda\end{aligned} \tag{5.5.21}$$

and

$$\begin{aligned}\int_0^{\tilde\lambda} \frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_1(\lambda)\,d\lambda &\le \tilde\lambda^{\frac{q}{p}}|Q_R| = c(n,\lambda_0)^{\frac{q}{p}} B^{\frac{s}{p^2}q}\left(\fint_{Q_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}} b\,\mathrm{d}x\right)^{\frac{q}{p}}|Q_R| \le \\ &\le c(n,\lambda_0)^{\frac{s}{p}} B^{\frac{s}{p^2}q}\left(\fint_{Q_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}} b\,\mathrm{d}x\right)^{\frac{q}{p}}|Q_R| = \\ &= c(p,k,\lambda_0,n,\|b\|_*,\mathscr{D}_K) B^{\frac{s}{p^2}q}\left(\fint_{Q_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}} b\,\mathrm{d}x\right)^{\frac{q}{p}}|Q_R|,\end{aligned}$$

where we assumed $c(n,\lambda_0)>1$. In a similar way we have

$$\int_{\tilde\lambda}^\infty \frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_1(\lambda)\,d\lambda = \sum_{n=0}^\infty \int_{(AB)^{\frac{n}{p}}\tilde\lambda}^{(AB)^{\frac{n+1}{p}}\tilde\lambda} q\lambda^{q-1}\mu_1(\lambda)\,d\lambda \le (AB\tilde\lambda)^{\frac{q}{p}}\sum_{n=0}^\infty (AB)^{\frac{nq}{p}}\mu_1((AB)^n\tilde\lambda).$$

Again,

$$\begin{aligned}(AB\tilde\lambda)^{\frac{q}{p}}\mu_1(\tilde\lambda) &\le (AB\tilde\lambda)^{\frac{q}{p}}\left|\left\{x\in Q_R : M^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}} b\right](x) > \tilde\lambda\lambda_0\right\}\right| \le \\ &\le c(n,\lambda_0)(AB)^{\frac{q}{p}}\tilde\lambda^{\frac{q}{p}-1}\int_{Q_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}} b\,\mathrm{d}x \le \\ &\le c(p,k,\lambda_0,n,\|b\|_*,\mathscr{D}_K)(A)^{\frac{s}{p}} B^{-\frac{s-q}{p}} B^{\frac{s}{p^2}q}|Q_{2R}|\left(\fint_{Q_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}} b\,\mathrm{d}x\right)^{\frac{q}{p}}.\end{aligned}$$

Joining the last three estimates to (5.5.21) yields

$$\begin{aligned}\int_{Q_R}(\mu^2+|Du|^2)^{\frac{q}{2}} b\,\mathrm{d}x &\le cB^{\frac{s}{p^2}q}\left(\fint_{Q_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}} b\,\mathrm{d}x\right)^{\frac{q}{p}}|Q_R| + \\ &\quad + (AB\tilde\lambda)^{\frac{q}{p}}\mu_1(\tilde\lambda) + (AB\tilde\lambda)^{\frac{q}{p}}\sum_{k=1}^\infty (AB)^{k\frac{q}{p}}\mu_1((AB)^k\tilde\lambda) \le \\ &\le cB^{\frac{s}{p^2}q}\left(\fint_{Q_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}} b\,\mathrm{d}x\right)^{\frac{q}{p}}|Q_R| + \\ &\quad + 2(AB\tilde\lambda)^{\frac{q}{p}}\mu_1(\tilde\lambda) + 2(AB\tilde\lambda)^{\frac{q}{p}}\sum_{k=0}^\infty (AB)^{k\frac{q}{p}}\mu_2((AB)^k\varepsilon_0\tilde\lambda) \le \\ &\le cB^{\frac{s}{p^2}q}\left(\fint_{Q_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}} b\,\mathrm{d}x\right)^{\frac{q}{p}}|Q_R| + cB^{\frac{2s}{p}}\tilde\lambda\sum_{k=0}^\infty (AB)^{k\frac{q}{p}}\mu_2((AB)^k\varepsilon_0\tilde\lambda),\end{aligned} \tag{5.5.22}$$

where $c=c(p,k,\lambda_0,n,s-q,\|b\|_*,\mathscr{D}_K)$. It remains to estimate the last series. To this aim, observe that, as before,

$$\begin{aligned}\int_{Q_R}\left|M^*\left[\left(\mu^2+|G|^2\right)^{\frac{p}{2}}\right](x)\right|^{\frac{q}{p}}\mathrm{d}x &= \int_0^\infty \frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_2(\lambda)\,d\lambda = \\ &= \int_0^{\varepsilon_0\tilde\lambda}\frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_2(\lambda)\,d\lambda + \int_{\varepsilon_0\tilde\lambda}^\infty \frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_2(\lambda)\,d\lambda.\end{aligned}$$

