Second-Order Γ-Limit for the Cahn–Hilliard Functional with Dirichlet Boundary Conditions, I

Irene Fonseca
Department of Mathematical Sciences,
Carnegie Mellon University,
Pittsburgh PA 15213-3890, USA

Leonard Kreutz
School of Computation, Information and Technology,
Technical University of Munich
Garching bei München, 85748, Germany

Giovanni Leoni
Department of Mathematical Sciences,
Carnegie Mellon University,
Pittsburgh PA 15213-3890, USA

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Abstract

This paper addresses the asymptotic development of order 2 by the Γ-convergence of the Cahn–Hilliard functional with Dirichlet boundary conditions. The Dirichlet data are assumed to be well separated from one of the two wells. In the case where there are no interfaces, it is shown that there is a transition layer near the boundary of the domain.

1 Introduction

In this paper, we study the second-order asymptotic development via Γ-convergence of the Cahn-Hilliard functional

$$F_\varepsilon(u) := \int_\Omega (W(u) + \varepsilon^2 |\nabla u|^2)\, dx, \quad u \in H^1(\Omega), \tag{1.1}$$

subject to the Dirichlet boundary condition

$$\operatorname{tr} u = g_\varepsilon \quad \text{on } \partial\Omega. \tag{1.2}$$

Here $W : \mathbb{R} \to [0, \infty)$ is a double-well potential with

$$W^{-1}(\{0\}) = \{a, b\}, \tag{1.3}$$

$\Omega \subset \mathbb{R}^N$ is an open, bounded set with a smooth boundary, $N \geq 2$, and $g_\varepsilon \in H^{1/2}(\partial\Omega)$.

We recall that, given a metric space X and a family of functions $\mathcal{F}_\varepsilon : X \to [-\infty, \infty]$ for $\varepsilon > 0$, *the asymptotic development of order* n via Γ-convergence is written as:

$$\mathcal{F}_\varepsilon = \mathcal{F}^{(0)} + \varepsilon \mathcal{F}^{(1)} + \cdots + \varepsilon^n \mathcal{F}^{(n)} + o(\varepsilon^n). \tag{1.4}$$

This expansion holds if we can find $\mathcal{F}^{(i)} : X \to [-\infty, \infty]$, $i = 0, \ldots, n$, such that the functions

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf_X \mathcal{F}^{(i-1)}}{\varepsilon}, \quad \mathcal{F}_\varepsilon^{(0)} := \mathcal{F}_\varepsilon$$

are well-defined and the family $\{\mathcal{F}_\varepsilon^{(i)}\}_\varepsilon$ Γ-converges to $\mathcal{F}^{(i)}$ as $\varepsilon \to 0^+$.

The notion of asymptotic expansion was introduced by Anzellotti and Baldo in 1993 [3]. Observe that if we define

$$\mathcal{U}_i := \{\text{minimizers of } \mathcal{F}^{(i)}\},$$

it can be shown that

$$\mathcal{F}^{(i)} = \infty \text{ on } X \setminus \mathcal{U}_{i-1}$$

and the sets of minimizers satisfy the nested relationship

$$\mathcal{U}_n \subseteq \mathcal{U}_{n-1} \subseteq \cdots \subseteq \mathcal{U}_0 = \{\text{limits of minimizers of } \mathcal{F}_\varepsilon\}.$$

In general, the above set inclusions can be shown to be strict. Therefore, leveraging the hierarchical structure of functionals $\mathcal{F}^{(i)}$, this framework provides a systematic selection criterion for the limits of the minimizers of functionals $\mathcal{F}_\varepsilon$.

In many cases, the powers of ε in the asymptotic development (1.4) may be replaced by more general scales $\delta_\varepsilon^{(i)}$, where $\delta_\varepsilon^{(i)} > 0$ for all $i = 1, \ldots, m$ and $\varepsilon > 0$, $\delta_\varepsilon^{(0)} := 1$, and $\sigma_\varepsilon^{(i)} := \delta_\varepsilon^{(i)} / \delta_\varepsilon^{(i-1)} \to 0$ as $\varepsilon \to 0^+$ for all $i = 1, \ldots, m$, and the asymptotic expansion takes the form

$$\mathcal{F}_\varepsilon = \mathcal{F}^{(0)} + \delta_\varepsilon^{(1)} \mathcal{F}^{(1)} + \cdots + \delta_\varepsilon^{(n)} \mathcal{F}^{(n)} + o(\delta_\varepsilon^{(n)}).$$

In this setting, the functions $\mathcal{F}_\varepsilon^{(i)}$ are defined by

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf_X \mathcal{F}^{(i-1)}}{\sigma_\varepsilon^{(i)}}, \quad \mathcal{F}_\varepsilon^{(0)} := \mathcal{F}_\varepsilon.$$

The second-order asymptotic expansion of the Cahn-Hilliard functional (1.1) subject to a mass constraint

$$\int_\Omega u(x)\, dx = m \tag{1.5}$$

was studied by the third author and Murray in [21], [22] in dimension $N \geq 2$. With $X := L^1(\Omega)$ and

$$\mathcal{G}_\varepsilon(u) := \begin{cases} \int_\Omega (W(u) + \varepsilon^2 |\nabla u|^2)\, dx & \text{if } u \in H^1(\Omega), \int_\Omega u\, dx = m, \\ \infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{1.6}$$

they proved that, under appropriate hypotheses on Ω and W, if W is quadratic near the wells, then

$$\mathcal{G}_\varepsilon^{(2)}(u) = \frac{1}{2}\frac{C_W^2(N-1)^2}{W''(a)(b-a)^2}\kappa_u^2 + (C_{\text{sym}} + C_W\tau_u)\kappa_u \,\mathrm{P}(\{u=a\};\Omega),$$

where u is a minimizer of the first-order functional (see [6], [18], [24], [23], [28]),

$$\mathcal{G}^{(1)}(v) := \begin{cases} C_W \,\mathrm{P}(\{v=a\};\Omega) & \text{if } v \in BV(\Omega;\{a,b\}),\ \int_\Omega v\,dx = m, \\ \infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

$\tau_u \in \mathbb{R}$ is a constant related to the mass constraint (1.5), κ_u and $\mathrm{P}(\{u=a\};\Omega)$ are the constant mean curvature and the perimeter of the set $\{u=a\}$ in Ω, respectively, the constants C_W and C_{sym} are given by[1]

$$C_W := 2\int_a^b W^{1/2}(\rho)\,d\rho, \tag{1.7}$$

and

$$C_{\text{sym}} := 2\int_{\mathbb{R}} W(z_c(t))t\,dt,$$

where c is the central zero of W' (see (2.4)), and for $\alpha \in \mathbb{R}$, z_α solves to the Cauchy problem

$$\begin{cases} z_\alpha' = W^{1/2}(z_\alpha), \\ z_\alpha(0) = \alpha. \end{cases} \tag{1.8}$$

The third author and Murray in [21], [22] also considered the case where W exhibits subquadratic growth near the wells. This scenario had been previously analyzed by the first and third authors together with Dal Maso in [13], where they assumed both zero Dirichlet boundary conditions and the mass constraint (1.5).

In the case of Dirichlet boundary conditions (1.2), we take $X := L^1(\Omega)$ and define

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_\Omega (W(u) + \varepsilon^2|\nabla u|^2)\,dx & \text{if } u \in H^1(\Omega),\ \operatorname{tr} u = g_\varepsilon \text{ on } \partial\Omega, \\ \infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Under suitable assumptions on Ω, W, and g_ε, Owen, Rubinstein, and Sternberg [26] showed that the first non-trivial scale is $\delta_\varepsilon^{(1)} = \varepsilon$, i.e.,

$$\mathcal{F}_\varepsilon^{(1)}(u) = \begin{cases} \int_\Omega (\frac{1}{\varepsilon}W(u) + \varepsilon|\nabla u|^2)\,dx & \text{if } u \in H^1(\Omega),\ \operatorname{tr} u = g_\varepsilon \text{ on } \partial\Omega, \\ \infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

and that the functionals $\{\mathcal{F}_\varepsilon^{(1)}\}_\varepsilon$ Γ-converge as $\varepsilon \to 0^+$ to

$$\mathcal{F}^{(1)}(u) := \begin{cases} C_W \,\mathrm{P}(\{u=a\};\Omega) \\ \quad + \int_{\partial\Omega} \mathrm{d}_W(\operatorname{tr} u, g)\,d\mathcal{H}^{N-1} & \text{if } u \in BV(\Omega;\{a,b\}), \\ \infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{1.9}$$

[1] Note that our constants c_W and c_{sym} differ by correspond to the constants $2c_W$ and $2c_{\text{sym}}$ in [21], [22].

where $g_\varepsilon \to g$ in $L^1(\partial\Omega)$, d_W is the geodesic distance determined by W

$$\mathrm{d}_W(r,s) := \begin{cases} 2\left|\int_r^s W^{1/2}(\rho)\,d\rho\right| & \text{if } r \in \{a,b\} \text{ or } s \in \{a,b\}, \\ \infty & \text{otherwise,} \end{cases} \tag{1.10}$$

and the constant C_W is given in (1.7). We also refer to the recent work by Cristoferi and Gravina [11], who addressed the vectorial case and considered potentials where the wells depend on the spatial variable x, and to the work by Gazoulis [19], who studied the vectorial case under different settings.

We aim to extend the results of Owen, Rubinstein, and Sternberg [26] by determining the second-order asymptotic expansion of $\mathcal{F}_\varepsilon$ via Γ-convergence, assuming the boundary data $g_\varepsilon : \overline{\Omega} \to \mathbb{R}$ stay away from one of the two wells a, b:

$$a < \alpha_- \leq g_\varepsilon(x) \leq b \tag{1.11}$$

for all $x \in \overline{\Omega}$, all $\varepsilon \in (0,1)$, and some constant α_-. In this article, we consider only the case where the Dirichlet boundary datum is close to the value b and far from a. The case where the boundary datum is close to a and far from b can be addressed using the same arguments. Under this hypothesis, when the constant α_- is sufficiently close to b, the only minimizer of $\mathcal{F}^{(1)}$ is the constant function b (see Proposition 2.5 below). Hence, we assume that

$$u_0 \equiv b \quad \text{is the unique minimizer of } \mathcal{F}^{(1)}. \tag{1.12}$$

In this case, due to (1.9), we have

$$\min \mathcal{F}^{(1)} = \int_{\partial\Omega} \mathrm{d}_W(b,g)\,d\mathcal{H}^{N-1}$$

and we define

$$\begin{aligned} \mathcal{F}_\varepsilon^{(2)}(u) &:= \frac{\mathcal{F}_\varepsilon^{(1)}(u) - \min \mathcal{F}^{(1)}}{\varepsilon} \\ &= \int_\Omega \left(\frac{1}{\varepsilon^2} W(u) + |\nabla u|^2\right) dx - \frac{1}{\varepsilon}\int_{\partial\Omega} \mathrm{d}_W(b,g)\,d\mathcal{H}^{N-1} \end{aligned} \tag{1.13}$$

if $u \in H^1(\Omega)$ and $\operatorname{tr} u = g_\varepsilon$ on $\partial\Omega$, and $\mathcal{F}_\varepsilon^{(2)}(u) := \infty$ otherwise in $L^1(\Omega)$.

The main result of this paper is the following theorem:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected set with boundary of class $C^{2,d}$, $0 < d \leq 1$. Assume that W satisfies* (2.1)-(2.4) *and that g_ε satisfy* (1.11), (2.12)-(2.14)*. Suppose also that* (1.12) *holds. Then*

$$\mathcal{F}^{(2)}(u) = \int_{\partial\Omega} \kappa(y) \int_0^\infty 2W^{1/2}(z_{g(y)}(s)) z'_{g(y)}(s) s\,ds\,d\mathcal{H}^{N-1}(y) \tag{1.14}$$

if $u = b$ and $\mathcal{F}^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$. Here, κ is the mean curvature of $\partial\Omega$ and z_α is the solution to the Cauchy problem (1.8) *with $\alpha = g(y)$.*

In particular, if $u_\varepsilon \in H^1(\Omega)$ is a minimizer of (1.1) *subject to the Dirichlet boundary condition* (1.2)*, then*

$$\int_\Omega (W(u_\varepsilon) + \varepsilon^2|\nabla u_\varepsilon|^2)\, dx = \varepsilon \int_{\partial\Omega} \mathrm{d}_W(b, g)\, d\mathcal{H}^{N-1} \tag{1.15}$$
$$+ \varepsilon^2 \int_{\partial\Omega} \kappa(y) \int_0^\infty 2W^{1/2}(z_{g(y)}(s)) z'_{g(y)}(s) s\, ds d\mathcal{H}^{N-1}(y) + o(\varepsilon^2).$$

Remark 1.2 *In the case where g is allowed to take the value a but* (1.12) *continues to hold, the scaling*

$$\mathcal{F}_\varepsilon^{(2)}(u) := \frac{\mathcal{F}_\varepsilon^{(1)}(u) - \min \mathcal{F}^{(1)}}{\varepsilon |\log \varepsilon|}$$

should replace the scaling in (1.13)*, as the latter becomes incorrect in this context. We address this problem in the paper [16].*

Remark 1.3 *The case where the minimizer u_0 of the functional $\mathcal{F}^{(1)}$ in* (1.9) *is not constant, the analysis becomes considerably more complex. By leveraging recent results from De Phillipis and Maggi [14], it can be shown that if Ω and g are sufficiently regular, then by modifying $E_0 := \{u_0 = a\}$ on a set of Lebesgue measure zero, E_0 is open and its trace $\partial E_0 \cap \partial\Omega$ has finite perimeter in $\partial\Omega$. Moreover, if $M = \overline{\partial E_0 \cap \Omega}$, then $\partial_{\partial\Omega}(\partial E_0 \cap \partial\Omega) = M \cap \partial\Omega$, and there exists a closed set $\Sigma \subseteq M$, with $\mathcal{H}^{N-2}(M \setminus \Sigma) = 0$, such that $M \setminus \Sigma$ is a $C^{1,1/2}$ hypersurface with boundary, $M \setminus \Sigma$ has zero mean curvature in Ω and satisfies the Young's law*

$$\nu_{E_0}(x) \cdot \nu_{\partial\Omega}(x) = \frac{1}{C_W}(\mathrm{d}_W(a, g(x)) - \mathrm{d}_W(b, g(x))) \tag{1.16}$$

for all $x \in (M \cap \partial\Omega) \setminus \Sigma$. Here ν_{E_0} and $\nu_{\partial\Omega}$ are the outward unit normals to E_0 and Ω, respectively. We are currently investigating this problem in dimension two. In this setting,

$$u_0 = a\chi_{E_0} + b\chi_{\Omega \setminus E_0},$$

where

$$\partial E_0 \cap \Omega = \bigcup_{i=1}^m \Sigma_i,$$

with Σ_i being disjoint segments that have endpoints P_i and Q_i on $\partial\Omega$ and form angles θ_i that satisfy Young's law (1.16). By adapting the techniques presented in this paper, we have constructed $u_\varepsilon \in H^1(\Omega)$ satisfying the Dirichlet boundary conditions (1.2) and converging to u_0 in $L^1(\Omega)$, such that

$$\limsup_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) \le \int_{\partial\Omega \cap \overline{\Omega}^b} \kappa(y) \int_0^\infty 2W^{1/2}(z_{g(y)}(s)) z'_{g(y)}(s) s\, ds d\mathcal{H}^1(y)$$
$$+ \int_{\partial\Omega \cap \overline{\Omega}^a} \kappa(y) \int_{-\infty}^0 2W^{1/2}(z_{g(y)}(s)) z'_{g(y)}(s) |s|\, ds d\mathcal{H}^1(y)$$
$$- \sum_{i=1}^m \frac{1 + \cos\theta_i}{\sin\theta_i} C_i - \sum_{i=1}^m \frac{1 - \cos\theta_i}{\sin\theta_i} D_i,$$

where

$$C_i := \int_0^\infty 2W^{1/2}(z_{g(P_i)}(s))z'_{g(P_i)}(s)s\,ds + \int_0^\infty 2W^{1/2}(z_{g(Q_i)}(s))z'_{g(Q_i)}(s)s\,ds,$$

$$D_i := \int_{-\infty}^0 2W^{1/2}(z_{g(P_i)}(s))z'_{g(P_i)}(s)|s|\,ds + \int_{-\infty}^0 2W^{1/2}(z_{g(Q_i)}(s))z'_{g(Q_i)}(s)|s|\,ds,$$

and z_α *solves the Cauchy problem* (1.8) *with* $\alpha = g(y)$*, and* $\Omega^r := \{x \in \Omega : u_0(x) = r\}$*,* $r \in \{a, b\}$*.*

Theorem 1.1 is in the same spirit as the work by Anzellotti, Baldo, and Orlandi [4], who considered the case $W(\rho) = \rho^2$ and derived a formula similar to (1.14). Our proof, however, takes a different approach and relies on the asymptotic development of order two by Γ-convergence of the weighted one-dimensional functional

$$G_\varepsilon(v) := \int_0^T (W(v(t)) + \varepsilon^2(v'(t))^2)\omega(t)\,dt, \quad v \in H^1(I), \tag{1.17}$$

subject to the Dirichlet boundary conditions

$$v(0) = \alpha_\varepsilon, \quad v(T) = \beta_\varepsilon, \tag{1.18}$$

where ω is a smooth positive weight, and

$$a < \alpha_\varepsilon,\, \beta_\varepsilon \le b. \tag{1.19}$$

The second-order asymptotic expansion of this functional was studied by the third author and Murray ([21], [22]) in the case where the Dirichlet boundary conditions (1.18) were replaced by the mass constraint:

$$\int_0^T v(t)\omega(t)\,dt = m. \tag{1.20}$$

The key difference in our proof of the Γ-liminf inequality is that in [21], [22], the authors utilized a rearrangement technique based on the isoperimetric function to reduce the functional (1.6) to the one-dimensional weighted problem (1.17) and (1.20). This approach, however, is not feasible in our case (except in the case of trivial boundary conditions). Instead, we adapt techniques from Sternberg and Zumbrum [29] and Caffarelli and Cordoba [10] to study the behavior of minimizers of (1.1) and (1.2) near the boundary and use slicing arguments.

The case $N = 1$ was previously addressed by Anzellotti and Baldo [3] under the assumption that W is zero in a neighborhood of a and b, and by Bellettini, Nayam, and Novaga [7] in the periodic case.

This paper is organized as follows. In Section 3, we characterize the asymptotic development of order two by the Γ convergence of the weighted one-dimensional family of functionals G_ε defined in (1.17). Section 4 explores the qualitative properties of critical points and minimizers of functional 1.1. Finally, in Section 5, we prove Theorem 1.1.

2 Preliminaries

We assume that the double-well potential $W : \mathbb{R} \to [0, \infty)$ satisfies the following hypotheses:

$$W \text{ is of class } C^{2,\alpha_0}(\mathbb{R}),\ \alpha_0 \in (0,1), \text{ and has precisely two zeros at } a \text{ and } b, \text{ with } a < b, \tag{2.1}$$

$$W''(a) > 0, \quad W''(b) > 0, \tag{2.2}$$

$$\lim_{s\to-\infty} W'(s) = -\infty, \quad \lim_{s\to\infty} W'(s) = \infty, \tag{2.3}$$

$$W' \text{ has exactly 3 zeros at } a, b, c \text{ with } a < c < b, \quad W''(c) < 0, \tag{2.4}$$

Let

$$a < \alpha_- < \min\left\{c, \frac{a+b}{2}\right\} \leq \max\left\{c, \frac{a+b}{2}\right\} < \beta_- < b. \tag{2.5}$$

Remark 2.1 *Since $W \in C^2(\mathbb{R})$, $W(a) = W'(a) = 0$, $W(b) = W'(b) = 0$, and $W''(a)$, $W''(b) > 0$, there exists a constant $\sigma > 0$ depending on α_- and β_- such that*

$$\sigma^2(b-s)^2 \leq W(s) \leq \frac{1}{\sigma^2}(b-s)^2 \quad \text{for all } \alpha_- \leq s \leq b+1, \tag{2.6}$$

$$\sigma^2(s-a)^2 \leq W(s) \leq \frac{1}{\sigma^2}(s-a)^2 \quad \text{for all } a-1 \leq s \leq \beta_-. \tag{2.7}$$

Proposition 2.2 *For $a < \alpha_- < b$ and $0 < \delta \leq \sigma^{-1}$, we have*

$$\begin{aligned} &-\frac{\sigma^{-1}}{2}\log(\sigma^{-2}\delta) + \sigma^{-1}\log(b-\alpha) - \sigma^{-1}\log(1 + 2\sigma^{-1}(b-\beta)/\delta^{1/2}) \\ &\leq \int_\alpha^\beta \frac{1}{(\delta + W(s))^{1/2}}\,ds \leq -\frac{\sigma}{2}\log(\sigma^2\delta) + \sigma\log(1 + 2(b-a)) \end{aligned} \tag{2.8}$$

for every $\alpha_- \leq \alpha \leq \beta \leq b$, where $\sigma > 0$ is the constant given in (2.6).

Proof. By (2.6),

$$\frac{\sigma^{-1}}{(\sigma^{-2}\delta + (b-s)^2)^{1/2}} \leq \frac{1}{(\delta + W(s))^{1/2}} \leq \frac{\sigma}{(\sigma^2\delta + (b-s)^2)^{1/2}}.$$

Hence, it suffices to estimate

$$\mathcal{A} := \int_\alpha^\beta \frac{1}{(r + (b-s)^2)^{1/2}}\,ds.$$

Consider the change of variables $r^{1/2}t = b - s$, so that $-r^{1/2}dt = ds$. Then

$$\begin{aligned}
\mathcal{A} &= \int_{\alpha}^{\beta} \frac{1}{(r + (b-s)^2)^{1/2}}\, ds = \frac{r^{1/2}}{r^{1/2}} \int_{(b-\beta)/r^{1/2}}^{(b-\alpha)/r^{1/2}} \frac{1}{(1+t^2)^{1/2}}\, dt \\
&= [\log[t + (t^2+1)^{1/2}]]_{(b-\beta)/r^{1/2}}^{(b-\alpha)/r^{1/2}} \\
&= -\frac{1}{2}\log r + \log(b - \alpha + [r + (b-\alpha)^2]^{1/2}) \\
&\quad - \log((b-\beta)/r^{1/2} + [1 + (b-\beta)^2/r]^{1/2}).
\end{aligned}$$

Hence, for $0 < r \le 1$, we have

$$\begin{aligned}
&-\frac{1}{2}\log r + \log(b-\alpha) - \log(1 + 2(b-\beta)/r^{1/2}) \\
&\qquad \le \mathcal{A} \le -\frac{1}{2}\log r + \log(1 + 2(b-a)).
\end{aligned}$$

■

Proposition 2.3 *Let $a \le \alpha_\varepsilon \le \beta_\varepsilon \le b$. Then there exists a constant $C > 0$ depending on σ such that*

$$\int_{\alpha_\varepsilon}^{\beta_\varepsilon} \left[\frac{2}{(\delta + W(s))^{1/2} + W^{1/2}(s)} - \frac{1}{(\delta + W(s))^{1/2}} \right] ds \le C \tag{2.9}$$

for all $0 < \delta < 1$, where $\sigma > 0$ is the constant given in (2.6).

Proof. For $A \ge 0$, we have

$$\begin{aligned}
\frac{2}{(\delta + A)^{1/2} + A^{1/2}} - \frac{1}{(\delta + A)^{1/2}} &= \frac{(\delta + A)^{1/2} - A^{1/2}}{[(\delta + A)^{1/2} + A^{1/2}](\delta + A)^{1/2}} \\
&= \frac{\delta}{[(\delta + A)^{1/2} + A^{1/2}]^2(\delta + A)^{1/2}} \ge 0.
\end{aligned}$$

Hence, the left side of (2.9) can be bounded from above by

$$\begin{aligned}
&\int_a^b \frac{\delta}{[(\delta + W(s))^{1/2} + W^{1/2}(s)]^2(\delta + W(s))^{1/2}} ds \\
&= \int_a^c \frac{\delta}{[(\delta + W(s))^{1/2} + W^{1/2}(s)]^2(\delta + W(s))^{1/2}} \\
&\quad + \int_c^b \frac{\delta}{[(\delta + W(s))^{1/2} + W^{1/2}(s)]^2(\delta + W(s))^{1/2}} \\
&=: \mathcal{A} + \mathcal{B}.
\end{aligned}$$

By (2.7) we have

$$\mathcal{A} \le \int_a^c \frac{\delta}{[(\delta+\sigma^2(s-a)^2)^{1/2}+\sigma(s-a)]^2(\delta+\sigma^2(s-a)^2)^{1/2}}ds$$
$$= \int_0^{(c-a)/\delta^{1/2}} \frac{\delta}{[(\delta+\sigma^2\delta t^2)^{1/2}+\sigma\delta^{1/2}t]^2(\delta+\sigma^2\delta t^2)^{1/2}}\delta^{1/2}dt$$
$$\le \int_0^\infty \frac{1}{[(1+\sigma^2t^2)^{1/2}+\sigma t]^2(1+\sigma^2t^2)^{1/2}}dt,$$

where we have made the change of variables $s-a=\delta^{1/2}t$, so that $ds=\delta^{1/2}dt$, and used the fact that $0<\delta<1$. A similar estimate holds for $\mathcal{B}$. ■

Next, we study the properties of the solutions to the Cauchy problem (1.8).

