Some fundamental results on Probabilities and Stochastic Processes in Infinite Dimensional Spaces and Manifolds

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Abstract

The study of stochastic processes in both vector spaces and manifolds is essential across numerous fields of Mathematics, Physics, and Applied Sciences. Altough significant research exists on infinite-dimensional vector spaces and finite-dimensional manifolds, the study of processes in infinite-dimensional manifolds is less developed. One key distinction in this setting is that infinite-dimensional manifolds are not locally compact, necessitating revisions to techniques commonly used in finite dimensions. This compendium gathers a comprehensive collection of results and examples—some widely known, others potentially novel—aimed at those interested in the field. These findings have been used in the paper *Tightness of Random Walks in Infinite Dimensional Spaces and Manifolds*, but may provide a useful reference for further exploration of these subjects.

Keywords: Stochastic processes, infinite dimensional manifolds, tightness, nets

1 Introduction

The study of stochastic processes in infinite-dimensional spaces and manifolds is fundamental across several branches of mathematics, physics, and applied sciences. These processes provide important insights into complex systems, particularly when analyzing their behavior and convergence properties.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X a Hausdorff topological space, $I \subseteq \mathbb{R}$ be an interval. We will concentrate on processes $\mathfrak{X} : I \times \Omega \to X$ whose paths are continuous; so $\mathfrak{X}$ can be viewed as a *random function* $\mathfrak{X} : \Omega \to C(I; X)$, hence their distributions are probability measures on $C(I; X)$.

Much research has been dedicated to the case when X is a finite-dimensional manifolds, or when X is an infinite-dimensional vector spaces. In contrast, the exploration of stochastic processes on infinite-dimensional manifolds remains relatively underdeveloped. The non-locally-compact nature of these manifolds requires new tools and techniques, distinct from those used in the finite-dimensional setting.

This compendium aims to provide a comprehensive reference for results and examples related to processes in infinite-dimensional spaces and manifolds. It serves both as a supplement to the paper *Tightness of Random Walks in Infinite Dimensional Spaces and Manifolds* [12] and as a standalone resource for researchers interested in the field. The document organizes theoretical tools and results essential for addressing the specific challenges of infinite-dimensional spaces, with a particular focus on continuous functions, measures, and probability theory.

In Section 2 we compare different families of continuous functions that are useful in Probability Theory; we recall results such as Urysohn's lemmas, and Riesz representation theorems. In Section 3 we discuss the relation between continuous functions and regular measure (or regular finitely additive bounded set functions, called `rba`). We discuss properties of a topological space S that ensure different degrees of regularity for Borel measures on S, such as Ulam's theorem. In Section 5 we list properties of spaces of continuous functions taking values in infinite dimensional spaces, such as a version of Ascoli–Arzelà Theorem. In Section 6 we discuss narrow convergence of measures, and its relation to tightness, with result such as Prokhorov's theorem. We apply all presented material to processes taking values in closed subsets of Banach spaces (that may be infinite dimensional manifolds), and propose a criterion for tightness. During the presentation, we extend the theory to converging "nets" of measures/probabilities, since in some applications "sequences" are not general enough. (Converging nets are defined in Section 4, for convenience of the reader). We also provide throughout several examples illustrating the relevance of these theoretical results.

Through this compendium, we hope to facilitate further exploration and development of the theory of stochastic processes on infinite-dimensional spaces and manifolds, thereby contributing to a deeper understanding of this area of study.

2 Continuous functions

In the following (S, τ) will be a Hausdorff topological space.

We may consider the following vector spaces of functions.

Definition 2.1. • The space $C_c(S)$ consists of compactly supported continuous functions $f : S \to \mathbb{R}$.

- The space $C_0(S)$ consists of continuous functions $f : S \to \mathbb{R}$ such that $\lim_{x\to\infty} f(x) = 0$ (the point ∞ is to be intended as in one-point Alexandroff compactification [1]).
- The space $C_l(S)$ consists of continuous functions $f : S \to \mathbb{R}$ such that $\lim_{x\to\infty} f(x)$ exists and is finite.
- The space $C_b(S)$ consists of bounded continuous functions $f : S \to \mathbb{R}$.

Obviously

$$C_c(S) \subseteq C_0(S) \subseteq C_l(S) \subseteq C_b(S) \quad ;$$

the last three are Banach spaces with the sup-norm.

Lemma 2.2. *If S is locally compact, then $C_0(S)$ is the closure of the space $C_c(S)$.*

This is primarily proved by the following result.

Lemma 2.3 (Urysohn's Lemma)**.** *Suppose that S is a locally compact space, V is open, K is compact, and $K \subset V$: then there exists $f \in C_c(S)$ such that*

$$\forall x \in S \ , \ \ 0 \le f(x) \le 1 \ , \ \ x \in V \Rightarrow f(x) = 0 \ , \ \ x \in K \Rightarrow f(x) = 1 \ .$$

The proof of 2.3 is in Chap. 2 in [14].

The proof of 2.2 is as follows.

Proof. Fix $g \in C_0(S)$ and $\varepsilon > 0$, then there exists $K \subseteq S$ compact such that $x \notin K \Rightarrow |g(x)| < \varepsilon$, and there exists f as in lemma 2.3. So $fg \in C_0(S)$ and $\|g - fg\| \le \varepsilon$. □

Another version of the above Lemma is as follows.

Definition 2.4. S is a normal space when every two disjoint closed sets have disjoint open neighborhoods.

Since we are assuming that S be Hausdorff, then "normal" is here equivalent to T_4.

Lemma 2.5 (Urysohn's Lemma)**.** *S is a normal space if, and only if: for any $A, B \subseteq S$ disjoint closed sets there exists $f \in C_b(S)$ such that*

$$\forall x \in S \ , \ \ 0 \le f(x) \le 1 \ , \ \ x \in A \Rightarrow f(x) = 0 \ , \ \ x \in B \Rightarrow f(x) = 1 \ .$$

Note that there are Hausdorff locally compact spaces that are not normal (such as the deleted Tychonoff plank [15]), and normal spaces that are not locally compact (such as an infinite dimensional Banach space).

2.1 Why?

Why are we interested in these spaces? There are many uses for those spaces in Probability Theory. Suppose that C is one of the above discussed spaces of continuous functions.

- Suppose that μ_n, μ are finite measures; we will in the following discuss *narrow convergence*, that is defined as

$$\forall f \in C \ , \ \lim_{n\to\infty} \int_S f(x)\,\mathrm{d}\mu_n(x) = \int_S f(x)\,\mathrm{d}\mu(x) \quad .$$

[1] It is known that, the one-point Alexandroff compactification S^* is Hausdorff if and only if S is Hausdorff and locally compact. But in the following, S will in general not be locally compact.

- Suppose that X_t is a stochastic process for $t \in \mathbb{R}^+$, solution of a SDE, then, under appropriate hypotheses, we can define the linear operators $T_t : C \to C$ by

$$(T_t f)(x) = \mathbb{E}[f(X_t)|X_0 = x]$$

and $(T_t)_t$ will be a semigroup.

All of the above follows from the fundamental consideration that the space of Probability Measures, or more in general the space of Finite Signed Measures, can be identified with a subset of the dual C^* of those spaces, using the identification

$$\mu \in \mathcal{M} \mapsto J_\mu \in C^*$$

where

$$J_\mu(f) \stackrel{\text{def}}{=} \int_S f(x)\,\mathrm{d}\mu(x) \quad .$$

The topology induced by this identification will be called *narrow topology*. (The term *narrow* seems to have originated in Bourbaki's texts.)

Remark 2.6. Changing the reference space C is not without consequences. Consider the case $S = \mathbb{R}$ and $\mu_n = \delta_n$, the Dirac delta centered at $n \in \mathbb{N}$; let $\mu \equiv 0$. It is easily verified that

$$\forall f \in C_0(S)\,,\ \lim_{n\to\infty} \int_S f(x)\,\mathrm{d}\mu_n(x) = \int_S f(x)\,\mathrm{d}\mu(x) \quad ,$$

so if the reference space is $C = C_0(S)$ then we can say that $\mu_n \to_n \mu$ narrowly. If instead the reference space is chosen to be $C = C_b(S)$, then this is not true anymore: just choose $f \equiv 1$.

Usually, for "finite dimensional spaces", the reference space is $C = C_0(S)$ in those definitions. We will in the following Theorem 2.7 show that this choice does not make sense in the setting of "infinite dimensional spaces", so we will forced to choose $C = C_b(S)$. This will have some consequences, though.

Theorem 2.7. *Suppose that (S, τ) is a topological space where*

- *Baire's Theorem holds, and*
- *for any $x \in S$ and any open set $V \ni x$, the closure $\overline{V}$ is not compact; or equivalently, S is a space where all compact sets have empty interior.* [2]

Then any function in $C_l(S)$ is constant. In particular, $C_c(S) = \{0\} = C_0(S)$.

