THE LIMIT OF A NONLOCAL $p-$LAPLACIAN OBSTACLE PROBLEM WITH NONHOMOGENEOUS TERM AS $p \to \infty$

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Abstract. The aim of this paper is to investigate the asymptotic behavior of the minimizers to the following problems related to the fractional $p-$Laplacian with nonhomogeneous term $h(u)$ in the presence of an obstacle ψ in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$,

$$\min\left\{\frac{1}{2}\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} + \int_{\Omega} h(u)^p \,:\, u \in W^{s,p}(\Omega),\ u \geq \psi \text{ on } \bar{\Omega},\ u = g \text{ on } \partial\Omega\right\}.$$

First, we show the convergence of the solutions to certain limit as $p \to \infty$ and identify the limit equation. More precisely, we show that the limit problem is closely related to the infinity fractional Laplacian. In the particular case when h is increasing, we study the Hölder regularity of any solution to the limit problem and we extend the existence result to the case when h is singular.

1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^N$ and g be a $\alpha-$Hölder boundary datum on $\partial\Omega$. From [1], it is well known that if u_p minimizes the functional

$$E_p[u] := \int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}}\,\mathrm{d}x\,\mathrm{d}y$$

among all functions u in the fractional Sobolev space $W^{s,p}(\Omega)$ such that $u_p = g$ on $\partial\Omega$ (with $s = \alpha - \frac{N}{p}$), then $u_p \to u$ as $p \to \infty$ where the limit function u solves the following equation (which is usually referred to as the infinity fractional Laplacian):

$$L_\infty u := L_\infty^+ u + L_\infty^- u = 0,$$

where

$$L_\infty^+ u = \sup_{y\in\Omega,\ y\neq x}\frac{u(y)-u(x)}{|y-x|^\alpha} \qquad \text{and} \qquad L_\infty^- u = \inf_{y\in\Omega,\ y\neq x}\frac{u(y)-u(x)}{|y-x|^\alpha}.$$

In fact, one of the most important motivations to analyse this kind of problems is the $\alpha-$Hölder extension of the function $g \in C^{0,\alpha}(\partial\Omega)$. In fact, one can show that the limit function u is the optimal Hölder extension to $\bar{\Omega}$ of the boundary datum g, i.e. the Hölder seminorm for u in Ω is always less than or equal to the one for the boundary datum given on $\partial\Omega$.

Given a continuous obstacle ψ defined on $\bar{\Omega}$, the authors in [7] follow the work in [1] and prove the existence of a super infinity fractional harmonic function constrained to lie above the obstacle and to take the datum on $\partial\Omega$. More precisely, they show that the following obstacle problem has a viscosity solution:

$$(1.1)\qquad \begin{cases} L_\infty u = 0 & \text{in } \{x\in\Omega:\ u(x) > \psi(x)\},\\ L_\infty u \leq 0 & \text{in } \{x\in\Omega:\ u(x) = \psi(x)\},\\ u(x) \geq \psi(x) & \text{if } x\in\Omega,\\ u(x) = g(x) & \text{if } x\in\partial\Omega.\end{cases}$$

In order to have a solution for this problem (1.1), it is necessary that $\psi(x) \le g(x)$, for all $x \in \partial\Omega$. So, in the sequel, we will assume the following natural condition on the obstacle ψ:

$$\psi \le g \ \text{ on } \ \partial\Omega.$$

The idea in [7] follows exactly the one in [1], where the authors approximate Problem (1.1) with a sequence of fractional $p-$Laplacian operators. To be more precise, they consider the following minimization problem:

$$\min\Big\{E_p[u] : u \in W^{s,p}(\Omega),\ u \ge \psi \ \text{ in } \ \bar{\Omega},\ \ u = g \ \text{ on } \partial\Omega\Big\}. \tag{1.2}$$

But, it is not difficult to check that the Euler-Lagrange equation associated to this functional is

$$\begin{cases} L_p u_p = 0 & \text{in } \{u_p > \psi\}, \\ L_p u_p \le 0 & \text{in } \{u_p = \psi\}, \end{cases} \tag{1.3}$$

where

$$L_p u := \int_\Omega \left(\frac{|u(x)-u(y)|}{|x-y|^\alpha}\right)^{p-1} \frac{1}{|x-y|^\alpha} \frac{u(y)-u(x)}{|u(y)-u(x)|}\,\mathrm{d}y.$$

Let us denote by L_p^+ and L_p^- the positive and negative parts of L_p, respectively. So, one has

$$L_p^+ u_p = L_p^- u_p \qquad \text{in } \ \{u_p > \psi\}.$$

Hence,

$$\left(\int_\Omega \left(\frac{[u_p(x)-u_p(y)]_+}{|x-y|^\alpha}\right)^{p-1} \frac{1}{|x-y|^\alpha}\,\mathrm{d}y\right)^{\frac{1}{p-1}} = \left(\int_\Omega \left(\frac{[u_p(x)-u_p(y)]_-}{|x-y|^\alpha}\right)^{p-1} \frac{1}{|x-y|^\alpha}\,\mathrm{d}y\right)^{\frac{1}{p-1}},$$

where $[z]_\pm := \max\{\pm z, 0\}$. Letting p goes to ∞, we may show that up to a subsequence $u_p \to u$. Formally, we get that

$$L_\infty^+ u = -L_\infty^- u$$

and so, $L_\infty u = 0$ in $\{u > \psi\}$. We note that this limit procedure only works when the right hand side in (1.3) is zero.

In this paper, we consider the minimization problem (1.2) but in the presence of an extra nonhomogeneous term:

$$\min\left\{\frac{E_p[u]}{2} + \int_\Omega h(u)^p \ : \ u \in W^{s,p}(\Omega),\ u \ge \psi \ \text{ on } \ \bar{\Omega},\ u = g \ \text{ on } \partial\Omega\right\}, \tag{1.4}$$

where h is a given C^1 function; in the sequel, we will denote by f the derivative of h. The main goal of this paper is to study the limit as $p \to \infty$ of the minimizers u_p to (1.4), prove their convergence up to a subsequence to a function u, and to identify the limit problem for u. In fact, we may assume that the limit function u solves the following problem:

$$\begin{cases} L_\infty u = h(u) & \text{in } \{u > \psi\}, \\ L_\infty u \le h(u) & \text{in } \{u = \psi\}. \end{cases} \tag{1.5}$$

However, we will see that this is not the case and the limit equation is different, so the presence of the nonhomogeneous term makes more delicate the analysis of our problem. This will also depends on the monotonicity of h.

In [2], the authors characterize the limit as $p \to \infty$ of the branches of solutions to the local $p-$Laplacian:

$$-\nabla \cdot [|\nabla u|^{p-2} \nabla u] = \lambda u^{\gamma(p)}, \qquad u > 0,$$

with $\lambda > 0$ and $\lim_{p\to\infty} \frac{\gamma(p)}{p-1} = \gamma_\star < 1$. They show that the limit set is a curve of positive viscosity solutions of the equation

$$\min\{-\Delta_\infty u, |\nabla u| - \Lambda u^{\gamma_\star}\} = 0,$$

where $\Delta_\infty u := D^2 u \, \nabla u \cdot \nabla u$ is the infinity Laplacian operator and $\Lambda > 0$. On the other hand, in [5], the problem of minimizing the fractional Rayleigh quotient has been considered

$$\min\left\{ \frac{\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}}}{\int_{\mathbb{R}^N} u^p} \ : \ u \in W^{s,p}(\Omega), \quad u = 0 \ \text{ on } \partial\Omega \right\}. \tag{1.6}$$

This problem leads to an interesting eigenvalue problem with the non-local Euler-Lagrange equation:

$$-\mathcal{L}_p u = \lambda |u|^{p-2} u,$$

where the operator $\mathcal{L}_p$ is defined exactly as L_p but with integration set $\mathbb{R}^N$ instead of Ω. The limit equation takes the form

$$\max\{\mathcal{L}_\infty u, \mathcal{L}_\infty^- u + \lambda u\} = 0 \qquad \text{in } \ \Omega.$$