Proposition 2.4 *Assume that W satisfies* (2.1)-(2.4) *and let $a<\alpha<b$. Then the Cauchy problem* (1.8) *admits a unique global solution $z_\alpha:\mathbb{R}\to\mathbb{R}$. The function z_α is increasing with*

$$a<z_\alpha(t)<b \quad \textit{for all } t\in\mathbb{R},$$

and

$$\lim_{t\to-\infty} z_\alpha(t)=a, \quad \lim_{t\to\infty} z_\alpha(t)=b. \tag{2.10}$$

Moreover, if $\alpha_-\le\alpha<b$, where α_- is given in (2.5)*, then*

$$(b-\alpha)e^{-\sigma^{-1}t}\le b-z_\alpha(t)\le(b-a)e^{-\sigma t} \tag{2.11}$$

for all $t\ge0$.

Proof. As $\sqrt{W}$ is Lipschitz continuous in $[a-1,b+1]$, the Cauchy problem (1.8) admits a unique local solution. As the constant functions a and b are solutions to the differential equation, by uniqueness, $a<z_\alpha(t)<b$ for all t in the interval of existence of z_α. This implies that z_α can be uniquely extended to the entire real line. Standard ODEs techniques show that (2.10) is valid.

If $\alpha_-\le\alpha<b$, since z_α is increasing, using (2.10), we can find $T_\alpha<0$ such that $z_\alpha(T_\alpha)=\alpha_-$ and $z_\alpha(t)>\alpha_-$ for all $t>T_\alpha$. In turn, by (2.6),

$$\sigma(b-z_\alpha(t))\le z_\alpha'(t)\le\sigma^{-1}(b-z_\alpha(t)) \quad \text{for all } t\ge T_\alpha.$$

Dividing by $b-z_\alpha(t)$ and integrating from 0 to t gives

$$-\sigma^{-1}t\le\log\left(\frac{b-z_\alpha(t)}{b-\alpha}\right)\le-\sigma t,$$

which implies (2.11). ■

We assume that $g_\varepsilon:\partial\Omega\to\mathbb{R}$ and $g:\partial\Omega\to\mathbb{R}$ satisfy the following hypotheses:

$$g_\varepsilon\in H^1(\partial\Omega), \tag{2.12}$$

$$(\varepsilon|\log\varepsilon|)^{1/2}\int_{\partial\Omega}|\nabla_\tau g_\varepsilon|^2d\mathcal{H}^{N-1}=o(1) \quad \text{as } \varepsilon\to0^+, \tag{2.13}$$

$$|g_\varepsilon(x)-g(x)|\le C\varepsilon^\gamma, \quad x\in\partial\Omega, \quad \gamma>1 \tag{2.14}$$

for all $\varepsilon \in (0,1)$ and for some constant $C > 0$. Here, ∇_τ denotes the tangential gradient.

Condition (2.13) is of a technical nature and ensures that, in the energy estimates for the recovery sequence in the Γ-limsup inequality, the tangential component of the gradient near the boundary of Ω does not contribute to the limiting energy (see (5.10) below). In particular, this condition is satisfied if $g_\varepsilon = g$ for all $\varepsilon > 0$ for some $g \in H^1(\partial\Omega)$.

Observe that the hypotheses (2.13) and (2.14) imply some regularity of g. In particular, when $N = 2$, we see that the functions g_ε are continuous, and since (2.14) implies uniform convergence, it follows that g must be continuous.

For $a \le \alpha \le b$, let

$$\begin{aligned}\phi(\alpha) &:= \mathrm{d}_W(a,\alpha) - \mathrm{d}_W(\alpha,b)\\ &= 2\int_a^\alpha W^{1/2}(\rho)\,d\rho - 2\int_\alpha^b W^{1/2}(\rho)\,d\rho,\end{aligned}$$

where d_W is defined in (1.10). Since $\phi(a) = -C_W$, $\phi(b) = C_W$, $\phi'(\alpha) = 4W^{1/2}(\alpha) > 0$ for $\alpha \in (a,b)$, there exists a unique $\bar\alpha \in (a,b)$ such that

$$\phi(\bar\alpha) = 0 \quad \text{and} \quad \phi(\alpha) > 0 \quad \text{for all } \bar\alpha < \alpha \le b. \tag{2.15}$$

Proposition 2.5 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected set with boundary of class $C^{2,d}$, $0 < d \le 1$. Assume that W satisfies* (2.1)-(2.4) *and that g_ε satisfy* (1.11), (2.12)-(2.14). *Suppose that*

$$g_- > \bar\alpha \tag{2.16}$$

where $\bar\alpha$ is given in (2.15). *Then the constant function b is the unique minimizer of the functional $\mathcal{F}^{(1)}$ defined in* (1.9).

Proof. Let $u \in BV(\Omega;\{a,b\})$. We have $\operatorname{tr} u(x) \in \{a,b\}$ for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$ and thus

$$\begin{aligned}\mathcal{F}^{(1)}(u) &\ge C_W\,\mathrm{P}(\{u=b\};\Omega) + \int_{\partial\Omega} \mathrm{d}_W(\operatorname{tr} u, g)\,d\mathcal{H}^{N-1}\\ &\ge \int_{\partial\Omega} \mathrm{d}_W(b,g)\,d\mathcal{H}^{N-1} = \mathcal{F}^{(1)}(b)\end{aligned}$$

provided

$$\int_{\partial\Omega\cap\{\operatorname{tr} u = a\}} \mathrm{d}_W(a,g)\,d\mathcal{H}^{N-1} \ge \int_{\partial\Omega\cap\{\operatorname{tr} u=a\}} \mathrm{d}_W(g,b)\,d\mathcal{H}^{N-1}.$$

By (1.11), (2.15), and (2.16), we obtain

$$\mathrm{d}_W(a,g(x)) > \mathrm{d}_W(g(x),b).$$

Hence, if $\mathrm{P}(\{u=b\};\Omega) > 0$ or $\mathcal{H}^{N-1}(\partial\Omega \cap \{\operatorname{tr} u = a\}) > 0$, we have that $\mathcal{F}^{(1)}(u) > \mathcal{F}^{(1)}(b)$, which shows that the constant function b is the unique minimizer of $\mathcal{F}^{(1)}$. ■

In what follows, given $z \in \mathbb{R}^N$, with a slight abuse of notation, we write

$$z = (z', z_N) \in \mathbb{R}^{N-1} \times \mathbb{R}, \tag{2.17}$$

where $z' := (z_1, \dots, z_{N-1})$. We also write

$$\nabla' := \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_{N-1}} \right). \tag{2.18}$$

Also, given $\delta > 0$ we define

$$\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}. \tag{2.19}$$

The following result is classical. We recall it and its proof for the reader's convenience.

Lemma 2.6 *Assume that $\Omega \subset \mathbb{R}^N$ is an open, bounded, connected set and that its boundary $\partial\Omega$ is of class $C^{2,d}$, $0 < d \leq 1$. If $\delta > 0$ is sufficiently small, the mapping*

$$\Phi : \partial\Omega \times [0, \delta] \to \overline{\Omega}_\delta$$

given by

$$\Phi(y, t) = y + t\nu(y),$$

where $\nu(y)$ is the unit inward normal vector to $\partial\Omega$ at y and Ω_δ is defined in (2.19)*, is a diffeomorphism of class $C^{1,d}$. Moreover, $\Omega \setminus \Omega_\delta$ is connected for all $\delta > 0$ sufficiently small. Finally,*

$$\det J_\Phi(y, 0) = 1 \quad \text{for all } y \in \partial\Omega \tag{2.20}$$

and

$$\frac{\partial}{\partial t} \det J_\Phi(y, t)|_{t=0} = \kappa(y) \quad \text{for all } y \in \partial\Omega, \tag{2.21}$$

where $\kappa(y)$ is the mean curvature of $\partial\Omega$ at y.

Proof. The fact that $\Phi : \partial\Omega \times [0, \delta] \to \overline{\Omega}_\delta$ is a diffeomorphism $\delta > 0$ is sufficiently small is classical (see, e.g. [20, Theorem 6.17]). Its inverse is given by

$$\Phi^{-1}(x) = (y(x), \operatorname{dist}(x, \partial\Omega)),$$

where we denote by $y(x) \in \partial\Omega$ the unique projection of x onto $\partial\Omega$, with

$$\operatorname{dist}(x, \partial\Omega) = |y(x) - x|.$$

Next, we show that $\Omega \setminus \Omega_\delta$ is pathwise connected. Let x_0 and x_1 be two points in $\Omega \setminus \Omega_\delta$. Since Ω is open and connected, there exists a continuous function $f : [0, 1] \to \Omega$ such that $f(0) = x_0$ and $f(1) = x_1$. Since Φ is a diffeomorphism and $\Phi(\partial\Omega \times \{\delta\}) = \partial(\Omega \setminus \Omega_\delta)$, the function

$$h(x) = y(x) + \delta\nu(y(x)), \quad x \in \overline{\Omega}_\delta,$$

is continuous, with $h(\overline{\Omega}_\delta) = \partial(\Omega \setminus \Omega_\delta)$. Note that if $x \in \partial(\Omega \setminus \Omega_\delta)$, then $h(x) = x$. Hence, if we extend h to be the identity in $\Omega \setminus \Omega_\delta$, we have a continuous function $h : \overline{\Omega} \to \Omega \setminus \Omega_\delta$. Then $h \circ f : [0,1] \to \Omega \setminus \Omega_\delta$ is continuous and $(h \circ f)(0) = x_0$ and $(h \circ f)(1) = x_1$, which shows that $\Omega \setminus \Omega_\delta$ is pathwise connected.

To prove (2.20) and (2.21), we fix $y_0 \in \partial\Omega$ and find a rigid motion $T : \mathbb{R}^N \to \mathbb{R}^N$, with $T(y_0) = 0$, $r > 0$, and a function $f : B_{N-1}(0,r) \to \mathbb{R}$ of class $C^{2,d}$ such that $f(0) = 0$, $\nabla' f(0) = 0$, and

$$T(B(y_0, r) \cap \Omega) = \{z \in \mathbb{R}^N : z_N > f(z'),\, z' \in B_{N-1}(0,r)\} =: V,$$

where we are using the notations (2.17) and (2.18) and $B_{N-1}(0,r)$ is the open ball centered at 0 and radius r in $\mathbb{R}^{N-1}$. The unit inward normal to ∂V at a point $(z', f(z'))$ is the vector

$$\nu = \frac{(-\nabla' f(z'), 1)}{(1 + |\nabla' f(z')|^2_{N-1})^{1/2}}$$

Hence, if we consider

$$\Psi(z', t) := (z', f(z')) + t\frac{(-\nabla' f(z'), 1)}{(1 + |\nabla' f(z')|^2_{N-1})^{1/2}},$$

we have that for $i, j = 1, \dots, N-1$,

$$\frac{\partial \Psi_j}{\partial z_i}(z', t) = \delta_{i,j} + t\frac{\partial}{\partial z_i}\left(\frac{\frac{\partial f}{\partial z_j}(z')}{\sqrt{1 + |\nabla' f(z')|^2_{N-1}}}\right),$$

$$\frac{\partial \Psi_N}{\partial z_i}(z', t) = \frac{\partial f}{\partial z_i}(z') - t\frac{\partial}{\partial z_i}\left(\frac{1}{\sqrt{1 + |\nabla' f(z')|^2_{N-1}}}\right),$$

$$\frac{\partial \Psi_j}{\partial t}(z', t) = \frac{-\frac{\partial f}{\partial z_j}(z')}{\sqrt{1 + |\nabla' f(z')|^2_{N-1}}}, \quad \frac{\partial \Psi_N}{\partial t}(z', t) = \frac{1}{\sqrt{1 + |\nabla' f(z')|^2_{N-1}}}.$$

In particular, since $\nabla' f(0) = 0$,

$$J_\Psi(0,0) = I_{N-1}.$$

This proves (2.20). As

$$\frac{\partial^2 \Psi_j}{\partial t \partial z_i}(z', t) = \frac{\partial}{\partial z_i}\left(\frac{\frac{\partial g}{\partial z_j}(z')}{\sqrt{1 + |\nabla' g(z')|^2_{N-1}}}\right),$$

$$\frac{\partial^2 \Psi_N}{\partial t \partial z_i}(z', t) = -\frac{\partial}{\partial z_i}\left(\frac{1}{\sqrt{1 + |\nabla' g(z')|^2_{N-1}}}\right),$$

$$\frac{\partial^2 \Psi_j}{\partial^2 t}(z', t) = 0, \quad \frac{\partial^2 \Psi_N}{\partial t^2}(z', t) = 0,$$

using Jacobi's formula, we obtain

$$\frac{\partial \det J_\Psi}{\partial t}(z',t) = \det J_\Psi(z',t)\,\mathrm{tr}\left(J_\Psi^{-1}(z',t)\frac{\partial J_\Psi(z',t)}{\partial t}\right)$$

In particular, taking $z'=0$ and using the fact that $J_\Psi(0,t) = I_{N-1}$ we get

$$\frac{\partial \det J_\Psi}{\partial t}(0,t) = \sum_{i=1}^{N-1} \frac{\partial}{\partial z_i}\left(\frac{\frac{\partial g}{\partial z_j}(z')}{\sqrt{1+|\nabla' g(z')|_{N-1}^2}}\right)\Bigg|_{z'=0} = \kappa(y_0).$$

By the arbitrariness of y_0, this concludes the proof of (2.21). ■

3 A 1D Functional Problem

Let

$$I := (0,T)$$

and consider a weight function

$$\omega \in C^{1,d}([0,T]), \quad \min_{[0,T]} \omega > 0. \tag{3.1}$$

The prototype we have in mind is

$$\omega(t) := 1 + t\kappa(t).$$

In this section, we study the second-order Γ-convergence of the family of functionals

$$G_\varepsilon(v) := \int_I (W(v(t)) + \varepsilon^2(v'(t))^2)\omega(t)\,dt, \quad v \in H^1(I),$$

subject to the Dirichlet boundary condition

$$v(0) = \alpha_\varepsilon, \quad v(T) = \beta_\varepsilon, \tag{3.2}$$

where $\alpha_\varepsilon, \beta_\varepsilon \in \mathbb{R}$. In what follows, we will use the weighted BV space $BV_\omega(I)$ given by all functions $v \in BV_{\mathrm{loc}}(I)$ for which the norm

$$\|v\|_{BV_\omega} := \int_I |v(t)|\omega(t)\,dt + \int_I \omega(t)\,d|Dv|(t)$$

is finite. For $v \in BV_\omega(I)$ we will also write the weighted total variation of the derivative as

$$|Dv|_\omega(E) := \int_E \omega(t)\,d|Dv|(t).$$

For a more detailed introduction to weighted BV spaces and their applications to phase-field models, we refer to [5, 17].

We will study the second-order Γ-convergence with respect to the metric in $L^1(I)$. This choice is motivated by the following compactness result.

Theorem 3.1 (Compactness) *Assume that W satisfies* (2.1)-(2.4)*, that ω satisfies* (3.1)*, and that $\alpha_\varepsilon \to \alpha$ and $\beta_\varepsilon \to \beta$ as $\varepsilon \to 0^+$ for some $\alpha, \beta \in \mathbb{R}$. Let $\varepsilon_n \to 0^+$ and $v_n \in H^1(I)$ be such that*

$$\sup_n \int_I \left(\frac{1}{\varepsilon_n} W(v_n(t)) + \varepsilon_n (v_n'(t))^2 \right) \omega(t)\, dt < \infty.$$

Then there exist a subsequence $\{v_{n_k}\}_k$ of $\{v_n\}_n$ and $v \in BV_\omega(I; \{a, b\})$ such that $v_{n_k} \to v$ in $L^1(I)$.

The proof is identical to the one of [21, Proposition 4.3] and so we omit it. In view of the previous theorem, we extend G_ε to $L^1(I)$ by setting

$$G_\varepsilon(v) := \begin{cases} \int_I (W(v(t)) + \varepsilon^2 (v'(t))^2) \omega(t)\, dt & \text{if } v \in H^1(I) \text{ satisfies (3.2)}, \\ \infty & \text{otherwise in } L^1(I). \end{cases} \tag{3.3}$$

3.1 Zeroth and First-Order Γ-limit of G_ε

We begin by establishing the zeroth-order Γ-limit of the functional G_ε.

Theorem 3.2 *Assume that W satisfies* (2.1)-(2.4)*, that ω satisfies* (3.1)*, and that $\alpha_\varepsilon \to \alpha$ and $\beta_\varepsilon \to \beta$ as $\varepsilon \to 0^+$ for some $\alpha, \beta \in \mathbb{R}$. Then the family $\{G_\varepsilon\}_\varepsilon$ Γ-converges to $G^{(0)}$ in $L^1(I)$ as $\varepsilon \to 0^+$, where*

$$G^{(0)}(v) := \int_I W(v(t)) \omega(t)\, dt.$$

Proof. To prove the liminf inequality, let $\varepsilon_n \to 0^+$ and $v_n \to v$ in $L^1(I)$. Write $\alpha_n := \alpha_{\varepsilon_n}$ and $\beta_n := \beta_{\varepsilon_n}$. Consider a subsequence $\{\varepsilon_{n_k}\}_k$ of $\{\varepsilon_n\}_n$ such that

$$\lim_{k\to\infty} G_{\varepsilon_{n_k}}(v_{n_k}) = \liminf_{n\to\infty} G_{\varepsilon_n}(v_n).$$

Since $v_n \to v$ in $L^1(I)$ and $\inf_I \omega > 0$, by selecting a further subsequence, not relabeled, we can assume that $v_{n_k}(t) \to v(t)$ for $\mathcal{L}^1$-a.e. $t \in I$. Hence, by Fatou's lemma and the continuity and nonnegativity of W, we have

$$\lim_{k\to\infty} G_{\varepsilon_{n_k}}(v_{n_k}) \geq \liminf_{k\to\infty} \int_I W(v_{n_k}(t)) \omega(t)\, dt \geq \int_I W(v(t)) \omega(t)\, dt.$$

To prove the limsup inequality, let $\varepsilon_n \to 0^+$ and $v \in L^1(I)$. Assume first that v is bounded. Let $\bar{v}$ be a representative of v and let φ_δ be a standard mollifier, where $\delta > 0$. Let $\delta_n \to 0^+$ to be chosen later on and define

$$\bar{v}_n(t) := \begin{cases} \alpha_n & \text{if } -1 < t < 2\delta_n, \\ \bar{v}(t) & \text{if } 2\delta_n \leq t \leq T - 2\delta_n, \\ \beta_n & \text{if } T - 2\delta_n < t < T + 1, \end{cases}$$

and $v_n := \varphi_{\delta_n} * \bar{v}_n$. Assuming that $\operatorname{supp} \varphi \subseteq (-1, 1)$, we have that $v_n(0) = (\varphi_{\delta_n} * \alpha_n)(0) = \alpha_n$ and $v_n(T) = (\varphi_{\delta_n} * \beta_n)(T) = \beta_n$. On the the other hand, if

$0 < t_0 < 1$ is a Lebesgue point of $\bar{v}$ then for all n sufficiently large, we have that $v_n(t_0) = (\varphi_{\delta_n} * \bar{v})(t_0) \to \bar{v}(t_0)$ by standard properties of mollifiers. Using the continuity of W, we may apply the Lebesgue dominated convergence theorem to obtain that $v_n \to v$ in $L^1(I)$ and

$$\lim_{n\to\infty} \int_I W(v_n(t))\omega(t)\,dt = \int_I W(v(t))\omega(t)\,dt.$$

On the other hand,

$$|v_n'(t)| = |(\varphi_{\delta_n}' * \bar{v}_n)(t)| \leq \frac{C}{\delta_n}.$$

Hence,

$$\int_I \varepsilon_n^2 (v_n'(t))^2 \omega(t)\,dt \leq C\frac{\varepsilon_n^2}{\delta_n} \int_I \omega(t)\,dt \to 0$$

provided we choose δ_n such that $\frac{\varepsilon_n^2}{\delta_n} \to 0$. It follows that

$$\lim_{n\to\infty} G_{\varepsilon_n}(v_n) = \int_I W(v(t))\omega(t)\,dt.$$

To complete the proof, we use the fact that the Γ-lim sup is lower semicontinuous (see [12, Proposition 6.8]) and that every $v \in L^1(I)$ can be approximated by an increasing sequence of bounded functions for which the convergence of the integral on the right-hand side above follows from the monotone convergence theorem. ■

Since $W^{-1}(\{0\}) = \{a, b\}$, we have

$$\inf_{v\in L^1(I)} G^{(0)}(v) = 0$$

and therefore

$$\begin{aligned} G_\varepsilon^{(1)}(v) &:= \frac{G_\varepsilon(v) - \inf_{L^1(I)} G^{(0)}}{\varepsilon} \\ &= \int_I \left(\frac{1}{\varepsilon}W(v(t)) + \varepsilon(v'(t))^2\right)\omega(t)\,dt \end{aligned} \tag{3.4}$$

if $v \in H^1(I)$ satisfies (3.2) and $G_\varepsilon^{(1)}(v) := \infty$ if $v \in L^1(I)\setminus H^1(I)$ or if the boundary condition (3.2) fails.

We now characterize the first-order Γ-limit of the family $\{G_\varepsilon\}_\varepsilon$.

