Quasiconvex Bulk and Surface Energies with subquadratic growth

Menita Carozza
Department of Engineering, University of Sannio
Corso Garibaldi 107, 82100 Benevento, Italy
e-mail: carozza@unisannio.it

Luca Esposito
Dipartimento di Matematica, University of Salerno
Via Giovanni Paolo II, 84084 Fisciano (SA), Italy
e-mail: luesposi@unisa.it

Lorenzo Lamberti
Institute Élie Cartan, Université de Lorraine, CNRS
F-54000 Nancy, France
e-mail: lorenzo.lamberti@univ-lorraine.fr

December 9, 2024

Abstract

We establish partial Hölder continuity of the gradient for equilibrium configurations of vectorial multidimensional variational problems, involving bulk and surface energies. The bulk energy densities are uniformly strictly quasiconvex functions with p-growth, $1 < p < 2$, without any further structure conditions. The anisotropic surface energy is defined by means of an elliptic integrand Φ not necessarily regular.

AMS Classifications. 49N15, 49N60, 49N99.

Key words. Regularity, nonlinear variational problem, free interfaces.

1 Introduction and statements

Let us consider a functional $\mathcal{F}$ with density energy discontinuous through an interface ∂A, inside an open bounded subset Ω of $\mathbb{R}^n$, of the form

$$\mathcal{F}(v, A) := \int_\Omega \left(F(Dv) + \mathbb{1}_A G(Dv)\right) dx + P(A, \Omega), \tag{1.1}$$

where $v \in W^{1,p}_{loc}(\Omega; \mathbb{R}^N)$, and $F, G : \mathbb{R}^{n\times N} \to \mathbb{R}$ are C^2-integrands. Assume that these integrands satisfy the following growth and uniformly strict p-quasiconvexity conditions, for $p > 1$ and positive constants ℓ_1, ℓ_2, L_1, L_2:

$$0 \le F(\xi) \le L_1(1+|\xi|^2)^{\frac{p}{2}}, \tag{F1}$$

$$\int_\Omega F(\xi + D\varphi)\,dx \ge \int_\Omega \left(F(\xi) + \ell_1|D\varphi|^2(1+|D\varphi|^2)^{\frac{p-2}{2}}\right) dx, \tag{F2}$$

$$0 \le G(\xi) \le L_2(1+|\xi|^2)^{\frac{p}{2}}, \tag{G1}$$

$$\int_\Omega G(\xi + D\varphi)\,dx \ge \int_\Omega \left(G(\xi) + \ell_2|D\varphi|^2(1+|D\varphi|^2)^{\frac{p-2}{2}}\right) dx, \tag{G2}$$

for every $\xi \in \mathbb{R}^{n\times N}$ and $\varphi \in C^1_0(\Omega;\mathbb{R}^N)$.
Existence and regularity results have been obtained initially in the scalar case ($N = 1$) in [4, 5, 10, 21, 22, 23, 24, 25, 28, 33, 34, 35]. In the vectorial case ($N > 1$), the authors in [11] proved the existence of local minimizers of (1.1), for any $p > 1$ under the quasiconvexity assumption quoted above. In the same paper, the $C^{1,\alpha}$ partial regularity is proved for minimal configurations outside a negligible set, in the quadratic case $p = 2$.
In [9] the same regularity result has been established in the general case $p \ge 2$, also addressing anisotropic surface energies. F.J. Almgren was the first to study such surface energies in his celebrated paper [3] (see also [8, 20, 26, 38, 39] for subsequent results). This kind of energies arises in many physical contexts such as the formation of crystals (see [6, 7]), liquid drops (see [16, 27]), capillary surfaces (see [18, 19]) and phase transitions (see [32]).
In this paper, we consider the same functional as in [9], given by

$$\mathcal{I}(v, A) := \int_\Omega \left(F(Dv) + \mathbb{1}_A G(Dv)\right)\,dx + \int_{\Omega\cap\partial^* A} \Phi(x, \nu_A(x))\,d\mathcal{H}^{n-1}(x), \tag{1.2}$$