In addition, an equivalent nonlocal version for the fractional $p-$Laplacian was studied in [3], where the authors were interested in describing the behaviour of the solutions to the following Dirichlet problem as $p \to \infty$:

$$\begin{cases} -\mathcal{L}_p u = |u|^{\gamma(p)-1} u & \text{in } \ \Omega, \\ u = g & \text{on } \ \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.7}$$

Inspired by [5], the authors of [3] prove that the limit problem of (1.7) is the following:

$$\begin{cases} \min\{-\mathcal{L}_\infty u, -\mathcal{L}_\infty^- u - |u|^{\gamma_\star}\} = 0 & \text{in } \ \Omega, \\ u = g & \text{on } \ \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.8}$$

2. Preliminaries

In order to study the minimization problem (1.4), we recall some basic theory of fractional Sobolev spaces. Assume Ω is a Lipschitz domain. Then, we define the fractional Sobolev space $W^{s,p}(\Omega)$ with $0 < s < 1$ and $1 < p < \infty$ as follows:

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega), \ [u]_{s,p}^p := \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} < \infty \right\}.$$

We may see $W^{s,p}(\Omega)$ as an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$||u||_{W^{s,p}(\Omega)} = \left[||u||_p^p + [u]_{s,p}^p \right]^{\frac{1}{p}}.$$

In order to obtain a Poincaré inequality in $W_0^{s,p}(\Omega)$ (where the space $W_0^{s,p}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $||\cdot||_{W^{s,p}(\Omega)}$) valid for p large, we consider again the fractional Rayleigh quotient:

$$\lambda_p = \min{(1.6)}.$$

In [5], the authors show that

$$(\lambda_p)^{\frac{1}{p}} \to \frac{1}{R^\alpha},$$

with $R = \max\{\mathrm{dist}(x, \partial\Omega) : x \in \Omega\}$ being the radius of the largest ball inscribed in Ω. As a consequence, we have

$$||u||_{L^p(\Omega)} \leq C(R,\alpha)\bigg(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}}\bigg)^{\frac{1}{p}}.$$

But,

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} = \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} + 2\iint_{\Omega\times(\mathbb{R}^N\setminus\Omega)} \frac{|u(x)|^p}{|x-y|^{\alpha p}}.$$

However,

$$\iint_{\Omega\times(\mathbb{R}^N\setminus\Omega)} \frac{|u(x)|^p}{|x-y|^{\alpha p}} = \int_\Omega |u(x)|^p \bigg(\int_{\{z:\, x+z\in\mathbb{R}^N\setminus\Omega\}} \frac{1}{|z|^{\alpha p}}\mathrm{d}z\bigg)\,\mathrm{d}x.$$

Using polar coordinates, one has

$$\int_{\{z:\, x+z\in\mathbb{R}^N\setminus\Omega\}} \frac{1}{|z|^{\alpha p}}\mathrm{d}z = \int_{\mathbb{S}^{N-1}} \int_{\{r>0:\, x+rw\notin\Omega\}} \frac{1}{r^{\alpha p-N+1}}\,\mathrm{d}r\,\mathrm{d}w.$$

For $w \in \mathbb{S}^{N-1}$, we define

$$d_{w,\Omega}(x) := \inf\{r > 0 : x + rw \notin \Omega\}.$$

Hence, we have

$$\int_{\{z:\, x+z\in\mathbb{R}^N\setminus\Omega\}} \frac{1}{|z|^{\alpha p}}\mathrm{d}z \leq \int_{\mathbb{S}^{N-1}} \int_{d_{w,\Omega}(x)}^\infty \frac{1}{r^{\alpha p-N+1}}\,\mathrm{d}r\,\mathrm{d}w = \frac{1}{\alpha p-N}\int_{\mathbb{S}^{N-1}} \frac{1}{d_{w,\Omega}(x)^{\alpha p-N}}\,\mathrm{d}w.$$

Thus, we get

$$\iint_{\Omega\times(\mathbb{R}^N\setminus\Omega)} \frac{|u(x)|^p}{|x-y|^{\alpha p}} \leq \frac{1}{\alpha p-N}\int_\Omega |u(x)|^p \bigg(\int_{\mathbb{S}^{N-1}} \frac{1}{d_{w,\Omega}(x)^{\alpha p-N}}\,\mathrm{d}w\bigg)\,\mathrm{d}x.$$

Thanks to [6, Theorem 1.2], if $sp > 1$ then we have the following fractional Hardy-type inequality:

$$\int_\Omega |u(x)|^p \bigg(\int_{\mathbb{S}^{N-1}} \frac{1}{d_{w,\Omega}(x)^{\alpha p-N}}\,\mathrm{d}w\bigg)\,\mathrm{d}x \leq C(N,p,\alpha)\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}}.$$

Finally, this yields that

$$||u||_{L^p(\Omega)} \leq C\bigg(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}}\bigg)^{\frac{1}{p}}, \qquad \text{for all } u \in W_0^{s,p}(\Omega).$$

On the other hand, one can show certain regularity properties for functions in $W^{s,p}(\Omega)$ when $sp > N$. From [8, Theorem 8.2], there exists a constant $C < \infty$ depending only on s, p, N such that

$$||u||_{C^{0,\beta}(\bar{\Omega})} \leq C[u]_{s,p}, \qquad \text{for all } u \in W_0^{s,p}(\Omega), \tag{2.1}$$

where $\beta = s - \frac{N}{p}$ and

$$||u||_{C^{0,\beta}(\bar{\Omega})} = ||u||_{L^\infty(\Omega)} + \sup\bigg\{\frac{|u(x)-u(y)|}{|x-y|^\beta} : x,\, y \in \bar{\Omega},\, x \neq y\bigg\}.$$

Since we are interested in what happens when $p \to \infty$, we want to diminish the dependence on p. Thus, it is useful to note that the constant C can be chosen independently of p such that the following inequality holds:

$$||u||_{L^\infty(\Omega)} \le C[u]_{s,p}, \qquad \text{for all } \ u \in W_0^{s,p}(\Omega). \tag{2.2}$$

3. The nonlocal fractional $p-$Laplacian with obstacle

3.1. **Existence & weak solution.** Let $h_p : \mathbb{R} \mapsto \mathbb{R}^+$ be a C^1 function and set $f_p = h_p'$. We consider the minimization problem:

$$\min\left\{\frac{1}{2p}\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(u) \ : \ u \in W^{s,p}(\Omega),\ u \ge \psi \text{ on } \bar\Omega,\ u = g \text{ on } \partial\Omega\right\}, \tag{3.1}$$

where $\alpha p = sp + N$. Assume that there is an extension $\tilde g \in W^{s,p}(\Omega)$ such that $\tilde g = g$ on $\partial\Omega$. For simplicity of notation, we will simply call it g instead of $\tilde g$.

Proposition 3.1. *Assume $\alpha > \frac{2N}{p}$. Then, there exists a minimizer u_p for Problem (3.1). Moreover, u_p is a weak solution to the following equation:*

$$\begin{cases} L_p u = f_p(u) & \text{in} \quad \{u > \psi\}, \\ L_p u \le f_p(u) & \text{in} \quad \{u = \psi\}, \end{cases}$$

where

$$L_p u = \int_\Omega \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{\alpha p}}[u(y)-u(x)]\,\mathrm{d}y.$$

Proof. Let $(u_n)_n$ be a minimizing sequence in Problem (3.1). So, there will be a constant $C < \infty$ such that

$$\frac{1}{2p}\iint_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(u_n) \le C, \qquad \text{for all } \ n.$$

Since $h_p \ge 0$, this implies that

$$\iint_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{\alpha p}} \le C.$$

But, we have

$$||u_n - g||_{C^{0,\beta}(\bar\Omega)} \le C[u_n - g]_{s,p} \le C([u_n]_{s,p} + [g]_{s,p}).$$

