ZERO DIFFUSION-DISPERSION LIMIT FOR THE BENJAMIN–ONO–BURGERS EQUATION

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Abstract. We prove that the solution of the viscous Benjamin–Ono equation converges, as diffusion and dispersion parameters tend to zero (under a suitable balance condition), to the unique entropy solution of the inviscid Burgers equation. The key tool in our proof is Schonbek's L^p-compensated compactness method. Specifically, we prove a uniform L^4-estimate using a modification of a conserved quantity for the inviscid Benjamin–Ono equation and a suitable differential inequality argument.

1. Introduction

The *Benjamin–Ono equation* was first formally derived by Brooke Benjamin in [4] (and, independently, by Russ E. Davis and Andreas Acrivos in [34]) and later by Hiroaki Ono in [57] to describe the propagation of long weakly nonlinear internal waves in a stratified fluid such that two layers of different densities are joined by a thin region where the density varies continuously, the lower layer being infinite. We refer to [59] for a recent detailed survey of the literature about this model (see also [47, Chapter 3] and [6]).

We shall focus on the *viscous Benjamin–Ono equation* (also known as *Benjamin–Ono–Burgers equation*):

$$\begin{cases} \partial_t u_{\varepsilon,\,\delta} + u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta} = \varepsilon\partial_x^2 u_{\varepsilon,\,\delta} - \delta\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}, & t>0,\ x\in\mathbb{R},\\ u_{\varepsilon,\,\delta}(0,x) = u_{0,\,\varepsilon,\,\delta}(x), & x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is the *diffusion coefficient*, $\delta > 0$ is the *dispersion coefficient*, and $\mathcal{H}$ is the *Hilbert transform*, which is defined as follows:

$$\mathcal{H}u(x) := \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{u(y)}{y-x}\,\mathrm{d}y = \mathcal{F}^{-1}(i\,\mathrm{sign}(\xi)\mathcal{F}(\xi)),$$

with $\mathcal{F}$ denoting the *Fourier transform*. By changing variables, the principal value integral above can be written explicitly as

$$\mathcal{H}u(x) = -\frac{1}{\pi}\lim_{\varepsilon\searrow 0}\int_{\varepsilon}^{\infty} \frac{u(x+y)-u(x-y)}{y}\,\mathrm{d}y.$$

If $u\in L^p(\mathbb{R})$, for $1<p<\infty$, then the Hilbert transform is well-defined: i.e., the limit defining the improper integral exists for almost every x; moreover, the limit function $\mathcal{H}u$ belongs to $L^p(\mathbb{R})$ as well.

The viscous Benjamin–Ono equation, whose physical relevance has been discussed in [35], is well-posed in $H^s(\mathbb{R})$, with $s\geqslant 1$, and in $H^{1/2}(\mathbb{R})$ (see [43, 55]), *uniformly* with respect to $\varepsilon\geqslant 0$.

In the present work, we are interested in studying the competition between dispersive and diffusive effects in the singular limit $\varepsilon,\,\delta\searrow 0$ of $\{u_{\varepsilon,\,\delta}\}_{\varepsilon,\,\delta>0}$.

This zero–dispersion limit problem was first addressed by Lax and Levermore, using inverse scattering theory, in the case of the *Korteweg–de Vries equation*,

$$\begin{cases} \partial_t u_\delta + u_\delta\partial_x u_\delta = -\delta\partial_x^3 u_\delta, & t>0,\ x\in\mathbb{R},\\ u_\delta(0,x) = u_{0,\delta}(x), & x\in\mathbb{R}, \end{cases} \tag{1.2}$$

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who showed that solutions *do not* converge in a strong topology (oscillatory effect of capillarity). On the other hand, in the presence of competing diffusion and dispersion, i.e., for the *KdV–Burgers equation*

$$\begin{cases} \partial_t u_{\varepsilon,\delta} + u_{\varepsilon,\delta}\partial_x u_{\varepsilon,\delta} = \varepsilon\partial_x^2 u_{\varepsilon,\delta} - \delta\partial_x^3 u_{\varepsilon,\delta}, & t>0,\ x\in\mathbb{R},\\ u_{\varepsilon,\delta}(0,x) = u_{0,\varepsilon,\delta}(x), & x\in\mathbb{R}, \end{cases} \tag{1.3}$$

Schonbek, in [60], used a novel L^p compensated compactness technique to prove that the strong convergence to the (unique) *entropy admissible* solution of the *Burgers equation*

$$\begin{cases} \partial_t u + u\partial_x u = 0, & t>0,\ x\in\mathbb{R},\\ u(0,x) = u_0(x), & x\in\mathbb{R}, \end{cases} \tag{1.4}$$

holds under a suitable balance condition between ε and δ. This convergence result has been later extended in various directions (cf., e.g., [48, 44, 1, 49, 46, 3, 2, 30, 31])[1]. In conclusion, depending on the relationship between ε and δ when taking the limit $\varepsilon,\delta \searrow 0$, different results are obtained: the family $\{u_{\varepsilon,\delta}\}_{\varepsilon,\delta>0}$ may converge to a weak solution of the Burgers equation, or to the entropy solution, or be highly oscillatory and not converge to a solution.

For the Benjamin–Ono equation, on the other hand, much less is known. If $\delta > 0$ is fixed, the vanishing viscosity limit, $\varepsilon \searrow 0$, was studied in [43, 55] (as a byproduct of the uniform well-posedness results mentioned above). For the *inviscid* Benjamin–Ono equation (i.e., in case $\varepsilon = 0$), the zero dispersion limit, $\delta \searrow 0$, was studied (either on the torus or on the real line) in [54, 53, 52, 40, 41] using inverse scattering theory for the Lax operator inherited from the integrability of the Benjamin–Ono equation. More recently, in [42], the zero-dispersion limit was identified by relying on an explicit solution formula and a combination of suitable approximation arguments and computations carried out in the special case of rational initial data (see also [8, 7, 5] for further studies). However, as in the case of the zero-dispersion limit for KdV, this weak limit is not a weak solution of the Burgers equation (1.4). Roughly speaking, the shock happening for the inviscid Burgers equation is converted into a dispersive shock wave for solutions to the Benjamin–Ono equation: for small dispersion parameters, it becomes strongly oscillatory in a localized region that is precisely where the inviscid Burgers solution is multivalued.

We are interested in showing that, in the presence of a vanishing viscosity, provided that δ is small enough compared to ε, the family $\{u_{\varepsilon,\delta}\}_{\varepsilon,\delta>0}$ of solutions of (1.1) converges to the (unique) entropy solution of the Burgers equation (1.4).

Going forward, we assume that the initial datum satisfies

$$u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}) \tag{1.5}$$

and that it is regularized as follows:

$$u_{0,\varepsilon,\delta} \in H^2(\mathbb{R}), \tag{1.6}$$

$$\|u_{0,\varepsilon,\delta}\|^4_{L^4(\mathbb{R})} + \varepsilon^2\|\partial_x u_{0,\varepsilon,\delta}\|^2_{L^2(\mathbb{R})} \leqslant C, \tag{1.7}$$

$$\|u_{0,\varepsilon,\delta}\|_{L^2(\mathbb{R})} \leqslant \|u_0\|_{L^2(\mathbb{R})}, \tag{1.8}$$

$$\|u_{0,\varepsilon,\delta}\|_{L^4(\mathbb{R})} \leqslant \|u_0\|_{L^4(\mathbb{R})}, \tag{1.9}$$

$$u_{0,\varepsilon,\delta} \longrightarrow u_0, \qquad \text{a.e. and in } L^p_{\mathrm{loc}}(\mathbb{R}),\ 1\leqslant p\leqslant 4, \quad \text{as } \varepsilon,\delta\searrow 0, \tag{1.10}$$

(for some constant $C > 0$ that does not depend on ε and δ). Owing to [43, Theorem 1], we already know that, if $u_{0,\varepsilon,\delta} \in H^s(\mathbb{R})$, with $s \geqslant 1$, then there exists a unique solution $u_{\varepsilon,\delta} \in C(\mathbb{R}_+; H^s(\mathbb{R}))$ of (1.1).