Proof. Fix $f \in C_l(S)$ and let $L = \lim_{x\to\infty} f(x)$; consider then, for $\theta \in \mathbb{Q}, \theta > 0$, the open sets

$$A_\theta = \{x \in S : |f(x) - L| < \theta\} \quad .$$

Each A_θ must contain an open set B_θ such that the complement B_θ^c is compact, so

$$K_\theta \stackrel{\text{def}}{=} A_\theta^c = \{x \in S : |f(x) - L| \geq \theta\} \subseteq B_\theta^c$$

[2] Note that, for such spaces, the one-point Alexandroff compactification is not Hausdorff.

is compact as well. Hence

$$Z \stackrel{\text{def}}{=} \{x \in S : f(x) \neq L\} = \bigcup_{\theta > 0} K_\theta \quad ;$$

by Baire's Theorem, this means that the interior of Z is empty; being f continuous, then $f \equiv L$. □

The hypotheses of the above Theorem are enjoyed by infinite dimensional Banach spaces, and by manifolds modeled on such spaces. So we will mostly concentrate on the space $C_b(S)$ in the following.

3 Measures

In the following (S, τ) will be a Hausdorff topological space.

We write $\overline{\mathbb{R}}$ for $\mathbb{R} \cup \{+\infty, -\infty\}$.

Definition 3.1. Let $\mathcal{B} = \mathcal{B}(S)$ be the Borel sigma algebra.

- we call **Borel measure** a $\mu : \mathcal{B} \to [0, \infty]$ that is countably-additive;
- we call **Borel signed measure** a $\mu : \mathcal{B} \to \overline{\mathbb{R}}$ that is countably-additive. We assume that μ takes at most one of the two values $\infty, -\infty$.

More in general, a *complex Borel measure* is a $\mu : \mathcal{B} \to \mathbb{C}$ that is countably-additive; we will only marginally deal with this case.

Before talking further about measures (in Sec. 3.3), we will discuss *finitely-additive* set functions.

3.1 RBA, RCA

Definition 3.2. Let $\mathcal{F}$ be an algebra of subsets of S. Denote by $C_b(S, \mathcal{F}) \subseteq C_b(S)$ the set of bounded continuous functions that can be uniformly approximated by simple functions φ, that is, functions

$$\varphi = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$$

that are finite linear combinations of characteristics of sets $A_i \in \mathcal{F}$, with $a_i \in \mathbb{R}$.

Definition 3.3. Let $\mathcal{F}_\tau$ be the algebra of subsets of S generated by open sets of S.

Lemma 3.4. *If $\mathcal{F}$ is an algebra containing the open sets, then $C_b(S) = C_b(S, \mathcal{F})$.*

For example, if $\mathcal{B} = \mathcal{B}(S)$ is the Borel sigma algebra, then $C_b(S) = C_b(S, \mathcal{B})$.

Hypotheses 3.5. In all of this section, we assume that

- the algebra $\mathcal{F}$ contains the open sets, equivalently $\mathcal{F} \supseteq \mathcal{F}_\tau$;
- $\mu : \mathcal{F} \to \overline{\mathbb{R}}$ will be a **finitely-additive** function.

Definition 3.6. We say that

- μ is **finite** if $\forall A \in \mathcal{F}, \mu(A) \in \mathbb{R}$;
- μ is **bounded** if

$$\sup_{A \in \mathcal{F}} |\mu(A)| < \infty \quad .$$

Bounded implies *finite*; but, for *finitely-additive* functions, the opposite is in general not true.

Example 3.7. *Let $S = \mathbb{N}$, and $\mathcal{F}_s$ the algebra generated by singletons; each $A \in \mathcal{F}_s$ is either finite, or is* cofinite *(*i.e. *A^c is finite). Let*

$$\mu(A) = \begin{cases} \#A & \text{when } A \text{ is finite;} \\ -\#(A^c) & \text{when is cofinite;} \end{cases}$$

where $\#A$ is the cardinality of A. In particular, $\mu(\emptyset) = \mu(S) = 0$. This μ is finitely additive, is finite, but not bounded, and not countably additive.

In some cases, though, finite *implies* bounded*:*

- *if μ is non-negative;*
- *if $\mathcal{F}$ is a σ-algebra and μ is countably additive* [3] .

Definition 3.8. The **total variation**[4] of μ on a subset $F \subseteq S$ is defined as

$$\|\mu\|(F) = \sup\left\{ \sum_{i=1}^{n} |\mu(A_i)| \,:\, n \in \mathbb{N},\ A_i \in \mathcal{F},\ A_i \subseteq F,\ A_i \text{ pairwise disjoint} \right\} \quad .$$

(Note that, for real-valued μ, we may set $n = 2$ in the above definition of $\|\mu\|(F)$. The general $n \in \mathbb{N}$ is only needed for complex-valued μ.)

Definition 3.9. μ is **regular** [5] if for each $E \in \mathcal{F}$ and $\varepsilon > 0$, there exist $F, G \in \mathcal{F}$ with $F \subseteq E \subseteq G$, F closed and G open and such that $\|\mu\|(G \setminus F) < \varepsilon$.

Although $\|\mu\|$ can be defined for any subset, it is mostly interesting for sets in $\mathcal{F}$.

Lemma 3.10. $\|\mu\| : \mathcal{F} \to [0, \infty]$ *is a* finitely-additive *function* [6] *and is regular* [7].

If μ is countably additive then $\|\mu\|$ is. [8]

Lemma 3.11. [9] *A finitely-additive function $\mu : \mathcal{F} \to \overline{\mathbb{R}}$ is* bounded *if and only if $\|\mu\|(S) < \infty$; and in this case $\|\mu\|(S) \le 4 \sup_{E \in \mathcal{F}} |\mu(E)|$.*

Remark 3.12. For bounded non-negative μ, regularity is equivalent to inner regularity

$$\forall E \in \mathcal{F}\ ,\ \mu(E) = \sup\{\mu(F) : F \subseteq E, F \text{ closed}\} \quad .$$

More can be said, see Thm. 3.24

Definition 3.13 (rba, rca). [10] Let $\mathcal{F}$ be an algebra of subsets of S.

- We call $\mathtt{rba}(\mathcal{F})$ the vector space of all $\mu : \mathcal{F} \to \mathbb{R}$ regular bounded finitely-additive functions;
- we call $\mathtt{rca}(\mathcal{F})$ the vector space of all $\mu : \mathcal{F} \to \mathbb{R}$ regular bounded countably-additive functions (that is, regular bounded signed measures);

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[3] See Theorem 6.4 in [14]; this result holds also for complex-valued measures.
[4] Definition 4 in Chapter III Section 1 in [6]
[5] Definition 11 in Chapter III Section 5 in [6], simplified since we assumed that $\mathcal{F} \supseteq \mathcal{F}_\tau$.
[6] Lemma 6 in Chapter III Section 1 in [6]
[7] Lemma 12 in Chapter III Section 5 in [6]
[8] See Theorem 6.2 in [14].
[9] Lemma 5 in Chapter III Section 1 in [6], or also [3] Chap. 3 Sec. 1. The value "4" covers also the case of complex valued μ.
[10] Definition 1 in Chapter IV Section 6 in [6]

when not specified, it will be intended that $\mathcal{F} = \mathcal{F}_\tau$, the algebra generated by open sets.

Theorem 3.14. [11] *$\mathtt{rba}$ and $\mathtt{rca}$ are Banach spaces when endowed with the total variation norm $\|\mu\| = \|\mu\|(S)$.*

If $\mu \in \mathtt{rba}$ and $f \in C_b(S)$, then (using Lemma 3.4) it is possible to show that the integral $\int_S f \,\mathrm{d}\mu$ is a well defined concept. More can be said.

Theorem 3.15. [12] *Suppose that S is normal (see definition 2.4), then there is a linear isometric isomorphism between $J \in C_b(S)^*$ and $\mu \in \mathtt{rba}$ such that*

$$\forall f \in C_b(S) \quad , \quad J(f) = \int_S f \,\mathrm{d}\mu \quad ;$$

and this isomorphism preserves order.

The above can be used to characterize non-negative μ, as follows.

Corollary 3.16. *Suppose that S is normal. For $\mu \in \mathtt{rba}(\mathcal{F})$, the following are equivalent*

$$\forall A \in \mathcal{F}\,, \mu(A) \geq 0$$
$$\forall f \in C_b(S)\,, f \geq 0 \Rightarrow \int f \,\mathrm{d}\mu \geq 0 \quad .$$

3.2 Riesz Representation Theorem

Again in this section we assume the Hypotheses in 3.5. We recall this result.

Theorem 3.17 (Alexandroff). [13] *Suppose that S is compact, then any $\mu \in \mathtt{rba}$ is countably additive.*

The combinations of the above Theorems 3.15 and 3.17 proves the Riesz Representation Theorem, in this form.

Theorem 3.18 (Riesz). *Suppose that S is a compact space, then there is a linear isometric isomorphism between $J \in C(S)^*$ and $\mu \in \mathtt{rca}$ such that*

$$\forall f \in C(S) \quad , \quad J(f) = \int_S f \,\mathrm{d}\mu \quad ;$$

and this isomorphism preserves order.

These examples illustrate what can happen when the space is not compact.