This yields that $(u_n)_n$ is bounded in $W^{s,p}(\Omega)$ and so, up to a subsequence, $u_n \rightharpoonup u_p$ in $W^{s,p}(\Omega)$ and so, $u_n \to u$ uniformly in $C^{0,\beta}(\bar\Omega)$ with $\beta = \alpha - \frac{2N}{p}$. By Fatou's Lemma, this yields that

$$\iint_{\Omega\times\Omega}\frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}} = \iint_{\Omega\times\Omega}\liminf_n\left[\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{\alpha p}}\right] \le \liminf_n \iint_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{\alpha p}}$$

and

$$\int_\Omega h_p(u_p) \le \liminf_n \int_\Omega h_p(u_n).$$

So, we get that

$$\frac{1}{2p}\iint_{\Omega\times\Omega}\frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(u_p) \le \liminf_n\left[\frac{1}{2p}\iint_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(u_n)\right].$$

Yet, $u_p \ge \psi$ on $\bar\Omega$ and $u_p = g$ on $\partial\Omega$. Hence, u_p minimizes (3.1). Now, we show the second part. Let ϕ be a smooth function such that $\mathrm{supp}(\phi) \subset \{u_p > \psi\}$. Thanks to the continuity of u_p, it is clear that $u_p + t\phi$ is admissible in (3.1), for all $t \in \mathbb{R}$ small enough. From the minimality of u_p, we have

$$\frac{1}{2p}\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(u_p)$$
$$\leq \frac{1}{2p}\iint_{\Omega\times\Omega} \frac{|u_p(x)+t\phi(x)-u_p(y)-t\phi(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(u_p+t\phi) := \mathcal{J}_\phi(t).$$

So, $\mathcal{J}_\phi$ has a minimum at $t = 0$. Therefore, we have $\mathcal{J}_\phi'(0) = 0$ and so, we get the following:

$$\frac{1}{2}\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^{p-1}}{|x-y|^{\alpha p}} \frac{[u_p(x)-u_p(y)]}{|u_p(x)-u_p(y)|}[\phi(x)-\phi(y)] + \int_\Omega f_p(u_p)\,\phi = 0.$$

By symmetry, this yields that

$$-\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^{p-2}}{|x-y|^{\alpha p}}[u_p(y)-u_p(x)]\,\phi(x) + \int_\Omega f_p(u_p)\,\phi = 0.$$

Finally, we note that for every $\phi \in C_0^\infty(\Omega)$ such that $\phi \geq 0$, the function $u_p + t\phi$ is admissible in (3.1), for all $t \in \mathbb{R}^+$. Hence, $\mathcal{J}_\phi'(0) \geq 0$ and so, one has

$$-\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^{p-2}}{|x-y|^{\alpha p}}[u_p(y)-u_p(x)]\,\phi(x) + \int_\Omega f_p(u_p)\,\phi \geq 0.$$

Then,

$$-\int_\Omega \frac{|u_p(x)-u_p(y)|^{p-2}}{|x-y|^{\alpha p}}[u_p(y)-u_p(x)]\,\mathrm{d}y + f_p(u_p) \geq 0 \qquad \text{in } \{u_p = \psi\}. \quad \square$$

3.2. **Viscosity solution.** The solutions in the previous subsection 3.1 were defined as weak solutions to the Euler-Lagrange equation in the usual way with test functions under the integral sign. In this subsection, we will see that they are also viscosity solutions of the equation

$$L_p u = f_p(u) \tag{3.2}$$

inside the noncoincidence set $\{u_p > \psi\}$ while it is a viscosity supersolution in the coincidence set $\{u_p = \psi\}$. We refer the reader to the book [4] for an introduction to the theory of viscosity solutions. Here, we give the definition of a viscosity supersolution (resp. subsolution).

Definition 3.1. *We will say that u is a viscosity supersolution in Ω of the equation (3.2) if the following holds: whenever $x_0 \in \Omega$ and $\phi \in C^1(\Omega) \cap C(\bar{\Omega})$ are such that*

$$\phi(x_0) = u(x_0) \ \text{ and } \ \phi(x) \leq u(x) \ \text{ for all } x \in \bar{\Omega},$$

then we have

$$\min\{-L_p\phi(x_0) + f_p(\phi(x_0)), \phi(x_0) - \psi(x_0)\} \geq 0.$$

The requirement for a viscosity subsolution is symmetric: the test function is touching from above and the inequality is reversed. Finally, a viscosity solution is defined as being both a viscosity supersolution and a viscosity subsolution.

In order to prove that weak solutions are viscosity solutions we need the following comparison principle (the proof follows in an analogous way the one in [5]):

Proposition 3.2. *Assume h_p is increasing on $\mathbb{R}$. Let u and v be two continuous functions belonging to $W^{s,p}(\Omega)$. Assume that $-L_pu+h_p(u)<-L_pv+h_p(v)$ in the weak sense on $B\subset\Omega$. If $u\le v$ on $\bar\Omega\backslash B$, then $u\le v$ in Ω.*

Proof. Assume $[u-v]_+\neq 0$ on B. Since $[u-v]_+=0$ on $\bar\Omega\backslash B$, then $[u-v]_+\in W^{s,p}_0(\Omega)$ and so, one has

$$\begin{aligned}&\iint_{B\times\Omega}\frac{|v(x)-v(y)|^{p-2}}{|x-y|^{\alpha p}}[v(x)-v(y)][u-v]_+(x)+\int_B h_p(v(x))[u-v]_+(x)\\ >&\iint_{B\times\Omega}\frac{|u(x)-u(y)|^{p-2}}{|x-y|^{\alpha p}}[u(x)-u(y)][u-v]_+(x)+\int_B h_p(u(x))[u-v]_+(x).\end{aligned}$$

Hence,

$$\begin{aligned}&\frac12\iint_{B\times\Omega}\frac{|v(x)-v(y)|^{p-2}}{|x-y|^{\alpha p}}[v(x)-v(y)]([u-v]_+(x)-[u-v]_+(y))+\int_B h_p(v(x))[u-v]_+(x)\\ >&\frac12\iint_{B\times\Omega}\frac{|u(x)-u(y)|^{p-2}}{|x-y|^{\alpha p}}[u(x)-u(y)]\left([u-v]_+(x)-[u-v]_+(y)\right)+\int_B h_p(u(x))[u-v]_+(x).\end{aligned}$$

Since h is increasing, we get

$$\begin{aligned}\frac12\iint_{B\times\Omega}\frac{1}{|x-y|^{\alpha p}}\,\Phi_1(x,y)\left([u-v]_+(x)-[u-v]_+(y)\right)&\\ >\int_B(h_p(u(x))-h_p(v(x)))[u-v]_+(x)\ge 0,&\end{aligned}\tag{3.3}$$

where

$$\Phi_1(x,y)=\Big[|v(x)-v(y)|^{p-2}[v(x)-v(y)]-|u(x)-u(y)|^{p-2}[u(x)-u(y)]\Big].$$

For a, $b\in\mathbb{R}$, one has

$$|b|^{p-2}b-|a|^{p-2}a=\int_0^1\frac{d}{dt}[|a+t(b-a)|^{p-2}(a+t(b-a))]=(p-1)\left(\int_0^1|a+t(b-a)|^{p-2}\,\mathrm{d}t\right)[b-a].$$

Then,

$$\begin{aligned}&\Phi_1(x,y)\\ =&(p-1)\left(\int_0^1|u(x)-u(y)+t(v(x)-v(y)-u(x)+u(y))|^{p-2}\,\mathrm{d}t\right)[v(x)-v(y)-u(x)+u(y)].\end{aligned}$$

So, we get

$$\begin{aligned}&\Phi_1(x,y)[[u-v]_+(x)-[u-v]_+(y)]\\ =&\Phi_2(x,y)[v(x)-v(y)-u(x)+u(y)][[u-v]_+(x)-[u-v]_+(y)]\end{aligned}$$

where

$$\Phi_2(x,y)=(p-1)\left(\int_0^1|u(x)-u(y)+t(v(x)-v(y)-u(x)+u(y))|^{p-2}\,\mathrm{d}t\right)\ge 0.$$