Theorem 3.3 *Assume that W satisfies* (2.1)-(2.4)*, that ω satisfies* (3.1)*, and that $\alpha_\varepsilon \to \alpha$ and $\beta_\varepsilon \to \beta$ as $\varepsilon \to 0^+$ for some $\alpha, \beta \in \mathbb{R}$. Then the family $\{G_\varepsilon^{(1)}\}_\varepsilon$ Γ-converges to $G^{(1)}$ in $L^1(I)$ as $\varepsilon \to 0^+$, where*

$$G^{(1)}(v) := \frac{C_W}{b-a}|Dv|_\omega(I) + \mathrm{d}_W(v(0),\alpha)\omega(0) + \mathrm{d}_W(v(T),\beta)\omega(T) \tag{3.5}$$

if $v \in BV_\omega(I;\{a,b\})$ and $G^{(1)}(v) := \infty$ otherwise in $L^1(I)$, where d_W and C_W are defined in (1.10) *and* (1.7)*, respectively.*

Proof. Step 1: To prove the Γ-liminf inequality, let $\varepsilon_n \to 0^+$ and $v_n \to v$ in $L^1(I)$. Write $\alpha_n := \alpha_{\varepsilon_n}$ and $\beta_n := \beta_{\varepsilon_n}$. Assume that

$$\liminf_{n\to\infty} G^{(1)}_{\varepsilon_n}(v_n) < \infty,$$

since otherwise there is nothing to prove, and consider a subsequence $\{\varepsilon_{n_k}\}_k$ of $\{\varepsilon_n\}_n$ such that

$$\lim_{k\to\infty} G^{(1)}_{\varepsilon_{n_k}}(v_{n_k}) = \liminf_{n\to\infty} G^{(1)}_{\varepsilon_n}(v_n).$$

Then for all k sufficiently large, $v_{n_k} \in H^1(I)$, $v_{n_k}(0) = \alpha_{n_k}$, and $v_{n_k}(T) = \beta_{n_k}$. Extend ω and v_{n_k} to $(-1, T+1)$, by setting

$$\bar\omega(t) := \begin{cases} \omega(0) & \text{if } t \le 0, \\ \omega(t) & \text{if } t \in I, \\ \omega(T) & \text{if } t \ge T, \end{cases} \qquad \bar v_{n_k}(t) := \begin{cases} \alpha_{n_k} & \text{if } t \le 0, \\ v_{n_k}(t) & \text{if } t \in I, \\ \beta_{n_k} & \text{if } t \ge T. \end{cases}$$

Define

$$W_1(s) := \min\{W(s), K\}, \quad \Phi_1(s) := \int_a^s 2W_1^{1/2}(\rho)\, d\rho,$$

where

$$K := \max_J W$$

and J is the smallest closed interval that contains $[a,b]$, $\{\alpha_{n_k}\}_k$, and $\{\beta_{n_k}\}_k$. Then

$$\begin{aligned} G^{(1)}_{\varepsilon_{n_k}}(v_{n_k}) &= \int_I \left(\frac{1}{\varepsilon_{n_k}} W(v_{n_k}(t)) + \varepsilon_{n_k}(v'_{n_k}(t))^2 \right) \omega(t)\, dt \\ &\ge \int_I 2W_1^{1/2}(v_{n_k}(t))|v'_{n_k}(t)|\omega(t)\, dt = \int_I |(\Phi_1 \circ v_{n_k})'(t)|\omega(t)\, dt \\ &= \int_{-1}^{T+1} |(\Phi_1 \circ \bar v_{n_k})'(t)|\bar\omega(t)\, dt, \end{aligned}$$

where in the last equality we used the fact that $(\Phi_1 \circ \bar v_{n_k})(t) \equiv \Phi(\alpha_{n_k})$ in $(-1,0)$ and $(\Phi_1 \circ \bar v_{n_k})(t) \equiv \Phi(\beta_{n_k})$ in $(T, T+1)$. Since Φ_1 is Lipschitz continuous, we have that $\Phi_1 \circ \bar v_{n_k} \to \Phi_1 \circ \bar v$ in $L^1((-1, T+1))$, where

$$\bar v(t) := \begin{cases} \alpha & \text{if } t \le 0, \\ v(t) & \text{if } t \in I, \\ \beta & \text{if } t \ge T. \end{cases}$$

Hence, by standard lower semicontinuity results,

$$\begin{aligned}\lim_{k\to\infty} G^{(1)}_{\varepsilon_{n_k}}(v_{n_k}) &\geq \liminf_{k\to\infty}\int_{-1}^{T+1} |(\Phi_1\circ \bar{v}_{n_k})'(t)|\bar{\omega}(t)\,dt\\ &\geq \int_{-1}^{T+1}\bar{\omega}\,d|D(\Phi_1\circ\bar{v})| = \int_{-1}^{T+1}\bar{\omega}\,d|D(\Phi\circ\bar{v})|\\ &= \int_I \omega\,d|D(\Phi\circ v)| + \mathrm{d}_W(\alpha, v(0))\omega(0) + \mathrm{d}_W(\beta, v(T))\omega(T)\\ &= \frac{C_W}{b-a}\int_I \omega\,d|Dv| + \mathrm{d}_W(\alpha, v(0))\omega(0) + \mathrm{d}_W(\beta, v(T))\omega(T)\\ &= G^{(1)}(v).\end{aligned}$$

Step 2: To prove the Γ-limsup inequality, assume first that v is of the form

$$v(t) = \begin{cases} a & \text{if } t\in[t_{2k}, t_{2k+1}),\\ b & \text{otherwise,}\end{cases}$$

where $0 = t_0 < t_1 < \cdots < t_{2\ell} = T$. Observe that

$$v(t) = \mathrm{sgn}_{a,b}(f(t)), \tag{3.6}$$

where

$$\mathrm{sgn}_{a,b}(t) := \begin{cases} a & \text{if } t\leq 0,\\ b & \text{if } t>0,\end{cases} \tag{3.7}$$

and

$$f(t) := \begin{cases} t - t_1 & \text{if } t\in[t_0,t_1),\\ -\min\{t-t_{2k}, t_{2k+1}-t\} & \text{if } t\in[t_{2k},t_{2k+1}),\ \text{and } k\geq 1,\\ \min\{t-t_{2k+1}, t_{2k+2}-t\} & \text{if } t\in(t_{2k+1},t_{2k+2}],\ \text{and } k<\ell-1,\\ t-t_{2\ell-1} & \text{if } t\in[t_{2\ell-1},t_{2\ell})\end{cases} \tag{3.8}$$

is the signed distance function of the set $E := \{t\in I : v(t) = a\}$ relative to I. We will construct smooth approximations of the function $\mathrm{sgn}_{a,b}$ that almost minimize the energy $G^{(1)}_\varepsilon$.

Since we expect each transition to happen in an infinitesimal interval and $\omega(t)\sim\omega(t_k)$ for t close to t_k, to construct an approximate solution, we consider minimizers of the functional $\int_0^{T_\varepsilon}\left(\frac{1}{\varepsilon}W(\phi)+\varepsilon(\phi')^2\right)dt$ with appropriate boundary conditions and where $T_\varepsilon\to 0^+$. The minimizers of this functional satisfy the Euler–Lagrange equations $2\varepsilon^2\phi'' = W'(\phi)$. If we multiply each side by ϕ' and integrate, we get

$$\varepsilon^2(\phi'(t))^2 = \varepsilon^2(\phi'(0))^2 - W(\phi(0)) + W(\phi(t)).$$

Thus,

$$\varepsilon\phi'(t) = \pm(\varepsilon^2(\phi'(0))^2 - W(\phi(0)) + W(\phi(t)))^{1/2}.$$

Setting $\delta_\varepsilon := \varepsilon^2(\phi'(0))^2 - W(\phi(0))$ we solve the differential equation

$$\varepsilon\phi'(t) = \pm(\delta_\varepsilon + W(\phi(t)))^{1/2},$$

where we determine the sign according to each transition. Provided $\delta_\varepsilon + W(\phi(t)) \neq 0$, this gives

$$\pm\int_{\phi(0)}^{\phi(t)} \frac{\varepsilon}{(\delta_\varepsilon + W(\rho))^{1/2}}\, d\rho = t.$$

We consider first the transition from α_ε to a. Let $\delta_\varepsilon > 0$ and introduce the function

$$\bar{\Psi}_\varepsilon(r) := \int_r^{\alpha_\varepsilon} \frac{\varepsilon}{(\delta_\varepsilon + W(\rho))^{1/2}}\, d\rho. \tag{3.9}$$

Define

$$\bar{T}_\varepsilon := \bar{\Psi}_\varepsilon(a)$$

and observe that since $W \geq 0$,

$$0 < \bar{T}_\varepsilon = \int_a^{\alpha_\varepsilon} \frac{\varepsilon}{(\delta_\varepsilon + W(\rho))^{1/2}}\, d\rho \leq (b-a)\frac{\varepsilon}{\delta_\varepsilon^{1/2}}. \tag{3.10}$$

The function $\bar{\Psi}_\varepsilon$ is strictly decreasing and differentiable. Let $\bar{\phi}_\varepsilon : [0, \bar{T}_\varepsilon] \to [a, \alpha_\varepsilon]$ be the inverse of $\bar{\Psi}_\varepsilon$ on the interval $[a, \alpha_\varepsilon]$. Then $\bar{\phi}_\varepsilon(0) = \alpha_\varepsilon$, $\bar{\phi}_\varepsilon(\bar{T}_\varepsilon) = a$, and

$$\bar{\phi}_\varepsilon'(s) = \frac{1}{\bar{\Psi}_\varepsilon'(\bar{\phi}_\varepsilon(s))} = -\frac{(\delta_\varepsilon + W(\bar{\phi}_\varepsilon(s)))^{1/2}}{\varepsilon}. \tag{3.11}$$

Extend $\bar{\phi}_\varepsilon$ to be a for $t > \bar{T}_\varepsilon$.

Similarly, to transition from a to b, we define

$$\Psi_\varepsilon(r) := \int_a^r \frac{\varepsilon}{(\delta_\varepsilon + W(\rho))^{1/2}}\, d\rho,$$

and

$$0 \leq T_\varepsilon := \Psi_\varepsilon(b) \leq (b-a)\frac{\varepsilon}{\delta_\varepsilon^{1/2}}. \tag{3.12}$$

Let $\phi_\varepsilon : [0, T_\varepsilon] \to [a, b]$ be the inverse of Ψ_ε on the interval $[a, b]$. Then $\phi_\varepsilon(0) = a$, $\phi_\varepsilon(T_\varepsilon) = b$, and

$$\phi_\varepsilon'(s) = \frac{(\delta_\varepsilon + W(\phi_\varepsilon(s)))^{1/2}}{\varepsilon}. \tag{3.13}$$

Extend ϕ_ε to be equal to a for $s < 0$ and b for $s > T_\varepsilon$.

Finally, to transition from β_ε to b, define

$$\hat{\Psi}_\varepsilon(r) := \int_{\beta_\varepsilon}^r \frac{\varepsilon}{(\delta_\varepsilon + W(\rho))^{1/2}}\, d\rho,$$

and

$$0 \leq \hat{T}_\varepsilon := \hat{\Psi}_\varepsilon(b) \leq (b-a)\frac{\varepsilon}{\delta_\varepsilon^{1/2}}. \tag{3.14}$$

Let $\hat{\phi}_\varepsilon : [0, \hat{T}_\varepsilon] \to [\beta_\varepsilon, b]$ be the inverse of $\hat{\Psi}_\varepsilon$ on the interval $[\beta_\varepsilon, b]$. Then $\hat{\phi}_\varepsilon(0) = b$, $\hat{\phi}_\varepsilon(\hat{T}_\varepsilon) = \beta_\varepsilon$, and

$$\hat{\phi}'_\varepsilon(s) = -\frac{(\delta_\varepsilon + W(\hat{\phi}_\varepsilon(s)))^{1/2}}{\varepsilon}. \tag{3.15}$$

Extend $\hat{\phi}_\varepsilon$ to be equal to b for $s < 0$. Assume that

$$\delta_\varepsilon \to 0, \quad \frac{\varepsilon}{\delta_\varepsilon^{1/2}} \to 0. \tag{3.16}$$

Taking ε be so small that transition layers do not overlap or leave $\overline{I}$, we can define

$$v_\varepsilon(t) := \begin{cases} \bar{\phi}_\varepsilon(t) & \text{if } 0 < t < \bar{T}_\varepsilon, \\ \phi_\varepsilon(f(t)) & \text{if } \bar{T}_\varepsilon \le t \le T - \hat{T}_\varepsilon, \\ \hat{\phi}_\varepsilon(t - T + \hat{T}_\varepsilon) & \text{if } T - \hat{T}_\varepsilon < t < T. \end{cases}$$

Since $\bar{T}_\varepsilon \to 0$, $T_\varepsilon \to 0$, and $\hat{T}_\varepsilon \to 0$ as $\varepsilon \to 0^+$ by (3.10), (3.12), and (3.14), respectively, in view of (3.6), (3.8), we have that $v_\varepsilon(t) \to v(t)$ for all $t \in I$. Moreover, $|v_\varepsilon(t)| \le C$ for all $t \in I$, all $\varepsilon > 0$, and for some constant $C > 0$. Hence, by the Lebesgue dominated convergence theorem, $v_\varepsilon \to v$ in $L^1(I)$.

By (3.11) and the change of variables $\rho = \bar{\phi}_\varepsilon(t)$, we have

$$\begin{aligned}
&\int_0^{\bar{T}_\varepsilon} \left(\frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon (v'_\varepsilon)^2\right) \omega \, dt = \int_0^{\bar{T}_\varepsilon} \left(\frac{1}{\varepsilon} W(\bar{\phi}_\varepsilon) + \varepsilon (\bar{\phi}'_\varepsilon)^2\right) \omega \, dt \\
&\le \max_{[0,\bar{T}_\varepsilon]} \omega \int_0^{\bar{T}_\varepsilon} \left(\frac{1}{\varepsilon}(\delta_\varepsilon + W(\bar{\phi}_\varepsilon)) + \varepsilon (\bar{\phi}'_\varepsilon)^2\right) dt \\
&= \max_{[0,\bar{T}_\varepsilon]} \omega \int_0^{\bar{T}_\varepsilon} 2(\delta_\varepsilon + W(\bar{\phi}_\varepsilon))^{1/2} |\bar{\phi}'_\varepsilon| \, dt = \max_{[0,\bar{T}_\varepsilon]} \omega \int_a^{\alpha_\varepsilon} 2(\delta_\varepsilon + W(\rho))^{1/2} \, d\rho.
\end{aligned}$$

On the other hand, by (3.15) and the changes of variables $t - T + \hat{T}_\varepsilon = s$, $\rho = \hat{\phi}_\varepsilon(s)$,

$$\begin{aligned}
&\int_{T-\hat{T}_\varepsilon}^{T} \left(\frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon (v'_\varepsilon)^2\right) \omega \, dt = \int_0^{\hat{T}_\varepsilon} \left(\frac{1}{\varepsilon} W(\hat{\phi}_\varepsilon) + \varepsilon (\hat{\phi}'_\varepsilon)^2\right) \omega(T - \hat{T}_\varepsilon + s) \, ds \\
&\le \max_{[T-\hat{T}_\varepsilon, T]} \omega \int_0^{\hat{T}_\varepsilon} \left(\frac{1}{\varepsilon}(\delta_\varepsilon + W(\hat{\phi}_\varepsilon)) + \varepsilon (\hat{\phi}'_\varepsilon)^2\right) dt \\
&= \max_{[T-\hat{T}_\varepsilon, T]} \omega \int_0^{\hat{T}_\varepsilon} 2(\delta_\varepsilon + W(\hat{\phi}_\varepsilon))^{1/2} |\hat{\phi}'_\varepsilon| \, dt = \max_{[T-\hat{T}_\varepsilon, T]} \omega \int_{\beta_\varepsilon}^{b} 2(\delta_\varepsilon + W(\rho))^{1/2} d\rho.
\end{aligned}$$

Similarly, by (3.13),

$$
\begin{aligned}
&\sum_{k=1}^{2\ell-1}\int_{t_{k-1}}^{t_k}\left(\frac{1}{\varepsilon}W(v_\varepsilon)+\varepsilon(v_\varepsilon')^2\right)\omega\,dt\\
&=\sum_{k=1}^{2\ell-1}\int_0^{T_\varepsilon}\left(\varepsilon(\phi_\varepsilon'(s))^2+\varepsilon^{-1}W(\phi_\varepsilon(s))\right)\omega(t_k+(-1)^{k+1}s)\,ds\\
&\le\sum_{k=1}^{2\ell-1}\int_0^{T_\varepsilon}2(\delta_\varepsilon+W(\phi_\varepsilon(s)))^{1/2}\phi_\varepsilon'(s)\omega(t_k+(-1)^{k+1}s)\,ds\\
&\le\sum_{k=1}^{2\ell-1}\sup\{\omega(t_k+(-1)^{k+1}r):\,r\in(0,T_\varepsilon)\}\int_0^{T_\varepsilon}2(\delta_\varepsilon+W(\phi_\varepsilon(s)))^{1/2}\phi_\varepsilon'(s)\,ds\\
&=\sum_{k=1}^{2\ell-1}\sup\{\omega(t_k+(-1)^{k+1}r):\,r\in(0,T_\varepsilon)\}\int_a^{b}2(\delta_\varepsilon+W(s))^{1/2}ds.
\end{aligned}
$$

Thus taking the limit as $\varepsilon\to0^+$ we find that

$$
\begin{aligned}
\limsup_{\varepsilon\to0^+}G_\varepsilon^{(1)}(v_\varepsilon)&\le\int_a^\alpha 2W^{1/2}(s)\,ds\omega(0)+\int_\beta^b 2W^{1/2}(s)ds\omega(T)+C_W\sum_{k=1}^{2\ell-1}\omega(t_k)\\
&=G^{(1)}(v).
\end{aligned}
$$

The cases where v starts or ends at values different from those we assumed above are treated analogously. ■

Next, we show that if ω is sufficiently close to $\omega(0)$, $a<\alpha<b$, and $\beta=b$, then the unique minimizer of $G^{(1)}$ is the constant function b. We recall that α and β appear in the definition of $G^{(1)}$ (see (3.5)).

Corollary 3.4 *Assume that W satisfies* (2.1)-(2.4) *and let $G^{(1)}$ be given by* (3.5) *with $a<\alpha<b$ and $\beta=b$. Suppose that ω satisfies* (3.1) *and that*

$$\omega(t)>\omega(0)-\omega_0\quad\text{for all } t\in(0,T],\tag{3.17}$$

where

$$0\le\omega_0<\frac{1}{2}\frac{C_W-\mathrm{d}_W(\alpha,b)}{C_W}\omega(0).\tag{3.18}$$

Then the unique minimizer of $G^{(1)}$ is the constant function b, with

$$\min_{L^1(I)}G^{(1)}(v)=G^{(1)}(b)=\mathrm{d}_W(\alpha,b)\omega(0).$$

Proof. Let $v\in BV_\omega(I;\{a,b\})$. If v has at least one jump point at $t_0\in I$, then by (3.17) and (3.18),

$$G^{(1)}(v)\ge\frac{C_W}{b-a}|Dv|_\omega(I)\ge C_W\omega(t_0)>C_W(\omega(0)-\omega_0)\ge\mathrm{d}_W(\alpha,b)\omega(0).$$

Hence, either $v \equiv b$ or $v \equiv a$. If $v = a$, then again by (3.17) and (3.18)

$$G^{(1)}(a) = \mathrm{d}_W(a,\alpha)\omega(0) + C_W\omega(T) > C_W(\omega(0) - \omega_0) \geq \mathrm{d}_W(\alpha, b)\omega(0).$$

This completes the proof. ■

Remark 3.5 *Note that condition* (3.17) *holds if either ω is strictly increasing, with $\omega_0 = 0$, or if T is sufficiently small, by continuity of ω.*

3.2 Second-Order Γ-limsup

Under the hypotheses of Corollary 3.4, we have

$$\min_{L^1(I)} G^{(1)}(v) = G^{(1)}(b) = \mathrm{d}_W(\alpha, b)\omega(0).$$

We define

$$\begin{aligned} G_\varepsilon^{(2)}(v) &:= \frac{G_\varepsilon^{(1)}(v) - \inf_{L^1(I)} G^{(1)}}{\varepsilon} \\ &= \int_I \left(\frac{1}{\varepsilon^2} W(v(t)) + (v'(t))^2\right)\omega(t)\,dt - \mathrm{d}_W(\alpha,b)\omega(0)\frac{1}{\varepsilon} \end{aligned} \tag{3.19}$$

if $v \in H^1(I)$ satisfies (3.2) and $G_\varepsilon^{(2)}(v) := \infty$ if $v \in L^1(I) \setminus H^1(I)$ or if the boundary condition (3.2) fails.

We study the second-order Γ-limsup of the family $\{G_\varepsilon\}_\varepsilon$.

Theorem 3.6 (Second-Order Γ-Limsup) *Assume that W satisfies* (2.1)-(2.4), *that α_- satisfies* (2.5), *and that ω satisfies* (3.1), (3.17), *where*

$$0 \leq \omega_0 < \frac{1}{2}\frac{\mathrm{d}_W(a,\alpha_-)}{C_W}\omega(0). \tag{3.20}$$

Let

$$\alpha_- \leq \alpha_\varepsilon,\, \beta_\varepsilon \leq b, \tag{3.21}$$

with

$$|\alpha_\varepsilon - \alpha| \leq A_0\varepsilon^\gamma, \quad |\beta_\varepsilon - b| \leq B_0\varepsilon^\gamma \tag{3.22}$$

for some α, β and where A_0, $B_0 > 0$, and $\gamma > 1$. Then there exist constants $0 < \varepsilon_0 < 1$, $C, C_0 > 0$, and $\gamma_0, \gamma_1 > 0$, depending only on α_-, A_0, B_0, T, ω, and W, and functions $v_\varepsilon \in H^1(I)$ satisfying (3.2), *$a \leq v_\varepsilon \leq b$, and $v_\varepsilon \to b$ in $L^1(I)$, such that*

$$G_\varepsilon^{(2)}(v_\varepsilon) \leq \int_0^l 2W(p_\varepsilon(t))t\,dt\,\omega'(0) + Ce^{-2\sigma l}(2\sigma l + 1) + C\varepsilon^{2\gamma}l + C\varepsilon^{\gamma_1}|\log\varepsilon|^{\gamma_0} \tag{3.23}$$

for all $0 < \varepsilon < \varepsilon_0$ and all $l > 0$, where $p_\varepsilon(t) = v_\varepsilon(\varepsilon t)$ is such that $p_\varepsilon \to z_\alpha$ pointwise in $[0,\infty)$, and where z_α solves the Cauchy problem (1.8). *In particular,*

$$\limsup_{\varepsilon \to 0^+} G_\varepsilon^{(2)}(v_\varepsilon) \leq \int_0^\infty 2W^{1/2}(z_\alpha(t))z_\alpha'(t)t\,dt\,\omega'(0). \tag{3.24}$$

Proof. Let $\delta_\varepsilon \to 0^+$ as $\varepsilon \to 0^+$, and define

$$\Psi_\varepsilon(r) := \begin{cases} \int_{\alpha_\varepsilon}^{r} \frac{\varepsilon}{(\delta_\varepsilon + W(s))^{1/2}}\, ds & \text{if } \alpha_\varepsilon \leq \beta_\varepsilon, \\ \int_{r}^{\alpha_\varepsilon} \frac{\varepsilon}{(\delta_\varepsilon + W(s))^{1/2}}\, ds & \text{if } \beta_\varepsilon < \alpha_\varepsilon, \end{cases} \tag{3.25}$$

and

$$0 \leq T_\varepsilon := \Psi_\varepsilon(\beta_\varepsilon). \tag{3.26}$$

Since $\alpha_- \leq \alpha_\varepsilon,\ \beta_\varepsilon \leq b$, by Proposition 2.2 we have

$$\begin{aligned} T_\varepsilon &\leq \int_{\alpha_-}^{b} \frac{\varepsilon}{(\delta_\varepsilon + W(s))^{1/2}}\, ds \\ &\leq -\frac{\sigma}{2}\varepsilon \log(\sigma^2 \delta_\varepsilon) + \sigma\varepsilon \log(1 + 2(b - a)). \end{aligned}$$

Since $\delta_\varepsilon \to 0^+$, there exist $C_0 > 0$ and $\varepsilon_0 > 0$, depending only on W, γ, and B, such that

$$T_\varepsilon \leq C_0 \varepsilon |\log \delta_\varepsilon| \tag{3.27}$$

for all $0 < \varepsilon < \varepsilon_0$.

On the other hand, if $\alpha < b$, by Proposition 2.2, and (3.22), we obtain

$$\begin{aligned} T_\varepsilon \geq &-\varepsilon \log(\sigma^{-2}\delta_\varepsilon) + \sigma^{-1}\varepsilon \log(b - \alpha - A\varepsilon^\gamma) \\ &- \sigma^{-1}\varepsilon \log(1 + 2\sigma^{-1} B\varepsilon^\gamma / \delta_\varepsilon^{1/2}) \end{aligned}$$

for all $0 < \varepsilon < \varepsilon_\alpha$. Since $\delta_\varepsilon \to 0^+$, by taking ε_α smaller if necessary, we can find $C_\alpha > 0$ such that

$$C_\alpha \varepsilon |\log \delta_\varepsilon| \leq T_\varepsilon \quad \text{if } \alpha < b \tag{3.28}$$

for all $0 < \varepsilon < \varepsilon_\alpha$.

Let $v_\varepsilon : [0, T_\varepsilon] \to [\alpha_\varepsilon, \beta_\varepsilon]$ be the inverse of Ψ_ε. Then $v_\varepsilon(0) = \alpha_\varepsilon$, $v_\varepsilon(T_\varepsilon) = \beta_\varepsilon$, and

$$v_\varepsilon'(t) = \pm \frac{(\delta_\varepsilon + W(v_\varepsilon(t)))^{1/2}}{\varepsilon}, \tag{3.29}$$

where we take the plus sign if $\alpha_\varepsilon \leq \beta_\varepsilon$ and the minus sign if $\beta_\varepsilon < \alpha_\varepsilon$. Extend v_ε to be equal to β_ε for $t > T_\varepsilon$.

