

# DIFFERENTIAL INCLUSIONS INVOLVING THE CURL OPERATOR

NURUN NESHA<sup>†,1</sup>

ABSTRACT. In this article, we study the existence of  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  satisfying

$$\operatorname{curl} \eta \in E \text{ a.e. in } \Omega,$$

where  $n \in \mathbb{N}, \Omega \subseteq \mathbb{R}^n$  is open, bounded and  $E \subseteq \Lambda^2$ .

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## 1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the following differential inclusion problem

$$\begin{aligned} \operatorname{curl} \eta &\in E \quad \text{a.e. in } \Omega \\ \text{and } \int_{\Omega} \eta &\neq 0 \end{aligned} \tag{1.1}$$

where  $\Omega \subseteq \mathbb{R}^n$  is open, bounded,  $E \subseteq \Lambda^2(\mathbb{R}^n)$ , and  $n \geq 4$ . This problem has been studied in Bandyopadhyay-Barroso-Dacorogna-Matias [2] and Bandyopadhyay-Dacorogna-Kneuss [3] in the lower dimensional cases, namely when  $\dim \operatorname{span} E = n - 1$  when  $n \geq 3$ ,  $\dim \operatorname{span} E = 3$ , when  $n = 3$ . In this article, we investigate the case when  $\dim \operatorname{span} E \geq n$ . The most fundamental case is, of course, the gradient case which has received notable attention, in particular, by Bressan-Flores [4], Cellina [5, 6], Dacorogna-Marcellini [12] and Friesecke [14]. An extensive study has been done in [10] on this topic. We prove a few existence as well as non-existence results in this regard. Our main result is the following which we will prove in section 4.

**Theorem 1.1.** *Let  $n \in \mathbb{N}, n \geq 5$  and  $E \subseteq \Lambda^2(\mathbb{R}^n)$  be such that*

$$\omega \wedge \omega' = 0 \text{ for all } \omega, \omega' \in E.$$

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<sup>†</sup>Indian Institute of Science Education and Research Kolkata, Campus Road, Mohanpur, West Bengal 741246, India; <sup>1</sup>nn16ip021@gmail.com.

Let  $\Omega \subseteq \mathbb{R}^n$  be an open, bounded set. Then there exists  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  such that

$$\begin{aligned} \operatorname{curl} \eta &\in E \text{ a.e. in } \Omega \\ \operatorname{meas}\{x \in \Omega : \operatorname{curl} \eta(x) = e\} &> 0 \text{ for all } e \in E \\ \text{and } \int_{\Omega} \eta &\neq 0 \end{aligned} \quad (1.2)$$

if and only if  $0 \in \operatorname{rico} E$  and  $\dim \operatorname{span} E = n - 1$ .

The next result is about non-existence of solution when  $\dim \operatorname{span} E = n$  which we will discuss in section 3.

**Theorem 1.2.** *Let  $n \in \mathbb{N}, n \geq 4$  and  $\Omega \subseteq \mathbb{R}^n$  be open, bounded. Then the following differential inclusion problem*

$$\operatorname{curl} \eta \in E \text{ a.e. in } \Omega \text{ and } \int_{\Omega} \eta \neq 0$$

has no solution  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  if  $\dim \operatorname{span} E = n$  and  $\operatorname{meas}\{x \in \Omega : \operatorname{curl} \eta(x) = e\} > 0$  for all  $e \in E$ .

In [1], Ball-James considered two gradient problem and found that rank of the difference of two gradients is less than or equal to 1 and in a similar way we can show that, in curl case the rank of the difference will be less than or equal to 2. So, in section 4, we will see the following theorem under taking the constraint on the set  $E$  that rank of difference of any two elements is less than or equal to 2.

**Theorem 1.3.** *Let  $n \in \mathbb{N}$  and  $\Omega \subseteq \mathbb{R}^n$  be open, bounded set. Let  $E \subseteq \Lambda^2(\mathbb{R}^n)$  be such that  $\operatorname{rank}[e - f] \leq 2$  for any  $e, f \in E$ , in other words, there exist  $x, y \in \mathbb{R}^n$  such that  $e - f = x \wedge y$ . Then there does not exist any  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  of the following problem*

$$\operatorname{curl} \eta \in E \text{ a.e. in } \Omega$$

if  $\dim \operatorname{span} E \geq n + 1$  and  $\operatorname{meas}\{x \in \Omega : \operatorname{curl} \eta(x) = e\} > 0$  for all  $e \in E$ .

In section 5, we will give one existence result of solution  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  at dimension  $(2n - 3)$  for the following differential inclusion problem

$$\begin{aligned} \operatorname{curl} \eta &\in E \text{ a.e. in } \Omega, \\ \operatorname{meas}\{x \in \Omega : \operatorname{curl} \eta(x) = e\} &> 0 \text{ for all } e \in E. \end{aligned}$$

Finally, in section 6, we show that

**Theorem 1.4.** *Let  $n \in \mathbb{N}, 1 \leq k \leq n - 3$ . Suppose  $f : \mathbb{R}^n \rightarrow \Lambda^k(\mathbb{R}^n)$  be continuous such that  $f(0) \neq 0$  and  $f(0)$  is  $k$ -divisible, i.e.,  $f(0) = c^1 \wedge \dots \wedge c^k$  for some  $c^i \in \mathbb{R}^n \setminus \{0\}$  for  $i = 1, \dots, k$ . Then*

$$\dim \operatorname{span}\{x \wedge f(x) : x \in \mathbb{R}^n\} \neq n - k + 1.$$

The results of differential inclusion problems can be applied, embracing the notions due to Cellina [5, 6] and Friesecke [14], to obtain solutions for a non-convex variational problem. In particular, according to [2], one can show that:

**Theorem 1.5.** *Let  $\Omega \subseteq \mathbb{R}^n$  be a bounded, open set,  $0 \leq k \leq n - 1$  and*

$$f : \Lambda^{k+1}(\mathbb{R}^n) \rightarrow \mathbb{R}_+$$

be lower semi-continuous. Let

$$\inf \left\{ \int_{\Omega} f(d\eta(x)) dx : \eta \in W_0^{1,\infty}(\Omega; \Lambda^k(\mathbb{R}^n)) \right\} \quad (1.3)$$

and  $K = \{\xi \in \Lambda^{k+1}(\mathbb{R}^n) : f^{**}(\xi) < f(\xi)\}$ , where  $f^{**}$  is the convex envelope of  $f$ . Assume that  $K$  is connected and  $0 \in K$ . If  $K$  is bounded and  $f^{**}$  is affine on  $K$  then (1.3) has a solution.

Many results as well as applications of differential inclusions can be observed in Dacorogna-Pisante [8], Dacorogna-Fonseca [11], Dacorogna-Marcellini [12], Blasi-Pianigiani [13], Sil [16] and Sychev [17].

## 2. NOTATIONS

We gather here some notations which will be used throughout this article. For more details on exterior algebra and differential forms see [7] and for convex analysis see [9] or [15].

(1) Let  $k, n$  be two integers.