But,

$$\begin{aligned}&[v(x)-v(y)-u(x)+u(y)][[u-v]_+(x)-[u-v]_+(y)]\\ =&-[u(x)-v(x)-(u(y)-v(y))][[u-v]_+(x)-[u-v]_+(y)]\\ =&-\Big[[u-v]_+^2(x)+[u-v]_+^2(y)-\Phi_3(x,y)\Big],\end{aligned}$$

where

$$\Phi_3(x,y) = [u(x)-v(x)][u(y)-v(y)]_+ + [u(y)-v(y)][u(x)-v(x)]_+.$$

For simplicity of notation, we set $s_\pm := [u(x)-v(x)]_\pm$ and $t_\pm := [u(y)-v(y)]_\pm$. Then, we have

$$\Phi_3(x,y) = [s_+ - s_-]t_+ + [t_+ - t_-]s_+ = 2s_+t_+ - s_-t_+ - t_-s_+.$$

Hence,

$$\begin{aligned} &[v(x)-v(y)-u(x)+u(y)][[u-v]_+(x)-[u-v]_+(y)] \\ &= -[s_+^2 + t_+^2 - 2s_+t_+ + s_-t_+ + t_-s_+] = -[(s_+-t_+)^2 + s_-t_+ + t_-s_+] \le 0. \end{aligned}$$

Thus, we get that

$$\Phi_1(x,y)[[u-v]_+(x)-[u-v]_+(y)] \le 0.$$

Finally, we infer that

$$\iint_{B\times\Omega} \frac{1}{|x-y|^{\alpha p}}\,\Phi_1(x,y)\left([u-v]_+(x)-[u-v]_+(y)\right) \le 0,$$

which is in contradiction with the strict inequality in (3.3). Hence, $[u-v]_+ = 0$ and so, $u \le v$ on B. □

Proposition 3.3. *Assume $\alpha \le 1 - \frac{1}{p}$. The weak solution u_p of Problem (3.1) is a viscosity solution to the equation:*

$$L_p u = f_p(u) \qquad \text{in } \{u > \psi\}. \tag{3.4}$$

In addition, u_p is a viscosity supersolution to the equation (3.2) on the coincidence set $\mathcal{S} := \{x \in \Omega : u(x) = \psi(x)\}$.

Proof. Assume u_p is not a viscosity subsolution in $\{u_p > \psi\}$, i.e. there is a point $x_0 \in \{u_p > \psi\}$ and a test function $\phi \in C^1(\Omega)\cap C(\overline{\Omega})$ such that $u_p \le \phi$ on $\bar{\Omega}$, $\phi(x_0) = u_p(x_0)$ and

$$L_p\phi(x_0) - f_p(\phi(x_0)) < 0.$$

Thanks to our assumption that $\alpha \le 1-\frac{1}{p}$, it is easy to see that $x \mapsto L_p\phi(x)$ is continuous on Ω. Hence, there is a $r > 0$ small enough such that

$$L_p\phi(x) - f_p(\phi(x_0)) < 0 \qquad \text{on } \ B(x_0,r).$$

Let η be a smooth cutoff function such that $\eta(x_0) = 1$ and $\eta = 0$ on $\Omega\backslash B(x_0,r)$. Then, we define

$$\phi_\varepsilon := \phi - \varepsilon\eta.$$

Clearly, $\phi_\varepsilon = \phi$ on $\Omega\backslash B(x_0,r)$. Moreover, one has

$$|\phi_\varepsilon(x)-\phi_\varepsilon(y)|^{p-2}[\phi_\varepsilon(x)-\phi_\varepsilon(y)] = |\phi(x)-\phi(y)-\varepsilon[\eta(x)-\eta(y)]|^{p-2}[\phi(x)-\phi(y)-\varepsilon[\eta(x)-\eta(y)]].$$

Yet,

$$\begin{aligned} &\Big||\phi_\varepsilon(x)-\phi_\varepsilon(y)|^{p-2}[\phi_\varepsilon(x)-\phi_\varepsilon(y)] - |\phi(x)-\phi(y)|^{p-2}[\phi(x)-\phi(y)]\Big| \\ &= \left|(p-1)\left(\int_0^1 |\phi(x)-\phi(y)-\varepsilon t[\eta(x)-\eta(y)]|^{p-2}\,\mathrm{d}t\right)[-\varepsilon[\eta(x)-\eta(y)]]\right| \\ &\le C\varepsilon|x-y|^{p-1}. \end{aligned}$$

Then, we get

$$|L_p\phi_\varepsilon(x) - L_p\phi(x)| \le C\varepsilon.$$

We recall that u_p and f are both continuous. For $\varepsilon > 0$ small enough, we have then

$$L_p\phi_\varepsilon(x) - f_p(u_p(x)) < 0 \qquad \text{on} \;\; B(x_0, r).$$

But, $\phi_\varepsilon = \phi \geq u_p$ on $\Omega \backslash B(x_0, r)$. By Proposition 3.2, we infer that $u_p \leq \phi_\varepsilon$ in $B(x_0, r)$. In particular, $u_p(x_0) = \phi(x_0) \leq \phi_\varepsilon(x_0) = \phi(x_0) - \varepsilon$, which is a contradiction. This concludes the proof that u_p is a viscosity subsolution in $\{u_p > \psi\}$.

The proof that u_p is a viscosity supersolution in Ω is similar and so, we omit some details. Assume by contradiction that there is a point $x_0 \in \Omega$ and a test function $\phi \in C^1(\Omega) \cap C(\bar{\Omega})$ such that $\phi \leq u_p$ on $\bar{\Omega}$ with equality at x_0 and

$$L_p\phi(x_0) - f_p(\phi(x_0)) > 0.$$

Now, set $\phi_\varepsilon := \phi + \varepsilon\eta$, where η is always a cutoff function such that $\eta(x_0) = 1$ and $\eta = 0$ outside $B(x_0, r)$. Then, we have $\phi_\varepsilon = \phi$ on $\Omega \backslash B(x_0, r)$. In addition, one can show as before that for every $\varepsilon > 0$ small enough,

$$L_p\phi_\varepsilon(x) - f_p(u_p(x)) > 0 \qquad \text{on} \;\; B(x_0, r).$$

Again, by Proposition 3.2, we infer that $u_p \geq \phi_\varepsilon$ in $B(x_0, r)$, which is a contradiction. $\square$

4. The limit problem as $p \to \infty$

In this section, we show that up to a subsequence the solutions u_p to (1.4) converge uniformly to a function u as p goes to infinity. Moreover, we will be interested in identifying the limit problem verified by u. First of all, let us remember the definition of the infinity fractional Laplacian

$$L_\infty u = \sup_{y \in \Omega,\, y \neq x} \frac{u(y) - u(x)}{|y - x|^\alpha} + \inf_{y \in \Omega,\, y \neq x} \frac{u(y) - u(x)}{|y - x|^\alpha}.$$

We decompose this operator as follows:

$$L_\infty^+ u = \sup_{y \in \Omega,\, y \neq x} \frac{u(y) - u(x)}{|y - x|^\alpha} \qquad \text{and} \qquad L_\infty^- u = \inf_{y \in \Omega,\, y \neq x} \frac{u(y) - u(x)}{|y - x|^\alpha}.$$

In the sequel, we will need the following technical result where the proof can be found in [1, Lemma 6.5].

