Second-Order Γ-Limit for the Cahn–Hilliard Functional with Dirichlet Boundary Conditions, II

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Abstract

This paper continues the study of the asymptotic development of order 2 by Γ-convergence of the Cahn–Hilliard functional with Dirichlet boundary conditions initiated in [7]. While in the first paper, the Dirichlet data are assumed to be well separated from one of the two wells, here this is no longer the case. In the case where there are no interfaces, it is shown that there is a transition layer near the boundary of the domain.

1 Introduction

In a recent paper [7] we began the study of the second-order asymptotic development via Γ-convergence of the Cahn-Hilliard functional

$$F_\varepsilon(u) := \int_\Omega (W(u) + \varepsilon^2 |\nabla u|^2)\, dx, \quad u \in H^1(\Omega), \tag{1.1}$$

subject to the Dirichlet boundary condition

$$\operatorname{tr} u = g_\varepsilon \quad \text{on } \partial\Omega. \tag{1.2}$$

Here $W : \mathbb{R} \to [0, \infty)$ is a double-well potential with

$$W^{-1}(\{0\}) = \{a, b\}, \tag{1.3}$$

$\Omega \subset \mathbb{R}^N$ is an open, bounded set with a smooth boundary, $N \geq 2$, and $g_\varepsilon \in H^{1/2}(\partial\Omega)$.

We recall that, given a metric space X and a family of functions $\mathcal{F}_\varepsilon : X \to [-\infty, \infty]$ for $\varepsilon > 0$, *the asymptotic development of order n via Γ-convergence*, written as

$$\mathcal{F}_\varepsilon = \mathcal{F}^{(0)} + \varepsilon\mathcal{F}^{(1)} + \cdots + \varepsilon^n\mathcal{F}^{(n)} + o(\varepsilon^n), \tag{1.4}$$

holds if we can find functions $\mathcal{F}^{(i)} : X \to [-\infty, \infty]$, $i = 0, \ldots, n$, such that the functions

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf_X \mathcal{F}^{(i-1)}}{\varepsilon}$$

are well-defined and the family $\{\mathcal{F}_\varepsilon^{(i)}\}_\varepsilon$ Γ-converges to $\mathcal{F}^{(i)}$ as $\varepsilon \to 0^+$ (see [1] and [2]). In many cases, the powers ε^k in the asymptotic development (1.4) may be replaced by more general scales, where $\delta_\varepsilon^{(i)} > 0$ for all $i = 1, \ldots, m$ and $\varepsilon > 0$, $\delta_\varepsilon^{(0)} := 1$ and $\sigma_\varepsilon^{(i)} := \delta_\varepsilon^{(i)}/\delta_\varepsilon^{(i-1)} \to 0$ as $\varepsilon \to 0^+$ for all $i = 1, \ldots, m$, and the asymptotic expansion takes the form:

$$\mathcal{F}_\varepsilon = \mathcal{F}^{(0)} + \delta_\varepsilon^{(1)}\mathcal{F}^{(1)} + \cdots + \delta_\varepsilon^{(n)}\mathcal{F}^{(n)} + o(\delta_\varepsilon^{(n)}),$$

where the functions $\mathcal{F}_\varepsilon^{(i)}$ are defined by

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf_X \mathcal{F}^{(i-1)}}{\sigma_\varepsilon^{(i)}}.$$

The first order asymptotic development of (1.1), (1.2) was studied by Owen, Rubinstein, and Sternberg [14] (see also [6] and [9]), who proved that the family of functionals

$$\mathcal{F}_\varepsilon^{(1)}(u) = \begin{cases} \int_\Omega (\frac{1}{\varepsilon}W(u) + \varepsilon|\nabla u|^2)\,dx & \text{if } u \in H^1(\Omega), \operatorname{tr} u = g_\varepsilon \text{ on } \partial\Omega, \\ \infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

Γ-converges as $\varepsilon \to 0^+$ in $L^1(\Omega)$ to

$$\mathcal{F}^{(1)}(u) := \begin{cases} C_W\,\mathrm{P}(\{u = b\}; \Omega) + \int_{\partial\Omega} \mathrm{d}_W(\operatorname{tr} u, g)\,d\mathcal{H}^{N-1} & \text{if } u \in BV(\Omega; \{a, b\}), \\ \infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{1.5}$$

where $\mathrm{P}(\{u = b\}; \Omega)$ is the perimeter of the set $\{u = b\}$ in Ω, $g_\varepsilon \to g$ in $L^1(\partial\Omega)$, d_W is the geodesic distance determined, to be precise, by W:

$$\mathrm{d}_W(r, s) := \begin{cases} 2\left|\int_r^s W^{1/2}(\rho)\,d\rho\right| & \text{if } r \in \{a, b\} \text{ or } s \in \{a, b\}, \\ \infty & \text{otherwise}, \end{cases} \tag{1.6}$$

and

$$C_W := 2\int_a^b W^{1/2}(\rho)\,d\rho. \tag{1.7}$$

In [7], we studied the second-order asymptotic expansion of (1.1), (1.2) under the hypothesis that the boundary data $g_\varepsilon : \overline{\Omega} \to \mathbb{R}$ stay away from one of the two wells a, b:

$$a < \alpha_- \leq g_\varepsilon(x) \leq b \tag{1.8}$$

for all $x \in \overline{\Omega}$, all $\varepsilon \in (0,1)$, and some constant α_-. If the constant α_- is sufficiently close to b, the only minimizer of $\mathcal{F}^{(1)}$ is the constant function b (see [7, Proposition 2.5]). Hence, it is natural to assume that

$$u_0 \equiv b \quad \text{is the unique minimizer of } \mathcal{F}^{(1)}. \tag{1.9}$$

Under this hypothesis, we define

$$\begin{aligned} \mathcal{F}_\varepsilon^{(2)}(u) &:= \frac{\mathcal{F}_\varepsilon^{(1)}(u) - \min \mathcal{F}^{(1)}}{\varepsilon} \\ &= \int_\Omega \left(\frac{1}{\varepsilon^2} W(u) + |\nabla u|^2 \right) dx - \frac{1}{\varepsilon} \int_{\partial\Omega} \mathrm{d}_W(b, g)\, d\mathcal{H}^{N-1} \end{aligned} \tag{1.10}$$

if $u \in H^1(\Omega)$ and $\operatorname{tr} u = g_\varepsilon$ on $\partial\Omega$, and $\mathcal{F}_\varepsilon^{(2)}(u) := \infty$ otherwise in $L^1(\Omega)$.

The main result in [7] is the following theorem.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected set with a boundary of class $C^{2,d}$, $0 < d \leq 1$. Assume that W satisfies (2.1)-(2.4) and that $g_\varepsilon \in H^1(\partial\Omega)$ is such that*

$$(\varepsilon|\log\varepsilon|)^{1/2} \int_{\partial\Omega} |\nabla_\tau g_\varepsilon|^2 d\mathcal{H}^{N-1} = o(1) \quad \text{as } \varepsilon \to 0^+,$$

and

$$|g_\varepsilon(x) - g(x)| \leq C\varepsilon^\gamma, \quad x \in \partial\Omega,$$

for all $\varepsilon \in (0,1)$ and for some constants $C > 0$ and $\gamma > 1$. Suppose also that (1.9) holds. Then

$$\mathcal{F}^{(2)}(u) = \int_{\partial\Omega} \kappa(y) \int_0^\infty 2W^{1/2}(z_{g(y)}(s)) z'_{g(y)}(s) s\, ds\, d\mathcal{H}^{N-1}(y) \tag{1.11}$$

if $u = b$ and $\mathcal{F}^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$, where κ is the mean curvature of $\partial\Omega$ and $z_{g(y)}$ is the solution to the Cauchy problem

$$\begin{cases} z'_\alpha = W^{1/2}(z_\alpha), \\ z_\alpha(0) = \alpha \end{cases} \tag{1.12}$$

with $\alpha = g(y)$.

Here, ∇_τ denotes the tangential gradient.

In the present paper, we relax the bound from below in (1.8) and allow g_ε to take the value a,

$$a \leq g_\varepsilon(x) \leq b, \tag{1.13}$$

while still assuming (1.9). We observe that this scenario can only happen if $\{g=a\} \subseteq \{\kappa \leq 0\}$ (see Theorem 4.1). If we assume that

$$\{g=a\} \subseteq \{\kappa < 0\}, \tag{1.14}$$

then the rescaling (1.10) should be replaced by

$$\begin{aligned} \mathcal{F}_\varepsilon^{(2)}(u) &:= \frac{\mathcal{F}_\varepsilon^{(1)}(u) - \min \mathcal{F}^{(1)}}{\varepsilon|\log\varepsilon|} \\ &= \frac{1}{\varepsilon|\log\varepsilon|}\int_\Omega \left(\frac{1}{\varepsilon}W(u) + \varepsilon|\nabla u|^2\right) dx - \frac{1}{\varepsilon|\log\varepsilon|}\int_{\partial\Omega} \mathrm{d}_W(b,g)\, d\mathcal{H}^{N-1} \end{aligned} \tag{1.15}$$

if $u \in H^1(\Omega)$ and $\operatorname{tr} u = g_\varepsilon$ on $\partial\Omega$, and $\mathcal{F}_\varepsilon^{(2)}(u) := \infty$ otherwise in $L^1(\Omega)$.

The main result of this paper is the following theorem.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected set with boundary of class $C^{2,d}$, $0 < d \leq 1$. Assume that W satisfies* (2.1)-(2.4) *and that g_ε satisfy* (1.13), (1.14), (2.14)-(2.16)*. Suppose also that* (1.9) *holds. Then*

$$\mathcal{F}^{(2)}(u) = \frac{C_W}{2^{1/2}(W''(a))^{1/2}}\int_{\partial\Omega\cap\{g=a\}} \kappa(y)\, d\mathcal{H}^{N-1}(y)$$

if $u = b$ and $\mathcal{F}^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$. Here, $\mathcal{F}^{(2)}$ is defined in (1.15)*, κ is the mean curvature of $\partial\Omega$, and C_W is the constant defined in* (1.7)*.*

In particular, if $u_\varepsilon \in H^1(\Omega)$ is a minimizer of (1.1) *subject to the Dirichlet boundary condition* (1.2)*, then*

$$\begin{aligned} \int_\Omega (W(u_\varepsilon) + \varepsilon^2|\nabla u_\varepsilon|^2)\, dx &= \varepsilon \int_{\partial\Omega} \mathrm{d}_W(b,g)\, d\mathcal{H}^{N-1} \\ &+ \varepsilon^2|\log\varepsilon|\frac{C_W}{2^{1/2}(W''(a))^{1/2}}\int_{\partial\Omega\cap\{g=a\}} \kappa(y)\, d\mathcal{H}^{N-1}(y) + o(\varepsilon^2|\log\varepsilon|). \end{aligned} \tag{1.16}$$

When the Dirichlet boundary conditions (1.2) are replaced by the mass constraint

$$\int_\Omega u(x)\, dx = m, \tag{1.17}$$

the first-order asymptotic expansion of the Cahn-Hilliard functional (1.1) was characterized in [3], [8], [13], [12], [15], while the second order asymptotic expansion was first proved by the third author and Murray in [10], [11] in dimension $N \geq 2$ (see also [4]).

As in [10], [11], our proof relies on the asymptotic development of order two by Γ-convergence of the weighted one-dimensional functional

$$G_\varepsilon(v) := \int_0^T (W(v(t)) + \varepsilon^2(v'(t))^2)\omega(t)\, dt, \quad v \in H^1(I), \tag{1.18}$$

subject to the Dirichlet boundary conditions

$$v(0) = \alpha_\varepsilon, \quad v(T) = \beta_\varepsilon, \tag{1.19}$$

where ω is a smooth positive weight, and

$$a \le \alpha_\varepsilon,\, \beta_\varepsilon \le b. \tag{1.20}$$

The key difference in our proof of the Γ-liminf inequality is that in [10], [11], the authors utilized a rearrangement technique based on the isoperimetric function to reduce the functional (1.1) to the one-dimensional weighted problem. This approach, however, seems to be difficult to implement in this new context except under trivial boundary conditions. Instead, we adopt techniques from Sternberg and Zumbrum [16] and Caffarelli and Cordoba [5] to analyze the behavior of minimizers of (1.1) and (1.2) near the boundaryy, leveraging slicing arguments in our study.

This paper is organized as follows. In Section 3, we characterize the asymptotic development of order two by Γ-convergence of the weighted one-dimensional family of functionals G_ε defined in (1.18). Section 4 explores qualitative properties of critical points and minimizers of the functional 1.1. Finally, in Section 5, we prove Theorem 1.2.

2 Preliminaries

We assume that the double-well potential $W : \mathbb{R} \to [0, \infty)$ satisfies the following hypotheses:

W is of class $C^{2,\alpha_0}(\mathbb{R})$, $\alpha_0 \in (0,1)$, and has precisely two zeros at a and b, with $a < b$, (2.1)

$$W''(a) > 0, \quad W''(b) > 0, \tag{2.2}$$

$$\lim_{s\to-\infty} W'(s) = -\infty, \quad \lim_{s\to\infty} W'(s) = \infty, \tag{2.3}$$

W' has exactly 3 zeros at a, b, c with $a < c < b$, $W''(c) < 0$, (2.4)

Let

$$a < \alpha_- < \min\left\{c, \frac{a+b}{2}\right\} \le \max\left\{c, \frac{a+b}{2}\right\} < \beta_- < b. \tag{2.5}$$

Remark 2.1 *Since $W \in C^2(\mathbb{R})$, $W(a) = W'(a) = 0$, $W(b) = W'(b) = 0$, and $W''(a)$, $W''(b) > 0$, there exists a constant $\sigma > 0$ depending on α_- and β_- such that*

$$\sigma^2(b-s)^2 \le W(s) \le \frac{1}{\sigma^2}(b-s)^2 \quad \textit{for all } \alpha_- \le s \le b+1, \tag{2.6}$$

$$\sigma^2(s-a)^2 \le W(s) \le \frac{1}{\sigma^2}(s-a)^2 \quad \textit{for all } a-1 \le s \le \beta_-. \tag{2.7}$$

Proposition 2.2 *For $a < \beta < \beta_-$, we have*

$$\lim_{\varepsilon\to 0^+} \frac{\int_a^\beta \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds}{|\log\varepsilon|} = \frac{1}{2^{1/2}(W''(a))^{1/2}}, \tag{2.8}$$

while for $a < \alpha < b$,

$$\lim_{\varepsilon\to 0^+} \frac{\int_\alpha^b \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds}{|\log\varepsilon|} = \frac{1}{2^{1/2}(W''(b))^{1/2}}.$$

In particular, there exists a constant $C > 0$ depending only on W such that

$$\int_a^b \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds \le C|\log\varepsilon| \tag{2.9}$$

for all $0 < \varepsilon < \varepsilon_0$, where $\varepsilon_0 > 0$ depends only W.

Proof. Step 1: Given $c_0 > 0$, we estimate

$$\mathcal{A} := \int_a^\beta \frac{1}{(\varepsilon + c_0(s-a)^2)^{1/2}}\,ds.$$

Consider the change of variables $\frac{\varepsilon^{1/2}}{c_0^{1/2}}t := s-a$, so that $\frac{\varepsilon^{1/2}}{c_0^{1/2}}dt = ds$. Then

$$\begin{aligned}\mathcal{A} &= \frac{1}{c_0^{1/2}}\int_0^{(\beta-a)c_0^{1/2}/\varepsilon^{1/2}} \frac{1}{(1+t^2)^{1/2}}\,dt = \frac{1}{c_0^{1/2}}[\log(t+(t^2+1)^{1/2})]_0^{(\beta-a)c_0^{1/2}/\varepsilon^{1/2}}\\ &= \frac{1}{c_0^{1/2}}\log\left((\beta-a)c_0^{1/2}/\varepsilon^{1/2} + ((\beta-a)^2c_0/\varepsilon+1)^{1/2}\right)\\ &= \frac{1}{2c_0^{1/2}}|\log\varepsilon| + \frac{1}{c_0^{1/2}}\log\left((\beta-a)c_0^{1/2} + ((\beta-a)^2c_0+\varepsilon)^{1/2}\right).\end{aligned}$$

Step 2: By (2.1) and (2.2), given $0 < \eta << 1$, we can find $0 < \delta_\eta < \alpha_- - a$ such that

$$\frac{1}{2}(1-\eta)W''(a)(s-a)^2 \le W(s) \le \frac{1}{2}(1+\eta)W''(a)(s-a)^2$$

for all $a \le s < a+\delta_\eta$. Hence,

$$\frac{1}{(\varepsilon+c_1(s-a)^2)^{1/2}} \le \frac{1}{(\varepsilon+W(s))^{1/2}} \le \frac{1}{(\varepsilon+c_2(s-a)^2)^{1/2}} \tag{2.10}$$

for all $a \le s < a+\delta_\eta$, where

$$c_1 = \frac{1}{2}(1+\eta)W''(a), \quad c_2 = \frac{1}{2}(1-\eta)W''(a). \tag{2.11}$$

Write

$$\int_a^\beta \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds = \int_a^{a+\delta_\eta} \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds + \int_{a+\delta_\eta}^\beta \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds.$$

Since $\min_{[a+s_\eta,\beta]} W = w_0 > 0$,

$$\frac{\int_{a+\delta_\eta}^\beta \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds}{|\log\varepsilon|} \le \frac{\frac{1}{w_0^{1/2}}(b-a)}{|\log\varepsilon|} \to 0 \tag{2.12}$$

as $\varepsilon \to 0^+$.