Example 3.19. *Let $S = [0, \infty) \subset \mathbb{R}$, and let $\mathcal{F}_f$ be the algebra generated by finite-length intervals. Note that $\mathcal{F}_f \subseteq \mathcal{F}_\tau$ and $\mathcal{F}_f \neq \mathcal{F}_\tau$ since there are open sets not in $\mathcal{F}_f$. It is easily proved that $C_l(S) = C_b(S, \mathcal{F}_f)$ (as defined in 3.2). Define $\mu : \mathcal{F}_f \to \mathbb{R}$ by*

- *$\mu(A) = 1$ if the right-most interval in A is infinite length,*
- *$\mu(A) = 0$ otherwise;*

[11] This is proven in Chapter III Section 7 in [6].
[12] Theorem 2 in Chapter IV Section 6 in [6]
[13] Theorem 13 in Chapter III Section 5 in [6]

then $\mu \in \mathscr{rba}(S, \mathcal{F}_f)$ but not countably additive; moreover

$$\forall f \in C_l \ , \quad \int_S f(x)\,\mathrm{d}\mu(x) = \lim_{x\to\infty} f(x) \quad .$$

Let $\Phi(f)$ that functional, then $\Phi \in C_l(S)^*$; by Hahn–Banach theorem, there is a continuous extension to $\widetilde{\Phi} \in C_b(S)^*$ such that $\|\Phi\| = \|\widetilde{\Phi}\|$, so by Theorem 3.15 there is an extension of μ to a $\tilde{\mu} \in \mathscr{rba}(S, \mathcal{F})$. This $\tilde{\mu}$ cannot be countably additive.

A different version of Riesz Theorem, for positive measures, is:

Theorem 3.20 (Riesz [14]). *Suppose that S is a locally compact space. There is a linear isometric isomorphism relating any positive [15] $J \in C_c(S)^*$ to a countably additive non-negative measure μ such that*

$$\forall f \in C_c(S) \quad , \quad J(f) = \int_S f\,\mathrm{d}\mu \quad .$$

That measure enjoys many regularity properties and is often is called *a Radon measure*; the complete statement and proof is in Theorem 2.14 in Chap. 2 in [14], or in Theorem 7.2 in [7]; but note that we will use a different Definition 3.26 of *Radon measure*.

For signed measures there is also this version.

Theorem 3.21 (Riesz). *Suppose that S is a locally compact Hausdorff space. There is a linear isometric isomorphism relating any $J \in C_0(S)^*$ to a signed Radon measure[16] μ, such that*

$$\forall f \in C_0(S) \quad , \quad J(f) = \int_S f\,\mathrm{d}\mu \quad .$$

The proof of this result is in Theorem 7.17 in [7]. [17]

3.3 Borel measures

A version of Carathéodory's extension theorem [18] assures that any $\mu \in \mathtt{rca}$ can be extended uniquely to a regular bounded signed Borel measure $\tilde{\mu} : \mathcal{B} \to \mathbb{R}$; so we will consider Borel measures in the following.

Definition 3.22. We denote by $\mathcal{M}(S)$ the set of all regular Borel bounded signed measures on S.

$\mathcal{M}(S)$ is again a Banach space with the norm of the total variation. If S is uncountable, $\mathcal{M}(S)$ is not separable.

Regularity "comes for free" in certain spaces.

Definition 3.23. We recall that a Hausdorff topological space (S, τ) is called *perfectly normal* if, for every two disjoint closed sets E and F, there is a continuous function $f : S \to [0,1]$ such that $f^{-1}(\{0\}) = E$ and $f^{-1}(\{1\}) = F$.

Theorem 3.24. *If S is perfectly normal, then any bounded Borel measure $\mu : \mathcal{B} \to \mathbb{R}$ is regular.*

This is stated in Cor. 7.1.9 in [3]. The proof is the same as that of Thm. 1.4.8 *ibidem*.

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[14] Also known as Riesz–Markov–Kakutani [17], although this attribution is debatable.
[15] "Positive" means $f \geq 0 \Rightarrow J(f) \geq 0$.
[16] "signed Radon measure" is defined [7].
[17] If S is a locally compact separable metric space, then see also Theorem 1.54 in [1].
[18] Theorem 14 in Chapter III Section 5 in [6]

3.4 Radon measures

Let again (S, τ) be a Hausdorff topological space. Unfortunately, there are different definitions of *a Radon measure*. We use the definition from [3, 4, 16].

Definition 3.25. Let $\mu : \mathcal{F} \to [0, \infty)$ be a finite, non negative, finitely-additive set function defined on an algebra $\mathcal{F}$ of S containing the open sets. It is called **tight** if, for each $\varepsilon > 0$ and $B \in \mathcal{F}$, there exists a a compact set $K \subseteq B$ such that $\mu(B \setminus K) < \varepsilon$.

Definition 3.26. A **Radon measure** in S is a finite non negative tight Borel measure $\mu : \mathcal{B}(S) \to [0, \infty)$.

Obviously a Radon measure is also regular.[19]

We recall that a *Polish space* is a topological space homeomorphic to a separable complete metric space.

Theorem 3.27 (Ulam). *If (S, τ) is a Polish space, then each Borel finite measure is Radon.*

(For the proof, see Theorem 3.1 in [16], or Theorem 7.1.7 in [3]).

Tightness and regularity are deeply connected. As aforementioned, any Radon measure is regular and tight; vice versa.

Theorem 3.28. *Any $\nu \in \mathscr{rba}$ that is non-negative, regular and tight has an unique extension to a Radon measure.*

This is proven in Theorem 3.2 in [16].

3.4.1 Dieudonné measure

There is an example, attributed to J. Dieudonné, of a Borel probability measure that is regular (in the sense of 3.9 above) but not Radon.

Example 3.29 (Dieudonné measure). *Let ω_1 be the first uncountable ordinal, and $X = [0, \omega_1)$. We endow X with the order topology. Then this topological space is first countable, not separable, completely normal (that is, $T_1 \wedge T_5$), not compact but sequentially compact (for further details, see example 42 in [15]).*

A set $C \subseteq X$ is called club set *when it is closed and unbounded. (Note that a subset of X is bounded iff it is countable iff it is not cofinal). It can be proven that the intersection of countably many club sets is also a club set.*

Let $\mathcal{B}$ be the Borel σ-algebra on X: then for any set $A \in \mathcal{B}$, either A or $X \setminus A$ contains a club set. Define the Borel probability measure μ_{ω_1} by $\mu_{\omega_1}(A) = 1$ if A contains a club set, otherwise $\mu_{\omega_1}(A) = 0$. μ_{ω_1} is regular but not Radon. (See 411Q in Volume 4 of [8], or example 7.1.3 in [3], for details).

Since in the above example X is normal and μ is regular, then (by Theorem 3.15) we can associate to it a functional $J \in C_b(X)$.

Proposition 3.30. *Any continuous $f : X \to \mathbb{R}$ is eventually constant, that is, $\exists a < \omega_1$ such that, $\forall b, a < b < \omega_1$, we have $f(b) = f(a)$ (again, see example 42 in [15]). We then define the functional $J(f) = f(a)$. So*

$$J(f) = \int_X f \, \mathrm{d}\mu_{\omega_1} \quad .$$

[19] See Definition 3.31.

3.4.2 Uniform tightness

Definition 3.31. Let $\mathfrak{M}$ be a family of Radon measures on S. It is called **uniformly tight** [20] if for every $\varepsilon > 0$ there is a compact set $K \subseteq S$ such that

$$\forall \mu \in \mathfrak{M} \ , \ \mu(S \setminus K) < \varepsilon \quad .$$

Note that any finite family is *uniformly tight*, by 3.26; and if some families $\mathfrak{M}_1, \dots \mathfrak{M}_v$ are *uniformly tight* then $\bigcup_{j=1}^{v} \mathfrak{M}_j$ is uniformly tight.

4 Dposets, Nets

Definition 4.1. *A set Θ with a reflexive and transitive binary relation $\leq$ is called* directed *[9] if*

$$\forall \theta_1, \theta_2 \in \Theta \ , \ \exists \theta_3 \in \Theta \ , \theta_1 \leq \theta_3, \theta_2 \leq \theta_3 \quad .$$

We will assume, without loss of generality, that $\leq$ is also antisymmetric, so that $(\Theta, \leq)$ is a partially ordered directed set; see Remark [2B3] from [11], and references therein, or the final section of Chapter 1 in [13].

Lemma 4.2. *Suppose that $(\Theta, \leq)$ is a partially ordered directed set, then the following are equivalent:*

- $(\Theta, \leq)$ *has no maximum;*
- $(\Theta, \leq)$ *has no maximals;*
- $$\forall \theta_1, \theta_2 \in \Theta \ , \ \exists \theta_3 \in \Theta \ , \theta_1 < \theta_3, \theta_2 < \theta_3 \quad . \tag{4.3}$$

(For the proof, see [06V] from [11].)

Let (S, τ_S) be a Hausdorff topological space.