Lemma 4.1. *Assume $\phi \in C^1(\Omega)$. Let $\{x_p\}_p \subset \Omega$ be such that $x_p \to x_0$. We define*

$$f_p(y) = \frac{\phi(y) - \phi(x_p)}{|y - x_p|^\alpha} \qquad \textit{and} \qquad f(y) = \frac{\phi(y) - \phi(x_0)}{|y - x_0|^\alpha}.$$

Then, we have

$$\lim_{p \to \infty} \left|\left| \frac{[f_p]_\pm}{|y - x_p|^{\frac{\alpha}{p}}} \right|\right|_{L^p(\Omega)} = ||[f]_\pm||_{L^\infty(\Omega)}.$$

Then, we have the following:

Proposition 4.2. *Suppose that $h : \mathbb{R} \mapsto \mathbb{R}^+$ is a C^1 function and $g \in C^{0,\alpha}(\partial\Omega)$. Moreover, assume that there is a constant $M < \infty$ such that*

$$\psi(x) \leq \min\{M|x - x_0|^\alpha + g(x_0) \,:\, x_0 \in \partial\Omega\}, \qquad \textit{for all } x \in \Omega. \tag{4.1}$$

For $h_p := \frac{h^p}{p}$, let u_p be a solution of Problem (3.4). Then, up to a subsequence, $u_p \to u$ uniformly in Ω. Moreover, $u \in C^{0,\alpha}(\bar{\Omega})$ and, u is a viscosity solution to the following problem:

$$\begin{cases} \min\{-L_\infty u, -L_\infty^- u - h(u)\} = 0 & \text{in } \{u > \psi\} \cap \{f(u) < 0\}, \\ \max\{-L_\infty u, -L_\infty^+ u + h(u)\} = 0 & \text{in } \{u > \psi\} \cap \{f(u) > 0\}, \\ \min\{-L_\infty u, -L_\infty^- u - h(u)\} \geq 0 & \text{in } \{u = \psi\} \cap \{f(u) < 0\}, \\ \max\{-L_\infty u, -L_\infty^+ u + h(u)\} \geq 0 & \text{in } \{u = \psi\} \cap \{f(u) > 0\}, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where $f = h'$.

Proof. First, we show that there is a function $\tilde{g} \in C^{0,\alpha}(\bar{\Omega})$ such that $\tilde{g} \geq \psi$ on $\bar{\Omega}$ and $\tilde{g} = g$ on $\partial\Omega$. For $\hat{x} \in \partial\Omega$ and $c \in \mathbb{R}$, we set

$$V_{\hat{x},c}(x) := C|x - \hat{x}|^\alpha + c, \qquad \text{for all } x \in \Omega,$$

where $C > 0$ is any large constant. If $c \geq ||g||_\infty$, then $V_{\hat{x},c} \geq \psi$ on $\bar{\Omega}$ and $V_{\hat{x},c} \geq g$ on $\partial\Omega$. Now, we define

$$\tilde{g}(x) = \inf\left\{V_{\hat{x},c}(x) :\, \hat{x} \in \partial\Omega,\, c \in \mathbb{R} \text{ such that } V_{\hat{x},c} \geq \psi \text{ on } \bar{\Omega},\, V_{\hat{x},c} \geq g \text{ on } \partial\Omega\right\}.$$

We clearly have $\tilde{g} \geq \psi$ on $\bar{\Omega}$ and $\tilde{g} \geq g$ on $\partial\Omega$. Now, fix a point $\hat{x}_0 \in \partial\Omega$ and set $c_0 = g(\hat{x}_0)$. By (4.1), one has

$$V_{\hat{x}_0,c_0}(x) = C|x - \hat{x}_0|^\alpha + g(\hat{x}_0) \geq \psi(x), \qquad \text{for every } x \in \Omega.$$

Thanks to the $\alpha-$Hölder regularity of g, then we also have

$$V_{\hat{x}_0,c_0}(x) = C|x - \hat{x}_0|^\alpha + g(\hat{x}_0) \geq g(x), \qquad \text{for every } x \in \partial\Omega.$$

But so,

$$\tilde{g}(\hat{x}_0) \leq V_{\hat{x}_0,c_0}(\hat{x}_0) = c_0 = g(\hat{x}_0).$$

This yields that $\tilde{g} = g$ on $\partial\Omega$. Moreover, it is clear that $\tilde{g} \in C^{0,\alpha}(\bar{\Omega})$. On the other hand, we have

$$\frac{1}{2p}\iint_{\Omega\times\Omega} \frac{|u_p(x) - u_p(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(u_p) \leq \frac{1}{2p}\iint_{\Omega\times\Omega} \frac{|\tilde{g}(x) - \tilde{g}(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(\tilde{g})$$
$$\leq \frac{C^p|\Omega|^2}{2p} + \frac{||h(\tilde{g})||_\infty^p|\Omega|}{p} \leq \frac{C^p}{p}.$$

We get that

$$\iint_{\Omega\times\Omega} \frac{|u_p(x) - u_p(y)|^p}{|x-y|^{\alpha p}} \leq C^p.$$

Hence, there is a uniform constant C (independent of p) such that we have the following bound:

$$\left[\iint_{\Omega\times\Omega} \frac{|u_p(x) - u_p(y)|^p}{|x-y|^{\alpha p}}\right]^{\frac{1}{p}} \leq C.$$

On the other side, we recall that $||u_p||_\infty \leq C([u]_{s,p} + [\tilde{g}]_{s,p}) + ||\tilde{g}||_\infty \leq C$ thanks to the fact that $\tilde{g} \in C^{0,\alpha}(\partial\Omega)$. Fix $m < p$, one has

$$\left[\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^m}{|x-y|^{\alpha m}}\right]^{\frac{1}{m}} \leq \left[\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}}\right]^{\frac{1}{p}} |\Omega|^{2(1-\frac{m}{p})} \leq C.$$

Consequently, $(u_p)_p$ is bounded in $W^{s,m}(\Omega)$ (with $s = \alpha - \frac{N}{m}$) and so, up to a subsequence, it converges uniformly to a function $u \in W^{s,m}(\Omega)$, for all m. In particular, u belongs to $C^{0,\alpha}(\bar{\Omega})$.

Now, fix $x_0 \in \{u > \psi\}$. We show that u is a viscosity subsolution at x_0 to equation (4.2). First, we consider the case when $f(u(x_0)) > 0$. Assume there is a function $\phi \in C^1(\Omega) \cap C(\overline{\Omega})$ such that $u \leq \phi$ on $\bar{\Omega}$, $u(x_0) = \phi(x_0)$ and,

$$\max\{-L_\infty\phi(x_0), -L_\infty^+\phi(x_0) + h(\phi(x_0))\} > 0.$$

In fact, one can assume that x_0 is the unique maximizer of $u - \phi$. To see this fact, fix $\delta > 0$ small enough and set $\phi_\delta(x) := \phi(x) + \delta|x-x_0|^2$, for every $x \in \Omega$. We have

$$L_\infty\phi_\delta(x_0) = \sup_{x\in\Omega,\, x\neq x_0} \frac{\phi_\delta(x)-\phi_\delta(x_0)}{|x-x_0|^\alpha} + \inf_{x\in\Omega,\, x\neq x_0} \frac{\phi_\delta(x)-\phi_\delta(x_0)}{|x-x_0|^\alpha}.$$

Yet,

$$[\phi_\delta(x)-\phi_\delta(x_0)] - [\phi(x)-\phi(x_0)] = \delta|x-x_0|^2.$$

Hence,

$$\frac{\phi_\delta(x)-\phi_\delta(x_0)}{|x-x_0|^\alpha} = \frac{\phi(x)-\phi(x_0)}{|x-x_0|^\alpha} + \delta|x-x_0|^{2-\alpha} \leq \frac{\phi(x)-\phi(x_0)}{|x-x_0|^\alpha} + C\delta.$$

Therefore, we get that

$$\left| L_\infty^\pm\phi_\delta(x_0) - L_\infty^\pm\phi(x_0) \right| \leq C\delta.$$

Then, $-L_\infty\phi_\delta(x_0) > 0$ or $-L_\infty^+\phi_\delta(x_0) + h(\phi_\delta(x_0)) > 0$ provided that $\delta > 0$ is small enough. This proves our claim.