Since $\omega \in C^{1,d}(I)$, by Taylor's formula, for $t \in [0, T]$ we have

$$\omega(t) = \omega(0) + \omega'(0)t + R_1(t),$$

where

$$|R_1(t)| = |\omega'(\theta t) - \omega'(0)| t \leq |\omega'|_{C^{0,d}} t^{1+d}. \tag{3.30}$$

Write

$$
\begin{aligned}
G^{(2)}_\varepsilon(v_\varepsilon) &= \left[\int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right)\, dt - \mathrm{d}_W(b,\alpha)\right]\frac{\omega(0)}{\varepsilon} \\
&+ \int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right) t\, dt \frac{\omega'(0)}{\varepsilon} \\
&+ \int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right) R_1\, dt \frac{1}{\varepsilon} \\
&+ \int_{T_\varepsilon}^{T} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right) \omega\, dt \frac{1}{\varepsilon} =: \mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}.
\end{aligned}
\tag{3.31}
$$

Step 1: We estimate $\mathcal{A}$ in (3.31). Assume first that $\alpha_\varepsilon \le \beta_\varepsilon$. By (3.25), (3.26), (3.29), the change of variables $s = v_\varepsilon(t)$, the fact that $a < \alpha_\varepsilon$, $\beta_\varepsilon \le b$, and the equality

$$
(A+B)^{1/2} - B^{1/2} = \frac{A}{(A+B)^{1/2} + B^{1/2}},
$$

we have

$$
\begin{aligned}
\int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right)\, dt &= \int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon}(\delta_\varepsilon + W(v_\varepsilon)) + \varepsilon(v_\varepsilon')^2\right)\, dt - \frac{T_\varepsilon\delta_\varepsilon}{\varepsilon} \\
&= \int_0^{T_\varepsilon} 2(\delta_\varepsilon + W(v_\varepsilon))^{1/2} v_\varepsilon'\, dt - \frac{T_\varepsilon\delta_\varepsilon}{\varepsilon} \\
&= \int_{\alpha_\varepsilon}^{\beta_\varepsilon} 2(\delta_\varepsilon + W(s))^{1/2}\, ds - \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{\delta_\varepsilon}{(\delta_\varepsilon + W(s))^{1/2}}\, ds \\
&= \int_{\alpha_\varepsilon}^{\beta_\varepsilon} 2W^{1/2}(s)\, ds + \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \left[\frac{2\delta_\varepsilon}{(\delta_\varepsilon + W(s))^{1/2} + W^{1/2}(s)} - \frac{\delta_\varepsilon}{(\delta_\varepsilon + W(s))^{1/2}}\right] ds.
\end{aligned}
$$

By Proposition 2.3,

$$
\int_{\alpha_\varepsilon}^{\beta_\varepsilon} \left[\frac{2\delta_\varepsilon}{(\delta_\varepsilon + W(s))^{1/2} + W^{1/2}(s)} - \frac{\delta_\varepsilon}{(\delta_\varepsilon + W(s))^{1/2}}\right] ds \le C\delta_\varepsilon
$$

for all $0 < \varepsilon < \varepsilon_0$, while, since $a \le \alpha_\varepsilon$, $\beta_\varepsilon \le b$,

$$
\int_{\alpha_\varepsilon}^{\beta_\varepsilon} 2W^{1/2}(s)\, ds \le \int_\alpha^b 2W^{1/2}(s).
$$

Hence, we obtain

$$
\int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2\right)\, dt \le \int_\alpha^b 2W^{1/2}(s)\, ds + C\delta_\varepsilon. \tag{3.32}
$$

The case $\beta_\varepsilon < \alpha_\varepsilon$ is similar. We omit the details.

It follows from (3.31) that

$$
\mathcal{A} \le C\frac{\delta_\varepsilon}{\varepsilon}
$$

for all $0 < \varepsilon < \varepsilon_0$.
Step 2: We estimate $\mathcal{B}$ in (3.31). By (3.29), (3.31), and the change of variables $t = \varepsilon s$ for $l > 0$,

$$\begin{aligned}\mathcal{B} &= \int_0^{T_\varepsilon} \frac{2}{\varepsilon^2} W(v_\varepsilon) t\, dt\, \omega'(0) + \frac{\delta_\varepsilon}{\varepsilon^2} \int_0^{T_\varepsilon} t\, dt\, \omega'(0) \\ &= \int_0^l 2W(p_\varepsilon(s)) s\, ds \omega'(0) + \int_l^{T_\varepsilon \varepsilon^{-1}} 2W(p_\varepsilon(s)) s\, ds \omega'(0) + \frac{1}{2}\omega'(0) \frac{\delta_\varepsilon T_\varepsilon^2}{\varepsilon^2} \\ &:= \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3.\end{aligned}$$

where $p_\varepsilon(s) := v_\varepsilon(\varepsilon s)$ solves the initial value problem

$$\begin{cases} p_\varepsilon'(s) = \pm(\delta_\varepsilon + W(p_\varepsilon\,(s)))^{1/2}, \\ p_\varepsilon(0) = \alpha_\varepsilon \end{cases} \tag{3.33}$$

in $[0, T_\varepsilon \varepsilon^{-1}]$, while $p_\varepsilon\,(s) = \beta_\varepsilon$ for $s > T_\varepsilon \varepsilon^{-1}$.

Let q_ε be the unique solution to (3.33) in $\mathbb{R}$. Since $\delta_\varepsilon \to 0$, and $\alpha_\varepsilon \to \alpha$, by standard results on the continuous dependence of solutions on a parameter (see, e.g. [30, Section 2.4]), it follows that $q_\varepsilon \to z_\alpha$ pointwise in $\mathbb{R}$, where z_α is given in (1.8).

If $\alpha < b$, then $T_\varepsilon \varepsilon^{-1} \geq C_\alpha |\log \delta_\varepsilon| \to \infty$ by (3.28), and so $p_\varepsilon \to z_\alpha$ pointwise in $[0, l]$. On the other hand, if $\alpha = b$, then $q_\varepsilon \to z_b = b$ pointwise in $\mathbb{R}$, while $p_\varepsilon\,(s) = \beta_\varepsilon$ for $s > T_\varepsilon \varepsilon^{-1}$. Hence,

$$|p_\varepsilon\,(s) - z_b(s)| = |p_\varepsilon\,(s) - b| \leq |q_\varepsilon\,(s) - b| + |\beta_\varepsilon - b| \to 0$$

for all $s \in [0, l]$.

Since $a \leq p_\varepsilon(s) \leq b$, by the Lebesgue dominated convergence theorem

$$\lim_{\varepsilon \to 0^+} \mathcal{B}_1 = \int_0^l 2W(z_\alpha(s)) s\, ds \omega'(0).$$

To estimate $\mathcal{B}_2$, observe that since $\delta_\varepsilon > 0$ and $\alpha_\varepsilon \leq p_\varepsilon(s) \leq \beta_\varepsilon < b$ for $0 \leq s \leq T_\varepsilon \varepsilon^{-1}$, by (2.6) we have

$$p_\varepsilon'(s) \geq (W(p_\varepsilon\,(s)))^{1/2} \geq \sigma(b - p_\varepsilon(s)) > 0,$$

and so

$$-\sigma \geq \frac{(b - p_\varepsilon(s))'}{b - p_\varepsilon(s)} = (\log(b - p_\varepsilon(s)))'.$$

Upon integration, we get

$$0 \leq b - p_\varepsilon(s) \leq (b - \alpha_\varepsilon) e^{-\sigma s} \leq (b - \alpha) e^{-\sigma s}.$$

In turn, again by (2.6), for $s \in [0, T_\varepsilon \varepsilon^{-1}]$,

$$W(p_\varepsilon(s)) \leq \sigma^{-2}(b - p_\varepsilon(s))^2 \leq \sigma^{-2}(b - \alpha)^2 e^{-2\sigma s}. \tag{3.34}$$

On the other hand, if $s \in [T_\varepsilon \varepsilon^{-1}, T\varepsilon^{-1}]$, then by (2.6) and (3.22),

$$W(p_\varepsilon(s)) = W(\beta_\varepsilon) \le \sigma^{-2}(b-\beta_\varepsilon)^2 \le C\varepsilon^{2\gamma}. \tag{3.35}$$

Hence, if $T_\varepsilon \varepsilon^{-1} \le l$, by (3.34),

$$\mathcal{B}_2 \le C\int_l^\infty e^{-2\sigma s} s\, ds = Ce^{-2\sigma l}\left(2\sigma l + 1\right),$$

while if $T_\varepsilon \varepsilon^{-1} \ge l$, by (3.35),

$$\mathcal{B}_2 \le C\varepsilon^{2\gamma} l.$$

Therefore,

$$\mathcal{B}_2 \le Ce^{-2\sigma l}\left(2\sigma l + 1\right) + C\varepsilon^{2\gamma} l$$

for all $0 < \varepsilon < \varepsilon_0$.

On the other hand, again by (3.27),

$$\mathcal{B}_3 \le C\frac{\delta_\varepsilon T_\varepsilon^2}{2\varepsilon^2} \le C\delta_\varepsilon \log^2 \delta_\varepsilon.$$

Step 3: To estimate $\mathcal{C}$ in (3.31), observe that by (3.32), (3.30), (3.27), and (3.31),

$$\begin{aligned}\mathcal{C} &\le |\omega'|_{C^{0,d}} \int_0^{T_\varepsilon} \left(\frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon (v_\varepsilon')^2\right) dt\, T_\varepsilon^{1+d}\frac{1}{\varepsilon} \\ &\le C\left(C_W + C\delta_\varepsilon |\log \delta_\varepsilon| + C\varepsilon^\gamma\right)\varepsilon^d |\log\delta_\varepsilon|^{1+d} \le C\varepsilon^d |\log \delta_\varepsilon|^{1+d}\end{aligned}$$

for all $0 < \varepsilon < \varepsilon_0$.

Step 4: We estimate $\mathcal{D}$ in (3.31). By (2.6), (3.22), and (3.31), for $t \ge T_\varepsilon$

$$\mathcal{D} = W(\beta_\varepsilon)\int_{T_\varepsilon}^T \omega\, dt \frac{1}{\varepsilon^2} \le \sigma^{-2}(b-\beta_\varepsilon)^2 \int_0^T \omega\, dt\frac{1}{\varepsilon^2} \le C\varepsilon^{2\gamma-2}.$$

Combining the estimates for $\mathcal{A}$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{C}$, and $\mathcal{D}$ and using (3.31) gives

$$\begin{aligned}G_\varepsilon^{(2)}(v_\varepsilon) \le C\frac{\delta_\varepsilon}{\varepsilon} + \int_0^l 2W(p_\varepsilon(s))s\, ds\omega'(0) + Ce^{-2\sigma l}\left(2\sigma l + 1\right) \\ + C\varepsilon^{2\gamma} l + C\delta_\varepsilon \log^2\delta_\varepsilon + C\varepsilon^d |\log\delta_\varepsilon|^{1+d} + C\varepsilon^{2\gamma-2}.\end{aligned} \tag{3.36}$$

By taking

$$\delta_\varepsilon := \varepsilon^m \quad \text{or} \quad \delta_\varepsilon := \frac{\varepsilon}{\log^m \varepsilon}, \quad m \ge 2, \tag{3.37}$$

we obtain (3.23). In turn, letting $\varepsilon \to 0^+$ in (3.23), we have

$$\limsup_{\varepsilon\to 0^+} G_\varepsilon^{(2)}(v_\varepsilon) \le \int_0^l 2W(z_\alpha(t))t\, dt\, \omega'(0) + Ce^{-2\sigma l}\left(2\sigma l + 1\right).$$

Since $b - z_\alpha$ decays exponentially as $t \to \infty$, by (2.11), using (2.6) and the Lebesgue dominated convergence theorem, we let $l \to \infty$ to obtain (3.24). ■

Remark 3.7 *Observe that if we take $\delta_\varepsilon := \varepsilon$, it follows from (3.36) that*

$$G_\varepsilon^{(2)}(v_\varepsilon) \le C + \int_0^l 2W(p_\varepsilon(s))s\,ds\omega'(0) + Ce^{-2\sigma l}(2\sigma l + 1) \\ + C\varepsilon^{2\gamma} l + C\varepsilon \log^2 \varepsilon + C\varepsilon^d |\log \varepsilon|^{1+d} + C\varepsilon^{2\gamma - 2}.$$

3.3 Properties of Minimizers of G_ε

In this subsection we study qualitative properties of the minimizers of the functional G_ε defined in (3.3):

$$G_\varepsilon(v) := \int_I (W(v(t)) + \varepsilon^2 (v'(t))^2)\omega(t)\,dt, \quad v \in H^1(I), \tag{3.38}$$

subject to the Dirichlet boundary conditions

$$v_\varepsilon(0) = \alpha_\varepsilon, \quad v_\varepsilon(T) = \beta_\varepsilon. \tag{3.39}$$

Theorem 3.8 *Assume that W satisfies* (2.1)-(2.4)*, that ω satisfies* (3.1)*, and that $a \le \alpha_\varepsilon$, $\beta_\varepsilon \le b$. Then the functional G_ε admits a minimizer $v_\varepsilon \in H^1(I)$. Moreover, $v_\varepsilon \in C^2([0,T])$, v_ε satisfies the Euler–Lagrange equations*

$$2\varepsilon^2 (v_\varepsilon'(t)\omega(t))' - W'(v_\varepsilon(t))\omega(t) = 0, \tag{3.40}$$

and $v_\varepsilon \equiv a$, or $v_\varepsilon \equiv b$, or

$$a < v_\varepsilon(t) < b \quad \textit{for all } t \in (0,T). \tag{3.41}$$

Proof. Since $\int_I (v')^2 \omega\,dt$ is convex, (3.1) holds, and $W \ge 0$, the existence of minimizers $v_\varepsilon \in H^1(I)$ of G_ε subject to the Dirichlet boundary conditions (3.39) follows from the direct method of the calculus of variations. Since $a \le \alpha_\varepsilon$, $\beta_\varepsilon \le b$, by replacing v_ε with the truncation

$$\bar{v}_\varepsilon(t) := \begin{cases} a & \text{if } v_\varepsilon(t) \le a, \\ v_\varepsilon(t) & \text{if } a < v_\varepsilon(t) < b, \\ b & \text{if } v_\varepsilon(t) \ge b, \end{cases}$$

without loss of generality, we may assume that v_ε satisfies $a \le v_\varepsilon \le b$.

As $dG_\varepsilon(v_\varepsilon) = 0$, we have

$$\int_I (W'(v_\varepsilon(t))\varphi(t) + 2\varepsilon^2 v_\varepsilon'(t)\varphi'(t))\omega(t)\,dt = 0$$

for all $\varphi \in C_c^1(I)$. This implies that $W'(v_\varepsilon)\omega$ is the weak derivative of $2\varepsilon^2 v_\varepsilon'\omega$. Hence,

$$2\varepsilon^2 v_\varepsilon'(t)\omega(t) = c + \int_a^s W'(v_\varepsilon)\omega\,ds.$$

Since the right-hand side is of class C^1 and $\omega \in C^{1,d}(I)$, we have that v_ε' is of class C^1 and so we can differentiate to obtain (3.40).

To prove (3.41), observe that if there exists $t_0 \in (0,T)$ such that $v_\varepsilon(t_0) = b$, then since $v_\varepsilon \le b$, the point t_0 is a point of local maximum, and so $v_\varepsilon'(t_0) = 0$. Since $W'(b) = 0$, it follows by uniqueness of the Cauchy problem (3.40) with initial data $v_\varepsilon(t_0) = b$, $v_\varepsilon'(t_0) = 0$, that the unique solution if $v_\varepsilon \equiv b$. Similarly, if $v_\varepsilon(t_0) = a$ for some $t_0 \in (0,T)$, then $v_\varepsilon \equiv a$. ■

Corollary 3.9 *Assume that W satisfies hypotheses* (2.1)-(2.4)*, that ω satisfies hypothesis* (3.1)*, and that that $a \le \alpha_\varepsilon$, $\beta_\varepsilon \le b$. Let v_ε be the minimizer of G_ε obtained in Theorem 3.8. Then there exists a constant $C_0 > 0$, depending only on ω, T, a, b, and W, such that*

$$|v_\varepsilon'(t)| \le \frac{C_0}{\varepsilon} \quad \textit{for all } t \in I$$

and for every $0 < \varepsilon < 1$.

Proof. In what follows $C_0 > 0$ is a constant that changes from line to line and depends only on ω, T, a, b, and W. Consider the function $v_0(t) := \alpha_\varepsilon \frac{(T-t)}{T} + \beta_\varepsilon \frac{t}{T}$. We have $a \le v_0 \le b$ and $|v_0'(t)| \le \frac{b-a}{T}$. Since v_ε is a minimizer of G_ε, it follows

$$\begin{aligned}\int_I (W(v_\varepsilon) + \varepsilon^2 (v_\varepsilon')^2)\omega\, dt &\le \int_I (W(v_0) + \varepsilon^2 (v_0')^2)\omega\, dt \\ &\le \left(\max_{[a,b]} W + \frac{(b-a)^2}{T^2} \right) \int_I \omega\, dt \le C_0.\end{aligned}$$

As v_ε' and ω are continuous, by the mean value theorem for integrals, there exists $t_\varepsilon \in [0,T]$ such that

$$\varepsilon^2 (v_\varepsilon'(t_\varepsilon))^2 \omega(t_\varepsilon) = \frac{1}{T} \int_I \varepsilon^2 (v_\varepsilon')^2 \omega\, dt \le C_0.$$

In turn, by (3.1),

$$\varepsilon |v_\varepsilon'(t_\varepsilon)| \le C_0.$$

By (3.40),

$$2\varepsilon^2 v_\varepsilon'(t)\omega(t) = 2\varepsilon^2 v_\varepsilon'(t_\varepsilon)\omega(t_\varepsilon) + \int_{t_\varepsilon}^t W'(v_\varepsilon)\omega\, ds \tag{3.42}$$

for every $t \in \overline{I}$. Since v_ε is bounded by (3.41), by (3.1) this implies that

$$2\varepsilon^2 |v_\varepsilon'(t)| \le C_0 \varepsilon + \frac{C_0}{\omega(t)} \int_0^T \omega\, ds \le C_0 \tag{3.43}$$

for all $t \in \overline{I}$. Rewrite (3.40) as

$$2\varepsilon^2 v_\varepsilon''(t) + 2\varepsilon^2 \frac{\omega'(t)}{\omega(t)} v_\varepsilon'(t) = W'(v_\varepsilon(t)). \tag{3.44}$$

Using (3.1) and (3.43), we obtain

$$2\varepsilon^2|v_\varepsilon''(t)| \le \frac{|\omega'(t)|}{\omega(t)} 2\varepsilon^2 |v_\varepsilon'(t)| + C_0 \le C_0. \tag{3.45}$$

Next, we use a classical interpolation result. Let $t \in I$ and consider $t_1 \in I$ such that $|t - t_1| = \varepsilon$. By the mean value theorem there exists θ between t and t_1 such that

$$v_\varepsilon(t) - v_\varepsilon(t_1) = v_\varepsilon'(\theta)(t - t_1).$$

In turn, by the fundamental theorem of calculus

$$v_\varepsilon'(t) = v_\varepsilon'(\theta) + \int_\theta^t v_\varepsilon''(s)\, ds = \frac{v_\varepsilon(t) - v_\varepsilon(t_1)}{t - t_1} + \int_\theta^t v_\varepsilon''(s)\, ds.$$

Using (3.41) and (3.45), we obtain

$$|v_\varepsilon'(t)| \le \frac{C_0}{\varepsilon} + \sup_I |v_\varepsilon''||t - \theta| \le \frac{C_0}{\varepsilon} + \frac{C_0}{\varepsilon^2}\varepsilon.$$

This concludes the proof. ∎

Another consequence of Theorem 3.8 is the following estimate.

Theorem 3.10 *Assume that W satisfies (2.1)-(2.4), that ω satisfies (3.1), and that $a \le \alpha_\varepsilon$, $\beta_\varepsilon \le b$. Let v_ε be the minimizer of G_ε obtained in Theorem 3.8, and for $k \in \mathbb{N}$ let*

$$B_\varepsilon^k := \{t \in [0, T] : \ \beta_- \le v_\varepsilon(t) \le \beta_\varepsilon - \varepsilon^k\}. \tag{3.46}$$

Then there exist $\mu > 0$ and $0 < \varepsilon_0 < 1$ depending only on β_-, T, ω, W, such that if $I_\varepsilon = [p_\varepsilon, q_\varepsilon]$ is a maximal subinterval of B_ε^k, then

$$b - v_\varepsilon(t) \le (b - v_\varepsilon(p_\varepsilon))e^{-\mu(t-p_\varepsilon)\varepsilon^{-1}} + (b - v_\varepsilon(q_\varepsilon))e^{-\mu(q_\varepsilon - t)\varepsilon^{-1}} \tag{3.47}$$

for all $t \in I_\varepsilon$ and all $0 < \varepsilon < \varepsilon_0$. In particular,

$$\operatorname{diam} I_\varepsilon \le C\varepsilon|\log \varepsilon| \tag{3.48}$$

for all $0 < \varepsilon < \varepsilon_0$, where the constants $0 < \varepsilon_0 < 1$ and $C > 0$ depend only on β_-, T, ω, W and k.

Proof. We claim that there exists $\mu > 0$ such that

$$-W'(s) \ge 2\mu^2(b - s) \quad \text{for all } s \in [\beta_-, b]. \tag{3.49}$$

Since $W''(b) > 0$, by the continuity of W'', we have that $W''(s) \ge 2\mu^2 > 0$ for all $s \in B(b, R_1)$ and for some $\mu > 0$ and $R_1 > 0$. Upon integration, it follows that

$$W'(s) = -\int_s^b W''(r)\, dr \le -2\mu^2(b - s)$$

for all $s \in B(b, R_1)$, with $s < b$. Using the fact that $W' < 0$ in (c, b), and by taking μ smaller, if necessary, we can assume that

$$W'(s) \leq -2\mu^2(b-s)$$

for all $s \in [\beta_-, b]$. Note that μ depends upon β_- but not on ε. This proves the claim.

Write $I_\varepsilon := [p_\varepsilon, q_\varepsilon]$ and define

$$\phi(t) := (b - v_\varepsilon(p_\varepsilon))e^{-\mu(t-p_\varepsilon)\varepsilon^{-1}} + (b - v_\varepsilon(q_\varepsilon))e^{-\mu(q_\varepsilon - t)\varepsilon^{-1}} \tag{3.50}$$

with μ fixed by (3.49). We note that ϕ satisfies the following differential inequality:

$$\begin{aligned}(\phi'\omega)' &= \frac{\mu^2}{\varepsilon^2}\phi\omega + \frac{\mu}{\varepsilon}\omega'\left(-(b - v_\varepsilon(p_\varepsilon))e^{-\mu(t-p_\varepsilon)\varepsilon^{-1}} + (b - v_\varepsilon(q_\varepsilon))e^{-\mu(q_\varepsilon - t)\varepsilon^{-1}}\right)\\ &\leq \frac{1}{\varepsilon^2}\left(\mu^2 + \varepsilon\frac{|\omega'|}{\omega}\mu\right)\phi\omega.\end{aligned}$$

On the other hand, $\omega(t) \geq \omega_0 > 0$ for all $t \in I$. Thus,

$$\varepsilon\frac{|\omega'(t)|}{\omega(t)} \leq \varepsilon\frac{\max|\omega'|}{\omega_0} \leq \mu$$

for all $t \in I$ and all ε sufficiently small. Therefore in I_ε

$$(\phi'\omega)' \leq 2\varepsilon^{-2}\mu^2\phi\omega. \tag{3.51}$$

We then set $g(t) := b - v_\varepsilon(t)$, and using (3.40) and (3.49) we have

$$(g'\omega)' = -\varepsilon^{-2}(W'(v_\varepsilon))\omega \geq 2\varepsilon^{-2}\mu^2 g\omega. \tag{3.52}$$

We define $\Psi := g - \phi$. By (3.50), (3.51) and (3.52), for ε small we obtain the following:

$$\begin{aligned}&(\Psi'\omega)' \geq 2\varepsilon^{-2}\mu^2\Psi\omega,\\ &\Psi(p_\varepsilon) \leq 0, \quad \Psi(q_\varepsilon) \leq 0.\end{aligned}$$

The maximum principle implies that $\Psi \leq 0$ for all $t \in I_\varepsilon$. Thus

$$b - v_\varepsilon(t) \leq (b - v_\varepsilon(p_\varepsilon))e^{-\mu(t-p_\varepsilon)\varepsilon^{-1}} + (b - v_\varepsilon(q_\varepsilon))e^{-\mu(q_\varepsilon - t)\varepsilon^{-1}},$$

which proves (3.47). In turn, for $t := \frac{p_\varepsilon + q_\varepsilon}{2}$, we have

$$\varepsilon^k \leq \beta_\varepsilon - v_\varepsilon(t) \leq b - v_\varepsilon(t) \leq 2be^{-\mu 2^{-1}(q_\varepsilon - p_\varepsilon)\varepsilon^{-1}},$$

which implies that $-\frac{\mu}{2}(q_\varepsilon - p_\varepsilon)\varepsilon^{-1} \geq k\log\varepsilon - \log 2b$, that is,

$$0 \leq q_\varepsilon - p_\varepsilon \leq 2\mu^{-1}k\varepsilon|\log\varepsilon| + 2\mu^{-1}\varepsilon\log 2b.$$

This asserts (3.48). ∎

Remark 3.11 *With a similar proof, one can show that if I_ε is a maximal subinterval of*

$$A_\varepsilon^k := \{t \in [0,T] : \ \alpha_\varepsilon + \varepsilon^k \le v_\varepsilon(t) \le \alpha_-\},$$

then

$$\operatorname{diam} I_\varepsilon \le C\varepsilon|\log\varepsilon|$$

for all $0 < \varepsilon < \varepsilon_0$, where the constants $0 < \varepsilon_0 < 1$ and $C > 0$ depend only on α_-, T, ω, W, and k.

