THE LIMIT OF A NONLOCAL $p-$LAPLACIAN OBSTACLE PROBLEM WITH NONHOMOGENEOUS TERM AS $p \to \infty$

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ABSTRACT. The aim of this paper is to investigate the asymptotic behavior of the minimizers to the following problems related to the fractional $p-$Laplacian with nonhomogeneous term $h_p(x,u)$ in the presence of an obstacle ψ in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$,

$$\min\left\{\frac{1}{2}\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(x,u)\ :\ u \in W^{s,p}(\Omega),\ u \geq \psi \text{ on } \bar{\Omega},\ u = g \text{ on } \partial\Omega\right\}.$$

In the case when $h_p(x,u) = \frac{h(x,u)^p}{p}$ and $h(x,u) \geq 0$, we show the convergence of the solutions to certain limit as $p \to \infty$ and identify the limit equation. More precisely, we show that the limit problem is closely related to the infinity fractional Laplacian. In the particular case when $\partial_s h > 0$, we study the Hölder regularity of any solution to the limit problem and we extend the existence result to the case when h is not smooth.

In addition, we will study the limit of this problem when the nonhomogeneous term $h_p(x,u)$ is not necessarily positive. To be more precise, we will consider the following two cases: $h_p(x,u) = h(x)\,u$ and $h_p(x,u) = h(x)\,\frac{|u|^\Lambda}{\Lambda}$ with $\Lambda := \Lambda(p) < p$.

1. INTRODUCTION

Let Ω be a bounded open set in $\mathbb{R}^N$ and g be a $\alpha-$Hölder boundary datum on $\partial\Omega$. From [1], it is well known that if u_p minimizes the functional

$$E_p[u] := \int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}}\,\mathrm{d}x\,\mathrm{d}y$$

among all functions u in the fractional Sobolev space $W^{s,p}(\Omega)$ such that $u_p = g$ on $\partial\Omega$ (with $s = \alpha - \frac{N}{p}$), then $u_p \to u$ as $p \to \infty$ where the limit function u solves the following equation (which is usually referred to as the infinity fractional Laplacian):

$$L_\infty u := L_\infty^+ u + L_\infty^- u = 0,$$

where

$$L_\infty^+ u = \sup_{y\in\Omega,\ y\neq x}\frac{u(y)-u(x)}{|y-x|^\alpha} \qquad \text{and} \qquad L_\infty^- u = \inf_{y\in\Omega,\ y\neq x}\frac{u(y)-u(x)}{|y-x|^\alpha}.$$

In fact, one of the most important motivations to analyse this kind of problems is the $\alpha-$Hölder extension of the function $g \in C^{0,\alpha}(\partial\Omega)$. In fact, one can show that the limit function u is the optimal Hölder extension to $\bar{\Omega}$ of the boundary datum g, i.e. the Hölder seminorm for u in Ω is always less than or equal to the one for the boundary datum given on $\partial\Omega$.

Given a continuous obstacle ψ, the authors in [9] follow the work in [1] and prove existence of a fractional harmonic function constrained to lie above the obstacle and to take the datum on $\partial\Omega$. More precisely, they show that the following obstacle problem has a viscosity solution:

$$\begin{cases} L_\infty u = 0 & \text{in } \{x \in \Omega : \, u(x) > \psi(x)\}, \\ L_\infty u \le 0 & \text{in } \{x \in \Omega : \, u(x) = \psi(x)\}, \\ u(x) \ge \psi(x) & \text{if } x \in \Omega, \\ u(x) = g(x) & \text{if } x \in \partial\Omega. \end{cases} \tag{1.1}$$

In order to have a solution for this problem (1.1), it is necessary that $\psi(x) \le g(x)$, for all $x \in \partial\Omega$. So, in the sequel, we will assume the following natural condition on the obstacle ψ:

$$\psi \le g \text{ on } \partial\Omega.$$

The idea in [9] follows exactly the one in [1], where the authors approximate Problem (1.1) with a sequence of fractional $p-$Laplacian operators. To be more precise, they consider the following minimization problem:

$$\min\left\{ E_p[u] : u \in W^{s,p}(\Omega), \; u \ge \psi \text{ in } \bar{\Omega}, \;\; u = g \text{ on } \partial\Omega \right\}. \tag{1.2}$$

But, it is not difficult to check that the Euler-Lagrange equation associated to this functional is

$$\begin{cases} L_p u_p = 0 & \text{in } \{u_p > \psi\}, \\ L_p u_p \le 0 & \text{in } \{u_p = \psi\}, \end{cases} \tag{1.3}$$

where

$$L_p u := \int_\Omega \left(\frac{|u(x) - u(y)|}{|x-y|^\alpha} \right)^{p-1} \frac{1}{|x-y|^\alpha} \frac{u(y) - u(x)}{|u(y) - u(x)|} \, \mathrm{d}y.$$

Let us denote by L_p^+ and L_p^- the positive and negative parts of L_p, respectively. So, one has

$$L_p^+ u_p = L_p^- u_p \qquad \text{in } \{u_p > \psi\}.$$

Hence,

$$\left(\int_\Omega \left(\frac{[u_p(x) - u_p(y)]_+}{|x-y|^\alpha} \right)^{p-1} \frac{1}{|x-y|^\alpha} \, \mathrm{d}y \right)^{\frac{1}{p-1}} = \left(\int_\Omega \left(\frac{[u_p(x) - u_p(y)]_-}{|x-y|^\alpha} \right)^{p-1} \frac{1}{|x-y|^\alpha} \, \mathrm{d}y \right)^{\frac{1}{p-1}},$$

where $[z]_\pm := \max\{\pm z, 0\}$. Letting p goes to ∞, we may show that up to a subsequence $u_p \to u$. Formally, we get that

$$L_\infty^+ u = -L_\infty^- u$$

and so, $L_\infty u = 0$ in $\{u > \psi\}$. We note that this limit procedure only works when the right hand side in (1.3) is zero.

In this paper, we consider the minimization problem (1.2) but in the presence of an extra nonhomogeneous term:

$$\min\left\{ \frac{E_p[u]}{2p} + \int_\Omega h_p(x,u) \, : \, u \in W^{s,p}(\Omega), \; u \ge \psi \text{ on } \bar{\Omega}, \; u = g \text{ on } \partial\Omega \right\}. \tag{1.4}$$

The main goal of this paper is to study the limit as $p \to \infty$ of the minimizers u_p to (1.4), prove their convergence up to a subsequence to a function u, and to identify the limit problem for

u. Assume $h \geq 0$ and $h_p = \frac{h^p}{p}$, we may assume that the limit function u solves the following problem:

$$\begin{cases} L_\infty u = h(x,u) & \text{in } \{u > \psi\}, \\ L_\infty u \leq h(x,u) & \text{in } \{u = \psi\}. \end{cases} \tag{1.5}$$

However, we will see in Section 3 that this is not the case and the limit equation is completely different, so the presence of the nonhomogeneous term makes the analysis of our problem more delicate. We note that this will also depends on the monotonicity of h. In Section 4, we will study the limit of (1.4) in the linear case, i.e. when $h_p(x,u) = h(x)u$, and show that the limit of (1.4) is equivalent to an optimal transport problem with import/export taxes. In Section 5, we will also consider the superlinear case, i.e. when $h_p(x,u) = h(x)\frac{|u|^\Lambda}{\Lambda}$ where $1 < \Lambda = \Lambda(p) < p$.

In [2], the authors characterize the limit as $p \to \infty$ of the branches of solutions to the local $p-$Laplacian:

$$-\nabla \cdot [|\nabla u|^{p-2}\nabla u] = \lambda\, u^{\gamma(p)}, \qquad u > 0,$$

with $\lambda > 0$ and $\lim_{p\to\infty} \frac{\gamma(p)}{p-1} = \gamma_\star < 1$. They show that the limit set is a curve of positive viscosity solutions of the equation

$$\min\{-\Delta_\infty u, |\nabla u| - Cu^{\gamma_\star}\} = 0,$$

where $\Delta_\infty u := D^2 u\, \nabla u \cdot \nabla u$ is the infinity Laplacian operator and $C > 0$. On the other hand, in [7], the problem of minimizing the fractional Rayleigh quotient has been considered

$$\min \left\{ \frac{\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}}}{\int_{\mathbb{R}^N} u^p} \;:\; u \in W^{s,p}(\Omega), \quad u = 0 \text{ on } \partial\Omega \right\}. \tag{1.6}$$

This problem leads to an interesting eigenvalue problem with the non-local Euler-Lagrange equation:

$$-\mathcal{L}_p u = \lambda |u|^{p-2} u,$$

where the operator $\mathcal{L}_p$ is defined exactly as L_p but with integration set $\mathbb{R}^N$ instead of Ω. The limit equation takes the form

$$\max\{\mathcal{L}_\infty u, \mathcal{L}_\infty^- u + \lambda u\} = 0 \qquad \text{in } \Omega.$$

In addition, an equivalent nonlocal version for the fractional $p-$Laplacian was studied in [5], where the authors were interested in describing the behaviour of the solutions to the following Dirichlet problem as $p \to \infty$:

$$\begin{cases} -\mathcal{L}_p u = |u|^{\gamma(p)-1} u & \text{in } \Omega, \\ u = g & \text{on } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.7}$$

Inspired by [7], the authors of [5] prove that the limit problem of (1.7) is the following problem:

$$\begin{cases} \min\{-\mathcal{L}_\infty u, -\mathcal{L}_\infty^- u - |u|^{\gamma_\star}\} = 0 & \text{in } \Omega, \\ u = g & \text{on } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.8}$$

2. Preliminaries

In order to study the minimization problem (1.4), we recall some basic theory of fractional Sobolev spaces. Assume Ω is a Lipschitz domain. Then, we define the fractional Sobolev space $W^{s,p}(\Omega)$ with $0 < s < 1$ and $1 < p < \infty$ as follows:

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega),\ [u]^p_{s,p} := \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} < \infty \right\}.$$

We may see $W^{s,p}(\Omega)$ as an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$||u||_{W^{s,p}(\Omega)} = \left[||u||^p_p + [u]^p_{s,p} \right]^{\frac{1}{p}}.$$

In order to obtain a Poincaré inequality in $W^{s,p}_0(\Omega)$ (where the space $W^{s,p}_0(\Omega)$ is defined as the closure of $C^\infty_0(\Omega)$ with respect to the norm $||\cdot||_{W^{s,p}(\Omega)}$) valid for p large, we consider again the fractional Rayleigh quotient:

$$\lambda_p = \min\,(1.6).$$

In [7], the authors show that

$$(\lambda_p)^{\frac{1}{p}} \to \frac{1}{R^\alpha},$$

with $R = \max\{\mathrm{dist}(x, \partial\Omega) : x \in \Omega\}$ being the radius of the largest ball inscribed in Ω. As a consequence, we have

$$||u||_{L^p(\Omega)} \le C(R,\alpha) \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} \right)^{\frac{1}{p}}.$$

But,

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} = \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} + 2\iint_{\Omega\times(\mathbb{R}^N\setminus\Omega)} \frac{|u(x)|^p}{|x-y|^{\alpha p}}.$$

However,

$$\iint_{\Omega\times(\mathbb{R}^N\setminus\Omega)} \frac{|u(x)|^p}{|x-y|^{\alpha p}} = \int_\Omega |u(x)|^p \left(\int_{\{z:\, x+z\in\mathbb{R}^N\setminus\Omega\}} \frac{1}{|z|^{\alpha p}} \mathrm{d}z \right) \mathrm{d}x.$$

Using polar coordinates, one has

$$\int_{\{z:\, x+z\in\mathbb{R}^N\setminus\Omega\}} \frac{1}{|z|^{\alpha p}} \mathrm{d}z = \int_{\mathbb{S}^{N-1}} \int_{\{r>0:\, x+rw\notin\Omega\}} \frac{1}{r^{\alpha p - N+1}} \,\mathrm{d}r\,\mathrm{d}w.$$