Using (2.10) with $\beta = a+\delta_\eta$ and c_0 replaced by c_1 and c_2, respectively,

$$\begin{aligned} &\frac{1}{2c_1^{1/2}}|\log\varepsilon| + \frac{1}{c_1^{1/2}}\log\left(\delta_\eta c_1^{1/2} + (\delta_\eta^2 c_1+\varepsilon)^{1/2}\right) \\ &\le \int_a^{a+\delta_\eta} \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds \\ &\le \frac{1}{2c_2^{1/2}}|\log\varepsilon| + \frac{1}{c_2^{1/2}}\log\left(\delta_\eta c_0^{1/2} + (\delta_\eta^2 c_0+\varepsilon)^{1/2}\right) \end{aligned}$$

Dividing by $|\log\varepsilon|$ and letting $\varepsilon \to 0^+$, we get

$$\frac{1}{2c_1^{1/2}} \le \liminf_{\varepsilon\to 0^+} \frac{\int_a^{a+\delta_\eta} \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds}{|\log\varepsilon|} \le \limsup_{\varepsilon\to 0^+} \frac{\int_a^{a+\delta_\eta} \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds}{|\log\varepsilon|} \le \frac{1}{2c_2^{1/2}}.$$

In turn, by (2.12),

$$\frac{1}{2c_1^{1/2}} \le \liminf_{\varepsilon\to 0^+} \frac{\int_a^{\beta} \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds}{|\log\varepsilon|} \le \limsup_{\varepsilon\to 0^+} \frac{\int_a^{\beta} \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds}{|\log\varepsilon|} \le \frac{1}{2c_2^{1/2}}.$$

Letting $\eta \to 0^+$ gives

$$\lim_{\varepsilon\to 0^+} \frac{\int_a^{\beta} \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds}{|\log\varepsilon|} = \frac{1}{2^{1/2}(W''(a))^{1/2}}.$$

Similarly, one can show that for every $a < \alpha < b$,

$$\lim_{\varepsilon\to 0^+} \frac{\int_\alpha^{b} \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds}{|\log\varepsilon|} = \frac{1}{2^{1/2}(W''(b))^{1/2}}.$$

Hence,

$$\begin{aligned} \frac{\int_a^{b} \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds}{|\log\varepsilon|} &= \frac{\int_c^{b} \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds}{|\log\varepsilon|} + \frac{\int_a^{c} \frac{1}{(\varepsilon+W(s))^{1/2}}\,ds}{|\log\varepsilon|} \\ &\to \frac{1}{2^{1/2}(W''(a))^{1/2}} + \frac{1}{2^{1/2}(W''(b))^{1/2}}. \end{aligned}$$

The inequality (2.9) now follows. ■

For the proof of the following proposition, we refer to [7, Proposition 2.3].

Proposition 2.3 *Let $a \leq \alpha_\varepsilon \leq \beta_\varepsilon \leq b$. Then, there exists a constant $C > 0$ such that*

$$\int_{\alpha_\varepsilon}^{\beta_\varepsilon} \left[\frac{2}{(\delta + W(s))^{1/2} + W^{1/2}(s)} - \frac{1}{(\delta + W(s))^{1/2}} \right] ds \leq C \tag{2.13}$$

for all $0 < \delta < 1$.

We assume that $g_\varepsilon : \partial\Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ satisfy the following hypotheses:

$$g_\varepsilon \in H^1(\partial\Omega), \tag{2.14}$$

$$\varepsilon \int_{\partial\Omega} |\nabla_\tau g_\varepsilon|^2 d\mathcal{H}^{N-1} = o(1) \quad \text{as } \varepsilon \to 0^+, \tag{2.15}$$

$$|g_\varepsilon(x) - g(x)| \leq C\varepsilon^\gamma, \quad x \in \partial\Omega, \quad \gamma > 1 \tag{2.16}$$

for all $\varepsilon \in (0, 1)$ and for some constant $C > 0$. Here, ∇_τ denotes the tangential gradient.

In what follows, given $z \in \mathbb{R}^N$, with a slight abuse of notation, we write

$$z = (z', z_N) \in \mathbb{R}^{N-1} \times \mathbb{R}, \tag{2.17}$$

where $z' := (z_1, \dots, z_{N-1})$. We also write

$$\nabla' := \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_{N-1}} \right). \tag{2.18}$$

In what follows, given $\delta > 0$ we define

$$\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}. \tag{2.19}$$

For the proof of the following lemma, we refer to [7, Lemma 2.6].

Lemma 2.4 *Assume that $\Omega \subset \mathbb{R}^N$ is an open, bounded, connected set and that its boundary $\partial\Omega$ is of class $C^{2,d}$, $0 < d \leq 1$. If $\delta > 0$ is sufficiently small, then the mapping*

$$\Phi : \partial\Omega \times [0, \delta] \to \overline{\Omega}_\delta$$

given by

$$\Phi(y, t) := y + t\nu(y),$$

where $\nu(y)$ is the unit inward normal vector to $\partial\Omega$ at y and Ω_δ is defined in (2.19)*, is a diffeomorphism of class $C^{1,d}$. Moreover, $\Omega \setminus \Omega_\delta$ is connected for all $\delta > 0$ sufficiently small. Finally,*

$$\det J_\Phi(y, 0) = 1 \quad \text{for all } y \in \partial\Omega \tag{2.20}$$

and

$$\frac{\partial}{\partial t} \det J_\Phi(y, t)|_{t=0} = \kappa(y) \quad \text{for all } y \in \partial\Omega, \tag{2.21}$$

where $\kappa(y)$ is the mean curvature of $\partial\Omega$ at y.

3 A 1D Functional Problem

Let

$$I := (0, T)$$

for some $T > 0$ and consider a weight function

$$\omega \in C^{1,d}([0,T]), \quad \min_{[0,T]} \omega > 0. \tag{3.1}$$

The prototype we have in mind is given by

$$\omega(t) := 1 + t\kappa(t).$$

In this section, we study the second-order Γ-convergence of the family of functionals

$$G_\varepsilon(v) := \int_I (W(v(t)) + \varepsilon^2 (v'(t))^2)\omega(t)\, dt, \quad v \in H^1(I),$$

subject to the Dirichlet boundary condition

$$v(0) = \alpha_\varepsilon, \quad v(T) = \beta_\varepsilon. \tag{3.2}$$

In what follows, we will need the weighted BV space $BV_\omega(I)$ given by all functions $v \in BV_{\text{loc}}(I)$ for which the norm

$$\|v\|_{BV_\omega} := \int_I |v(t)|\omega(t)\, dt + \int_I \omega(t)\, d|Dv|(t)$$

is finite. For $v \in BV_\omega(I)$ we will also write the weighted total variation of the derivative in the following manner

$$|Dv|_\omega(E) := \int_E \omega(t)\, d|Dv|(t).$$

For a more detailed introduction to weighted BV spaces and their applications to phase field models, we refer to [**?**, **?**].

We will study the second-order Γ-convergence with respect to the metric in $L^1(I)$. This choice is motivated by the following compactness result.

Theorem 3.1 (Compactness) *Assume that W satisfies* (2.1)-(2.4)*, that ω satisfies* (3.1)*, and that $\alpha_\varepsilon \to \alpha$ and $\beta_\varepsilon \to \beta$ as $\varepsilon \to 0^+$ for some $\alpha, \beta \in \mathbb{R}$. Let $\varepsilon_n \to 0^+$ and $v_n \in H^1(I)$ be such that*

$$\sup_n \int_I \left(\frac{1}{\varepsilon_n} W(v_n(t)) + \varepsilon_n (v_n'(t))^2 \right) \omega(t)\, dt < \infty.$$

Then there exist a subsequence $\{v_{n_k}\}_k$ of $\{v_n\}_n$ and $v \in BV_\omega(I; \{a, b\})$ such that $v_{n_k} \to v$ in $L^1(I)$.

The proof is identical to the one of [10, Proposition 4.3] and so we omit it. In view of the previous theorem, we extend G_ε to $L^1(I)$ by setting

$$G_\varepsilon(v) := \begin{cases} \int_I (W(v(t)) + \varepsilon^2 (v'(t))^2)\omega(t)\, dt & \text{if } v \in H^1(I) \text{ satisfies (3.2)} \\ \infty & \text{otherwise in } L^1(I). \end{cases} \tag{3.3}$$

3.1 Zeroth and First-Order Γ-limit of G_ε

For the proof of the results in this subsection, we refer to [7]. We begin by establishing the zeroth order Γ-limit of the functional G_ε.

Theorem 3.2 *Assume that W satisfies* (2.1)-(2.4)*, that ω satisfies* (3.1)*, and that $\alpha_\varepsilon \to \alpha$ and $\beta_\varepsilon \to \beta$ as $\varepsilon \to 0^+$ for some $\alpha, \beta \in \mathbb{R}$. Then the family $\{G_\varepsilon\}_\varepsilon$ Γ-converges to $G^{(0)}$ in $L^1(I)$ as $\varepsilon \to 0^+$, where*

$$G^{(0)}(v) := \int_I W(v(t))\omega(t)\,dt.$$

Since $W^{-1}(\{0\}) = \{a, b\}$, it follows that

$$\inf_{v\in L^1(I)} G^{(0)}(v) = 0.$$

Therefore,

$$\begin{aligned} G_\varepsilon^{(1)}(v) &:= \frac{G_\varepsilon(v) - \inf_{L^1(I)} G^{(0)}}{\varepsilon} \\ &= \int_I \left(\frac{1}{\varepsilon} W(v(t)) + \varepsilon (v'(t))^2 \right) \omega(t)\,dt \end{aligned} \tag{3.4}$$

if $v \in H^1(I)$ satisfies (3.2) and $G_\varepsilon^{(1)}(v) := \infty$ if $v \in L^1(I) \setminus H^1(I)$ or if the boundary condition (3.2) fails.

We now characterize the first-order Gamma limit of the family $\{G_\varepsilon\}_\varepsilon$.

Theorem 3.3 *Assume that W satisfies hypotheses* (2.1)-(2.4)*, that ω satisfies hypothesis* (3.1)*, and that $\alpha_\varepsilon \to \alpha$ and $\beta_\varepsilon \to \beta$ as $\varepsilon \to 0^+$ for some $\alpha, \beta \in \mathbb{R}$. Then the family $\{G_\varepsilon^{(1)}\}_\varepsilon$ Γ-converges to $G^{(1)}$ in $L^1(I)$ as $\varepsilon \to 0^+$, where*

$$G^{(1)}(v) := \begin{cases} \dfrac{C_W}{b-a} |Dv|_\omega(I) + \mathrm{d}_W(v(0), \alpha)\omega(0) & \\ \qquad\qquad + \mathrm{d}_W(v(T), \beta)\omega(T) & \text{if } v \in BV_\omega(I; \{a, b\}), \\ \infty & \text{otherwise in } L^1(I), \end{cases}$$

where d_W and C_W are defined in (1.6) *and* (1.7)*, respectively.*

Next we show that if ω is sufficiently close to $\omega(0)$ or strictly increasing, then the unique minimizer of $G^{(1)}$ is the constant function b.

Corollary 3.4 *Assume that W satisfies* (2.1)-(2.4) *and let $a < \alpha < b$ and $\beta = b$. Suppose that ω satisfies* (3.1) *and that*

$$\omega(t) > \omega(0) - \omega_0 \quad \text{for all } t \in (0, T], \tag{3.5}$$

where

$$0 \le \omega_0 < \frac{1}{2} \frac{C_W - \mathrm{d}_W(\alpha, b)}{C_W} \omega(0) \tag{3.6}$$

if $a < \alpha$, while ω is strictly increasing if $\alpha = a$. Then the unique minimizer of $G^{(1)}$ is the constant function b, with

$$\min_{L^1_\omega(I)} G^{(1)}(v) = G^{(1)}(b) = \mathrm{d}_W(\alpha, b)\omega(0).$$

Proof. Step 1: Assume that $a < \alpha < b$. Let $v \in BV_\omega(I; \{a, b\})$. If v has at least one jump point at $t_0 \in I$, then by (3.5) and (3.6),

$$G^{(1)}(v) \geq \frac{C_W}{b-a}|Dv|_\omega(I) \geq C_W\omega(t_0) > C_W(\omega(0) - \omega_0) \geq \mathrm{d}_W(\alpha, b)\omega(0).$$

Hence, either $v \equiv b$ or $v \equiv a$. If $v \equiv a$, then again by (3.5) and (3.6)

$$G^{(1)}(a) = \mathrm{d}_W(a, \alpha)\omega(0) + C_W\omega(T) > C_W(\omega(0) - \omega_0) \geq \mathrm{d}_W(\alpha, b)\omega(0).$$

Step 2: Assume that $\alpha = a$ and $\beta = b$. Let $v \in BV_\omega(I; \{a, b\})$. If v has at least one jump point at $t_0 \in I$, then since ω is strictly increasing

$$G^{(1)}(v) \geq \frac{C_W}{b-a}|Dv|_\omega(I) \geq C_W\omega(t_0) > C_W\omega(0).$$

Hence, either $v \equiv b$ or $v \equiv a$. If $v \equiv a$, then again by (3.5) and (3.6)

$$G^{(1)}(a) = C_W\omega(T) > C_W\omega(0).$$

This completes the proof. ■

Remark 3.5 *Note that condition* (3.5) *holds if either ω is strictly increasing, with $\omega_0 = 0$, or if T is sufficiently small, by continuity of ω.*

3.2 Second-Order Γ-limsup

The scaling of the second-order asymptotic development via Γ-convergence of G_ε changes depending on whether $a < \alpha$ and $a = a$. When $a < \alpha$, under the hypotheses of Corollary 3.4, we have

$$\min_{L^1(I)} G^{(1)}(v) = G^{(1)}(b) = \mathrm{d}_W(\alpha, b)\omega(0).$$

In this case, we define

$$\begin{aligned} G^{(2)}_\varepsilon(v) &:= \frac{G^{(1)}_\varepsilon(v) - \inf_{L^1(I)} G^{(1)}}{\varepsilon} \\ &= \int_I \left(\frac{1}{\varepsilon^2}W(v(t)) + (v'(t))^2\right)\omega(t)\,dt - \mathrm{d}_W(\alpha, b)\omega(0)\frac{1}{\varepsilon} \end{aligned} \tag{3.7}$$

if $v \in H^1(I)$ satisfies (3.2) and $G^{(2)}_\varepsilon(v) := \infty$ if $v \in L^1(I)\setminus H^1(I)$ or if the boundary condition (3.2) fails. For the proof of the following theorem, we refer to [7].