Since $u_p \to u$ uniformly in Ω, then there is a point $x_p \in \{u_p > \psi\}$ such that $u_p - \phi$ has a maximum at x_p and $x_p \to x_0$ (since x_0 is the unique maximizer of $u - \phi$). In the sequel, we set $M_p := \max_\Omega[u_p - \phi]$; we note that $M_p \to 0$, $u_p \leq \phi + M_p$ and $u_p(x_p) = \phi(x_p) + M_p$. But, u_p is a viscosity solution to equation (3.4). Hence,

$$-L_p[\phi + M_p](x_p) + f_p(\phi(x_p) + M_p) \leq 0,$$

where $f_p = h_p'$. So, we get

$$-L_p\phi(x_p) + f_p(\phi(x_p) + M_p) \leq 0.$$

Recalling the definition of L_p, one has

$$-\int_\Omega \frac{|\phi(x)-\phi(x_p)|^{p-1}}{|x-x_p|^{\alpha p}} \frac{\phi(x)-\phi(x_p)}{|\phi(x)-\phi(x_p)|}\,\mathrm{d}x + f_p(\phi(x_p)+M_p) \leq 0. \tag{4.3}$$

Then,

$$\int_\Omega \frac{[\phi(x)-\phi(x_p)]_+^{p-1}}{|x-x_p|^{\alpha p}}\,\mathrm{d}x \geq \int_\Omega \frac{[\phi(x)-\phi(x_p)]_-^{p-1}}{|x-x_p|^{\alpha p}}\,\mathrm{d}x + f_p(\phi(x_p)+M_p).$$

Set

$$A_p[\phi] := \left[\int_\Omega \frac{[\phi(x)-\phi(x_p)]_+^{p-1}}{|x-x_p|^{\alpha p}}\,\mathrm{d}x\right]^{\frac{1}{p-1}}$$

and

$$B_p[\phi] := \left[\int_\Omega \frac{[\phi(x) - \phi(x_p)]_-^{p-1}}{|x - x_p|^{\alpha p}} \,\mathrm{d}x \right]^{\frac{1}{p-1}}.$$

We have

$$A_p[\phi]^{p-1} \geq B_p[\phi]^{p-1} + f_p(\phi(x_p) + M_p).$$

Assume $A_p[\phi] > 0$. Hence,

$$\frac{B_p[\phi]^{p-1}}{A_p[\phi]^{p-1}} + \frac{f_p(\phi(x_p) + M_p)}{A_p[\phi]^{p-1}} \leq 1. \tag{4.4}$$

Therefore, we have

$$\frac{B_p[\phi]}{A_p[\phi]} \leq 1 \qquad \text{and} \qquad \frac{f_p(\phi(x_p) + M_p)^{\frac{1}{p-1}}}{A_p[\phi]} = \frac{h(\phi(x_p) + M_p)\, f(\phi(x_p) + M_p)^{\frac{1}{p-1}}}{A_p[\phi]} \leq 1$$

since otherwise, at least one of the two terms in (4.4) goes to ∞, which is a contradiction. Thanks to Lemma 4.1, we have that $A_p[\phi] \to L_\infty^+ \phi$ and $B_p[\phi] \to -L_\infty^- \phi$. Passing to the limit when $p \to \infty$, this yields that

$$-L_\infty \phi(x_0) \leq 0 \qquad \text{and} \qquad -L_\infty^+ \phi(x_0) + h(\phi(x_0)) \leq 0.$$

If $f(u(x_0)) < 0$, we assume that there is a function $\phi \in C^1(\Omega) \cap C(\overline{\Omega})$ such that $u \leq \phi$ on $\bar{\Omega}$, $u(x_0) = \phi(x_0)$ and,

$$\min\{-L_\infty \phi(x_0), -L_\infty^- \phi(x_0) - h(\phi(x_0))\} > 0.$$

Recalling (4.3), we have

$$\int_\Omega \frac{[\phi(x) - \phi(x_p)]_-^{p-1}}{|x - x_p|^{\alpha p}} \,\mathrm{d}x \leq \int_\Omega \frac{[\phi(x) - \phi(x_p)]_+^{p-1}}{|x - x_p|^{\alpha p}} \,\mathrm{d}x - f_p(\phi(x_p) + M_p).$$

Following the same steps as before, we arrive to a contradiction and so, u is a viscosity subsolution to the following equation:

$$\min\{-L_\infty u, -L_\infty^- u - h(u)\} \leq 0 \qquad \text{in } \{u > \psi\}.$$

Let us prove that u is also a viscosity supersolution in Ω to equation (4.2) in the case when $f(u(x_0)) > 0$. Our aim is to show that for every function $\phi \in C^1(\Omega) \cap C(\overline{\Omega})$ such that $u \geq \phi$ on $\bar{\Omega}$ and $\phi(x_0) = u(x_0)$, we have

$$\max\{-L_\infty \phi(x_0), -L_\infty^+ \phi(x_0) + h(\phi(x_0))\} \geq 0.$$

Assume this is not the case. Thanks to the uniform convergence of u_p to u, there is a point $x_p \in \Omega$ such that $x_p \to x_0$ and $u_p - \phi$ has a minimum at x_p. We denote by $m_p := \min_\Omega [u_p - \phi] \to 0$. Since u_p is a viscosity solution to (3.4), then one has

$$-L_p[\phi](x_p) + f_p(\phi(x_p) + m_p) \geq 0.$$

So, we have

$$B_p[\phi]^{p-1} + f_p(\phi(x_p) + m_p) \geq A_p[\phi]^{p-1}.$$

In particular, we get

$$\frac{B_p[\phi]}{A_p[\phi]} \geq 1 \qquad \text{or} \qquad \frac{f_p(\phi(x_p) + m_p)}{A_p[\phi]} \geq 1.$$

Hence,

$$-L_\infty \phi(x_0) \geq 0$$

or

$$-L_\infty^+ \phi(x_0) + h(\phi(x_0)) \geq 0.$$

Consequently,

$$\max\{-L_\infty \phi(x_0), -L_\infty^+ \phi(x_0) + h(\phi(x_0))\} \geq 0.$$

Finally, if $f(u(x_0)) < 0$ then one can show similarly that u is a viscosity supersolution in Ω to the following equation:

$$\min\{-L_\infty u, -L_\infty^- u - h(u)\} \geq 0.$$

This concludes the proof. □

5. Regularity

In this section, we assume that $h' > 0$ over $\mathbb{R}$. Under this assumption, we recall that the limit problem is the following:

$$\begin{cases} \max\{-L_\infty u, -L_\infty^+ u + h(u)\} = 0 & \text{in } \{u > \psi\}, \\ \max\{-L_\infty u, -L_\infty^+ u + h(u)\} \geq 0 & \text{in } \{u = \psi\}, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

In order to study the regularity of a viscosity solution u to Problem (5.1), we start by the following comparison principle:

Proposition 5.1. *Let u be a viscosity solution of (5.1). Let $\phi \in C^1(\Omega) \cap C(\bar{\Omega})$ be a strict viscosity supersolution in $\{u > \psi\}$ such that $u \leq \phi$ on $\{u = \psi\} \cup \partial\Omega$. Then, we have $u \leq \phi$ in Ω.*

Proof. Assume there exists a point $x^\star \in \{x \in \Omega : u(x) > \psi(x)\}$ such that $u(x^\star) - \phi(x^\star) = \max_\Omega [u - \phi] = M > 0$. So, we have $u \leq \phi + M$ on $\bar{\Omega}$ and $u(x^\star) = \phi(x^\star) + M$. Since u is a viscosity solution, then one has

$$\max\{-L_\infty \phi(x^\star), -L_\infty^+ \phi(x^\star) + h(\phi(x^\star) + M)\} \leq 0.$$

As $h' > 0$, then

$$\max\{-L_\infty \phi(x^\star), -L_\infty^+ \phi(x^\star) + h(\phi(x^\star))\} \leq 0,$$

which is a contradiction since ϕ is a strict viscosity supersolution in $\{u > \psi\}$. □

Lemma 5.2. *Fix $x_0 \in \{u > \psi\}$. If u is a viscosity solution of Problem (5.1) in Ω, then u is a viscosity solution of (5.1) in $\Omega \backslash \{x_0\}$.*