- We write  $\Lambda^k(\mathbb{R}^n)$  (or simply  $\Lambda^k$ ) to denote the vector space of all alternating  $k$ -linear maps  $f : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k\text{-times}} \rightarrow \mathbb{R}$ . For  $k = 0$ , we set  $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$ . Note

that  $\Lambda^k(\mathbb{R}^n) = \{0\}$  for  $k > n$  and, for  $k \leq n$ ,  $\dim(\Lambda^k(\mathbb{R}^n)) = \binom{n}{k}$ .

- $\wedge, \lrcorner, \langle ; \rangle$  and, respectively,  $*$  denote the exterior product, interior product, the scalar product and, respectively, the Hodge star operator.
- For  $b \in \Lambda^k$ ,  $\text{rank}[b]$  denotes the rank of the exterior  $k$ -form  $b$ .
- If  $\{e^1, \dots, e^n\}$  is a basis of  $\mathbb{R}^n$ , then, identifying  $\Lambda^1$  with  $\mathbb{R}^n$ ,

$$\{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$$

is a basis of  $\Lambda^k$ .

- For  $E \subseteq \Lambda^k$ ,  $\text{span } E$  denotes the subspace spanned by  $E$ .
- Let  $W$  be a subspace of  $\Lambda^k$ . We write  $\dim W$  to denote the dimension of  $W$  and  $W^\perp$  to denote the orthogonal complement of  $W$ .
- For  $b \in \Lambda^k$ , we write, identifying again  $\Lambda^1$  with  $\mathbb{R}^n$ ,

$$\mathbb{R}^n \wedge b = \Lambda^1 \wedge b = \{x \wedge b : x \in \Lambda^1\} \subseteq \Lambda^{k+1}.$$

(2) Let  $\Omega \subseteq \mathbb{R}^n$  be a bounded open set.

- The spaces  $C^1(\Omega; \Lambda^k)$ ,  $W^{1,p}(\Omega; \Lambda^k)$  and  $W_0^{1,p}(\Omega; \Lambda^k)$ ,  $1 \leq p \leq \infty$  are defined in the usual way.
- For  $\eta \in W^{1,p}(\Omega; \Lambda^k)$ ,  $\int_{\Omega} \eta$  denotes the exterior  $k$ -form obtained by integrating componentwise the differential form  $\eta$ . Explicitly, for  $1 \leq i_1 < \cdots < i_k \leq n$ ,

$$\left( \int_{\Omega} \eta \right)_{i_1 \cdots i_k} = \int_{\Omega} \eta_{i_1 \cdots i_k}.$$

- For  $\eta \in W^{1,p}(\Omega; \Lambda^k)$ , the exterior derivative  $d\eta$  belongs to  $L^p(\Omega; \Lambda^{k+1})$  and is defined by

$$(d\eta)_{i_1 \cdots i_{k+1}} = \sum_{j=1}^{k+1} (-1)^{j+1} \frac{\partial \eta_{i_1 \cdots i_{j-1} i_{j+1} \cdots i_{k+1}}}{\partial x_{i_j}},$$

for  $1 \leq i_1 < \cdots < i_{k+1} \leq n$ . If  $k = 0$ , then  $d\eta \simeq \text{grad } \eta$ . If  $k = 1$ , then for  $1 \leq i < j \leq n$ ,

$$(d\eta)_{ij} = \frac{\partial \eta_j}{\partial x_i} - \frac{\partial \eta_i}{\partial x_j}$$

i.e.,  $d\eta \simeq \text{curl } \eta$ .

(3) For subsets  $C, V \subseteq \Lambda^k$ ,

- $\text{co } C$  denotes the convex hull of  $C$ ;

- $\text{int}_V C$  denotes the interior of  $C$  with respect to the topology relative to  $V$ .
- (4) For a convex set  $C \subseteq \Lambda^k$ ,
- $\text{aff } C$  denotes the affine hull of  $C$  which is the intersection of all affine subsets of  $\Lambda^k$  containing  $C$ ;
  - $\text{ri } C$  denotes the relative interior of  $C$  which is the interior of  $C$  with respect to the topology relative to affine hull of  $C$ . Equivalently  $\text{ri } C = \text{int}_{\text{aff } C} C$ ;
  - $\text{rbd } C$  denotes the relative boundary of  $C$  which is  $\overline{C} \setminus \text{ri } C$ .
- (5) For a set  $A \subseteq \mathbb{R}^n$   $\text{meas}(A)$  denotes the Lebesgue measure of  $A$ .
- (6)  $\mathbb{R}_+$  denotes the set of all non-negative real numbers.

### 3. NON EXISTENCE OF SOLUTION: DIMENSIONALITY OF E

In this section, we will prove a non-existence result, namely, that there is no  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  satisfying

$$\text{curl } \eta \in E, \text{ a.e. in } \Omega, \int_{\Omega} \eta \neq 0,$$

when  $\dim \text{span } E = n$ . The following lemma plays the main role.

**Lemma 3.1.** *Let  $n \in \mathbb{N}$ ,  $n \geq 4$  and let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be continuous with  $f(0) \neq 0$ . Then*

$$\dim \text{span}\{x \wedge f(x) : x \in \mathbb{R}^n\} \neq n.$$

**Remark 3.2.** *Lemma 3.1 is not true when  $n = 3$ . Let us define  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  by*

$$f(x) := (e^1 \otimes e^1)x + e^2, \text{ for all } x \in \mathbb{R}^3.$$

*Then  $\dim \text{span}\{x \wedge f(x) : x \in \mathbb{R}^3\} = 3$ . To see this, we note that  $e^1 \wedge f(e^1) = e^1 \wedge e^2$ ,  $e^3 \wedge f(e^3) = -e^2 \wedge e^3$ , and  $(e^1 + e^3) \wedge f(e^1 + e^3) = e^1 \wedge e^2 - e^2 \wedge e^3 - e^1 \wedge e^3$ .*

*Proof.* Let us set

$$\mathcal{S} := \text{span}\{x \wedge f(x) : x \in \mathbb{R}^n\}.$$

We prove by contradiction. Let us suppose to the contrary that  $\dim \mathcal{S} = n$ . Note that,  $\mathbb{R}^n \wedge f(0) \subseteq \mathcal{S}$ , see Proposition 2.2 of [3]. Furthermore, we can find  $x_0 \in \mathbb{R}^n$  such that  $x_0 \wedge f(x_0) \wedge f(0) \neq 0$ . Indeed, if this was not the case, we would have

$$x \wedge f(x) \wedge f(0) = 0 \text{ for all } x \in \mathbb{R}^n.$$

Cartan's lemma, see Theorem 2.42 of [7], then guarantees the existence of  $u_x \in \mathbb{R}^n$  satisfying

$$x \wedge f(x) = u_x \wedge f(0) \text{ for all } x \in \mathbb{R}^n,$$

which implies that

$$\mathcal{S} = \mathbb{R}^n \wedge f(0).$$

This is a contradiction as  $\dim(\mathbb{R}^n \wedge f(0)) = n - 1$  [ Lemma 2.1 of [3] ], whereas  $\dim \mathcal{S} = n$ . Therefore, we indeed have a  $x_0 \in \mathbb{R}^n \setminus \{0\}$  such that

$$x_0 \wedge f(x_0) \wedge f(0) \neq 0.$$

Since  $f$  is continuous at  $x_0$ , there exists an  $\epsilon > 0$  such that

$$x \wedge f(x) \wedge f(0) \neq 0 \text{ for all } x \in B_{\epsilon}(x_0).$$

Let us find a basis  $\{a^1, \dots, a^n\}$  of  $\mathbb{R}^n$  inside  $B_{\epsilon}(x_0)$ . Then

$$a^i \wedge f(a^i) \wedge f(0) \neq 0 \text{ for all } i = 1, \dots, n. \quad (3.1)$$