Theorem 3.6 (Second-Order Limsup, $a < \alpha$) *Assume that W satisfies* (2.1)-(2.4)*, that α_- satisfies* (2.5)*, and that ω satisfies* (3.1), (3.5)*, where*

$$0 \le \omega_0 < \frac{1}{2}\frac{\mathrm{d}_W(a,\alpha_-)}{C_W}\omega(0). \tag{3.8}$$

Let

$$\alpha_- \le \alpha_\varepsilon,\, \beta_\varepsilon \le b,$$

with

$$|\alpha_\varepsilon - \alpha| \le A_0\varepsilon^\gamma, \quad |\beta_\varepsilon - b| \le B_0\varepsilon^\gamma \tag{3.9}$$

for some α, β and where A_0, $B_0 > 0$, and $\gamma > 1$. Then there exist constants $0 < \varepsilon_0 < 1$, $C, C_0 > 0$, and $\gamma_0, \gamma_1 > 0$, depending only on α_-, A_0, B_0, T, ω, and W, and functions $v_\varepsilon \in H^1(I)$ satisfying (3.2)*, $a \le v_\varepsilon \le b$, and $v_\varepsilon \to b$ in $L^1(I)$, such that*

$$G_\varepsilon^{(2)}(v_\varepsilon) \le \int_0^l 2W(p_\varepsilon(t))t\,dt\,\omega'(0) + Ce^{-2\sigma l}\,(2\sigma l + 1) + C\varepsilon^{2\gamma}l + C\varepsilon^{\gamma_1}|\log\varepsilon|^{\gamma_0} \tag{3.10}$$

for all $0 < \varepsilon < \varepsilon_0$ and all $l > 0$, where $p_\varepsilon(t) := v_\varepsilon(\varepsilon t)$ is such that $p_\varepsilon \to z_\alpha$ pointwise in $[0,\infty)$, where z_α solves the Cauchy problem (1.12) *and $G_\varepsilon^{(2)}$ is defined in* (3.7)*. In particular,*

$$\limsup_{\varepsilon\to 0^+} G_\varepsilon^{(2)}(v_\varepsilon) \le \int_0^\infty 2W^{1/2}(z_\alpha(t))z_\alpha'(t)t\,dt\,\omega'(0).$$

Remark 3.7 *The function v_ε is constructed as the inverse function of the function*

$$\Psi_\varepsilon(r) := \int_{\alpha_\varepsilon}^r \frac{\varepsilon}{(\delta_\varepsilon + W(s))^{1/2}}\,ds,$$

where $\delta_\varepsilon \to 0^+$ goes to zero faster than ε. Observe that if we take $\delta_\varepsilon = \varepsilon$, then (3.10) *should be replaced by*

$$G_\varepsilon^{(2)}(v_\varepsilon) \le C + \int_0^l 2W(p_\varepsilon(s))s\,ds\omega'(0) + Ce^{-2\sigma l}\,(2\sigma l + 1) + C\varepsilon^{2\gamma}l + C\varepsilon\log^2\varepsilon + C\varepsilon^d|\log\varepsilon|^{1+d} + C\varepsilon^{2\gamma-2}.$$

On the other hand, when $\alpha = a$, again under the hypotheses of Corollary 3.4, we have

$$\min_{L^1(I)} G^{(1)}(v) = G^{(1)}(b) = C_W\omega(0).$$

In this case, we define

$$\begin{aligned} G_\varepsilon^{(2)}(v) &:= \frac{G_\varepsilon^{(1)}(v) - \inf_{L^1(I)} G^{(1)}}{\varepsilon|\log\varepsilon|} \\ &= \frac{1}{\varepsilon|\log\varepsilon|}\int_I \left(\frac{1}{\varepsilon}W(v(t)) + \varepsilon(v'(t))^2\right)\omega(t)\,dt - C_W\omega(0)\frac{1}{\varepsilon|\log\varepsilon|} \end{aligned} \tag{3.11}$$

if $v \in H^1(I)$ satisfies (3.2) and $G_\varepsilon^{(2)}(v) := \infty$ if $v \in L^1(I)\setminus H^1(I)$ or if the boundary condition (3.2) fails.

We study the second-order Γ-limsup of the family $\{G_\varepsilon\}_\varepsilon$.

Theorem 3.8 (Second-Order Γ-Limsup, $\alpha = a$) *Assume that W satisfies (2.1)-(2.4) and that ω satisfies (3.1) and is strictly increasing with $\omega'(0) > 0$. Let $a \le \alpha_\varepsilon \le \beta_\varepsilon < b$ with*

$$|\alpha_\varepsilon - a| \le A_0\varepsilon^\gamma, \quad |\beta_\varepsilon - b| \le B_0\varepsilon^\gamma, \tag{3.12}$$

where A_0, $B_0 > 0$, and $\gamma > 1$. There exist $v_\varepsilon \in H^1_\omega(I)$ satisfying (3.2), such that $v_\varepsilon \to b$ in $L^1_\omega(I)$ and, for every $0 < \eta < 1$,

$$G_\varepsilon^{(2)}(v_\varepsilon) \le (1+\eta)\frac{C_W}{2^{1/2}(W''(a))^{1/2}}\omega'(0) + \frac{C}{|\log\varepsilon|} \tag{3.13}$$

for all $0 < \varepsilon < \varepsilon_\eta$, for some $0 < \varepsilon_\eta < 1$ depending on η, A_0, B_0, T, ω, and W, and for some constant $C > 0$, depending on A_0, B_0, T, ω, and W, and where $G_\varepsilon^{(2)}$ is defined in (3.11). In particular,

$$\limsup_{\varepsilon\to 0^+} G_\varepsilon^{(2)}(v_\varepsilon) \le \frac{C_W}{2^{1/2}(W''(a))^{1/2}}\omega'(0). \tag{3.14}$$

Proof. In this proof, ε_0 and C depend only on A_0, B_0, T, ω, W. In what follows, we will take ε_0 smaller and C larger, if necessary, preserving the same dependence on the parameters.

Define

$$\Psi_\varepsilon(r) := \int_{\alpha_\varepsilon}^r \frac{\varepsilon}{(\varepsilon + W(s))^{1/2}}\,ds.$$

Let

$$0 \le L_\varepsilon := \Psi_\varepsilon(c) < T_\varepsilon := \Psi_\varepsilon(\beta_\varepsilon). \tag{3.15}$$

By (2.9) and the fact that $a \le \alpha_\varepsilon, \beta_\varepsilon \le b$, we have

$$L_\varepsilon \le T_\varepsilon \le \int_a^b \frac{\varepsilon}{(\varepsilon + W(s))^{1/2}}\,ds \le C\varepsilon|\log\varepsilon| \tag{3.16}$$

for all $0 < \varepsilon < \varepsilon_0$.

Let $v_\varepsilon : [0, T_\varepsilon] \to [\alpha_\varepsilon, \beta_\varepsilon]$ be the inverse of Ψ_ε. Then $v_\varepsilon(0) = \alpha_\varepsilon$, $v_\varepsilon(T_\varepsilon) = \beta_\varepsilon$, and

$$v_\varepsilon'(t) = \frac{(\varepsilon + W(v_\varepsilon(t)))^{1/2}}{\varepsilon}. \tag{3.17}$$

Extend v_ε to be equal to β_ε for $t > T_\varepsilon$.

Since $\omega \in C^{1,d}(I)$, by Taylor's formula, for $t \in [0, T]$,

$$\omega(t) = \omega(0) + \omega'(0)t + R_1(t),$$

where

$$|R_1(t)| = |\omega'(\theta t) - \omega'(0)|t \le |\omega'|_{C^{0,d}} t^{1+d}. \tag{3.18}$$

Write

$$G_\varepsilon^{(2)}(v_\varepsilon) = \left[\int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon (v_\varepsilon')^2 \right) dt - C_W \right] \frac{\omega(0)}{\varepsilon |\log \varepsilon|}$$
$$+ \int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon (v_\varepsilon')^2 \right) t \, dt \frac{\omega'(0)}{\varepsilon |\log \varepsilon|} \tag{3.19}$$
$$+ \int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon (v_\varepsilon')^2 \right) R_1 \, dt \frac{1}{\varepsilon |\log \varepsilon|}$$
$$+ \int_{T_\varepsilon}^{T} \left(\frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon (v_\varepsilon')^2 \right) \omega \, dt \frac{1}{\varepsilon |\log \varepsilon|} =: \mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}.$$

Step 1. We estimate $\mathcal{A}$. By (3.17), the change of variables $s = v_\varepsilon(t)$, and the equality

$$(A+B)^{1/2} - B^{1/2} = \frac{A}{(A+B)^{1/2} + B^{1/2}},$$

we have

$$\int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon (v_\varepsilon')^2 \right) dt = \int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon} (\varepsilon + W(v_\varepsilon)) + \varepsilon (v_\varepsilon')^2 \right) dt - T_\varepsilon$$
$$= \int_0^{T_\varepsilon} 2(\varepsilon + W(v_\varepsilon))^{1/2} v_\varepsilon' \, dt - T_\varepsilon$$
$$= \int_{\alpha_\varepsilon}^{\beta_\varepsilon} 2(\varepsilon + W(s))^{1/2} \, ds - \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{\varepsilon}{(\varepsilon + W(s))^{1/2}} \, ds$$
$$= \int_{\alpha_\varepsilon}^{\beta_\varepsilon} 2W^{1/2}(s) \, ds + \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \left[\frac{2\varepsilon}{(\varepsilon + W(s))^{1/2} + W^{1/2}(s)} - \frac{\varepsilon}{(\varepsilon + W(s))^{1/2}} \right] ds.$$

By Proposition 2.3,

$$\int_{\alpha_\varepsilon}^{\beta_\varepsilon} \left[\frac{2\varepsilon}{(\varepsilon + W(s))^{1/2} + W^{1/2}(s)} - \frac{\varepsilon}{(\varepsilon + W(s))^{1/2}} \right] ds \le C\varepsilon$$

for all $0 < \varepsilon < \varepsilon_0$. Hence, using also the fact that $a < \alpha_\varepsilon < \beta_\varepsilon < b$, we obtain

$$\int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon (v_\varepsilon')^2 \right) dt \le C_W + C\varepsilon, \tag{3.20}$$

and so

$$\mathcal{A} \le C \frac{1}{|\log \varepsilon|}$$

for all $0 < \varepsilon < \varepsilon_0$.

Step 2. We estimate $\mathcal{B}$ in (3.19). By (3.17) and the change of variables $t := r + L_\varepsilon$,

$$\mathcal{B} = \int_{-L_\varepsilon}^{T_\varepsilon - L_\varepsilon} \left(\frac{1}{\varepsilon} W(\bar{v}_\varepsilon) + \varepsilon (\bar{v}_\varepsilon')^2 \right) dr \frac{\omega'(0) L_\varepsilon}{\varepsilon |\log \varepsilon|}$$
$$+ \int_{-L_\varepsilon}^{T_\varepsilon - L_\varepsilon} \left(\frac{1}{\varepsilon} W(\bar{v}_\varepsilon) + \varepsilon (\bar{v}_\varepsilon')^2 \right) r \, dr \frac{\omega'(0)}{\varepsilon |\log \varepsilon|} =: \mathcal{B}_1 + \mathcal{B}_2,$$

where $\bar{v}_\varepsilon(r) := v_\varepsilon(r + L_\varepsilon)$. By (3.20), (3.16) and the fact that $\omega'(0) > 0$,

$$\begin{aligned}\mathcal{B}_1 &\le C_W \frac{\omega'(0) L_\varepsilon}{\varepsilon |\log \varepsilon|} + C \frac{L_\varepsilon}{|\log \varepsilon|} \\ &\le C_W \frac{\omega'(0)}{|\log \varepsilon|} \int_a^c \frac{1}{(\varepsilon + W(\rho))^{1/2}} \, d\rho + C\varepsilon\end{aligned}$$

for all $0 < \varepsilon < \varepsilon_0$. By (2.8), given $0 < \eta < 1$, there exists $0 < \varepsilon_\eta < 1$ such that

$$C_W \frac{\omega'(0)}{|\log \varepsilon|} \int_a^{a+\eta} \frac{1}{(\varepsilon + W(\rho))^{1/2}} \, d\rho \le (1+\eta) \frac{C_W \omega'(0)}{2^{1/2} (W''(a))^{1/2}}$$

for all $0 < \varepsilon < \varepsilon_\eta$.

On the other hand, by the change of variables $r := \varepsilon s$,

$$\begin{aligned}\mathcal{B}_2 &= \int_{-L_\varepsilon}^{T_\varepsilon - L_\varepsilon} 2W(\bar{v}_\varepsilon) r \, dr \frac{\omega'(0)}{\varepsilon^2 |\log \varepsilon|} + \int_{-L_\varepsilon}^{T_\varepsilon - L_\varepsilon} r \, dr \frac{\varepsilon \omega'(0)}{\varepsilon^2 |\log \varepsilon|} \\ &= \int_{-L_\varepsilon \varepsilon^{-1}}^{(T_\varepsilon - L_\varepsilon)\varepsilon^{-1}} 2W(p_\varepsilon(s)) s \, ds \frac{\omega'(0)}{|\log \varepsilon|} + \frac{\varepsilon \omega'(0) [(T_\varepsilon - L_\varepsilon)^2 - L_\varepsilon^2]}{2\varepsilon^2 |\log \varepsilon|} \\ &:= \mathcal{B}_{2,1} + \mathcal{B}_{2,2},\end{aligned}$$

where $p_\varepsilon(s) := \bar{v}_\varepsilon(\varepsilon s) = v_\varepsilon(\varepsilon s + L_\varepsilon)$ solves the Cauchy problem

$$\begin{cases} p_\varepsilon'(s) = (\varepsilon + W(p_\varepsilon(s)))^{1/2}, \\ p_\varepsilon(0) = c, \end{cases}$$

in $[-L_\varepsilon \varepsilon^{-1}, (T_\varepsilon - L_\varepsilon)\varepsilon^{-1}]$. Since $c \le p_\varepsilon(s) \le \beta_\varepsilon < b$ for $0 \le s \le (T_\varepsilon - L_\varepsilon)\varepsilon^{-1}$, by (2.6) we have that

$$p_\varepsilon'(s) \ge (W(p_\varepsilon(s)))^{1/2} \ge \sigma(b - p_\varepsilon(s)) > 0,$$

and so

$$-\sigma \ge \frac{(b - p_\varepsilon(s))'}{b - p_\varepsilon(s)} = (\log(b - p_\varepsilon(s)))'.$$

Upon integration, we get

$$0 \le b - p_\varepsilon(s) \le (b-c) e^{-\sigma s} \le (b-c) e^{-\sigma s}.$$

In turn, again by (2.6), for $s \in [0, (T_\varepsilon - L_\varepsilon)\varepsilon^{-1}]$,

$$W(p_\varepsilon(s)) \le \sigma^{-2} (b - p_\varepsilon(s))^2 \le \sigma^{-2} (b-c)^2 e^{-2\sigma s}. \tag{3.21}$$

On the other hand, we claim that there exists $C > 0$ such that

$$-C \int_{-L_\varepsilon \varepsilon^{-1}}^0 e^{2\sigma s} |s| \, ds \le \int_{-L_\varepsilon \varepsilon^{-1}}^0 2W(p_\varepsilon(s)) s \, ds \le 0. \tag{3.22}$$

As $W \geq 0$ and $s \leq 0$ it is immediate that

$$\int_{-L_\varepsilon \varepsilon^{-1}}^{0} 2W(p_\varepsilon(s))s\,ds \leq 0.$$

Additionally, by (2.7) for $-L_\varepsilon \varepsilon^{-1} \leq s \leq 0$, we have that

$$p_\varepsilon'(s) \geq (W(p_\varepsilon(s)))^{1/2} \geq \sigma(p_\varepsilon(s) - a) > 0,$$

and so

$$(\log(p_\varepsilon(s) - a))' = \frac{(p_\varepsilon(s) - a)'}{p_\varepsilon(s) - a} \geq \sigma.$$

Upon integration, we get

$$\log \frac{c - a}{p_\varepsilon(s) - a} \geq \sigma(0 - s)$$

and so

$$c - a \geq (p_\varepsilon(s) - a)e^{-s\sigma},$$

which gives

$$0 \leq p_\varepsilon(s) - a \leq (c - a)e^{\sigma s}.$$

In turn, again by (2.7), for $s \in [-L_\varepsilon \varepsilon^{-1}, 0]$,

$$W(p_\varepsilon(s)) \leq \sigma^2 (p_\varepsilon(s) - a)^2 \leq \sigma^2 (c - a)^2 e^{2\sigma s}.$$

This implies (3.22) and therefore, using (3.21), we obtain

$$\mathcal{B}_{2,1} \leq C \int_0^\infty e^{-2\sigma s} s\,ds \frac{\omega'(0)}{|\log \varepsilon|} \leq \frac{C}{|\log \varepsilon|}$$

for all $0 < \varepsilon < \varepsilon_0$. By (3.16) and the fact that $\omega'(0) > 0$,

$$\mathcal{B}_{2,2} \leq C \frac{\varepsilon T_\varepsilon^2}{\varepsilon^2 |\log \varepsilon|} \leq C\varepsilon^2 |\log^2 \varepsilon|$$

for all $0 < \varepsilon < \varepsilon_0$.