For $w \in \mathbb{S}^{N-1}$, we define

$$d_{w,\Omega}(x) := \inf\{r > 0 : x + rw \notin \Omega\}.$$

Hence, we have

$$\int_{\{z:\, x+z\in\mathbb{R}^N\setminus\Omega\}} \frac{1}{|z|^{\alpha p}} \mathrm{d}z \le \int_{\mathbb{S}^{N-1}} \int_{d_{w,\Omega}(x)}^\infty \frac{1}{r^{\alpha p-N+1}} \,\mathrm{d}r\,\mathrm{d}w = \frac{1}{\alpha p - N} \int_{\mathbb{S}^{N-1}} \frac{1}{d_{w,\Omega}(x)^{\alpha p - N}} \,\mathrm{d}w.$$

Thus, we get

$$\iint_{\Omega\times(\mathbb{R}^N\setminus\Omega)} \frac{|u(x)|^p}{|x-y|^{\alpha p}} \le \frac{1}{\alpha p - N} \int_\Omega |u(x)|^p \left(\int_{\mathbb{S}^{N-1}} \frac{1}{d_{w,\Omega}(x)^{\alpha p-N}} \,\mathrm{d}w \right) \mathrm{d}x.$$

Thanks to [8, Theorem 1.2], if $sp > 1$ then we have the following fractional Hardy-type inequality:

$$\int_\Omega |u(x)|^p \left(\int_{\mathbb{S}^{N-1}} \frac{1}{d_{w,\Omega}(x)^{\alpha p - N}} \, \mathrm{d}w \right) \mathrm{d}x \le C(N,p,\alpha) \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}}.$$

Finally, this yields that

$$||u||_{L^p(\Omega)} \le C \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} \right)^{\frac{1}{p}}, \qquad \text{for all } \; u \in W_0^{s,p}(\Omega).$$

On the other hand, one can show certain regularity properties for functions in $W^{s,p}(\Omega)$ when $sp > N$. From [10, Theorem 8.2], there exists a constant $C < \infty$ depending only on s, p, N such that

$$||u||_{C^{0,\beta}(\bar{\Omega})} \le C[u]_{s,p}, \qquad \text{for all } \; u \in W_0^{s,p}(\Omega), \tag{2.1}$$

where $\beta = s - \frac{N}{p}$ and

$$||u||_{C^{0,\beta}(\bar{\Omega})} = ||u||_{L^\infty(\Omega)} + \sup\left\{ \frac{|u(x)-u(y)|}{|x-y|^\beta} \, : \, x,\, y \in \bar{\Omega},\, x \neq y \right\}.$$

Since we are interested in what happens when $p \to \infty$, we want to diminish the dependence on p. Thus, it is useful to note that the constant C can be chosen independently of p such that the following inequality holds:

$$||u||_{L^\infty(\Omega)} \le C[u]_{s,p}, \qquad \text{for all } \; u \in W_0^{s,p}(\Omega). \tag{2.2}$$

3. The case of a nonnegative nonhomogeneous term

3.1. **Existence of solutions to the fractional $p-$Laplacian problem.** Let $h_p : \Omega \times \mathbb{R} \mapsto \mathbb{R}^+$ be a nonnegative continuous in (x,s) and C^1 function with respect to the second variable s and set $f_p = \partial_s h_p$. We consider the minimization problem:

$$\min\left\{ \frac{1}{2p} \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(x,u) \, : \, u \in W^{s,p}(\Omega),\, u \ge \psi \text{ on } \bar{\Omega},\, u = g \text{ on } \partial\Omega \right\}, \tag{3.1}$$

where $\alpha p = sp + N$. Assume that there is an extension $\tilde{g} \in W^{s,p}(\Omega)$ such that $\tilde{g} = g$ on $\partial\Omega$. For simplicity of notation, we will simply call it g instead of $\tilde{g}$.

Proposition 3.1. *Assume $\alpha > \frac{2N}{p}$. Then, there exists a minimizer u_p for Problem (3.1). Moreover, u_p is a weak solution to the following problem:*

$$\begin{cases} L_p u = f_p(x,u) & \text{in } \; \{u > \psi\}, \\ L_p u \le f_p(x,u) & \text{in } \; \{u = \psi\}, \end{cases}$$

where

$$L_p u = \int_\Omega \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{\alpha p}} [u(y) - u(x)] \, \mathrm{d}y.$$

Proof. Let $(u_n)_n$ be a minimizing sequence in Problem (3.1). So, there will be a constant $C < \infty$ such that

$$\frac{1}{2p} \iint_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(x,u_n) \le C, \qquad \text{for all } \; n.$$

Since $h_p \geq 0$, this implies that

$$\iint_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{\alpha p}} \leq C.$$

But, we have

$$||u_n - g||_{C^{0,\beta}(\bar{\Omega})} \leq C[u_n - g]_{s,p} \leq C([u_n]_{s,p} + [g]_{s,p}).$$

This yields that $(u_n)_n$ is bounded in $W^{s,p}(\Omega)$ and so, up to a subsequence, $u_n \rightharpoonup u_p$ in $W^{s,p}(\Omega)$ and so, $u_n \to u_p$ uniformly in $C^{0,\beta}(\bar{\Omega})$ with $\beta = \alpha - \frac{2N}{p}$. By Fatou's Lemma, this yields that

$$\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}} = \iint_{\Omega\times\Omega} \liminf_n \left[\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{\alpha p}}\right] \leq \liminf_n \iint_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{\alpha p}}$$

and

$$\int_\Omega h_p(x,u_p) \leq \liminf_n \int_\Omega h_p(x,u_n).$$

So, we get that

$$\frac{1}{2p}\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(x,u_p) \leq \liminf_n \left[\frac{1}{2p}\iint_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(x,u_n)\right].$$

Yet, $u_p \geq \psi$ on $\bar{\Omega}$ and $u_p = g$ on $\partial\Omega$. Hence, u_p minimizes (3.1). Now, we show the second part. Let ϕ be a smooth function such that $\mathrm{supp}(\phi) \subset \{u_p > \psi\}$. Thanks to the continuity of u_p, it is clear that $u_p + t\phi$ is admissible in (3.1), for all $t \in \mathbb{R}$ small enough. From the minimality of u_p, we have

$$\frac{1}{2p}\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(x,u_p)$$
$$\leq \frac{1}{2p}\iint_{\Omega\times\Omega} \frac{|u_p(x)+t\phi(x)-u_p(y)-t\phi(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(x,u_p+t\phi) := \mathcal{J}_\phi(t).$$

So, $\mathcal{J}_\phi$ has a minimum at $t=0$. Therefore, we have $\mathcal{J}'_\phi(0) = 0$ and so, we get the following:

$$\frac{1}{2}\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^{p-1}}{|x-y|^{\alpha p}} \frac{[u_p(x)-u_p(y)]}{|u_p(x)-u_p(y)|}[\phi(x)-\phi(y)] + \int_\Omega f_p(x,u_p)\,\phi = 0.$$

By symmetry, this yields that

$$-\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^{p-2}}{|x-y|^{\alpha p}}[u_p(y)-u_p(x)]\,\phi(x) + \int_\Omega f_p(x,u_p)\,\phi = 0.$$

Finally, we note that for every $\phi \in C_0^\infty(\Omega)$ such that $\phi \geq 0$, the function $u_p + t\phi$ is admissible in (3.1), for all $t \in \mathbb{R}^+$. Hence, $\mathcal{J}'_\phi(0) \geq 0$ and so, one has

$$-\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^{p-2}}{|x-y|^{\alpha p}}[u_p(y)-u_p(x)]\,\phi(x) + \int_\Omega f_p(x,u_p)\,\phi \geq 0.$$

Then,

$$-\int_\Omega \frac{|u_p(x)-u_p(y)|^{p-2}}{|x-y|^{\alpha p}}[u_p(y)-u_p(x)]\,\mathrm{d}y + f_p(x,u_p) \geq 0 \qquad \text{in } \{u_p = \psi\}. \quad \square$$

The solutions in the previous Proposition 3.1 were defined as weak solutions to the Euler-Lagrange equation in the usual way with test functions under the integral sign. In the sequel, we will see that they are also viscosity solutions of the equation

$$L_p u = f_p(x,u) \tag{3.2}$$

inside the noncoincidence set $\{u_p > \psi\}$ while it is a viscosity supersolution in the coincidence set $\{u_p = \psi\}$. We refer the reader to the book [6] for an introduction to the theory of viscosity solutions. Here, we give the definition of a viscosity supersolution (resp. subsolution).

Definition 3.1. *We will say that u is a viscosity supersolution in Ω of the equation (3.2) if the following holds: whenever $x_0 \in \Omega$ and $\phi \in C^1(\Omega) \cap C(\bar{\Omega})$ are such that*

$$\phi(x_0) = u(x_0) \ \text{ and } \ \phi(x) \le u(x) \ \text{ for all } x \in \bar{\Omega},$$

then we have

$$\min\{-L_p\phi(x_0) + f_p(x_0, \phi(x_0)), \phi(x_0) - \psi(x_0)\} \ge 0.$$

The requirement for a viscosity subsolution is symmetric: the test function is touching from above and the inequality is reversed. Finally, a viscosity solution is defined as being both a viscosity supersolution and a viscosity subsolution.

In order to prove that weak solutions are viscosity solutions we need the following comparison principle (the proof follows in an analogous way the one in [7]):

Proposition 3.2. *Let u and v be two continuous functions belonging to $W^{s,p}(\Omega)$. Assume that $-L_p u < -L_p v$ in the weak sense on $B \subset \Omega$. If $u \le v$ on $\bar{\Omega}\backslash B$, then $u \le v$ in Ω.*

Proof. Assume $[u-v]_+ \neq 0$ on B. Since $[u-v]_+ = 0$ on $\bar{\Omega}\backslash B$, then $[u-v]_+ \in W_0^{s,p}(\Omega)$ and so, one has

$$\iint_{B\times\Omega} \frac{|v(x)-v(y)|^{p-2}}{|x-y|^{\alpha p}}[v(x)-v(y)][u-v]_+(x) > \iint_{B\times\Omega} \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{\alpha p}}[u(x)-u(y)][u-v]_+(x).$$

Hence,

$$\begin{aligned}&\iint_{B\times\Omega} \frac{|v(x)-v(y)|^{p-2}}{|x-y|^{\alpha p}}[v(x)-v(y)]([u-v]_+(x) - [u-v]_+(y)) \\ &> \iint_{B\times\Omega} \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{\alpha p}}[u(x)-u(y)]\left([u-v]_+(x) - [u-v]_+(y)\right)\end{aligned}$$

Then, we get

$$\frac{1}{2}\iint_{B\times\Omega} \frac{1}{|x-y|^{\alpha p}}\,\Phi_1(x,y)\left([u-v]_+(x) - [u-v]_+(y)\right) > 0 \tag{3.3}$$

where

$$\Phi_1(x,y) = \Big[|v(x)-v(y)|^{p-2}[v(x)-v(y)] - |u(x)-u(y)|^{p-2}[u(x)-u(y)]\Big].$$

For a, $b \in \mathbb{R}$, one has

$$|b|^{p-2}b - |a|^{p-2}a = \int_0^1 \frac{d}{dt}[|a+t(b-a)|^{p-2}(a+t(b-a))] = (p-1)\left(\int_0^1 |a+t(b-a)|^{p-2}\,\mathrm{d}t\right)[b-a].$$

Then,

$$\Phi_1(x,y)$$

$$= (p-1)\left(\int_0^1 |u(x)-u(y)+t(v(x)-v(y)-u(x)+u(y))|^{p-2}\,\mathrm{d}t\right)[v(x)-v(y)-u(x)+u(y)].$$

So, we get

$$\begin{aligned}&\Phi_1(x,y)[[u-v]_+(x)-[u-v]_+(y)]\\ &= \Phi_2(x,y)[v(x)-v(y)-u(x)+u(y)][[u-v]_+(x)-[u-v]_+(y)]\end{aligned}$$

where

$$\Phi_2(x,y) = (p-1)\left(\int_0^1 |u(x)-u(y)+t(v(x)-v(y)-u(x)+u(y))|^{p-2}\,\mathrm{d}t\right) \geq 0.$$

But,

$$\begin{aligned}&[v(x)-v(y)-u(x)+u(y)][[u-v]_+(x)-[u-v]_+(y)]\\ &= -[u(x)-v(x)-(u(y)-v(y))][[u-v]_+(x)-[u-v]_+(y)]\\ &= -\Big[[u-v]_+^2(x)+[u-v]_+^2(y)-\Phi_3(x,y)\Big],\end{aligned}$$

where

$$\Phi_3(x,y) = [u(x)-v(x)][u(y)-v(y)]_+ + [u(y)-v(y)][u(x)-v(x)]_+.$$

For simplicity of notation, we set $s_\pm := [u(x)-v(x)]_\pm$ and $t_\pm := [u(y)-v(y)]_\pm$. Then, we have

$$\Phi_3(x,y) = [s_+-s_-]t_+ + [t_+-t_-]s_+ = 2s_+t_+ - s_-t_+ - t_-s_+.$$

Hence,

$$\begin{aligned}&[v(x)-v(y)-u(x)+u(y)][[u-v]_+(x)-[u-v]_+(y)]\\ &= -[s_+^2+t_+^2-2s_+t_+ + s_-t_+ + t_-s_+] = -[(s_+-t_+)^2 + s_-t_+ + t_-s_+] \leq 0.\end{aligned}$$

Thus, we get that

$$\Phi_1(x,y)[[u-v]_+(x)-[u-v]_+(y)] \leq 0.$$

Finally, we infer that

$$\iint_{B\times\Omega} \frac{1}{|x-y|^{\alpha p}}\,\Phi_1(x,y)\,([u-v]_+(x)-[u-v]_+(y)) \leq 0,$$

which is in contradiction with the strict inequality in (3.3). Hence, $[u-v]_+ = 0$ and so, $u \leq v$ on B. □

Proposition 3.3. *Assume $\alpha \leq 1-\frac{1}{p}$. The weak solution u_p of Problem (3.1) is a viscosity solution to the equation:*

$$L_p u = f_p(x,u) \qquad \text{in } \{u>\psi\}. \tag{3.4}$$

In addition, u_p is a viscosity supersolution to the equation (3.2) on the coincidence set $\mathcal{S} := \{x\in\Omega : u(x)=\psi(x)\}$.