Let us write

$$\mathcal{S} = [\mathbb{R}^n \wedge f(0)] \oplus [\mathbb{R}^n \wedge f(0)]^{\perp} = [\mathbb{R}^n \wedge f(0)] \oplus \text{span}\{\omega\},$$

where  $\omega \in [\mathbb{R}^n \wedge f(0)]^{\perp} \setminus \{0\}$ . For each  $i = 1, \dots, n$ , we have

$$a^i \wedge f(a^i) = c_i \wedge f(0) + \beta_i \omega, \quad (3.2)$$

for some  $c_i \in \mathbb{R}^n$  and  $\beta_i \in \mathbb{R}$ . Note that, thanks to equation 3.1, we have  $\beta_i \neq 0$ , for all  $i = 1, \dots, n$ . It follows from Equation 3.2 that, for all  $i = 1, \dots, n$ ,

$$\beta_i \omega \wedge f(0) \wedge a_i = 0.$$

Since  $\beta_i \neq 0$  for every  $i = 1, \dots, n$ , we have

$$\omega \wedge f(0) \wedge a^i = 0 \text{ for every } i = 1, \dots, n,$$

which implies that  $\omega \wedge f(0) = 0$  as  $\{a^1, \dots, a^n\}$  is a basis of  $\mathbb{R}^n$ . Using Proposition 2.16 of [7], we also note that

$$\begin{aligned} \langle f(0) \lrcorner \omega; x \rangle &= (-1)^{(1+1)} \langle \omega; f(0) \wedge x \rangle \\ &= 0, \text{ for all } x \in \mathbb{R}^n, \end{aligned}$$

as  $\omega \in \mathcal{S} \cap [\mathbb{R}^n \wedge f(0)]^\perp$ . It follows that  $f(0) \lrcorner \omega = 0$ . This, combined with  $\omega \wedge f(0) = 0$  and Proposition 2.16 of [7] implies that

$$\|f(0)\|^2 \omega = f(0) \lrcorner (f(0) \wedge \omega) + f(0) \wedge (f(0) \lrcorner \omega) = 0.$$

Since  $f(0) \neq 0$ , we have  $\omega = 0$ , which is a contradiction. Therefore  $\dim \text{span}\{x \wedge f(x) : x \in \mathbb{R}^n\} \neq n$ .  $\square$

**Theorem 3.3.** *Let  $n \in \mathbb{N}$  with  $n \geq 4$ , let  $\Omega \subseteq \mathbb{R}^n$  be open, bounded and let  $E \subseteq \Lambda^2(\mathbb{R}^n)$ . Then there is no  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  satisfying*

$$\text{curl } \eta \in E \text{ a.e. in } \Omega \text{ and } \int_{\Omega} \eta \neq 0 \quad (3.3)$$

if

- (i)  $\dim \text{span } E = n$ , and
- (ii)  $\text{meas}\{x \in \Omega : \text{curl } \eta(x) = e\} > 0$  for all  $e \in E$

**Remark 3.4.** *Theorem 3.3 is not true when  $n = 3$ , see Theorem 4.15 of [2]. The solution  $\eta$  constructed in the proof of Theorem 4.15 of [2] has the property that  $\int_{\Omega} \eta \neq 0$ .*

**Remark 3.5.** *The case  $n \leq 3$  has been done completely in [2]. We don't need to take the case  $\int_{\Omega} \eta \neq 0$  for  $n \leq 3$ .*

*Proof.* Let

$$\mathcal{P} : \Lambda^2(\mathbb{R}^n) \rightarrow \Lambda^2(\mathbb{R}^n)$$

be the projection onto the orthogonal complement of  $\text{span } E$ . Since  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  extending  $\eta$  by 0 to  $\mathbb{R}^n$ , it follows that

$$\mathcal{P}(\text{curl } \eta) = 0 \text{ a.e. in } \mathbb{R}^n.$$

Applying the Fourier transform, we obtain

$$\mathcal{P}(x \wedge \hat{\eta}(x)) = 0 \text{ for all } x \in \mathbb{R}^n$$

which implies that

$$x \wedge \hat{\eta}(x) \in \text{span } E \text{ for all } x \in \mathbb{R}^n,$$

where  $\hat{\eta}(x) = \int_{\mathbb{R}^n} \eta(y) \cos(2\pi \langle x; y \rangle) dy$ . Together with the Proposition 2.2 of [3] and the above we can conclude that

$$\mathbb{R}^n \wedge \hat{\eta}(0) \subseteq \text{span}\{x \wedge \hat{\eta}(x) : x \in \mathbb{R}^n\} \subseteq \text{span } E. \quad (3.4)$$

We will now show that

$$\text{span}\{x \wedge \hat{\eta}(x) : x \in \mathbb{R}^n\} = \text{span } E.$$

Suppose not, i.e.,  $\text{span}\{x \wedge \hat{\eta}(x) : x \in \mathbb{R}^n\} \subsetneq \text{span } E$ . Let  $m \in \text{span}\{x \wedge \hat{\eta}(x) : x \in \mathbb{R}^n\}^\perp$ . Then

$$\langle x \wedge \hat{\eta}(x); m \rangle = 0 \text{ for all } x \in \mathbb{R}^n.$$

Using Plancherel Theorem, this implies that

$$\langle \text{curl } \eta(x); m \rangle = 0 \text{ for all } x \in \mathbb{R}^n \text{ a.e.}$$

and hence

$$\langle e; m \rangle = 0 \text{ for all } e \in E.$$

This gives us that  $m \in (\text{span } E)^\perp$ . So,  $\text{span } E \subseteq \text{span}\{x \wedge \hat{\eta}(x) : x \in \mathbb{R}^n\}$ . Therefore

$$\dim \text{span}\{x \wedge \hat{\eta}(x) : x \in \mathbb{R}^n\} = n.$$

But it can not happen because of lemma 3.1. Thus there does not exist any solution  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  of the problem (3.3) if  $\dim \text{span } E = n$ .  $\square$

#### 4. RESTRICTIONS ON THE CURL SET

In this section, we will see that  $\dim \text{span } E \leq n$  if we add one constraint on  $E$  that  $\text{rank}[e - f] \leq 2$  for any  $e, f \in E$ , where  $E \subseteq \Lambda^2(\mathbb{R}^n)$  and  $n \geq 4$ . In lemma 4.1, we will prove it for  $n = 4$  and in lemma 4.6, we will do it for  $n \geq 5$ . Let us notice here one thing that between two statements ‘ $\text{rank}[e - f] \leq 2$  for any  $e, f \in E$ ’ and ‘ $e \wedge f = 0$  for all  $e, f \in E$ ’, the later one will always imply the first one but the converse may not be true. We have given one example in remark 4.5(ii) in this respect. Finally, we will establish Theorem 4.8. Let us first prove the lemma below.