We remark that assumption (1.7) prescribes how the regularization $u_{0,\varepsilon,\delta}$ blows up in $\dot{H}^1$ as $\varepsilon,\delta \searrow 0$, namely, as δ^{-1} and will play a key role in Lemma 2.2 below.

Our main result is as follows.

Theorem 1.1 (Zero diffusion-dispersion limit for the Benjamin–Ono equation)**.** *Let us assume that $\varepsilon,\delta > 0$ and $\delta = O(\varepsilon^{\frac{3}{2}})$. Let us consider an initial datum u_0, satisfying (1.5), and a family $\{u_{0,\varepsilon,\delta}\}_{\varepsilon,\delta>0}$ such that (1.6)–(1.10) holds.*

[1] Schonbek's L^p compensated compactness method has been also applied to the study of other kinds of higher-order approximations for scalar conservation laws (cf., e.g., [9, 21, 25, 24, 13, 23, 14, 12, 27, 17, 29, 19, 20, 16, 11, 15, 18, 26, 10, 22, 28, 39, 38, 32, 37, 36, 45, 61]).

Let $u_{\varepsilon,\delta}$ be the (unique) solution of the Cauchy problem (1.1). Then, with u the (unique) entropy solution of (1.4), we have that

$$u_{\varepsilon,\delta} \longrightarrow u \quad \text{strongly in } L^p_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}), \text{ for every } 1 \leqslant p < 4$$

when $\varepsilon \to 0$.

1.1. **Outline of the proof of Theorem 1.1.** The key ingredient of the proof of Theorem 1.1 is Schonbek's L^p-compensated compactness framework (see [60, 50]).

Lemma 1.2 (L^p compensated compactness)**.** *Let Ω be a bounded open subset of $\mathbb{R}_+ \times \mathbb{R}$. Let $f \in C^2(\mathbb{R})$ satisfy*

$$|f(u)| \leqslant C|u|^{s+1} \text{ for } u \in \mathbb{R}, \qquad |f'(u)| \leqslant C|u|^s \text{ for } u \in \mathbb{R},$$

for some $s \geqslant 0$, and

$$\operatorname{meas}\{u \in \mathbb{R} : f''(u) = 0\} = 0.$$

Given $\ell > 0$, let us define functions I_ℓ, f_ℓ, $F_\ell : \mathbb{R} \to \mathbb{R}$ as follows:

$$\begin{cases} I_\ell \in C^2(\mathbb{R}), \quad |I_\ell(u)| \leqslant |u| \text{ for } u \in \mathbb{R}, \quad |I'_\ell(u)| \leqslant 2 \text{ for } u \in \mathbb{R}, \\ |I_\ell(u)| \leqslant |u| \text{ for } |u| \leqslant l, \quad I_\ell(u) = 0 \text{ for } |u| \geqslant 2l \end{cases}$$

and

$$f_\ell(u) = \int_0^u I'_\ell(\zeta) f'(\zeta)\, \mathrm{d}\zeta, \qquad F_\ell(u) = \int_0^u f'_\ell(\zeta) f'(\zeta)\, \mathrm{d}\zeta$$

Let us suppose $\{u_n\}_{n=1}^\infty \subset L^{2(s+1)}(\Omega)$ is such that the two sequences

$$\{\partial_t I_\ell(u_n) + \partial_x f_\ell(u_n)\}_{n=1}^\infty, \qquad \{\partial_t f_\ell(u_n) + \partial_x F_\ell(u_n)\}_{n=1}^\infty$$

of distributions belong to a compact subset of $H^{-1}_{\text{loc}}(\Omega)$, for each fixed $l > 0$. Then there exists a subsequence of $\{u_n\}_{n=1}^\infty$ that converges to a function $u \in L^{2(s+1)}(\Omega)$ strongly in $L^r(\Omega)$ for any $1 \leqslant r < 2(s+1)$.

In order to apply Lemma 1.2, we need to establish a *uniform* L^4_{loc}-estimate and to prove that the entropy production (for a compactly supported entropy) is compact in H^{-1}_{loc}.

First, in Section 2, we establish a uniform energy estimate. The form of the energy dissipation will ultimately yield (when combined to a suitable balance condition on the ratio δ/ε) the compactness of the entropy production. Second, we deduce a uniform L^4_{loc}-estimate assuming now that $\delta = O(\varepsilon^{\frac{3}{2}})$.

2. A priori estimates

In this section, we obtain several a priori estimates that will be needed to gain strong compactness in $L^2(\mathbb{R})$ of the family $\{u_{\varepsilon,\delta}\}_{\varepsilon,\delta>0}$.

We start by recalling some properties of the Hilbert transform that will be needed in the proofs (see [62, 58]):

- (H-i) *Anti-involution*: $\mathcal{H}^2 u = -u$;
- (H-ii) *Anti-self adjointness*: $\int_{\mathbb{R}} u\mathcal{H}v\, \mathrm{d}x = -\int_{\mathbb{R}} v\mathcal{H}u\, \mathrm{d}x$ if $u \in L^p(\mathbb{R})$ and $v \in L^q(\mathbb{R})$, with $1 < p, q < \infty$ Hölder conjugates;
- (H-iii) *Corollary I of anti-involution and anti-self adjointness*: $\int_{\mathbb{R}} u\mathcal{H}u\, \mathrm{d}x = 0$ if $u \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$, with $1 < p, q < \infty$ Hölder conjugates;
- (H-iv) *Corollary II of anti-involution and anti-self adjointness*: $\int_{\mathbb{R}} \mathcal{H}u\mathcal{H}v\, \mathrm{d}x = \int_{\mathbb{R}} uv\, \mathrm{d}x$ if $u \in L^p(\mathbb{R})$ and $v \in L^q(\mathbb{R})$ with $1 < p, q < \infty$ Hölder conjugates;
- (H-v) *Product rule*: $\mathcal{H}(uv) = u\mathcal{H}v + v\mathcal{H}u + \mathcal{H}(\mathcal{H}u\mathcal{H}v)$;
- (H-vi) *Behavior with respect to differentiation:* $\mathcal{H}\partial_x^k u = \partial_x^k \mathcal{H}u$ if $u \in W^{k,p}(\mathbb{R})$ with $1 < p < \infty$.
- (H-vii) *Relationship with the fractional Laplacian*: $\int_{\mathbb{R}} u\mathcal{H}\partial_x u\, \mathrm{d}x = \int_{\mathbb{R}} |(-\Delta)^{1/4} u|^2\, \mathrm{d}x = \|u\|^2_{H^{1/2}(\mathbb{R})}$ if $u \in L^p(\mathbb{R})$ and $\partial_x u \in L^q(\mathbb{R})$ with $1 < p, q < \infty$ Hölder conjugates.

As a first step, we prove the following L^2-estimate. The dissipative term in it will play a key role in showing that the entropy production $\{\partial_t\eta(u_{\varepsilon,\delta})+\partial_x q(u_{\varepsilon,\delta})\}_{\varepsilon,\delta>0}$ is compact in H^{-1}_{loc}.