Then

$$\int_0^{\varepsilon_0\tilde{\lambda}} \frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_2(\lambda)\,d\lambda \geq (\varepsilon_0\tilde{\lambda})^{\frac{q}{p}}\mu_2(\varepsilon_0\tilde{\lambda}),$$

and

$$\begin{aligned}
\int_{\varepsilon_0\tilde{\lambda}}^{\infty} \frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_2(\lambda)\,d\lambda &= \sum_{k=0}^{\infty}\int_{(AB)^k\varepsilon_0\tilde{\lambda}}^{(AB)^{k+1}\varepsilon_0\tilde{\lambda}} \frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_2(\lambda)\,d\lambda \geq \\
&\geq \sum_{k=0}^{\infty}\mu_2((AB)^{k+1}\varepsilon_0\tilde{\lambda})\left[((AB)^{k+1}\varepsilon_0\tilde{\lambda})^{\frac{q}{p}} - ((AB)^k\varepsilon_0\tilde{\lambda})^{\frac{q}{p}}\right] = \\
&= (\varepsilon_0\tilde{\lambda})^{\frac{q}{p}}\sum_{k=0}^{\infty}(AB)^{(k+1)\frac{q}{p}}\mu_2((AB)^{k+1}\varepsilon_0\tilde{\lambda})\left[1-(AB)^{-\frac{q}{p}}\right] \geq \\
&\geq \frac{1}{2}(\varepsilon_0\tilde{\lambda})^{\frac{q}{p}}\sum_{k=1}^{\infty}(AB)^{k\frac{q}{p}}\mu_2((AB)^k\varepsilon_0\tilde{\lambda}).
\end{aligned}$$

Combining the last estimates with the maximal inequality we finally get

$$\begin{aligned}
&\frac{p}{2q}(\varepsilon_0\tilde{\lambda})^{\frac{q}{p}}\sum_{k=1}^{\infty}(AB)^{k\frac{q}{p}}\mu_2((AB)^k\varepsilon_0\tilde{\lambda}) + \frac{p(\varepsilon_0\tilde{\lambda})^{\frac{q}{p}}}{q}\mu_2(\varepsilon_0\tilde{\lambda}) \leq \\
&\leq \frac{p}{q}(\varepsilon_0\tilde{\lambda})^{\frac{q}{p}}\sum_{k=0}^{\infty}(AB)^{k\frac{q}{p}}\mu_2((AB)^k\varepsilon_0\tilde{\lambda}) \leq \frac{2p}{q}\int_{Q_R}\left|M^*\left[\left(\mu^2+|G|^2\right)^{\frac{p}{2}}\right](x)\right|^{\frac{q}{p}}\,\mathrm{d}x \leq \\
&\leq c(n,p,s,\|b\|_*)\int_{Q_{2R}}\left(\mu^2+|G|^2\right)^{\frac{q}{2}}\,\mathrm{d}x \leq c(n,p,s,\lambda_0,\|b\|_*)\int_{Q_{2R}}\left(\mu^2+|G|^2\right)^{\frac{q}{2}}b\,\mathrm{d}x.
\end{aligned}$$

Using this estimate in (5.5.22) and passing to averages we have

$$\begin{aligned}
\left(\fint_{Q_R}\left(\mu^2+|Du|^2\right)^{\frac{q}{2}}b\,\mathrm{d}x\right)^{\frac{1}{q}} &\leq c\left(\fint_{Q_{2R}}\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}b\,\mathrm{d}x\right)^{\frac{1}{p}} + \\
&\quad + c\left(\fint_{Q_{2R}}\left(\mu^2+|G|^2\right)^{\frac{q}{2}}b\,\mathrm{d}x\right)^{\frac{1}{q}},
\end{aligned}$$

□

5.6 Globality

The goal of this section is to prove global versions of the estimates in Theorem 5.1.1 and Theorem 5.1.2 when we consider solutions of the corresponding Dirichlet problem on a bounded domain $\Omega \subset \mathbb{R}^n$ with C^2 boundary. In the following we assume that

$$\Omega \subset\subset B_{2R} \subset Q_0,$$

where, without loss of generality, we suppose that the ball B_{2R} and the cube Q_0 are centered in the origin.
Let conditions (5.1.2) - (5.1.6) hold in $\overline{B}_{2R}$. Set $A(x,\xi) = 0$ for any $x \in \mathbb{R}^n \setminus Q_0$, we consider a standard mollifier $\varrho : \mathbb{R}^n \to [0,\infty)$ with compact support contained in $B_1 \subset \mathbb{R}^n$. If $0 < \varepsilon < \min\{R, 1\}$, for any $x \in B_{2R-\varepsilon}$ and $\xi \in \mathbb{R}^{N\times n}$ we consider

$$A_\varepsilon(x,\xi) := \int_{B_1} A(x+\varepsilon y,\xi)\,\varrho(y)\,dy,$$

$$K_\varepsilon(x) := \int_{B_1} K(x+\varepsilon y)\,\varrho(y)\,dy,$$

$$b_\varepsilon(x) := \int_{B_1} b(x+\varepsilon y)\,\varrho(y)\,dy,$$

$$F_\varepsilon(x) := \int_{B_1} F(x+\varepsilon y)\varrho(y)\, dy.$$

It is easy to verify that for a.e. $x, y \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^{N\times n}$

$$|A_\varepsilon(x,\xi) - A_\varepsilon(x,\eta)| \le k b_\varepsilon(x)|\xi-\eta|\,(\mu^2+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}, \tag{5.6.1}$$

$$\frac{1}{k} b_\varepsilon(x)|\xi-\eta|^2\,(\mu^2+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} \le \langle A_\varepsilon(x,\xi) - A_\varepsilon(x,\eta), \xi-\eta\rangle, \tag{5.6.2}$$

$$|A_\varepsilon(x,\eta) - A_\varepsilon(y,\eta)| \le |x-y|\,\left[K_\varepsilon(x)+K_\varepsilon(y)\right]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}}, \tag{5.6.3}$$

$$A_\varepsilon(x,0) = 0, \tag{5.6.4}$$

$$|b_\varepsilon(x) - b_\varepsilon(y)| \le |x-y|\,[K_\varepsilon(x)+K_\varepsilon(y)]. \tag{5.6.5}$$