Proof. Assume u_p is not a viscosity subsolution in $\{u_p>\psi\}$, i.e. there is a point $x_0\in\{u_p>\psi\}$ and a test function $\phi\in C^1(\Omega)\cap C(\overline{\Omega})$ such that $u_p\leq\phi$ on $\bar\Omega$, $\phi(x_0)=u_p(x_0)$ and

$$L_p\phi(x_0) - f_p(x_0,\phi(x_0)) < 0.$$

Thanks to our assumption that $\alpha\leq 1-\frac{1}{p}$, it is easy to see that $x\mapsto L_p\phi(x)$ is continuous on Ω. Hence, there is a $r>0$ small enough such that

$$L_p\phi(x) - f_p(x_0,\phi(x_0)) < 0 \qquad \text{on } B(x_0,r).$$

Let η be a smooth cutoff function such that $\eta(x_0) = 1$ and $\eta = 0$ on $\Omega\backslash B(x_0, r)$. Then, we define

$$\phi_\varepsilon := \phi - \varepsilon\eta.$$

Clearly, $\phi_\varepsilon = \phi$ on $\Omega\backslash B(x_0, r)$. Moreover, one has

$$|\phi_\varepsilon(x)-\phi_\varepsilon(y)|^{p-2}[\phi_\varepsilon(x)-\phi_\varepsilon(y)] = |\phi(x)-\phi(y)-\varepsilon[\eta(x)-\eta(y)]|^{p-2}[\phi(x)-\phi(y)-\varepsilon[\eta(x)-\eta(y)]].$$

Yet,

$$\begin{aligned} &\Big||\phi_\varepsilon(x) - \phi_\varepsilon(y)|^{p-2}[\phi_\varepsilon(x) - \phi_\varepsilon(y)] - |\phi(x) - \phi(y)|^{p-2}[\phi(x) - \phi(y)]\Big| \\ &= \left|(p-1)\bigg(\int_0^1 |\phi(x) - \phi(y) - \varepsilon t[\eta(x) - \eta(y)]|^{p-2}\,\mathrm{d}t\bigg)[-\varepsilon[\eta(x) - \eta(y)]]\right| \\ &\leq C\varepsilon|x-y|^{p-1}. \end{aligned}$$

Then, we get

$$|L_p\phi_\varepsilon(x) - L_p\phi(x)| \leq C\varepsilon.$$

We recall that u_p and f are both continuous. For $\varepsilon > 0$ small enough, we have then

$$L_p\phi_\varepsilon(x) - f_p(x, u_p(x)) < 0 \qquad \text{on} \quad B(x_0, r).$$

But, $\phi_\varepsilon = \phi \geq u_p$ on $\Omega\backslash B(x_0, r)$. By Proposition 3.2, we infer that $u_p \leq \phi_\varepsilon$ in $B(x_0, r)$. In particular, $u_p(x_0) = \phi(x_0) \leq \phi_\varepsilon(x_0) = \phi(x_0) - \varepsilon$, which is a contradiction. This concludes the proof that u_p is a viscosity subsolution in $\{u_p > \psi\}$.

The proof that u_p is a viscosity supersolution in Ω is similar and so, we omit some details. Assume by contradiction that there is a point $x_0 \in \Omega$ and a test function $\phi \in C^1(\Omega) \cap C(\bar{\Omega})$ such that $\phi \leq u_p$ on $\bar{\Omega}$ with equality at x_0 and

$$L_p\phi(x_0) - f_p(x_0, \phi(x_0)) > 0.$$

Now, set $\phi_\varepsilon := \phi + \varepsilon\eta$, where η is always a cutoff function such that $\eta(x_0) = 1$ and $\eta = 0$ outside $B(x_0, r)$. Then, we have $\phi_\varepsilon = \phi$ on $\Omega\backslash B(x_0, r)$. In addition, one can show as before that for every $\varepsilon > 0$ small enough,

$$L_p\phi_\varepsilon(x) - f_p(x, u_p(x)) > 0 \qquad \text{on} \quad B(x_0, r).$$

Again, by Proposition 3.2, we infer that $u_p \geq \phi_\varepsilon$ in $B(x_0, r)$, which is a contradiction. □

3.2. **The limit problem as $p \to \infty$.** In this section, we show that up to a subsequence the solutions u_p to (1.4) converge uniformly to a function u as p goes to infinity. Moreover, we will be interested in identifying the limit problem verified by u. First of all, let us remember the definition of the infinity fractional Laplacian

$$L_\infty u = \sup_{y\in\Omega,\ y\neq x} \frac{u(y) - u(x)}{|y-x|^\alpha} + \inf_{y\in\Omega,\ y\neq x} \frac{u(y) - u(x)}{|y-x|^\alpha}.$$

We decompose this operator as follows:

$$L_\infty^+ u = \sup_{y\in\Omega,\ y\neq x} \frac{u(y) - u(x)}{|y-x|^\alpha} \qquad \text{and} \qquad L_\infty^- u = \inf_{y\in\Omega,\ y\neq x} \frac{u(y) - u(x)}{|y-x|^\alpha}.$$

In the sequel, we will need the following technical result where the proof can be found in [1, Lemma 6.5].

Lemma 3.4. *Assume $\phi \in C^1(\Omega)$. Let $\{x_p\}_p \subset \Omega$ be such that $x_p \to x_0$. So, we define*

$$f_p(y) = \frac{\phi(y) - \phi(x_p)}{|y - x_p|^\alpha} \qquad \textit{and} \qquad f(y) = \frac{\phi(y) - \phi(x_0)}{|y - x_0|^\alpha}.$$

Then, one has

$$\lim_{p\to\infty} \left|\left| \frac{[f_p]_\pm}{|y - x_p|^{\frac{\alpha}{p}}} \right|\right|_{L^p(\Omega)} = ||[f]_\pm||_{L^\infty(\Omega)}.$$

Hence, we have the following:

Proposition 3.5. *Suppose that $h : \overline{\Omega} \times \mathbb{R} \mapsto \mathbb{R}^+$ is a nonnegative continuous function in (x, s) and that it is C^1 with respect to the second variable s and $g \in C^{0,\alpha}(\partial\Omega)$. Moreover, assume that there is a constant $M < \infty$ such that*

$$\psi(x) \leq \min\{M|x - x_0|^\alpha + g(x_0) \,:\, x_0 \in \partial\Omega\}, \qquad \textit{for all } x \in \Omega. \tag{3.5}$$

For $h_p(x, s) := \frac{h(x,s)^p}{p}$, let u_p be a solution of Problem (3.4). Then, up to a subsequence, $u_p \to u$ uniformly in Ω. Moreover, $u \in C^{0,\alpha}(\bar{\Omega})$ and, u is a viscosity solution to the following problem:

(3.6)

$$\begin{cases} \min\{-L_\infty u, -L_\infty^- u - h(x, u)\} = 0 & \textit{in } \{u > \psi\} \cap \{f(x, u) < 0\}, \\ \max\{-L_\infty u, -L_\infty^+ u + redh(x, u)\} = 0 & \textit{in } \{u > \psi\} \cap \{f(x, u) > 0\}, \\ \min\{-L_\infty u, -L_\infty^- u - h(x, u)\} \geq 0 & \textit{in } \{u = \psi\} \cap \{f(x, u) < 0\}, \\ \max\{-L_\infty u, -L_\infty^+ u + h(x, u)\} \geq 0 & \textit{in } \{u = \psi\} \cap \{f(x, u) > 0\}, \\ L_\infty u = 0 & \textit{in } \{u > \psi\} \cap \Omega \backslash \overline{\{f(x, u) \neq 0\}}, \\ L_\infty u \leq 0 & \textit{in } \{u = \psi\} \cap \Omega \backslash \overline{\{f(x, u) \neq 0\}}, \\ L_\infty u \geq 0 & \textit{in } \{u > \psi\} \cap \partial\{f(x, u) > 0\} \backslash \partial\{f(x, u) < 0\}, \\ L_\infty u \leq 0 & \textit{in } \Omega \cap \partial\{f(x, u) < 0\} \backslash \partial\{f(x, u) > 0\}, \\ u = g & \textit{on } \partial\Omega, \end{cases}$$

where $f(x, s) = \partial_s h(x, s)$.

Proof. First, we show that there is a function $\tilde{g} \in C^{0,\alpha}(\bar{\Omega})$ such that $\tilde{g} \geq \psi$ on $\bar{\Omega}$ and $\tilde{g} = g$ on $\partial\Omega$. For $\hat{x} \in \partial\Omega$ and $c \in \mathbb{R}$, we set

$$V_{\hat{x},c}(x) := C|x - \hat{x}|^\alpha + c, \qquad \text{for all } x \in \Omega,$$

where $C = \max\{[g]_{C^{0,\alpha}(\partial\Omega)}, M\} > 0$. If $c \geq ||g||_\infty$, then $V_{\hat{x},c} \geq \psi$ on $\bar{\Omega}$ and $V_{\hat{x},c} \geq g$ on $\partial\Omega$. Now, we define

$$\tilde{g}(x) = \inf\left\{V_{\hat{x},c}(x) : \hat{x} \in \partial\Omega, c \in \mathbb{R} \text{ such that } V_{\hat{x},c} \geq \psi \text{ on } \bar{\Omega}, V_{\hat{x},c} \geq g \text{ on } \partial\Omega\right\}.$$

We clearly have $\tilde{g} \geq \psi$ on $\bar{\Omega}$ and $\tilde{g} \geq g$ on $\partial\Omega$. Now, fix a point $\hat{x}_0 \in \partial\Omega$ and set $c_0 = g(\hat{x}_0)$. By (3.5), one has

$$V_{\hat{x}_0,c_0}(x) = C|x - \hat{x}_0|^\alpha + g(\hat{x}_0) \geq \psi(x), \qquad \text{for every } x \in \Omega.$$

Thanks to the $\alpha-$Hölder regularity of g, then we also have

$$V_{\hat{x}_0,c_0}(x) = C|x - \hat{x}_0|^\alpha + g(\hat{x}_0) \geq g(x), \qquad \text{for every } x \in \partial\Omega.$$

But so,

$$\tilde{g}(\hat{x}_0) \leq V_{\hat{x}_0,c_0}(\hat{x}_0) = c_0 = g(\hat{x}_0).$$

This yields that $\tilde{g} = g$ on $\partial\Omega$. Moreover, it is clear that $\tilde{g} \in C^{0,\alpha}(\bar{\Omega})$. On the other hand, we have

$$\frac{1}{2p}\iint_{\Omega\times\Omega}\frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(x,u_p) \leq \frac{1}{2p}\iint_{\Omega\times\Omega}\frac{|\tilde{g}(x)-\tilde{g}(y)|^p}{|x-y|^{\alpha p}} + \int_\Omega h_p(x,\tilde{g})$$
$$\leq \frac{C^p|\Omega|^2}{2p} + \frac{||h(\cdot,\tilde{g})||_\infty^p|\Omega|}{p} \leq \frac{C^p}{p}.$$

We get that

$$\iint_{\Omega\times\Omega}\frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}} \leq C^p.$$

Hence, there is a uniform constant C (independent of p) such that we have the following bound:

$$\left[\iint_{\Omega\times\Omega}\frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}}\right]^{\frac{1}{p}} \leq C.$$

On the other side, we recall that $||u_p||_\infty \leq C([u]_{s,p} + [\tilde{g}]_{s,p}) + ||\tilde{g}||_\infty \leq C$ thanks to the fact that $\tilde{g} \in C^{0,\alpha}(\partial\Omega)$. Fix $m < p$, one has

$$\left[\iint_{\Omega\times\Omega}\frac{|u_p(x)-u_p(y)|^m}{|x-y|^{\alpha m}}\right]^{\frac{1}{m}} \leq \left[\iint_{\Omega\times\Omega}\frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}}\right]^{\frac{1}{p}}|\Omega|^{2(\frac{p-m}{pm})} \leq C. \tag{3.7}$$

Consequently, $(u_p)_p$ is bounded in $W^{s,m}(\Omega)$ (with $s = \alpha - \frac{N}{m}$) and so, up to a subsequence, it converges uniformly to a function $u \in W^{s,m}(\Omega)$, for all m. In particular, u belongs to $C^{0,\alpha}(\bar{\Omega})$.