**Lemma 4.1.** *Let  $E \subseteq \Lambda^2(\mathbb{R}^4)$  be such that  $\text{rank}[e - f] \leq 2$  for all  $e, f \in E$ . Then  $\dim \text{span } E \leq 4$ .*

*Proof.* Let  $V := \Lambda^2(\mathbb{R}^4)$ . Let us define a bilinear map  $B : V \times V \rightarrow \mathbb{R}$  by

$$B(u, v) := c(u \wedge v) \text{ for all } u, v \in V,$$

where  $c(u \wedge v) \in \mathbb{R}$  is such that

$$u \wedge v = c(u \wedge v)e^1 \wedge e^2 \wedge e^3 \wedge e^4.$$

Clearly,  $B$  is symmetric and non-degenerate. For any subspace  $F \subseteq V$ , let us define

$$\tilde{F} := \{v \in V : B(v, f) = 0 \text{ for all } f \in F\}.$$

As  $B$  is non-degenerate, the map

$$B_1 : V \rightarrow V^*, \text{ defined by } v \mapsto B(v, \cdot)$$

is one-one and hence onto. Therefore, the map

$$B_2 : V \rightarrow F^*, \text{ defined by } v \mapsto B(v, \cdot)$$

is also onto because it can be written as  $B_2 = \psi \circ B_1$ , where  $\psi : V^* \rightarrow F^*$ . Hence

$$\dim V = \dim \ker B_2 + \dim F^*.$$

Clearly,  $\ker B_2 = \tilde{F}$  and  $\dim F^* = \dim F$ . So

$$\dim F + \dim \tilde{F} = \dim V. \tag{4.1}$$

Now suppose that  $F \subseteq V$  is an isotropic subspace, i.e.,  $B|_{F \times F} = 0$ . In other words,  $F \subseteq \tilde{F}$ . In this case, we can say from (4.1) that

$$\begin{aligned} \dim V &= \dim F + \dim \tilde{F} \\ &\geq \dim F + \dim F, \end{aligned}$$

i.e.,  $\dim F \leq \frac{\dim V}{2} = 3$ . Therefore, if  $F \subseteq V$  is any isotropic subspace of  $V$ , then  $\dim F \leq 3$ .

Now suppose that  $E \subseteq V$  is such that for any  $e, f \in E$ , there exist  $x, y \in \mathbb{R}^4$  such that  $e - f = x \wedge y$ , i.e.,  $\text{rank}[e - f] \leq 2$ . We will show that  $\dim \text{span } E \leq 4$ .

In contrary, let us suppose that  $E$  contains five linearly independent elements  $\zeta^0, \zeta^1, \zeta^2, \zeta^3, \zeta^4$  and let

$$\xi^i := \zeta^i - \zeta^0 \text{ for } i = 1, 2, 3, 4.$$

As every  $\xi^i$  has rank less than or equals to 2,

$$\xi^i \wedge \xi^i = 0 \text{ for all } i = 1, 2, 3, 4 \quad [\text{see proposition 2.37(iii) of [7]}]$$

i.e.,

$$B(\xi^i, \xi^i) = 0 \text{ for all } i = 1, 2, 3, 4.$$

Now  $\xi^i - \xi^j = \zeta^i - \zeta^j$  and  $(\zeta^i - \zeta^j) \wedge (\zeta^i - \zeta^j) = 0$  so  $(\xi^i - \xi^j) \wedge (\xi^i - \xi^j) = 0$ , i.e.,  $B(\xi^i - \xi^j, \xi^i - \xi^j) = 0$  for  $i, j \in \{1, 2, 3, 4\}$ . This implies that

$$\xi^i \wedge \xi^j = 0 \text{ for all } i, j \in \{1, 2, 3, 4\}.$$

If we take

$$F = \{\xi^i : i = 1, 2, 3, 4\} \text{ and } F' := \text{span } F$$

then

$$B|_{F' \times F'} = 0, \text{ i.e., } F' \text{ is an isotropic subspace of } V$$

and  $\dim F' = 4$ . It contradicts that  $\dim F' \leq 3$ . Therefore,  $E$  can not contain 5 linearly independent elements. Hence  $\dim \text{span } E \leq 4$ .  $\square$

We will state two trivial lemmas below, the proofs of which are straightforward. We will use these lemmas 4.2 and 4.3 in the proofs of lemmas 4.4 and 4.6.

**Lemma 4.2.** *Let  $n \in \mathbb{N}$  and  $n \geq 3$ . Let  $\{\omega, \omega'\}$  be a linearly independent subset of  $\Lambda^2(\mathbb{R}^n)$  such that  $\text{rank}[\omega] = 2$ ,  $\text{rank}[\omega'] = 2$  and  $\omega \wedge \omega' = 0$ . Then  $\dim[\ker\{\omega\}^\perp \cap \ker\{\omega'\}^\perp] = 1$ . Also  $\dim[\ker\{\omega\} \cap \ker\{\omega'\}] = n - 3$ .*

*Proof.* As  $\text{rank}[\omega] = \text{rank}[\omega'] = 2$ , let us suppose that

$$\omega = x \wedge y \text{ and } \omega' = x' \wedge y' \text{ where } x, y, x', y' \in \mathbb{R}^n.$$

Again since  $\omega \wedge \omega' = 0$ , the set  $\{x, y, x', y'\}$  is linearly dependent [see Theorem 2.3, [7]]. This gives us that

$$\ker\{\omega\}^\perp \cap \ker\{\omega'\}^\perp \neq \{0\},$$

because  $\ker\{\omega\}^\perp \cap \ker\{\omega'\}^\perp = \text{span}\{x, y\} \cap \text{span}\{x', y'\} = \{0\}$  implies that  $\{x, y, x', y'\}$  is linearly independent, which is a contradiction. Hence

$$\dim[\ker\{\omega\}^\perp \cap \ker\{\omega'\}^\perp] \geq 1.$$

Now if  $\dim[\ker\{\omega\}^\perp \cap \ker\{\omega'\}^\perp] = 2$  then clearly,

$$\ker\{\omega\}^\perp \cap \ker\{\omega'\}^\perp = \ker\{\omega\}^\perp = \ker\{\omega'\}^\perp,$$

since  $\dim \ker\{\omega\}^\perp = \dim \ker\{\omega'\}^\perp = 2$ . This implies that

$$\text{span}\{x, y\} = \text{span}\{x', y'\}, \text{ i.e., } \{\omega, \omega'\} \text{ is linearly dependent,}$$

which is a contradiction. Therefore  $\dim[\ker\{\omega\}^\perp \cap \ker\{\omega'\}^\perp] = 1$ .