Global differentiability The following lemma holds

Lemma 5.6.1. *Let U a bounded Lipschitz domain such that, if we denote by Q_r a cube centered in the origin and with side of length r and*

$$Q_r^+ = Q_r \cap \{x_n > 0\},$$

then

$$Q_{4d}^+ \subset U \subset Q_1^+,$$

with $d > 0$. Let $A_\varepsilon : U \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ satisfy assumptions (5.6.1) - (5.6.5) for $x \in U$, $p \ge 2$ and $b_\varepsilon \in L^\infty(U)$. Let $F \in W^{1,2}(b, U; \mathbb{R}^{N\times n})$. Consider the problem

$$\begin{cases} \operatorname{div} A_\varepsilon(x, Du_\varepsilon) = \operatorname{div} F_\varepsilon & \text{in } U \\ u_\varepsilon = 0 & \text{on } \partial U \cap \{x_n = 0\}. \end{cases} \tag{5.6.6}$$

If $\alpha_1 > 0$ is the constant, depending on p, n, λ_0, μ and k, in Theorem 5.1.1 and if

$$\mathscr{D}_K \equiv \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_1,$$

then

$$\int_{Q_{2d}^+} |D(V_\mu(Du_\varepsilon))|^2 b_\varepsilon\, \mathrm{d}x \le c \int_{Q_{4d}^+} \left(\left(1+\frac{1}{d^2}\right)(\mu^2+|Du_\varepsilon|^2)^{\frac{p}{2}} + (\mu^2+|DF_\varepsilon|^2)\right) b_\varepsilon, \tag{5.6.7}$$

for a constant c depending on p, k, λ_0, n, μ and $\mathscr{D}_K$.

Regarding the proof of Lemma 5.6.1, we refer to [DKM07, Theorem 2.3]. Actually, we can firstly repeat the proof of Theorem 5.1.1 by using the standard difference quotient method in the tangential directions. This allows to prove the existence of $D_s(V_\mu(Du_\varepsilon))$, $s = 1, \dots, n-1$, in L^2. Secondly, we can use the definition of (5.6.6) to bound the L^2-norm of $D_n(V_\mu(Du_\varepsilon))$ by the L^2-norm of the tangential derivatives.
Now let $u \in W_0^{1,p}(b, \Omega; \mathbb{R}^{N\times n})$ be the unique solution of the problem

$$\begin{cases} \operatorname{div} A(x, Du) = \operatorname{div} F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $F \in W^{1,2}(b, \Omega; \mathbb{R}^{N\times n})$. Then

Theorem 5.6.2. *There exists $\alpha_3 > 0$, depending on p, n, λ_0, μ, k and Ω, such that, if*

$$\mathscr{D}_K \equiv \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_3,$$

then $D(V_\mu(Du)) \in L^2(b, \Omega; \mathbb{R}^{N\times n})$ and

$$\int_\Omega |D(V_\mu(Du))|^2 b\, \mathrm{d}x \le c \int_\Omega \left((\mu^2+|Du|^2)^{\frac{p}{2}} + (\mu^2+|DF|^2)\right) b\, \mathrm{d}x, \tag{5.6.8}$$

where $c = c(p, k, \lambda_0, n, \mu, \mathscr{D}_K, \Omega)$.

Proof. Firstly we prove (5.6.8) when $b(x) \in L^\infty(\Omega)$ with a constant $c = c(p, k, \lambda_0, n, \mu, \mathscr{D}_K, \Omega)$. If $b(x) \in L^\infty(\Omega)$, let $A_\varepsilon(x, \xi)$ satisfy (5.6.1) - (5.6.5) and let u_ε be the unique solution of the system

$$\begin{cases} \operatorname{div} A_\varepsilon(x, Du_\varepsilon) = \operatorname{div} F_\varepsilon & \text{in} \quad \Omega \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

In a standard way (see for example [Cam87], [Gro02], [Ham07], [Min06], [MP21]), we cover Ω by a family of open sets $\Omega', \Omega'', U_1, \dots, U_m, V_1, \dots, V_m$ such that

- $\Omega' \subset\subset \Omega'' \subset\subset \Omega$;
- U_l, V_l are cubes centered in $x_l \in \partial\Omega$, with $l = 1, \dots m$;
- $V_l \subset\subset U_l$, with $l = 1, \dots, m$;
- $\cup_{l=1}^m V_l \supsetneq \partial\Omega$;
- $\Omega \subsetneq \cup_{l=1}^m V_l \cup \Omega'$.