Now, fix $x_0 \in \{u > \psi\}$. We show that u is a viscosity subsolution at x_0 to equation (3.6). First, we consider the case when $f(x_0, u(x_0)) > 0$. Assume there is a function $\phi \in C^1(\Omega)\cap C(\overline{\Omega})$ such that $u \leq \phi$ on $\bar{\Omega}$, $u(x_0) = \phi(x_0)$ and,

$$\max\{-L_\infty\phi(x_0), -L_\infty^+\phi(x_0) + h(x_0,\phi(x_0))\} > 0.$$

In fact, one can assume that x_0 is the unique maximizer of $u - \phi$. To see this fact, fix $\delta > 0$ small enough and set $\phi_\delta(x) := \phi(x) + \delta|x - x_0|^2$, for every $x \in \Omega$. We have

$$L_\infty\phi_\delta(x_0) = \sup_{x\in\Omega,\, x\neq x_0}\frac{\phi_\delta(x)-\phi_\delta(x_0)}{|x-x_0|^\alpha} + \inf_{x\in\Omega,\, x\neq x_0}\frac{\phi_\delta(x)-\phi_\delta(x_0)}{|x-x_0|^\alpha}.$$

Yet,

$$[\phi_\delta(x) - \phi_\delta(x_0)] - [\phi(x) - \phi(x_0)] = \delta|x - x_0|^2.$$

Hence,

$$\frac{\phi_\delta(x)-\phi_\delta(x_0)}{|x-x_0|^\alpha} = \frac{\phi(x)-\phi(x_0)}{|x-x_0|^\alpha} + \delta|x-x_0|^{2-\alpha} \leq \frac{\phi(x)-\phi(x_0)}{|x-x_0|^\alpha} + C\delta.$$

Therefore, we get that

$$\left|L_\infty^\pm\phi_\delta(x_0) - L_\infty^\pm\phi(x_0)\right| \leq C\delta.$$

Then, $-L_\infty\phi_\delta(x_0) > 0$ or $-L_\infty^+\phi_\delta(x_0) + h(x_0, \phi_\delta(x_0)) > 0$ provided that $\delta > 0$ is small enough. This proves our claim.

Since $u_p \to u$ uniformly in Ω, then there is a point $x_p \in \{u_p > \psi\}$ such that $u_p - \phi$ has a maximum at x_p and $x_p \to x_0$ (since x_0 is the unique maximizer of $u - \phi$). In the sequel, we set $M_p := \max_\Omega[u_p - \phi]$; we note that $M_p \to 0$, $u_p \leq \phi + M_p$ and $u_p(x_p) = \phi(x_p) + M_p$. But, u_p is a viscosity solution to equation (3.4). Hence,

$$-L_p[\phi + M_p](x_p) + f_p(x_p, \phi(x_p) + M_p) \leq 0,$$

where $f_p = \partial_s h_p$. So, we get

$$-L_p\phi(x_p) + f_p(x_p, \phi(x_p) + M_p) \le 0.$$

Recalling the definition of L_p, one has

$$-\int_\Omega \frac{|\phi(x)-\phi(x_p)|^{p-1}}{|x-x_p|^{\alpha p}} \frac{\phi(x)-\phi(x_p)}{|\phi(x)-\phi(x_p)|}\,\mathrm{d}x + f_p(x_p, \phi(x_p)+M_p) \le 0. \tag{3.8}$$

Then,

$$\int_\Omega \frac{[\phi(x)-\phi(x_p)]_+^{p-1}}{|x-x_p|^{\alpha p}}\,\mathrm{d}x \ge \int_\Omega \frac{[\phi(x)-\phi(x_p)]_-^{p-1}}{|x-x_p|^{\alpha p}}\,\mathrm{d}x + f_p(x_p,\phi(x_p)+M_p).$$

Set

$$A_p[\phi] := \left[\int_\Omega \frac{[\phi(x)-\phi(x_p)]_+^{p-1}}{|x-x_p|^{\alpha p}}\,\mathrm{d}x\right]^{\frac{1}{p-1}}$$

and

$$B_p[\phi] := \left[\int_\Omega \frac{[\phi(x)-\phi(x_p)]_-^{p-1}}{|x-x_p|^{\alpha p}}\,\mathrm{d}x\right]^{\frac{1}{p-1}}.$$

We have

$$A_p[\phi]^{p-1} \ge B_p[\phi]^{p-1} + f_p(x_p,\phi(x_p)+M_p).$$

Without loss of generality, we may assume that $A_p[\phi] > 0$. Hence, dividing by $A_p[\phi]$, we get:

$$\frac{B_p[\phi]^{p-1}}{A_p[\phi]^{p-1}} + \frac{f_p(x_p,\phi(x_p)+M_p)}{A_p[\phi]^{p-1}} \le 1. \tag{3.9}$$

Therefore, we have

$$\frac{B_p[\phi]}{A_p[\phi]} \le 1 \qquad \text{and} \qquad \frac{f_p(x_p,\phi(x_p)+M_p)^{\frac{1}{p-1}}}{A_p[\phi]} = \frac{h(x_p,\phi(x_p)+M_p)\,f(x_p,\phi(x_p)+M_p)^{\frac{1}{p-1}}}{A_p[\phi]} \le 1$$

since otherwise, at least one of the two terms in (3.9) goes to ∞, which is a contradiction. Thanks to Lemma 3.4, we have that $A_p[\phi] \to L_\infty^+\phi$ and $B_p[\phi] \to -L_\infty^-\phi$. Passing to the limit when $p \to \infty$, this yields that

$$-L_\infty\phi(x_0) \le 0 \qquad \text{and} \qquad -L_\infty^+\phi(x_0) + h(x_0,\phi(x_0)) \le 0.$$

If $f(x_0,u(x_0)) < 0$, we assume that there is a function $\phi \in C^1(\Omega)\cap C(\overline{\Omega})$ such that $u \le \phi$ on $\bar{\Omega}$, $u(x_0) = \phi(x_0)$ and,

$$\min\{-L_\infty\phi(x_0), -L_\infty^-\phi(x_0) - h(x_0,\phi(x_0))\} > 0.$$

Recalling (3.8), we have

$$\int_\Omega \frac{[\phi(x)-\phi(x_p)]_-^{p-1}}{|x-x_p|^{\alpha p}}\,\mathrm{d}x \le \int_\Omega \frac{[\phi(x)-\phi(x_p)]_+^{p-1}}{|x-x_p|^{\alpha p}}\,\mathrm{d}x - f_p(x_p,\phi(x_p)+M_p).$$

Following the same steps as before, we arrive to a contradiction and so, u is a viscosity subsolution to the following equation:

$$\min\{-L_\infty u, -L_\infty^- u - h(x,u)\} \le 0 \qquad \text{in } \{u > \psi\}.$$

Let us prove that u is also a viscosity supersolution in Ω to equation (3.6) in the case when $f(x_0, u(x_0)) > 0$. Our aim is to show that for every function $\phi \in C^1(\Omega) \cap C(\overline{\Omega})$ such that $u \geq \phi$ on $\bar{\Omega}$ and $\phi(x_0) = u(x_0)$, we have

$$\max\{-L_\infty \phi(x_0), -L_\infty^+ \phi(x_0) + h(x_0, \phi(x_0))\} \geq 0.$$

Assume this is not the case. Thanks to the uniform convergence of u_p to u, there is a point $x_p \in \Omega$ such that $x_p \to x_0$ and $u_p - \phi$ has a minimum at x_p. We denote by $m_p := \min_\Omega [u_p - \phi] \to 0$. Since u_p is a viscosity solution to (3.4), then one has

$$-L_p[\phi](x_p) + f_p(x_p, \phi(x_p) + m_p) \geq 0.$$

So, we have

$$B_p[\phi]^{p-1} + f_p(x_p, \phi(x_p) + m_p) \geq A_p[\phi]^{p-1}.$$

In particular, we get

$$\frac{B_p[\phi]}{A_p[\phi]} \geq 1 \qquad \text{or} \qquad \frac{f_p(x_p, \phi(x_p) + m_p)}{A_p[\phi]} \geq 1.$$

Hence,

$$-L_\infty \phi(x_0) \geq 0$$

or

$$-L_\infty^+ \phi(x_0) + h(x_0, \phi(x_0)) \geq 0.$$

Consequently,

$$\max\{-L_\infty \phi(x_0), -L_\infty^+ \phi(x_0) + h(x_0, \phi(x_0))\} \geq 0.$$

Finally, if $f(x_0, u(x_0)) < 0$ then one can show similarly that u is a viscosity supersolution in Ω to the following equation:

$$\min\{-L_\infty u, -L_\infty^- u - h(\cdot, u)\} \geq 0.$$

This concludes the proof. □

3.3. **Regularity.** In this section, we assume that $\partial_s h > 0$ over $\Omega \times \mathbb{R}$. Under this assumption, we recall that the limit problem is the following:

$$\begin{cases} \max\{-L_\infty u, -L_\infty^+ u + h(x,u)\} = 0 & \text{in } \{u > \psi\}, \\ \max\{-L_\infty u, -L_\infty^+ u + h(x,u)\} \geq 0 & \text{in } \{u = \psi\}, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

From the previous section, we know that Problem (3.10) has a solution u which can be obtained by approximation with the fractional $p-$Laplacian problem (1.4), and this solution u belongs to $C^{0,\alpha}(\overline{\Omega})$. The goal of this section is to show that any viscosity solution to (3.10) is in fact $\alpha-$Hölderian. In order to study the regularity of a solution u, we start by the following comparison principle:

Proposition 3.6. *Let u be a viscosity solution of (3.10). Let $\phi \in C^1(\Omega) \cap C(\bar{\Omega})$ be a strict viscosity supersolution in $\{u > \psi\}$ such that $u \leq \phi$ on $\{u = \psi\} \cup \partial\Omega$. Then, we have $u \leq \phi$ in Ω.*

Proof. Assume there exists a point $x^\star \in \{x \in \Omega : u(x) > \psi(x)\}$ such that $u(x^\star) - \phi(x^\star) = \max_\Omega[u-\phi] = M > 0$. So, we have $u \le \phi + M$ on $\bar\Omega$ and $u(x^\star) = \phi(x^\star) + M$. Since u is a viscosity solution, then one has

$$\max\{-L_\infty\phi(x^\star), -L_\infty^+\phi(x^\star) + h(x^\star, \phi(x^\star) + M)\} \le 0.$$

As $\partial_s h > 0$, then

$$\max\{-L_\infty\phi(x^\star), -L_\infty^+\phi(x^\star) + h(x^\star, \phi(x^\star))\} \le 0,$$

which is a contradiction since ϕ is a strict viscosity supersolution in $\{u > \psi\}$. □