For the second part,

$$\begin{aligned} \dim[\ker\{\omega\} \cap \ker\{\omega'\}] &= n - \dim[\{\ker\{\omega\} \cap \ker\{\omega'\}\}^\perp] \\ &= n - [\dim \ker\{\omega\}^\perp + \dim \ker\{\omega'\}^\perp] \\ &= n - [\dim \ker\{\omega\}^\perp + \dim \ker\{\omega'\}^\perp - \dim\{\ker\{\omega\}^\perp \cap \ker\{\omega'\}^\perp\}] \\ &= n - [2 + 2 - 1], \quad \text{as } \dim\{\ker\{\omega\}^\perp \cap \ker\{\omega'\}^\perp\} = 1 \\ &= n - 3. \end{aligned}$$

Therefore  $\dim[\ker\{\omega\} \cap \ker\{\omega'\}] = n - 3$ .  $\square$

**Lemma 4.3.** *Let  $n \in \mathbb{N}$  and  $b \in \mathbb{R}^n \setminus \{0\}$ . Let  $\{\omega_1, \omega_2, \dots, \omega_m\} \subseteq \mathbb{R}^n \wedge b$  be a linearly independent subset, where  $m \in \mathbb{N}, m \leq n - 1$ . If  $\omega_i = x_i \wedge b$  for some  $x_i \in \mathbb{R}^n, i = 1, \dots, m$  then  $\{b, x_1, \dots, x_m\}$  is linearly independent.*

*Proof.* Let  $\alpha b + \alpha_1 x_1 + \dots + \alpha_m x_m = 0$ , where  $\alpha, \alpha_1, \dots, \alpha_m \in \mathbb{R}$ . Then

$$\begin{aligned} 0 &= 0 \wedge b = (\alpha b + \alpha_1 x_1 + \dots + \alpha_m x_m) \wedge b \\ &= \alpha_1 x_1 \wedge b + \dots + \alpha_m x_m \wedge b \\ &= \alpha_1 \omega_1 + \dots + \alpha_m \omega_m. \end{aligned}$$

As  $\{\omega_1, \dots, \omega_m\}$  is linearly independent,  $\alpha_i = 0$  for  $i = 1, \dots, m$ . So  $\alpha b = 0$  and this gives  $\alpha = 0$ . Therefore  $\{b, x_1, \dots, x_m\}$  is linearly independent.  $\square$

Now let us consider  $n \geq 5$ . Using this lemma 4.4 below we will prove another main lemma 4.6 of this section which states that  $\dim \text{span } E \leq n$  if  $\text{rank}[e - f] \leq 2$  for any  $e, f \in E$ .

**Lemma 4.4.** *Let  $E \subseteq \Lambda^2(\mathbb{R}^n) \setminus \{0\}$ ,  $n \geq 5$ . Let  $\dim \text{span } E = n - 1$  and  $\omega \wedge \omega' = 0$  for any  $\omega, \omega' \in E$ , then*

$$\bigcap_{\omega \in E} \ker\{\omega\}^\perp \neq \{0\}.$$

*Proof.* Let  $\{\omega_1, \omega_2, \dots, \omega_{n-1}\}$  be a basis of  $\text{span } E$  and

$$A_i := \ker\{\omega_i\}, \quad i = 1, 2, \dots, n - 1.$$

Now two cases may arise:

*Case 1:* In this case, let there exists  $1 \leq i < j < k \leq n - 1$  such that

$$\dim(A_i^\perp \cap A_j^\perp \cap A_k^\perp) = 1.$$

Without loss of generality, let

$$\dim(A_1^\perp \cap A_2^\perp \cap A_3^\perp) = 1.$$

If  $\bigcap_{i=1}^4 A_i^\perp = \{0\}$ , then together with  $\dim(A_1^\perp \cap A_2^\perp \cap A_3^\perp) = 1$  and  $\omega_4 \wedge \omega_4 = 0$ , we can say that

$$(A_1^\perp \cap A_2^\perp \cap A_3^\perp) = \text{span}\{b\}$$

and

$$\omega_4 = \alpha \wedge \beta \text{ with } \dim \text{span}\{\alpha, \beta, b\} = 3 \text{ for some } \alpha, \beta, b \in \mathbb{R}^n \setminus \{0\}.$$

Now let

$$\omega_1 = x \wedge b, \omega_2 = y \wedge b, \omega_3 = z \wedge b \text{ for some } x, y, z \in \mathbb{R}^n \setminus \{0\},$$

the existence of  $x, y, z$  follows from Cartan's lemma, see Theorem 2.42 of [7]. As  $\{\omega_1, \omega_2, \omega_3\}$  is linearly independent, it follows from lemma 4.3 that

$$\{x, y, z, b\} \text{ is linearly independent.}$$

If possible, let all of  $\{\alpha, \beta, b, x\}$ ,  $\{\alpha, \beta, b, y\}$  and  $\{\alpha, \beta, b, z\}$  are linearly dependent. Then we see that

$$x, y, z \in \text{span}\{\alpha, \beta, b\}$$

and therefore

$$\text{span}\{x, y, z\} = \text{span}\{\alpha, \beta, b\}.$$

This gives

$$b \in \text{span}\{x, y, z\},$$

which is a contradiction because  $\{x, y, z, b\}$  is linearly independent. Now without loss of generality let  $\{\alpha, \beta, x, b\}$  is linearly independent then clearly

$$\omega_4 \wedge \omega_1 \neq 0$$



which contradicts our hypothesis that

$$\omega_i \wedge \omega_j = 0 \text{ for all } 1 \leq i < j \leq 4.$$

Therefore  $\bigcap_{i=1}^4 A_i^\perp \neq \{0\}$  and hence we can say that

$$\bigcap_{i=1}^4 \ker\{\omega_i\}^\perp = \bigcap_{i=1}^3 \ker\{\omega_i\}^\perp, \quad (4.2)$$

as  $\dim \bigcap_{i=1}^3 \ker\{\omega_i\}^\perp = 1$ . By similar argument, we can again show that

$$\bigcap_{i=1, i \neq 4}^5 \ker\{\omega_i\}^\perp = \bigcap_{i=1}^3 \ker\{\omega_i\}^\perp. \quad (4.3)$$

From equations (4.2) and (4.3) we can say that

$$\begin{aligned} \bigcap_{i=1}^5 \ker\{\omega_i\}^\perp &= \left[ \bigcap_{i=1}^4 \ker\{\omega_i\}^\perp \right] \cap \left[ \bigcap_{i=1, i \neq 2}^5 \ker\{\omega_i\}^\perp \right] \\ &= \bigcap_{i=1}^3 \ker\{\omega_i\}^\perp. \end{aligned} \quad (4.4)$$

Thus we can assert that

$$\bigcap_{i=1}^3 A_i^\perp = A_1^\perp \cap A_2^\perp \cap A_3^\perp \cap A_k^\perp \text{ for all } k \in \{4, 5, \dots, n-1\}.$$

Therefore

$$\bigcap_{i=1}^{n-1} A_i^\perp = \bigcap_{i=4}^{n-1} \left( A_1^\perp \cap A_2^\perp \cap A_3^\perp \cap A_i^\perp \right) \quad (4.5)$$

$$= A_1^\perp \cap A_2^\perp \cap A_3^\perp. \quad (4.6)$$

This implies that

$$\dim \left( \bigcap_{i=1}^{n-1} A_i^\perp \right) = 1.$$

Therefore

$$\bigcap_{i=1}^{n-1} A_i^\perp = \text{span}\{b\} \text{ for some } b \in \mathbb{R}^n \setminus \{0\}$$

and hence

$$\text{span } E = \text{span}\{\omega_1, \omega_2, \dots, \omega_{n-1}\} = \mathbb{R}^n \wedge b.$$