Proof. We show that u is a viscosity subsolution in $\Omega \backslash \{x_0\} \cap \{u > \psi\}$. Let $x^\star \in \Omega \backslash \{x_0\} \cap \{u > \psi\}$ and $\phi \in C^1(\Omega \setminus \{x_0\}) \cap C(\bar{\Omega})$ be such that $u \leq \phi$ on $\bar{\Omega}$ and $\phi(x^\star) = u(x^\star)$. Assume that $u(x_0) < \phi(x_0)$. Let $\phi_n \in C^1(\Omega) \cap C(\bar{\Omega})$ be such that $\phi_n = \phi$ on $\Omega \backslash B(x_0, \frac{1}{n})$, $\phi \leq \phi_n$ and ϕ_n converges uniformly to ϕ. So, we have $u \leq \phi_n$ on $\bar{\Omega}$ and $u(x^\star) = \phi_n(x^\star)$. Since u is a viscosity solution on Ω, then we must have

$$\max\{-L_\infty \phi_n(x^\star), -L_\infty^+ \phi_n(x^\star) + h(\phi_n(x^\star))\} \leq 0. \tag{5.2}$$

Fix $0<\varepsilon<|x_0-x^\star|$. For $n\in\mathbb{N}^\star$ large enough, one has the following:

$$L^+_\infty\phi_n(x^\star)=\max\left\{\sup_{y\in\bar\Omega\setminus\bar B(x_0,\varepsilon),\,y\neq x^\star}\frac{\phi_n(y)-\phi_n(x^\star)}{|y-x^\star|^\alpha},\sup_{y\in\bar B(x_0,\varepsilon),\,y\neq x^\star}\frac{\phi_n(y)-\phi_n(x^\star)}{|y-x^\star|^\alpha}\right\}$$
$$=\max\left\{\sup_{y\in\bar\Omega\setminus\bar B(x_0,\varepsilon),\,y\neq x^\star}\frac{\phi(y)-\phi(x^\star)}{|y-x^\star|^\alpha},\sup_{y\in\bar B(x_0,\varepsilon),\,y\neq x^\star}\frac{\phi_n(y)-\phi_n(x^\star)}{|y-x^\star|^\alpha}\right\}.$$

Yet, it is clear that

$$\frac{\phi_n(y)-\phi_n(x^\star)}{|y-x^\star|^\alpha}\to\frac{\phi(y)-\phi(x^\star)}{|y-x^\star|^\alpha}\quad\text{uniformly in }\ \bar B(x_0,\varepsilon).$$

Hence, $\lim_{n\to\infty}L^+\phi_n(x^\star)=L^+\phi(x^\star)$. In the same way, we show that $L^-\phi_n(x^\star)\to L^-\phi(x^\star)$. Passing to the limit in (5.2) as $n\to\infty$, we get that

$$\max\{-L_\infty\phi(x^\star),-L^+_\infty\phi(x^\star)+h(\phi(x^\star))\}\le 0.$$

Finally, assume that $u(x_0)=\phi(x_0)$. For every $\delta>0$, we define $\phi_\delta:=\phi+\delta|x-x^\star|^2$. We have $\phi_\delta(x^\star)=u(x^\star)$ and $u\le\phi_\delta$ on $\bar\Omega$. Hence,

$$\max\{-L_\infty\phi_\delta(x^\star),-L^+_\infty\phi_\delta(x^\star)+h(\phi_\delta(x^\star))\}\le 0.$$

But, we recall that

$$|L^\pm_\infty\phi_\delta(x)-L^\pm_\infty\phi(x)|\le C\delta.$$

Passing to the limit when $\delta\to 0^+$, we conclude the proof. In the same way, we show that u is a viscosity supersolution in $\Omega\setminus\{x_0\}$. □

Proposition 5.3. *Any viscosity solution u of Problem (5.1) is bounded. Moreover, we have $||u||_\infty\le\max\{||g||_\infty,||\psi||_\infty\}$.*

Proof. Set $\phi=M\ge\max\{||g||_\infty,||\psi||_\infty\}$. We have $u\le\phi$ on $\{u=\psi\}\cup\partial\Omega$. Fix $x\in\Omega\cap\{u>\psi\}$, then we clearly have

$$\max\{-L\phi(x),-L^+\phi(x)+h(\phi(x))\}=h(M)>0.$$

Hence, ϕ is a strict viscosity supersolution. Thanks to the comparison principle 5.1, this yields that $u\le\phi$ on Ω. □

Proposition 5.4. *Let u be a viscosity solution of (5.1). Then, u is locally $\alpha-$Hölderian in $\{u>\psi\}$. Moreover, we have the following estimate:*

$$[u]_\alpha\le\frac{2||u||_\infty}{\operatorname{dist}(\omega,\{u=\psi\}\cup\partial\Omega)},\qquad\textit{for every }\ \omega\subset\subset\Omega\cap\{u>\psi\},$$

where $[u]_\alpha$ denotes the $\alpha-$Hölder constant of u.

Proof. Fix $x_0\in\omega\cap\{u>\psi\}$. Assume $\alpha<1$. Set $\Psi_{x_0}(x)=|x-x_0|^\alpha$, for all $x\in\Omega$. For $x\in\Omega\setminus\{x_0\}$, it is easy to see that

$$L^-\Psi_{x_0}(x)=\inf_{y\in\bar\Omega,\,y\neq x}\frac{\Psi(y)-\Psi(x)}{|y-x|^\alpha}=\inf_{y\in\bar\Omega,\,y\neq x}\frac{|y-x_0|^\alpha-|x-x_0|^\alpha}{|y-x|^\alpha}\le -1.$$

On the other hand,

$$L^+\Psi_{x_0}(x)=\sup_{y\in\bar\Omega,\,y\neq x}\frac{\Psi(y)-\Psi(x)}{|y-x|^\alpha}\le\sup_{y\in\bar\Omega,\,y\neq x}\frac{|y-x_0|^\alpha-|x-x_0|^\alpha}{||y-x_0|-|x-x_0||^\alpha}\le\sup_{1<r<\frac{\operatorname{diam}(\Omega)}{|x-x_0|}}\Psi(r),$$

with $\Psi(r) = \dfrac{r^\alpha - 1}{(r-1)^\alpha}$. Yet, it is easy to check that Ψ is increasing on $(1, +\infty)$ and so, since $\alpha < 1$, we have

$$L^+\Psi_{x_0}(x) \leq \Psi\left(\frac{\operatorname{diam}(\Omega)}{|x-x_0|}\right) = \frac{\left(\dfrac{\operatorname{diam}(\Omega)}{|x-x_0|}\right)^\alpha - 1}{\left(\dfrac{\operatorname{diam}(\Omega)}{|x-x_0|} - 1\right)^\alpha} < 1.$$

Consequently, we get that

$$L\Psi_{x_0}(x) \leq \frac{\left(\dfrac{\operatorname{diam}(\Omega)}{|x-x_0|}\right)^\alpha - 1}{\left(\dfrac{\operatorname{diam}(\Omega)}{|x-x_0|} - 1\right)^\alpha} - 1 < 0.$$

Now, we define $\phi(x) = u(x_0) + C|x-x_0|^\alpha$. We have $\phi \in C^1(\Omega \backslash \{x_0\}) \cap C(\bar{\Omega})$. Moreover,

$$L_\infty \phi(x) = CL\Psi_{x_0}(x) < 0.$$

In particular, one has

$$\max\{-L_\infty\phi(x), -L^+_\infty\phi(x) + h(\phi(x))\} > 0.$$

If we choose the constant C large enough, one can get that $u \leq \phi$ on $\{u = \psi\} \cup \partial\Omega \cup \{x_0\}$. Indeed, for every $x \in \{u = \psi\} \cup \partial\Omega$, we have

$$u(x) - \phi(x) = u(x) - u(x_0) - C|x-x_0|^\alpha \leq 2||u||_\infty - C\operatorname{dist}(x_0, \{u=\psi\} \cup \partial\Omega)^\alpha \leq 0$$

as soon as

$$C \geq \frac{2||u||_\infty}{\operatorname{dist}(x_0, \{u = \psi\} \cup \partial\Omega)^\alpha}.$$