Covering $\bar{\Omega}'$ by a finite number of balls, by Theorem 5.1.1 we have that

$$\int_{\Omega'} |D(V_\mu(Du_\varepsilon))|^2 b_\varepsilon \, dx \le c \int_\Omega \left((\mu^2 + |Du_\varepsilon|^2)^{\frac{p}{2}} + (\mu^2 + |DF_\varepsilon|^2) \right) b_\varepsilon \, dx,$$

with $c = c(p, k, \lambda_0, n, \mu, \mathscr{D}_K, \Omega)$. Regarding the boundary regularity of the solution, on every U_l we can consider a diffeomorphism Φ which maps $\Omega_l \equiv U_l \cap \Omega$ to an open set of $\mathbb{R}^n$ and such that

$$\Phi(U_l \cap \Omega) \subset \{y \in \mathbb{R}^n : y_n > 0\}, \qquad \Phi\{U_l \cap \partial\Omega\} \subset \{y \in \mathbb{R}^n : y_n = 0\}.$$

If $\tilde{u}_\varepsilon$ is such that $u_\varepsilon(x) = (\tilde{u}_\varepsilon \circ \Phi)(x)$, with $x \in U_l \cap \bar{\Omega}$, then $\tilde{u}_\varepsilon$ solves in $\tilde{\Omega}_l \equiv \Phi(U_l \cap \Omega)$ a system

$$\begin{cases} \operatorname{div} \tilde{A}_\varepsilon(x, D\tilde{u}_\varepsilon) = \operatorname{div} \tilde{F}_\varepsilon & \text{in} \quad \tilde{\Omega}_l \\ \tilde{u}_\varepsilon = 0 & \text{on} \quad \{y_n = 0\} \cap \partial\tilde{\Omega}_l, \end{cases}$$

where $\tilde{A}_\varepsilon$ satisfies conditions similar to (5.6.1) - (5.6.5) with new constants $\tilde{\lambda}_0$ and $\tilde{k}$ depending on Φ. This diffeomorphism preserves the BMO norm and the distance (see [Ast83], [BBC75]), that is

$$\|b_\varepsilon \circ \Phi\|_* \le \|b_\varepsilon\|_* \le \|b\|_*,$$

$$\mathscr{D}_{K_\varepsilon \circ \Phi} \le \mathscr{D}_{K_\varepsilon} \le \mathscr{D}_K$$

and we apply Lemma 5.6.1 in $\tilde{\Omega}_l$, giving (5.6.7) with a new constant $c = c(\tilde{\lambda}_0, \tilde{k}, p, n, \mu, \mathscr{D}_K)$. Coming back to the original variables and summing on l, we get that, for any $0 < \varepsilon < 1$, $|DV_\mu(Du_\varepsilon)| \in L^2(\Omega)$ and

$$\int_\Omega |DV_\mu(Du_\varepsilon)|^2 b_\varepsilon \, dx \le c \left(\int_\Omega (\mu^2 + |Du_\varepsilon|^2)^{\frac{p}{2}} b_\varepsilon \, dx + \int_\Omega (\mu^2 + |DF_\varepsilon|^2) b_\varepsilon \, dx \right). \tag{5.6.9}$$

Now we prove that $Du_\varepsilon \to Du$ in $L^p(b, \Omega; \mathbb{R}^{N\times n})$. From (5.6.2)

$$\begin{aligned} &\frac{1}{k} \int_\Omega (\mu^2 + |Du_\varepsilon|^2 + |Du|^2)^{\frac{p-2}{2}} |Du - Du_\varepsilon|^2 b_\varepsilon \, dx \le \\ &\le \int_\Omega \langle A_\varepsilon(x, Du) - A_\varepsilon(x, Du_\varepsilon), Du - Du_\varepsilon \rangle \, dx = \\ &= \int_\Omega \langle F - F_\varepsilon, Du - Du_\varepsilon \rangle \, dx + \int_\Omega \langle A_\varepsilon(x, Du) - A(x, Du), Du - Du_\varepsilon \rangle \, dx \le \\ &\le \frac{\nu}{p} \|Du - Du_\varepsilon\|_p^p + \frac{c}{\nu} \|F - F_\varepsilon\|_{p'}^{p'} + \frac{c}{\nu} \int_\Omega |A_\varepsilon(x, Du) - A(x, Du)|^{p'} \, dx. \end{aligned} \tag{5.6.10}$$

We remark that from (5.6.3) we deduce that $A_\varepsilon(x, Du) \to A(x, Du)$ a.e. Moreover (5.6.1) and (5.6.4) give

$$|A_\varepsilon(x, Du)|^{\frac{p}{p-1}} \le k^{\frac{p}{p-1}} \|b\|_{L^\infty}^{\frac{p}{p-1}} (\mu^2 + |Du|^2)^{\frac{p}{2}},$$

and by dominated convergence Theorem 2.1.11 we obtain that $A_\varepsilon(x, Du) \to A(x, Du)$ in $L^{\frac{p}{p-1}}$. Then, from (5.6.10) with a suitable choice of v, we get that $Du_\varepsilon \to Du$ in L^p. From (5.6.9) and the semicontinuity of the norm with respect to weak convergence, we get (5.6.8) for $b(x) \in L^\infty$.
Now let $A_j(x, \xi)$, $j \in \mathbb{N}$, be the operators defined in Section 5.4. We consider the problem

$$\begin{cases} \operatorname{div} A_j(x, Du_j) = \operatorname{div} F & \text{in} \quad \Omega \\ u_j = u & \text{on} \quad \partial\Omega, \end{cases} \tag{5.6.11}$$

Since (5.6.8) holds for $b_j(x)$, we get that for any $j \in \mathbb{N}$

$$\int_\Omega |DV_\mu(Du_j)|^2 b_j \,\mathrm{d}x \le c \left(\int_\Omega (\mu^2 + |Du_j|^2)^{\frac{p}{2}} b_j \,\mathrm{d}x + \int_\Omega (\mu^2 + |DF|^2) b \,\mathrm{d}x \right). \tag{5.6.12}$$