*Case 2:* In this case we let

$$A_i^\perp \cap A_j^\perp \cap A_k^\perp = \{0\} \text{ for any } 1 \leq i < j < k \leq n-1.$$

As  $n \geq 5$ , there exists  $l \in \{1, 2, \dots, n-1\} \setminus \{i, j, k\}$  such that

$$\dim A_l^\perp = 2 \text{ and } A_i^\perp \cap A_j^\perp \cap A_k^\perp \subseteq A_l^\perp.$$

This gives us  $\dim(A_l^\perp + (A_i^\perp \cap A_j^\perp \cap A_k^\perp)) = 2$  and so

$$\dim(A_l \cap (A_i + A_j + A_k)) = n-2. \quad (4.7)$$

Because of  $A_i^\perp \cap A_j^\perp \cap A_k^\perp = \{0\}$ ,

$$A_i^\perp \cap A_j^\perp \cap A_k^\perp \cap A_l^\perp = \{0\}$$

and it follows that

$$\dim(A_i + A_j + A_k + A_l) = n.$$

Now

$$\begin{aligned} \dim(A_i + A_j + A_k + A_l) &= \dim A_i + \dim A_j + \dim A_k + \dim A_l \\ &\quad - \dim(A_j \cap A_k) - \dim(A_j \cap A_l) - \dim(A_k \cap A_l) \\ &\quad + \dim(A_j \cap A_k \cap A_l) - \dim(A_i \cap (A_j + A_k + A_l)) \\ &= (n-2) \times 4 - (n-3) \times 3 + \dim(A_j \cap A_k \cap A_l) - (n-2). \\ &\quad \text{[using lemma 4.2 and equation 4.7]} \\ &= \dim(A_j \cap A_k \cap A_l). \end{aligned}$$

That is,  $\dim(A_j \cap A_k \cap A_l) = n$ . But  $n \geq 5$ , so  $\dim(A_j \cap A_k \cap A_l) \geq 5$ . Also

$$A_j \cap A_k \cap A_l \subseteq A_j \cap A_k$$

and  $\dim(A_j \cap A_k) = n-3$  by lemma 4.2. So  $\dim(A_i \cap A_j \cap A_k)$  can not be equal to  $n$  for  $n \geq 5$ . Thus we are getting contradiction and it follows that case 2 can not happen.  $\square$

**Remark 4.5.**

- (i) We can not apply the proof of the above lemma 4.4 for  $\Lambda^2(\mathbb{R}^4)$ , i.e., if  $E \subset \Lambda^2(\mathbb{R}^4)$  such that  $\dim \text{span } E = 3$  and  $\omega \wedge \omega' = 0$  for any  $\omega, \omega' \in E$  then it may not happen that  $\text{span } E = \mathbb{R}^4 \wedge b$  for some  $b \in \mathbb{R}^4 \setminus \{0\}$  because if we take the set  $E$  as  $\{e^1 \wedge e^2, e^1 \wedge e^3, e^2 \wedge e^3\}$  then  $E$  can not be written as a subset of  $\mathbb{R}^4 \wedge b$  for any  $b \in \mathbb{R}^4 \setminus \{0\}$ . Importantly, case 2 of the above lemma is true for this example.
- (ii) The aforementioned lemma is not true if we replace the case  $\omega \wedge \omega' = 0$  for all  $\omega, \omega' \in E$  with  $\text{rank}[\omega - \omega'] \leq 2$  for any  $\omega, \omega' \in E$ . For example, if we take  $E$  as

$$E = \{e^2 \wedge e^3, e^2 \wedge e^3 + e^1 \wedge e^2, e^2 \wedge e^3 + e^1 \wedge e^3, \dots, e^2 \wedge e^3 + e^1 \wedge e^{n-1}\}$$

for  $n \in \mathbb{N}$  and  $n \geq 5$ , then  $\dim \text{span } E = n-1$  but  $\text{span } E$  can not be written as  $\mathbb{R}^n \wedge b$  for any  $b \in \mathbb{R}^n$ . For  $n = 4$ , we can simply take the set  $\{e^2 \wedge e^3, e^2 \wedge e^3 + e^1 \wedge e^2, e^2 \wedge e^3 + e^1 \wedge e^4\}$ .

We will use the following lemma in Theorem 4.8.

**Lemma 4.6.** Let  $n \in \mathbb{N}, n \geq 5$  and  $E \subset \Lambda^2(\mathbb{R}^n)$  be such that  $\text{rank}[e - f] \leq 2$  for any  $e, f \in E$ . Then  $\dim \text{span } E \leq n$ .

**Remark 4.7.** For  $n = 4$ , we have done separate proof of this lemma in 4.1 because we will use lemma 4.4 in the proof below which may not hold for  $n = 4$ .

*Proof.* We will show that  $\dim \text{span } E \leq n$  if we take  $\text{rank}[e - f] \leq 2$  for any  $e, f \in E$ . Let us suppose that  $E$  contains  $n+1$  linearly independent elements  $\omega_0, \omega_1, \dots, \omega_n$ . Let

$$\psi_i := \omega_i - \omega_0, \quad i = 1, \dots, n.$$

The set  $\{\psi_i : i = 1, \dots, n\}$  is linearly independent set with

$$\psi_i \wedge \psi_j = 0 \text{ for any } i, j.$$

Indeed,  $\psi_i - \psi_j = \omega_i - \omega_j$  and  $(\omega_i - \omega_j) \wedge (\omega_i - \omega_j) = 0$ , i.e.,  $(\psi_i - \psi_j) \wedge (\psi_i - \psi_j) = 0$ . Therefore from lemma 4.4 we can say that there exists  $b \in \mathbb{R}^n \setminus \{0\}$  such that

$$\text{span}\{\psi_1, \dots, \psi_{n-1}\} = \mathbb{R}^n \wedge b.$$

Similarly there exists  $b' \in \mathbb{R}^n \setminus \{0\}$  such that

$$\text{span}\{\psi_2, \dots, \psi_n\} = \mathbb{R}^n \wedge b'.$$

As  $n \geq 5$ , we can say from lemma 4.2 that

$$\ker\{\psi_2\}^\perp \cap \ker\{\psi_3\}^\perp = \bigcap_{i=1}^{n-1} \ker\{\psi_i\}^\perp$$

and

$$\ker\{\psi_2\}^\perp \cap \ker\{\psi_3\}^\perp = \bigcap_{i=2}^n \ker\{\psi_i\}^\perp$$

as  $\dim[\ker\{\psi_2\}^\perp \cap \ker\{\psi_3\}^\perp] = 1$ . Therefore

$$\mathbb{R}^n \wedge b = \text{span}\{\psi_1, \dots, \psi_{n-1}\} = \mathbb{R}^n \wedge b' = \text{span}\{\psi_2, \dots, \psi_n\},$$

which is a contradiction because  $\{\psi_1, \dots, \psi_n\}$  is a linearly independent set.