in the case of sub-quadratic growth, $1 < p < 2$. We achieve analogous regularity results as those established in [9], thereby completing the answer to the problem for all $p > 1$.
In this setting $A \subset \Omega$ is a set of finite perimeter, $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, $\mathbb{1}_A$ is the characteristic function of the set A, $\partial^* A$ denotes the reduced boundary of A in Ω and ν_A is the measure-theoretic outer unit normal to A. Moreover, Φ is an elliptic integrand on Ω (see Definition 2.8), i.e. $\Phi : \overline{\Omega} \times \mathbb{R}^n \to [0,\infty]$ is lower semicontinuous, $\Phi(x,\cdot)$ is convex and positively one-homogeneous, $\Phi(x, t\nu) = t\Phi(x,\nu)$ for every $t \ge 0$, and the anisotropic surface energy of a set A of finite perimeter in Ω is defined as follows

$$\mathbf{\Phi}(A; B) := \int_{B\cap\partial^* A} \Phi(x, \nu_A(x))\,d\mathcal{H}^{n-1}(x), \tag{1.3}$$

for every Borel set $B \subset \Omega$. The further assumption

$$\frac{1}{\Lambda} \le \Phi(x,\nu) \le \Lambda, \tag{1.4}$$

with $\Lambda > 1$, allows to compare the surface energy introduced in (1.3) with the usual perimeter.
Let us recall that in the vectorial setting, as in the previously cited papers, the regularity we can expect for the gradient of the minimal deformation $u : \Omega \to \mathbb{R}^N$, ($N > 1$), even in absence of a surface term, is limited to a partial regularity result.

We say that a pair (u,E) is a local minimizer of $\mathcal{I}$ in Ω, if for every open set $U \Subset \Omega$ and every pair (v,A), where $v-u \in W_0^{1,p}(U;\mathbb{R}^N)$ and A is a set of finite perimeter with $A\Delta E \Subset U$, we have

$$\int_U (F(Du) + \mathbb{1}_E G(Du))\,dx + \mathbf{\Phi}(E;U) \leq \int_U (F(Dv) + \mathbb{1}_A G(Dv))\,dx + \mathbf{\Phi}(A;U).$$

Existence and regularity results for local minimizers of integral functionals with uniformly strict p-quasiconvex integrand, also in the non autonomous case, have been widely investigated (see [1, 2, 12, 13, 14, 15, 30, 37] and [29, 31]).
Regarding the functional (1.2), the existence of local minimizers is guaranteed by the following theorem, proved in [9].

Theorem 1.1. *Let $p > 1$ and assume that $(F1)$, $(F2)$, $(G1)$, $(G2)$ hold. Then, if $v \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $A \subset \Omega$ is a set of finite perimeter in Ω, for every sequence $\{(v_k, A_k)\}_{k\in\mathbb{N}}$ such that $\{v_k\}$ weakly converges to v in $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $\mathbb{1}_{A_k}$ strongly converges to $\mathbb{1}_A$ in $L^1_{\mathrm{loc}}(\Omega)$, we have*

$$\mathcal{I}(v,A) \leq \liminf_{k\to\infty} \mathcal{I}(v_k, A_k).$$

In particular, $\mathcal{I}$ admits a minimal configuration $(u, \mathbb{1}_E) \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N) \times BV_{\mathrm{loc}}(\Omega;[0,1])$

We emphasize that, in particular, the previous theorem implies the semicontinuity of the anisotropic perimeter functional (1.3).
In this paper, we obtain a $C^{1,\alpha}$ regularity result for the minimizers of (1.2) in the case of sub-quadratic growth, $1<p<2$. If we further assume a closeness condition on F and G (see (H) in Theorem 1.2), we prove that $u \in C^{1,\gamma}(\Omega_1)$ for every $\gamma \in (0, \frac{1}{p'})$ on a full measure set $\Omega_1 \subset \Omega$. Furthermore, we do not assume any regularity on Φ in order to get the regularity of u.