Step 3. We estimate $\mathcal{C}$ in (3.19). Observe that by (3.20), (3.18), and (3.16),

$$\begin{aligned} \mathcal{C} &\leq \int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon (v_\varepsilon')^2 \right) dt \frac{|\omega'|_{C^{0,d}} T_\varepsilon^{1+d}}{\varepsilon |\log \varepsilon|} \\ &\leq C\varepsilon^d |\log \varepsilon|^d \left(C_W + C\varepsilon |\log \varepsilon| \right) \leq C\varepsilon^d |\log \varepsilon|^d \end{aligned}$$

for all $0 < \varepsilon < \varepsilon_0$.

Step 4. We estimate $\mathcal{D}$ in (3.19). By (2.6) and (3.12), for $t \geq T_\varepsilon$,

$$\mathcal{D} = W(\beta_\varepsilon) \int_{T_\varepsilon}^{T} \omega\,dt \frac{1}{\varepsilon^2 |\log \varepsilon|} \leq \sigma^{-2} (b - \beta_\varepsilon)^2 \int_0^T \omega\,dt \frac{1}{\varepsilon^2 |\log \varepsilon|} \leq C \frac{\varepsilon^{2\gamma - 2}}{|\log \varepsilon|}$$

for all $0 < \varepsilon < \varepsilon_0$.

Combining the estimates for $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ gives (3.13). In turn, letting first $\varepsilon \to 0^+$ and then $\eta \to 0^+$ in (3.13) proves (3.14). ∎

3.3 Properties of Minimizers of G_ε

In this subsection, we study some qualitative properties of the minimizers of the functional G_ε defined in (3.3):

$$G_\varepsilon(v) := \int_I (W(v(t)) + \varepsilon^2 (v'(t))^2)\omega(t)\, dt, \quad v \in H^1(I), \tag{3.23}$$

subject to the Dirichlet boundary conditions

$$v_\varepsilon(0) = \alpha_\varepsilon, \quad v_\varepsilon(T) = \beta_\varepsilon. \tag{3.24}$$

Theorem 3.9, Corollary 3.10, Theorem 3.11, and Theorem 3.12 have been proven in [7], cf. [7, Theorem 3.8], [7, Corollary 3.9], [7, Theorem 3.12], and [7, Theorem 3.10]. We state them for the convenience of the reader.

Theorem 3.9 *Assume that W satisfies* (2.1)-(2.4)*, that ω satisfies* (3.1)*, and that $a \le \alpha_\varepsilon$, $\beta_\varepsilon \le b$. Then the functional G_ε admits a minimizer $v_\varepsilon \in H^1(I)$. Moreover, $v_\varepsilon \in C^2([0,T])$, v_ε satisfies the Euler–Lagrange equations*

$$2\varepsilon^2 (v_\varepsilon'(t)\omega(t))' - W'(v_\varepsilon(t))\omega(t) = 0, \tag{3.25}$$

and $v_\varepsilon \equiv a$, or $v_\varepsilon \equiv b$, or

$$a < v_\varepsilon(t) < b \quad \textit{for all } t \in (0,T). \tag{3.26}$$

Corollary 3.10 *Assume that W satisfies* (2.1)-(2.4)*, that ω satisfies* (3.1)*, and that $a \le \alpha_\varepsilon$, $\beta_\varepsilon \le b$. Let v_ε be the minimizer of G_ε obtained in Theorem 3.9. Then there exists a constant $C_0 > 0$, depending only on ω, T, a, b, and W, such that*

$$|v_\varepsilon'(t)| \le \frac{C_0}{\varepsilon} \quad \textit{for all } t \in I$$

and for every $0 < \varepsilon < 1$.

Next, we recall some differential inequalities for v_ε. To this end, we introduce two auxiliary values

$$\hat{\alpha}_- := \frac{1}{2}\left(a + \min\left\{c, \frac{a+b}{2}\right\}\right), \quad \hat{\beta}_- := \frac{1}{2}\left(b + \max\left\{c, \frac{a+b}{2}\right\}\right). \tag{3.27}$$

Note that these values only depend on (the zeros of the derivative of) W. These are used together with some of the statements in [7] in order to obtain a dependence on only W in these statements.

Theorem 3.11 *Assume that W satisfies* (2.1)-(2.4)*, that ω satisfies* (3.1)*, and that $a \le \alpha_\varepsilon$, $\beta_\varepsilon \le b$. Let v_ε be the minimizer of G_ε obtained in Theorem 3.9 and let $\hat{\alpha}_-, \hat{\beta}_-$ be given as in* (3.27)*. Then there exists a constant $C > 0$ such that*

$$\varepsilon (v_\varepsilon'(0))^2 - \frac{1}{\varepsilon} W(\alpha_\varepsilon) \le C \tag{3.28}$$

for all $0 < \varepsilon < 1$*. Moreover, there exist a constant* $\tau_0 > 0$*, depending only on* ω*,* T*,* a*,* b*, and* W*, such that*

$$\frac{1}{2}\sigma^2(v_\varepsilon(t)-a)^2 \leq \varepsilon^2(v_\varepsilon'(t))^2 \leq \frac{3}{2}\sigma^{-2}(v_\varepsilon(t)-a)^2 \tag{3.29}$$

whenever $a+\tau_0\varepsilon^{1/2} \leq v_\varepsilon(t) \leq \hat{\beta}_-$ *and*

$$\frac{1}{2}\sigma^2(b-v_\varepsilon(t))^2 \leq \varepsilon^2(v_\varepsilon'(t))^2 \leq \frac{3}{2}\sigma^{-2}(b-v_\varepsilon(t))^2 \tag{3.30}$$

whenever $\hat{\alpha}_- \leq v_\varepsilon(t) \leq b-\tau_0\varepsilon^{1/2}$*, where* $\sigma > 0$ *is the constant given in Remark 2.1.*

Theorem 3.12 *Assume that* W *satisfies* (2.1)-(2.4)*, that* ω *satisfies* (3.1)*, and that that* $a \leq \alpha_\varepsilon$*,* $\beta_\varepsilon \leq b$*. Let* v_ε *be the minimizer of* G_ε *obtained in Theorem 3.9 and for* $k \in \mathbb{N}$ *let*

$$A_\varepsilon^k := \{t \in [0,T] :\ \alpha_\varepsilon + \varepsilon^k \leq v_\varepsilon(t) \leq \hat{\alpha}_-\}, \tag{3.31}$$

$$B_\varepsilon^k := \{t \in [0,T] :\ \hat{\beta}_- \leq v_\varepsilon(t) \leq \beta_\varepsilon - \varepsilon^k\}. \tag{3.32}$$

Then there exist $C > 0$ *and* $0 < \varepsilon_0 < 1$ *depending only on* T*,* ω*,* W*,*k *such that if* I_ε *is a maximal subinterval of* A_ε^k *or* B_ε^k*, then*

$$\operatorname{diam} I_\varepsilon \leq C\varepsilon|\log\varepsilon| \tag{3.33}$$

for all $0 < \varepsilon < \varepsilon_0$*.*

Next, we strengthen the hypotheses on the Dirichlet data α_ε and β_ε and derive additional properties of minimizers.

Given $0 < \eta < \frac{1}{4}$, by Taylor's formula and the fact that $W''(a) > 0$, we can find $\delta_\eta > 0$ such such that

$$\frac{1}{2}W''(a)(1-\eta)(s-a)^2 \leq W(s) \leq \frac{1}{2}W''(a)(1+\eta)(s-a)^2 \tag{3.34}$$

for all $a \leq s \leq a+\delta_\eta$.

Theorem 3.13 *Assume that* W *satisfies* (2.1)-(2.4) *and that* ω *satisfies* (3.1) *and is strictly increasing with* $\omega'(0) > 0$*. Let* $a \leq \alpha_\varepsilon$*,* $\beta_\varepsilon \leq b$ *satisfy* (3.12) *and let* v_ε *be the minimizer of* G_ε *obtained in Theorem 3.9. Given* $k \in \mathbb{N}$ *with* $k \geq \gamma$*, there exist* $0 < \varepsilon_0 < 1$*,* $C > 0$ *depending only on* k*,* A_0*,* B_0*,* T*,* ω*,* W*, such that, for all* $0 < \varepsilon < \varepsilon_0$*, the following properties hold:*

(i) If T_ε *is the first time such that* $v_\varepsilon = \beta_\varepsilon - \varepsilon^k$*, then*

$$T_\varepsilon \leq C\varepsilon|\log\varepsilon|. \tag{3.35}$$

(ii) Let $0 < \eta < \frac{1}{4}$, let δ_η be as in (3.34), and let $S_{\varepsilon,\eta}$ be the first time such that $v_\varepsilon = a + \delta_\eta$. Then there exists a constant $C_\eta > 0$, depending on η, k, A_0, B_0, T, ω, W, such that

$$S_{\varepsilon,\eta} \geq \frac{1}{2^{1/2}(W''(a))^{1/2}} (\varepsilon|\log\varepsilon| - \eta\varepsilon|\log\varepsilon|) - C_\eta\varepsilon. \tag{3.36}$$

Proof. In this proof, ε_0 and the constants C, C_0, and C_1 depend only on A_0, B_0, T, ω, W. In what follows, we will take ε_0 smaller and C, C_0, and C_1 larger, if necessary, preserving the same dependence on the parameters.

By Theorem 3.8,

$$G_\varepsilon^{(1)}(v_\varepsilon) \leq C_W\omega(0) + C_1\varepsilon|\log\varepsilon| \tag{3.37}$$

for all $0 < \varepsilon < \varepsilon_0$.

Let

$$t_1^\varepsilon < t_2^\varepsilon < t_3^\varepsilon < t_4^\varepsilon$$

be the first time such that v_ε equals $\alpha_\varepsilon + \varepsilon^k$, $\hat{\alpha}_-$, $\hat{\beta}_-$, and $\beta_\varepsilon - \varepsilon^k$, respectively.

Step 1: We claim that there exist $0 < \varepsilon_0 < 1$ and $C > 0$ such that

$$t_2^\varepsilon - t_1^\varepsilon \leq C\varepsilon|\log\varepsilon| \tag{3.38}$$

for all $0 < \varepsilon < \varepsilon_0$. To see this, observe that since $v_\varepsilon(0) = \alpha_\varepsilon < \alpha_\varepsilon + \varepsilon^k$, we have that $v_\varepsilon'(t_1^\varepsilon) \geq 0$. Using (2.4) and (3.25),

$$2\varepsilon^2(v_\varepsilon'(t)\omega(t))' = W'(v_\varepsilon(t))\omega(t) > 0$$

for all $a < v_\varepsilon(t) < c$. In particular, since $\hat{\alpha}_- < c$, we have that $v_\varepsilon'(t) > 0$ for all $t_1^\varepsilon < t \leq t_2^\varepsilon$. It follows that $[t_1^\varepsilon, t_2^\varepsilon]$ is a maximal interval of the set A_ε defined in (3.31), and so by Theorem 3.12, the claim (3.38) follows.

Step 2: We claim that there exist $0 < \varepsilon_0 < 1$ and $C > 0$ such that

$$t_3^\varepsilon - t_2^\varepsilon \leq C\varepsilon \tag{3.39}$$

for all $0 < \varepsilon < \varepsilon_0$. Indeed, since $v_\varepsilon'(t_2^\varepsilon) > 0$, by (3.29) and (3.30), we have that $v_\varepsilon'(t) > 0$ for all $t \geq t_2^\varepsilon$ such that $v_\varepsilon(t) \leq b - \tau_0\varepsilon^{1/2}$. It follows that $\hat{\alpha}_- \leq v_\varepsilon(t) \leq \hat{\beta}_-$ for all $t \in [t_2^\varepsilon, t_3^3]$. Since ω is increasing, by (3.37),

$$C \geq G_\varepsilon(v_\varepsilon) \geq \frac{\omega(0)}{\varepsilon}\int_{t_2^\varepsilon}^{t_3^\varepsilon} W(v_\varepsilon)\,dt \geq \min_{[\alpha_-,\beta_-]} W\frac{\omega(0)}{\varepsilon}(t_3^\varepsilon - t_2^\varepsilon),$$

which proves (3.39).

Step 3: We claim that there exist $0 < \varepsilon_0 < 1$ and $C > 0$ such that

$$t_4^\varepsilon - t_3^\varepsilon \leq C\varepsilon|\log\varepsilon| \tag{3.40}$$

for all $0 < \varepsilon < \varepsilon_0$. Since $v_\varepsilon'(t) > 0$ for all $t \geq t_3^\varepsilon$ such that $v_\varepsilon(t) \leq b - \tau_0\varepsilon^{1/2}$, there are two possible scenarios. Either $v_\varepsilon(t) \geq \hat{\beta}_-$ for all $t \in [t_3^\varepsilon, t_4^\varepsilon]$, in which case (3.40) follows from Theorem 3.12, or there exists a last time $t_3^\varepsilon < t_\varepsilon < t_4^\varepsilon$ such that $v_\varepsilon = \beta_-$and $\beta_- \le v_\varepsilon(t) \le t_4^\varepsilon$. We claim that this latter case cannot happen.

Since $v_\varepsilon'(t) > 0$ for all $t \ge t_3^\varepsilon$ such that $v_\varepsilon(t) \le b - \tau_0\varepsilon^{1/2}$, there exists $\tau_\varepsilon \in (t_3^\varepsilon, t_\varepsilon)$ such that $v_\varepsilon(\tau_\varepsilon) = b - \tau_0\varepsilon^{1/2}$. It follows that $v_\varepsilon([t_3^\varepsilon, \tau_\varepsilon]) = [\hat\beta_-, b - \tau_0\varepsilon^{1/2}]$, while $v_\varepsilon([t_\varepsilon, t_4^\varepsilon]) = [\hat\beta_-, b - \tau_0\varepsilon^{1/2}]$. Then by (3.37), and (3.5), and the fact that ω is increasing

$$\begin{aligned} C_W\omega(0) + C_1\varepsilon|\log\varepsilon| &\ge G_\varepsilon^{(1)}(v_\varepsilon) \ge G_\varepsilon^{(1)}(v; [0, \tau_\varepsilon] \cup [t_\varepsilon, t_4^\varepsilon]) \\ &\ge \omega(0) \int_{[0,\tau_\varepsilon]\cup[t_\varepsilon,t_4^\varepsilon]} 2W^{1/2}(v_\varepsilon)|v_\varepsilon'|\,dt \\ &= \Big(\mathrm{d}_W(\alpha_\varepsilon, b - \tau_0\varepsilon^{1/2}) + \mathrm{d}_W(\hat\beta_-, \beta_\varepsilon - \varepsilon^k)\Big)\,\omega(0). \end{aligned}$$

Using the fact that $\mathrm{d}_W(\cdot, r)$ and $\mathrm{d}_W(s, \cdot)$ are Lipschitz continuous and (3.12), it follows that

$$C_W\omega(0) + C_1\varepsilon|\log\varepsilon| \ge (C_W + \mathrm{d}_W(\beta_-, b) - L(A_0\varepsilon^\gamma + 2\tau_0\varepsilon^{1/2}))\omega(0),$$

or, equivalently,

$$C(\varepsilon|\log\varepsilon| + \varepsilon^\gamma + \varepsilon^{1/2}) \ge \mathrm{d}_W(\beta_-, b)\omega(0),$$

which is a contradiction provided we take $0 < \varepsilon < \varepsilon_0$ with ε_0 sufficiently small.