Lemma 2.1 (L^2-estimate)**.** *Let $u_{\varepsilon,\delta}$ be the solution of (1.1). Then the following estimates hold:*

$$\int_{\mathbb{R}} |u_{\varepsilon,\delta}(t,x)|^2\,\mathrm{d}x + 2\varepsilon\int_0^t\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant \int_{\mathbb{R}} |u_0(x)|^2\,\mathrm{d}x, \tag{2.1}$$

$$\int_{\mathbb{R}} |\mathcal{H}u_{\varepsilon,\delta}(t,x)|^2\,\mathrm{d}x + 2\varepsilon\int_0^t\int_{\mathbb{R}} |\mathcal{H}\partial_x u_{\varepsilon,\delta}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant \int_{\mathbb{R}} |u_0(x)|^2\,\mathrm{d}x. \tag{2.2}$$

Proof. We compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{R}} u_{\varepsilon,\delta}^2\,\mathrm{d}x &= \int_{\mathbb{R}} u_{\varepsilon,\delta}\partial_t u_{\varepsilon,\delta}\,\mathrm{d}x\\
&= \int_{\mathbb{R}} u_{\varepsilon,\delta}(\varepsilon\partial_x^2 u_{\varepsilon,\delta} - \delta\mathcal{H}\partial_x^2 u_{\varepsilon,\delta} + u_{\varepsilon,\delta}\partial_x u_{\varepsilon,\delta})\,\mathrm{d}x\\
&= -\varepsilon\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}|^2\,\mathrm{d}x - \delta\underbrace{\int_{\mathbb{R}} u_{\varepsilon,\delta}\mathcal{H}\partial_x^2 u_{\varepsilon,\delta}\,\mathrm{d}x}_{=0} - \underbrace{\int_{\mathbb{R}} u_{\varepsilon,\delta}^2\partial_x u_{\varepsilon,\delta}\,\mathrm{d}x}_{=0}\\
&= -\varepsilon\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}|^2\,\mathrm{d}x.
\end{aligned}$$

Integrating on $(0,t)$, and recalling (1.8), this yields (2.1). The claim in (2.2) is equivalent to (2.1) owing to property (H-iv) of the Hilbert transform. □

Secondly, we need to prove (uniform) L^4_{loc}-integrability of $\{u_{\varepsilon,\delta}\}_{\varepsilon,\delta>0}$. As noted in [51], the quantity

$$\mathcal{I}_4 := \int_{\mathbb{R}} \left(\frac{1}{4}|u_{\varepsilon,\delta}(t,x)|^4 + \frac{3}{2}\delta\, u_{\varepsilon,\delta}^2\mathcal{H}\partial_x u_{\varepsilon,\delta}(t,x) + 2\delta^2\,|\partial_x u_{\varepsilon,\delta}(t,x)|^2\right)\mathrm{d}x$$

is conserved by the *inviscid* Benjamin–Ono equation (but not by the *viscous* one). In the following lemma, we are able to use a suitable perturbation of $\mathcal{I}_4$ to obtain a uniform bound on the L^4-norm of $u_{\varepsilon,\delta}$ provided that $\delta = O(\varepsilon^{\frac{3}{2}})$ and that assumption (1.7) holds on the regularized initial data.

Lemma 2.2 (L^4-estimate)**.** *Let $u_{\varepsilon,\delta}$ be the solution of (1.1). Let us suppose that $\delta = O(\varepsilon^{\frac{3}{2}})$. Then, there exists $K>0$ (depending only on the constant C from (1.7)) such that*

$$\int_{\mathbb{R}} u_{\varepsilon,\delta}^4(t,x)\,\mathrm{d}x \leqslant K.$$

Proof of Lemma 2.2. We introduce the functional

$$\mathcal{I}_{4,\varepsilon} := \int_{\mathbb{R}} \left(\frac{1}{4}|u_{\varepsilon,\delta}|^4 + \frac{3}{2}\delta\, u_{\varepsilon,\delta}^2\mathcal{H}\partial_x u_{\varepsilon,\delta} + 2\varepsilon^2\,|\partial_x u_{\varepsilon,\delta}|^2\right)\mathrm{d}x,$$

which is equal to the conserved quantity $\mathcal{I}_4$, up to replacing the δ^2 in the last term by ε^2. We will compute its time-evolution in three pieces and then put them together and estimate them to deduce a suitable differential inequality.

Step 1. *L^4-norm contribution:*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{4}\int_{\mathbb{R}} u_{\varepsilon,\delta}^4\,\mathrm{d}x &= \int_{\mathbb{R}} u_{\varepsilon,\delta}^3\partial_t u_{\varepsilon,\delta}\,\mathrm{d}x\\
&= \int_{\mathbb{R}} u_{\varepsilon,\delta}^3\left(\varepsilon\partial_x^2 u_{\varepsilon,\delta} - \delta\mathcal{H}\partial_x^2 u_{\varepsilon,\delta} - u_{\varepsilon,\delta}\partial_x u_{\varepsilon,\delta}\right)\mathrm{d}x\\
&= -3\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\delta}^2|\partial_x u_{\varepsilon,\delta}|^2\,\mathrm{d}x + 3\delta\int_{\mathbb{R}} u_{\varepsilon,\delta}^2\partial_x u_{\varepsilon,\delta}\mathcal{H}\partial_x u_{\varepsilon,\delta}\,\mathrm{d}x - \underbrace{\int_{\mathbb{R}} u_{\varepsilon,\delta}^4\partial_x u_{\varepsilon,\delta}\,\mathrm{d}x}_{=0}\\
&= -3\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\delta}^2|\partial_x u_{\varepsilon,\delta}|^2\,\mathrm{d}x + 3\delta\int_{\mathbb{R}} u_{\varepsilon,\delta}^2\partial_x u_{\varepsilon,\delta}\mathcal{H}\partial_x u_{\varepsilon,\delta}\,\mathrm{d}x;
\end{aligned}$$

Step 2. $\dot{H}^1$*-norm contribution:*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}2\varepsilon^2\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x &= 4\varepsilon^2\int_{\mathbb{R}}\partial_x u_{\varepsilon,\,\delta}\partial^2_{tx}u_{\varepsilon,\,\delta}\,\mathrm{d}x\\
&= 4\varepsilon^2\int_{\mathbb{R}}\partial_x u_{\varepsilon,\,\delta}\partial_x\left(\varepsilon\partial_x^2 u_{\varepsilon,\,\delta}-\delta\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}-u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\right)\\
&= -4\varepsilon^2\int_{\mathbb{R}}\partial_x^2 u_{\varepsilon,\,\delta}\left(\varepsilon\partial_x^2 u_{\varepsilon,\,\delta}-\delta\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}-u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\right)\\
&= -4\varepsilon^2\varepsilon\int_{\mathbb{R}}|\partial_x^2 u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x+4\delta\varepsilon^2\underbrace{\int_{\mathbb{R}}\partial_x^2 u_{\varepsilon,\,\delta}\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x}_{=0}\\
&\quad+4\varepsilon^2\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x\\
&= -4\varepsilon^3\int_{\mathbb{R}}|\partial_x^2 u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x+4\varepsilon^2\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x;
\end{aligned}$$