Next, we prove some differential inequalities for v_ε.

Theorem 3.12 *Assume that W satisfies* (2.1)-(2.4)*, that ω satisfies* (3.1)*, and that that $a \le \alpha_\varepsilon$, $\beta_\varepsilon \le b$. Let v_ε be the minimizer of G_ε obtained in Theorem 3.8 and let α_-, β_- be given as in* (2.5)*. Then there exists a constant $C > 0$ such that*

$$\varepsilon(v_\varepsilon'(0))^2 - \frac{1}{\varepsilon}W(\alpha_\varepsilon) \le C$$

for all $0 < \varepsilon < 1$. Moreover, there exist a constant $\tau_0 > 0$, depending only on ω, T, a, b, α_-, β_- and W, such that

$$\frac{1}{2}\sigma^2(v_\varepsilon(t) - a)^2 \le \varepsilon^2(v_\varepsilon'(t))^2 \le \frac{3}{2}\sigma^{-2}(v_\varepsilon(t) - a)^2 \tag{3.53}$$

whenever $a + \tau_0\varepsilon^{1/2} \le v_\varepsilon(t) \le \beta_-$, and

$$\frac{1}{2}\sigma^2(b - v_\varepsilon(t))^2 \le \varepsilon^2(v_\varepsilon'(t))^2 \le \frac{3}{2}\sigma^{-2}(b - v_\varepsilon(t))^2 \tag{3.54}$$

whenever $\alpha_- \le v_\varepsilon(t) \le b - \tau_0\varepsilon^{1/2}$, where $\sigma > 0$ is the constant given in Remark 2.1.

Proof. Step 1: We claim that

$$\varepsilon(v_\varepsilon'(0))^2 - \frac{1}{\varepsilon}W(\alpha_\varepsilon) \le C$$

for all $0 < \varepsilon < 1$ and for some constant $C > 0$ independent of ε. By Theorem 3.3,

$$\sup_{0<\varepsilon<1} \int_I \left(\frac{1}{\varepsilon}W(v) + \varepsilon(v')^2\right)\omega\,dt \le C \tag{3.55}$$

for all $0 < \varepsilon < 1$ and for some constant $C > 0$ independent of ε. Subdivide I into $\lfloor\varepsilon^{-1}\rfloor$ equal subintervals I_i of equal length. Since

$$\sum_{i=1}^{\lfloor\varepsilon^{-1}\rfloor} \int_{I_i} \left(\frac{1}{\varepsilon}W(v) + \varepsilon(v')^2\right)\omega\,dt \le C$$

there exists $i_\varepsilon \in \{1, \ldots, \lfloor\varepsilon^{-1}\rfloor\}$ such that

$$\int_{I_{i_\varepsilon}} \left(\frac{1}{\varepsilon}W(v) + \varepsilon(v')^2\right)\omega\,dt \le C\varepsilon.$$

In turn, there exists $t_\varepsilon \in I_{i_\varepsilon}$ such that

$$\left(\frac{1}{\varepsilon}W(v(t_\varepsilon)) + \varepsilon(v'(t_\varepsilon))^2\right)\omega(t_\varepsilon) \le C. \tag{3.56}$$

Multiply (3.44) by $\frac{1}{\varepsilon}v_\varepsilon'(t)$ to get

$$\varepsilon((v_\varepsilon'(t))^2)' - \frac{1}{\varepsilon}(W(v_\varepsilon(t)))' = -2\varepsilon\frac{\omega'(t)}{\omega(t)}(v_\varepsilon'(t))^2. \tag{3.57}$$

Integrating between 0 and t_ε, we have

$$\varepsilon(v_\varepsilon'(0))^2 - \frac{1}{\varepsilon}W(\alpha_\varepsilon) = \varepsilon(v_\varepsilon'(t_\varepsilon))^2 - \frac{1}{\varepsilon}W(v_\varepsilon(t_\varepsilon)) - 2\varepsilon\int_0^{t_\varepsilon}\frac{\omega'(t)}{\omega(t)}(v_\varepsilon'(t))^2 dt \le C$$

where we used (3.56) and the fact that

$$2\varepsilon\int_0^{t_\varepsilon}\frac{|\omega'(t)|}{\omega(t)}(v_\varepsilon'(t))^2 dt \le \int_I \varepsilon(v_\varepsilon'(t))^2\omega(t)dt \le C \tag{3.58}$$

since $\omega \in C^1([0,T])$, $\inf_I \omega > 0$, and (3.55).

Step 2: Integrating (3.57) between t and 0 and using Step 1 and (3.58) gives

$$\left|\varepsilon(v_\varepsilon'(t))^2 - \frac{1}{\varepsilon}W(v_\varepsilon(t))\right| \le \varepsilon(v_\varepsilon'(0))^2 + \frac{1}{\varepsilon}W(\alpha_\varepsilon) + 2\varepsilon\int_0^T\frac{|\omega'|}{\omega}(v_\varepsilon')^2 dt \le C_1.$$

In turn,

$$W(v_\varepsilon(t)) - C_1\varepsilon \le \varepsilon^2(v_\varepsilon'(t))^2 \le W(v_\varepsilon(t)) + C_1\varepsilon \tag{3.59}$$

for all $t \in I$, for all $0 < \varepsilon < 1$, and for some constant $C_1 > 0$ independent of ε. By Remark 2.1,

$$\sigma^2(s-a)^2 \le W(s) \le \frac{1}{\sigma^2}(s-a)^2$$

for all $s \in [a, \beta_-]$. Hence,

$$\begin{aligned}&\frac{1}{2}\sigma^2(s-a)^2 \le \sigma^2(s-a)^2 - C_1\varepsilon \le W(s) - C_1\varepsilon,\\ &W(s) + C_1\varepsilon \le \frac{1}{\sigma^2}(s-a)^2 + C_1\varepsilon \le \frac{3}{2}\frac{1}{\sigma^2}(s-a)^2\end{aligned}$$

for all $s \in [a + \tau_0\varepsilon^{1/2}, \beta_-]$, where $\tau_0 := \sqrt{2}\sigma^{-1}C_1^{1/2}$ and we are assuming that $0 < \sigma < 1$. In turn, by (3.59), we obtain (3.53). Estimate (3.54) can be obtained similarly. We omit the details. ■

Remark 3.13 *The proof of* (3.53) *and* (3.54) *is adapted from [25].*

Next, we strengthen the hypotheses on the Dirichlet data α_ε and β_ε and derive additional properties of minimizers. In particular, we assume that $\beta_\varepsilon \to b$ (the case $\beta_\varepsilon \to a$ is similar).

Theorem 3.14 *Assume that W satisfies* (2.1)-(2.4)*, that α_-, β_- satisfy* (2.5)*, and that ω satisfies hypotheses* (3.1)*,* (3.17)*, where*

$$0 \le \omega_0 < \frac{1}{2C_W} \min\{\mathrm{d}_W(a, \alpha_-), \mathrm{d}_W(\beta_-, b)\}\omega(0). \tag{3.60}$$

Let $\alpha_- \le \alpha_\varepsilon$, $\beta_\varepsilon \le b$ satisfy (3.22) *and let v_ε be the minimizer of G_ε obtained in Theorem 3.8. Given $k \in \mathbb{N}$ there exist $0 < \varepsilon_0 < 1$, $\mu > 0$, and $C > 0$ depending only on α_-, β_-, k, A_0, B_0, T, ω, W, such that, for all $0 < \varepsilon < \varepsilon_0$, the following properties hold:*

(i) Either $v_\varepsilon > \beta_-$ in I or if R_ε is the first time in $[0, T]$ such that $v_\varepsilon = \beta_-$, then

$$R_\varepsilon \le C\varepsilon. \tag{3.61}$$

(ii) Either $v_\varepsilon \ge \beta_\varepsilon - \varepsilon^k$ in I or if T_ε is the first time such that $v_\varepsilon = \beta_\varepsilon - \varepsilon^k$, then

$$T_\varepsilon \le C\varepsilon|\log\varepsilon|. \tag{3.62}$$

Moreover, if R_ε exists, then $R_\varepsilon < T_\varepsilon$ and $v_\varepsilon(t) \in [\beta_-, \beta_\varepsilon - \varepsilon^k]$ for $t \in [R_\varepsilon, T_\varepsilon]$.

(iii) If T_ε exists, then $v_\varepsilon \ge b - \tau_0\varepsilon^{1/2}$ in $[T_\varepsilon, T]$, where τ_0 is the constant in Theorem 3.12.

By Corollary 3.4 and the fact that $\beta = b$, we have that the minimizer of $G^{(1)}$ is given by the constant function b. In turn, by Theorem 3.3 and by standard properties of Γ-convergence (see [9, Theorem 1.21]), minimizers v_ε of G_ε converge in $L^1(I)$ to b as $\varepsilon \to 0^+$, i.e.,

$$v_\varepsilon \to b \quad \text{in } L^1(I) \quad \text{as } \varepsilon \to 0^+. \tag{3.63}$$

We are now prepared to prove Theorem 3.14. For every measurable subset $E \subseteq I$ and for every $v \in H^1(I)$ satisfying (3.2), we define the localized energy

$$G_\varepsilon^{(1)}(v; E) := \int_E \left(\frac{1}{\varepsilon}W(v) + \varepsilon(v')^2\right) \omega \, dt. \tag{3.64}$$

Proof of Theorem 3.14. Throughout the proof, the constants $0 < \varepsilon_0 < 1$ and $C > 0$ depend only on α_-, β_-, k, A_0, B_0, T, ω, W. By Theorem 3.6 there exists $\tilde{v}_\varepsilon \in H^1(I)$ satisfying (3.2) such that

$$G_\varepsilon^{(2)}(\tilde{v}_\varepsilon) \le \int_0^l 2W(p_\varepsilon(t))t \, dt \, \omega'(0) + Ce^{-2\sigma l}(2\sigma l + 1) + C\varepsilon^{2\gamma} l + C\varepsilon^{\gamma_1}|\log\varepsilon|^{\gamma_0}$$

for all $0 < \varepsilon < \varepsilon_0$. Fixing l and using the fact that v_ε is a minimizer of G_ε, we have that

$$G_\varepsilon^{(1)}(v_\varepsilon) \le G_\varepsilon^{(1)}(\tilde{v}_\varepsilon) \le G^{(1)}(b) + C\varepsilon = \mathrm{d}_W(\alpha, b)\omega(0) + C\varepsilon \tag{3.65}$$

for all $0 < \varepsilon < \varepsilon_0$, where $C > 0$ is independent of ε. We extend v_ε to be α_ε for $t < 0$ and β_ε for $t > T$.

Step 1: Since

$$\mathrm{d}_W(a, \alpha_-) = \lim_{\delta \to 0^+} 2 \int_{a+\delta}^{\alpha_-} W^{1/2}(r)\, dr,$$

we can find $0 < \delta_1 < \alpha_- - a$ so small that

$$\mathrm{d}_W(a + \delta, \alpha_-) = 2 \int_{a+\delta}^{\alpha_-} W^{1/2}(r)\, dr \geq \mathrm{d}_W(a, \alpha_-) - \epsilon_0, \tag{3.66}$$

for all $0 < \delta \leq \delta_1$, where

$$\epsilon_0 < \frac{1}{4}\, \mathrm{d}_W(a, \alpha_-). \tag{3.67}$$

We claim that there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the set

$$A_\varepsilon := \{t \in [0, T] :\ a \leq v_\varepsilon(t) \leq a + \delta_1\}$$

is empty. To see this, assume by contradiction that there exists $t_\varepsilon \in (0, T)$ such that $v_\varepsilon(t_\varepsilon) = a + \delta_1$. Since v_ε is continuous, $v_\varepsilon([0, T]) \supseteq [a + \delta_1, \alpha_\varepsilon \vee \beta_\varepsilon]$. Hence, we can find a closed interval I_ε such that $v_\varepsilon(I_\varepsilon) = [a + \delta_1, \alpha_\varepsilon \wedge \beta_\varepsilon]$ and a closed interval J_ε such that $v_\varepsilon(J_\varepsilon) = [\alpha_\varepsilon \wedge \beta_\varepsilon, \alpha_\varepsilon \vee \beta_\varepsilon]$. Then by (3.65), and (3.17),

$$\begin{aligned}
\mathrm{d}_W(\alpha, b)\omega(0) + C\varepsilon &\geq G_\varepsilon^{(1)}(v_\varepsilon) \geq G_\varepsilon^{(1)}(v; I_\varepsilon \cup J_\varepsilon) \\
&\geq \int_{I_\varepsilon \cup J_\varepsilon} 2W^{1/2}(v_\varepsilon)|v_\varepsilon'|\omega\, dt \geq \int_{I_\varepsilon \cup J_\varepsilon} 2W^{1/2}(v_\varepsilon)|v_\varepsilon'|\, dt(\omega(0) - \omega_0) \\
&= \left(\mathrm{d}_W(a + \delta_1, \alpha_\varepsilon \wedge \beta_\varepsilon) + \mathrm{d}_W(\alpha_\varepsilon \wedge \beta_\varepsilon, \alpha_\varepsilon \vee \beta_\varepsilon)\right)(\omega(0) - \omega_0).
\end{aligned}$$

In view of (1.10), $\mathrm{d}_W(\cdot, r)$ and $\mathrm{d}_W(s, \cdot)$ are Lipschitz continuous with Lipschitz constant $L = \max_{[a,b]} \sqrt{W}$. Hence, by (3.22),

$$\mathrm{d}_W(\alpha_\varepsilon \wedge \beta_\varepsilon, \alpha_\varepsilon \vee \beta_\varepsilon) \geq \mathrm{d}_W(\alpha, \beta) - L(A_0 \varepsilon^\gamma + B_0 \varepsilon^\gamma),$$

and so, using the fact that $\alpha_- < \alpha_\varepsilon$, β_ε, we have

$$\mathrm{d}_W(\alpha, b)\omega(0) + C\varepsilon \geq \left(\mathrm{d}_W(a + \delta_1, \alpha_-) + \mathrm{d}_W(\alpha, b)\right)(\omega(0) - \omega_0) - C\varepsilon^\gamma$$

or, equivalently,

$$\begin{aligned}
\mathrm{d}_W(a + \delta_1, \alpha_-)\omega(0) &\leq \left(\mathrm{d}_W(a + \delta_1, \alpha_-) + \mathrm{d}_W(\alpha, b)\right)\omega_0 + C\varepsilon \\
&\leq C_W \omega_0 + C\varepsilon < \frac{1}{2}\, \mathrm{d}_W(a, \alpha_-)\omega(0),
\end{aligned}$$

for $0 < \varepsilon < \varepsilon_0$, where in the last inequality we used (3.20), and where ε_0 is so small that

$$C\varepsilon_0 \leq \frac{1}{4}\, \mathrm{d}_W(a, \alpha_-)\omega(0).$$

Hence, also by (3.66),

$$\mathrm{d}_W(a,\alpha_-) - \epsilon_0 \le \mathrm{d}_W(a+\delta_1,\alpha_-) < \frac{1}{2}\,\mathrm{d}_W(a,\alpha_-),$$

which contradicts (3.67).

Step 2: To prove (3.61), observe that by Step 1, for all $t \in [0, R_\varepsilon)$ we have that $v_\varepsilon(t) \in [a+\delta_1, \beta_-)$. Hence,

$$G_\varepsilon^{(1)}(v_\varepsilon;[0,R_\varepsilon]) \ge \varepsilon^{-1} R_\varepsilon \min_{[0,T]} \omega \min_{[a+\delta_1,\beta_-]} W,$$

and so (3.61) follows from (3.65).

Step 3: To prove Item (ii), we consider three separate cases.

Substep 3 a. Assume first that either $\alpha_\varepsilon < \beta_-$ or $\alpha_\varepsilon = \beta_-$ and $v_\varepsilon'(0) > 0$. If $\alpha_\varepsilon < \beta_-$, since $v_\varepsilon(t) \in [a+\delta_1, \beta_-)$ for all $t \in [0, R_\varepsilon)$, we have that $v_\varepsilon'(R_\varepsilon) \ge 0$. On the other hand, by (3.54), $v_\varepsilon'(R_\varepsilon) > 0$, which in turn implies, again by (3.54), that $v_\varepsilon'(t) > 0$ for all $t \ge R_\varepsilon$ such that $v_\varepsilon(t) \le b - \tau_0\varepsilon^{1/2}$. Similarly, if $\alpha_\varepsilon = \beta_-$ and $v_\varepsilon'(0) > 0$, then by (3.54) that $v_\varepsilon'(t) > 0$ for all $t \ge R_\varepsilon = 0$ such that $v_\varepsilon(t) \le b - \tau_0\varepsilon^{1/2}$.

Hence, in both cases, there exists a maximal interval I_ε of the set B_ε defined in (3.46) whose left endpoint is R_ε. Let S_ε be the right endpoint of I_ε. If $v_\varepsilon(S_\varepsilon) = \beta_\varepsilon - \varepsilon^k$, then $S_\varepsilon = T_\varepsilon$ and so (3.62) follows from (3.48) and (3.61).

If $v_\varepsilon(S_\varepsilon) = \beta_-$, then since $v_\varepsilon'(t) > 0$ for all $t \ge R_\varepsilon$ such that $v_\varepsilon(t) \le b - \tau_0\varepsilon^{1/2}$, there exists $P_\varepsilon \in (R_\varepsilon, S_\varepsilon)$ such that $v_\varepsilon(P_\varepsilon) = b - \tau_0\varepsilon^{1/2}$. It follows that $v_\varepsilon([R_\varepsilon, P_\varepsilon]) = [\beta_-, b-\tau_0\varepsilon^{1/2}]$, while $v_\varepsilon([P_\varepsilon, S_\varepsilon]) \supseteq [\beta_-, b-\tau_0\varepsilon^{1/2}]$. Then by (3.65), and (3.17), we have

$$\begin{aligned}
\mathrm{d}_W(\alpha,b)\omega(0) + C\varepsilon &\ge G_\varepsilon^{(1)}(v_\varepsilon) \ge G_\varepsilon^{(1)}(v;[0,P_\varepsilon]\cup[P_\varepsilon,S_\varepsilon]) \\
&\ge \int_{[0,P_\varepsilon]\cup[P_\varepsilon,S_\varepsilon]} 2W^{1/2}(v_\varepsilon)|v_\varepsilon'|\omega\,dt \\
&\ge \int_{[0,P_\varepsilon]\cup[P_\varepsilon,S_\varepsilon]} 2W^{1/2}(v_\varepsilon)|v_\varepsilon'|\,dt(\omega(0)-\omega_0) \\
&= \left(\mathrm{d}_W(\alpha_\varepsilon, b-\tau_0\varepsilon^{1/2}) + \mathrm{d}_W(\beta_-, b-\tau_0\varepsilon^{1/2})\right)(\omega(0)-\omega_0).
\end{aligned}$$

As in Step 1, using the fact that $\mathrm{d}_W(\cdot,r)$ and $\mathrm{d}_W(s,\cdot)$ are Lipschitz continuous and (3.22), it follow that

$$\mathrm{d}_W(\alpha,b)\omega(0) + C\varepsilon \ge (\mathrm{d}_W(\alpha,b) + \mathrm{d}_W(\beta_-,b) - L(A_0\varepsilon^\gamma + 2\tau_0\varepsilon^{1/2}))(\omega(0)-\omega_0),$$

or, equivalently,

$$[\mathrm{d}_W(\alpha,b) + \mathrm{d}_W(\beta_-,b)]\omega_0 + C(\varepsilon^\gamma + \varepsilon^{1/2}) \ge \mathrm{d}_W(\beta_-,b)\omega(0),$$

which contradicts (3.60), provided we take $0 < \varepsilon < \varepsilon_0$ with ε_0 sufficiently small (depending only on β_- and W).

On the other hand, if $v_\varepsilon(t) > \beta_-$ for all $t \in I$, then $I_\varepsilon = [0, T_\varepsilon]$ is a maximal interval of the set B_ε defined in (3.46), and so (3.62) follows from (3.48).

Substep 3 b. Assume first that $\alpha_\varepsilon = \beta_-$ and $v_\varepsilon'(0) \le 0$. Then

$$(\omega v_\varepsilon')'(0) = \omega(0)W'(\beta_-) < 0$$

and so in both cases $v_\varepsilon'(t) < 0$ for all $t > 0$ small. It follows from (3.53), that $v_\varepsilon'(t) < 0$ for all $t > 0$ such that $v_\varepsilon(t) \ge a + \tau_0\varepsilon^{1/2}$. Since $v_\varepsilon(T) = \beta_\varepsilon$, this implies that there exist $L_\varepsilon < M_\varepsilon < N_\varepsilon$ such that $v_\varepsilon([0, L_\varepsilon]) \supseteq [a + \tau_0\varepsilon^{1/2}, \beta_-]$ and $v_\varepsilon([M_\varepsilon, N_\varepsilon]) \supseteq [\beta_-, \beta_\varepsilon]$. Then by (3.65), and (3.17), we obtain

$$\begin{aligned}
\mathrm{d}_W(\alpha, b)\omega(0) + C\varepsilon = \mathrm{d}_W(\beta_-, b)\omega(0) + C\varepsilon &\ge G_\varepsilon^{(1)}(v_\varepsilon) \ge G_\varepsilon^{(1)}(v; [0, L_\varepsilon] \cup [M_\varepsilon, N_\varepsilon]) \\
&\ge \int_{[0,L_\varepsilon]\cup[M_\varepsilon,N_\varepsilon]} 2W^{1/2}(v_\varepsilon)|v_\varepsilon'|\omega\, dt \\
&\ge \int_{[0,L_\varepsilon]\cup[M_\varepsilon,N_\varepsilon]} 2W^{1/2}(v_\varepsilon)|v_\varepsilon'|\, dt(\omega(0) - \omega_0) \\
&= \Big(\mathrm{d}_W(a + \tau_0\varepsilon^{1/2}, \beta_-) + \mathrm{d}_W(\beta_-, \beta_\varepsilon)\Big)(\omega(0) - \omega_0).
\end{aligned}$$

Using the fact that $\mathrm{d}_W(\cdot, r)$ and $\mathrm{d}_W(s, \cdot)$ are Lipschitz continuous and (3.22), it follow that

$$\mathrm{d}_W(\beta_-, b)\omega(0) + C\varepsilon \ge (\mathrm{d}_W(a, \beta_-) + \mathrm{d}_W(\beta_-, b) - L(B_0\varepsilon^\gamma + 2\tau_0\varepsilon^{1/2}))(\omega(0) - \omega_0),$$

or, equivalently,

$$[\mathrm{d}_W(a, \beta_-) + \mathrm{d}_W(\beta_-, b)]\omega_0 + C(\varepsilon^\gamma + \varepsilon^{1/2}) \ge \mathrm{d}_W(a, \beta_-)\omega(0) \ge \mathrm{d}_W(a, \alpha_-)\omega(0),$$

which contradicts (3.60), provided we take $0 < \varepsilon < \varepsilon_0$ with ε_0 sufficiently small (depending only on α_- and W).

This contradiction shows that if $\alpha = \beta_-$, then $v_\varepsilon'(0) > 0$ and so we are back to Substep 3 a.

Substep 3 c. Finally, we consider the case in which $\alpha_\varepsilon > \beta_-$. We claim that $v_\varepsilon > \beta_-$ in I. Indeed, assume by contradiction that R_ε exists. Then $v_\varepsilon'(R_\varepsilon) \le 0$ and so by (3.53), $v_\varepsilon'(t) < 0$ for all $t > 0$ such that $v_\varepsilon(t) \ge a + \tau_0\varepsilon^{1/2}$. Since $v_\varepsilon(T) = \beta_\varepsilon$, this implies that there exist $R_\varepsilon < L_\varepsilon < M_\varepsilon < N_\varepsilon$ such that $v_\varepsilon([R_\varepsilon, L_\varepsilon]) \supseteq [a + \tau_0\varepsilon^{1/2}, \beta_-]$ and $v_\varepsilon([M_\varepsilon, N_\varepsilon]) \supseteq [\beta_-, \beta_\varepsilon]$. Then by (3.65), and (3.17), we have

$$\begin{aligned}
\mathrm{d}_W(\alpha, b)\omega(0) + C\varepsilon &\ge G_\varepsilon^{(1)}(v_\varepsilon) \ge G_\varepsilon^{(1)}(v; [R, L_\varepsilon] \cup [M_\varepsilon, N_\varepsilon]) \\
&\ge \int_{[0,L_\varepsilon]\cup[M_\varepsilon,N_\varepsilon]} 2W^{1/2}(v_\varepsilon)|v_\varepsilon'|\omega\, dt \\
&\ge \int_{[0,L_\varepsilon]\cup[M_\varepsilon,N_\varepsilon]} 2W^{1/2}(v_\varepsilon)|v_\varepsilon'|\, dt(\omega(0) - \omega_0) \\
&= \Big(\mathrm{d}_W(a + \tau_0\varepsilon^{1/2}, \beta_-) + \mathrm{d}_W(\beta_-, \beta_\varepsilon)\Big)(\omega(0) - \omega_0).
\end{aligned}$$

Using the fact that $\mathrm{d}_W(\cdot, r)$ and $\mathrm{d}_W(s, \cdot)$ are Lipschitz continuous and (3.22), it follow that

$$\mathrm{d}_W(\alpha, b)\omega(0) + C\varepsilon \ge (\mathrm{d}_W(a, \beta_-) + \mathrm{d}_W(\beta_-, b) - L(B_0\varepsilon^\gamma + 2\tau_0\varepsilon^{1/2}))(\omega(0) - \omega_0),$$

or, equivalently,

$$[\mathrm{d}_W(a,\beta_-)+\mathrm{d}_W(\beta_-,b)]\omega_0+C(\varepsilon^\gamma+\varepsilon^{1/2})\geq \mathrm{d}_W(a,\alpha)\omega(0)\geq \mathrm{d}_W(a,\alpha_-)\omega(0),$$

which contradicts (3.60), provided we take $0<\varepsilon<\varepsilon_0$ with ε_0 sufficiently small (depending only on α_- and W).