Definition 4.4 (dposet,net)**.** *A partially ordered directed set with no maximum will be abbreviated to* ***dposet*** *in the following.*

Functions $f : \Theta \to S$, where the domain Θ is a dposet, are called ***S-valued nets***.

(We will simply write "nets" in the following, when the codomain is the topological space S). Nets generalize sequences (as $\mathbb{N}$ is a dposet). The concept of *subsequence* is replaced by the concept of *subnet*.

Definition 4.5. Let $f : \Theta \to S$ be a net. Suppose that $(H, \leq_H)$ is a *dposet*, and is cofinal in $(\Theta, \leq)$ via a monotone map $i : H \to \Theta$; this last statement means that

$$(\forall h_1, h_2 \in H, h_1 \leq_H h_2 \Rightarrow i(h_1) \leq i(h_2)) \wedge (\forall j \in \Theta \ \exists h \in H, \ i(h) \geq j) \quad ; \tag{4.6}$$

then $h = f \circ i$ is a **subnet of the net** f.

If in the above $H = \mathbb{N}$ with the usual ordering, then we will say that h **is a subsequence of the net** f.

[20] Definition 3.8.3 from [4].

4.1 Nets and random walks

The following example is of fundamental relevance in the study of convergence of discrete-time to continuous-time processes (as in [12]).

Example 4.7. *Let*

$$\theta = \{t_0 = 0 < t_1 < t_2 \ldots\} \subset \mathbb{Q}$$

be such that

$$\lim_{n\to\infty} t_n = \infty \quad , \quad \sup_n (t_{n+1} - t_n) \le 1 \quad .$$

Let Θ be the dposet of all such θ, ordered by inclusion.

There does not exist a cofinal sequence $i : \mathbb{N} \to \Theta$. (The proof is in Appendix A).

We can interpret the values in θ as the discrete times for a random walk $\mathfrak{X}^\theta = (X_t^\theta)_{t\in\theta}$ of random variables $X_t^\theta : \Omega \to S$. We are interested in the existence of limit points of the net of random processes $\mathfrak{X}^\theta, \theta \in \Theta$: these will be continuous time random walks. More details are in [12]. The above example shows that, to study these limit points in full generality, we cannot simply rely on sequences, we have to develop the base theory for nets.

4.2 Properties

Most definitions and results that are valid for *sequences* can be reformulated for *nets*. Let (S, τ_S) be a Hausdorff topological space.

Definition 4.8.
- Let $f : \Theta \to S$; we define

  $$\lim_{\theta\in\Theta} f(\theta) = x \in S$$

  if, for all $A \in \tau_S$ with $x \in A$, there exists $\hat\theta$ such that

  $$\forall \theta \ge \hat\theta \quad , \quad f(\theta) \in A \quad ;$$

  or, more concisely, if for all $A \in \tau_S$ with $x \in A$, $f(\theta) \in A$ eventually for $\theta \in \Theta$.
- For $f : \Theta \to \mathbb{R}$ we define as follows:

  $$\limsup_{\theta\in\Theta} f(\theta) \stackrel{\text{def}}{=} \inf_{\hat\theta\in\Theta} \sup_{\theta\ge\hat\theta} f(\theta)$$

  and symmetrically for lim inf.

Proposition 4.9.
- *$C \subseteq S$ is closed iff, for any $x \in S$, for any net $f : \Theta \to C$ converging to a point $x \in S$, we have $x \in C$.*
- *Suppose that (X, τ_X) is a Hausdorff topological space, $\varphi : S \to X$; let $s \in S$. The following are equivalent.*

  1. *φ is continuous at s;*
  2. *for each dposet Θ and each net $f : \Theta \to S$ such that*

  $$\lim_{\theta\in\Theta} f(\theta) = s$$

we have

$$\lim_{\theta\in\Theta} \varphi(f(\theta)) = \varphi(s) \quad .$$

Note that if s is an isolated point, then any function φ is continuous at s, and, simultaneously, for any net $f : \Theta \to S$ with $\lim_{\theta\in\Theta} f(\theta) = s$ we have that actually $f(\theta) = s$ eventually in θ.

In proving the above results, this Lemma is a key result.

Lemma 4.10. *Consider a point $x \in S$ that is not an isolated point; let Θ be the set of all neighborhoods of x: then Θ is a dposet, when ordered by descending inclusion.*

Remark 4.11. *All of the above can be formulated for partially ordered directed sets Θ that have a maximum $\tilde{\theta}$, but then it is quite trivial:* $\lim_{\theta\in\Theta} f(\theta) = f(\tilde{\theta})$ *and so on.*

Remark 4.12. We again remark that in [9] and other texts, a *net* is a function $f : \Theta \to S$ whose domain is a directed set; but, all results that we will need are equally valid for this definition of *net*.

Some results are actually more intuitive with nets. The following theorem is of fundamental importance in topology (and in particular in connection with Prokhorov's Theorem, in the form presented in Theorem 6.20 later on).

Theorem 4.13. *Let (S, τ_S) be a Hausdorff topological space, $K \subseteq S$; the following are equivalent.*

- *K is pre-compact[21];*
- *for any dposet Θ and any net $f : \Theta \to S$ there is a converging subnet.*

For a proof, see Chapter 5 in [9] or, [0K8] from [11].

4.3 Examples

These examples stress the fundamental difference between "net" and "sequence", as well as "subnet" and "subsequence".

Example 4.14. *Suppose that I is a set with continuum cardinality; to fix the ideas, let $I = \{0,1\}^{\mathbb{N}}$; and that $X = \{0,1\}^I$ is endowed with the product topology of the discrete topology on $\{0,1\}$: then X is compact, by Tychonoff Theorem. In this space there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq X$ that does not admit a converging subsequence. A simple example is $f_n(x) = x(n)$ (recall that $x \in I$ is itself a sequence $x : \mathbb{N} \to \{0,1\}$). At the same time, due to theorem above, this sequence admits a converging subnet.*

Example 4.15. *There is an example of a real valued injective net, that converges to zero, but has no subsequence converging to zero.*

Let X be the interval of all ordinals up to the first uncountable ordinal, excluded (as was in example 3.29). Let $J = \mathbb{N} \times X$ be ordered by the product order $\preceq$, i.e.

$$(n_0, \alpha_0) \preceq (n_1, \alpha_1) \iff \Big(n_0 \le n_1 \wedge \alpha_0 \le \alpha_1\Big) \quad ;$$

then $(J, \preceq)$ is a dposet. Define the function $g : J \to \mathbb{R}$ by mapping injectively

$$\alpha \in X \to g(n, \alpha) \in \left(\frac{1}{n+1}, \frac{1}{n+2}\right) \quad ,$$

[21] This means that the closure of K is compact.

for each $n \in \mathbb{N}$. This net converges to zero.

Any sequence

$$n \mapsto g(n, \alpha)$$

for fixed α converges to zero, but it is not a "subsequence" of the net, since it is not cofinal. More in general, no map $i : \mathbb{N} \to J$ can be cofinal: indeed, writing $i(m) = (n_m, \alpha_m)$, we know α_m is upper bounded, let $\alpha > \sup_m \alpha_m$, thus there is no m such that $i(m) \geq (0, \alpha)$.

Example 4.16. *If (M, ρ) is a metric space and $a : \mathbb{N} \to M$ is a sequence convergent to $x \in M$, then its image $a(\mathbb{N})$ is pre-compact , and its closure is $a(\mathbb{N}) \cup \{x\}$ that is compact. Instead for nets the situation is different.*

1. *If $f : \Theta \to X$ is a converging net then its image $f(\Theta)$ may fail to be pre-compact as in this simple example:*
$$f : \mathbb{Z} \to \mathbb{R} \ , \ \ n \mapsto 2^{-n} \ .$$
2. *Also, the net $g : J \to \mathbb{R}$ in the Example 4.15 has the property that, if $i : \Theta \to J$ is cofinal and $h = g \circ i$ is the subnet, then the closure of the image $h(\Theta)$ is larger than $h(\Theta) \cup \{0\}$. Indeed, frequently in n, the set*
$$\{\theta \in \Theta : i_1(\theta) = n\}$$
is uncountable (otherwise we may get a contradiction, as in 4.15).

5 Spaces of continuous paths

Before moving into Probability and Stochastic Processes, we need to settle some definitions regarding spaces of continuous paths.

For X a Hausdorff topological space and $I \subseteq \mathbb{R}$ an interval, let $C(I; X)$ be the set of continuous functions $x : I \to X$.

Definition 5.1. *Suppose that H is a Banach space.*

- *If I is not compact, then $C(I; H)$ is a Fréchet space where the topology[22] is defined by the seminorms*
$$[f]_{I_k,\infty} \ \text{where} \ [f]_{I,\infty} = \sup_{t \in I} |f(t)|_H$$
and I_k are compact intervals, $I_k \subset I_{k+1}, \bigcup_k I_k = I$; the convergence in this space is "uniform convergence on compact sets"*.*
- *If I is compact then $C(I; H) = C_b(I; H)$ is the usual Banach space with norm*
$$\|f\|_\infty = \sup_{t \in I} |f(t)|_H \quad .$$

If $M \subseteq H$ is a closed subset of H then $C(I; M) \subseteq C(I; H)$ is also a closed subset.