Our main theorem is the following,

Theorem 1.2. *Let (u,E) be a local minimizer of $\mathcal{I}$. Let the bulk density energies F and G satisfy $(F1)$, $(F2)$, $(G1)$, $(G2)$, with $1<p<2$, and let the surface energy $\mathbf{\Phi}$ be of general type (1.3) with Φ satisfying (1.4). Then there exist an exponent $\beta \in (0,1)$ and an open set $\Omega_0 \subset \Omega$ with full measure such that $u \in C^{1,\beta}(\Omega_0;\mathbb{R}^N)$. If we assume in addition that*

$$\frac{L_2}{\ell_1 + \ell_2} < 1, \tag{H}$$

there exists an open set $\Omega_1 \subset \Omega$ with full measure such that $u \in C^{1,\gamma}(\Omega_1;\mathbb{R}^N)$ for every $\gamma \in \left(0, \frac{1}{p'}\right)$.

The proof of the regularity of u is based on a blow-up argument aimed to establish a decay estimate for the excess function

$$U(x_0,r) := \fint_{B_r(x_0)} \left|V(Du) - V\big((Du)_{x_0,r}\big)\right|^2 dx + \frac{P(E,B_r(x_0))}{r^{n-1}} + r,$$

where

$$V(\xi) = (1+|\xi|^2)^{(p-2)/4}\xi, \quad \forall \xi \in \mathbb{R}^k.$$

To this aim, we use a comparison argument between the blow-up sequence v_h at small scale in the balls $B_{r_h}(x_h)$ and the solution v of a suitable linearized system. The challenging part of the argument, as usual, is to prove that the 'good' decay estimates available for the function v (see Proposition 2.1), are inherited by the v_h as $h \to \infty$.
To achieve this result, the main tool is a Caccioppoli type inequality that we prove for minimizers of perturbed rescaled functionals (see (3.16)) involving the function $V(Dv_h)$ and the perimeter of the rescaled minimal set E_h. The Caccioppoli inequality combined with the Sobolev Poincarè inequality will lead us to a contradiction (see Step 6 of Proposition 3.1). In this final step, the issue to deal with the function $V(Du)$ in the sub-quadratic case, is overcome by using a suitable Sobolev Poincarè inequality involving $V(Du)$ (see Theorem 2.6), whose proof is due to [12].

2 Preliminaries

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, $u : \Omega \to \mathbb{R}^N$, $N > 1$. We denote by $B_r(x) := \{y \in \mathbb{R}^n : |y-x| < r\}$ the open ball centered at $x \in \mathbb{R}^n$ of radius $r > 0$, $\mathbb{S}^{n-1}$ represents the unit sphere of $\mathbb{R}^n$, c a generic constant that may vary.
For $B_r(x_0) \subset \mathbb{R}^n$ and $u \in L^1(B_r(x_0); \mathbb{R}^N)$ we denote

$$(u)_{x_0,r} := \fint_{B_r(x_0)} u(x)\, dx$$

and we will omit the dependence on the center when it is clear from the context.

$$\langle \xi, \eta \rangle := \mathrm{trace}(\xi^T \eta),$$

for the usual inner product of ξ and η, and accordingly $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$. If $F \colon \mathbb{R}^{n\times N} \to \mathbb{R}$ is sufficiently differentiable, we write

$$DF(\xi)\eta := \sum_{\alpha=1}^{N} \sum_{i=1}^{n} \frac{\partial F}{\partial \xi_i^\alpha}(\xi)\eta_i^\alpha \quad \text{and} \quad D^2F(\xi)\eta\eta := \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} \frac{\partial F}{\partial \xi_i^\alpha \partial \xi_j^\beta}(\xi)\eta_i^\alpha \eta_j^\beta,$$

for ξ, $\eta \in \mathbb{R}^{n\times N}$.
It is well known that for quasiconvex C^1 integrands the assumptions $(F1)$ and $(G1)$ yield the upper bounds

$$|D_\xi F(\xi)| \leq c_1 L_1 (1+|\xi|^2)^{\frac{p-1}{2}} \quad \text{and} \quad |D_\xi G(\xi)| \leq c_2 L_2 (1+|\xi|^2)^{\frac{p-1}{2}} \tag{2.1}$$