Step 3. $u^2_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}$ *contribution:*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{3}{2}\delta\int_{\mathbb{R}}u^2_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\,\mathrm{d}x = 3\delta\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\partial_t u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}+\frac{3}{2}\delta\int_{\mathbb{R}}u^2_{\varepsilon,\,\delta}\mathcal{H}\partial^2_{tx}u_{\varepsilon,\,\delta}\,\mathrm{d}x\\
&= 3\delta\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\left(\varepsilon\partial_x^2 u_{\varepsilon,\,\delta}-\delta\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}-u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\right)\,\mathrm{d}x\\
&\quad-3\delta\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\left(\varepsilon\partial_x^2 u_{\varepsilon,\,\delta}-\delta\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}-u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\right)\,\mathrm{d}x\\
&= 3\delta\varepsilon\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\partial_x^2 u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\,\mathrm{d}x-3\delta^2\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x-3\delta\int_{\mathbb{R}}u^2_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\,\mathrm{d}x\\
&\quad-3\delta\varepsilon\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x-3\delta^2\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x+3\delta\underbrace{\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}(u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta})\,\mathrm{d}x}_{=0}\\
&= \underbrace{3\delta\varepsilon\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\partial_x^2 u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\,\mathrm{d}x}_{I_1}\underbrace{-3\delta^2\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x}_{I_2}-3\delta\int_{\mathbb{R}}u^2_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\,\mathrm{d}x\\
&\quad-3\delta\varepsilon\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x-3\delta^2\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x.
\end{aligned}$$

To deal with I_2, we use the formula $\mathcal{H}(u\mathcal{H}u)=\frac{1}{2}\big((\mathcal{H}u)^2-u^2\big)$ and compute as follows:

$$\begin{aligned}
-3\delta^2\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x &= -\frac{3}{2}\delta^2\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\partial_x((\mathcal{H}\partial_x u_{\varepsilon,\,\delta})^2)\,\mathrm{d}x\\
&= \frac{3}{2}\delta^2\int_{\mathbb{R}}\partial_x u_{\varepsilon,\,\delta}(\mathcal{H}\partial_x u_{\varepsilon,\,\delta})^2\,\mathrm{d}x\\
&= -\frac{3}{2}\delta^2\int_{\mathbb{R}}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}(\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta})\,\mathrm{d}x\\
&= -\frac{3}{4}\delta^2\int_{\mathbb{R}}\partial_x u_{\varepsilon,\,\delta}(\mathcal{H}\partial_x u_{\varepsilon,\,\delta})^2\,\mathrm{d}x+\frac{3}{4}\delta^2\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\,\delta})^3\,\mathrm{d}x\\
&= \frac{3}{2}\delta^2\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x-\frac{3}{2}\delta^2\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x,
\end{aligned}$$

i.e.,

$$\left(-3-\frac{3}{2}\right)\delta^2\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x = -\frac{3}{2}\delta^2\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x,$$

which means

$$I_2 = -\delta^2\int_{\mathbb{R}}u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x.$$

For I_1, integration by parts yields

$$I_1 = -3\delta\varepsilon \int_{\mathbb{R}} (\partial_x u_{\varepsilon,\,\delta})^2 \mathcal{H}\partial_x u_{\varepsilon,\,\delta}\,\mathrm{d}x - 3\delta\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x.$$

Regrouping, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{3}{2}\delta \int_{\mathbb{R}} u_{\varepsilon,\delta}^2 \mathcal{H}\partial_x u_{\varepsilon,\,\delta}\,\mathrm{d}x &= -3\delta \int_{\mathbb{R}} u_{\varepsilon,\,\delta}^2 \partial_x u_{\varepsilon,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\,\mathrm{d}x - 4\delta^2 \int_{\mathbb{R}} u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x \\ &\quad - 6\delta\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x^2 u_{\varepsilon,\delta}\,\mathrm{d}x - 3\delta\varepsilon \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\,\delta}|^2 \mathcal{H}\partial_x u_{\varepsilon,\,\delta}\,\mathrm{d}x.\end{aligned}$$

Step 4. *Interpolation.* In conclusion, summing up the three contributions, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_{4,\varepsilon} &= -3\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\,\delta}^2 |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x - 4\varepsilon^3 \int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x \\ &\quad + 4(\varepsilon^2-\delta^2)\int_{\mathbb{R}} u_{\varepsilon,\delta}\partial_x u_{\varepsilon,\,\delta}\partial_x^2 u_{\varepsilon,\delta}\,\mathrm{d}x - 6\delta\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x^2 u_{\varepsilon,\delta}\,\mathrm{d}x \\ &\quad - 3\delta\varepsilon \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}|^2 \mathcal{H}\partial_x u_{\varepsilon,\,\delta}\,\mathrm{d}x.\end{aligned} \tag{2.3}$$

By Cauchy–Schwarz and Young's inequalities, we estimate

$$\begin{aligned}\left|\int_{\mathbb{R}} u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\delta}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x\right| &\leqslant \left(\int_{\mathbb{R}} u_{\varepsilon,\,\delta}^2 |\partial_x u_{\varepsilon,\delta}|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon,\,\delta}|^2\right)^{\frac{1}{2}} \\ &\leqslant \frac{1}{2\varepsilon}\int_{\mathbb{R}} u_{\varepsilon,\delta}^2 |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x + \frac{\varepsilon}{2}\int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x,\end{aligned}$$

as well as

$$\begin{aligned}\left|\int_{\mathbb{R}} u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\delta}\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x\right| &\leqslant \left(\int_{\mathbb{R}} u_{\varepsilon,\,\delta}^2 |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leqslant \left(\int_{\mathbb{R}} u_{\varepsilon,\,\delta}^2 |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leqslant \frac{1}{12\delta}\int_{\mathbb{R}} u_{\varepsilon,\delta}^2 |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x + 3\delta\int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x.\end{aligned}$$

Therefore, assuming for now only that $\delta = o_{\varepsilon\to 0}(\varepsilon)$, for $\varepsilon > 0$ small enough,

$$\begin{aligned}&\left|4(\varepsilon^2-\delta^2)\int_{\mathbb{R}} u_{\varepsilon,\delta}\partial_x u_{\varepsilon,\,\delta}\partial_x^2 u_{\varepsilon,\delta}\,\mathrm{d}x\right| + \left|6\delta\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\,\delta}\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta}\,\mathrm{d}x\right| \\ &\leqslant \left(2\frac{\varepsilon^2-\delta^2}{\varepsilon} + \frac{\varepsilon}{2}\right)\int_{\mathbb{R}} u_{\varepsilon,\,\delta}^2 |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x + (2(\varepsilon^2-\delta^2)\varepsilon + 18\,\delta^2\varepsilon)\int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x \\ &\leqslant \frac{11}{4}\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\delta}^2 |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x + 3\varepsilon^3\int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x.\end{aligned} \tag{2.4}$$

For the remaining term in (2.3), we use first Cauchy–Schwarz' inequality, then Gargliardo–Nirenberg's inequality (in the form $\|v\|_{L^4(\mathbb{R})} \leqslant C\|\partial_x v\|_{L^2(\mathbb{R})}^{1/4}\|v\|_{L^2(\mathbb{R})}^{3/4}$, with $v = \partial_x u_{\varepsilon,\,\delta}$), and finally Young's inequality to deduce

$$\begin{aligned}&3\left|\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\,\delta}|^2 \mathcal{H}\partial_x u_{\varepsilon,\,\delta}\,\mathrm{d}x\right| \\ &\leqslant 3\left(\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}|^4\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leqslant C\left(\left(\int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\right)^{\frac{3}{2}}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leqslant C\left(\int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon,\delta}|^2\,\mathrm{d}x\right)^{\frac{1}{4}}\left(\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\right)^{\frac{5}{4}}\end{aligned}$$

$$\leqslant \frac{\varepsilon^2\delta^{-1}}{2}\int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x + C\varepsilon^{-\frac{2}{3}}\delta^{\frac{1}{3}}\left(\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}|^2\,\mathrm{d}x\right)^{\frac{5}{3}}.$$

This implies that

$$\left|3\delta\varepsilon\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}|^2 \mathcal{H}\partial_x u_{\varepsilon,\,\delta}\,\mathrm{d}x\right| \leqslant \frac{\varepsilon^3}{2}\int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x + C\delta^{\frac{4}{3}}\varepsilon^{\frac{1}{3}}\left(\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\right)^{\frac{5}{3}}. \tag{2.5}$$