Lemma 3.7. *Fix $x_0 \in \{u > \psi\}$. If u is a viscosity solution of Problem (3.10) in Ω, then u is a viscosity solution of (3.10) in $\Omega\backslash\{x_0\}$.*

Proof. We show that u is a viscosity subsolution in $\Omega\backslash\{x_0\}\cap\{u > \psi\}$. Let $x^\star \in \Omega\backslash\{x_0\}\cap\{u > \psi\}$ and $\phi \in C^1(\Omega \setminus \{x_0\}) \cap C(\bar\Omega)$ be such that $u \le \phi$ on $\bar\Omega$ and $\phi(x^\star) = u(x^\star)$. Assume that $u(x_0) < \phi(x_0)$. Let $\phi_n \in C^1(\Omega) \cap C(\bar\Omega)$ be such that $\phi_n = \phi$ on $\Omega\backslash B(x_0, \frac{1}{n})$, $\phi \le \phi_n$ and ϕ_n converges uniformly to ϕ. So, we have $u \le \phi_n$ on $\bar\Omega$ and $u(x^\star) = \phi_n(x^\star)$. Since u is a viscosity solution on Ω, then we must have

$$\max\{-L_\infty\phi_n(x^\star), -L_\infty^+\phi_n(x^\star) + h(x^\star, \phi_n(x^\star))\} \le 0. \tag{3.11}$$

Fix $0 < \varepsilon < |x_0 - x^\star|$. For $n \in \mathbb{N}^\star$ large enough, one has the following:

$$\begin{aligned} L_\infty^+\phi_n(x^\star) &= \max\left\{\sup_{y\in\bar\Omega\backslash\bar B(x_0,\varepsilon),\, y\neq x^\star} \frac{\phi_n(y)-\phi_n(x^\star)}{|y-x^\star|^\alpha}, \sup_{y\in\bar B(x_0,\varepsilon),\, y\neq x^\star} \frac{\phi_n(y)-\phi_n(x^\star)}{|y-x^\star|^\alpha}\right\} \\ &= \max\left\{\sup_{y\in\bar\Omega\backslash\bar B(x_0,\varepsilon),\, y\neq x^\star} \frac{\phi(y)-\phi(x^\star)}{|y-x^\star|^\alpha}, \sup_{y\in\bar B(x_0,\varepsilon),\, y\neq x^\star} \frac{\phi_n(y)-\phi_n(x^\star)}{|y-x^\star|^\alpha}\right\}. \end{aligned}$$

Yet, it is clear that

$$\frac{\phi_n(y)-\phi_n(x^\star)}{|y-x^\star|^\alpha} \to \frac{\phi(y)-\phi(x^\star)}{|y-x^\star|^\alpha} \quad \text{uniformly in } \bar B(x_0,\varepsilon).$$

Hence, $\lim_{n\to\infty} L^+\phi_n(x^\star) = L^+\phi(x^\star)$. In the same way, we show that $L^-\phi_n(x^\star) \to L^-\phi(x^\star)$. Passing to the limit in (3.11) as $n \to \infty$, we get that

$$\max\{-L_\infty\phi(x^\star), -L_\infty^+\phi(x^\star) + h(x^\star, \phi(x^\star))\} \le 0.$$

Finally, assume that $u(x_0) = \phi(x_0)$. For every $\delta > 0$, we define $\phi_\delta := \phi + \delta|x - x^\star|^2$. We have $\phi_\delta(x^\star) = u(x^\star)$ and $u \le \phi_\delta$ on $\bar\Omega$. Hence,

$$\max\{-L_\infty\phi_\delta(x^\star), -L_\infty^+\phi_\delta(x^\star) + h(x^\star, \phi_\delta(x^\star))\} \le 0.$$

But, we recall that

$$|L_\infty^\pm\phi_\delta(x) - L_\infty^\pm\phi(x)| \le C\delta.$$

Passing to the limit when $\delta \to 0^+$, we conclude the proof. In the same way, we show that u is a viscosity supersolution in $\Omega\backslash\{x_0\}$. □

Proposition 3.8. *Any viscosity solution u of Problem (3.10) is bounded. Moreover, we have $||u||_\infty \le \max\{||g||_\infty, ||\psi||_\infty\}$.*

Proof. Set $\phi = M \ge \max\{||g||_\infty, ||\psi||_\infty\}$. We have $u \le \phi$ on $\{u = \psi\} \cup \partial\Omega$. Fix $x \in \Omega \cap \{u > \psi\}$, then we clearly have

$$\max\{-L\phi(x), -L^+\phi(x) + h(x, \phi(x))\} = h(x, M) > 0.$$

Hence, ϕ is a strict viscosity supersolution. Thanks to the comparison principle 3.6, this yields that $u \le \phi$ on Ω. □

Proposition 3.9. *Let u be a viscosity solution of (3.10). Then, u is locally $\alpha-$Hölderian in $\{u > \psi\}$. Moreover, we have the following estimate:*

$$[u]_\alpha \le \frac{2||u||_\infty}{\mathrm{dist}(\omega, \{u = \psi\} \cup \partial\Omega)}, \qquad \textit{for every } \ \omega \subset\subset \Omega \cap \{u > \psi\},$$

where $[u]_\alpha$ denotes the $\alpha-$Hölder constant of u.

Proof. Fix $x_0 \in \omega \cap \{u > \psi\}$. Assume $\alpha < 1$. Set $\Psi_{x_0}(x) = |x - x_0|^\alpha$, for all $x \in \Omega$. For $x \in \Omega\backslash\{x_0\}$, it is easy to see that

$$L^-\Psi_{x_0}(x) = \inf_{y\in\bar{\Omega},\, y\neq x} \frac{\Psi(y) - \Psi(x)}{|y - x|^\alpha} = \inf_{y\in\bar{\Omega},\, y\neq x} \frac{|y - x_0|^\alpha - |x - x_0|^\alpha}{|y - x|^\alpha} \le -1.$$

On the other hand,

$$L^+\Psi_{x_0}(x) = \sup_{y\in\bar{\Omega},\, y\neq x} \frac{\Psi(y) - \Psi(x)}{|y - x|^\alpha} \le \sup_{y\in\bar{\Omega},\, y\neq x} \frac{|y - x_0|^\alpha - |x - x_0|^\alpha}{||y - x_0| - |x - x_0||^\alpha} \le \sup_{1<r<\frac{\mathrm{diam}(\Omega)}{|x-x_0|}} \Psi(r),$$

with $\Psi(r) = \dfrac{r^\alpha - 1}{(r-1)^\alpha}$. Yet, it is easy to check that Ψ is increasing on $(1, +\infty)$ and so, since $\alpha < 1$, we have

$$L^+\Psi_{x_0}(x) \le \Psi\left(\frac{\mathrm{diam}(\Omega)}{|x - x_0|}\right) = \frac{\left(\dfrac{\mathrm{diam}(\Omega)}{|x - x_0|}\right)^\alpha - 1}{\left(\dfrac{\mathrm{diam}(\Omega)}{|x - x_0|} - 1\right)^\alpha} < 1.$$

Consequently, we get that

$$L\Psi_{x_0}(x) \le \frac{\left(\dfrac{\mathrm{diam}(\Omega)}{|x - x_0|}\right)^\alpha - 1}{\left(\dfrac{\mathrm{diam}(\Omega)}{|x - x_0|} - 1\right)^\alpha} - 1 < 0.$$

Now, we define $\phi(x) = u(x_0) + C|x - x_0|^\alpha$. We have $\phi \in C^1(\Omega\backslash\{x_0\}) \cap C(\bar{\Omega})$. Moreover,

$$L_\infty\phi(x) = CL\Psi_{x_0}(x) < 0.$$

In particular, one has

$$\max\{-L_\infty\phi(x), -L^+_\infty\phi(x) + h(x, \phi(x))\} > 0.$$

If we choose the constant C large enough, one can get that $u \le \phi$ on $\{u = \psi\} \cup \partial\Omega \cup \{x_0\}$. Indeed, for every $x \in \{u = \psi\} \cup \partial\Omega$, we have

$$u(x) - \phi(x) = u(x) - u(x_0) - C|x - x_0|^\alpha \le 2||u||_\infty - C\,\mathrm{dist}(x_0, \{u = \psi\} \cup \partial\Omega)^\alpha \le 0$$

as soon as

$$C \ge \frac{2||u||_\infty}{\mathrm{dist}(x_0, \{u = \psi\} \cup \partial\Omega)^\alpha}.$$

Thanks to the comparison principle 3.6 and since u is a viscosity solution while ϕ is a strict viscosity supersolution in $\Omega\backslash\{x_0\} \cap \{u > \psi\}$, this implies that $u < \phi$ in $\Omega\backslash\{x_0\}$. Consequently,

$$u(x) \le u(x_0) + C|x - x_0|^\alpha.$$

Finally, assume $\alpha = 1$. Fix $\varepsilon > 0$ small enough. Then, we define $\Psi_{x_0}(x) = |x-x_0| - \varepsilon|x-x_0|^2$. Again, we have

$$L^-\Psi_{x_0}(x) = \inf_{y\in\bar{\Omega},\, y\neq x} \frac{\Psi_{x_0}(y)-\Psi_{x_0}(x)}{|y-x|} = \inf_{y\in\bar{\Omega},\, y\neq x} \frac{|y-x_0| - \varepsilon|y-x_0|^2 - |x-x_0| + \varepsilon|x-x_0|^2}{|y-x|}$$
$$\leq -1 + \varepsilon|x-x_0|.$$

Moreover,

$$L^+\Psi_{x_0}(x) = \sup_{y\in\bar{\Omega},\, y\neq x} \frac{\Psi_{x_0}(y)-\Psi_{x_0}(x)}{|y-x|} \leq \sup_{y\in\bar{\Omega},\, y\neq x} \frac{|y-x_0| - \varepsilon|y-x_0|^2 - |x-x_0| + \varepsilon|x-x_0|^2}{||y-x_0|-|x-x_0||}$$
$$= 1 - \varepsilon|x-x_0| - \varepsilon \min_{|y-x_0|>|x-x_0|} |y-x_0| = 1 - 2\varepsilon|x-x_0|.$$

Hence, we get

$$L\Psi_{x_0}(x) \leq -\varepsilon|x-x_0| < 0, \quad \text{for all } x \in \Omega\backslash\{x_0\}.$$

Now, set $\phi(x) = u(x_0) + C[|x-x_0| - \varepsilon|x-x_0|^2]$. So, we have $L\phi(x) = CL\Psi_{x_0}(x) < 0$, for every $x \in \Omega\backslash\{x_0\}$. In addition, one has

$$u(x) - \phi(x) = u(x) - u(x_0) - C[|x-x_0| - \varepsilon|x-x_0|^2]$$
$$\leq 2||u||_\infty - C\text{dist}(w, \{u=\psi\}\cup\partial\Omega) + C\varepsilon\text{diam}(\Omega)^2 \leq 0$$

as soon as

$$C \geq \frac{2||u||_\infty}{\text{dist}(w, \{u=\psi\}\cup\partial\Omega) - \varepsilon\text{diam}(\Omega)^2}.$$

Then,

$$|u(x) - u(x_0)| \leq \frac{2||u||_\infty}{\text{dist}(w, \{u=\psi\}\cup\partial\Omega) - \varepsilon\text{diam}(\Omega)^2}[|x-x_0| - \varepsilon|x-x_0|^2].$$

Letting $\varepsilon \to 0$, we get

$$|u(x) - u(x_0)| \leq \frac{2||u||_\infty}{\text{dist}(w, \{u=\psi\}\cup\partial\Omega)}|x-x_0|.$$

□

Proposition 3.10. *Assume $\psi \in C^{0,\alpha}(\bar{\Omega})$. Then, any viscosity solution u of (3.10) belongs to $C^{0,\alpha}(\bar{\Omega})$. Moreover, we have*

$$[u]_\alpha \leq C(||g||_\infty, ||\psi||_\infty, [g]_\alpha, [\psi]_\alpha).$$

Proof. Fix $x_0 \in \partial\{u > \psi\}\backslash\partial\Omega$. Set $\phi(x) = \psi(x_0) + C|x-x_0|^\alpha$. From Proposition 3.9, we recall that ϕ is a strict viscosity supersolution in $\{u > \psi\}$. Since $\psi \in C^{0,\alpha}(\bar{\Omega})$, we have

$$u(x) - \phi(x) = \psi(x) - \psi(x_0) - C|x-x_0|^\alpha \leq 0, \qquad \text{for all } x \in \{u=\psi\}$$

and

$$u(x) - \phi(x) = g(x) - \psi(x_0) - C|x-x_0|^\alpha \leq ||g||_\infty + ||\psi||_\infty - C\,\text{dist}(x_0,\partial\Omega)^\alpha \leq 0, \quad \text{for all } x \in \partial\Omega,$$

as soon as

$$C \geq \frac{||g||_\infty + ||\psi||_\infty}{\text{dist}(x_0,\partial\Omega)^\alpha}.$$

Hence, by Proposition 3.6, we infer that $u \leq \phi$ in $\{u > \psi\}$. Thanks to Proposition 3.9, this implies that $u \in C^{0,\alpha}_{loc}(\Omega)$.