If moreover H is separable, then $C(I; H)$ and $C(I; M)$ are Polish spaces.

A typical case for Stochastic Processes may be

$$I = \mathbb{R}^+ \ , \ I_k = [0, k] \ .$$

[22] The topology does not depend on the choice of the sequence I_n.

Remark 5.2. Most of what follows may be generalized to a more general context. Let (I, d_I) be a σ-compact metric space, and (X, d_X) be a complete metric space; let I_k be compact subsets such that $I_k \subset I_{k+1}, \bigcup_k I_k = I$. Define a distance on $C(I;X)$ as

$$d_C(f,g) = \sum_{k\in\mathbb{N}} 2^{-n}\varphi\left(\sup_{t\in I_k} d_X(f(t),g(t))\right)$$

where $\varphi(s) = \min\{1,s\}$ or $\varphi(s) = s/(1+s)$. [23] Then $(C(I;X), d_C)$ is a complete metric spaces, where the convergence is again the "*uniform convergence on compact sets*". This more general case is though not currently useful for the intended applications, so we will restrict the discussion to the cases presented in the previous Definition.

Remark 5.3. Consider the restriction map

$$r_n : C(I;H) \to C(I_n;H) \tag{5.4}$$

given by $r_n f = f_{|_{I_n}}$; then the topology on $C(I;H)$ is the initial topology [24] with respect to the maps r_n and the Banach spaces $C(I_n;H)$. It is also a "projective limit" since the restriction maps

$$r_{n,m} : C(I_n;H) \to C(I_m;H)$$

are continuous and satisfy the property

$$r_{n,m} \circ r_n = r_m$$

for $m < n$.

Hence the following result can be applied, by setting $W = C(I;H), W_n = C(I_n;H)$.

Proposition 5.5. *Let W be a set and $r_n : W \to W_n$ be separating functions where W_n are Hausdorff topological spaces, for $n \in \mathbb{N}$; endow W with the initial topology. Let $Z = \prod_n W_n$ with the product topology. A set $K \subseteq W$ is compact if and only if*

1. *for each $n \in \mathbb{N}$, $r_n(K)$ is compact in W_n, and*
2. *the image $P(K)$ of K under the product map*

$$P : W \to Z \ , \ \ x \mapsto (r_n(x))_n \tag{5.6}$$

is closed.

The above result was inspired by Exercise 2.48 in [10].

Proof. We prove only the inverse statement. The map P defined in equation (5.6) is injective (since the maps r_n are assumed to be "separating"); so it is a bijection with its image $I = P(W)$. Moreover the map $P : W \to I$ is a homeomorphism: indeed

[23] In general any $\varphi(s) : \mathbb{R}^+ \to \mathbb{R}^+$ that is bounded and concave with $\varphi(0) = 0$ will do.

[24] The *initial topology* is the coarsest topology on $C(I;H)$ that makes those functions r_n continuous. It is also known as: induced topology, strong topology, limit topology or projective topology.

- the topology of I is generated by the pre-base of sets of the form

$$A = I \cap \prod_n A_n$$

where each A_n is open in W_n, and $A_n = W_n$ but for exactly one n;
- the topology of W is generated by the pre-base of sets of the form $B = r_n^{-1}(A_n)$ where A_n is open in W_n;

and it is easily seen that, to each B in the second clause there corresponds an A as in the first clause such that $B = r_n^{-1}(A_n)$, and vice versa.

Suppose each $C_n = r_n(K)$ is compact, so $C = \prod_n C_n$ is compact in Z; since $P(K)$ is contained in C then $P(K)$ is compact as well, so K is compact. □

Proposition 5.7. *Define W_n, W, P, Z as in the previous proposition. Assume that the image $I = P(W)$ of W under the product map P is closed. Fix $O \subseteq W$. The following are equivalent.*

1. *For each $n \in \mathbb{N}$, $r_n(O)$ is pre-compact in W_n.*
2. *O is pre-compact in W.*

Proof. Let $Z = \prod_n W_n$. Let $I = P(W)$ be the image of the map P defined in equation (5.6). Consider (Z, τ_Z) with the product topology, and (I, τ_I) with the induced topology. Both spaces are Hausdorff.

We now prove the two implications.

- If O is precompact then let C be its closure, that is compact; so $r_n(O) \subseteq r_n(C)$ that is compact.
- We first remark these facts.
  - The map P is injective (since the maps r_n are assumed to be "separating"), and it is a homeomorphism with its image $I = P(W)$.
  - For a given $A \subseteq I$, let C be the closure of A in (I, τ_I) and D be the closure of A in (Z, τ_Z), then $C = I \cap D$.
  - So for any $O \subseteq W$ we have that

$$P(\overline{O}) = I \cap \overline{P(O)} \quad . \tag{5.8}$$

  Suppose each $r_n(O)$ is precompact, let C_n be its closure, that is compact; so $\prod_n C_n$ is compact in Z; but $\prod_n C_n$ contains $P(O)$ so it contains its closure $\overline{P(O)}$; hence $\overline{P(O)}$ is compact. By (5.8), $P(\overline{O})$ is compact, so so $\overline{O}$ is compact.

□

The above results can conveniently be applied to the cases of interest, due to this result.

Proposition 5.9. *Let H be a Banach space. Let $W = C(I; H)$, $W_n = C(I_n; H)$ and r_n be as in eqn. (5.4). Define $Z = \prod_n W_n$ and P as in Prop. 5.5. Let $C \subseteq W$ be a closed subset of W, then the image $P(C)$ of C under the product map P is closed in Z.*

Proof. W is a Frechét space, and the spaces W_n are Banach, so we will use sequences. Suppose that $(\hat{f}_k) \subseteq P(C)$ is a sequence converging to $\hat{g}$ in Z; then $\hat{f}_k = (f_{n,k})_n$ with $f_{n,k} : I_n \to H$

all being restrictions of one $f_k : I \to H$ such that $f_k \in C$; and $\hat{g} = (g_n)_n$ with $g_n : I_n \to H$. By the properties of the the product topology we know that $\lim_k f_{n,k} = g_n$ in W_n. Note that, for $m < n$, $f_{m,k}$ is the restriction of $f_{n,k}$, so g_m is the restriction of g_n: we can then define a continuous function $g : I \to H$ by

$$g(t) = g_n(t) \text{ if } \quad t \in I_n \quad .$$

By the definition of initial topology we get that f_k converges to g according to the topology of W; since C is closed then $g \in C$ and $\hat{g} = P(g) \in P(C)$ so $P(C)$ is closed. □

5.1 Pre-compact subsets

In the following, for $\psi : \mathbb{R} \to \mathbb{R}$, "*monotonic*" means *monotonically weakly increasing* that is $s \le t \Rightarrow \psi(s) \le \psi(t)$.

Definition 5.10. Let $I \subseteq \mathbb{R}$ an interval, E a normed vector space, for $x : I \to E$ uniformly continuous and $\eta \ge 0$ we define *the modulus of continuity*

$$\omega_{I,E}(x, \eta) \stackrel{\text{def}}{=} \sup\{\|x(t) - x(s)\|_E : t, s \in I, |t - s| \le \eta\} \quad ; \tag{5.11}$$

note that $\omega_{I,E}(x, \cdot)$ is continuous, sub-additive, monotonic, and $\omega_{I,E}(x, 0) = 0$.

Remark 5.12. Note that if $I_1 \subseteq I_2$ then $\omega_{I_1,E} \le \omega_{I_2,E}$

(Pre-)compactness can be verified using this version of Ascoli-Arzelà Theorem. (Recall that if I is compact then $C(I; S) = C_b(I; S)$)

Theorem 5.13. *Suppose H is a Banach space. Let $I \subseteq \mathbb{R}$ be a compact interval. Let $F \subseteq C(I; H)$ be a family of continuous functions $x : I \to H$. Consider these two clauses:*

- *there is $J \subseteq I$ countable dense subset such that for each $t \in J$ there exists a pre-compact set $C_t \subset H$ such that $\forall x \in F, x(t) \in C_t$;*
- $$\lim_{\eta \to 0} \sup_{x \in F} \omega_{I,H}(x, \eta) = 0 \quad . \tag{5.14}$$

The above two clauses hold if and only if F is pre-compact in $C(I; H)$.

The above result holds also if we replace "pre-compact" with "compact" everywhere.

Combining with the above results, we obtain this.

Theorem 5.15. *Suppose H is a Banach space. Let $I \subseteq \mathbb{R}$ be a non-compact interval. Let $F \subseteq C(I; H)$ be a family of continuous functions $x : I \to H$. Consider these two clauses:*

- *there is $J \subseteq I$ countable dense subset such that for each $t \in J$ there exists a pre-compact set $C_t \subset H$ such that $\forall x \in F, x(t) \in C_t$;*
- *For any compact interval $K \subset I$,*

$$\lim_{\eta \to 0} \sup_{x \in F} \omega_{K,H}(x, \eta) = 0 \quad . \tag{5.16}$$

The above two clauses hold if and only if F is pre-compact in $C(I; H)$.