Now we prove that $Du_j \to Du$ in $L^p(b, \Omega; \mathbb{R}^{N\times n})$. From (5.1.3) we get

$$\begin{aligned} \frac{1}{k} \int_\Omega (\mu^2 + |Du|^2 + |Du_j|^2)^{\frac{p-2}{2}} |Du - Du_j|^2 b \,\mathrm{d}x &\le \\ \le \int_\Omega \left\langle A(x, Du) - A(x, Du_j), Du - Du_j \right\rangle \,\mathrm{d}x &= \\ = \int_\Omega \left\langle A_j(x, Du_j) - A(x, Du_j), Du - Du_j \right\rangle \,\mathrm{d}x &\le \\ \le k \int_\Omega \left(1 - \frac{b_j}{b}\right) (\mu^2 + |Du_j|^2)^{\frac{p-1}{2}} |Du - Du_j| b \,\mathrm{d}x &= \\ = k \int_\Omega (b - b_j)(\mu^2 + |Du_j|^2)^{\frac{p-1}{2}} |Du - Du_j| \,\mathrm{d}x&. \end{aligned}$$

Then from Young inequality we get

$$\begin{aligned} \int_\Omega |Du - Du_j| b \,\mathrm{d}x &\le c \int_\Omega (b - b_j)^{\frac{p}{p-1}} (\mu^2 + |Du_j|^2)^{\frac{p}{2}} \,\mathrm{d}x \le \\ &\le c \left(\int_\Omega (b - b_j)^r \,\mathrm{d}x \right)^{\frac{1}{2}} \cdot \left(\int_\Omega (\mu^2 + |Du_j|^2)^{\frac{np}{n-2}} \,\mathrm{d}x \right)^{\frac{n-2}{2n}}, \end{aligned} \tag{5.6.13}$$

where $r = \frac{p}{p-1} \cdot \frac{2n}{n+1}$. The last term goes to zero as $j \to +\infty$ thanks to (5.6.12), the embedding Sobolev Theorem and the convergence of b_j to b in every Lebesgue space L^q with $1 \le q < n$. Now, from (5.6.12), by using (5.6.11) and (5.4.1), we obtain that $\{|DV_\mu(Du_j)|\}$ is a bounded sequence in $L^2(b_j, \Omega)$. Then, by the semicontinuity of the norm with respect to the weak convergence, we get the result. □

Global integrability For $G \in L^p(b, Q_{2R}; \mathbb{R}^{N\times n})$, consider the problem

$$\begin{cases} \operatorname{div} A_\varepsilon(x, Du_\varepsilon) = \operatorname{div} |G_\varepsilon|^{p-2} G_\varepsilon & \text{in} \quad Q_{2R}^+ \\ u_\varepsilon = 0 & \text{on} \quad Q_{2R} \cap \{x_n = 0\}. \end{cases}$$

Lemma 5.6.3. *If $\alpha_2 > 0$ is the constant, depending on p, n, λ_0 and k, in Theorem 5.1.2, if*

$$\mathscr{D}_K \equiv \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_2$$

and if $G \in L^q(b, Q_{2R}; \mathbb{R}^{N\times n})$, for $q \in (p, s)$, then

$$\begin{aligned} \left(\fint_{Q_R^+} (\mu^2 + |Du_\varepsilon|^2)^{\frac{q}{2}} b_\varepsilon \,\mathrm{d}x \right)^{\frac{1}{q}} &\le c \left(\fint_{Q_{2R}^+} (\mu^2 + |Du_\varepsilon|^2)^{\frac{p}{2}} b_\varepsilon \,\mathrm{d}x \right)^{\frac{1}{p}} + \\ &+ c \left(\fint_{Q_{2R}^+} (\mu^2 + |G_\varepsilon|^2)^{\frac{q}{2}} b_\varepsilon \,\mathrm{d}x \right)^{\frac{1}{q}}, \end{aligned}$$

where $c = c(p, n, \lambda_0, k, \mathscr{D}_K, \|b\|_*)$.

Proof. We proceed as in the proof of Theorem 5.1.2. We consider in Lemma 5.5.3 the comparison map defined as the unique solution of the problem

$$\begin{cases} \operatorname{div} A_\varepsilon(x, Dw_\varepsilon) = 0 & \text{in} \quad 3Q^+ \\ w_\varepsilon - u_\varepsilon \in W_0^{1,p}(3Q^+; \mathbb{R}^N). \end{cases}$$

Moreover $M^*_{b_\varepsilon} \equiv M^*_{b_\varepsilon, Q^+_{2R}}$ and we continue arguing as in [KM06, Lemma 7.5]. □

Now we consider the problem

$$\begin{cases} \operatorname{div} A(x, Du) = \operatorname{div} |G|^{p-2} G & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

we prove the following

Theorem 5.6.4. *Let $u \in W^{1,p}(b, \Omega, \mathbb{R}^N)$ be the solution of (5.6). There exists $\alpha_4 > 0$, depending on p, n, λ_0, k and Ω, such that, if*

$$\mathscr{D}_K \equiv \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_2$$

and $G \in L^q(b, \Omega; \mathbb{R}^{N\times n})$, for $q \in (p, s)$, then

$$\left(\fint_\Omega (\mu^2 + |Du|^2)^{\frac{q}{2}} b \, dx \right)^{\frac{1}{q}} \le c \left(\fint_\Omega (\mu^2 + |G|^2)^{\frac{q}{2}} b \, dx \right)^{\frac{1}{q}},$$

where $c = c(p, n, \lambda_0, k, \mathscr{D}_K, \|b\|_*, \Omega)$.