Now, fix $x_0 \in \partial\Omega$. Again, we define $\phi(x) = g(x_0) + C|x - x_0|^\alpha$. So, ϕ is a strict viscosity supersolution in $\{u > \psi\}$. Moreover, one has

$$u(x) - \phi(x) = g(x) - g(x_0) - C|x - x_0|^\alpha \leq 0, \qquad \text{for all } x \in \partial\Omega$$

and

$$u(x) - \phi(x) = \psi(x) - g(x_0) - C|x - x_0|^\alpha \leq 0, \quad \text{for all } x \in \{u = \psi\},$$

provided that $C \geq \max\{[g]_\alpha, M\}$; we note that in the last inequality we have used that $\psi(x) \leq \min\{M|x - x_0|^\alpha + g(x_0) : x_0 \in \partial\Omega\}$. Consequently, $u \in C^{0,\alpha}(\Omega)$. □

We conclude this section by the following existence result but in the case when $h = h(s)$ is not smooth.

Proposition 3.11. *Assume h is a nonnegative nondecreasing continuous function. Then, Problem (3.10) has a solution.*

Proof. Let h_n be a sequence of smooth functions such that $h_n' > 0$ and $h_n \to h$ locally uniformly on $\mathbb{R}$. For every $n \in \mathbb{N}$, let u_n be a solution to Problem (3.10). Thanks to Proposition 3.10, we have

$$[u_n]_\alpha \leq C(||g||_\infty, ||\psi||_\infty, [g]_\alpha, [\psi]_\alpha).$$

Moreover,

$$||u_n||_\infty \leq \max\{||g||_\infty, ||\psi||_\infty\}.$$

Hence, up to a subsequence, $u_n \to u$ uniformly in $\bar{\Omega}$. In particular, we have $u = g$ on $\partial\Omega$ and $u \geq \psi$ on $\bar{\Omega}$. Now, let us show that u is a viscosity solution of (3.10). First, we show that u is a viscosity subsolution in $\{u > \psi\}$. Fix $x_0 \in \{u > \psi\}$ and let $\varphi \in C^1(\Omega) \cap C(\bar{\Omega})$ be such that $u \leq \varphi$ on $\bar{\Omega}$ and $u(x_0) = \varphi(x_0)$, we recall that one can assume x_0 to be the unique maximizer of $u - \varphi$. For every n, let x_n be a maximizer of $u_n - \varphi$ and set $M_n := \max_{\bar{\Omega}}[u_n - \varphi]$. Then, $x_n \to x_0$ and $M_n \to 0$. Since u_n is a viscosity solution, then one has

$$\max\{-L_\infty\varphi(x_n), -L_\infty^+\varphi(x_n) + h_n(x_n, \varphi(x_n) + M_n)\} \leq 0.$$

But, $L_\infty\varphi \in C(\Omega)$ since $\varphi \in C^1(\Omega)$ (see [1, Lemma 3.5]). Passing to the limit when $n \to +\infty$, we get

$$\max\{-L_\infty\varphi(x_0), -L_\infty^+\varphi(x_0) + h(x_0, \varphi(x_0))\} \leq 0.$$

In the same way, we show that u is a viscosity supersolution in Ω. This concludes the proof that u is a viscosity solution. □

4. The case of a linear nonhomogeneous term

In this section, we consider the case when the nonhomogeneous term is linear in s, i.e. $h_p(x, s) = h(x) \cdot s$, where $h \in L^q(\Omega)$ with $q > 1$. For $p \geq q/q - 1$, we minimize the following problem

$$\min\left\{\frac{1}{2p}\iint_{\Omega\times\Omega}\frac{|u(x) - u(y)|^p}{|x - y|^{\alpha p}} - \int_\Omega h(x)\,u \,:\, u \in W^{s,p}(\Omega),\, u \geq \psi \text{ on } \bar{\Omega},\, u = g \text{ on } \partial\Omega\right\}. \tag{4.1}$$

Notice that the nonhomogeneous term has no sign now and so, we cannot use Proposition 3.1 to get existence of solution to Problem (4.1). However, we still have the following:

Proposition 4.1. *Problem (4.1) has a minimizer u_p. Moreover, u_p is a weak solution to the following problem:*

$$\begin{cases} -L_p u = h & \text{in} \quad \{u > \psi\}, \\ -L_p u \le h & \text{in} \quad \{u = \psi\}. \end{cases} \tag{4.2}$$

In addition, if h is continuous and $p \ge 1/1-\alpha$ then u_p is also a viscosity solution to Problem (4.2).

Proof. Let $(u_n)_n$ be a minimizing sequence. Then, there is a uniform constant $C < \infty$ such that

$$\frac{1}{2p} \iint_{\Omega\times\Omega} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{\alpha p}} - \int_\Omega h(x)\, u_n \le C, \qquad \text{for all } \; n.$$

Using both Hölder and Poincaré inequalities, we have

$$\int_\Omega h(x)\, u_n \le |\Omega|^{1-\frac{1}{p}-\frac{1}{q}} \, ||h||_{L^q(\Omega)} \, ||u_n||_{L^p(\Omega)} \le |\Omega|^{1-\frac{1}{p}-\frac{1}{q}} \, ||h||_{L^q(\Omega)} \, ||u_n||_{L^p(\Omega)}$$

$$\le |\Omega|^{1-\frac{1}{p}-\frac{1}{q}} \, ||h||_{L^q(\Omega)} (C[u_n - g]_{s,p} + ||g||_{L^p(\Omega)}) \le C([u_n]_{s,p} + 1) \tag{4.3}$$

where the constant C is uniform in n. In particular, this implies that

$$[u_n]_{s,p}^p - C[u_n]_{s,p} \le C.$$

Hence, $(u_n)_n$ is bounded in $W^{s,p}(\Omega)$ and so, up to a subsequence, $u_n \rightharpoonup u_p$ in $W^{s,p}(\Omega)$ and $u_n \to u_p$ uniformly in $C^{0,\beta}(\bar\Omega)$, $\beta = \alpha - \frac{2N}{p}$, with $u_p \ge \psi$ on $\overline{\Omega}$ and $u_p = g$ on $\partial\Omega$. So, we get that

$$\iint_{\Omega\times\Omega} \frac{|u_p(x) - u_p(y)|^p}{|x-y|^{\alpha p}} \le \liminf_n \iint_{\Omega\times\Omega} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{\alpha p}}$$

and

$$\int_\Omega h(x)\, u_n \to \int_\Omega h(x)\, u_p.$$

Hence, u_p minimizes Problem (4.1). Now, let ϕ be a smooth function such that $\text{supp}(\phi) \subset \{u_p > \psi\}$. Then, for all $t \in \mathbb{R}$ small enough, we have

$$\frac{1}{2p} \iint_{\Omega\times\Omega} \frac{|u_p(x) - u_p(y)|^p}{|x-y|^{\alpha p}} - \int_\Omega h(x) u_p$$

$$\le \frac{1}{2p} \iint_{\Omega\times\Omega} \frac{|u_p(x) + t\phi(x) - u_p(y) - t\phi(y)|^p}{|x-y|^{\alpha p}} - \int_\Omega h(x)[u_p + t\phi].$$

Thus, we get

$$-\iint_{\Omega\times\Omega} \frac{|u_p(x) - u_p(y)|^{p-2}}{|x-y|^{\alpha p}} [u_p(y) - u_p(x)]\, \phi(x) - \int_\Omega h(x)\, \phi(x) \; = 0.$$

The fact that $L_p u_p \le h$ on the coincidence set $\{u_p = \psi\}$ can be treated similarly. Following the proof of Proposition 3.3, one can also show that u_p is a viscosity solution to (4.2) provided that h is continuous. This concludes the proof. □

Proposition 4.2. *Up to a subsequence, $u_p \to u$ uniformly in $\overline{\Omega}$ and u belongs to $C^{0,\alpha}(\overline{\Omega})$. In addition, assume $[g]_{C^{0,\alpha}(\partial\Omega)} \leq 1$ and that the obstacle ψ satisfies the following inequality:*

$$\psi(x) \leq \min\{|x-x_0|^\alpha + g(x_0) \,:\, x_0 \in \partial\Omega\}, \qquad \text{for all } x \in \Omega. \tag{4.4}$$

Then, u maximizes the following problem:

(4.5)

$$\max\left\{ \int_\Omega \phi(x)\, h(x)\, \mathrm{d}x \,:\, \phi \in C^{0,\alpha}(\overline{\Omega}),\ [\phi]_{C^{0,\alpha}(\overline{\Omega})} \leq 1,\ \phi \geq \psi \ \text{ in } \overline{\Omega} \ \text{ and } \phi = g \ \text{ on } \partial\Omega \right\}$$

where

$$[\phi]_{C^{0,\alpha}(\overline{\Omega})} = \sup_{x,\, y \in \overline{\Omega}} \frac{|\phi(x)-\phi(y)|}{|x-y|^\alpha}.$$

Moreover, this limit function u is a viscosity solution to the following limit problem:

$$\begin{cases} L_\infty^+ u = 1 & \text{in } \{u>\psi\}\cap\{h<0\},\\ L_\infty^- u = -1 & \text{in } \{u>\psi\}\cap\{h>0\},\\ L_\infty u = 0 & \text{in } \{u>\psi\}\cap(\Omega\backslash \overset{\circ}{\operatorname{spt}}\, h),\\ L_\infty u \leq 0 & \text{in } \{u=\psi\}\cap(\Omega\backslash \overset{\circ}{\operatorname{spt}}\, h),\\ L_\infty u \leq 0 & \text{in } \Omega\cap\partial\{h>0\}\backslash\partial\{h<0\},\\ L_\infty u \geq 0 & \text{in } \{u>\psi\}\cap\partial\{h<0\}\backslash\partial\{h>0\},\\ L_\infty^+ u \leq 1 & \text{in } \Omega,\\ L_\infty^- u \geq -1 & \text{in } \Omega,\\ u = g & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

Proof. Recalling the proof of Proposition 3.5, there is a function $\tilde{g} \in C^{0,\alpha}(\bar{\Omega})$ such that $\tilde{g} \geq \psi$ on $\bar{\Omega}$ and $\tilde{g} = g$ on $\partial\Omega$. Thanks to (4.12) and the fact that $[g]_{C^{0,\alpha}(\partial\Omega)} \leq 1$, it is not difficult to check that $\tilde{g}$ satisfies $[\tilde{g}]_{C^{0,\alpha}(\overline{\Omega})} \leq 1$. From the optimality of u_p in (4.1), we have

$$\begin{aligned} \frac{1}{2p}\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}} - \int_\Omega h(x) u_p &\leq \frac{1}{2p}\iint_{\Omega\times\Omega} \frac{|\tilde{g}(x)-\tilde{g}(y)|^p}{|x-y|^{\alpha p}} - \int_\Omega h(x)\tilde{g} \\ &\leq \frac{|\Omega|^2\,[\tilde{g}]^p_{C^{0,\alpha}(\overline{\Omega})}}{2p} + ||\tilde{g}||_{L^\infty(\Omega)}\,||h||_{L^1(\Omega)} \leq C\bigg(\frac{[\tilde{g}]^p_{C^{0,\alpha}(\overline{\Omega})}}{p} + 1\bigg). \end{aligned}$$