The above result holds also if we replace "pre-compact" with "compact" everywhere.

Remark 5.17. Define

$$\overline{\omega}_{I,F}(\eta) \stackrel{\text{def}}{=} \sup_{x\in F} \omega_{I,H}(x,\eta) = \sup\{\|x(t)-x(s)\|_H : t,s\in I, |t-s|\le\eta, x\in F\} \quad ;$$

where equality follows from (5.11); then $\overline{\omega}_{I,F}$ is sub-additive and monotonic. The condition (5.14) then becomes

$$\lim_{\eta\to 0} \overline{\omega}_{I,F}(\eta) = 0 \tag{5.18}$$

and implies that $\overline{\omega}_{I,F}$ is a modulus of continuity for all $x \in F$. So when (5.14) holds, we will say that the family F is *equicontinuous*.

Similarly (5.16) means that that the family F is *equicontinuous on compact sets*.

For example, for $I = \mathbb{R}$, the family

$$F = \{t \mapsto e^{\theta t} : \theta \in [-1,1]\}$$

is *equicontinuous on compact sets*.

6 Probability Theory

In this section, for convenience of the reader, we recall some definitions and results in Probability Theory from the literature. Let (S, τ_S) be a Hausdorff topological space.

Definition 6.1. We denote by $\mathcal{P}(S)$ the set of all regular Borel probability measures on S.

$\mathcal{P}(S)$ is a closed convex set within the space $\mathcal{M}(S)$ (see 3.22).

Remark 6.2. We summarize some of the ideas above. Suppose that S is normal (see definition 2.4). By Theorem 3.15, $C_b(S)^*$ can be isometrically identified with $\mathtt{rba}$. The set

$$\mathtt{prba} = \left\{\mu \in \mathtt{rba} : \mu(S) = 1, \forall f \in C_b(S), f \ge 0 \Rightarrow \int f \,\mathrm{d}\mu \ge 0\right\}$$

is bounded and is closed according to the weak-* topology, so by Banach–Alaoglu theorem $\mathtt{prba}$ is compact in the weak-* topology. By Lemma 3.16, $\mathcal{P}(S) \subseteq \mathtt{prba}$, and more precisely,

$$\mathcal{P}(S) = \mathtt{prba} \cap \{\mu \textit{ is countably additive}\} \quad .$$

Remark 6.3. Again we refer to the weak-* topology on $\mathcal{P}(S)$ as the "**narrow topology**" to avoid confusion with the weak topology on S when S is a Hilbert space.

Example 6.4. *Let δ_n be Dirac's delta centered at $n \in \mathbb{N}$, then it is a Radon probability measure on $\mathbb{R}$. The family $\{\delta_n\}$ is not tight. Fix any subsequence n_k, then there is an $f \in C_b(\mathbb{R})$ such that the subsequence*

$$\delta_{n_k}(f) = f(n_k)$$

does not admit limit. This means that the sequence $\{\delta_n\}$ does not admit a subsequence that converges narrowly. So $\mathtt{prba}$, when endowed with the narrow topology, is another example of space that is compact but not sequentially compact.

(Compare Examples 3.19 and 4.14.)

So we expect that, in general, $\mathtt{prba}$ cannot be metrized. The space of probability measures, instead can be metrized when S is a Polish space.

Theorem 6.5. *If S is a Polish space, the topological space $\mathcal{P}(S)$ with the narrow topology is a Polish space.*

See for instance Chap. 5 in [2] for the proof. Note also that, by Ulam's Theorem 3.27, in that case all probabilities in $\mathcal{P}(S)$ are Radon.

Example 6.6. *Consider again the example 3.29. Let $J = X$, this is a dposet. For each $x \in J$, consider the Dirac measure δ_x: then the net $(\delta_x)_{x\in J}$ narrowly converges to μ_{ω_1}, but it doesn't have a subsequence converging to μ_{ω_1}. Hence the narrow topology on $\mathcal{P}(X)$ is not metrizable.*

6.1 Narrow Convergence

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, (S, τ_S) a Hausdorff topological space, we associate to S the Borel σ-algebra.

Definition 6.7. If $Z : \Omega \to S$ is a random variable, then we denote by $\mu = Z_\sharp \mathbb{P}$ the *law* of the random variable, that is the Borel probability measure defined by

$$\mu(B) = \mathbb{P}(Z^{-1}(B))$$

for each $B \subseteq S$ Borel subset.

In all of the following $\alpha \in J$, where J is a dposet.

Definition 6.8 (Narrow Convergence)**.** Given a net of Borel measures μ_α and μ on S, we will say that $\lim_{\alpha\in J} \mu_\alpha = \mu$ **narrowly** if

$$\forall f \in C_b(S) \ , \ \lim_{\alpha\in J} \int_S f(x)\, \mathrm{d}\mu_\alpha(x) = \int_S f(x)\, \mathrm{d}\mu(x) \quad .$$

The same definition can be stated when $\mu_\alpha, \mu \in$`rba`.

Definition 6.9. If Z_α, Z are random variables taking values in S we will say that $\lim_{\alpha\in A} Z_\alpha = Z$ *narrowly* when $\lim_{\alpha\in J} \mu_\alpha = \mu$ *narrowly* where $\mu_\alpha = Z_{\alpha\sharp}\mathbb{P}, \mu = Z_\sharp \mathbb{P}$, *i.e.* if

$$\forall f \in C_b(S) \ , \ \lim_{\alpha\in J} \mathbb{E}[f(Z_\alpha)] = \mathbb{E}[f(Z)] \quad .$$

Remark 6.10. In some texts ([4], [16]...) this convergence is called *weak convergence*, but this may create confusion when S is a Hilbert space, where *weak convergence* usually means: convergence of a sequence $(x_n)_n \subset H$ to $x \in H$ in the duality with continuous linear functions:

$$\forall v \in H \, , \ \lim_{n\in\mathbb{N}} \langle x_n, v\rangle_H = \langle x, v\rangle_H \quad .$$

(There is though an important connection, see Corollary 3.8.5 in [4]). In other texts it is called *distributional convergence*, but this may cause confusion with the *Schwartz distributions* [25].

We recall this fact from Probability Theory.

Theorem 6.11 (Alexandrov Theorem)**.** *Suppose that S is a Polish space. Let μ_α, μ be probability measures on S; then these are equivalent*

- *narrow convergence of μ_α to μ;*

[25]But, note that the two concepts are equivalent when $S = \mathbb{R}^n$, see Remark 5.1.6 in [2].

- $$\limsup_{\alpha\in J}\mu_\alpha(C)\le\mu(C)$$
  for all closed sets $C\subseteq S$;
- $$\liminf_{\alpha\in J}\mu_\alpha(A)\ge\mu(A)$$
  for all open sets $A\subseteq S$.

Proof. By Prop. 3.1 in [16], then μ_α,μ are τ-smooth; so we can apply Alexandrov's Theorem in the form in Theorem 3.5 in [16]. □

This can be applied to nets of random variables $Z_\alpha, Z : \Omega\to S$, as explained in Definition 6.9.

Some implications in the above Theorem hold also in a more general context (as can be seen by reading the proof of Theorem 3.5 in [16]); as in this proposition.

Proposition 6.12. *Suppose that S is normal (see definition 2.4); suppose that $\mu_\alpha\to\mu$* narrowly, *where μ_α,μ are in $\mathcal{rba}$ and non-negative; then*

- $$\limsup_{\alpha\in J}\mu_\alpha(C)\le\mu(C)$$
  for all closed sets $C\subseteq S$;
- $$\liminf_{\alpha\in J}\mu_\alpha(A)\ge\mu(A)$$
  for all open sets $A\subseteq S$.

Proof. Since S is normal then Urysohn's Lemma 2.5 holds in S. Given $C\subseteq A\subseteq S$ where C is closed and A is open, there exists a continuous function $f : S\to[0,1]$ such that

$$\mathbb{1}_C\le f\le\mathbb{1}_A \tag{6.13}$$

so

$$\limsup_\alpha\mu_\alpha(C)\le\int f\,\mathrm{d}\mu\le\liminf_\alpha\mu_\alpha(A) \tag{6.14}$$

and then in particular

$$\limsup_\alpha\mu_\alpha(C)\le\mu(A)$$
$$\mu(C)\le\liminf_\alpha\mu_\alpha(A).$$

Using the fact that μ is regular then we conclude.

□

(Note that this is a fundamental step in the proof the Riesz representation theorem 3.20).

6.1.1 Properties

In all of the following $\alpha \in J$, where J is a dposet.

Lemma 6.15. *Let μ_α, μ be probability measures on S. If $\mu_\alpha \to \mu$ narrowly, if $f : S \to \mathbb{R}$ is continuous and μ-integrable and*

$$\lim_{R\to\infty} \sup_\alpha \int_{|f|\geq R} |f| \, \mathrm{d}\mu_\alpha = 0 \tag{6.16}$$

then

$$\lim_\alpha \int_S f \, \mathrm{d}\mu_\alpha(x) = \int_S f \, \mathrm{d}\mu(x) \quad .$$

(The proof is the same as Lemma 3.8.7 from [4], where though it is stated for sequences and not nets).