Step 4. *Differential inequality argument.* Putting (2.4)–(2.5) into (2.3), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_{4,\varepsilon} \leqslant C\delta^{\frac{4}{3}}\varepsilon^{\frac{1}{3}}\left(\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\right)^{\frac{5}{3}} \leqslant C\delta^{\frac{4}{3}}\varepsilon^{-3}\left(\varepsilon^2\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\right)^{\frac{5}{3}}. \tag{2.6}$$

Recalling that

$$\mathcal{I}_{4,\varepsilon} = \int_{\mathbb{R}} \frac{u_{\delta,\varepsilon}^4}{4}\,\mathrm{d}x + \frac{3}{2}\delta\int_{\mathbb{R}} u_{\varepsilon,\,\delta}^2\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\,\mathrm{d}x + 2\varepsilon^2\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x,$$

we use Cauchy–Schwarz' inequality, still assuming that $\delta = o_{\varepsilon\to 0}(\varepsilon)$ and taking $\varepsilon > 0$ small enough, to deduce

$$\begin{aligned}
\left|\int_{\mathbb{R}} \frac{3}{2}\delta u_{\varepsilon,\,\delta}^2\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\,\mathrm{d}x\right| &\leqslant \frac{3}{2}\delta\left(\int_{\mathbb{R}} u_{\varepsilon,\,\delta}^4\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} |\mathcal{H}\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\\
&\leqslant \frac{3}{2}\delta\left(\int_{\mathbb{R}} u_{\delta,\varepsilon}^4\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\\
&\leqslant \frac{1}{8}\int_{\mathbb{R}} u_{\varepsilon,\,\delta}^4\,\mathrm{d}x + C\delta^2\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\\
&\leqslant \frac{1}{8}\int_{\mathbb{R}} u_{\varepsilon,\,\delta}^4\,\mathrm{d}x + \varepsilon^2\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x,
\end{aligned}$$

and, therefore,

$$\mathcal{I}_{4,\varepsilon} \geqslant \frac{1}{8}\int_{\mathbb{R}} u_{\varepsilon,\,\delta}^4\,\mathrm{d}x + \varepsilon^2\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x. \tag{2.7}$$

We also remark that

$$\int_{\mathbb{R}} u_{\varepsilon,\,\delta}^4\,\mathrm{d}x \leqslant 8\,\mathcal{I}_{4,\varepsilon}. \tag{2.8}$$

Combining (2.7) with (2.6) gives

$$\sigma_{\varepsilon,\delta}(t) := \varepsilon^2\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}(t,x)|^2\mathrm{d}x \leqslant \mathcal{I}_{4,\,\varepsilon}(0) + \frac{C\delta^{\frac{4}{3}}}{\varepsilon^3}\int_0^t \sigma_{\varepsilon,\delta}(s)^{\frac{5}{3}}\mathrm{d}s. \tag{2.9}$$

Now, if $\delta \leqslant M\varepsilon^{\frac{3}{2}}$, for some constant M, then σ is bounded uniformly in ε,δ. To see this, we argue as follows: By continuity of $\sigma_{\varepsilon,\delta}$ it suffices to bound $\sigma_{\varepsilon,\delta}(T)$ for any $T \geqslant 0$ satisfying $\sigma_{\varepsilon,\delta}(T) \geqslant \max_{t\in[0,T]}\sigma_{\varepsilon,\delta}(t)$. For such a T, we use (2.9) and compute

$$\begin{aligned}
\sigma_{\varepsilon,\delta}(T) &\leqslant \mathcal{I}_{4,\,\varepsilon}(0) + \frac{CM}{\varepsilon}\left(\int_0^T \sigma_{\varepsilon,\delta}(s)\mathrm{d}s\right)\sigma_{\varepsilon,\delta}(T)^{\frac{2}{3}}\\
&\leqslant \mathcal{I}_{4,\,\varepsilon}(0) + \frac{CM}{2}\|u_0\|_{L^2(\mathbb{R})}^2\sigma_{\varepsilon,\delta}(T)^{\frac{2}{3}}\\
&\leqslant \mathcal{I}_{4,\,\varepsilon}(0) + \frac{1}{3}\left(\frac{CM}{2}\|u_0\|_{L^2(\mathbb{R})}^2\right)^3 + \frac{2\sigma_{\varepsilon,\delta}(T)}{3},
\end{aligned}$$

where the second inequality used Lemma 2.1. Rearranging yields

$$\sigma_{\varepsilon,\delta}(T) \leqslant \frac{3\mathcal{I}_{4,\,\varepsilon}(0)}{2} + \frac{(CM)^3}{16}\|u_0\|_{L^2(\mathbb{R})}^6 =: \tilde{M}.$$

Thus, returning to (2.6), we find

$$\mathcal{I}_{4,\,\varepsilon}(t) \leqslant \mathcal{I}_{4,\,\varepsilon}(0) + \frac{CM}{\varepsilon}\int_0^t \sigma^{\frac{5}{3}}(s)\,\mathrm{d}s \leqslant \mathcal{I}_{4,\,\varepsilon}(0) + \frac{CM\tilde{M}^{\frac{2}{3}}\|u_0\|_{L^2(\mathbb{R})}}{2}. \tag{2.10}$$

In conclusion,

$$\int_{\mathbb{R}} u_{\varepsilon,\,\delta}^4 \,\mathrm{d}x \leqslant 8\,\mathcal{I}_{4,\,\varepsilon} \leqslant 8\left(\mathcal{I}_{4,\,\varepsilon}(0) + \frac{CM\tilde{M}^{\frac{2}{3}}\|u_0\|_{L^2(\mathbb{R})}}{2}\right).$$

□

Remark 2.3 (On the assumption $\delta = O(\varepsilon^{\frac{3}{2}})$). It is natural to conjecture (also in light of the computations in Section 3; cf. Remark 3.2) that one should require $\delta = O(\varepsilon)$ to obtain a uniform L^4_{loc}-bound. Indeed, from the computation of the L^4-norm contribution and the beginning of the proof, we see that if we could remove the $\mathcal{H}$ in

$$3\delta \int_{\mathbb{R}} u_{\varepsilon,\,\delta}^2 \partial_x u_{\varepsilon,\,\delta} \mathcal{H} \partial_x u_{\varepsilon,\,\delta} \,\mathrm{d}x,$$

we directly have a uniform bound on the L^4 norm assuming only $\delta = O(\varepsilon)$. However, we find $\delta = O(\varepsilon^{\frac{3}{2}})$ instead. The loss of sharpness in the estimates comes from the use of Gagliardo–Nirenberg's inequality in estimating the term

$$3\delta\varepsilon \int_{\mathbb{R}} (\partial_x u_{\varepsilon,\,\delta})^2 \mathcal{H} \partial_x u_{\varepsilon,\,\delta} \,\mathrm{d}x,$$

where there is, in a sense, a loss of derivatives. As such, we do not believe that $\delta = O(\varepsilon^{\frac{3}{2}})$ is the optimal condition.

3. L^p compensated compactness and proof of the convergence result

In order to verify the entropy dissipation assumption in Lemma 1.2, we shall use Murat's compact embedding (see [56] or and [33, Lemma 17.2.2]).

Lemma 3.1 (Murat's compact embedding)**.** *Let Ω be a bounded open subset of $\mathbb{R}^N, N \geqslant 2$. Suppose the sequence $\{\mathcal{L}_n\}_{n=1}^{\infty}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_n = \mathcal{L}_n^1 + \mathcal{L}_n^2,$$

where $\{\mathcal{L}_n^1\}_{n=1}^{\infty}$ lies in a compact subset of $H^{-1}_{\mathrm{loc}}(\Omega)$ and $\{\mathcal{L}_n^2\}_{n=1}^{\infty}$ lies in a bounded subset of $\mathcal{M}_{\mathrm{loc}}(\Omega)$. Then $\{\mathcal{L}_n\}_{n=1}^{\infty}$ lies in a compact subset of $H^{-1}_{\mathrm{loc}}(\Omega)$.