Step 4: We claim that there exist $0 < \varepsilon_0 < 1$ and $C_0 > 0$ such that

$$t_1^\varepsilon \le C_0\varepsilon|\log\varepsilon|. \tag{3.41}$$

Fix $C_0 > 0$ such that

$$\frac{1}{2}\omega'(0)C_0C_W > 2C_1, \tag{3.42}$$

where C_1 is the constant in (3.37) and let $0 < \varepsilon < \varepsilon_0$, where ε_0 was .... Assume by contradiction that

$$t_1^\varepsilon > C_0\varepsilon|\log\varepsilon| =: t_0^\varepsilon.$$

Since ω is increasing, we have

$$\begin{aligned} G_\varepsilon(v_\varepsilon) &\ge \int_{t_1^\varepsilon}^T \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right)\omega\,dt \ge \omega(t_0^\varepsilon)\int_{t_1^\varepsilon}^T \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right)\,dt \\ &\ge \omega(t_0^\varepsilon)\int_{\alpha_\varepsilon+\varepsilon^k}^{\beta_\varepsilon} 2W^{1/2}(s)\,ds \ge \omega(t_0^\varepsilon)(C_W - C\varepsilon^{2\gamma}). \end{aligned}$$

By Taylor's formula, for $0 < t < t_0$ for some t_0 small.

$$\omega(t_0^\varepsilon) = \omega(0) + \omega'(0)t_0^\varepsilon + o(t_0^\varepsilon) \ge \omega(0) + \frac{1}{2}\omega'(0)t_0^\varepsilon$$

for all $0 < \varepsilon < \varepsilon_0$ provided ε_0 is taken even smaller (depending on C_0). Then by (3.37),

$$C_W\omega(0) + C_1\varepsilon|\log\varepsilon| \geq G_\varepsilon^{(1)}(v_\varepsilon) \geq \left(\omega(0) + \frac{1}{2}\omega'(0)C_0\varepsilon|\log\varepsilon|\right)(C_W - C\varepsilon^{2\gamma}),$$

and so

$$\frac{1}{2}\omega'(0)C_0C_W\varepsilon|\log\varepsilon| \leq C_1\varepsilon|\log\varepsilon| + \left(\omega(0) + \frac{1}{2}\omega'(0)C_0\varepsilon|\log\varepsilon|\right)C\varepsilon^{2\gamma} \leq 2C_1\varepsilon|\log\varepsilon|$$

provided ε_0 is taken even smaller (depending on C_0). This contradicts (3.42).

Combining Steps 1-4 proves (3.35).

Step 5: In this step, we prove item (ii). Rewrite (3.25) as

$$2\varepsilon^2 v_\varepsilon''(t) - W'(v_\varepsilon(t)) + 2\varepsilon^2\frac{\omega'(t)}{\omega(t)}v_\varepsilon'(t) = 0. \tag{3.43}$$

Multiply (3.43) by $\frac{1}{\varepsilon}v_\varepsilon'(t)$ to get

$$\varepsilon((v_\varepsilon'(t))^2)' - \frac{1}{\varepsilon}(W(v_\varepsilon(t)))' + 2\varepsilon\frac{\omega'(t)}{\omega(t)}(v_\varepsilon'(t))^2 = 0. \tag{3.44}$$

Integrating between 0 and t and we have

$$\varepsilon(v_\varepsilon'(t))^2 - \frac{1}{\varepsilon}W(v_\varepsilon(t)) + 2\varepsilon\int_0^t \frac{\omega'}{\omega}(v_\varepsilon')^2 dt = \varepsilon(v_\varepsilon'(0))^2 - \frac{1}{\varepsilon}W(\alpha_\varepsilon). \tag{3.45}$$

By (3.28) and the fact that $\omega' \geq 0$, we have

$$\varepsilon(v_\varepsilon'(t))^2 - \frac{1}{\varepsilon}W(v_\varepsilon(t)) \leq C.$$

Hence,

$$\varepsilon^2(v_\varepsilon'(t))^2 \leq W(v_\varepsilon(t)) + C\varepsilon \tag{3.46}$$

for all $t \in I$. Let δ_η be as in (3.34) and let $S_{\varepsilon,\eta}$ be the first time such that $v_\varepsilon = a + \delta_\eta$. Then, by (3.34),

$$\begin{aligned}\varepsilon^2(v_\varepsilon'(t))^2 &\leq \frac{1}{2}W''(a)(1+\eta)(v_\varepsilon(t)-a)^2 + C\varepsilon \\ &\leq \frac{1}{2}W''(a)(1+2\eta)(v_\varepsilon(t)-a)^2\end{aligned}$$

provided $a + c_\eta\varepsilon^{1/2} \leq v_\varepsilon(t) \leq a + \delta_\eta$ and $t \leq S_{\varepsilon,\eta}$, where

$$c_\eta := \left(\frac{2C}{W''(a)\eta}\right)^{1/2}.$$

In turn,

$$\frac{\varepsilon v_\varepsilon'(t)}{v_\varepsilon(t) - a} \leq \left(\frac{1}{2}W''(a)(1+2\eta)\right)^{1/2} := L_\eta.$$

Let $R_{\varepsilon,\eta}$ be the first time such that $v_\varepsilon = a+c_\eta\varepsilon^{1/2}$. Note that by (3.29), $v'_\varepsilon(t) \neq 0$ whenever $a + \tau_0\varepsilon^{1/2} \leq v_\varepsilon(t) \leq \hat{\beta}_-$. By taking η smaller, if necessary, we can assume that $c_\eta \geq \tau_0$. Hence, $v_\varepsilon(t) \in [a + c_\eta\varepsilon^{1/2}, a + \delta_\eta]$ for all $t \in [R_{\varepsilon,\eta}, S_{\varepsilon,\eta}]$. Integrating in $[R_{\varepsilon,\eta}, S_{\varepsilon,\eta}]$ and using the change of variables $\rho = v_\varepsilon(t)$ gives

$$\varepsilon \log \delta_\eta - \varepsilon \log(c_\eta\varepsilon^{1/2}) = \int_{R_{\varepsilon,\eta}}^{S_{\varepsilon,\eta}} \frac{\varepsilon v'_\varepsilon(t)}{v_\varepsilon(t) - a}\, dt \leq L_\eta(S_{\varepsilon,\eta} - R_{\varepsilon,\eta}).$$

Therefore,

$$\frac{1}{2}\varepsilon|\log\varepsilon| + \varepsilon\log\delta_\eta - \varepsilon\log c_\eta \leq L_\eta(S_{\varepsilon,\eta} - R_{\varepsilon,\eta}).$$

■

Corollary 3.14 *Assume that W satisfies (2.1)-(2.4) and that ω satisfies (3.1) and is strictly increasing with $\omega'(0) > 0$. Let $a \leq \alpha_\varepsilon$, $\beta_\varepsilon \leq b$ satisfy (3.12) and let v_ε be the minimizer of G_ε obtained in Theorem 3.9. There exist $0 < \varepsilon_0 < 1$, $C > 0$ depending only on A_0, B_0, T, ω, W, such that*

$$|v'_\varepsilon(t)| \geq \frac{C}{\varepsilon^{1/2}}$$

for all $0 < \varepsilon < \varepsilon_0$ and for all t such that $\alpha_\varepsilon \leq v_\varepsilon(t) \leq a + \tau_0\varepsilon^{1/2}$, where τ_0 is the constant given in Theorem 3.11.

Proof. In this proof, ε_0 and C depend only on A_0, B_0, T, ω, W. Since ω is increasing

$$\begin{aligned}\int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v'_\varepsilon)^2\right)\omega\, dt &\geq 2\omega(0)\int_0^{T_\varepsilon} W^{1/2}(v_\varepsilon)v'_\varepsilon\, dt \\ &= 2\omega(0)\int_{\alpha_\varepsilon}^{\beta_\varepsilon-\varepsilon^k} W^{1/2}(\rho)\, d\rho\end{aligned}$$

and so

$$\begin{aligned}\int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v'_\varepsilon)^2\right)\omega\, dt - C_W\omega(0) &\geq -2\omega(0)\int_{\beta_\varepsilon-\varepsilon^k}^{\beta_k} W^{1/2}(\rho)\, d\rho \\ &\quad - 2\omega(0)\int_a^{\alpha_\varepsilon} W^{1/2}(\rho)\, d\rho \geq -C\varepsilon^{2\gamma}.\end{aligned}$$

Hence, also by (3.37),

$$\begin{aligned}\int_{T_\varepsilon}^{T} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v'_\varepsilon)^2\right)\omega\, dt &= \int_0^{T} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v'_\varepsilon)^2\right)\omega\, dt \\ &\quad - \int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v'_\varepsilon)^2\right)\omega\, dt \\ &\leq C_W\omega(0) + C_1\varepsilon|\log\varepsilon| - C_W\omega(0) + C\varepsilon^{2\gamma} \leq C\varepsilon|\log\varepsilon|\end{aligned}$$

since $\gamma > 1$. Therefore,

$$\int_{T_\varepsilon}^{T} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right) \omega\, dt \leq C\varepsilon|\log\varepsilon|.$$

Since

$$\begin{aligned}\frac{2}{T}\int_{T/2}^{T} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right) dt &\leq \frac{2}{T\min\omega}\int_{T_\varepsilon}^{T} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right) \omega\, dt \\ &\leq C\varepsilon|\log\varepsilon|,\end{aligned}$$

by the mean value theorem, there exists t_ε such that

$$\frac{1}{\varepsilon}W(v_\varepsilon(t_\varepsilon)) + \varepsilon(v_\varepsilon'(t_\varepsilon))^2 = \frac{2}{T}\int_{T/2}^{T} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right) dt \leq C\varepsilon|\log\varepsilon|.$$

Integrating (3.44) from t to t_ε we get

$$\begin{aligned}\varepsilon(v_\varepsilon'(t))^2 - \frac{1}{\varepsilon}W(v_\varepsilon(t_\varepsilon)) &= \varepsilon(v_\varepsilon'(t_\varepsilon))^2 - \frac{1}{\varepsilon}W(v_\varepsilon(t_\varepsilon)) + \int_t^{t_\varepsilon} 2\varepsilon(v_\varepsilon')^2\frac{\omega'}{\omega}\, dr \\ &\geq -C\varepsilon|\log\varepsilon| + \int_t^{t_\varepsilon} 2\varepsilon(v_\varepsilon')^2\frac{\omega'}{\omega}\, dr. \qquad (3.47)\end{aligned}$$

Since $\omega'(0) > 0$ and ω' is continuous, there exists $\tau_0 > 0$ such that $\omega'(t) \geq \frac{1}{2}\omega'(0)$ for all $0 < t \leq \tau_0$. By taking ε_0 even smaller, we can assume that $T_\varepsilon < \tau_0$ for all $0 < \varepsilon < \varepsilon_0$. It follows that

$$\int_t^{t_\varepsilon} 2\varepsilon(v_\varepsilon')^2\frac{\omega'}{\omega}\, dr \geq \varepsilon\frac{\omega'(0)}{\min\omega}\int_t^{T_\varepsilon} 2\varepsilon(v_\varepsilon')^2\, dt. \qquad (3.48)$$

Let P_ε be the first time such that $v_\varepsilon = a + \tau_0\varepsilon^{1/2}$, where τ_0 is the constant in Theorem 3.11. Then by Theorem 3.11, and the properties of W,

$$\varepsilon(v_\varepsilon')^2 \geq \frac{1}{2}\sigma^2(v_\varepsilon - a)^2 \geq CW(v_\varepsilon)$$

for all t such that $a + \tau_0\varepsilon^{1/2} \leq v_\varepsilon \leq c$. Let $J \subseteq [P_\varepsilon, T_\varepsilon]$ be a maximal interval such that $a + \tau_0\varepsilon^{1/2} \leq v_\varepsilon \leq c$. It follows that

$$\begin{aligned}\int_t^{T_\varepsilon} 2\varepsilon(v_\varepsilon')^2\omega\, dr &\geq \int_J [\varepsilon(v_\varepsilon')^2 + CW(v_\varepsilon)]\, dr \geq C\int_J 2W^{1/2}(v_\varepsilon)v_\varepsilon'\, dr \\ &= C\int_{a+\tau_0\varepsilon^{1/2}}^{c} 2W^{1/2}(s)\, ds \geq C\int_{\frac{a+c}{2}}^{c} 2W^{1/2}(s)\, ds =: C_1\end{aligned}$$

for all $0 \leq t \leq P_\varepsilon$. In turn, from (3.47) and (3.48),

$$\varepsilon(v_\varepsilon'(t))^2 - \frac{1}{\varepsilon}W(v_\varepsilon(t)) \geq -C\varepsilon|\log\varepsilon| + \frac{\omega'(0)}{\min\omega}C_1.$$

Hence,

$$\varepsilon(v_\varepsilon'(t))^2 \geq C_2 > 0$$

for all $0 \leq t \leq P_\varepsilon$. ■

Remark 3.15 *Note that this corollary, together with Theorem 3.11, implies that* v'_ε *does not vanish as long as* $v_\varepsilon(t) \le b - \tau_0\varepsilon^{1/2}$*. Hence* $v'_\varepsilon(t) > 0$ *as long as* $v_\varepsilon(t) \le b - \tau_0\varepsilon^{1/2}$*.*

3.4 Second-Order Γ-liminf

In this subsection, we present the Γ-liminf counterparts of Theorems 3.6 and 3.8. We recall that when $a < \alpha$, $G_\varepsilon^{(2)}$ is defined as in (3.7). For the proof of the following theorem, we refer to [7, Theorem 3.15].

Theorem 3.16 (Second-Order Liminf, $a < \alpha$**)** *Assume that* W *satisfies* (2.1)-(2.4)*, that that* α_- *satisfies* (2.5)*, and that* ω *satisfies* (3.1)*,* (3.5)*, and* (3.8)*. Let* $\alpha_- \le \alpha_\varepsilon, \beta_\varepsilon \le b$ *satisfy* (3.12) *and let* v_ε *be the minimizer of* G_ε *obtained in Theorem 3.9. Then there exist* $0 < \varepsilon_0 < 1$*,* $C > 0$*, and* $l_0 > 1$*, depending only on* α_-*,* A_0*,* B_0*,* T*,* ω*, and* W*, such that*

$$G_\varepsilon^{(2)}(v_\varepsilon) \ge 2\omega'(0)\int_0^l W^{1/2}(w_\varepsilon)w'_\varepsilon s\,ds - Ce^{-l\mu}(l\mu+1) - C\varepsilon^{1/2}l - C\varepsilon^{\gamma_1}|\log\varepsilon|^{2+\gamma_0}$$

for all $0 < \varepsilon < \varepsilon_0$ *and* $l > l_0$*, where* $G_\varepsilon^{(2)}$ *is defined in* (3.7)*,* $w_\varepsilon(s) := v_\varepsilon(\varepsilon s)$ *for* $s \in [0, T\varepsilon^{-1}]$ *satisfies*

$$\lim_{\varepsilon\to 0^+}\int_0^l W^{1/2}(w_\varepsilon)w'_\varepsilon s\,ds = \int_0^l W^{1/2}(z_\alpha)z'_\alpha s\,ds$$

for every $l > 0$*, and where* z_α *solves the Cauchy problem* (1.12)*. In particular,*

$$\liminf_{\varepsilon\to 0^+} G_\varepsilon^{(2)}(v_\varepsilon) \ge 2\omega'(0)\int_0^\infty W^{1/2}(z_\alpha)z'_\alpha s\,ds.$$

When $\alpha = a$, $G_\varepsilon^{(2)}$ is defined as in (3.11).