Recalling estimate (4.3), we get

$$\int_\Omega h(x)\, u_p \leq |\Omega|^{1-\frac{1}{p}-\frac{1}{q}}\, ||h||_{L^q(\Omega)}(C[u_p-g]_{s,p} + ||g||_{L^p(\Omega)})$$

where the Poincaré constant C is independent of p (see Section 2). Hence, we infer that

$$\int_\Omega h(x)\, u_p \leq C([u_p]_{s,p}+1).$$

Hence,

$$\frac{1}{2p}[u_p]^p_{s,p} - C[u_p]_{s,p} \leq C\bigg(\frac{[\tilde{g}]^p_{C^{0,\alpha}(\overline{\Omega})}}{p}+1\bigg).$$

Using Young inequality, one has

$$C[u_p]_{s,p} \leq \frac{\varepsilon}{p}\,[u_p]^p_{s,p} + \frac{\varepsilon^{\frac{-1}{p-1}}}{\frac{p}{p-1}}\, C^{\frac{p}{p-1}}, \qquad \text{for any } \varepsilon > 0.$$

Therefore, we get

$$\frac{1}{2p}[u_p]_{s,p}^p - \frac{\varepsilon}{p}\,[u_p]_{s,p}^p - \frac{\varepsilon^{\frac{-1}{p-1}}}{\frac{p}{p-1}}\,C^{\frac{p}{p-1}} \leq C\Big(\frac{[\tilde{g}]^p_{C^{0,\alpha}(\overline{\Omega})}}{p} + 1\Big).$$

Choose $\varepsilon = \frac{1}{4}$. Then, we infer that

$$[u_p]_{s,p}^p \leq Cp\Big(\frac{[\tilde{g}]^p_{C^{0,\alpha}(\overline{\Omega})}}{p} + 1\Big). \tag{4.7}$$

Consequently,

$$\Big[\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}}\Big]^{\frac{1}{p}} \leq C.$$

Yet,

$$||u_p||_\infty \leq C([u]_{s,p} + [\tilde{g}]_{s,p}) + ||\tilde{g}||_\infty \leq C.$$

Recalling (3.7), we may show that $(u_p)_p$ is bounded in $W^{\alpha-\frac{N}{m},m}(\Omega)$, for any $m \in \mathbb{N}$. Hence, up to a subsequence, $u_p \to u$ uniformly in $\overline{\Omega}$ where $u \in C^{0,\alpha}(\overline{\Omega})$, $u \geq \psi$ in $\overline{\Omega}$ and $u = g$ on $\partial\Omega$. Recalling (4.7), since $[\tilde{g}]_{C^{0,\alpha}(\overline{\Omega})} \leq 1$, then we have

$$[u_p]_{s,p} \leq [Cp]^{\frac{1}{p}}. \tag{4.8}$$

Fix $m < p$. Thanks again to (3.7), one has

$$\Big[\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^m}{|x-y|^{\alpha m}}\Big]^{\frac{1}{m}} \leq [Cp]^{\frac{1}{p}}|\Omega|^{2(\frac{p-m}{pm})}. \tag{4.9}$$

By Fatou's Lemma, we get

$$\liminf_{m\to\infty}\Big[\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^m}{|x-y|^{\alpha m}}\Big]^{\frac{1}{m}} \leq \liminf_{m\to\infty}\liminf_{p\to\infty}\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^m}{|x-y|^{\alpha m}}$$

$$\leq \liminf_{m\to\infty}\liminf_{p\to\infty}[Cp]^{\frac{1}{p}}|\Omega|^{2(\frac{p-m}{pm})} = \liminf_{m\to\infty}\,|\Omega|^{\frac{2}{m}} = 1.$$

Now, let $\phi \in C^{0,\alpha}(\overline{\Omega})$ with $\phi \geq \psi$ in $\overline{\Omega}$, $\phi = g$ on $\partial\Omega$ and $[\phi]_{C^{0,\alpha}(\overline{\Omega})} \leq 1$. Since u_p minimizes (4.1) and ϕ is admissible, then we have

$$\frac{1}{2p}\iint_{\Omega\times\Omega}\frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}} - \int_\Omega h(x)u_p(x) \leq \frac{1}{2p}\iint_{\Omega\times\Omega}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{\alpha p}} - \int_\Omega h(x)\phi(x).$$

Hence,

$$-\int_\Omega h(x)u_p(x) \leq \frac{|\Omega|^2}{2p} - \int_\Omega h(x)\phi(x).$$

Passing to the limit when p goes to ∞, we get

$$\int_\Omega h(x)u(x) \geq \int_\Omega h(x)\phi(x).$$

Fix $x_0 \in \Omega$. Let $\phi \in C^1(\Omega)\cap C(\overline{\Omega})$ be such that $u \leq \phi$ on $\bar{\Omega}$ and $u(x_0) = \phi(x_0)$. Then, we claim that

$$-L_\infty^-\phi(x_0) \leq 1.$$

By definition of L^-_∞, we have

$$-L^-_\infty\phi(x_0) = \sup_{x\in\Omega,\ x\neq x_0} \frac{\phi(x_0)-\phi(x)}{|x-x_0|^\alpha} \le \sup_{x\in\Omega,\ x\neq x_0} \frac{u(x_0)-u(x)}{|x-x_0|^\alpha} \le 1$$

where in the last two inequalities we have used that $\phi(x_0)=u(x_0)$, $u\le\phi$ in $\overline{\Omega}$ and $[u]_{C^{0,\alpha}}\le 1$.

Now, assume $\phi\in C^1(\Omega)\cap C(\overline{\Omega})$ is such that $u\ge\phi$ on $\bar{\Omega}$ and $\phi(x_0)=u(x_0)$. From the definition of L^+_∞, one has

$$L^+_\infty\phi(x_0) = \sup_{x\in\Omega,\ x\neq x_0} \frac{\phi(x)-\phi(x_0)}{|x-x_0|^\alpha} \le \sup_{x\in\Omega,\ x\neq x_0} \frac{u(x)-u(x_0)}{|x-x_0|^\alpha} \le 1.$$

Hence,

$$-L^+_\infty\phi(x_0)+1\ge 0.$$

Fix $x_0\in\{u>\psi\}\cap\{h<0\}$. So, we will show that u is a viscosity subsolution at x_0 to (4.6). Assume by contradiction that there is a function $\phi\in C^1(\Omega)\cap C(\overline{\Omega})$ with $u\le\phi$ on $\bar{\Omega}$ and $u(x_0)=\phi(x_0)$ such that

$$-L^+_\infty\phi(x_0)+1>0.$$

From the proof of Proposition 3.3, we recall that by considering $\phi_\delta(x):=\phi(x)+\delta|x-x_0|^2$ instead of ϕ, one can assume that x_0 is the unique maximizer of $u-\phi$. Since $u_p\to u$ uniformly in $\overline{\Omega}$, then there is a point $x_p\in\{u_p>\psi\}$ such that $u_p-\phi$ has a maximum at x_p and $x_p\to x_0$. Set $M_p:=\max_\Omega[u_p-\phi]$; we note that $M_p\to 0$, $u_p\le\phi+M_p$ and $u_p(x_p)=\phi(x_p)+M_p$. But, u_p is a viscosity solution to equation (4.2). Hence,

$$-L_p\phi(x_p)-h(x_p)\le 0.$$

Hence,

$$-\int_\Omega \frac{|\phi(x)-\phi(x_p)|^{p-1}}{|x-x_p|^{\alpha p}}\frac{\phi(x)-\phi(x_p)}{|\phi(x)-\phi(x_p)|}\,\mathrm{d}x - h(x_p)\le 0.$$

Thus, we get

$$A_p[\phi]^{p-1}\ge B_p[\phi]^{p-1}-h(x_p), \tag{4.10}$$

where we recall that

$$A_p[\phi]:=\left[\int_\Omega \frac{[\phi(x)-\phi(x_p)]_+^{p-1}}{|x-x_p|^{\alpha p}}\,\mathrm{d}x\right]^{\frac{1}{p-1}} \quad\text{and}\quad B_p[\phi]:=\left[\int_\Omega \frac{[\phi(x)-\phi(x_p)]_-^{p-1}}{|x-x_p|^{\alpha p}}\,\mathrm{d}x\right]^{\frac{1}{p-1}}.$$

Then,

$$\frac{B_p[\phi]^{p-1}}{A_p[\phi]^{p-1}}-\frac{h(x_p)}{A_p[\phi]^{p-1}}\le 1.$$

Since $h(x_0)<0$, then we have

$$\frac{B_p[\phi]}{A_p[\phi]}\le 1 \qquad\text{and}\qquad \frac{[-h(x_p)]^{\frac{1}{p-1}}}{A_p[\phi]}\le 1.$$

Recalling Lemma 3.4, we have that $A_p[\phi]\to L^+_\infty\phi$ and $B_p[\phi]\to -L^-_\infty\phi$. Passing to the limit when $p\to\infty$, we get

$$-L_\infty\phi(x_0)\le 0 \qquad\text{and}\qquad -L^+_\infty\phi(x_0)+1\le 0$$

which is a contradiction.

Fix $x_0\in\{u>\psi\}\cap\{h>0\}$. Assume there is a function $\phi\in C^1(\Omega)\cap C(\overline{\Omega})$ with $u\ge\phi$ on $\bar{\Omega}$ and $u(x_0)=\phi(x_0)$ such that

$$-L^-_\infty\phi(x_0)-1<0.$$

Again we may assume that x_0 is the unique minimizer of $u-\phi$. Thanks to the uniform convergence of u_p to u, there is a point $x_p \in \Omega$ such that $x_p \to x_0$ and $u_p - \phi$ has a minimum at x_p. We denote by $m_p := \min_\Omega[u_p-\phi] \to 0$. Since u_p is a viscosity solution to (4.2), then one has

$$-L_p[\phi](x_p) - h(x_p) \geq 0.$$

So, we have

$$B_p[\phi]^{p-1} \geq A_p[\phi]^{p-1} + h(x_p).$$

In particular, we get

$$\frac{A_p[\phi]}{B_p[\phi]} \leq 1 \qquad \text{and} \qquad \frac{h(x_p)^{\frac{1}{p-1}}}{B_p[\phi]} \leq 1.$$

Hence,

$$-L_\infty\phi(x_0) \geq 0 \qquad \text{and} \qquad -L^-_\infty\phi(x_0) - 1 \geq 0,$$

which is a contradiction.

If the point $x_0 \in \{u > \psi\} \cap (\Omega \backslash \overset{\circ}{\operatorname{spt}}\, h)$ then one can show exactly as above that u is a viscosity solution at x_0 to the following equation

$$L_\infty u = 0. \tag{4.11}$$

If $x_0 \in \{u = \psi\} \cap (\Omega \backslash \overset{\circ}{\operatorname{spt}}\, h)$, then u is a viscosity supersolution to (4.11) at x_0. In the same way, we show the remaining equations in (4.6) when $x_0 \in \partial\{h>0\}\backslash\partial\{h<0\}$ or $x_0 \in \partial\{h<0\}\backslash\partial\{h>0\}$. This concludes the proof. $\square$

Finally, we note that if $\alpha = 1$ then the maximization problem (4.5) without both the obstacle $u \geq \psi$ in $\overline{\Omega}$ and the boundary condition $u = g$ on $\partial\Omega$ is nothing else than the dual of the classical Monge-Kantorovich problem. In [3, 4], the authors studied a mass transportation problem between two masses h^+ and h^- (which do not have a priori the same total masses) with the possibility of transporting some mass to/from the boundary, paying the transport cost $c(x,y)$ plus an extra cost $g(y)$ for each unit of mass that comes out from a pointy $y \in \partial\Omega$ (the export taxes) or $-g(x)$ for each unit of mass that enters at the point $x \in \partial\Omega$ (the import taxes); this means that $\partial\Omega$ can be used as an infinite reserve/repository, we can take as much mass as we wish from the boundary, or send back as much mass as we want, provided that we pay the transportation cost plus the import/export taxes. In this case, the dual problem will be complemented with the boundary condition $u = g$ on $\partial\Omega$. In addition, assume that we can import mass through any point $x \in \Omega$ but we have to pay again an import tax given by $-\psi(x)$. Then, the obstacle condition $u \geq \psi$ in Ω will be added now to the dual problem. So, we conclude this section with the following connection with optimal transport theory (the proof of this duality will be similar to the one in [3]).