We are now ready to combine the ingredients prepared in Section 2 to prove Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into several steps.

Step 1. H^{-1}_{loc}*-Compactness of the entropy production.* Let (η, q) be a compactly supported entropy–entropy flux pair. By Lemma 2.2, we already know that $u_{\varepsilon,\,\delta}$ is uniformly bounded in $L^4(\mathbb{R}_+ \times \mathbb{R})$. Therefore, in order apply Lemma 1.2 and conclude that

$$u_{\varepsilon,\,\delta} \longrightarrow u_{\varepsilon,\,\delta} \quad \text{strongly in } L^p_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}), \text{ for } 1 \leqslant p < 4,$$

we only need to show that the entropy production $\{\partial_t \eta(u_{\varepsilon,\,\delta}) + \partial_x q(u_{\varepsilon,\,\delta})\}_{\varepsilon,\,\delta>0}$ is compact in $H^{-1}_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R})$.

The entropy production is given by

$$\partial_t \eta(u_{\varepsilon,\,\delta}) + \partial_x q(u_{\varepsilon,\,\delta}) = \eta'(u_{\varepsilon,\,\delta})\partial_x^2 u_{\varepsilon,\,\delta} - \delta\eta'(u_{\varepsilon,\,\delta})\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta},$$

where the terms on the right-hand side can be rewritten as follows:

$$\varepsilon\eta'(u_{\varepsilon,\,\delta})\partial_x^2 u_{\varepsilon,\,\delta} = \underbrace{\partial_x(\varepsilon\eta'(u_{\varepsilon,\,\delta})\partial_x u_{\varepsilon,\,\delta})}_{E^1_{\varepsilon,\,\delta}} - \underbrace{\varepsilon\eta''(u_{\varepsilon,\,\delta})(\partial_x u_{\varepsilon,\,\delta})^2}_{E^2_{\varepsilon,\,\delta}};$$

$$\delta\eta'(u_{\varepsilon,\,\delta})\mathcal{H}\partial_x^2 u_{\varepsilon,\,\delta} = \underbrace{\partial_x(\delta\eta'(u_{\varepsilon,\,\delta})\mathcal{H}\partial_x u_{\varepsilon,\,\delta})}_{E^3_{\varepsilon,\,\delta}} - \underbrace{\delta\eta''(u_{\varepsilon,\,\delta})\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}}_{E^4_{\varepsilon,\,\delta}}.$$

Since η is compactly supported, we have that $\{\partial_t \eta(u_{\varepsilon,\,\delta}) + \partial_x q(u_{\varepsilon,\,\delta})\}_{\varepsilon,\,\delta>0}$ is uniformly bounded in $W^{-1,\infty}(\mathbb{R}_+ \times \mathbb{R})$. Therefore, due to Lemma 3.1, it suffices to prove that $E^1_{\varepsilon,\,\delta}$ and $E^3_{\varepsilon,\,\delta}$ lie in a compact set of $H^{-1}_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R})$ and that $E^2_{\varepsilon,\,\delta}$ and $E^4_{\varepsilon,\,\delta}$ lie in a bounded set of $\mathcal{M}(\mathbb{R}_+ \times \mathbb{R})$ to conclude that the entropy production is compact in $H^{-1}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R})$ and $E^2_{\varepsilon,\,\delta}$. To this end, we fix a compact set $\Omega := [0,T] \times K \subset \mathbb{R}_+ \times \mathbb{R}$ and compute (relying on Lemma 2.1)

$$\begin{aligned}\left\|\varepsilon\eta'\left(u_{\varepsilon,\,\delta}\right)\partial_x u_{\varepsilon,\,\delta}\right\|^2_{L^2(\Omega)} &\leqslant \left\|\eta'\right\|^2_{L^\infty(\mathbb{R})}\varepsilon^2\int_0^T \left\|\partial_x u_{\varepsilon,\,\delta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \\ &\leqslant \left\|\eta'\right\|^2_{L^\infty(\mathbb{R})}\varepsilon\left\|u_0\right\|^2_{L^2(\mathbb{R})} \longrightarrow 0 \quad \text{as } \varepsilon,\,\delta \searrow 0;\end{aligned} \tag{3.1}$$

$$\begin{aligned}\left\|\varepsilon\eta''\left(u_{\varepsilon,\,\delta}\right)\left(\partial_x u_{\varepsilon,\,\delta}\right)^2\right\|_{L^1(\Omega)} &\leqslant \left\|\eta''\right\|_{L^\infty(\mathbb{R})}\varepsilon\int_0^T \left\|\partial_x u_{\varepsilon,\,\delta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \\ &\leqslant \frac{1}{2}\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\left\|u_0\right\|^2_{L^2(\mathbb{R})};\end{aligned}$$

$$\begin{aligned}\|\delta\eta'(u_{\varepsilon,\,\delta})\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\|^2_{L^2(\Omega)} &\leqslant \|\eta'\|_{L^\infty(\mathbb{R})}\delta^2\int_0^T \|\mathcal{H}\partial_x u_{\varepsilon,\,\delta}(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \\ &\leqslant \frac{\delta^2}{\varepsilon}\|\eta'\|_{L^\infty(\mathbb{R})}\|u_0\|_{L^2(\mathbb{R})} \longrightarrow 0 \quad \text{as } \varepsilon,\,\delta \searrow 0;\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&\|\delta\eta''(u_{\varepsilon,\,\delta})\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\|_{L^1(\Omega)} \\ &\leqslant \delta\|\eta''\|_{L^\infty(\mathbb{R})}\int_0^T\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}|\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \frac{\delta}{\varepsilon}\|\eta''\|_{L^\infty(\mathbb{R})}\left(\frac{\varepsilon}{2}\int_0^T\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon}{2}\int_0^T\int_{\mathbb{R}}|\mathcal{H}\partial_x u_{\varepsilon,\,\delta}|^2\,\mathrm{d}x\,\mathrm{d}t\right) \\ &\leqslant \frac{\delta}{\varepsilon}\|\eta''\|_{L^\infty(\mathbb{R})}\|u_0\|_{L^2(\mathbb{R})} = O(1)\|\eta''\|_{L^\infty(\mathbb{R})}\|u_0\|_{L^2(\mathbb{R})}.\end{aligned} \tag{3.3}$$

This concludes the proof that $\{\partial_t\eta(u_{\varepsilon,\,\delta}) + \partial_x q(u_{\varepsilon,\,\delta})\}_{\varepsilon,\,\delta>0}$ is compact in $H^{-1}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R})$. We stress that we used the assumption $\delta = O(\varepsilon)$ in (3.2) and (3.3).

Step 2. *Strong convergence of $\{u_{\varepsilon,\,\delta}\}_{\varepsilon,\,\delta>0}$ to a limit point.* Using the result of Step 1 and L^4_{loc}-bound from Lemma 2.2, which requires $\delta = O(\varepsilon^{\frac{3}{2}})$, we can apply Lemma 1.2 (with $s = 1$) and conclude that there exists a subsequence of $\{u_{\varepsilon,\,\delta}\}_{\varepsilon,\,\delta>0}$ that converges strongly in $L^p(\mathbb{R})$, with $1 \leqslant p < 4$, to a limit function u.

Step 3. *Convergence to a weak solution.* By applying Lebesgue's dominated convergence theorem, we can deduce that u is a weak solution of (1.4).