Thanks to the comparison principle 5.1 and since u is a viscosity solution while ϕ is a strict viscosity supersolution in $\Omega\backslash\{x_0\} \cap \{u > \psi\}$, this implies that $u < \phi$ in $\Omega\backslash\{x_0\}$. Consequently,

$$u(x) \leq u(x_0) + C|x-x_0|^\alpha.$$

Finally, assume $\alpha = 1$. Fix $\varepsilon > 0$ small enough. Then, we define $\Psi_{x_0}(x) = |x-x_0| - \varepsilon|x-x_0|^2$. Again, we have

$$\begin{aligned} L^-\Psi_{x_0}(x) = \inf_{y\in\bar{\Omega},\, y\neq x} \frac{\Psi_{x_0}(y) - \Psi_{x_0}(x)}{|y-x|} &= \inf_{y\in\bar{\Omega},\, y\neq x} \frac{|y-x_0| - \varepsilon|y-x_0|^2 - |x-x_0| + \varepsilon|x-x_0|^2}{|y-x|} \\ &\leq -1 + \varepsilon|x-x_0|. \end{aligned}$$

Moreover,

$$\begin{aligned} L^+\Psi_{x_0}(x) = \sup_{y\in\bar{\Omega},\, y\neq x} \frac{\Psi_{x_0}(y) - \Psi_{x_0}(x)}{|y-x|} &\leq \sup_{y\in\bar{\Omega},\, y\neq x} \frac{|y-x_0| - \varepsilon|y-x_0|^2 - |x-x_0| + \varepsilon|x-x_0|^2}{||y-x_0| - |x-x_0||} \\ &= 1 - \varepsilon|x-x_0| - \varepsilon \min_{|y-x_0|>|x-x_0|} |y-x_0| = 1 - 2\varepsilon|x-x_0|. \end{aligned}$$

Hence, we get

$$L\Psi_{x_0}(x) \leq -\varepsilon|x-x_0| < 0, \quad \text{for all } x \in \Omega\backslash\{x_0\}.$$

Now, set $\phi(x) = u(x_0) + C[|x - x_0| - \varepsilon|x - x_0|^2]$. So, we have $L\phi(x) = CL\Psi_{x_0}(x) < 0$, for every $x \in \Omega\backslash\{x_0\}$. In addition, one has

$$u(x) - \phi(x) = u(x) - u(x_0) - C[|x - x_0| - \varepsilon|x - x_0|^2]$$
$$\leq 2||u||_\infty - C\mathrm{dist}(w, \{u = \psi\} \cup \partial\Omega) + C\varepsilon\mathrm{diam}(\Omega)^2 \leq 0$$

as soon as

$$C \geq \frac{2||u||_\infty}{\mathrm{dist}(w, \{u = \psi\} \cup \partial\Omega) - \varepsilon\mathrm{diam}(\Omega)^2}.$$

Then,

$$|u(x) - u(x_0)| \leq \frac{2||u||_\infty}{\mathrm{dist}(w, \{u = \psi\} \cup \partial\Omega) - \varepsilon\mathrm{diam}(\Omega)^2}[|x - x_0| - \varepsilon|x - x_0|^2].$$

Letting $\varepsilon \to 0$, we get

$$|u(x) - u(x_0)| \leq \frac{2||u||_\infty}{\mathrm{dist}(w, \{u = \psi\} \cup \partial\Omega)}|x - x_0|.$$

□

Proposition 5.5. *Assume $\psi \in C^{0,\alpha}(\bar{\Omega})$. Then, any viscosity solution u of (5.1) belongs to $C^{0,\alpha}(\bar{\Omega})$. Moreover, we have*

$$[u]_\alpha \leq C(||g||_\infty, ||\psi||_\infty, [g]_\alpha, [\psi]_\alpha).$$

Proof. Fix $x_0 \in \partial\{u > \psi\}\backslash\partial\Omega$. Set $\phi(x) = \psi(x_0) + C|x - x_0|^\alpha$. From Proposition 5.4, we recall that ϕ is a strict viscosity supersolution in $\{u > \psi\}$. Since $\psi \in C^{0,\alpha}(\bar{\Omega})$, we have

$$u(x) - \phi(x) = \psi(x) - \psi(x_0) - C|x - x_0|^\alpha \leq 0, \qquad \text{for all } x \in \{u = \psi\}$$

and

$$u(x) - \phi(x) = g(x) - \psi(x_0) - C|x - x_0|^\alpha \leq ||g||_\infty + ||\psi||_\infty - C\,\mathrm{dist}(x_0, \partial\Omega)^\alpha \leq 0, \quad \text{for all } x \in \partial\Omega,$$

as soon as

$$C \geq \frac{||g||_\infty + ||\psi||_\infty}{\mathrm{dist}(x_0, \partial\Omega)^\alpha}.$$

Hence, by Proposition 5.1, we infer that $u \leq \phi$ in $\{u > \psi\}$. Thanks to Proposition 5.4, this implies that $u \in C^{0,\alpha}_{loc}(\Omega)$.

Now, fix $x_0 \in \partial\Omega$. Again, we define $\phi(x) = g(x_0) + C|x - x_0|^\alpha$. So, ϕ is a strict viscosity supersolution in $\{u > \psi\}$. Moreover, one has

$$u(x) - \phi(x) = g(x) - g(x_0) - C|x - x_0|^\alpha \leq 0, \qquad \text{for all } x \in \partial\Omega$$

and

$$u(x) - \phi(x) = \psi(x) - g(x_0) - C|x - x_0|^\alpha \leq 0, \quad \text{for all } x \in \{u = \psi\},$$

provided that $C \geq \max\{[g]_\alpha, M\}$; we note that in the last inequality we have used that $\psi(x) \leq \min\{M|x - x_0|^\alpha + g(x_0) : x_0 \in \partial\Omega\}$. Consequently, $u \in C^{0,\alpha}(\Omega)$. □

We conclude the paper by the following existence result in the case when h is not smooth.

Proposition 5.6. *Assume h is a nonnegative increasing continuous function on $\mathbb{R}$. Then, Problem (5.1) has a solution.*

Proof. Let h_n be a sequence of smooth functions such that $h_n' > 0$ and $h_n \to h$ locally uniformly on $\mathbb{R}$. For every $n \in \mathbb{N}$, let u_n be a solution to Problem (5.1). Thanks to Proposition 5.5, we have

$$[u_n]_\alpha \leq C(||g||_\infty, ||\psi||_\infty, [g]_\alpha, [\psi]_\alpha).$$

Moreover,

$$||u_n||_\infty \leq \max\{||g||_\infty, ||\psi||_\infty\}.$$

Hence, up to a subsequence, $u_n \to u$ uniformly in $\bar{\Omega}$. In particular, we have $u = g$ on $\partial\Omega$ and $u \geq \psi$ on $\bar{\Omega}$. Now, let us show that u is a viscosity solution of (5.1). First, we show that u is a viscosity subsolution in $\{u > \psi\}$. Fix $x_0 \in \{u > \psi\}$ and let $\varphi \in C^1(\Omega) \cap C(\bar{\Omega})$ be such that $u \leq \varphi$ on $\bar{\Omega}$ and $u(x_0) = \varphi(x_0)$, we recall that one can assume x_0 to be the unique maximizer of $u - \varphi$. For every n, let x_n be a maximizer of $u_n - \varphi$ and set $M_n := \max_{\bar{\Omega}}[u_n - \varphi]$. Then, $x_n \to x_0$ and $M_n \to 0$. Since u_n is a viscosity solution, then one has

$$\max\{-L_\infty\varphi(x_n), -L_\infty^+\varphi(x_n) + h_n(\varphi(x_n) + M_n)\} \leq 0.$$

But, $L_\infty\varphi \in C(\Omega)$ since $\varphi \in C^1(\Omega)$ (see [1, Lemma 3.5]). Passing to the limit when $n \to +\infty$, we get

$$\max\{-L_\infty\varphi(x_0), -L_\infty^+\varphi(x_0) + h(\varphi(x_0))\} \leq 0.$$

In the same way, we show that u is a viscosity supersolution in Ω. This concludes the proof that u is a viscosity solution. □

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