Step 4. To prove Item (iii), assume that T_ε exists. Assume by contradiction that there exists a first time $Q_\varepsilon > T_\varepsilon$ such that $v_\varepsilon = b-\tau_0\varepsilon^{1/2}$. Then $v'_\varepsilon(Q_\varepsilon)\leq 0$, but so by (3.53) and (3.54), $v'_\varepsilon(t)<0$ for all $t\geq Q_\varepsilon$ such that $v_\varepsilon(t)\geq a+\tau_0\varepsilon^{1/2}$. Since $v_\varepsilon(T)=\beta_\varepsilon$, there exists a first time S_ε such that $v_\varepsilon = a+\tau_0\varepsilon^{1/2}$. Hence, $v_\varepsilon([0,T_\varepsilon])\supseteq[\alpha_\varepsilon,\beta_\varepsilon-\varepsilon^k]$ and $v_\varepsilon([Q_\varepsilon,S_\varepsilon])\supseteq[a+\tau_0\varepsilon^{1/2}, b-\tau_0\varepsilon^{1/2}]$. Then by (3.65), and (3.17), we have

$$\begin{aligned}\mathrm{d}_W(\alpha,b)\omega(0)+C\varepsilon &\geq G^{(1)}_\varepsilon(v_\varepsilon)\geq G^{(1)}_\varepsilon(v;[0,T_\varepsilon]\cup[Q_\varepsilon,S_\varepsilon])\\ &\geq \int_{[0,T_\varepsilon]\cup[Q_\varepsilon,S_\varepsilon]} 2W^{1/2}(v_\varepsilon)|v'_\varepsilon|\omega\,dt\\ &\geq \int_{[0,T_\varepsilon]\cup[Q_\varepsilon,S_\varepsilon]} 2W^{1/2}(v_\varepsilon)|v'_\varepsilon|\,dt(\omega(0)-\omega_0)\\ &= \left(\mathrm{d}_W(\alpha_\varepsilon,\beta_\varepsilon-\varepsilon^k)+\mathrm{d}_W(a+\tau_0\varepsilon^{1/2}, b-\tau_0\varepsilon^{1/2})\right)(\omega(0)-\omega_0).\end{aligned}$$

Using the fact that $\mathrm{d}_W(\cdot,r)$ and $\mathrm{d}_W(s,\cdot)$ are Lipschitz continuous and (3.22), it follow that

$$\mathrm{d}_W(\alpha,b)\omega(0)+C\varepsilon\geq(\mathrm{d}_W(\alpha,b)+\mathrm{d}_W(a,b)-C(\varepsilon^k+\varepsilon^\gamma+\varepsilon^{1/2}))(\omega(0)-\omega_0),$$

or, equivalently,

$$[\mathrm{d}_W(a,\beta_-)+\mathrm{d}_W(\beta_-,b)]\omega_0+C(\varepsilon^k+\varepsilon^\gamma+\varepsilon^{1/2})\geq \mathrm{d}_W(a,b)\omega(0)\geq \mathrm{d}_W(a,\alpha_-)\omega(0),$$

which contradicts (3.60), provided we take $0<\varepsilon<\varepsilon_0$ with ε_0 sufficiently small (depending only on α_-, β_-, and W). ∎

3.4 Second-Order Γ-liminf

In this subsection, we prove the lim inf counterpart of Theorem 3.6.

Theorem 3.15 (Second-Order Γ-Liminf) *Assume that W satisfies* (2.1)-(2.4), *that that α_- satisfies* (2.5), *and that ω satisfies* (3.1), (3.17), *and* (3.20). *Let $\alpha_-\leq\alpha_\varepsilon$, $\beta_\varepsilon\leq b$ satisfy* (3.22) *and let v_ε be the minimizer of G_ε obtained in Theorem 3.8. Then there exist $0<\varepsilon_0<1$, $C>0$, and $l_0>1$, depending only on α_-, A_0, B_0, T, ω, and W, such that*

$$G^{(2)}_\varepsilon(v_\varepsilon)\geq 2\omega'(0)\int_0^l W^{1/2}(w_\varepsilon)w'_\varepsilon s\,ds - Ce^{-l\mu}(l\mu+1)-C\varepsilon^{1/2}l^2-C\varepsilon^{\gamma_1}|\log\varepsilon|^{2+\gamma_0}$$

for all $0<\varepsilon<\varepsilon_0$ and $l>l_0$, where $w_\varepsilon(s):=v_\varepsilon(\varepsilon s)$ for $s\in[0,T\varepsilon^{-1}]$ satisfies

$$\lim_{\varepsilon\to0^+}\int_0^l W^{1/2}(w_\varepsilon)w'_\varepsilon s\,ds=\int_0^l W^{1/2}(z_\alpha)z'_\alpha s\,ds$$

for every $l > 0$ and where z_α solves the Cauchy problem (1.8)*. In particular,*

$$\liminf_{\varepsilon\to 0^+} G_\varepsilon^{(2)}(v_\varepsilon) \ge 2\omega'(0)\int_0^\infty W^{1/2}(z_\alpha)z'_\alpha s\,ds.$$

Note that Theorems 3.6 and 3.15 together provide the second-order asymptotic development by Γ-convergence for the functionals G_ε defined in (3.38). To prove Theorem 3.15, it is convenient to rescale the functionals G_ε. We define

$$H_\varepsilon(w) := \int_0^{T\varepsilon^{-1}} (W(w(s)) + (w'(s))^2)\omega_\varepsilon(s)\,ds \tag{3.68}$$

for all $w \in H^1((0, T\varepsilon^{-1}))$ such that

$$w(0) = \alpha_\varepsilon, \quad w(T\varepsilon^{-1}) = \beta_\varepsilon, \tag{3.69}$$

where

$$\omega_\varepsilon(s) := \omega(\varepsilon s). \tag{3.70}$$

Note that if v_ε is the minimizers of G_ε obtained in Theorem 3.8, then

$$w_\varepsilon(s) := v_\varepsilon(\varepsilon s), \quad s \in [0, T\varepsilon^{-1}] \tag{3.71}$$

is a minimizer of H_ε.

We prove that the functions w_ε necessarily converge.

Lemma 3.16 *Assume that the hypotheses of Theorem 3.15 hold. Let w_ε be as in* (3.71)*. Then $w_\varepsilon \rightharpoonup z_\alpha$ in $H^1((0,l))$ for every $l \in \mathbb{N}$, where z_α solves the Cauchy problem* (1.8)*. Moreover, the family*

$$|w'_\varepsilon(t)| \le C \quad \text{for all } t \in (0, T\varepsilon^{-1})$$

and all $0 < \varepsilon < 1$, where the constant $C > 0$ depends only on ω, T, a, b, and W.

Proof. Extend w_ε to be β_ε for $t \ge T\varepsilon^{-1}$. The fact that the family $\{w'_\varepsilon\}_\varepsilon$ is uniformly bounded in $L^\infty(\mathbb{R}_+)$ follows from Corollary 3.9. Furthermore, we have that the functions w_ε are bounded in $L^\infty(\mathbb{R}_+)$ by (3.41). Let $\varepsilon_n \to 0^+$. After a diagonalization argument, we can find a subsequence $\{\varepsilon_{n_k}\}_k$ of $\{\varepsilon_n\}_n$ and $w_0 \in H^1_{\text{loc}}(\mathbb{R}_+)$ such that

$$w_{\varepsilon_{n_k}} \rightharpoonup w_0 \text{ in } H^1_{\text{loc}}(\mathbb{R}_+). \tag{3.72}$$

For simplicity, in what follows, we write ε in place of ε_{n_k}.

Since $w_\varepsilon(0) = \alpha_\varepsilon \to \alpha$, we have that that $w_0(0) = \alpha$. By Theorem 3.8 and (3.71), we obtain

$$\begin{cases} 2(w'_\varepsilon\omega_\varepsilon)' - W'(w_\varepsilon)\omega_\varepsilon = 0 & \text{on } (0, T\varepsilon^{-1}), \\ w_\varepsilon(0) = \alpha_\varepsilon, \quad w_\varepsilon(T\varepsilon^{-1}) = \beta_\varepsilon. \end{cases} \tag{3.73}$$

Hence for every $\phi \in C_c^\infty(\mathbb{R}_+)$ and for ε small enough we find that

$$\int_0^{T\varepsilon^{-1}} 2w_\varepsilon' \omega_\varepsilon \phi' + W'(w_\varepsilon)\omega_\varepsilon \phi \, ds = 0.$$

Letting $\varepsilon \to 0$ and using (3.70) and (3.72) gives

$$\int_{\mathbb{R}} 2w_0' \omega(0)\phi' + W'(w_0)\omega(0)\phi \, ds = 0,$$

which then shows that w_0 solves the initial value problem

$$\begin{cases} 2w_0'' = W'(w_0) & \text{in } \mathbb{R}_+, \\ w_0(0) = \alpha. \end{cases} \tag{3.74}$$

Furthermore, by (3.41) we know that $a \leq w_0 \leq b$, which by (3.74) implies that $|w_0''| \leq C$. Also, by (3.65), the fact that $H_\varepsilon(w_\varepsilon) = G_\varepsilon(v_\varepsilon)$,

$$\begin{aligned} \omega(0) \int_0^l ((w_0')^2 + W(w_0)) \, ds &\leq \lim_{\varepsilon \to 0} \int_0^l ((w_\varepsilon')^2 + W(w_\varepsilon))\omega_\varepsilon \, ds \\ &\leq \lim_{\varepsilon \to 0^+} H_\varepsilon(w_\varepsilon) = \mathrm{d}_W(\alpha, b)\omega(0) \end{aligned}$$

for every $l \in \mathbb{N}$, and thus

$$\int_0^\infty ((w_0')^2 + W(w_0)) \, ds \leq \mathrm{d}_W(\alpha, b). \tag{3.75}$$

If $\alpha = b$, then this inequality implies that $w_0 \equiv b$. Otherwise, if $\alpha < b$, (3.75) combined with the fact that $|w_0''(t)| \leq C$ for all $t \in \mathbb{R}_+$ (by (3.74)) implies that $\lim_{s \to +\infty} w_0'(s) = 0$. In turn, $|w_0'(t)| \leq C$ for all $t \in \mathbb{R}_+$, and since

$$\liminf_{t \to \infty} W(w_0) = 0$$

in view (3.75), we have that

$$\text{either} \quad \lim_{t \to \infty} w_0(t) = a \quad \text{or} \quad \lim_{t \to \infty} w_0(t) = b.$$

By integrating (3.74) we find that

$$(w_0')^2 = W(w_0). \tag{3.76}$$

We now distinguish two cases. If $\beta_- < \alpha < b$, we define $R_\varepsilon = 0$. On the other hand, if $\alpha \leq \beta_-$, then by Theorem 3.14, we have that $R_\varepsilon \leq C\varepsilon$, where R_ε is the first time in $[0, T_\varepsilon]$ such that $v_\varepsilon = \max\{\alpha_\varepsilon, \beta_-\}$. Hence, in both cases $w_\varepsilon(\varepsilon^{-1}R_\varepsilon) = v_\varepsilon(R_\varepsilon) \geq \beta_-$. Since $\varepsilon^{-1}R_\varepsilon \leq C$, by extracting a subsequence, we can assume that $\varepsilon^{-1}R_\varepsilon \to s_0$. In turn, $w_0(s_0) = \max\{a, \beta_-\}$. It follows from (3.76), that $w_0'(s_0) = W^{1/2}(\max\{a, \beta_-\}) > 0$. Hence, w_0 is increasing after s_0 and so it is the unique solution to the Cauchy problem

$$\begin{cases} w_0' = W^{1/2}(w_0), \\ w_0(s_0) = \max\{a, \beta_-\}. \end{cases}$$

By uniqueness, it follows that $a < w_0(s) < b$ for all s, which means that w_0 is strictly increasing. In turn,

$$\begin{cases} w_0' = W^{1/2}(w_0), \\ w_0(0) = \alpha, \end{cases}$$

and

$$\lim_{t\to\infty} w_0(t) = b.$$

This shows that $w_0 = z_\alpha$. Using the fact that $\{\varepsilon_n\}_n$ was an arbitrary sequence, the statement of the lemma follows. ■

Next, we will use the previous lemmas to derive a second-order liminf inequality, which immediately implies Theorem 3.15.

Proof of Theorem 3.15. By Theorem 3.14, we have $v_\varepsilon(T_\varepsilon) \geq \beta_\varepsilon - \varepsilon^k$ for all $0 < \varepsilon < \varepsilon_0$, where $T_\varepsilon = 0$ if $v_\varepsilon > \beta_\varepsilon - \varepsilon^k$ in I and otherwise T_ε is the first time that $v_\varepsilon = \beta_\varepsilon - \varepsilon^k$, and we have $T_\varepsilon \leq C\varepsilon|\log\varepsilon|$. Moreover, $v_\varepsilon(t) \in [\beta_-, \beta_\varepsilon - \varepsilon^k]$ for $t \in [R_\varepsilon, T_\varepsilon]$, where $R_\varepsilon < T_\varepsilon$ is either the first time such that $v_\varepsilon = \beta_-$ or $R_\varepsilon = 0$ and $v_\varepsilon > \beta_-$ in I.

Here, ε_0 and C depend only on α_-, β_-, A_0, B_0, T, ω, W. In what follows, we will take ε_0 smaller and C larger, if necessary, preserving the same dependence on the parameters of the problem.

Setting $l_\varepsilon := \varepsilon^{-1}T_\varepsilon$, we have that

$$w_\varepsilon(l_\varepsilon) \geq \beta_\varepsilon - \varepsilon^k, \tag{3.77}$$

for $0 < \varepsilon < \varepsilon_0$, where

$$l_\varepsilon \leq C|\log\varepsilon|. \tag{3.78}$$

By (3.68) we have

$$\begin{aligned}
&\frac{H_\varepsilon(w_\varepsilon) - \mathrm{d}_W(\alpha,b)\omega(0)}{\varepsilon} \\
&= \varepsilon^{-1}\int_0^{l_\varepsilon} (W^{1/2}(w_\varepsilon) - w_\varepsilon')^2\omega_\varepsilon\,ds + 2\varepsilon^{-1}\int_0^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'(\omega_\varepsilon - \omega(0))\,ds \\
&\quad + \varepsilon^{-1}\int_{l_\varepsilon}^{T\varepsilon^{-1}} \left(W(w_\varepsilon) + (w_\varepsilon')^2\right)\omega_\varepsilon\,ds + \varepsilon^{-1}\omega(0)\left(2\int_0^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'\,ds - \mathrm{d}_W(\alpha,b)\right) \\
&\geq 2\varepsilon^{-1}\int_0^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'(\omega_\varepsilon - \omega(0))\,ds + \varepsilon^{-1}\omega(0)\left(2\int_0^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'\,ds - \mathrm{d}_W(\alpha,b)\right) \\
&=: \mathcal{A} + \mathcal{B}.
\end{aligned}$$

To estimate $\mathcal{B}$, observe that by the change of variables $r = w_\varepsilon(s)$, we obtain

$$\begin{aligned}
2\int_0^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'\,ds &= 2\int_{\alpha_\varepsilon}^{w_\varepsilon(l_\varepsilon)} W^{1/2}(r)\,dr \\
&= 2\int_\alpha^b W^{1/2}(r)\,dr - 2\int_{\alpha_\varepsilon}^{\alpha} W^{1/2}(r)\,dr - 2\int_{w_\varepsilon(l_\varepsilon)}^b W^{1/2}(r)\,dr.
\end{aligned}$$

By (3.22),

$$2\left|\int_{\alpha_\varepsilon}^{\alpha} W^{1/2}(r)\,dr\right| \le 2\max_{[a,b]} W^{1/2}|\alpha_\varepsilon - \alpha| \le C\varepsilon^\gamma.$$

On the other hand, by (2.6), (3.22), and (3.77),

$$2\int_{w_\varepsilon(l_\varepsilon)}^{b} W^{1/2}(r)\,dr \le C\int_{w_\varepsilon(l_\varepsilon)}^{b} (b-r)\ dr = C(b - w_\varepsilon(l_\varepsilon))^2 \le C(\varepsilon^{2k} + \varepsilon^{2\gamma}).$$

Hence,

$$\mathcal{B} \ge -C(\varepsilon^\gamma + \varepsilon^{2k}).$$

To estimate $\mathcal{A}$, we use Taylor's formula and assumption (3.1) to get

$$|\omega_\varepsilon(s) - \omega(0) - \varepsilon s\omega'(0)| \le |\omega'|_{C^{0,d}}\varepsilon^{1+d}|s|^{1+d}.$$

Using (3.41), Lemma 3.16 and (3.78), we have

$$\left|\varepsilon^{-1}\int_0^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'\varepsilon^{1+d}|s|^{1+d}\,ds\right| \le C\varepsilon^d|\log\varepsilon|^{2+d}.$$

Thus, we find that

$$\begin{aligned}\mathcal{A} &\ge 2\omega'(0)\int_0^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon' s\,ds - C\varepsilon^d|\log\varepsilon|^{2+d} \qquad (3.79)\\ &= 2\omega'(0)\int_0^{l} W^{1/2}(w_\varepsilon)w_\varepsilon' s\,ds + 2\omega'(0)\int_l^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon' s\,ds - C\varepsilon^d|\log\varepsilon|^{2+d}\\ &=: \mathcal{A}_1 + \mathcal{A}_2 - C\varepsilon^d|\log\varepsilon|^{2+d},\end{aligned}$$

where l is fixed.

To estimate $\mathcal{A}_2$, we distinguish two cases. If $l_\varepsilon \le l$, then we use the fact that $v_\varepsilon(t) \ge b - \tau_0\varepsilon^{1/2}$ for all $t \in [T_\varepsilon, T]$ to obtain

$$0 \le b - v_\varepsilon(t) \le \tau\varepsilon^{1/2}.$$

In turn, by (2.6),

$$W^{1/2}(w_\varepsilon(s)) \le \sigma^{-1}(b - w_\varepsilon(s)) \le C\varepsilon^{1/2}$$

for all $s \in [l_\varepsilon, l]$. Hence, also by Lemma 3.16,

$$\mathcal{A}_2 \ge -2|\omega'(0)|\int_{l_\varepsilon}^{l} W^{1/2}(w_\varepsilon)|w_\varepsilon'|s\,ds \ge -C\varepsilon^{1/2}l^2.$$

On the other hand, if $l_\varepsilon > l$, since $v_\varepsilon(t) \in [\beta_-, \beta_\varepsilon - \varepsilon^k]$ for $t \in [R_\varepsilon, T_\varepsilon]$, where $R_\varepsilon < T_\varepsilon$ is either the first time such that $v_\varepsilon = \beta_-$ or $R_\varepsilon = 0$ and $v_\varepsilon > \beta_-$ in I, we have that $[R_\varepsilon, T_\varepsilon]$ is a maximal interval of the set B_ε^k defined in (3.46), and so by Theorem 3.10, and (3.22),

$$\begin{aligned}
b - v_\varepsilon(t) &\le (b - v_\varepsilon(R_\varepsilon))e^{-\mu(t-R_\varepsilon)\varepsilon^{-1}} + (b - v_\varepsilon(T_\varepsilon))e^{-\mu(T_\varepsilon - t)\varepsilon^{-1}} \\
&\le (b - v_\varepsilon(R_\varepsilon))e^{-\mu(t-R_\varepsilon)\varepsilon^{-1}} + (b - \beta_\varepsilon + \varepsilon^k) \\
&\le b e^{-\mu(t-R_\varepsilon)\varepsilon^{-1}} + B_0\varepsilon^\gamma + \varepsilon^k
\end{aligned}$$

for $t \in [R_\varepsilon, T_\varepsilon]$. By Theorem 3.14, we have that $R_\varepsilon \le C\varepsilon$. It follows that $r_\varepsilon := \varepsilon^{-1}R_\varepsilon \le C$ and

$$0 \le b - w_\varepsilon(s) \le b e^{-\mu(s-r_\varepsilon)} + B_0\varepsilon^\gamma + \varepsilon^k$$

for all $s \in [r_\varepsilon, l_\varepsilon]$ and all $0 < \varepsilon < \varepsilon_0$. Using (2.6) and Lemma 3.16, we have

$$\begin{aligned}
\mathcal{A}_2 &\ge -C\int_l^{l_\varepsilon} (b - w_\varepsilon)s\,ds \ge -C\rho\int_l^\infty e^{-\mu(s-r_\varepsilon)}s\,ds + C(\varepsilon^\gamma + \varepsilon^k)l_\varepsilon^2 \\
&\ge -Ce^{\mu r_\varepsilon}e^{-l\mu}\,(l\mu + 1) - C(\varepsilon^\gamma + \varepsilon^k)l_\varepsilon^2 \\
&\ge -Ce^{-l\mu}\,(l\mu + 1) - C(\varepsilon^\gamma + \varepsilon^k)\log^2\varepsilon
\end{aligned}$$

where we used (3.78) and the fact that $r_\varepsilon \le C$ and we take $l > C \ge r_\varepsilon$. Using this estimate in (3.79) gives

$$\mathcal{A} \ge 2\omega'(0)\int_0^l W^{1/2}(w_\varepsilon)w_\varepsilon' s\,ds - Ce^{-l\mu}\,(l\mu + 1) - Cl^2\varepsilon^{1/2} - C\varepsilon^{\gamma_1}|\log\varepsilon|^{2+d},$$

where $\gamma_1 = \min\{d, \gamma, k\}$. Combining the estimates for $\mathcal{A}$ and $\mathcal{B}$ gives

$$\begin{aligned}
\frac{H_\varepsilon(w_\varepsilon) - \mathrm{d}_W(\alpha, b)\omega(0)}{\varepsilon} \ge 2\omega'(0)\int_0^l W^{1/2}(w_\varepsilon)w_\varepsilon' s\,ds - Ce^{-l\mu}\,(l\mu + 1) \\
- Cl^2\varepsilon^{1/2} - C\varepsilon^{\gamma_1}|\log\varepsilon|^{2+d}
\end{aligned} \tag{3.80}$$

for all $0 < \varepsilon < \varepsilon_0$ and all $l > C$.

By (3.72), we can write

$$\lim_{\varepsilon\to 0^+}\int_0^l W^{1/2}(w_\varepsilon)w_\varepsilon' s\,ds = \int_0^l W^{1/2}(z_\alpha)z_\alpha' s\,ds.$$

Taking first $\varepsilon \to 0^+$ and then $l \to \infty$ and using the Lebesgue dominated convergence theorem and (2.11) gives

$$\liminf_{\varepsilon\to 0^+}\frac{H_\varepsilon(w_\varepsilon) - \mathrm{d}_W(\alpha, b)\omega(0)}{\varepsilon} \ge 2\int_0^\infty W^{1/2}(z_\alpha)z_\alpha' s\,ds\,\omega'(0). \tag{3.81}$$

Since $H(w_\varepsilon) = G_\varepsilon^{(1)}(v_\varepsilon)$, this concludes the proof. ■

4 Properties of Minimizers of F_ε

In this section, we study qualitative properties of critical points and minimizers of the functional F_ε given in (1.1) and subject to the Dirichlet boundary conditions (1.2).

The following theorem is the analog of Lemma 4.3 in Sternberg and Zumbrun [29]. Here, we replace the mass constraint with Dirichlet boundary conditions.

Recall (2.19).