Theorem 3.17 (Second-Order Liminf, $a = \alpha$**)** *Assume that* W *satisfies* (2.1)-(2.4) *and that* ω *satisfies* (3.1) *and is strictly increasing with* $\omega'(0) > 0$*. Let* $a \le \alpha_\varepsilon, \beta_\varepsilon \le b$ *satisfy* (3.12) *and let* v_ε *be the minimizer of* G_ε *obtained in Theorem 3.9. Then for every* $0 < \eta < \frac{1}{4}$ *there exist a constant* $C_\eta > 0$*, depending on* η*,* A_0*,* B_0*,* T*,* ω*,* W*, such that*

$$G_\varepsilon^{(2)}(v_\varepsilon) \ge \frac{C_W\omega'(0)}{2^{1/2}(W''(a))^{1/2}}(1-\eta) - \frac{C_\eta}{|\log\varepsilon|}$$

for all $0 < \varepsilon < \varepsilon_\eta$*, where* $\varepsilon_\eta > 0$ *depends on* η*,* A_0*,* B_0*,* T*,* ω*,* W*, and where* $G_\varepsilon^{(2)}$ *is defined in* (3.11)*.*

Proof. In this proof, the constants ε_0, C, and C_1 depend only on A_0, B_0, T, ω, W, while ε_η and C_η depend on all these parameters but also on η. Since $\omega \in C^{1,d}(I)$, by Taylor's formula, for $t \in [0,T]$,

$$\omega(t) = \omega(0) + \omega'(0)t + R_1(t),$$

where

$$|R_1(t)| = |\omega'(\theta t) - \omega'(0)|t \le |\omega'|_{C^{0,d}} t^{1+d}. \tag{3.49}$$

Let T_ε be the first time such that $v_\varepsilon = \beta_\varepsilon - \varepsilon^k$, and write

$$\begin{aligned} G_\varepsilon^{(2)}(v_\varepsilon) &= \left[\int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right) dt - C_W\right] \frac{\omega(0)}{\varepsilon|\log\varepsilon|} \\ &+ \int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right) t\,dt \frac{\omega'(0)}{\varepsilon|\log\varepsilon|} \\ &+ \int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right) R_1\,dt \frac{1}{\varepsilon|\log\varepsilon|} \\ &+ \int_{T_\varepsilon}^{T} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right) \omega\,dt \frac{1}{\varepsilon|\log\varepsilon|} =: \mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}. \end{aligned} \tag{3.50}$$

Step 1: We estimate $\mathcal{A}$. By the change of variables $\rho = v_\varepsilon(t)$, (2.7), (2.6), and (3.12), we have

$$\begin{aligned} \int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right) dt &\ge 2\int_0^{T_\varepsilon} 2W^{1/2}(v_\varepsilon)v_\varepsilon'\,dt = 2\int_{\alpha_\varepsilon}^{\beta_\varepsilon-\varepsilon^k} W^{1/2}(\rho)\,d\rho \\ &= C_W - 2\int_a^{\alpha_\varepsilon} W^{1/2}(\rho)\,d\rho - 2\int_{\beta_\varepsilon-\varepsilon^k}^{b} W^{1/2}(\rho)\,d\rho \\ &= C_W - C\varepsilon^{2\gamma} \end{aligned} \tag{3.51}$$

for all $0 < \varepsilon < \varepsilon_0$. Hence,

$$\mathcal{A} \ge -C\frac{\varepsilon^{2\gamma-1}}{|\log\varepsilon|}$$

for all $0 < \varepsilon < \varepsilon_0$.

Step 2: We estimate $\mathcal{B}$ in (3.50). Let $0 < \eta < \frac{1}{4}$, let δ_η be as in (3.34), and let $S_{\varepsilon,\eta}$ be the first time such that $v_\varepsilon = a + \delta_\eta$. By the change of variables $t = r + S_{\varepsilon,\eta}$,

$$\begin{aligned} \mathcal{B} &= \int_{-S_{\varepsilon,\eta}}^{T_\varepsilon - S_{\varepsilon,\eta}} \left(\frac{1}{\varepsilon}W(\bar v_\varepsilon) + \varepsilon(\bar v_\varepsilon')^2\right) dr \frac{\omega'(0)S_{\varepsilon,\eta}}{\varepsilon|\log\varepsilon|} \\ &+ \int_{-S_{\varepsilon,\eta}}^{T_\varepsilon - S_{\varepsilon,\eta}} \left(\frac{1}{\varepsilon}W(\bar v_\varepsilon) + \varepsilon(\bar v_\varepsilon')^2\right) r\,dr \frac{\omega'(0)}{\varepsilon|\log\varepsilon|} \\ &=: \mathcal{B}_1 + \mathcal{B}_2, \end{aligned}$$

where $\bar v_\varepsilon(r) := v_\varepsilon(r + S_{\varepsilon,\eta})$. By the change of variables $r := t - S_{\varepsilon,\eta}$, (3.51), and (3.36),

$$\begin{aligned} \mathcal{B}_1 &\ge (C_W - C\varepsilon^{2\gamma})\frac{\omega'(0)S_{\varepsilon,\eta}}{\varepsilon|\log\varepsilon|} \\ &\ge \frac{C_W\omega'(0)}{2^{1/2}(W''(a))^{1/2}}(1-\eta) - C\varepsilon^{2\gamma} - \frac{C_\eta}{|\log\varepsilon|} \end{aligned}$$

for all $0 < \varepsilon < \varepsilon_\eta$.

Define

$$p_\varepsilon(s) := v_\varepsilon(\varepsilon s + S_{\varepsilon,\eta}).$$

Then

$$\begin{aligned}\mathcal{B}_2 &= \frac{\omega'(0)}{|\log\varepsilon|}\int_{-S_{\varepsilon,\eta}\varepsilon^{-1}}^{(T_\varepsilon - S_{\varepsilon,\eta})\varepsilon^{-1}} \left(W(p_\varepsilon(s)) + (p_\varepsilon'(s))^2\right) s\, ds \\ &\geq -\frac{\omega'(0)}{|\log\varepsilon|}\int_{-S_{\varepsilon,\eta}\varepsilon^{-1}}^{0} \left(W(p_\varepsilon(s)) + (p_\varepsilon'(s))^2\right) |s|\, ds\end{aligned}$$

By (3.46), we have that

$$(p_\varepsilon'(s))^2 \leq W(p_\varepsilon(s)) + C\varepsilon.$$

Hence, by (3.35),

$$\begin{aligned}\mathcal{B}_2 &\geq -\frac{\omega'(0)}{|\log\varepsilon|}\int_{-S_{\varepsilon,\eta}\varepsilon^{-1}}^{0} 2\left(W(p_\varepsilon(s)) + C\varepsilon\right) |s|\, ds \\ &\geq -\frac{\omega'(0)}{|\log\varepsilon|}\int_{-S_{\varepsilon,\eta}\varepsilon^{-1}}^{0} 2W(p_\varepsilon(s))|s|\, ds - C\varepsilon|\log\varepsilon|\end{aligned} \tag{3.52}$$

for all $0 < \varepsilon < \varepsilon_\eta$. By Corollary 3.14 and Remark 3.15,

$$v_\varepsilon'(t) \geq \frac{C}{\varepsilon^{1/2}}$$

for all t such that $v_\varepsilon(t) \leq a + \tau_0\varepsilon^{1/2}$. Therefore,

$$\varepsilon^2(v_\varepsilon'(t))^2 \geq C\varepsilon \geq \frac{C}{\tau_0^2}(v_\varepsilon(t) - a)^2.$$

Together with Theorem 3.11, this implies that

$$\varepsilon v_\varepsilon'(t) \geq \sigma_0(v_\varepsilon(t) - a)$$

for all $t \geq 0$ such that $v_\varepsilon(t) \leq c$, where $\sigma_0 > 0$. In turn,

$$p_\varepsilon'(s) \geq \sigma_0(p_\varepsilon(s) - a)$$

for all $-S_{\varepsilon,\eta}\varepsilon^{-1} \leq s \leq 0$. Hence,

$$(\log(p_\varepsilon(s) - a))' = \frac{(p_\varepsilon(s) - a)'}{p_\varepsilon(s) - a} \geq \sigma_0.$$

Upon integration, we get

$$\log\frac{\delta_\eta}{p_\varepsilon(s) - a} \geq \sigma_0(0 - s)$$

and so

$$c - a \geq \delta_\eta \geq (p_\varepsilon(s) - a)e^{-s\sigma_0},$$

which gives

$$0 \leq p_\varepsilon(s) - a \leq (c-a)e^{\sigma_0 s}.$$

In turn, again by (2.7), for $s \in [-L_\varepsilon \varepsilon^{-1}, 0]$,

$$W(p_\varepsilon(s)) \leq C(p_\varepsilon(s) - a)^2 \leq Ce^{2\sigma_0 s}.$$

Hence,

$$\int_{-S_{\varepsilon,\eta}\varepsilon^{-1}}^0 2W(p_\varepsilon(s))|s|\,ds \leq C\int_{-S_{\varepsilon,\eta}\varepsilon^{-1}}^0 e^{2\sigma_0 s}|s|\,ds \leq C\int_{-\infty}^0 e^{2\sigma_0 s}|s|\,ds.$$

By (3.52),

$$\mathcal{B}_2 \geq -\frac{C}{|\log\varepsilon|} - C\varepsilon|\log\varepsilon|$$

for all $0 < \varepsilon < \varepsilon_\eta$. .

Step 3: We estimate $\mathcal{C}$ in (3.50). By Theorem 3.8,

$$G_\varepsilon^{(1)}(v_\varepsilon) \leq C_W\omega(0) + C_1\varepsilon|\log\varepsilon|$$

for all $0 < \varepsilon < \varepsilon_0$. We have,

$$\begin{aligned}
|\mathcal{C}| &\leq C\varepsilon^d|\log\varepsilon|^d \int_0^{T_\varepsilon}\left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right)dt \\
&\leq C\varepsilon^d|\log\varepsilon|^d \frac{1}{\min\omega}\int_0^{T_\varepsilon}\left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right)\omega\,dt \\
&\leq C\varepsilon^d|\log\varepsilon|^d \frac{1}{\min\omega}(C_W\omega(0) + C_1\varepsilon|\log\varepsilon|) \leq C\varepsilon^d|\log\varepsilon|^d
\end{aligned}$$

for all $0 < \varepsilon < \varepsilon_\eta$, where we used (3.35) and (3.49).

Step 4: To estimate $\mathcal{D}$ in (3.50), observe that $\mathcal{D} \geq 0$.

Combining the estimates in Steps 1-4 and using (3.50) gives

$$G_\varepsilon^{(2)}(v_\varepsilon) \geq \frac{C_W\omega'(0)}{2^{1/2}(W''(a))^{1/2}}(1-\eta) - \frac{C_\eta}{|\log\varepsilon|}$$

for all $0 < \varepsilon < \varepsilon_\eta$. ■

4 Properties of Minimizers of F_ε

In this section, we study qualitative properties of critical points and minimizers of the functional F_ε given in (1.1) and subject to the Dirichlet boundary conditions (1.2).

Theorem 4.1 *Assume that $\partial\Omega$ is of class C^2 and that $g : \mathbb{R}^N \to \mathbb{R}$ is a function of class C^1 such that*

$$a \le g(x) \le b \quad \text{for all } x \in \partial\Omega,$$

and that there exists $x_0 \in \partial\Omega$ such that

$$\kappa(x_0) > 0 \quad \text{and} \quad g(x_0) = a$$

Then the constant function b is not a minimizer of the functional $\mathcal{F}^{(1)}$ given in (1.5).

Proof. Since the boundary of Ω is of class C^2, without loss of generality by a translation and a rotation we can assume that $x_0 = 0$ and that there exist $r_0 > 0$ and a function $f : \mathbb{R}^{N-1} \to \mathbb{R}$ of class C^3 such that $f(0) = 0$, $\nabla' f(0) = 0$ and

$$Q(0, r_0) \cap \Omega = \{x \in Q(0, r_0) : \, x_N > f(x')\}, \tag{4.1}$$

where, with a slight abuse of notation, we are writing $x := (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, $Q'(0, r) := (-r, r)^{N-1}$, and $Q(0, r) := (-r, r)^N$. Let $\varphi \in C_c^\infty(Q'(0, 1)) \to [0, 1]$ be such that $\int_{Q'(0,1)} \varphi(y') \, dy' = 1$. For $0 < r \le r_0$, define

$$\varphi_r(x') := \varphi(x'/r).$$

Consider the function $u_0 : \Omega \to \mathbb{R}$ given by

$$u_0(x) := \begin{cases} a & \text{if } x \in Q(0, r) \cap \Omega \text{ and } x_N \le f(x') + r^3 \varphi_r(x'), \\ b & \text{elsewhere in } \Omega. \end{cases}$$

Define

$$\Gamma_f := \{(x', f(x')) : \, x' \in Q'(0, r)\},$$
$$\Gamma_{f + r^3 \varphi_r} := \{(x', f(x') + r^3 \varphi_r(x')) : \, x' \in Q'(0, r)\},$$

By contradiction, assume that b is a minimizer of $\mathcal{F}^{(1)}$. Then

$$\begin{aligned} \mathcal{F}^{(1)}(b) = \int_{\partial\Omega} \mathrm{d}_W(b, g) \, d\mathcal{H}^{N-1} \le \mathcal{F}^{(1)}(u_0) = C_W \mathcal{H}^{N-1}(\Omega \cap \Gamma_{f + r^3 \varphi_r}) \\ + \int_{(\partial\Omega \setminus Q(0,r)) \cup (\partial\Omega \cap \Gamma_{f + r^3 \varphi_r})} \mathrm{d}_W(b, g) \, d\mathcal{H}^{N-1} \\ + \int_{(\partial\Omega \cap Q(0,r)) \setminus \Gamma_{f + r^3 \varphi_r}} \mathrm{d}_W(a, g) \, d\mathcal{H}^{N-1}, \end{aligned}$$

which is equivalent to writing

$$\int_{(\partial\Omega \cap Q(0,r)) \setminus \Gamma_{f + r^3 \varphi_r}} (\mathrm{d}_W(b, g) - \mathrm{d}_W(a, g)) \, d\mathcal{H}^{N-1} \le C_W \mathcal{H}^{N-1}(\Omega \cap \Gamma_{f + r^3 \varphi_r}). \tag{4.2}$$

Define $\bar{g}(x') := g(x', f(x'))$, $x' \in \mathbb{R}^{N-1}$. Since $\bar{g}(0) = a$ and $a \le \bar{g}(x')$ for all x' small, 0 is a point of local minimum, and so $\nabla' \bar{g}(0) = 0$. Since $W^{1/2}(\rho) \sim (\rho - a)$ as $\rho \to a$, by Taylor's formula applied to the function $x' \mapsto \int_a^{\bar{g}(x')} W^{1/2}(\rho)\, d\rho$, we can write

$$\begin{aligned} \mathrm{d}_W(b, \bar{g}(x')) - \mathrm{d}_W(a, \bar{g}(x')) &= 2\int_{\bar{g}(x')}^{b} W^{1/2}(\rho)\, d\rho - 2\int_a^{\bar{g}(x')} W^{1/2}(\rho)\, d\rho \\ &= C_W - 4\int_a^{\bar{g}(x')} W^{1/2}(\rho)\, d\rho \\ &= C_W + O(|x'|^4). \end{aligned}$$

Then (4.1) and (4.2) imply

$$\begin{aligned} &\int_{Q'(0,r)\cap\{\varphi_r>0\}} (C_W + O(|x'|^4))(1 + |\nabla' f(x')|^2)^{1/2} dx' \\ &\le C_W \int_{Q'(0,r)\cap\{\varphi_r>0\}} (1 + |\nabla' f(x') + r^3 \nabla' \varphi_r(x')|^2)^{1/2} dx', \end{aligned}$$

or, equivalently,

$$\begin{aligned} &C_W \int_{Q'(0,r)\cap\{\varphi_r>0\}} \left((1 + |\nabla' f|^2)^{1/2} - (1 + |\nabla' f + r^3 \nabla' \varphi_r|^2)^{1/2} \right) dx' \\ &\le C r^4 \int_{Q'(0,r)\cap\{\varphi_r>0\}} (1 + |\nabla' f|^2)^{1/2} dx'. \end{aligned} \tag{4.3}$$

Using the fact that $(1+t)^{1/2} \le 1 + \frac{1}{2}t$ for $t \ge -1$,we have

$$\begin{aligned} (1 + |\nabla' f + r^3 \nabla' \varphi_r|^2)^{1/2} &= (1 + |\nabla' f|^2)^{1/2} \left(1 + \frac{2r^3 \nabla' f \cdot \nabla' \varphi_r}{1 + |\nabla' f|^2} + r^6 \frac{|\nabla' \varphi_r|^2}{1 + |\nabla' f|^2} \right)^{1/2} \\ &\le (1 + |\nabla' f|^2)^{1/2} + \frac{r^3 \nabla' f \cdot \nabla' \varphi_r}{(1 + |\nabla' f|^2)^{1/2}} + r^6 \frac{|\nabla' \varphi_r|^2}{(1 + |\nabla' f|^2)^{1/2}}. \end{aligned}$$

Hence,

$$\begin{aligned} &-C_W r^3 \int_{Q'(0,r)\cap\{\varphi_r>0\}} \frac{\nabla' f \cdot \nabla' \varphi_r}{(1 + |\nabla' f|^2)^{1/2}} dx' \\ &\le r^6 C_W \int_{Q'(0,r)\cap\{\varphi_r>0\}} \frac{|\nabla' \varphi_r|^2}{(1 + |\nabla' f|^2)^{1/2}} dx' \\ &\quad + C r^4 \int_{Q'(0,r)\cap\{\varphi_r>0\}} (1 + |\nabla' f|^2)^{1/2} dx'. \end{aligned}$$

Integrating by parts the first integral and using the fact that $\|\nabla' \varphi_r\|_\infty \le \frac{C}{r}$ gives

$$C_W r^3 \int_{Q'(0,r)} \mathrm{div}_{x'} \left(\frac{\nabla' f}{(1 + |\nabla' f|^2)^{1/2}} \right) \varphi_r \, dx' \le C r^{N+3}$$

Dividing this inequality by r^{N+2}, and considering the change of variables $y' := r^{-1}x'$, gives

$$C_W \int_{Q'(0,1)} \kappa(ry')\varphi(y')\,dy' \leq Cr.$$

Letting $r \to 0^+$ and recalling that $\int_{Q'(0,1)} \varphi(y')\,dy' = 1$, we have

$$C_W\kappa(0) \leq 0,$$

which is a contradiction. ■

For the proof of the following theorem, we refer to [7, Theorem 4.9].