for all $\xi \in \mathbb{R}^{n\times N}$, with c_1 and c_2 constants depending only on p (see [31, Lemma 5.2] or [37]).
Furthermore, if F and G are C^2, then $(F2)$ and $(G2)$ imply the following strong Legendre-Hadamard conditions

$$\sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} \frac{\partial F}{\partial \xi_i^\alpha \partial \xi_j^\beta}(Q)\lambda_i\lambda_j\mu^\alpha\mu^\beta \geq c_3|\lambda|^2|\mu|^2 \quad \text{and} \quad \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} \frac{\partial G}{\partial \xi_i^\alpha \partial \xi_j^\beta}(Q)\lambda_i\lambda_j\mu^\alpha\mu^\beta \geq c_4|\lambda|^2|\mu|^2,$$

for all $Q \in \mathbb{R}^{n\times N}$, $\lambda \in \mathbb{R}^n$, $\mu \in \mathbb{R}^N$, where $c_3 = c_3(p, \ell_1)$ and $c_4 = c_4(p, \ell_2)$ are positive constants (see [31, Proposition 5.2]).

We will need the following quite standard regularity result (see [12] for its proof).

Proposition 2.1. *Let $v \in W^{1,1}(\Omega;\mathbb{R}^N)$ be such that*

$$\int_\Omega Q^{ij}_{\alpha\beta} D_i v^\alpha D_j \varphi^\beta \, dx = 0,$$

for every $\varphi \in C^\infty_c(\Omega;\mathbb{R}^N)$, where $Q = \{Q^{ij}_{\alpha\beta}\}$ is a constant matrix satisfying $|Q^{ij}_{\alpha\beta}| \le L$ and the strong Legendre-Hadamard condition

$$Q^{ij}_{\alpha\beta}\lambda_i\lambda_j\mu^\alpha\mu^\beta \ge \ell|\lambda|^2|\mu|^2,$$

for all $\lambda \in \mathbb{R}^n$, $\mu \in \mathbb{R}^N$ and for some positive constants $\ell, L > 0$. Then $v \in C^\infty$ and, for any $B_R(x_0) \subset \Omega$, the following estimate holds

$$\sup_{B_{R/2}} |Dv| \le \frac{c}{R^n}\int_{B_R} |Dv|\, dx,$$

where $c = c(n, N, \ell, L) > 0$.

We assume that $1 < p < 2$ and we refer to the auxiliary function

$$V(\xi) = (1 + |\xi|^2)^{(p-2)/4}\xi, \quad \forall \xi \in \mathbb{R}^k, \tag{2.2}$$

whose useful properties are listed in the following lemma (see [12] for the proof).

Lemma 2.2. *Let $1 < p < 2$ and let $V : \mathbb{R}^k \to \mathbb{R}^k$ be the function defined in* (2.2)*, then for any $\xi, \eta \in \mathbb{R}^k$ and $t > 0$ the following inequalities hold:*

(i) $2^{(p-2)/4}\min\{|\xi|, |\xi|^{p/2}\} \le \big|V(\xi)\big| \le \min\{|\xi|, |\xi|^{p/2}\}$,

(ii) $\big|V(t\xi)\big| \le \max\{t, t^{p/2}\}\big|V(\xi)\big|$,

(iii) $\big|V(\xi+\eta)\big| \le c(p)\big[\big|V(\xi)\big| + \big|V(\eta)\big|\big]$,

(iv) $\frac{p}{2}|\xi-\eta| \le \big(1 + |\xi|^2 + |\eta|^2\big)^{(2-p)/4}\big|V(\xi) - V(\eta)\big| \le c(k,p)|\xi - \eta|$,

(v) $\big|V(\xi) - V(\eta)\big| \le c(k,p)\big|V(\xi-\eta)\big|$,

(vi) $\big|V(\xi-\eta)\big| \le c(p,M)\big|V(\xi) - V(\eta)\big|$, *if* $|\eta| \le M$.