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected set with boundary of class $C^{2,d}$, $0 < d \leq 1$. Assume that W satisfies (2.1)-(2.4) and that g_ε satisfy (1.11), (2.12)-(2.14). Let $u_\varepsilon \in H^1(\Omega)$ be a critical point of (1.1) subject to the Dirichlet boundary condition (1.2). Then*

$$a \leq u_\varepsilon(x) \leq b \quad \text{for all } x \in \Omega. \tag{4.1}$$

Moreover, for every

$$0 < \rho < b - c, \tag{4.2}$$

there exist $\mu_\rho > 0$ and $C_\rho > 0$, independent of ε, such that for all ε sufficiently small the following estimates hold

$$0 \leq b - u_\varepsilon(x) \leq C_\rho e^{-\mu_\rho \operatorname{dist}(x,K_\rho)/(2\varepsilon)} \quad \text{for } x \in \Omega \setminus K_\rho, \tag{4.3}$$

where $K_\rho := \{x \in \Omega : \, u_\varepsilon(x) \leq b - \rho\} \cup \Omega_{\varepsilon|\log\varepsilon|}$.

The proof relies on the following proposition, which is essentially due to Sternberg and Zumbrun [29, Proposition 4.1].

Proposition 4.2 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected set with C^1 boundary and let $K \subset \Omega$ be a compact set. Suppose that $v : \overline{\Omega} \to \mathbb{R}$ is a function in $C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfying the conditions*

$$\begin{cases} \varepsilon^2 \Delta v \geq \mu^2 v & \text{in } \Omega \setminus K, \\ v \leq M & \text{on } \partial K, \end{cases} \tag{4.4}$$

where $\mu > 0$ and M is a positive constant (not necessarily independent of ε). Then there exists a constant C_0 independent of ε such that

$$v(x) \leq C_0 M e^{-\mu \operatorname{dist}(x,K)/(2\varepsilon)} \quad \text{for } x \in \Omega \setminus K$$

for all $\varepsilon > 0$ sufficiently small.

Proof. By the maximum principle, $v \leq M$ in $\Omega \setminus K$. Let

$$K_\varepsilon := \{x \in \Omega : \, \operatorname{dist}(x, K) \leq \varepsilon\}.$$

Consider the radial function $\phi(x) := e^{-\mu|x|/\varepsilon}$. Letting $r = |x|$, we have that

$$\varepsilon^2 \Delta\phi = \varepsilon^2 \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{N-1}{r} \frac{\partial \phi}{\partial r} \right) = \mu^2 e^{-\mu r/\varepsilon} - \frac{N-1}{r} \varepsilon \mu e^{-\mu r/\varepsilon} \leq \mu^2 \phi. \tag{4.5}$$

Define

$$q(x) := M_1\varepsilon^{-N}\int_{K_\varepsilon} e^{-\mu|x-y|/\varepsilon}dy, \quad x\in\overline{\Omega\setminus K_\varepsilon},$$

where $M_1 > 0$ is to be determined. If $x \in \Omega\setminus K_\varepsilon$, then we can differentiate under the integral sign and use (4.5) to find

$$\varepsilon^2\Delta q \le \mu^2 q \quad \text{in } \Omega\setminus K_\varepsilon. \tag{4.6}$$

If $x_0 \in \overline{\Omega}$ and $\operatorname{dist}(x_0, K) \ge \varepsilon$, then there exists $y_0 \in K$ such that $|x_0 - y_0| = \operatorname{dist}(x_0, K) \ge \varepsilon$. In particular, $\Omega\cap B(y_0, \varepsilon) \subseteq K_\varepsilon$, and hence,

$$q(x_0) \ge M_1\varepsilon^{-N}\int_{\Omega\cap B(y_0,\varepsilon)} e^{-\mu|x_0-y|/\varepsilon}dy \ge M_1\varepsilon^{-N}e^{-\mu\operatorname{dist}(x,K)/\varepsilon}e^{-\mu}|\Omega\cap B(y_0,\varepsilon)|, \tag{4.7}$$

where we used the fact that if $y\in B(y_0,\varepsilon)$, then $|y-x_0|\le|y-y_0|+|y_0-x_0| < \operatorname{dist}(x_0,K)+\varepsilon$. If $B(y_0,\varepsilon/2)\subseteq\Omega$, then $|\Omega\cap B(y_0,\varepsilon)| \ge |B(y_0,\varepsilon/2)| = \alpha_N 2^{-N}\varepsilon^N$, where $\alpha_N = |B(0,1)|$. Otherwise, there exists $y_1\in B(y_0,\varepsilon/2)\cap\partial\Omega$. Since $\partial\Omega$ is of class C^1, it is Lipschitz continuous, hence, by taking by taking ε sufficiently small, we can find a cone $K_{y_1,\varepsilon}$ with vertex y_1 and vertex angle depending on the Lipschitz constant associated to Ω such that $K_{y_1,\varepsilon}\cap B(y_1,\varepsilon/2)\subseteq\Omega\cup\{y_1\}$. Since $y_1\in B(y_0,\varepsilon/2)$, we have that $K_{y_1,\varepsilon}\cap B(y_1,\varepsilon/2)\subseteq(\Omega\cap B(y_0,\varepsilon))\cup\{y_1\}$, and so

$$\mathcal{L}^N(\Omega\cap B(y_0,\varepsilon)) \ge \mathcal{L}^N(K_{y_1,\varepsilon}\cap B(y_1,\varepsilon/2)) = c_0\varepsilon^N.$$

This shows that $\mathcal{L}^N(\Omega\cap B(y_0,\varepsilon)) \ge \min\{c_0,\alpha_N 2^{-N}\}\varepsilon^N$. Take

$$M_1 := Me^{2\mu}/\min\{c_0,\alpha_N 2^{-N}\}.$$

Observe that if $x_0\in\partial K_\varepsilon\cap\Omega$, then $\operatorname{dist}(x_0,K)=\varepsilon$, and so by (4.7),

$$q(x_0) \ge M_1 e^{-2\mu}\min\{c_0,\alpha_N 2^{-N}\} \ge M, \tag{4.8}$$

while if $x_0\in\partial\Omega\setminus K_\varepsilon$, then

$$q(x_0) \ge M_1 e^{-\mu\operatorname{dist}(x,K)/\varepsilon}e^{-\mu}\min\{c_0,\alpha_N 2^{-N}\} \ge Me^{\mu-\mu\operatorname{dist}(x,K)/\varepsilon} \tag{4.9}$$

Next we estimate q from above on $\Omega\setminus K_\varepsilon$. If $x_0\in\Omega\setminus K_\varepsilon$, then $|x_0-y|\ge\operatorname{dist}(x_0,K_\varepsilon)$ for all $y\in K_\varepsilon$, and so,

$$\begin{aligned}
q(x_0) &= M_1\varepsilon^{-N}\int_{K_\varepsilon} e^{-\mu|x_0-y|/(2\varepsilon)}e^{-\mu|x_0-y|/(2\varepsilon)}dy \\
&\le M_1\varepsilon^{-N}e^{-\mu\operatorname{dist}(x_0,K_\varepsilon)/(2\varepsilon)}\int_{K_\varepsilon} e^{-\mu|x_0-y|/(2\varepsilon)}dy \\
&\le M_1\varepsilon^{-N}e^{-\mu\operatorname{dist}(x_0,K_\varepsilon)/(2\varepsilon)}\int_{\mathbb{R}^N\setminus B(x_0,\operatorname{dist}(x_0,K_\varepsilon))} e^{-\mu|x_0-y|/(2\varepsilon)}dy \\
&= M_1\varepsilon^{-N}e^{-\mu\operatorname{dist}(x_0,K_\varepsilon)/(2\varepsilon)}\beta_N\int_{\operatorname{dist}(x_0,K_\varepsilon)}^\infty e^{-\mu r/(2\varepsilon)}r^{N-1}dr \\
&\le M_1 e^{-\mu\operatorname{dist}(x_0,K_\varepsilon)/(2\varepsilon)}\beta_N\int_0^\infty e^{-\mu t/2}t^{N-1}dt =: MC_0 e^{-\mu\operatorname{dist}(x_0,K_\varepsilon)/(2\varepsilon)},
\end{aligned} \tag{4.10}$$

where we set $\beta_N := \mathcal{H}^{N-1}(\mathbb{S}^{N-1})$, we used spherical coordinates, and made the change of variables $t := r/\varepsilon$.

If we now define $w := v - q$, by (4.4) and (4.6), we have that $\varepsilon^2 \Delta w \geq \mu^2 w$ in in $\Omega \setminus K_\varepsilon$, while by the fact that $v \leq M$ in $\Omega \setminus K$ and (4.8), $w \leq 0$ in $\partial K_\varepsilon \cap \Omega$. Finally, if $x \in \partial\Omega \setminus K_\varepsilon$, by (4.4) and (4.9),

$$w(x) \leq M e^{-\mu \operatorname{dist}(x,K)/(2\varepsilon)} - M e^{\mu - \mu \operatorname{dist}(x,K)/\varepsilon} \leq 0.$$

Hence, we have shown that

$$\begin{cases} \varepsilon^2 \Delta w \geq \mu^2 w & \text{in } \Omega \setminus K_\varepsilon, \\ w \leq 0 & \text{on } \partial(\Omega \setminus K_\varepsilon). \end{cases}$$

It follows from the maximum principle that $w \leq 0$ in $\Omega \setminus K_\varepsilon$, that is,

$$v(x) \leq q(x) \leq M C_0 e^{-\mu \operatorname{dist}(x,K_\varepsilon)/(2\varepsilon)}, \quad x \in \Omega \setminus K_\varepsilon.$$

Finally, observe that if $x \in \Omega \setminus K$, then $\operatorname{dist}(x, K_\varepsilon) \leq \operatorname{dist}(x, K) + \varepsilon$, and so

$$v(x) \leq M C_0 e^{-\mu/2} e^{-\mu \operatorname{dist}(x,K)/(2\varepsilon)}, \quad x \in \Omega \setminus K_\varepsilon.$$

On the other hand, if $x \in K \setminus K_\varepsilon$, then $\operatorname{dist}(x, K) \leq \varepsilon$ and so, using the fact that $v \leq M$ in $\Omega \setminus K$, we have

$$v(x) \leq M \leq M \frac{e^{-\mu \operatorname{dist}(x,K)/(2\varepsilon)}}{e^{-\mu/2}},$$

which concludes the proof. ■

We turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. To prove (4.1), assume that there exists $x_0 \in \Omega$ such that $u_\varepsilon(x_0) \geq b$. Assume first that $u_\varepsilon(x_0) > b$. Since $W'(s) > 0$ for $s > b$ and $u_\varepsilon \leq b$ on $\partial\Omega$, we can assume that u_ε achieves its maximum at x_0. But then,

$$0 \geq \Delta u_\varepsilon(x_0) = \frac{1}{2\varepsilon^2} W'(u_\varepsilon(x_0)) > 0.$$

Similarly, we can conclude that $u_\varepsilon \geq a$.

Next, let $v := b - u_\varepsilon$. In $\Omega \setminus K_\rho$, we have that

$$\varepsilon^2 \Delta v = -\frac{1}{2} \frac{W'(u_\varepsilon)}{b - u_\varepsilon} v = \frac{1}{2} \frac{W'(b) - W'(u_\varepsilon)}{b - u_\varepsilon} v \geq \mu_\rho^2 v,$$

where

$$\mu_\rho^2 := \frac{1}{2} \sup_{b-\rho \leq s < b} \frac{W'(b) - W'(s)}{b - s} > 0$$

by (2.2), (2.4), and (4.2). Taking $M := b$, we can apply Proposition 4.2 to obtain (4.3). ■

Remark 4.3 *If W is symmetric with respect to c, or, more generally, if $c := \frac{a+b}{2}$, then*

$$a < u_\varepsilon(x) < b$$

for all $x \in \Omega$. To see this, suppose first that $a = -1$, $c = 0$, and $b = 1$. Assume that there exists $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = \pm 1$. Let

$$v_\varepsilon := u_\varepsilon^2 - 1.$$

Then for all $x \in \Omega$ such that $-1 < u_\varepsilon(x) < 1$,

$$\begin{aligned}\Delta v_\varepsilon = \Delta(u_\varepsilon^2) &= 2\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(u_\varepsilon \frac{\partial u_\varepsilon}{\partial x_i}\right) = 2\sum_{i=1}^{N}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2 + 2u_\varepsilon \Delta u_\varepsilon \\ &= 2|\nabla u_\varepsilon|^2 + \frac{u_\varepsilon}{\varepsilon^2} W'(u_\varepsilon) \geq \frac{u_\varepsilon}{\varepsilon^2}\frac{W'(u_\varepsilon)}{u_\varepsilon^2 - 1} v_\varepsilon.\end{aligned}$$

Since $W'(s) > 0$ for $-1 < s < 0$ and $W'(s) < 0$ for $0 < s < 1$, we have that $\frac{sW'(s)}{s^2-1} \geq 0$. Moreover,

$$\frac{sW'(s)}{s^2-1} = \frac{s}{s+1}\frac{W'(s) - W'(1)}{s-1} \to \frac{1}{2}W''(1) > 0$$

as $s \to 1^-$ and

$$\frac{sW'(s)}{s^2-1} = \frac{s}{s-1}\frac{W'(s) - W'(-1)}{s+1} \to \frac{1}{2}W''(-1) > 0$$

as $s \to -1^+$. Hence, by defining

$$c_\varepsilon(x) := \begin{cases} \frac{u_\varepsilon}{\varepsilon^2}\frac{W'(u_\varepsilon)}{u_\varepsilon^2-1} & \text{if } -1 < u_\varepsilon(x) < 1, \\ \frac{1}{2\varepsilon^2}W''(1) & \text{if } u_\varepsilon(x) = 1, \\ \frac{1}{2\varepsilon^2}W''(-1) & \text{if } u_\varepsilon(x) = -1, \end{cases}$$

we have that

$$\Delta v_\varepsilon \geq c_\varepsilon(x) v_\varepsilon(x),$$

where $c_\varepsilon(x) \geq 0$. Moreover, $v_\varepsilon = u_\varepsilon^2 - 1 = g_\varepsilon^2 - 1 < 0$ on $\partial\Omega$. Since $v_\varepsilon(x_0) = 0$, it follows from [15, Theorem 4, Chapter 6] that v_ε is constant in Ω, which is a contradiction.

To remove the additional condition that $a = -1$ and $b = 1$, it suffices to replace W with

$$\bar{W}(r) := W\left(\frac{b-a}{2}r + \frac{a+b}{2}\right)$$

and u_ε with $\bar{u}_\varepsilon := \frac{2}{b-a}u_\varepsilon - \frac{a+b}{b-a}$.

Theorem 4.4 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected set with boundary of class $C^{2,d}$, $0 < d \leq 1$. Assume that W satisfies (2.1)-(2.4) and that g_ε satisfy (1.11), (2.12)-(2.14). Let $u_\varepsilon \in H^1(\Omega)$ be a minimizer of (1.1) subject to the Dirichlet boundary condition (1.2). Then there exists a constant $C > 0$ depending only on N such that*

$$|\nabla u_\varepsilon(x)| \leq \frac{C}{\varepsilon} \quad \text{for all } x \in \Omega \setminus \Omega_\varepsilon.$$

Moreover, under the additional hypothesis that $g_\varepsilon \in C^2(\overline{\Omega})$, with

$$\|\nabla^2 g_\varepsilon\|_{L^\infty(\Omega)} \leq \frac{M}{\varepsilon^2},$$

for some constant $M > 0$, there exists a constant $C > 0$ depending only on Ω and M, such that

$$|\nabla u_\varepsilon(x)| \leq \frac{C}{\varepsilon} \quad \text{for all } x \in \Omega.$$

The following lemma is due to Bethuel, Brezis, and Hélein [8, Lemma A.1, Lemma A.2].

Lemma 4.5 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set, and let $f \in L^\infty(\Omega)$. Assume that $u \in H^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution to*

$$\Delta u = f \quad \text{in } \Omega.$$

Then for every $x \in \Omega$,

$$|\nabla u(x)|^2 \leq C\left(\|u\|_{L^\infty(\Omega)}\|f\|_{L^\infty(\Omega)} + \frac{1}{\operatorname{dist}^2(x, \partial\Omega)}\|u\|^2_{L^\infty(\Omega)}\right),$$

where $C > 0$ is a constant depending only on N.

Moreover, if $u \in H^1_0(\Omega)$, then

$$\|\nabla u\|^2_{L^\infty(\Omega)} \leq C\|u\|_{L^\infty(\Omega)}\|f\|_{L^\infty(\Omega)},$$

where $C > 0$ is a constant depending only on Ω.

We turn to the proof of Theorem 4.4.

Proof of Theorem 4.4. By Lemma 4.5, for every $x \in \Omega \setminus \Omega_\varepsilon$,

$$\begin{aligned}|\nabla u_\varepsilon(x)|^2 &\leq C\left(\|u_\varepsilon\|_{L^\infty(\Omega)}\left\|\frac{1}{2\varepsilon^2}W'(u_\varepsilon)\right\|_{L^\infty(\Omega)} + \frac{1}{\operatorname{dist}^2(x, \partial\Omega)}\|u_\varepsilon\|^2_{L^\infty(\Omega)}\right)\\ &\leq C\left(\max\{|a|,|b|\}\max_{[a,b]}|W'|\frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon^2}\max\{|a|^2,|b|^2\}\right),\end{aligned}$$

where we used the facts that $a \leq u_\varepsilon \leq b$ (4.1).

To prove the last statement, observe that the function $v_\varepsilon := u_\varepsilon - g_\varepsilon \in H_0^1(\Omega)$ is a weak solution to

$$\begin{cases} \Delta v_\varepsilon = \frac{1}{2\varepsilon^2} W'(u_\varepsilon) - \Delta g_\varepsilon & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

It now suffices to apply the second part of Lemma 4.5.

■

Given the functional

$$F(u) := \int_\Omega (W(u) + |\nabla u|^2)\, dx,$$

we say that a function $u \in H^1(\Omega)$ is a *local minimizer* of F if for every $U \Subset \Omega$ and all $w \in H^1(\Omega)$ with support contained in U, we have that $F(u+v) \geq F(u)$. The following theorem is a special case of a result of Caffarelli and Cordoba [10] (we refer to the paper for the general statement).

Theorem 4.6 *Assume that W satisfies hypotheses (2.1)-(2.4). Let $u \in H^1(B(0,R))$, with $a \leq u \leq b$, $R > 2$, be a local minimizer of*

$$F(v) := \int_{B(0,R)} (W(v) + |\nabla v|^2)\, dx, \quad v \in H^1(B(0,R)), \tag{4.11}$$

and assume that for $a < \lambda < b$ there exists $c_\lambda > 0$ such that

$$\mathcal{L}^N(B(0,1) \cap \{u > \lambda\}) > c_0.$$

Then there exists $c_1 > 0$ (depending only on λ, c_0, N, and W) such that

$$\mathcal{L}^N(B(0,r) \cap \{u > \lambda\}) > c_1 r^N$$

for every $1 < r < R$.

Remark 4.7 *A similar estimate continues to hold if we replace $\{u > \lambda\}$ with $\{u < \lambda\}$ in Theorem 4.6. To see this, define $\bar{W}(s) := W(a+b-s)$, and observe that if $u \in H^1(B(0,R))$ is a local minimizer of (4.11), then $v := -u + a + b$, is a local minimizer of*

$$\bar{F}(w) := \int_{B(0,R)} (\bar{W}(w) + |\nabla w|^2)\, dx, \quad v \in H^1(B(0,R)).$$

Moreover, $\{u < \lambda\} = \{v > a+b-\lambda\}$, where $a+b-\lambda \in (a,b)$. Hence, it suffices to apply Theorem 4.6 to $\bar{F}$.

Theorem 4.8 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected set with boundary of class $C^{2,d}$, $0 < d \leq 1$. Assume that W satisfies (2.1)-(2.4) and that g_ε satisfy (1.11), (2.12)-(2.14). Suppose also that (1.12) holds. Let $u_\varepsilon \in H^1(\Omega)$ be a minimizer of (1.1) subject to the Dirichlet boundary condition (1.2). Then*

$$u_\varepsilon \to b \quad \textit{in } L^1(\Omega).$$

Moreover, for every $a < \lambda < b$ and for every $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that

$$\{u_\varepsilon \le \lambda\} \subseteq \Omega_\delta \tag{4.12}$$

for all $0 < \varepsilon < \varepsilon_\delta$.

Proof. The fact that $u_\varepsilon \to b$ in $L^1(\Omega)$ follows from (1.12) and standard properties of Γ-convergence (see [9, Theorem 1.21]). Next, we prove (4.12). Given $a < \lambda < b$ and $R > 0$, assume by contradiction that there exist $\varepsilon_n \to 0^+$ and $x_n \in \Omega \setminus \Omega_{2R}$ such that $u_{\varepsilon_n}(x_n) \le \lambda$. By compactness, we can assume that $x_n \to x_0$. Define $v_n(y) := u_{\varepsilon_n}(x_n + \varepsilon_n y)$, $y \in B(0, R/\varepsilon_n)$. By a change of variables, and the minimality of u_{ε_n}, we have that v_n is a local minimizer of

$$F(v) := \int_{B(0,R/\varepsilon_n)} (W(v) + |\nabla v|^2)\, dx.$$

By Theorem 4.4 applied to u_ε in $\Omega \setminus \Omega_\varepsilon$, there exists $C_0 > 0$ such that

$$|\nabla v_n(y)| \le C_0 \quad \text{for all } y \in B(0, R/\varepsilon_n)$$

provided $0 < \varepsilon < 2R$. Given $\lambda < \lambda_1 < b$, since $v_n(0) \le \lambda$, it follows that

$$v_n(y) \le v_n(0) + C_0|y| \le \lambda + C_0|y| < \lambda_1$$

for all $y \in B(0, (\lambda_1 - \lambda)/C_0)$, where, without loss of generality, we assume that $(\lambda_1 - \lambda)/C_0 < 1$. Hence,

$$\mathcal{L}^N(B(0,1) \cap \{v_n < \lambda_1\}) \ge \mathcal{L}^N(B(0, (\lambda - \lambda_1)/C_0)) = c_0.$$

It follows from Remark 4.7 that there exists $c_1 > 0$ (depending only on λ, c_0, N, and W) such that

$$\mathcal{L}^N(B(0,r) \cap \{v_n < \lambda_1\}) > c_1 r^N$$

for every $1 < r < R/\varepsilon_n$. By the change of variables $x := x_n + \varepsilon_n y$, we find

$$\mathcal{L}^N(B(x_n, \varepsilon_n r) \cap \{u_{\varepsilon_n} < \lambda_1\}) > c_1 (\varepsilon_n r)^N$$

for every $1 < r < R/\varepsilon_n$. As a consequence,

$$\mathcal{L}^N(B(x_n, R) \cap \{u_{\varepsilon_n} < \lambda_1\}) \ge c_1 R^N.$$

But this is a contradiction, since $B(x_n, R) \subseteq B(x_0, 2R) \subseteq \Omega$, and $u_\varepsilon \to b$ in $L^1(\Omega)$. ∎

Theorem 4.9 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected set with boundary of class $C^{2,d}$, $0 < d \le 1$. Assume that W satisfies (2.1)-(2.4) and that g_ε satisfy (1.11), (2.12)-(2.14). Let $0 < \delta << 1$ and suppose that (1.12) holds. Then there exist $\mu > 0$ and $C > 0$, independent of ε and δ, such that for all ε sufficiently small the following estimate holds*

$$0 \le b - u_\varepsilon(x) \le C e^{-\mu\delta/\varepsilon} \quad \textit{for } x \in \Omega \setminus \Omega_{2\delta}. \tag{4.13}$$

Proof. Fix ρ as in (4.2). By Theorem 4.1, there exist $\mu > 0$ and $C > 0$, independent of ε, such that for all ε sufficiently small the following estimates hold

$$0 \le b - u_\varepsilon(x) \le Ce^{-\mu\,\mathrm{dist}(x,K_\rho)/(2\varepsilon)} \quad \text{for } x \in \Omega \setminus K_\rho, \tag{4.14}$$

where $K_\rho := \{x \in \overline{\Omega} : u_\varepsilon(x) \le b - \rho\} \cup \Omega_{\varepsilon|\log\varepsilon|}$. By Theorem 4.8, there exists $\varepsilon_{\delta,\rho} > 0$ such that

$$\{u_\varepsilon \le b - \rho\} \subseteq \Omega_\delta \tag{4.15}$$

for all $0 < \varepsilon < \varepsilon_{\delta,\rho}$. Thus,

$$0 \le b - u_\varepsilon(x) \le Ce^{-\mu\delta/(2\varepsilon)}$$

for all $x \in \Omega \setminus \Omega_{2\delta}$. ■

5 Second-Order Γ-Limit

In this section, we finally prove Theorem 1.1.