We assume that S is a Polish space; consequently, by Alexandrov's theorem [26], we state.

Lemma 6.17. *Let μ_α, μ be probability measures on S. If $\mu_\alpha \to \mu$ narrowly, if $f : S \to [0, \infty]$ is lower semi continuous and non negative, then*

$$\liminf_\alpha \int_S f \, \mathrm{d}\mu_n \geq \int_S f \, \mathrm{d}\mu \quad .$$

Corollary 6.18. *Let μ_α, μ be probability measures on S. If $\mu_\alpha \to \mu$ narrowly, if $f : S \to \mathbb{R}$ is continuous and if there is $\varepsilon > 0$ such that*

$$s = \sup_\alpha \int_S |f|^{1+\varepsilon} \, \mathrm{d}\mu_\alpha < \infty$$

then

$$\int_S |f|^{1+\varepsilon} \, \mathrm{d}\mu \leq s$$

and

$$\lim_\alpha \int_S f(x) \, \mathrm{d}\mu_\alpha(x) = \int_S f(x) \, \mathrm{d}\mu(x) \quad .$$

The proof is in Appendix A.

Theorem 6.19. [27] *Let $p_1, p_2 \in [1, \infty)$ with $p_1 < p_2$. Assume that $Z_\alpha, Z : \Omega \to H$ are random variables taking values in a Hilbert separable space H, such that $Z_\alpha \to Z$ narrowly and that*

$$\sup_\alpha \mathbb{E}[\|Z_\alpha\|_H^{p_2}] < \infty \quad ;$$

then

$$\lim_\alpha \mathbb{E}[\|Z_\alpha\|_H^{p_1}] = \mathbb{E}[\|Z\|_H^{p_1}] \quad .$$

[26] See note at Theorem 3.5 in [16].
[27] This seems a standard result, but we could not find a reference for it.

Proof. Let $\varepsilon = p_2 - p_1$; set $Y_\alpha = \|Z_\alpha\|_H, Y = \|Z\|_H$, then $\mu_\alpha = Y_{\alpha\sharp}\mathbb{P}, \mu = Y_\sharp \mathbb{P}$ and

$$f : \mathbb{R} \to \mathbb{R}, f(t) = |t|^{p_1}$$

so

$$\mathbb{E}[\|Z_\alpha\|_H^{p_2}] = \int_{\mathbb{R}} f(t)^{1+\varepsilon}\, \mathrm{d}\mu_\alpha \quad , \mathbb{E}[\|Z_\alpha\|_H^{p_1}] = \int_{\mathbb{R}} f(t)\, \mathrm{d}\mu_\alpha \quad ,$$

and apply the previous results. □

6.2 Tightness and compactness

Let (S, τ) be a Hausdorff topological space.

We endow the family $\mathcal{R}(S)$ of all Radon probabilities on S with the weak topology induced by the duality with $C_b(S; \mathbb{R})$; for coherence with the above, we call this topology *narrow topology*.

In this case the following version of Prokhorov's Theorem holds (here expressed in the form of Theorem 3.6 in [16]).

Theorem 6.20 (Prokhorov's Theorem)**.** *Let $\mathfrak{M} \subseteq \mathcal{R}(S)$.*

1. *If S is a completely regular Hausdorff topological space and $\mathfrak{M}$ is tight, then it is pre-compact in the* narrow topology.
2. *If S is a* Polish space *and $\mathfrak{M}$ is pre-compact in the* narrow topology*, then $\mathfrak{M}$ is tight.*

We agree on this (non standard) definition.

Definition 6.21. Let Θ be a dposet. Let μ_θ be a net of Radon measures on S. It is **tight** if $\forall \varepsilon > 0$ there is a compact set $C \subseteq S$ such that

$$\limsup_{\theta \in \Theta} \mu_\theta(S \setminus C) \leq \varepsilon \quad .$$

Let $\mathfrak{x}^\theta$ a net of random variables taking values in S, for $\theta \in \Theta$. It is **tight** if the net of laws $\mu_\theta = \mathfrak{x}^\theta_\sharp \mathbb{P}$ is tight, namely $\forall \varepsilon > 0$ there is a compact set $C \subseteq S$ such that

$$\limsup_{\theta \in \Theta} \mathbb{P}(\mathfrak{x}^\theta \notin C) \leq \varepsilon \quad .$$

Definition 6.22. For $f : \Theta \to \mathcal{R}(S)$ we define the **narrow limit points** $L \subseteq S$ by

$$L = \bigcap_{\hat\theta \in \Theta} \overline{\{f(\theta) : \hat\theta \leq \theta\}} \quad . \tag{6.23}$$

where "closure" is in the narrow topology of $\mathcal{R}(S)$.

Remark 6.24. Do not confuse "*narrow limit points*" with "*accumulation points of the image of the net*". Even for injective nets, they are different, as shown in second point in Example 4.16.

Proposition 6.25. *The above L is also the set of all possible limits of subnets of the net f. (For a proof, see Chapter 5 in [9])*

Theorem 6.26. *Let S be a Polish space, let Θ be a dposet, let $(\mu_\alpha)_{\alpha\in\Theta}$ be a tight net of Radon probabilities on S. Then it has a subnet narrowly converging to a Radon probability. In particular the set of narrow limit points is not empty.*

Proof. A possible proof can be obtained by adapting the proof of Theorem 3.6 in [16]; we present a slightly different proof.

By Theorem 4.13, and what was noted in Remark 6.2, there is a subnet that converges narrowly to a $\nu \in \mathtt{prba}$. We denote the subnet by $(\mu_\beta)_{\beta\in\tilde\Theta}$. We just need to prove that ν can be extended to a Radon measure $\tilde\nu$, since then

$$\forall f \in C_b(S)\ ,\ \int_S f\,\mathrm{d}\nu = \int_S f\,\mathrm{d}\tilde\nu \quad .$$

As aforementioned in 3.28, Theorem 3.2 in [16] asserts that any $\nu \in \mathtt{prba}$ that is tight has an unique extension to a Radon measure. We easily check that ν is tight. By hypothesis $\forall\varepsilon > 0$ there is a compact set $C_\varepsilon \subseteq S$ such that

$$\limsup_{\beta\in\tilde\Theta} \mu_\beta(S \setminus C_\varepsilon) \le \varepsilon \quad ,$$

so (since $(S \setminus C_\varepsilon)$ is open, and using the generalization 6.12 of Alexandroff Theorem)

$$\nu(S \setminus C_\varepsilon) \le \varepsilon \quad . \qquad \square$$

Corollary 6.27. *If S is a Polish space and if $\mathfrak{X}^\theta$ a tight net of random variables taking values in S, for $\theta \in \Theta$; then the net of laws $\mathfrak{X}^\theta_\sharp\mathbb{P}$ admits narrowly converging subnets.*

6.3 Processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X a Hausdorff topological space.

Let $I \subseteq \mathbb{R}$ be an interval. A measurable function $\mathfrak{X} : I \times \Omega \to X$ is called a *process*. For any fixed $\omega \in \Omega$ the function

$$t \mapsto \mathfrak{X}(t, \omega)$$

is called a *path* or a *trajectory*.

Suppose that each path is continuous: then $\mathfrak{X} : \Omega \to C(I; X)$ hence we can call it *random function*. As explained in Section 5, when X is a separable Banach space H or a closed subset of it, then the space $S = C(I; X)$ is a Polish space, so we can view the process $\mathfrak{X}$ simply as a random variable taking values in S, and we can apply to such processes the methods presented in previous Sections. [28]

Theorem 6.28. *Suppose X is closed subset of a separable Banach space H. Let $I \subseteq \mathbb{R}$ be an interval. Suppose that $\mathfrak{X}_\alpha : \Omega \to C(I; X)$ is a net of processes (with $\alpha \in A$, a dposet) satisfying the following two clauses.*

[28] In the general case when S is assumed to be a locally convex topological vector space, then there are measurability issues in the definition of a random variable $\mathfrak{X} : \Omega \to S$, that are explained in [4], see in particular Remark 3.1.3. These problems are not present when $S = C(I; X)$ is a separable metric space.