We will also use the following iteration lemma (see [31, Lemma 6.1])

Lemma 2.3. *Let $0 < \rho < R$ and let $\psi\colon [\rho, R] \to \mathbb{R}$ be a bounded non negative function. Assume that for all $\rho \le s < t \le R$ we have*

$$\psi(s) \le \vartheta\psi(t) + A + \frac{B}{(s-t)^\alpha} + \frac{C}{(s-t)^\beta}$$

where $\vartheta \in [0, 1)$, $\alpha > \beta > 0$ and $A, B, C \ge 0$ are constants. Then there exists a constant $c = c(\vartheta, \alpha) > 0$ such that

$$\psi(\rho) \le c\left(A + \frac{B}{(R-\rho)^\alpha} + \frac{C}{(R-\rho)^\beta}\right).$$

An easy extension of this result can be obtained by replacing homogeneity with condition (ii) of Lemma 2.2.

Lemma 2.4. *Let $R > 0$ and let $\psi\colon [R/2, R] \to [0, +\infty)$ be a bounded function. Assume that for all $R/2 \le s < t \le R$ we have*

$$\psi(s) \le \vartheta\psi(t) + A\int_{B_R}\left|V\left(\frac{h(x)}{t-s}\right)\right|^2 dx + B,$$

where $h \in L^p(B_r)$, $A, B > 0$, and $0 < \vartheta < 1$. Then there exists a constant $c(\vartheta) > 0$ such that

$$\psi\left(\frac{R}{2}\right) \le c(\vartheta)\left(A\int_{B_R}\left|V\left(\frac{h(x)}{R}\right)\right|^2 dx + B\right).$$

Given a C^1 function $f : \mathbb{R}^k \to \mathbb{R}$, $Q \in \mathbb{R}^k$ and $\lambda > 0$, we set

$$f_{Q,\lambda}(\xi) := \frac{f(Q+\lambda\xi) - f(Q) - Df(Q)\lambda\xi}{\lambda^2}, \quad \forall\xi \in \mathbb{R}^k.$$

In the next sections we will use the following lemma about the growth of $f_{Q,\lambda}$ and $Df_{Q,\lambda}$.

Lemma 2.5. *Let $1 < p < \infty$, and let f be a $C^2(\mathbb{R}^k)$ function such that*

$$|f(\xi)| \le L\big(1+|\xi|^p\big) \quad \textit{and} \quad |Df(\xi)| \le L\big(1+|\xi|^2\big)^{(p-1)/2},$$

for any $\xi \in \mathbb{R}^k$ and for some $L > 0$. Then for every $M > 0$ there exists a constant $c = c(p, L, M) > 0$ such that, for every $Q \in \mathbb{R}^k$, $|Q| \le M$ and $\lambda > 0$, it holds

$$|f_{Q,\lambda}(\xi)| \le c\big(1+|\lambda\xi|^2\big)^{(p-2)/2}|\xi|^2 \quad \textit{and} \quad |Df_{Q,\lambda}(\xi)| \le c\big(1+|\lambda\xi|^2\big)^{(p-2)/2}|\xi|, \tag{2.3}$$

for all $\xi \in \mathbb{R}^k$.

Proof. Applying Taylor's formula, there exists $\theta \in [0, 1]$ such that, for every $\xi \in \mathbb{R}^k$,

$$f_{Q,\lambda}(\xi) = \frac{1}{2}D^2 f(Q+\theta\lambda\xi)\xi\xi,$$

$$Df_{Q,\lambda}(\xi) = \frac{1}{\lambda}\big(Df(Q+\lambda\xi) - Df(Q)\big) = \int_0^1 D^2 f(Q+\theta\lambda\xi)\xi\, d\theta.$$

If we denote $K_M := \max\big\{|D^2 f(\xi)| : |\xi| \le M+1\big\}$, we have

$$|f_{Q,\lambda}(\xi)| \le \frac{1}{2}K_M|\xi|^2, \quad |Df_{Q,\lambda}(\xi)| \le K_M|\xi|, \quad \text{if } |\lambda\xi| \le 1. \tag{2.4}$$

On the other hand, using growth condition (2.3) and the definitions of $f_{Q,\lambda}$ and $Df_{Q,\lambda}$, we get

$$|f_{Q,\lambda}(\xi)| \le c(p, L, M)\lambda^{p-2}|\xi|^p, \quad |Df_{Q,\lambda}(\xi)| \le c(L, M)\lambda^{p-2}|\xi|^{p-1}, \quad \text{whereas } |\lambda\xi| > 1. \tag{2.5}$$

We get the result by combining (2.4) and (2.5) . □

A fundamental tool in order to handle the subquadratic case is the following Sobolev-Poincaré inequality related to the function V, proved in [12].