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected set with boundary of class $C^{2,d}$, $0 < d \leq 1$. Assume that W satisfies hypotheses* (2.1)*-*(2.4) *and that g_ε satisfy hypotheses* (1.13)*,* (2.14)*-*(2.16)*. Suppose also that* (1.9) *holds. Let $0 < \delta << 1$, then there exist $\mu > 0$ and $C > 0$, independent of ε and δ, such that for all ε sufficiently small the following estimate holds*

$$0 \leq b - u_\varepsilon(x) \leq Ce^{-\mu\delta/\varepsilon} \quad \textit{for } x \in \Omega \setminus \Omega_{2\delta}. \tag{4.4}$$

5 Second-Order Γ-Limit

In this section, we finally prove Theorem 1.2.

Theorem 5.1 (Second-Order Γ-Limsup) *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected set with boundary of class $C^{2,d}$, $0 < d \leq 1$. Assume that W satisfies* (2.1)*-*(2.4) *and that g_ε satisfy* (1.13)*,* (1.14)*,* (2.14)*-*(2.16)*. Suppose also that* (1.9) *holds. Then there exists $\{u_\varepsilon\}_\varepsilon$ in $H^1(\Omega)$ such that $\operatorname{tr} u_\varepsilon = g_\varepsilon$ on $\partial\Omega$, $u_\varepsilon \to b$ in $L^1(\Omega)$, and*

$$\limsup_{\varepsilon\to0^+} \mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) \leq \frac{C_W}{2^{1/2}(W''(a))^{1/2}} \int_{\partial\Omega\cap\{g=a\}} \kappa(y)\,d\mathcal{H}^{N-1}(y).$$

Here, $\mathcal{F}^{(2)}$ is defined in (1.15)*, κ is the mean curvature of $\partial\Omega$, and C_W is the constant defined in* (1.7)*.*

Proof. By Lemma 2.4, for $\delta > 0$ sufficiently small, the function $\Phi : \partial\Omega\times[0,\delta] \to \overline{\Omega}_\delta$ is of class $C^{1\,d}$. In turn, the function

$$\omega(y,t) := \det J_\Phi(y,t)$$

is of class $C^{1,d}$,

$$\omega_1 := \min_{y\in\partial\Omega} \omega(y,0) > 0, \quad \omega(y,0) = 1 \quad \text{for all } y \in \partial\Omega, \tag{5.1}$$

and

$$\frac{\partial\omega}{\partial t}(y,0) = \kappa(y) \quad \text{for all } y \in \partial\Omega, \tag{5.2}$$

where $\kappa(y)$ is the mean curvature of $\partial\Omega$ at y.

In view of (1.14), $\operatorname{dist}(\{g=a\}, \partial\{\kappa<0\}) =: \rho_0 > 0$. Let

$$K_1 := \{x \in \partial\Omega :\ \operatorname{dist}(x, \{g=a\}) \geq \rho_0/2\},$$
$$K_2 := \{x \in \partial\Omega :\ \operatorname{dist}(x, \{g=a\}) \leq \rho_0/2\}.$$

Then

$$\min_{K_1} g =: g_- > a.$$

Fix

$$0 < \omega_0 < \frac{1}{4}\frac{C_W - \mathrm{d}_W(a, g_-)}{C_W}\omega_1. \tag{5.3}$$

By taking $\delta > 0$ sufficiently small, we can assume that

$$|\omega(y, t_1) - \omega(y, t_2)| \leq \omega_0 \tag{5.4}$$

for all $y \in \partial\Omega$ and all $t_1, t_2 \in [0, \delta]$. Since ω is of class $C^{1,d}$ and $\kappa < 0$ in K_2, by (5.2) and taking δ even smaller, we can assume that

$$\frac{\partial\omega}{\partial t}(y, t) < 0 \tag{5.5}$$

for all $y \in K_2$ and $t \in [0, \delta]$.

For each $y \in \overline{\Omega}$, define

$$\Psi_\varepsilon(y, r) := \int_{g_\varepsilon(y)}^{r} \frac{\varepsilon}{(\varepsilon + W(s))^{1/2}}\, ds, \tag{5.6}$$

and

$$0 < T_\varepsilon(y) := \Psi_\varepsilon(y, b). \tag{5.7}$$

Note that $T_\varepsilon \in C^1(\overline{\Omega})$, with

$$T_\varepsilon(y) \leq \int_a^b \frac{\varepsilon}{(\varepsilon + W(s))^{1/2}}\, ds \leq C_0\varepsilon|\log\varepsilon| \tag{5.8}$$

for all $0 < \varepsilon < \varepsilon_0$ and all $y \in \partial\Omega$ by (2.9), where $C_0 > 0$ and $\varepsilon_0 > 0$ depend only on W

For each fixed $y \in \partial\Omega$, let $v_\varepsilon(y, \cdot) : [0, T_\varepsilon(y)] \to [g_\varepsilon(y), b]$ be the inverse of $\Psi_\varepsilon(y, \cdot)$. Then $v_\varepsilon(y, 0) = g_\varepsilon(y)$, $v_\varepsilon(y, T_\varepsilon(y)) = b$, and

$$\frac{\partial v_\varepsilon}{\partial t}(y, t) = \frac{(\varepsilon + W(v_\varepsilon(y, t)))^{1/2}}{\varepsilon} \tag{5.9}$$

for $t \in [0, T_\varepsilon(y)]$. Assume first that $g_\varepsilon \in C^1(\partial\Omega)$. Then, by standard results on the smooth dependence of solutions on a parameter (see, e.g. [17, Section 2.4]), we see that v_ε is of class C^1 in the variables (y, t). Extend $v_\varepsilon(y, t)$ to be equal to b for $t > T_\varepsilon(y)$.

We have

$$v_\varepsilon(y,\Psi_\varepsilon(y,r)) = r$$

for all $g_\varepsilon(y) \le r \le b$. For every $y \in \partial\Omega$ and every tangent vector τ to $\partial\Omega$ at y, differentiating in the direction τ gives

$$\frac{\partial v_\varepsilon}{\partial \tau}(y,\Psi_\varepsilon(y,r)) + \frac{\partial v_\varepsilon}{\partial t}(y,\Psi_\varepsilon(y,r))\frac{\partial \Psi_\varepsilon}{\partial \tau}(y,r) = 0.$$

Hence,

$$\frac{\partial v_\varepsilon}{\partial \tau}(y,t) + \frac{\partial v_\varepsilon}{\partial t}(y,t)\frac{\partial \Psi_\varepsilon}{\partial \tau}(y,r) = 0$$

for all $y \in \partial\Omega$ and $t \in [0, T_\varepsilon(y))$.

By (5.6),

$$\frac{\partial \Psi_\varepsilon}{\partial \tau}(y,r) = -\frac{\varepsilon}{(\varepsilon + W(g_\varepsilon(y)))^{1/2}}\frac{\partial g_\varepsilon}{\partial \tau}(y),$$

and so by (5.9), we have

$$\begin{aligned}\frac{\partial v_\varepsilon}{\partial \tau}(y,t) &= -\frac{\partial v_\varepsilon}{\partial t}(y,t)\frac{\partial \Psi_\varepsilon}{\partial \tau}(y,r)\\ &= \frac{(\varepsilon + W(v_\varepsilon(y,t)))^{1/2}}{(\varepsilon + W(g_\varepsilon(y)))^{1/2}}\frac{\partial g_\varepsilon}{\partial \tau}(y)\end{aligned}$$

for $t \in [0, T_\varepsilon(y))$, while $\frac{\partial v_\varepsilon}{\partial \tau}(y,t) = 0$ for $t > T_\varepsilon(y)$. Observe that if $g_\varepsilon(y) \ge c$, then since W is decreasing for $c \le s \le s$ and $v_\varepsilon(y,\cdot)$ is increasing, we have $W(v_\varepsilon(y,t)) \le W(g_\varepsilon(y))$. Thus, $\left|\frac{\partial v_\varepsilon}{\partial \tau}(y,t)\right| \le \left|\frac{\partial g_\varepsilon}{\partial \tau}(y)\right|$. On the other hand, if $g_\varepsilon(y) \le c$, then by (1.8),

$$(\varepsilon + W(g_\varepsilon(y)))^{1/2} \ge \min_{[g_-,c]} W^{1/2} =: W_0 > 0.$$

Since $a \le v_\varepsilon(y,t) \le b$, in both cases, we have

$$\left|\frac{\partial v_\varepsilon}{\partial \tau}(y,t)\right| \le \begin{cases} C\left|\frac{\partial g_\varepsilon}{\partial \tau}(y)\right| & \text{if } y \in \partial\Omega \text{ and } t \in [0, T_\varepsilon(y)),\\ 0 & \text{if } y \in \partial\Omega \text{ and } t \in (T_\varepsilon(y), \delta]. \end{cases} \tag{5.10}$$

If $g_\varepsilon \in H^1(\partial\Omega)$, a density argument shows that $v_\varepsilon \in H^1(\partial\Omega \times (0,\delta))$ and that (5.9) and (5.10) continues to hold a.e.

Set

$$u_\varepsilon(x) := \begin{cases} v_\varepsilon(\Phi^{-1}(x)) & \text{if } x \in \Omega_\delta,\\ b & \text{if } x \in \Omega \setminus \Omega_\delta, \end{cases}. \tag{5.11}$$

Then $u_\varepsilon \in H^1(\Omega)$, with

$$|\nabla u_\varepsilon(x)|^2 \le \left|\frac{\partial v_\varepsilon}{\partial t}(\Phi^{-1}(x))\right|^2 + C\|\nabla y\|^2_{L^\infty(\Omega_\delta)}\left|\nabla_\tau v_\varepsilon(\Phi^{-1}(x))\right|^2, \tag{5.12}$$

where we used the facts that $\Phi^{-1}(x) = (y(x), \operatorname{dist}(x,\partial\Omega))$, $|\nabla \operatorname{dist}(x,\partial\Omega)| = 1$, and $\tau \cdot \nabla \operatorname{dist}(x,\partial\Omega) = 0$ for every vector τ such that $\tau \cdot \nu(y) = 0$.

In view of Lemma 2.4, we can use the change of variables $x := \Phi(y,t)$ and Tonelli's theorem to write

$$\begin{aligned}
\mathcal{F}^{(2)}_\varepsilon(u_\varepsilon) &= \frac{1}{\varepsilon|\log\varepsilon|}\int_{\partial\Omega}\int_0^\delta \left(\frac{1}{\varepsilon}W(u_\varepsilon(\Phi(y,t))) + \varepsilon|\nabla u_\varepsilon(\Phi(y,t))|^2\right)\omega(y,t)\,dtd\mathcal{H}^{N-1}(y)\\
&\quad - \frac{1}{\varepsilon|\log\varepsilon|}\int_{\partial\Omega} \mathrm{d}_W(g(y),b)\,d\mathcal{H}^{N-1}(y) \qquad (5.13)\\
&\le \frac{1}{\varepsilon|\log\varepsilon|}\left(\int_{\partial\Omega}\int_0^\delta\left(\frac{1}{\varepsilon}W(v_\varepsilon(y,t)) + \varepsilon\left|\frac{\partial v_\varepsilon}{\partial t}(y,t)\right|^2\right)\omega(y,t)\,dtd\mathcal{H}^{N-1}(y)\right.\\
&\quad \left. - \frac{1}{\varepsilon|\log\varepsilon|}\int_{\partial\Omega}\mathrm{d}_W(g(y),b)\,d\mathcal{H}^{N-1}(y)\right)\\
&\quad + \frac{C}{|\log\varepsilon|}\|\nabla y\|^2_{L^\infty(\Omega_\delta)}\int_{\partial\Omega}\int_0^\delta |\nabla_\tau v_\varepsilon(y,t)|^2\,\omega(y,t)\,dtd\mathcal{H}^{N-1}(y) =: \mathcal{A}+\mathcal{B}.
\end{aligned}$$

To estimate $\mathcal{A}$, we consider two cases.

Case 1: $g(y) = a$**.** Fix $0 < \eta < \frac{1}{4}$, and let $y \in \partial\Omega$ be such $g(y) = a$, then by (5.5), $\frac{\partial\omega}{\partial t}(y,t) < 0$ for all $t \in [0,\delta]$. Thus, also by (5.1), we can apply Theorem 3.8 to obtain

$$\begin{aligned}
&\frac{1}{\varepsilon|\log\varepsilon|}\int_0^\delta\left(\frac{1}{\varepsilon}W(v_\varepsilon(y,t)) + \varepsilon\left|\frac{\partial v_\varepsilon}{\partial t}(y,t)\right|^2\right)\omega(y,t)\,dt - \frac{C_W}{\varepsilon|\log\varepsilon|}\\
&\le (1+\eta)\frac{C_W}{2^{1/2}(W''(a))^{1/2}}\frac{\partial\omega}{\partial t}(y,0) + \frac{C}{|\log\varepsilon|}
\end{aligned}$$

for all $0 < \varepsilon < \varepsilon_\eta$, for some $0 < \varepsilon_\eta < 1$ depending on η, A_0, B_0, δ, ω, and W, and for some constant $C > 0$, depending on A_0, B_0, T, δ, and W. Integrating over the set $\{g = a\}$ gives

$$\begin{aligned}
\mathcal{A}_1 &:= \frac{1}{\varepsilon|\log\varepsilon|}\int_{\partial\Omega\cap\{g=a\}}\int_0^\delta\left(\frac{1}{\varepsilon}W(v_\varepsilon(y,t)) + \varepsilon\left|\frac{\partial v_\varepsilon}{\partial t}(y,t)\right|^2\right)\omega(y,t)\,dtd\mathcal{H}^{N-1}(y)\\
&\quad - \frac{1}{\varepsilon|\log\varepsilon|}\mathcal{H}^{N-1}(\partial\Omega\cap\{g=a\})\\
&\le (1+\eta)\frac{C_W}{2^{1/2}(W''(a))^{1/2}}\int_{\partial\Omega\cap\{g=a\}}\kappa(y)\,d\mathcal{H}^{N-1}(y)\\
&\quad + \frac{C}{|\log\varepsilon|}\mathcal{H}^{N-1}(\partial\Omega\cap\{g=a\}).
\end{aligned}$$

Letting $\varepsilon \to 0^+$ gives

$$\limsup_{\varepsilon\to 0^+}\mathcal{A}_1 \le (1+\eta)\frac{C_W}{2^{1/2}(W''(a))^{1/2}}\int_{\partial\Omega\cap\{g=a\}}\kappa(y)\,d\mathcal{H}^{N-1}(y).$$