Step 4. *Convergence to the entropy solution.* Assuming $\delta = o(\varepsilon)$, we can improve (3.3) and deduce that

$$\|\delta\eta''(u_{\varepsilon,\,\delta})\partial_x u_{\varepsilon,\,\delta}\mathcal{H}\partial_x u_{\varepsilon,\,\delta}\|_{L^1(\Omega)} \longrightarrow 0 \qquad \text{as } \varepsilon,\delta \searrow 0. \tag{3.4}$$

From (3.1), (3.2), and (3.4), we infer that $E^1_{\varepsilon,\,\delta}$, $E^3_{\varepsilon,\,\delta}$, $E^4_{\varepsilon,\,\delta} \longrightarrow 0$ in the sense of distributions as $\varepsilon,\,\delta \searrow 0$. Combining this with the observation that $-\varepsilon\eta''\left(u_{\varepsilon,\,\delta}\right)\left(\partial_x u_{\varepsilon,\,\delta}\right)^2 \leqslant 0$, we can conclude that u is the entropy solution of (1.4).

Finally, owing to Urysohn's subsequence principle, from the uniqueness of the entropy solution of (1.4), we also obtain that the whole family $\{u_{\varepsilon,\,\delta}\}_{\varepsilon,\,\delta>0}$ converges to u (not just up to extracting a subsequence). □

Remark 3.2. In the proof of Theorem 1.1, we need $\delta = O(\varepsilon)$ to show the H^{-1}_{loc}-compactness of the entropy production and $\delta = o(\varepsilon)$ to prove that the limit point of the family $\{u_{\varepsilon,\,\delta}\}_{\varepsilon,\,\delta>0}$ (provided it exists) is the entropy solution of the conservation law (1.4). The only point where we needed the assumption $\delta = O(\varepsilon^{\frac{3}{2}})$ is for the uniform L^4_{loc}-bound needed to apply Schonbek's compensated compactness lemma (cf. Lemma 1.2).

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References

[1] J. Aleksić, D. Mitrovic, and S. Pilipović. "Hyperbolic conservation laws with vanishing nonlinear diffusion and linear dispersion in heterogeneous media". In: *J. Evol. Equ.* 9.4 (2009), pp. 809–828.

[2] N. Bedjaoui and J. M. C. Correia. "A note on nonlinear KdV-type equations". In: *Bol. Soc. Port. Mat.* Special Issue (2012), pp. 27–30.

[3] N. Bedjaoui, J. M. C. Correia, and Y. Mammeri. "On a limit of perturbed conservation laws with diffusion and non-positive dispersion". In: *Commun. Math. Sci.* 14.6 (2016), pp. 1501–1516.

[4] T. B. Benjamin. "Internal waves of permanent form in fluids of great depth". English. In: *J. Fluid Mech.* 29 (1967), pp. 559–592.

[5] E. Blackstone, L. Gassot, P. Gérard, and P. D. Miller. *The Benjamin-Ono equation in the zero-dispersion limit for rational initial data: generation of dispersive shock waves.* 2024. arXiv: 2410.17405 [math.AP].

[6] E. Blackstone, L. Gassot, P. Gérard, and P. D. Miller. "Zero-dispersion limit for the Benjamin-Ono equation". en. In: *Séminaire Laurent Schwartz — EDP et applications* (2023-2024). talk:6, pp. 1–16.

[7] E. Blackstone, L. Gassot, and P. D. Miller. *On Strong Zero-Dispersion Asymptotics for Benjamin-Ono Soliton Ensembles.* 2024. arXiv: 2311.05785 [math.AP].

[8] E. Blackstone, P. D. Miller, and M. D. Mitchell. *Universality in the Small-Dispersion Limit of the Benjamin-Ono Equation.* 2024. arXiv: 2410.21581 [math.AP].

[9] G. M. Coclite and K. H. Karlsen. "A singular limit problem for conservation laws related to the Camassa-Holm shallow water equation". In: *Comm. Partial Differential Equations* 31.7-9 (2006), pp. 1253–1272.

[10] G. M. Coclite and L. di Ruvo. "A note on convergence of the solutions of Benjamin-Bona-Mahony type equations". In: *Nonlinear Anal. Real World Appl.* 40 (2018), pp. 64–81.

[11] G. M. Coclite and L. di Ruvo. "A note on the convergence of the solution of the high order Camassa-Holm equation to the entropy ones of a scalar conservation law". In: *Discrete Contin. Dyn. Syst.* 37.3 (2017), pp. 1247–1282.

[12] G. M. Coclite and L. di Ruvo. "A note on the convergence of the solutions of the Camassa-Holm equation to the entropy ones of a scalar conservation law". In: *Discrete Contin. Dyn. Syst.* 36.6 (2016), pp. 2981–2990.

[13] G. M. Coclite and L. di Ruvo. "A singular limit problem for conservation laws related to the Kawahara equation". In: *Bull. Sci. Math.* 140.3 (2016), pp. 303–338.
[14] G. M. Coclite and L. di Ruvo. "A singular limit problem for conservation laws related to the Kawahara-Korteweg-de Vries equation". In: *Netw. Heterog. Media* 11.2 (2016), pp. 281–300.
[15] G. M. Coclite and L. di Ruvo. "A singular limit problem for conservation laws related to the Rosenau equation". In: *J. Abstr. Differ. Equ. Appl.* 8.1 (2017), pp. 24–47.
[16] G. M. Coclite and L. di Ruvo. "A singular limit problem for conservation laws related to the Rosenau–Korteweg–de Vries equation". In: *J. Math. Pures Appl. (9)* 107.3 (2017), pp. 315–335.
[17] G. M. Coclite and L. di Ruvo. "A singular limit problem for the Ibragimov-Shabat equation". In: *Discrete Contin. Dyn. Syst. Ser. S* 9.3 (2016), pp. 661–673.
[18] G. M. Coclite and L. di Ruvo. "A singular limit problem for the Kudryashov-Sinelshchikov equation". In: *ZAMM Z. Angew. Math. Mech.* 97.9 (2017), pp. 1020–1033.
[19] G. M. Coclite and L. di Ruvo. "A singular limit problem for the Rosenau-Korteweg–de Vries–regularized long wave and Rosenau-regularized long wave equations". In: *Adv. Nonlinear Stud.* 16.3 (2016), pp. 421–437.
[20] G. M. Coclite and L. di Ruvo. "Convergence of the Kuramoto-Sinelshchikov equation to the Burgers one". In: *Acta Appl. Math.* 145 (2016), pp. 89–113.
[21] G. M. Coclite and L. di Ruvo. "Convergence of the Ostrovsky equation to the Ostrovsky-Hunter one". In: *J. Differential Equations* 256.9 (2014), pp. 3245–3277.
[22] G. M. Coclite and L. di Ruvo. "Convergence of the regularized short pulse equation to the short pulse one". In: *Math. Nachr.* 291.5-6 (2018), pp. 774–792.
[23] G. M. Coclite and L. di Ruvo. "Convergence of the solutions on the generalized Korteweg–de Vries equation". In: *Math. Model. Anal.* 21.2 (2016), pp. 239–259.
[24] G. M. Coclite and L. di Ruvo. "Convergence results related to the modified Kawahara equation". In: *Boll. Unione Mat. Ital.* 8.4 (2016), pp. 265–286.
[25] G. M. Coclite and L. di Ruvo. "Dispersive and diffusive limits for Ostrovsky-Hunter type equations". In: *NoDEA Nonlinear Differential Equations Appl.* 22.6 (2015), pp. 1733–1763.
[26] G. M. Coclite and L. di Ruvo. "On the convergence of the modified Rosenau and the modified Benjamin-Bona-Mahony equations". In: *Comput. Math. Appl.* 74.5 (2017), pp. 899–919.
[27] G. M. Coclite and L. di Ruvo. "On the well-posedness of the exp-Rabelo equation". In: *Ann. Mat. Pura Appl. (4)* 195.3 (2016), pp. 923–933.
[28] G. M. Coclite and L. di Ruvo. "Well-posedness and dispersive/diffusive limit of a generalized Ostrovsky-Hunter equation". In: *Milan J. Math.* 86.1 (2018), pp. 31–51.
[29] G. M. Coclite and L. di Ruvo. "Well-posedness of the Ostrovsky-Hunter equation under the combined effects of dissipation and short-wave dispersion". In: *J. Evol. Equ.* 16.2 (2016), pp. 365–389.
[30] J. M. Correia and P. G. LeFloch. "Nonlinear hyperbolic conservation laws". In: *Nonlinear evolution equations and their applications (Macau, 1998)*. World Sci. Publ., River Edge, NJ, 1999, pp. 21–44.
[31] J. M. C. Correia. "Zero limit for multi-D conservation laws with nonlinear dissipation and dispersion". In: *Modeling, dynamics, optimization and bioeconomics. II.* Vol. 195. Springer Proc. Math. Stat. Springer, Cham, 2017, pp. 147–163.
[32] J. M. C. Correia and P. G. Lefloch. "Nonlinear diffusive-dispersive limits for multidimensional conservation laws". In: *Advances in nonlinear partial differential equations and related areas (Beijing, 1997)*. World Sci. Publ., River Edge, NJ, 1998, pp. 103–123.
[33] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*. Fourth. Vol. 325. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2016, pp. xxxviii+826.
[34] R. E. Davis and A. Acrivos. "Solitary internal waves in deep water". English. In: *J. Fluid Mech.* 29 (1967), pp. 593–607.
[35] P. M. Edwin and B. Roberts. "The Benjamin-Ono-Burgers equation: An application in solar physics". In: *Wave Motion* 8 (1986), pp. 151–158.
[36] N. Fujino and M. Yamazaki. "Vanishing at most seventh-order terms of scalar conservation laws". In: *Hyperbolic problems: theory, numerics, applications*. Springer, Berlin, 2008, pp. 1093–1100.