Proposition 4.3. *Under the assumptions that $[g]_{C^{0,\alpha}(\partial\Omega)} \leq 1$ and that the obstacle ψ satisfies the following inequality:*

$$\psi(x) \leq \min\{|x-x_0|^\alpha + g(x_0) \,:\, x_0 \in \partial\Omega\}, \qquad \textit{for all } x \in \Omega, \tag{4.12}$$

we have the following duality:

$$\max\left\{ \int_\Omega \phi(x)\,h(x)\,\mathrm{d}x \,:\, \phi \in C^{0,\alpha}(\overline{\Omega}),\ [\phi]_{C^{0,\alpha}(\overline{\Omega})} \leq 1,\ \phi \geq \psi \ \textit{ in } \overline{\Omega} \ \textit{ and } \phi = g \ \textit{ on } \partial\Omega \right\}$$

$$= \min\left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|^\alpha\,\mathrm{d}\gamma - \int_\Omega \psi\,\mathrm{d}[(\Pi_x)_\#\gamma - h^+] + \int_{\partial\Omega} g\,\mathrm{d}[(\Pi_y)_\#\gamma - (\Pi_x)_\#\gamma] \,:\, \gamma \in \Pi(h^+,h^-) \right\}$$

where

$$\Pi(h^+,h^-) := \{\gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}),\ (\Pi_x)_\#\gamma \geq h^+ \ \textit{ and } \ (\Pi_y)_\#\gamma \geq h^-\}.$$

5. The case of a superlinear nonhomogeneous term

This section is devoted to study the limit of Problem (3.6) when $p \to \infty$ but in the case when the nonhomogeneous term $h_p(x,u) = h(x)\frac{|u|^\Lambda}{\Lambda}$, where the constant $\Lambda = \Lambda(p) > 1$ depends on p and the function h is in $L^q(\Omega)$ for some $q > 1$. Then, we consider the following minimization problem:

(5.1)
$$\min\left\{\frac{1}{2p}\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} - \frac{1}{\Lambda}\int_\Omega h(x)\,|u|^\Lambda \;:\; u \in W^{s,p}(\Omega),\, u \geq \psi \text{ on } \bar{\Omega},\, u = g \text{ on } \partial\Omega\right\}.$$

Notice that this case was not covered in Section 3 since here the nonhomogeneous term is not necessarily nonnegative. Some of the proofs are similar to those in Section 3; therefore, we will omit certain details and focus on the main differences. We begin by proving the following existence result.

Proposition 5.1. *Under the assumption that $\Lambda < (1-\frac{1}{q})\,p$, Problem (5.1) has a minimizer u_p. In addition, u_p is a weak solution to the following problem:*

$$\begin{cases} -L_p u = h\,|u|^{\Lambda-2}\,u & \text{in } \{u > \psi\}, \\ -L_p u \leq h\,|u|^{\Lambda-2}\,u & \text{in } \{u = \psi\}. \end{cases} \tag{5.2}$$

Moreover, assume h is continuous and $p \geq 1/1-\alpha$. Then, u_p is also a viscosity solution to Problem (5.2).

Proof. Let $(u_n)_n$ be a minimizing sequence in Problem (5.1). Then, there is a constant $C < \infty$ such that
$$\frac{1}{2p}\iint_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{\alpha p}} - \frac{1}{\Lambda}\int_\Omega h(x)\,|u_n|^\Lambda \leq C.$$
Since $\Lambda < (1-\frac{1}{q})\,p$, then one has
$$\frac{1}{2p}\iint_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{\alpha p}} \leq \frac{1}{\Lambda}\,|\Omega|^{1-\frac{1}{q}-\frac{\Lambda}{p}}||h||_{L^q(\Omega)}\,||u_n||^\Lambda_{L^p(\Omega)} + C.$$
Hence,
$$[u_n]^p_{s,p} \leq C(||u_n||^\Lambda_{L^p(\Omega)} + 1).$$

Therefore, $(u_n)_n$ is bounded in $W^{s,p}(\Omega)$. But, this implies (see the proof of Proposition 4.1) that up to a subsequence, $u_n \to u_p$ uniformly in $\overline{\Omega}$ and u_p minimizes Problem (5.1). Finally, we note that if ϕ is a smooth function such that $\text{supp}(\phi) \subset \{u_p > \psi\}$ then, for all t small enough, one has
$$\begin{aligned}&\frac{1}{2p}\iint_{\Omega\times\Omega}\frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}} - \frac{1}{\Lambda}\int_\Omega h(x)|u_p|^\Lambda \\ &\leq \frac{1}{2p}\iint_{\Omega\times\Omega}\frac{|u_p(x)+t\phi(x)-u_p(y)-t\phi(y)|^p}{|x-y|^{\alpha p}} - \frac{1}{\Lambda}\int_\Omega h(x)|u_p+t\phi|^\Lambda.\end{aligned}$$
Hence,
$$-\iint_{\Omega\times\Omega}\frac{|u_p(x)-u_p(y)|^{p-2}}{|x-y|^{\alpha p}}[u_p(y)-u_p(x)]\,\phi(x) - \int_\Omega h(x)\,|u_p|^{\Lambda-2}\,u_p\,\phi(x) = 0.$$

□

Finally, we conclude the paper by the following:

Proposition 5.2. *Up to a subsequence, $u_p \to u$ uniformly in $\overline{\Omega}$. In addition, assume that*

$$\lim_{p\to\infty} \frac{\Lambda(p)}{p} = \Lambda^\star < 1 - \frac{1}{q}.$$

Then, the limit function u solves (in the viscosity sense) the following problem:

$$\begin{cases} \max\{-L_\infty u, -L^+_\infty u + |u|^{\Lambda^\star}\} = 0 & \text{in } \{u > \psi\} \cap \{h \cdot u > 0\}, \\ \min\{-L_\infty u, -L^-_\infty u - |u|^{\Lambda^\star}\} = 0 & \text{in } \{u > \psi\} \cap \{h \cdot u < 0\}, \\ \max\{-L_\infty u, -L^+_\infty u + |u|^{\Lambda^\star}\} \geq 0 & \text{in } \{u = \psi\} \cap \{h \cdot u > 0\}, \\ \min\{-L_\infty u, -L^-_\infty u - |u|^{\Lambda^\star}\} \geq 0 & \text{in } \{u = \psi\} \cap \{h \cdot u < 0\}, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

Proof. Since u_p minimizes (5.1), then we have

$$\frac{1}{2p}\iint_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}} - \frac{1}{\Lambda}\int_\Omega h(x)|u_p|^\Lambda \leq \frac{1}{2p}\iint_{\Omega\times\Omega} \frac{|\tilde g(x)-\tilde g(y)|^p}{|x-y|^{\alpha p}} - \frac{1}{\Lambda}\int_\Omega h(x)|\tilde g|^\Lambda$$

$$\leq \frac{|\Omega|^2\,[\tilde g]^p_{C^{0,\alpha}(\overline{\Omega})}}{2p} + \frac{1}{\Lambda}||\tilde g||^\Lambda_{L^\infty(\Omega)}\,||h||_{L^1(\Omega)} \leq C\left(\frac{[\tilde g]^p_{C^{0,\alpha}(\overline{\Omega})}}{p} + \frac{||\tilde g||^\Lambda_{L^\infty(\Omega)}}{\Lambda}\right) \leq \frac{C^p}{p}.$$

Yet,

$$\int_\Omega h(x)|u_p|^\Lambda \leq |\Omega|^{1-\frac{1}{q}-\frac{\Lambda}{p}}\,||h||_{L^q(\Omega)}\,(C[u_p-g]_{s,p} + |\Omega|^{\frac{1}{p}}\,||g||_{L^\infty(\Omega)})^\Lambda.$$

Therefore, we get that

$$\int_\Omega h(x)|u_p|^\Lambda \leq C^\Lambda([u_p]^\Lambda_{s,p} + 1).$$

Hence,

$$\frac{1}{2p}[u_p]^p_{s,p} - \frac{C^\Lambda}{\Lambda}[u_p]^\Lambda_{s,p} \leq \frac{C^p}{p}.$$

Then, we get

$$\frac{1}{2p}[u_p]^p_{s,p} - \frac{\varepsilon}{p}\,[u_p]^p_{s,p} - \frac{\varepsilon^{\frac{-\Lambda}{p-\Lambda}}}{\frac{p\Lambda}{p-\Lambda}}\,C^{\frac{p\Lambda}{p-\Lambda}} \leq \frac{C^p}{p}.$$

Consequently,

$$\left(\frac{1}{2} - \varepsilon\right)[u_p]^p_{s,p} \leq C^p + \frac{\varepsilon^{\frac{-\Lambda}{p-\Lambda}}}{\frac{\Lambda}{p-\Lambda}}\,C^{\frac{p\Lambda}{p-\Lambda}}.$$

Since $\Lambda < (1-\frac{1}{q})\,p$, then we have

$$[u_p]_{s,p} \leq C + \left(\frac{\varepsilon^{\frac{-\Lambda}{p-\Lambda}}}{\frac{\Lambda}{p-\Lambda}}\right)^{\frac{1}{p}} C^{\frac{\Lambda}{p-\Lambda}} \leq C.$$

Yet,

$$||u_p||_\infty \leq C([u]_{s,p} + [\tilde g]_{s,p}) + ||\tilde g||_\infty \leq C.$$

Thus, we infer that up to a subsequence, $u_p \to u$ uniformly in $\overline{\Omega}$ and $u \in C^{0,\alpha}(\overline{\Omega})$ with $u \geq \psi$ in $\overline{\Omega}$ and $u = g$ on $\partial\Omega$.

Finally, we show briefly that u solves (5.3) in the viscosity sense. Fix $x_0 \in \{u > \psi\}$. Then, we will show that u is a viscosity subsolution at x_0 to (5.3). Assume that $h(x_0)\cdot u(x_0) > 0$.

Let us assume that there is a function $\phi \in C^1(\Omega) \cap C(\overline{\Omega})$ such that $u \leq \phi$ on $\bar{\Omega}$, $u(x_0) = \phi(x_0)$ and,

$$\max\{-L_\infty \phi(x_0), -L_\infty^+ \phi(x_0) + |\phi(x_0)|^{\Lambda^\star}\} > 0.$$

Since $u_p \to u$ uniformly in Ω, then there is a point $x_p \in \{u_p > \psi\}$ such that $u_p - \phi$ has a maximum at x_p and $x_p \to x_0$. Set $M_p := \max_\Omega [u_p - \phi]$. So, one has $M_p \to 0$, $u_p \leq \phi + M_p$ and $u_p(x_p) = \phi(x_p) + M_p$. Yet, u_p is a viscosity solution to equation (5.2). Hence,

$$-\int_\Omega \frac{|\phi(x) - \phi(x_p)|^{p-1}}{|x - x_p|^{\alpha p}} \frac{\phi(x) - \phi(x_p)}{|\phi(x) - \phi(x_p)|} \,\mathrm{d}x + h(x_p)|\phi(x_p) + M_p|^{\Lambda - 2}[\phi(x_p) + M_p] \leq 0. \tag{5.4}$$

Then, we have

$$\frac{B_p[\phi]^{p-1}}{A_p[\phi]^{p-1}} + \frac{h(x_p)|\phi(x_p) + M_p|^{\Lambda-2}[\phi(x_p) + M_p]}{A_p[\phi]^{p-1}} \leq 1.$$

Therefore, we have

$$\frac{B_p[\phi]}{A_p[\phi]} \leq 1 \qquad \text{and} \qquad \frac{|\phi(x_p) + M_p|^{\frac{\Lambda-2}{p-1}}[h(x_p)(\phi(x_p) + M_p)]^{\frac{1}{p-1}}}{A_p[\phi]} \leq 1.$$

Recalling Lemma 3.4, one has $A_p[\phi] \to L_\infty^+ \phi$ and $B_p[\phi] \to -L_\infty^- \phi$. Passing to the limit when $p \to \infty$, we get

$$-L_\infty \phi(x_0) \leq 0 \qquad \text{and} \qquad -L_\infty^+ \phi(x_0) + |\phi(x_0)|^{\Lambda^\star} \leq 0.$$

Finally, we note that the other equations in (5.3) can be treated similarly. This concludes the proof. □

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