Theorem 2.6. *If $1 < p < 2$, there exist $2/p < \alpha < 2$ and $\sigma > 0$ such that if $u \in W^{1,p}(B_{3R}(x_0), \mathbb{R}^N)$, then*

$$\left(\fint_{B_R(x_0)} \left| V\left(\frac{u - u_{x_o,R}}{R} \right) \right|^{2(1+\sigma)} dx \right)^{\frac{1}{2(1+\sigma)}} \leq C \left(\fint_{B_{3R}(x_0)} \left| V\left(Du\right) \right|^{\alpha} dx \right)^{\frac{1}{\alpha}},$$

where the positive constant $C = C(n, N, p)$ is independent of R and u.

2.1 Sets of finite perimeter and anisotropic surface energies

In this subsection we recall some elementary definitions and well-known properties of sets of finite perimeter. We introduce the notion of anisotropic perimeter as well.

Given a set $E \subset \mathbb{R}^n$ and $t \in [0,1]$, we define the set of points of E of density t as

$$E^{(t)} = \left\{ x \in \mathbb{R}^n : \ |E \cap B_r(x)| = t|B_r(x)| + o(r^n) \text{ as } r \to 0^+ \right\}.$$

Let U be an open subset U of $\mathbb{R}^n$. A Lebesgue measurable set $E \subset \mathbb{R}^n$ is said to be a set of locally finite perimeter in U if there exists a $\mathbb{R}^n$-valued Radon measure μ_E on U (called the Gauss-Green measure of E) such that

$$\int_E \nabla\phi \; dx = \int_U \phi \, d\mu_E, \quad \forall \phi \in C_c^1(U).$$

Moreover, we denote the perimeter of E relative to $G \subset U$ by $P(E, G) = |\mu_E|(G)$.

It is well known that the support of μ_E can be characterized by

$$\text{spt}\mu_E = \left\{ x \in U : 0 < |E \cap B_r(x)| < \omega_n r^n, \, \forall r > 0 \right\} \subset U \cap \partial E, \tag{2.6}$$

(see [36, Proposition 12.19]). If E is of finite perimeter in U, the *reduced boundary* $\partial^* E \subset U$ of E is the set of those $x \in U$ such that

$$\nu_E(x) := \lim_{r \to 0^+} \frac{\mu_E(B_r(x))}{|\mu_E|(B_r(x))}$$

exists and belongs to $\mathbb{S}^{n-1}$. The *essential boundary* of E is defined as $\partial^e E := \mathbb{R}^n \setminus (E^0 \cup E^1)$. It is well-understood that

$$\partial^* E \subset U \cap \partial^e E \subset \text{spt}\mu_E \subset U \cap \partial E, \qquad U \cap \overline{\partial^* E} = \text{spt}\mu_E.$$

Furthermore, Federer's criterion (see for instance [36, Theorem 16.2]) ensures that

$$\mathcal{H}^{n-1}((U \cap \partial^e E) \setminus \partial^* E) = 0.$$

By De Giorgi's rectifiability theorem (see [36, Theorem 15.9]), the Gauss-Green measure μ_E is completely characterized as follows:

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E, \quad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E.$$

The equality holds in the class of Borel sets compactly contained in U. Here, we have denoted $\mu \llcorner \partial^* E(F) = \mu(\partial^* E \cap F)$, for any subset F of $\mathbb{R}^n$.