Hence  $\dim \text{span } E \leq n$ .  $\square$

Let us prove the main theorem on differential inclusions of this section using the previous lemma 4.6 and Theorem 3.3.

**Theorem 4.8.** *Let  $n \in \mathbb{N}$ ,  $n \geq 5$  and  $E \subseteq \Lambda^2(\mathbb{R}^n)$  be such that*

$$\omega \wedge \omega' = 0 \text{ for all } \omega, \omega' \in E.$$

*Let  $\Omega \subseteq \mathbb{R}^n$  be an open, bounded set. Then there exists  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  such that*

$$\begin{aligned} &\text{curl } \eta \in E \text{ a.e. in } \Omega \\ &\text{meas}\{x \in \Omega : \text{curl } \eta(x) = e\} > 0 \text{ for all } e \in E \\ &\text{and } \int_{\Omega} \eta \neq 0, \end{aligned} \tag{4.8}$$

*if and only if  $0 \in \text{rico } E$  and  $\dim \text{span } E = n - 1$ .*

*Proof.* If there exists  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  such that

$$\begin{aligned} &\text{curl } \eta \in E \text{ a.e. in } \Omega \\ &\text{meas}\{x \in \Omega : \text{curl } \eta(x) = e\} > 0 \text{ for all } e \in E \\ &\text{and } \int_{\Omega} \eta \neq 0, \end{aligned} \tag{4.9}$$

then from Theorem 2.5 of [3] we can say that

$$\dim \text{span } E \geq n - 1.$$

As  $E$  has the property that  $\omega \wedge \omega' = 0$  for any  $\omega, \omega' \in E$ , it implies that

$$\text{rank}[\omega - \omega'] \leq 2 \text{ for any } \omega, \omega' \in E.$$

Hence from lemma 4.6 it follows that

$$\dim \text{span } E \leq n.$$

Now for  $\dim \text{span } E = n$ , there does not exist any solution  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  of the problem (4.9) which directly follows from Theorem 3.3. Hence  $\dim \text{span } E = n - 1$ . Using lemma 2.4 of [3], it holds that  $0 \in \text{rico } E$ .

Conversely, if  $\dim \text{span } E = n - 1$ , then using lemma 4.4 there exists  $b \in \mathbb{R}^n \setminus \{0\}$  such that  $\text{span } E = \mathbb{R}^n \wedge b$ . Now a solution  $\eta$  exists satisfying (4.8) from corollary 3.9 of [3].  $\square$

**Remark 4.9.** *For  $n = 4$ , the necessary part of the above theorem is true and it follows from Theorem 3.3 and lemma 4.1. But the converse part is not true because  $\text{span } E$  may not be written as  $\mathbb{R}^4 \wedge b$  for some  $b \in \mathbb{R}^4 \setminus \{0\}$  always.*

5. EXISTENCE OF SOLUTION: DIMENSION  $2n-3$ 

In this section, we will see one existence result of solution  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  at dimension  $(2n-3)$  for the following differential inclusion problem

$$\begin{aligned} \operatorname{curl} \eta &\in E \text{ a.e. in } \Omega, \\ \operatorname{meas}\{x \in \Omega : \operatorname{curl} \eta(x) = e\} &> 0 \text{ for all } e \in E. \end{aligned}$$

Let  $n \in \mathbb{N}, n \geq 4$ . Let  $E \subseteq \Lambda^2(\mathbb{R}^n) \setminus \{0\}$  and  $\dim \operatorname{span} E = 2n-3$  where  $E = \{e^1 \wedge e^2, e^1 \wedge e^3, \dots, e^1 \wedge e^n, -e^1 \wedge e^2, -e^1 \wedge e^3, \dots, -e^1 \wedge e^n, e^2 \wedge e^3, e^2 \wedge e^4, \dots, e^2 \wedge e^n, -e^2 \wedge e^3, -e^2 \wedge e^4, \dots, -e^2 \wedge e^n\}$ . Clearly,  $\operatorname{span} E = \mathbb{R}^n \wedge e^1 + \mathbb{R}^n \wedge e^2$ . Let us write  $E = E_1 \cup E_2$ , where

$$E_1 = \{e^1 \wedge e^2, e^1 \wedge e^3, \dots, e^1 \wedge e^n, -e^1 \wedge e^2, -e^1 \wedge e^3, \dots, -e^1 \wedge e^n\}$$

and

$$E_2 = \{e^2 \wedge e^1, e^2 \wedge e^3, \dots, e^2 \wedge e^n, -e^2 \wedge e^1, -e^2 \wedge e^3, \dots, -e^2 \wedge e^n\}.$$

Then,  $\operatorname{span} E_1 = \mathbb{R}^n \wedge e^1$ ,  $\operatorname{span} E_2 = \mathbb{R}^n \wedge e^2$  and  $0 \in \operatorname{rico} E_1 \cap \operatorname{rico} E_2$ .

**Step-1:** Let  $G = I_1 \times I_2 \times \dots \times I_n$  be open unit cube in  $\mathbb{R}^n$ , where  $I_i = (0, 1)$  for each  $i = 1, \dots, n$ . Let us divide the domain  $G$  into two parts as follows:

$$G_1 = \left(0, \frac{1}{2}\right) \times I_2 \times \dots \times I_n$$

and

$$G_2 = \left(\frac{1}{2}, 1\right) \times I_2 \times \dots \times I_n.$$

Clearly,  $G_1, G_2$  are open, bounded sets in  $\mathbb{R}^n$ . Then there exist  $\overline{\eta}_1 \in W_0^{1,\infty}(G_1; \mathbb{R}^n)$  and  $\overline{\eta}_2 \in W_0^{1,\infty}(G_2; \mathbb{R}^n)$  such that

$$\operatorname{curl} \overline{\eta}_1 \in E_1 \text{ a.e. in } G_1$$

and

$$\operatorname{curl} \overline{\eta}_2 \in E_2 \text{ a.e. in } G_2.$$

Let us define a mapping  $\overline{\eta} \in W_0^{1,\infty}(G; \mathbb{R}^n)$  by

$$\overline{\eta}(x) = \begin{cases} \overline{\eta}_1(x), & \text{if } x \in G_1 \text{ a.e.} \\ \overline{\eta}_2(x), & \text{if } x \in G_2 \text{ a.e.} \end{cases}$$

Then

$$\operatorname{curl} \overline{\eta} \in E \text{ a.e. in } G,$$

i.e., there exists  $\overline{\eta} \in W_0^{1,\infty}(G; \mathbb{R}^n)$  such that

$$\begin{aligned} \operatorname{curl} \overline{\eta} &\in E \text{ a.e. in } G, \\ \operatorname{meas}\{x \in \Omega : \operatorname{curl} \overline{\eta}(x) = e\} &> 0 \text{ for all } e \in E, \end{aligned}$$

where  $\dim \operatorname{span} E = 2n-3$  and  $\operatorname{span} E = \mathbb{R}^n \wedge e^1 + \mathbb{R}^n \wedge e^2$ .