Proof. Firstly, assuming $b(x) \in L^\infty(\Omega)$, we consider for any $0 < \varepsilon < 1$ the system

$$\begin{cases} \operatorname{div} A_\varepsilon(x, Du_\varepsilon) = \operatorname{div} |G_\varepsilon|^{p-2} G_\varepsilon & \text{in} \quad \Omega \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{5.6.14}$$

Following the lines of Theorem 5.6.2, by using Lemma 5.6.3, since u_ε solve the system (5.6.14), we obtain that

$$\begin{aligned} \int_\Omega (\mu^2 + |Du_\varepsilon|^2)^{\frac{q}{2}} b_\varepsilon \, dx \le c \int_\Omega (\mu^2 + |G_\varepsilon|^2)^{\frac{q}{2}} b_\varepsilon \, dx \le \\ \le c \int_\Omega (\mu^2 + |G|^2)^{\frac{q}{2}} b \, dx, \end{aligned} \tag{5.6.15}$$

where $c = c(p, n, k, \lambda_0, \mathscr{D}_K, \|b\|_*)$. Arguing as in (5.6.10) we get that $Du_\varepsilon \to Du$ in $L^p(b, \Omega; \mathbb{R}^{N\times n})$. Then the result follows by (5.6.15). In order to study the case $b(x) \in BMO$, as in Theorem 5.6.2 we consider for any $j \in \mathbb{N}$ the problems

$$\begin{cases} \operatorname{div} A_j(x, Du_j) = \operatorname{div} |G|^{p-2} G & \text{in} \quad \Omega \\ u_j = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and we apply (5.6.15) to get

$$\int_\Omega (\mu^2 + |Du_j|^2)^{\frac{q}{2}} b_j \, dx \le c \int_\Omega (\mu^2 + |G|^2)^{\frac{q}{2}} b \, dx. \tag{5.6.16}$$

As in Theorem 5.6.2 we get that $Du_j \to Du$ in $L^p(b, \Omega; \mathbb{R}^{N\times n})$. Indeed in (5.6.13) we have

$$\begin{aligned} \int_\Omega |Du - Du_j|^p b \, dx \le c \int_\Omega (b - b_j)^{\frac{p}{p-1}} \cdot (\mu^2 + |Du_j|^2)^{\frac{p}{2}} \, dx \le \\ \le c \left(\int_\Omega (b - b_j)^{\frac{pr}{p-1}} \, dx \right)^{\frac{1}{r}} \cdot \left(\int_\Omega (\mu^2 + |Du_j|^2)^{\frac{q}{2}} \, dx \right)^{\frac{p}{q}}, \end{aligned}$$

where $r = \frac{p}{p-q}$, and from (5.6.16) $\|Du_j - Du\| \to 0$ as $j \to +\infty$. Now, from the semicontinuity of the norm with respect to weak convergence, (5.6.16) gives the conclusion. □

5.7 A result on regular points

The aim of this section is to prove the following

Theorem 5.7.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 boundary. Let $u \in u_0 + W^{1,2}(b,\Omega;\mathbb{R}^N)$ be a weak solution to the Dirichlet problem*

$$\begin{cases} \operatorname{div} A(x,Du) = 0 & \text{in} \quad \Omega \\ u - u_0 \in W_0^{1,2}(b,\Omega;\mathbb{R}^N) \end{cases} \tag{5.7.1}$$

under the assumptions (5.1.2), (5.1.3), (5.1.4) and (5.1.5) and with $u_0 \in C^2\left(\overline{\Omega};\mathbb{R}^N\right)$. There exists $\tilde{\alpha} > 0$, depending on p, λ_0, μ, k, u_0 and Ω, such that, if

$$\mathcal{D}_K := \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \tilde{\alpha},$$

then $\mathcal{H}^{1+\varepsilon}$-almost every boundary point is a regular point for u for every $\varepsilon > 0$.

We recall the following

Proposition 5.7.2 ([Min03]). *Let $v \in W^{\vartheta,q}_{loc}(\Omega;\mathbb{R}^N)$ where $\vartheta \in (0,1]$, $q > 1$ and set*

$$A := \left\{ x \in \Omega \ : \ \limsup_{\varrho \searrow 0} \fint_{B_\varrho(x)} \left| v(y) - (v)_{B_\varrho(x)} \right|^q dy > 0 \right\},$$

$$B := \left\{ x \in \Omega \ : \ \limsup_{\varrho \searrow 0} \left| (v)_{B_\varrho(x)} \right| = \infty \right\}.$$

Then

$$\dim_{\mathcal{H}}(A) \le n - \vartheta q \quad \text{and} \quad \dim_{\mathcal{H}}(B) \le n - \vartheta q.$$

Now we are ready to prove Theorem 5.7.1.