- $\forall \varepsilon > 0$, *there exist, a countable set* $J = \{a_0, a_1, \dots a_j \dots\}$ *dense in* I, *and compact sets* $C_j \subset C$, *such that*

$$\forall \alpha \in A \ \forall j \in \mathbb{N}, \ \mathbb{P}\{\mathfrak{X}_\alpha(a_j) \notin C_j\} \le \varepsilon 2^{-j} \quad . \tag{6.29}$$

- *For all* $K \subseteq I$ *compact interval,* $\forall \varepsilon_0 > 0$, $\forall \varepsilon_1 > 0$, $\exists \eta > 0$, $\exists \alpha_0 \in A$ *such that*

$$\forall \alpha \in A, \alpha \ge \alpha_0 \Rightarrow \mathbb{P}\left\{\omega_{K,H}(\mathfrak{X}_\alpha, \eta) \ge \varepsilon_0\right\} \le \varepsilon_1 \quad . \tag{6.30}$$

Then the net $\mathfrak{X}_\alpha$ *is tight*[29] *in* $C(I;X)$. *So it has a converging subnet (in the narrow topology), that is, a limit point (by Cor. 6.27).*

Note that if I *is compact, then the second clause must be verified only for* $K = I$, *since* $\omega_{I,H}$ *is monotonic in* I *(Remark 5.12).*

Proof. Fix K, $\varepsilon > 0$, $\varepsilon_0 > 0, \varepsilon_1 > 0$, let $F \subseteq C(I;X)$ be the set of all x such that

$$\forall j, \ x(a_j) \in C_j$$

and $\omega_{K,H}(x, \eta) < \varepsilon_0$. Then $\forall \alpha \ge \alpha_0$ we have

$$\mathbb{P}(\mathfrak{X}_\alpha \notin F) \le 2\varepsilon + \varepsilon_1 \quad .$$

If I is not compact, then we use Prop. 5.5 and 5.9. □

Remark 6.31. The second hypothesis (6.30) may be reformulated as

$$\forall \varepsilon_0 > 0 \ , \ \lim_{\eta \to 0} \limsup_\alpha \mathbb{P}\left\{\omega_{K,H}(\mathfrak{X}_\alpha, \eta) \ge \varepsilon_0\right\} = 0 \tag{6.32}$$

since ω is monotonic in η.

We can interpret the fact that the process takes values in X in two equivalent ways.

Proposition 6.33. *Suppose again that* X *is closed subset of a separable Banach space* H. *Suppose that* $\mathfrak{X}_\alpha : \Omega \to C(I;H)$ *is a net of processes, and we know that, for all* α, *almost certainly the process* $\mathfrak{X}_\alpha(t) \in X$, *for all* t. *Suppose that* $\mathfrak{X}$ *is a limit point for the net of processes. Then almost certainly* $\mathfrak{X}(t) \in X$, *for all* t.

Proof. We know that

$$\mathbb{P}(\{\mathfrak{X}_\alpha \in C(I;X)\}) = 1$$

and we know that $C(I;X)$ is closed in $C(I;H)$ so we can apply Alexandrov's Theorem 6.11 to conclude that

$$\mathbb{P}(\{\mathfrak{X} \in C(I;X)\}) = 1 \quad .$$

□

In the above, narrow convergence of nets can be interpreted in two equivalent ways.

Theorem 6.34. *Suppose again that* $I \subseteq \mathbb{R}$ *is an interval and* I_n *are compact intervals,* $I_n \subset I_{n+1}$, $\bigcup_n I_n = I$; *define*

$$r_n : C(I;H) \to C(I_n;H) \qquad \text{(as in (5.4))}$$

the restriction map. Let $W = C(I;H)$, $W_n = C(I_n;H)$ *for simplicity. Suppose* μ_α, μ *are Radon probability measures on* W, *these are equivalent:*

---

[29] See Definition 6.21.

- $\lim_{\alpha \in J} \mu_\alpha = \mu$ *narrowly in* W,
- *for each* $n \in \mathbb{N}$, $\lim_{\alpha \in J} r_{n\sharp}\mu_\alpha = r_{n\sharp}\mu$ *narrowly in* W_n .

The proof is in Appendix A.

We conclude by citing a standard method to ensure that the paths of a process can be assumed to be continuous: the Kolmogorov test (Theorem 3.3 in [5]).

Theorem 6.35. *Let* $I \subseteq \mathbb{R}$ *be a compact interval, and let* Z *be a process taking values in a complete metric space* (X, ρ). *Define* $Z_t(\omega) = Z(t, \omega)$ *for* $t \in I$, *so that* Z_t *is a random variable* $Z_t : \Omega \to X$. *Suppose that* $\exists C > 0, \delta > 0, \varepsilon > 0$

$$\forall t, s \in I\,,\ \mathbb{E}[\rho(Z_t, Z_s)^\delta] \le C|t-s|^{1+\varepsilon}$$

then Z *has a version with Hölder continuous paths, with an arbitrary exponent smaller than* ε/δ.

A Proofs

Proof of the statement in Example 4.7. By contradiction, suppose there is one; up to substituting $f(n)$ with $\bigcup_{j=0}^n f(j)$ we can suppose that f is monotonic. We will build iteratively a sequence

$$t_0 = 0 < t_1 < t_2 \dots$$

of rational numbers, such that

$$\theta = \{t_0 = 0 < t_1 < t_2 \dots\} \in \Theta$$

and, for all n we have $\theta \not\subseteq f(n)$. To this end, we will also build a (non decreasing) sequence $n_m \in \mathbb{N}$ such that $n_m \to_m \infty$. Let $t_0 = 0, t_1 = 1, n_0 = n_1 = 0$. For $m \ge 1$, having chosen t_m and n_m, we look for $k > n_m$ such that there is a $t \in f(k) \setminus f(n_m) \wedge t \ge t_m + 1/2$;

- if there is no such k, we stop the iterative process by adding to θ an arbitrary sequence $t_{m+1} < t_{m+2} < \dots$ with $t_{m+j} \notin \bigcup_k f(k)$ and $1/2 < t_{m+j+1} - t_{m+j} < 1$; we set $n_{m+j} = n_m + j$; this, for all $j \ge 0$.
- If there is such k, t, we add to θ an arbitrary sequence

  $$t_{m+1} < t_{m+2} < \dots < t_{m+l} = t$$

  such that

  $$1/2 < t_{m+j+1} - t_{m+j} < 1 \ \text{ for } j = 0, \dots l-1\,;$$

  then we set $n_{m+1} = \dots = n_{m+l} = k$; then we repeat the iteration using $m + l$ as the new m.

In any case, we obtain that for infinitely many m, there is a l such that $t_{m+l} \notin f(n_m)$. □

Proof of 6.18.

Proof. We check that (6.16) is satisfied. Setting

$$\nu_\alpha = |f|_\sharp \mu_\alpha$$

then

$$\int_S |f|^{1+\varepsilon}\,\mathrm{d}\mu_\alpha = \int_{\mathbb{R}} |t|^{1+\varepsilon}\,\mathrm{d}\nu_\alpha$$

and

$$\int_{|f|\ge R} |f|\,\mathrm{d}\mu_\alpha = \nu_\alpha\{|t| \ge R\}$$

so by Markov inequality

$$\int_{|f|\ge R} |f|\,\mathrm{d}\mu_\alpha \le \frac{\int_{\mathbb{R}} |t|^{1+\varepsilon}\,\mathrm{d}\nu_\alpha}{R} \le \frac{s}{R} \quad .$$

We have that $f_\sharp\mu_\alpha \to f_\sharp\mu$ narrowly, so by the hypothesis and the previous Lemma

$$s \ge \liminf_{n\to\infty} \int_S |f|^{1+\varepsilon}\,\mathrm{d}\mu_\alpha = \liminf_{n\to\infty} \int_{\mathbb{R}} |t|^{1+\varepsilon}\,\mathrm{d}f_\sharp\mu_\alpha \ge \int_{\mathbb{R}} |t|^{1+\varepsilon}\,\mathrm{d}f_\sharp\mu = \int_S |f|^{1+\varepsilon}\,\mathrm{d}\mu$$

in particular f is μ-integrable. So we can apply the previous Lemma 6.15 □

Here is the proof of Theorem 6.34.

Proof. One implication is trivial. We prove that the second clause implies the first. The balls (for $\varepsilon > 0, f \in W_n$)

$$B^{W_n}(f,\varepsilon) = \{g :\in W_n \,:\, [f-g]_{I_n,\infty} < \varepsilon\}$$

are a base for the topology in W^n; since each f, g can be extended constantly, we can say that

$$\{g \in W \,:\, [f-g]_{I_n,\infty} < \varepsilon\} \quad \text{for} \quad \varepsilon > 0, f \in W, n \in \mathbb{N}$$

are a base for the topology in W. Since W is separable, let f_k a countable dense sequence, then

$$B(k,m,n) \stackrel{\text{def}}{=} \{g \in W \,:\, [f_k - g]_{I_n,\infty} < 1/m\} \qquad \text{for } n,m,k \in \mathbb{N}$$

is a countable base.

For each $A \subseteq W$ open there are sequences k_j, m_j, n_j

$$A = \bigcup_{j=1}^{\infty} B(k_j, m_j, n_j)$$

let

$$A_l = \bigcup_{j=1}^{l} B(k_j, m_j, n_j)$$

then there are $B_l \in W_{N_l}$ open such that

$$A_l = r_{N_l}^{-1}(B_l)$$

with $N_l = \max_{j \le l} n_j$. We eventually write

$$\liminf_\alpha \mu_\alpha(A) \ge \liminf_\alpha \mu_\alpha(A_l) = \liminf_\alpha r_{N_l \sharp}\mu_\alpha(B_l) \ge r_{N_l \sharp}\mu(B_l) = \mu(A_l)$$

and then pass to the limit on RHS. We conclude by Alexandrov's Theorem 6.11. □

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