Remark 2.7 (Minimal topological boundary)**.** *If $E \subset \mathbb{R}^n$ is a set of locally finite perimeter in U and $F \subset \mathbb{R}^n$ is such that $|(E\Delta F)\cap U| = 0$, then F is a set of locally finite perimeter in U and $\mu_E = \mu_F$. In the rest of the paper, the topological boundary ∂E must be understood by considering the suitable representative of E in order to have that $\overline{\partial^* E} = \partial E \cap U$. We will choose $E^{(1)}$ as representative of E. With such a choice it can be easily verified that*

$$U \cap \partial E = \big\{x \in U : 0 < |E \cap B_r(x)| < \omega_n r^n, \forall r > 0\big\}.$$

Therefore, by (2.6)*,*

$$\overline{\partial^* E} = \mathrm{spt}\mu_E = \partial E \cap U.$$

In what follows, we give the definition of anisotropic surface energies and we recall some properties.

Definition 2.8 (Elliptic integrands)**.** *Given an open subset Ω of $\mathbb{R}^n$, $\Phi : \overline{\Omega} \times \mathbb{R}^n \to [0, \infty]$ is said to be an elliptic integrand on Ω if it is lower semicontinuous, with $\Phi(x, \cdot)$ convex and positively one-homogeneous for any $x \in \overline{\Omega}$, i.e. $\Phi(x, t\nu) = t\Phi(x, \nu)$ for every $t \geq 0$. Accordingly, the anisotropic surface energy of a set E of finite perimeter in Ω is defined as*

$$\mathbf{\Phi}(E; B) := \int_{B\cap\partial^* E} \Phi(x, \nu_E(x))\, d\mathcal{H}^{n-1}(x), \tag{2.7}$$

for every Borel set $B \subset \Omega$.

In order to prove the regularity of minimizers of anisotropic surface energies, it is well known that a C^k-dependence of the integrand Φ on the variable ν, and a continuity condition with respect to the variable x, must be assumed (see the seminal paper [3]). In fact, one more condition is essential, that is a non-degeneracy type condition for the integrand Φ. More precisely, we have to assume that there exists a constant $\Lambda > 1$ such that

$$\frac{1}{\Lambda} \leq \Phi(x, \nu) \leq \Lambda, \tag{2.8}$$

for any $x \in \Omega$ and $\nu \in \mathbb{S}^{n-1}$. We emphasize that (2.8) is the only assumption we make for the elliptic integrand Φ. We observe that, if the elliptic integrand Φ satisfies the previous condition, then the anisotropic surface energy (2.7) satisfies the following comparability condition to the perimeter:

$$\frac{1}{\Lambda}\mathcal{H}^{n-1}(B \cap \partial^* E) \leq \mathbf{\Phi}(E; B) \leq \Lambda\mathcal{H}^{n-1}(B \cap \partial^* E),$$

for any set E of finite perimeter in Ω and any Borel set $B \subset \Omega$. A useful relation is given by proposition below proved in [9].

Proposition 2.9. *Let $U \subset \mathbb{R}^n$ be an open set and let $E, F \subset U$ be two sets of finite perimeter in U. It holds that*

$$\mathbf{\Phi}(E \cup F; U) = \mathbf{\Phi}(E; F^{(0)}) + \mathbf{\Phi}(F; E^{(0)}) + \mathbf{\Phi}(E; \{\nu_E = \nu_F\}).$$

3 Decay Estimates

In this section we prove decay estimates for minimizers of functionals (1.2) by using a well-known blow-up technique involving a suitable excess function. We consider the bulk excess function defined as

$$U(x_0,r) := \fint_{B_r(x_0)} \big|V(Du) - V\big((Du)_{x_0,r}\big)\big|^2\,dx, \tag{3.1}$$

for $B_r(x_0) \subset \Omega$.

When the assumption (H) is in force, we refer to the following "hybrid" excess

$$U_*(x_0,r) := U(x_0,r) + \frac{P(E, B_r(x_0))}{r^{n-1}} + r.$$

Proposition 3.1. *Let (u,E) be a local minimizer of the functional $\mathcal{I}$ in (1.2) and let the assumptions $(F1)$, $(F2)$, $(G1)$, $(G2)$ and (H) hold. For every $M>0$ and every $0<\tau<\frac{1}{4}$, there exist two constants $\varepsilon_0 = \varepsilon_0(\tau, M) > 0$ and $c_* = c_*(n,p,\ell_1,\ell_2,L_1,L_2,\Lambda,M) > 0$ such that if in $B_r(x_0) \Subset \Omega$ it holds*