Case 2: $g(y) > a$**.** Now let $y \in \partial\Omega$ be such that $g(y) > a$. If $y \in K_1$, then $\omega(y,\cdot)$ satisfies (3.8) by (5.3) and (5.4), while if $y \in K_2$, then $\omega(y,\cdot)$ is strictly increasing in $[0,\delta]$ by (5.5), and so it satisfies (3.8) with $\omega_0 = 0$. Thus, in both cases, also by (5.1), given $l > 0$, we can apply Remark 3.7 to get

$$\frac{1}{\varepsilon|\log\varepsilon|}\int_0^\delta \left(\frac{1}{\varepsilon}W(v_\varepsilon(y,t)) + \varepsilon\left|\frac{\partial v_\varepsilon}{\partial t}(y,t)\right|^2\right)\omega(y,t)\,dt - \frac{\mathrm{d}_W(g(y),b)}{\varepsilon|\log\varepsilon|}$$
$$\leq \frac{C}{|\log\varepsilon|} + \frac{1}{|\log\varepsilon|}\int_0^l 2W(p_\varepsilon(y,s))s\,ds\frac{\partial\omega}{\partial t}(y,0) + Ce^{-2\sigma l}(2\sigma l+1)\frac{1}{|\log\varepsilon|}$$
$$+ \frac{C\varepsilon^{2\gamma}l}{|\log\varepsilon|} + C\varepsilon|\log\varepsilon| + C\varepsilon^d|\log\varepsilon|^d + C\frac{\varepsilon^{2\gamma-2}}{|\log\varepsilon|}$$

for all $0 < \varepsilon < \varepsilon_0$, where $p_\varepsilon(y,s) := v_\varepsilon(y,\varepsilon s)$ and the constants C and $\varepsilon_0 > 0$ depend on A_0, B_0, δ, ω, and W. Since $a \leq p_\varepsilon \leq b$,

$$\int_0^l 2W(p_\varepsilon(y,s))s\,ds \leq l^2 \max_{[a,b]} W,$$

by integrating over $\partial\Omega \setminus \{g = a\}$ and taking ε_0 smaller if necessary (depending on l), we obtain

$$\mathcal{A}_2 := \frac{1}{\varepsilon|\log\varepsilon|}\int_{\partial\Omega\setminus\{g=a\}}\int_0^\delta\left(\frac{1}{\varepsilon}W(v_\varepsilon(y,t)) + \varepsilon\left|\frac{\partial v_\varepsilon}{\partial t}(y,t)\right|^2\right)\omega(y,t)\,dtd\mathcal{H}^{N-1}(y)$$
$$- \frac{1}{\varepsilon|\log\varepsilon|}\int_{\partial\Omega\setminus\{g=a\}}\mathrm{d}_W(g(y),b)\,d\mathcal{H}^{N-1}(y)$$
$$\leq \frac{C}{|\log\varepsilon|}\mathcal{H}^{N-1}(\partial\Omega\setminus\{g=a\})$$

for all $0 < \varepsilon_0 < 1$. Letting $\varepsilon \to 0^+$ gives

$$\limsup_{\varepsilon\to 0^+}\mathcal{A}_2 \leq 0.$$

In conclusion, we have shown that

$$\limsup_{\varepsilon\to 0^+}\mathcal{A} \leq (1+\eta)\frac{C_W}{2^{1/2}(W''(a))^{1/2}}\int_{\partial\Omega\cap\{g=a\}}\kappa(y)\,d\mathcal{H}^{N-1}(y) \tag{5.14}$$

for every $0 < \eta < 1$. We now let $\eta \to 0^+$.

On the other hand, by (2.15), (5.8), and (5.10),

$$\mathcal{B} \leq \frac{C}{|\log\varepsilon|}\|\nabla y\|^2_{L^\infty(\Omega_\delta)}\int_{\partial\Omega}|\nabla_\tau g_\varepsilon(y)|^2\int_0^{T_\varepsilon(y)}\omega(y,t)\,dtd\mathcal{H}^{N-1}(y)$$
$$\leq C\varepsilon\|\omega\|_{L^\infty(\partial\Omega\times[0,\delta])}\int_{\partial\Omega}|\partial_\tau g_\varepsilon(y)|^2\,d\mathcal{H}^{N-1}(y) = o(1). \tag{5.15}$$

By (5.13), (5.14), (5.15), we have

$$\limsup_{\varepsilon\to 0^+}\mathcal{F}^{(2)}_\varepsilon(u_\varepsilon) \leq \frac{C_W}{2^{1/2}(W''(a))^{1/2}}\int_{\partial\Omega\cap\{g=a\}}\kappa(y)\,d\mathcal{H}^{N-1}(y).$$

Step 2: We claim that

$$u_\varepsilon \to b \quad \text{in } L^1(\Omega).$$

In view of Lemma 2.4, we can use the change of variables $x := \Phi(y,t)$ and Tonelli's theorem to write

$$\begin{aligned}\int_\Omega |u_\varepsilon - b|\,dx &= \int_{\partial\Omega}\int_0^\delta |u_\varepsilon(\Phi(y,t))) - b|\omega(y,t)\,dtd\mathcal{H}^{N-1}(y)\\ &= \int_{\partial\Omega}\int_0^{T_\varepsilon(y)} |v_\varepsilon(y,t) - b|\omega(y,t)\,dtd\mathcal{H}^{N-1}(y)\\ &\le C\varepsilon|\log\varepsilon|,\end{aligned}$$

where we used the fact that $v_\varepsilon(y,t) = b$ for $t \ge T_\varepsilon(y)$ and (5.8). ■

In the next proof, we use the localized energy

$$E_\varepsilon(u,E) := \frac{1}{\varepsilon|\log\varepsilon|}\int_E \left(\frac{1}{\varepsilon}W(u) + \varepsilon|\nabla u|^2\right)dx, \quad u \in H^1(\Omega),$$

defined for measurable sets $E \subseteq \Omega$.

Theorem 5.2 (Second-Order Γ-Liminf) *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected set with a boundary of class $C^{2,d}$, $0 < d \le 1$. Assume that W satisfies* (2.1)-(2.4) *and that g_ε satisfy* (1.13), (1.14), (2.14)-(2.16)*. Suppose also that* (1.9) *holds. Then*

$$\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) \ge \frac{C_W}{2^{1/2}(W''(a))^{1/2}}\int_{\partial\Omega\cap\{g=a\}} \kappa(y)\,d\mathcal{H}^{N-1}(y).$$

Proof. We define ω and $\delta > 0$ as in the first part of the proof of Theorem 5.1.

By Theorem 4.2 (with Ω_δ and $\Omega_{2\delta}$ replaced by $\Omega_{\delta/2}$ and Ω_δ, respectively), we can assume that

$$0 \le b - u_\varepsilon(x) \le Ce^{-\mu\delta/\varepsilon} \quad \text{for } x \in \Omega\setminus\Omega_\delta \tag{5.16}$$

for all $0 < \varepsilon < \varepsilon_\delta$.

Write

$$\begin{aligned}\mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) &= E_\varepsilon(u_\varepsilon, \Omega\setminus\Omega_\delta)\\ &\quad + \left(E_\varepsilon(u_\varepsilon,\Omega_\delta) - \frac{1}{\varepsilon|\log\varepsilon|}\int_{\partial\Omega} \mathrm{d}_W(g,b)\,d\mathcal{H}^{N-1}\right)\\ &=: \mathcal{A} + \mathcal{B}.\end{aligned}$$

Since $\mathcal{A} \ge 0$, it remains to evaluate $\mathcal{B}$. In view of Lemma 2.4, we can use the change of variables $x := \Phi(y,t)$ and Tonelli's theorem to write

$$E_\varepsilon(u_\varepsilon,\Omega_\delta) = \frac{1}{\varepsilon|\log\varepsilon|}\int_{\partial\Omega}\int_0^\delta \left(\frac{1}{\varepsilon}W(u_\varepsilon(\Phi(y,t))) + \varepsilon|\nabla u_\varepsilon(\Phi(y,t))|^2\right)\omega(y,t)\,dtd\mathcal{H}^{N-1}(y).$$

Since $u_\varepsilon \in C^1(\overline{\Omega})$, if we define

$$\tilde{u}_\varepsilon(y,t) := u_\varepsilon(y + t\nu(y)),$$

we have that

$$\frac{\partial \tilde{u}_\varepsilon}{\partial t}(y,t) = \frac{\partial u_\varepsilon}{\partial \nu(y)}(y + t\nu(y)),$$

and so,

$$\begin{aligned}\mathcal{B} \geq \int_{\partial\Omega} \Bigg[\frac{1}{\varepsilon|\log\varepsilon|} \int_0^\delta \left(\frac{1}{\varepsilon} W(\tilde{u}_\varepsilon(y,t)) + \varepsilon \left| \frac{\partial \tilde{u}_\varepsilon}{\partial t}(y,t) \right|^2 \right) \omega(y,t)\, dt \\ - \frac{1}{\varepsilon|\log\varepsilon|} \mathrm{d}_W(g(y), b) \Bigg] d\mathcal{H}^{N-1}(y)\end{aligned} \tag{5.17}$$

For $y \in \partial\Omega$, in view of (5.16) we have that

$$b - C_\rho e^{-\mu_\rho \delta/(2\varepsilon)} \leq \tilde{u}_\varepsilon(y,\delta) \leq b. \tag{5.18}$$

Let $v_\varepsilon^y \in H^1([0,\delta])$ be the minimizer of the functional

$$v \mapsto \int_0^\delta \left(\frac{1}{\varepsilon} W(v(t)) + \varepsilon |v'(t)|^2 \right) \omega(y,t)\, dt$$

defined for all $v \in H^1([0,\delta])$ such that $v(0) = g_\varepsilon(y)$ and $v(\delta) = \tilde{u}_\varepsilon(y,\delta)$.

There are now two cases. If $y \in \partial\Omega$ is such $g(y) = a$, then by (5.5), $\frac{\partial\omega}{\partial t}(y,t) < 0$ for all $t \in [0,\delta]$. Thus, also by (5.1), given $0 < \eta < \frac{1}{4}$, we can apply Theorem 3.17 to obtain

$$\begin{aligned}&\frac{1}{\varepsilon|\log\varepsilon|} \int_0^\delta \left(\frac{1}{\varepsilon} W(\tilde{u}_\varepsilon(y,t)) + \varepsilon \left| \frac{\partial \tilde{u}_\varepsilon}{\partial t}(y,t) \right|^2 \right) \omega(y,t)\, dt - \frac{C_W}{\varepsilon|\log\varepsilon|} \\ &\geq \frac{1}{\varepsilon|\log\varepsilon|} \int_0^\delta \left(\frac{1}{\varepsilon} W(v_\varepsilon^y(t)) + \varepsilon \left|(v_\varepsilon^y)'(t)\right|^2 \right) \omega(y,t)\, dt - \frac{C_W}{\varepsilon|\log\varepsilon|} \\ &\geq (1-\eta) \frac{C_W}{2^{1/2}(W''(a))^{1/2}} \frac{\partial\omega}{\partial t}(y,0) - \frac{C_\eta}{|\log\varepsilon|}\end{aligned}$$

for all $0 < \varepsilon < \varepsilon_\eta$, for some $0 < \varepsilon_\eta < 1$ and $C_\eta > 0$ depending on η, A_0, B_0, δ, ω, and W. Integrating over the set $\{g = a\}$ gives

$$\begin{aligned}\mathcal{B}_1 &:= \frac{1}{\varepsilon|\log\varepsilon|} \int_{\partial\Omega\cap\{g=a\}} \int_0^\delta \left(\frac{1}{\varepsilon} W(\tilde{u}_\varepsilon(y,t)) + \varepsilon \left| \frac{\partial \tilde{u}_\varepsilon}{\partial t}(y,t) \right|^2 \right) \omega(y,t)\, dt \\ &\quad - \frac{1}{\varepsilon|\log\varepsilon|} \mathcal{H}^{N-1}(\partial\Omega \cap \{g = a\}) \\ &\geq (1-\eta) \frac{C_W}{2^{1/2}(W''(a))^{1/2}} \int_{\partial\Omega\cap\{g=a\}} \kappa(y)\, d\mathcal{H}^{N-1}(y) \\ &\quad - \frac{C_\eta}{|\log\varepsilon|} \mathcal{H}^{N-1}(\partial\Omega \cap \{g = a\}).\end{aligned}$$

On the other hand, if $y \in \partial\Omega$ is such $g(y) > a$, then there are two cases. If $y \in K_1$, then $\omega(y, \cdot)$ satisfies (3.8) by (5.3) and (5.4), while if $y \in K_2$, then $\omega(y, \cdot)$ is strictly increasing in $[0, \delta]$ by (5.5), and so it satisfies (3.8) with $\omega_0 = 0$. Thus, in both cases, , also by (5.1), given $l > 0$, we can apply Theorem 3.16 to find $0 < \varepsilon_0 < 1$, $C > 0$, and $l_0 > 1$, depending only on $g_\pm$, k, a, b, δ, ω, and W such that

$$\begin{aligned}
&\frac{1}{\varepsilon|\log\varepsilon|}\int_0^\delta \left(\frac{1}{\varepsilon}W(\tilde{u}_\varepsilon(y,t)) + \varepsilon\left|\frac{\partial \tilde{u}_\varepsilon}{\partial t}(y,t)\right|^2\right)\omega(y,t)\,dt - \frac{1}{\varepsilon|\log\varepsilon|}\,\mathrm{d}_W(b, g(y)) \\
&\geq \frac{1}{\varepsilon|\log\varepsilon|}\int_0^\delta \left(\frac{1}{\varepsilon}W(v_\varepsilon^y(t)) + \varepsilon|(v_\varepsilon^y)'(t)|^2\right)\omega(y,t)\,dt - \frac{1}{\varepsilon|\log\varepsilon|}\,\mathrm{d}_W(b, g(y)) \\
&\geq \frac{2}{|\log\varepsilon|}\frac{\partial\omega}{\partial t}(y,0)\int_0^l W^{1/2}(w_\varepsilon)w_\varepsilon' s\,ds - \frac{Ce^{-l\mu}\,(l\mu+1)}{|\log\varepsilon|} - \frac{Cl\varepsilon^{1/2}}{|\log\varepsilon|} - C\varepsilon^{\gamma_1}|\log\varepsilon|^{1+\gamma_0}
\end{aligned}$$

for all $0 < \varepsilon < \varepsilon_0$ and $l > l_0$, where $w_\varepsilon(s) := v_\varepsilon^y(\varepsilon s)$ for $s \in [0, \delta\varepsilon^{-1}]$. Since $a \leq w_\varepsilon \leq b$, and $\|w_\varepsilon'\|_\infty \leq C_0$, by Corollary 3.10,

$$\int_0^l W^{1/2}(w_\varepsilon)|w_\varepsilon'|s\,ds \leq l^2 C_0 \max_{[a,b]} W^{1/2}.$$

Hence, by integrating over $\partial\Omega \setminus \{g = a\}$, we obtain

$$\begin{aligned}
\mathcal{B}_2 := &\frac{1}{\varepsilon|\log\varepsilon|}\int_{\partial\Omega\setminus\{g=a\}}\int_0^\delta \left(\frac{1}{\varepsilon}W(\tilde{u}_\varepsilon(y,t)) + \varepsilon\left|\frac{\partial\tilde{u}_\varepsilon}{\partial t}(y,t)\right|^2\right)\omega(y,t)\,dt d\mathcal{H}^{N-1}(y) \\
&- \frac{1}{\varepsilon|\log\varepsilon|}\int_{\partial\Omega\setminus\{g=a\}} \mathrm{d}_W(g(y), b)\,d\mathcal{H}^{N-1}(y) \\
&\geq -\frac{C_l}{|\log\varepsilon|}\mathcal{H}^{N-1}(\partial\Omega\setminus\{g=a\}).
\end{aligned}$$

By combining the estimates for $\mathcal{B}_1$ and $\mathcal{B}_2$, we have

$$\begin{aligned}
\mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) \geq \mathcal{B} \geq (1-\eta)&\frac{C_W}{2^{1/2}(W''(a))^{1/2}}\int_{\partial\Omega\cap\{g=a\}} \kappa(y)\,d\mathcal{H}^{N-1}(y) \\
&- \frac{C_\eta}{|\log\varepsilon|}\mathcal{H}^{N-1}(\partial\Omega\cap\{g=a\}) - \frac{C_l}{|\log\varepsilon|}\mathcal{H}^{N-1}(\partial\Omega\setminus\{g=a\}).
\end{aligned}$$

In turn,

$$\liminf_{\varepsilon\to 0^+}\mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) \geq (1-\eta)\frac{C_W}{2^{1/2}(W''(a))^{1/2}}\int_{\partial\Omega\cap\{g=a\}}\kappa(y)\,d\mathcal{H}^{N-1}(y).$$

We conclude by letting $\eta \to 0^+$. ■

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