**Step 2:** Let  $\Omega \subseteq \mathbb{R}^n$  be an open, bounded set. Using Vitali's covering theorem, there exists a sequence  $\{G_k : k \in \mathbb{N}\}$ , where  $G_k$ 's are translated and dilated sets of  $\overline{G}$  (closure of  $G$ ) and  $G$  is as defined in step-1 above such that

$$\begin{aligned} G_k &\subseteq \Omega \text{ for each } k \in \mathbb{N}, \\ G_h \cap G_k &= \emptyset \text{ for all } h, k \in \mathbb{N}, h \neq k, \end{aligned}$$

$$\operatorname{meas} \left( \Omega \setminus \bigcup_{k \in \mathbb{N}} G_k \right) = 0.$$

Let  $G_k := a_k + t_k \overline{G}$ , where  $a_k \in \mathbb{R}^n$  and  $t_k \in \mathbb{R} \setminus \{0\}$ . Let us define a map  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  by

$$\eta(z) = \begin{cases} \overline{\eta} \left( \frac{z - a_k}{t_k} \right), & \text{if } z \in G_k \\ 0, & \text{if } z \in \Omega \setminus \bigcup_{k \in \mathbb{N}} G_k. \end{cases}$$

Clearly,  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  and

$$\operatorname{curl} \eta \in E \text{ a.e. in } \Omega.$$

Thus we are getting a solution  $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$  for the following differential inclusion problem

$$\begin{aligned} \operatorname{curl} \eta &\in E \text{ a.e. in } \Omega, \\ \operatorname{meas}\{z \in \Omega : \operatorname{curl} \eta(z) = e\} &> 0 \text{ for all } e \in E. \end{aligned}$$

## 6. THE CASE OF A K-FORM

In this section, we will generalize lemma 3.1 to exterior  $k$ -form.

**Theorem 6.1.** *Let  $n \in \mathbb{N}$ ,  $1 \leq k \leq n - 3$ . Suppose  $f : \mathbb{R}^n \rightarrow \Lambda^k(\mathbb{R}^n)$  be continuous such that  $f(0) \neq 0$  and  $f(0)$  is  $k$ -divisible, i.e.,  $f(0) = c^1 \wedge \dots \wedge c^k$  for some  $c^i \in \mathbb{R}^n \setminus \{0\}$  for  $i = 1, \dots, k$ . Then*

$$\dim \operatorname{span}\{x \wedge f(x) : x \in \mathbb{R}^n\} \neq n - k + 1.$$

*Proof.* Let

$$\mathcal{S} := \operatorname{span}\{x \wedge f(x) : x \in \mathbb{R}^n\}.$$

Let us suppose to the contrary that  $\dim \mathcal{S} = n - k + 1$ . We know that

$$\mathbb{R}^n \wedge f(0) \subseteq \mathcal{S}, \tag{6.1}$$

from proposition 2.2 of [3]. Therefore

$$\mathcal{S} = (\mathbb{R}^n \wedge f(0)) \oplus \operatorname{span}(\omega),$$

for some  $\omega \in \Lambda^{k+1}(\mathbb{R}^n) \setminus \{0\}$ . Using proposition 2.16 of [7], we can write

$$\begin{aligned} \langle f(0) \lrcorner \omega; x \rangle &= (-1)^{k+1} \langle \omega; f(0) \wedge x \rangle \\ &= 0, \text{ because } \omega \in [\mathbb{R}^n \wedge f(0)]^\perp \text{ for all } x \in \mathbb{R}^n. \end{aligned}$$

Therefore

$$f(0) \lrcorner \omega = 0. \tag{6.2}$$

Let us choose  $x^0 \in \mathbb{R}^n$  such that  $x^0 \wedge f(x^0) \in \Lambda^{k+1} \setminus (\mathbb{R}^n \wedge f(0))$ . Such an  $x^0$  exists because  $\mathbb{R}^n \wedge f(0) \subsetneq \mathcal{S}$ . Since  $\Lambda^{k+1} \setminus (\mathbb{R}^n \wedge f(0))$  is open, it follows from the continuity of  $f$  that, for some  $\epsilon > 0$

$$x \wedge f(x) \in \Lambda^{k+1} \setminus (\mathbb{R}^n \wedge f(0)),$$

for all  $x \in B(x^0, \epsilon)$ .

Let us choose a basis  $\{a^1, \dots, a^n\}$  of  $\mathbb{R}^n$  within  $B(x^0, \epsilon)$ . Then for each  $j \in \{1, 2, \dots, n\}$ , we find  $\beta^j \in \mathbb{R}$  and  $b^j \in \mathbb{R}^n$  such that

$$a^j \wedge f(a^j) = b^j \wedge f(0) + \beta^j \omega.$$

Note that,  $\beta^j \neq 0$  for all  $j \in \{1, 2, \dots, n\}$ .

We have  $f(0) = c^1 \wedge \dots \wedge c^k$ . Let  $1 \leq r \leq k$  be fixed. Then for all  $j \in \{1, \dots, n\}$ ,

$$\beta^j (\omega \wedge c^r \wedge a^j) = 0$$

implies that

$$(\omega \wedge c^r) \wedge a^j = 0 \text{ for all } j = 1, \dots, n.$$

This gives us

$$\omega \wedge c^r = 0 \text{ for all } r \in \{1, \dots, k\}.$$

Therefore, we have

$$\omega = (c^1 \wedge \dots \wedge c^k) \wedge \omega'$$

for some  $\omega' \in \mathbb{R}^n$ , where  $c^i \lrcorner \omega' = 0$  for all  $i = 1, \dots, k$ .

Since  $(c^1 \wedge \dots \wedge c^k) \lrcorner \omega = 0$  [from equation (6.2)], we have  $\omega' = 0$ .

Hence  $\omega = 0$ , a contradiction. This proves the theorem.  $\square$

**Remark 6.2.** For  $k = n - 2$  and  $n \geq 4$ , there exists a continuous function such that the above Theorem 6.1 fails. For example, let us take  $f : \mathbb{R}^n \rightarrow \Lambda^{n-2}(\mathbb{R}^n)$  defined by

$$f(x_1, \dots, x_n) = (x_1 + 1)e^1 \wedge \dots \wedge e^{n-2} + x_2 e^1 \wedge \dots \wedge e^{n-3} \wedge e^{n-1},$$

for all  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ . Then

$$\begin{aligned} x \wedge f(x) &= (x_1 e^1 + \dots + x_n e^n) \wedge f(x) \\ &= ((-1)^{n-3} x_{n-2} x_2 + (-1)^{n-2} x_{n-1} (x_1 + 1)) e^1 \wedge \dots \wedge e^{n-1} \\ &\quad + (-1)^{n-2} x_n (x_1 + 1) e^1 \wedge \dots \wedge e^{n-2} \wedge e^n \\ &\quad + (-1)^{n-2} x_n x_2 e^1 \wedge \dots \wedge e^{n-3} \wedge e^{n-1} \wedge e^n. \end{aligned}$$

Clearly,  $\dim \text{span}\{x \wedge f(x) : x \in \mathbb{R}^n\}$  has dimension  $n - (n - 2) + 1 = 3$ .

For  $k = n - 1$ , the above Theorem 6.1 is trivially true.

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