[37] N. Fujino. “Scalar conservation laws with vanishing and highly nonlinear diffusive-dispersive terms”. In: *Publ. Res. Inst. Math. Sci.* 43.4 (2007), pp. 1005–1022.
[38] N. Fujino and M. Yamazaki. “A result on the equation of conservation laws having second and third order terms”. In: *Nonlinear dispersive equations.* Vol. 26. GAKUTO Internat. Ser. Math. Sci. Appl. Gakkotosho, Tokyo, 2006, pp. 19–33.
[39] N. Fujino and M. Yamazaki. “Hyperbolic conservation laws with nonlinear diffusion and nonlinear dispersion”. In: *J. Differential Equations* 228.1 (2006), pp. 171–190.
[40] L. Gassot. “Lax eigenvalues in the zero-dispersion limit for the Benjamin-Ono equation on the torus”. In: *SIAM J. Math. Anal.* 55.5 (2023), pp. 5782–5822.
[41] L. Gassot. “Zero-dispersion limit for the Benjamin-Ono equation on the torus with bell shaped initial data”. In: *Commun. Math. Phys.* 401.3 (2023), pp. 2793–2843.
[42] P. Gérard. *The zero dispersion limit for the Benjamin–Ono equation on the line.* 2023. arXiv: 2307.12768 [math.AP].
[43] Z. Guo, L. Peng, B. Wang, and Y. Wang. “Uniform well-posedness and inviscid limit for the Benjamin-Ono-Burgers equation”. In: *Adv. Math.* 228.2 (2011), pp. 647–677.
[44] H. Holden, K. H. Karlsen, and D. Mitrovic. “Zero diffusion-dispersion-smoothing limits for a scalar conservation law with discontinuous flux function”. In: *Int. J. Differ. Equ.* (2009), Art. ID 279818, 33.
[45] S. Hwang. “Singular limit problem of the Camassa-Holm type equation”. In: *J. Differential Equations* 235.1 (2007), pp. 74–84.
[46] S. Hwang and A. E. Tzavaras. “Kinetic decomposition of approximate solutions to conservation laws: application to relaxation and diffusion-dispersion approximations”. In: *Comm. Partial Differential Equations* 27.5-6 (2002), pp. 1229–1254.
[47] C. Klein and J.-C. Saut. *Nonlinear dispersive equations—inverse scattering and PDE methods.* Vol. 209. Applied Mathematical Sciences. Springer, Cham, [2021] ©2021, pp. xx+580.
[48] C. I. Kondo and P. G. Lefloch. “Zero diffusion-dispersion limits for scalar conservation laws”. In: *SIAM J. Math. Anal.* 33.6 (2002), pp. 1320–1329.
[49] P. G. LeFloch and R. Natalini. “Conservation laws with vanishing nonlinear diffusion and dispersion”. In: *Nonlinear Anal.* 36.2, Ser. A: Theory Methods (1999), pp. 213–230.
[50] Y. G. Lu. “Convergence of solutions to nonlinear dispersive equations without convexity conditions”. In: *Appl. Anal.* 31.4 (1989), pp. 239–246.
[51] J. D. Meiss and N. R. Pereira. “Internal wave solitons”. English. In: *Phys. Fluids* 21 (1978), pp. 700–702.
[52] P. D. Miller and A. N. Wetzel. “The scattering transform for the Benjamin-Ono equation in the small-dispersion limit”. In: *Physica D* 333 (2016), pp. 185–199.
[53] P. D. Miller and Z. Xu. “On the zero-dispersion limit of the Benjamin-Ono Cauchy problem for positive initial data”. In: *Comm. Pure Appl. Math.* 64.2 (2011), pp. 205–270.
[54] P. D. Miller and Z. Xu. “The Benjamin-Ono hierarchy with asymptotically reflectionless initial data in the zero-dispersion limit”. In: *Commun. Math. Sci.* 10.1 (2012), pp. 117–130.
[55] L. Molinet. “A note on the inviscid limit of the Benjamin-Ono-Burgers equation in the energy space”. In: *Proc. Amer. Math. Soc.* 141.8 (2013), pp. 2793–2798.
[56] F. Murat. “L’injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$”. In: *J. Math. Pures Appl. (9)* 60.3 (1981), pp. 309–322.
[57] H. Ono. “Algebraic solitary waves in stratified fluids”. In: *J. Phys. Soc. Japan* 39.4 (1975), pp. 1082–1091.
[58] J. N. Pandey. *The Hilbert transform of Schwartz distributions and applications.* Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996, pp. xvi+262.
[59] J.-C. Saut. “Benjamin-Ono and intermediate long wave equations: modeling, IST and PDE”. English. In: *Nonlinear dispersive partial differential equations and inverse scattering. Papers from the focus program on “Nonlinear Dispersive Partial Differential Equations and Inverse Scattering”, Fields Institute, July 31 – August 18, 2017.* New York, NY: Springer; Toronto, ON: The Fields Institute for Research in Mathematical Scienes, 2019, pp. 95–160.
[60] M. E. Schonbek. “Convergence of solutions to nonlinear dispersive equations”. In: *Comm. Partial Differential Equations* 7.8 (1982), pp. 959–1000.

[61] E. Tadmor. "Burgers' equation with vanishing hyper-viscosity". In: *Commun. Math. Sci.* 2.2 (2004), pp. 317–324.

[62] E. C. Titchmarsh. *Introduction to the theory of Fourier integrals*. Third. Chelsea Publishing Co., New York, 1986, pp. x+394.

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