Proof of Theorem 5.7.1. For a fixed $\gamma \in (0,1)$ to be determined later, let

$$\Omega_u^B := \left\{ x \in \partial\Omega \ : \ u \in C^{0,\gamma}\left(\overline{\Omega \cap A};\mathbb{R}^N\right) \text{ for some neighborhood } A \text{ of } x \right\}.$$

We shall denote by $\Sigma_u^B := \partial\Omega \setminus \Omega_u^B$ the set of singular boundary points of u; our aim is to prove that

$$\dim_{\mathcal{H}}(\Sigma_u^B) \le 1. \tag{5.7.2}$$

A standard flattening-of-the-boundary procedure allows us to reduce the study of problems of the type (5.7.1) to the study of those of the type

$$\begin{cases} \operatorname{div} A(x,Du) = 0 & \text{in} \quad Q_d^+ \\ u = u_0 & \text{on} \quad \Gamma_d, \end{cases} \tag{5.7.3}$$

where $\Gamma_R := Q_R \cap \{x \in \mathbb{R}^3 \ : \ x_3 = 0\}$ and $Q_R^+ := Q_R \cap \{x \in \mathbb{R}^3 \ : \ x_3 > 0\}$. We can locally flatten the boundary around any point $x_0 \in \partial\Omega$, with a C^2 chart (ϱ, C), whose regularity is determined by that of $\partial\Omega$, in such a way that $\varrho : Q_d \to C$, $\varrho(Q_d^+) = \Omega \cap C$, $\varrho(\Gamma_d) = \partial\Omega \cap C$ and $\varrho(0) = x_0$. The map $\tilde{u} := u \circ \varrho$ is then a solution of a problem of the type (5.7.3), for a new vector field $\tilde{A} : Q_d^+ \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$, satisfying the assumptions (5.1.2), (5.1.3), (5.1.4) and (5.1.5) for new values of k and λ_0. Since we can cover $\partial\Omega$ by a finite number of charts, these new values of k and λ_0 can be chosen independently of the chart (ϱ, C). It follows that $y \in \Gamma_d$ is a regular point of $D\tilde{u}$ if and only if $\varrho(y) \in \partial\Omega$ is a regular point for Du. Since the Hausdorff dimension is invariant under bi-Lipschitz transformations by Proposition 2.1.29, one checks by a standard covering argument that an estimate of Hausdorff dimension of the set of singular boundary points in $\partial\Omega$ follows from an analogous estimate of the singular points in Γ_d for a solution of a problem of the type considered in (5.7.3). With a little abuse of notation, such a solution will be denoted by the same letter u and its set of singular boundary points by Σ_u^B. Therefore it is sufficient to prove that

$$\dim_{\mathcal{H}}(\Sigma_u^B \cap \Gamma_d) \le 1, \tag{5.7.4}$$

for a fixed $d > 0$. The global estimate (5.7.2) is then recovered from the local estimates (5.7.4).
Moreover, a standard procedure allows to reduce the study of the Dirichlet problem (5.7.3) with non-zero Dirichlet data $u_0 \in C^2$ to the homogeneous case $u_0 \equiv 0$. Let $w := u - u_0 \in W_0^{1,2}$, then it is easy to check that w is a solution of the homogeneous Dirichlet problem

$$\begin{cases} \operatorname{div} \bar{A}(x, Dw) = 0 & \text{in} \quad Q_d^+ \\ w = 0 & \text{on} \quad \Gamma_d, \end{cases}$$

where

$$\bar{A}(x, \xi) := A(x, Du_0(x) + \xi)$$

for every $(x, \xi) \in \Omega \times \mathbb{R}^{N \times n}$. The vector field $\bar{A}$ satisfies the same growth conditions of the vector field A, with new constants depending on the initial ones and the data u_0. Up to a further flattening of the domain, since $u_0 \in C^2$, we can also assume that $A(x, 0) = 0$. For this reason we can assume without loss of generality that $u_0 \equiv 0$ and, by a little abuse of notation, we denote with same letter u such a solution.

Therefore it remains to prove (5.7.4). Let us denote $\Omega := Q_d^+ \cup \Gamma_d$. We shall estimate

$$U(x, \varrho) = \fint_{B_\varrho(x) \cap \Omega} \left| u - (u)_{B_\varrho(x_0) \cap \Omega} \right|^2 b \, \mathrm{d}x.$$

The average is made with respect to the measure

$$b(E) := \int_E b \, \mathrm{d}x,$$

that is, if $E \subset \mathbb{R}^n$ and $f \in L^1(b, E)$, then

$$(f)_E := \fint_E f b \, \mathrm{d}y = \frac{1}{b(E)} \int_E f b \, \mathrm{d}y.$$

By weighted imbedding Theorem 2.2.11 and weighted Sobolev – Poincaré inequality 2.2.12 with $k = 1$ and $p = 2$, if $\varrho < \frac{R}{4}$ by Proposition 5.4.1 we have

$$\begin{aligned} \int_{B_\varrho(x) \cap \Omega} \left| u - (u)_{B_\varrho(x_0) \cap \Omega} \right|^2 b \, \mathrm{d}x &\le c(\|b\|_*) \varrho^2 \int_{B_\varrho(x) \cap \Omega} |\nabla u|^2 b \, \mathrm{d}x \\ &\le c(\|b\|_*) \varrho^4 \int_{B_\varrho(x) \cap \Omega} |\nabla^2 u|^2 b \, \mathrm{d}x \le c(\|b\|_*) \varrho^4 \int_{B_{\frac{R}{4}}(x) \cap \Omega} |\nabla^2 u|^2 b \, \mathrm{d}x \\ &\le c(p, k, \lambda_0, R, \|b\|_*, \mathscr{D}_K) \varrho^4 \int_{B_{\frac{R}{2}}(x) \cap \Omega} |\nabla u|^2 b \, \mathrm{d}x \\ &\le c(p, k, \lambda_0, R, \|b\|_*, \mathscr{D}_K) \left(\frac{\varrho}{R} \right)^4 \int_{B_{\frac{R}{2}}(x) \cap \Omega} |\nabla u|^2 b \, \mathrm{d}x. \end{aligned}$$