Theorem 5.1 (Second-Order Γ-Limsup) *Assume that $\Omega \subset \mathbb{R}^N$ is an open, bounded, connected set and that its boundary $\partial\Omega$ is of class $C^{2,d}$, $0 < d \le 1$. Assume that W satisfies* (2.1)-(2.4) *and that g_ε satisfy* (1.11), (2.12)-(2.14). *Suppose also that* (1.12) *holds. Then there exists $\{u_\varepsilon\}_\varepsilon$ in $H^1(\Omega)$ such that $\operatorname{tr} u_\varepsilon = g_\varepsilon$ on $\partial\Omega$, $u_\varepsilon \to b$ in $L^1(\Omega)$, and*

$$\limsup_{\varepsilon\to0^+} \mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) \le \int_{\partial\Omega} \kappa(y) \int_0^\infty 2W^{1/2}(z_{g(y)}(s)) z'_{g(y)}(s) s \, ds d\mathcal{H}^{N-1}(y)$$

where $z_{g(y)}$ solves the Cauchy problem (1.8) *with $\alpha = g(y)$.*

Proof. By Lemma 2.6, for $\delta > 0$ sufficiently small the function $\Phi : \partial\Omega \times [0,\delta] \to \overline{\Omega}_\delta$ is of class $C^{1\,d}$. In turn, the function

$$\omega(y,t) := \det J_\Phi(y,t)$$

is of class $C^{1,d}$ and

$$\omega_1 := \min_{y\in\partial\Omega} \omega(y,0) > 0.$$

Fix

$$0 < \omega_0 < \frac{1}{4}\frac{C_W - \mathrm{d}_W(a,\alpha_-)}{C_W}\omega_1. \tag{5.1}$$

By taking $\delta > 0$ sufficiently small, we can assume that

$$|\omega(y,t_1) - \omega(y,t_2)| \le \omega_0 \tag{5.2}$$

for all $y \in \partial\Omega$ and all $t_1, t_2 \in [0,\delta]$.

Let $\delta_\varepsilon \to 0^+$ as $\varepsilon \to 0^+$, and for each $y \in \overline{\Omega}$ define

$$\Psi_\varepsilon(y,r) := \int_{g_\varepsilon(y)}^r \frac{\varepsilon}{(\delta_\varepsilon + W(s))^{1/2}} \, ds, \tag{5.3}$$

and

$$0 \leq T_\varepsilon(y) := \Psi_\varepsilon(y, b). \tag{5.4}$$

Note that $T_\varepsilon \in C^1(\overline{\Omega})$ with

$$\begin{aligned} T_\varepsilon(y) &\leq \int_{g_-}^{b} \frac{\varepsilon}{(\delta_\varepsilon + W(s))^{1/2}}\, ds \\ &\leq -\frac{\sigma}{2}\varepsilon \log(\sigma^2 \delta_\varepsilon) + \sigma\varepsilon \log(1 + 2(b-a)) \end{aligned}$$

by (1.11) and Proposition 2.2. Hence, there exist $C_0 > 0$ and $\varepsilon_0 > 0$, depending only on W such that

$$T_\varepsilon(y) \leq C_0 \varepsilon |\log \delta_\varepsilon| \tag{5.5}$$

for all $0 < \varepsilon < \varepsilon_0$ and all $y \in \partial\Omega$.

For each fixed $y \in \partial\Omega$, let $v_\varepsilon(y, \cdot) : [0, T_\varepsilon(y)] \to [g_\varepsilon(y), b]$ be the inverse of $\Psi_\varepsilon(y, \cdot)$. Then $v_\varepsilon(y, 0) = g_\varepsilon(y)$, $v_\varepsilon(y, T_\varepsilon(y)) = b$, and

$$\frac{\partial v_\varepsilon}{\partial t}(y, t) = \frac{(\delta_\varepsilon + W(v_\varepsilon(y, t)))^{1/2}}{\varepsilon} \tag{5.6}$$

for $t \in [0, T_\varepsilon(y)]$. Assume first that $g_\varepsilon \in C^1(\partial\Omega)$. Then by standard results on the smooth dependence of solutions on a parameter (see, e.g. [30, Section 2.4]), we have that v_ε is of class C^1 in the variables (y, t). Extend $v_\varepsilon(y, t)$ to be equal to b for $t > T_\varepsilon(y)$.

We have

$$v_\varepsilon(y, \Psi_\varepsilon(y, r)) = r$$

for all $g_\varepsilon(y) \leq r \leq b$. For every $y \in \partial\Omega$ and every tangent vector τ to $\partial\Omega$ at y, differentiating in the direction τ gives

$$\frac{\partial v_\varepsilon}{\partial \tau}(y, \Psi_\varepsilon(y, r)) + \frac{\partial v_\varepsilon}{\partial t}(y, \Psi_\varepsilon(y, r)) \frac{\partial \Psi_\varepsilon}{\partial \tau}(y, r) = 0.$$

Hence,

$$\frac{\partial v_\varepsilon}{\partial \tau}(y, t) + \frac{\partial v_\varepsilon}{\partial t}(y, t) \frac{\partial \Psi_\varepsilon}{\partial \tau}(y, r) = 0$$

for all $y \in \partial\Omega$ and $t \in [0, T_\varepsilon(y))$.

By (5.3),

$$\frac{\partial \Psi_\varepsilon}{\partial \tau}(y, r) = -\frac{\varepsilon}{(\delta_\varepsilon + W(g_\varepsilon(y)))^{1/2}} \frac{\partial g_\varepsilon}{\partial \tau}(y),$$

and so by (5.6), we have

$$\begin{aligned} \frac{\partial v_\varepsilon}{\partial \tau}(y, t) &= -\frac{\partial v_\varepsilon}{\partial t}(y, t) \frac{\partial \Psi_\varepsilon}{\partial \tau}(y, r) \\ &= \frac{(\delta_\varepsilon + W(v_\varepsilon(y, t)))^{1/2}}{(\delta_\varepsilon + W(g_\varepsilon(y)))^{1/2}} \frac{\partial g_\varepsilon}{\partial \tau}(y) \end{aligned}$$

for $t \in [0, T_\varepsilon(y))$, while $\frac{\partial v_\varepsilon}{\partial \tau}(y,t) = 0$ for $t > T_\varepsilon(y)$. Observe that if $g_\varepsilon(y) \geq c$, then since W is decreasing for $c \leq s \leq b$ and $v_\varepsilon(y,\cdot)$ is increasing, we have that $W(v_\varepsilon(y,t)) \leq W(g_\varepsilon(y))$. Hence, $\left|\frac{\partial v_\varepsilon}{\partial \tau}(y,t)\right| \leq \left|\frac{\partial g_\varepsilon}{\partial \tau}(y)\right|$. On the other hand, if $g_\varepsilon(y) \leq c$, then by (1.11),

$$(\delta_\varepsilon + W(g_\varepsilon(y)))^{1/2} \geq \min_{[g_-,c]} W^{1/2} =: W_0 > 0.$$

Since $a \leq v_\varepsilon(y,t) \leq b$, in both cases, we have

$$\left|\frac{\partial v_\varepsilon}{\partial \tau}(y,t)\right| \leq \begin{cases} C\left|\frac{\partial g_\varepsilon}{\partial \tau}(y)\right| & \text{if } y \in \partial\Omega \text{ and } t \in [0, T_\varepsilon(y)), \\ 0 & \text{if } y \in \partial\Omega \text{ and } t \in (T_\varepsilon(y), \delta]. \end{cases} \tag{5.7}$$

If $g_\varepsilon \in H^1(\partial\Omega)$, a density argument shows that $v_\varepsilon \in H^1(\partial\Omega \times (0,\delta))$ and that (5.6) and (5.7) continues to hold a.e.

Set

$$u_\varepsilon(x) := \begin{cases} v_\varepsilon(\Phi^{-1}(x)) & \text{if } x \in \Omega_\delta, \\ b & \text{if } x \in \Omega \setminus \Omega_\delta, \end{cases} . \tag{5.8}$$

Then $u_\varepsilon \in H^1(\Omega)$, with

$$|\nabla u_\varepsilon(x)|^2 \leq \left|\frac{\partial v_\varepsilon}{\partial t}(\Phi^{-1}(x))\right|^2 + C\|\nabla y\|^2_{L^\infty(\Omega_\delta)} \left|\nabla_\tau v_\varepsilon(\Phi^{-1}(x))\right|^2, \tag{5.9}$$

where we used the facts that $\Phi^{-1}(x) = (y(x), \operatorname{dist}(x,\partial\Omega))$, $|\nabla \operatorname{dist}(x,\partial\Omega)| = 1$, and $\tau \cdot \nabla \operatorname{dist}(x,\partial\Omega) = 0$ for every vector τ such that $\tau \cdot \nu(y) = 0$.

In view of Lemma 2.6, we can use the change of variables $x = \Phi(y,t)$ and Tonelli's theorem to write

$$\begin{aligned} \mathcal{F}^{(2)}_\varepsilon(u_\varepsilon) &= \int_{\partial\Omega}\int_0^\delta \left(\frac{1}{\varepsilon^2}W(u_\varepsilon(\Phi(y,t))) + |\nabla u_\varepsilon(\Phi(y,t))|^2\right)\omega(y,t)\,dt d\mathcal{H}^{N-1}(y) \\ &\quad - \frac{1}{\varepsilon}\int_{\partial\Omega} d(g(y),b)\,d\mathcal{H}^{N-1}(y) \\ &\leq \left(\int_{\partial\Omega}\int_0^\delta \left(\frac{1}{\varepsilon^2}W(v_\varepsilon(y,t)) + \left|\frac{\partial v_\varepsilon}{\partial t}(y,t)\right|^2\right)\omega(y,t)\,dt d\mathcal{H}^{N-1}(y)\right. \\ &\quad \left. -\frac{1}{\varepsilon}\int_{\partial\Omega} d(g(y),b)\,d\mathcal{H}^{N-1}(y)\right) \\ &\quad + C\|\nabla y\|^2_{L^\infty(\Omega_\delta)}\int_{\partial\Omega}\int_0^\delta |\nabla_\tau v_\varepsilon(y,t)|^2\,\omega(y,t)\,dt d\mathcal{H}^{N-1}(y) =: \mathcal{A} + \mathcal{B}. \end{aligned}$$

Taking δ_ε as in (3.37), by Theorem 3.6, there exist constants $0 < \varepsilon_0 < 1$, $C, C_0 > 0$, and $\gamma_0, \gamma_1 > 0$, depending only on α_-, A_0, B_0, T, ω, and W, such that

$$\begin{aligned} &\int_0^\delta \left(\frac{1}{\varepsilon^2}W(v_\varepsilon(y,t)) + \left|\frac{\partial v_\varepsilon}{\partial t}(y,t)\right|^2\right)\omega(y,t)\,dt - \frac{1}{\varepsilon}\,\mathrm{d}_W(b,g(y)) \\ &\leq \int_0^l 2W(p_\varepsilon(y,t))t\,dt\,\frac{\partial \omega}{\partial t}(y,0) + Ce^{-2\sigma l}(2\sigma l + 1) + C\varepsilon^{2\gamma} l + C\varepsilon^{\gamma_1}|\log\varepsilon|^{\gamma_0} \end{aligned}$$

for all $0 < \varepsilon < \varepsilon_0$ and all $l > 0$, where $p_\varepsilon(y,t) = v_\varepsilon(y, \varepsilon t)$, $p_\varepsilon(y,\cdot) \to z_{g(y)}$ pointwise in $[0,\infty)$, where z_α solves the Cauchy problem (1.8). Hence, by Lemma 2.6,

$$\mathcal{A} \le \int_{\partial\Omega} \kappa(y) \int_0^l 2W(p_\varepsilon(y,t))t\,dtd\mathcal{H}^{N-1}(y) + Ce^{-2\sigma l}(2\sigma l + 1) + C\varepsilon^{2\gamma} l + C\varepsilon^{\gamma_1} |\log \varepsilon|^{\gamma_0}$$

for all $0 < \varepsilon < \varepsilon_0$ and all $l > 0$. Since $p_\varepsilon(y,t) \to z_{g(y)}(t)$ for all $t \in [0,l]$ and $a \le p_\varepsilon(y,t) \le b$, we can apply the Lebesgue dominated convergence theorem to obtain

$$\lim_{\varepsilon \to 0^+} \int_{\partial\Omega} \kappa(y) \int_0^l 2W(p_\varepsilon(y,t))t\,dt\,d\mathcal{H}^{N-1}(y) = \int_{\partial\Omega} \kappa(y) \int_0^l 2W(z_{g(y)}(t))t\,dt\,d\mathcal{H}^{N-1}(y).$$

Hence,

$$\limsup_{\varepsilon \to 0^+} \mathcal{A} \le \int_{\partial\Omega} \kappa(y) \int_0^l 2W(z_{g(y)}(t))t\,dt\,d\mathcal{H}^{N-1}(y) + Ce^{-2\sigma l}(2\sigma l + 1).$$

By (2.11) and the Lebesgue dominated convergence theorem, the right-hand side converges to

$$\int_{\partial\Omega} \kappa(y) \int_0^\infty 2W(z_{g(y)}(t))t\,dt\,d\mathcal{H}^{N-1}(y).$$

On the other hand, by (5.5) and (5.7),

$$\begin{aligned}\mathcal{B} &\le C\|\nabla y\|^2_{L^\infty(\Omega_\delta)} \int_{\partial\Omega} |\nabla_\tau g_\varepsilon(y)|^2 \int_0^{T_\varepsilon(y)} \omega(y,t)\,dt\,d\mathcal{H}^{N-1}(y) \\ &\le C\varepsilon |\log\varepsilon| \|\omega\|_{L^\infty(\partial\Omega\times[0,\delta])} \int_{\partial\Omega} |\partial_\tau g_\varepsilon(y)|^2\,d\mathcal{H}^{N-1}(y) = o(1) \qquad (5.10)\end{aligned}$$

by (2.13).

In conclusion, we have shown that

$$\mathcal{F}^{(2)}_\varepsilon(u_\varepsilon) \le \int_{\partial\Omega} \kappa(y) \int_0^\infty 2W^{1/2}(z_{g(y)}(s)) z'_{g(y)}(s) s\,ds\,d\mathcal{H}^{N-1}(y) + o(1).$$

Step 2: We claim that

$$u_\varepsilon \to u_0 \quad \text{in } L^1(\Omega).$$

In view of Lemma 2.6, we can use the change of variables $x := \Phi(y,t)$ and

Tonelli's theorem to write

$$\begin{aligned}\int_\Omega |u_\varepsilon - u_0|\,dx &= \int_{\partial\Omega}\int_0^\delta |u_\varepsilon(\Phi(y,t))) - b|\omega(y,t)\,dtd\mathcal{H}^{N-1}(y)\\ &= \int_{\partial\Omega}\int_0^{T_\varepsilon(y)} |v_\varepsilon(y,t) - b|\omega(y,t)\,dtd\mathcal{H}^{N-1}(y)\\ &\le C\varepsilon|\log\varepsilon|,\end{aligned}$$

where we used the fact that $v_\varepsilon(y,t) = b$ for $t \ge T_\varepsilon(y)$ and (5.5). ■

For every measurable set $E \subseteq \Omega$, we define the localized energy

$$E_\varepsilon(u,E) := \int_E \left(\frac{1}{\varepsilon^2}W(u) + |\nabla u|^2\right)\,dx, \quad u \in H^1(\Omega).$$

Theorem 5.2 (Second-Order Γ-Liminf) *Assume that* $\Omega \subset \mathbb{R}^N$ *is an open, bounded, connected set and that its boundary* $\partial\Omega$ *is of class* $C^{2,d}$, $0 < d \le 1$. *Assume that* W *satisfies* (2.1)-(2.4) *and that* g_ε *satisfy* (1.11), (2.12)-(2.14). *Suppose also that* (1.12) *holds. Then*

$$\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) \ge \int_{\partial\Omega}\kappa(y)\int_0^\infty 2W^{1/2}(z_{g(y)}(s))z'_{g(y)}(s)s\,dsd\mathcal{H}^{N-1}(y),$$

where z_α *solves the Cauchy problem* (1.8) *with* $\alpha = g(y)$.

Proof. We choose ω and δ as in the proof of Theorem 5.1. By Theorem 4.9 (with Ω_δ and $\Omega_{2\delta}$ replaced by $\Omega_{\delta/2}$ and Ω_δ, respectively), we can assume that

$$0 \le b - u_\varepsilon(x) \le Ce^{-\mu\delta/\varepsilon} \quad \text{for } x \in \Omega\setminus\Omega_\delta \tag{5.11}$$

for all $0 < \varepsilon < \varepsilon_\delta$.

Write

$$\begin{aligned}F_\varepsilon^{(2)}(u_\varepsilon) &= E_\varepsilon(u_\varepsilon, \Omega\setminus\Omega_\delta)\\ &\quad + \left(E_\varepsilon(u_\varepsilon,\Omega_\delta) - \frac{1}{\varepsilon}\int_{\partial\Omega} \mathrm{d}_W(g,b)\,d\mathcal{H}^{N-1}\right)\\ &=: \mathcal{A} + \mathcal{B}.\end{aligned}$$

Since $\mathcal{A} \ge 0$, it remains to study $\mathcal{B}$. In view of Lemma 2.6, we can use the change of variables $x = \Phi(y,t)$ and Tonelli's theorem to write

$$E_\varepsilon(u_\varepsilon,\Omega_\delta) = \int_{\partial\Omega}\int_0^\delta \left(\frac{1}{\varepsilon^2}W(u_\varepsilon(\Phi(y,t))) + |\nabla u_\varepsilon(\Phi(y,t))|^2\right)\omega(y,t)\,dtd\mathcal{H}^{N-1}(y).$$

Since $u_\varepsilon \in C^1(\overline{\Omega})$, if we define

$$\tilde{u}_\varepsilon(y,t) := u_\varepsilon(y + t\nu(y)),$$

we have that

$$\frac{\partial \tilde{u}_\varepsilon}{\partial t}(y,t) = \frac{\partial u_\varepsilon}{\partial \nu(y)}(y + t\nu(y)),$$

and so,

$$\begin{aligned} &E_\varepsilon(u_\varepsilon, \Omega_\delta) - \frac{1}{\varepsilon}\int_{\partial\Omega} \mathrm{d}_W(g,b)\, d\mathcal{H}^{N-1} \qquad (5.12)\\ &\geq \int_{\partial\Omega}\left[\int_0^\delta \left(\frac{1}{\varepsilon^2}W(\tilde{u}_\varepsilon(y,t)) + \left|\frac{\partial \tilde{u}_\varepsilon}{\partial t}(y,t)\right|^2\right)\omega(y,t)\,dt - \frac{1}{\varepsilon}\mathrm{d}_W(g(y),b)\right] d\mathcal{H}^{N-1}(y). \end{aligned}$$

For $y \in \partial\Omega$, in view of (5.11), we have that

$$b - C_\rho e^{-\mu_\rho \delta/(2\varepsilon)} \leq \tilde{u}_\varepsilon(y,\delta) \leq b. \tag{5.13}$$

Let $v_\varepsilon^y \in H^1([0,\delta])$ be the minimizer of the functional

$$v \mapsto \int_0^\delta \left(\frac{1}{\varepsilon^2}W(v(t)) + |v'(t)|^2\right)\omega(y,t)\,dt$$

defined for all $v \in H^1([0,\delta])$ such that $v(0) = g_\varepsilon(y)$ and $v(\delta) = \tilde{u}_\varepsilon(y,\delta)$. In view of (2.14) and (5.13), we can apply Theorem 3.15 to find $0 < \varepsilon_0 < 1$, $C > 0$, and $l_0 > 1$, depending only on α_-, a, b, δ, ω, and W such that

$$\begin{aligned} \psi_\varepsilon(y) &:= \int_0^\delta \left(\frac{1}{\varepsilon^2}W(\tilde{u}_\varepsilon(y,t)) + \left|\frac{\partial \tilde{u}_\varepsilon}{\partial t}(y,t)\right|^2\right)\omega(y,t)\,dt - \frac{1}{\varepsilon}\mathrm{d}_W(b,g(y))\\ &\geq \int_0^\delta \left(\frac{1}{\varepsilon^2}W(v_\varepsilon^y(t)) + |(v_\varepsilon^y)'(t)|^2\right)\omega(y,t)\,dt - \frac{1}{\varepsilon}\mathrm{d}_W(b,g(y))\\ &\geq 2\frac{\partial\omega}{\partial t}(y,0)\int_0^l W^{1/2}(w_\varepsilon)w_\varepsilon' s\,ds - Ce^{-l\mu}\,(l\mu+1)\\ &\qquad\qquad - Cl^2\varepsilon^{1/2} - C\varepsilon^{\gamma_1}|\log\varepsilon|^{2+\gamma_0} =: \phi_\varepsilon(y), \end{aligned}$$

for all $0 < \varepsilon < \varepsilon_0$ and $l > l_0$, where $w_\varepsilon(s) := v_\varepsilon(\varepsilon s)$ for $s \in [0, \delta\varepsilon^{-1}]$ satisfies

$$\lim_{\varepsilon\to 0^+}\int_0^l W^{1/2}(w_\varepsilon)w_\varepsilon' s\,ds = \int_0^l W^{1/2}(z_{g(y)})z_{g(y)}' s\,ds \tag{5.14}$$

for every $l > 0$ and where $z_{g(y)}$ solves the Cauchy problem (1.8) with $\alpha = g(y)$. By Corollary 3.9, there exists a constant $C > 0$ depending only on α_-, a, b, δ, ω, and W such that $|w_\varepsilon(t)| \leq C$ for all $t \in [0, \delta\varepsilon^{-1}]$ and for all $0 < \varepsilon < \varepsilon_0$. Hence, $|\phi_\varepsilon(y)| \leq C_l$ for all $y \in \partial\Omega$ and for all $0 < \varepsilon < \varepsilon_0$. Since $\psi_\varepsilon - C_l \geq 0$, we can apply Fatou's lemma to obtain

$$\begin{aligned} \liminf_{\varepsilon\to 0^+}\int_{\partial\Omega}\psi_\varepsilon(y)d\mathcal{H}^{N-1}(y) &\geq \int_{\partial\Omega}\liminf_{\varepsilon\to 0^+}\psi_\varepsilon(y)d\mathcal{H}^{N-1}(y)\\ &\geq \int_{\partial\Omega}\liminf_{\varepsilon\to 0^+}\phi_\varepsilon(y)\,d\mathcal{H}^{N-1}(y)\\ = \int_{\partial\Omega} 2\kappa(y)&\left(\int_0^l W^{1/2}(z_{g(y)})z_{g(y)}' s\,ds - Ce^{-l\mu}\,(l\mu+1)\right) d\mathcal{H}^{N-1}(y). \end{aligned}$$

Letting $l \to \infty$ and using the Lebesgue monotone convergence theorem for the first term gives

$$\liminf_{\varepsilon\to 0^+} \int_{\partial\Omega} \psi_\varepsilon(y) d\mathcal{H}^{N-1}(y)$$
$$\geq \int_{\partial\Omega} 2\kappa(y) \int_0^\infty W^{1/2}(z_{g(y)}(s)) z'_{g(y)}(s) s\, ds d\mathcal{H}^{N-1}(y).$$

Recalling the definition of ψ_ε concludes the proof. ■

6 Note Added to Proof

When this paper was almost complete, we became aware of the paper by Alikakos and Fusco [1] (and consequently of [2], [19], [27]), where they studied the case $g_\varepsilon \equiv z_0$, where $z_0 \notin W^{-1}(\{0\})$, in the vectorial case, that is, when $W : \mathbb{R}^m \to [0, \infty)$ with $m \geq 1$, and W has a finite number of wells. In [1, Lemma 3.1 and Theorem 3.3], the authors proved that there exists $z_1 \in W^{-1}(\{0\})$ such that minimizers u_ε of F_ε satisfy the bound

$$\varepsilon\sigma^+ \mathcal{H}^{N-1}(\partial\Omega)(1 - C_1 \varepsilon^{1/3}) \leq F_\varepsilon(u_\varepsilon) \leq \varepsilon\sigma^+ \mathcal{H}^{N-1}(\partial\Omega) + C_2 \varepsilon^2, \tag{6.1}$$

where σ^+ is the vectorial version of $\mathrm{d}_W(z_0, z_1)$ and C_1 and C_2 are positive constants independent of ε. Using this estimate, they were able to show that

$$|u_\varepsilon(x) - z_1| \leq K e^{-k(\mathrm{dist}(x,\partial\Omega) - C\varepsilon^{1/[3(N-1)]})_+/\varepsilon}, \quad x \in \Omega, \tag{6.2}$$

where C, K, k are positive constants independent of ε.

In the scalar case $m = 1$ we are able to replace (6.1) with the sharp bound (1.15).

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