By (5.2.14) with $\varepsilon = 0$ we deduce

$$\int_{B_\varrho(x) \cap \Omega} \left| u - (u)_{B_\varrho(x_0) \cap \Omega} \right|^2 b \, \mathrm{d}x \le c \left(p, k, \lambda_0, R, \|b\|_*, \mathscr{D}_K \right) \left(\frac{\varrho}{R} \right)^4 \int_{B_R(x) \cap \Omega} \left| u - (u)_{B_R(x_0) \cap \Omega} \right|^2 b \, \mathrm{d}x.$$

In particular, if $U(x_0, R) < \tilde{\varepsilon}$ with $x_0 \in \Gamma_d$ then

$$U(x, \varrho) \le c \left(p, k, \lambda_0, R, \|b\|_*, \mathscr{D}_K \right) \tilde{\varepsilon} \left(\frac{\varrho}{R} \right)^4$$

for every x in a neighborhood B of x_0, so that by [Cam63, Teorema I.2] u is Hölder-continuous in B with a suitable exponent γ. By construction we have that

$$\Sigma_u^B \subset \left\{ x \in \Omega \ : \ \liminf_{\varrho \searrow 0} \fint_{B_\varrho(x) \cap \Omega} \left| u(y) - (u)_{B_\varrho(x) \cap \Omega} \right|^2 b \, \mathrm{d}y > 0 \text{ or } \limsup_{\varrho \searrow 0} \left| (u)_{B_\varrho(x) \cap \Omega} \right| = \infty \right\}. \tag{5.7.5}$$

By (5.7.5) the conclusion (5.7.4) easily follows. Indeed, since $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ then we use Proposition 5.7.2 to deduce that

$$\dim_{\mathcal{H}} \left(\Sigma_u^B \cap \Gamma_d \right) \le n - 2 = 1. \tag{5.7.6}$$

□

Appendix

In this section we prove (5.4.3). Assume for the moment that $A(x,\eta) \geq 0$ and $A(y,\eta) \geq 0$. We note that

- If $b(x) \leq j$ and $b(y) \leq j$, then

$$
\begin{aligned}
|A_j(x,\eta) - A_j(y,\eta)| = |A(x,\eta) - A(y,\eta)| \leq \\
\leq |x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \leq \\
\leq (k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}}.
\end{aligned}
$$

- If $b(y) \geq b(x) > j$, then

$$
\begin{aligned}
&-(k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \leq \\
&\leq -\frac{j}{b(y)}\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \leq \\
&\leq \frac{j}{b(y)}\,[A(x,\eta)-A(y,\eta)] \leq A_j(x,\eta) - A_j(y,\eta) = \\
&= \frac{j}{b(x)}\,[A(x,\eta)-A(y,\eta)] + \left(\frac{j}{b(x)} - \frac{j}{b(y)}\right) A(y,\eta) \leq \\
&\leq \frac{j}{b(x)}\,|x-y|\,[K(x)+K(y)](\mu^2+|\eta|^2)^{\frac{p-1}{2}} + \\
&+ \frac{j}{b(x)} k\,|x-y|\,[K(x)+K(y)](\mu^2+|\eta|^2)^{\frac{p-1}{2}} \leq \\
&\leq (k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}}.
\end{aligned}
$$

- If $b(x) > b(y) > j$, then

$$
\begin{aligned}
&-(k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \leq \\
&\leq -\frac{j}{b(x)}(k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} = \\
&= -\frac{j}{b(x)}|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} + \\
&- \frac{j}{b(x)} k\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \leq \\
&\leq \frac{j}{b(x)}\,[A(x,\eta)-A(y,\eta)] - j\left(\frac{b(x)-b(y)}{b(y)b(x)}\right) A(y,\eta) = \\
&= A_j(x,\eta) - A_j(y,\eta) \leq \frac{j}{b(y)}\,[A(x,\eta)-A(y,\eta)] \leq \\
&\leq (k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}}.
\end{aligned}
$$

- If $b(y) > j \geq b(x)$ then

$$
\begin{aligned}
&-(k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \leq \\
&\leq -\frac{j}{b(y)}|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \leq \\
&\leq \frac{j}{b(y)}\,[A(x,\eta)-A(y,\eta)] \leq A_j(x,\eta) - A_j(y,\eta) = \\
&= A(x,\eta) - A(y,\eta) + \left(1 - \frac{j}{b(y)}\right) A(y,\eta) \leq \\
&\leq |x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} + \left(\frac{b(y)-b(x)}{b(y)}\right) k b(y)(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
&\leq (k+1)|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}}.
\end{aligned}
$$

- If $b(x) > j \geq b(y)$ then

$$
\begin{aligned}
&-(k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \leq \\
&\leq -\frac{j}{b(x)}\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} + \\
&-\frac{j}{b(x)}k\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \leq \\
&\leq -\frac{j}{b(x)}\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} + \\
&-\frac{b(y)}{b(x)}k|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \leq \\
&\leq \frac{j}{b(x)}\,[A(x,\eta)-A(y,\eta)] - \left(\frac{b(x)-b(y)}{b(x)}\right) A(y,\eta) = \\
&= A_j(x,\eta) - A_j(y,\eta) \leq [A(x,\eta)-A(y,\eta)] \leq \\
&\leq (k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}}.
\end{aligned}
$$

The proof of the remaining cases is analogous, therefore (5.4.3) is proved.

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