$$|(Du)_{x_0,r}| \le M \quad \text{and} \quad U_*(x_0,r) \le \varepsilon_0,$$

then

$$U_*(x_0,\tau r) \le c_* \tau U_*(x_0,r). \tag{3.2}$$

Proof. In order to prove (3.2), we argue by contradiction. Let $M>0$ and $\tau \in (0,1/4)$ be such that for every $h \in \mathbb{N}$, $C_*>0$, there exists a ball $B_{r_h}(x_h) \Subset \Omega$ such that

$$|(Du)_{x_h,r_h}| \le M, \quad U_*(x_h,r_h) \to 0 \tag{3.3}$$

and

$$U_*(x_h,\tau r_h) \ge C_* \tau U_*(x_h,r_h). \tag{3.4}$$

The constant C_* will be determined later. We remark that we can confine ourselves to the case in which $E \cap B_{r_h}(x_h) \neq \emptyset$, since the case in which $B_{r_h}(x_h) \subset \Omega \setminus E$ is easier, being $U = U_*$.

Step 1. *Blow-up.*

We set $\lambda_h^2 := U_*(x_h,r_h)$, $A_h := (Du)_{x_h,r_h}$, $a_h := (u)_{x_h,r_h}$, and we define

$$v_h(y) := \frac{u(x_h + r_h y) - a_h - r_h A_h y}{\lambda_h r_h}, \quad \forall y \in B_1. \tag{3.5}$$

One can easily check that $(Dv_h)_{0,1} = 0$ and $(v_h)_{0,1} = 0$. We set

$$E_h := \frac{E - x_h}{r_h}, \quad E_h^* := \frac{E - x_h}{r_h} \cap B_1.$$

By using (ii) and (vi) of Lemma 2.2, we deduce

$$\fint_{B_1} |V(Dv_h(y))|^2\,dy \le \fint_{B_{r_h}(x_h)} \left|V\left(\frac{Du(x) - (Du)_{x_h,r_h}}{\lambda_h}\right)\right|^2 dx$$

$$\leq \frac{c(M)}{\lambda_h^2} \fint_{B_{r_h}(x_h)} \left|V(Du(x)) - V\big((Du)_{x_h,r_h}\big)\right|^2 dx.$$

Then, since

$$\lambda_h^2 = U_*(x_h, r_h) = \fint_{B_1} \left|V\big(Du(x_h + r_h y)\big) - V\big(A_h\big)\right|^2 dy + \frac{P(E, B_{r_h}(x_h))}{r_h^{n-1}} + r_h, \tag{3.6}$$

it follows that $r_h \to 0$, $P(E_h, B_1) \to 0$, and

$$\frac{r_h}{\lambda_h^2} \leq 1, \quad \fint_{B_1} \left|V\big(Dv_h(y)\big)\right|^2 dy \leq c(M), \quad \frac{P(E_h, B_1)}{\lambda_h^2} \leq 1. \tag{3.7}$$

Therefore, by (3.3) and (3.7), there exist a (not relabeled) subsequence of $\{v_h\}_{h \in \mathbb{N}}$, $A \in \mathbb{R}^{n \times N}$ and $v \in W^{1,p}(B_1; \mathbb{R}^N)$, such that

$$\begin{aligned} &v_h \rightharpoonup v \quad \text{weakly in } W^{1,p}(B_1; \mathbb{R}^N), \quad v_h \to v \quad \text{strongly in } L^p(B_1; \mathbb{R}^N), \\ &A_h \to A, \quad \lambda_h D v_h \to 0 \quad \text{in } L^p(B_1; \mathbb{R}^{n \times N}) \text{ and pointwise a.e. in } B_1, \end{aligned} \tag{3.8}$$

where we have used the fact that $(v_h)_{0,1} = 0$. Moreover, by (3.7) and (3.3), we have

$$\lim_{h\to\infty} \frac{(P(E_h, B_1))^{\frac{n}{n-1}}}{\lambda_h^2} \leq \lim_{h\to\infty} (P(E_h, B_1))^{\frac{1}{n-1}} \limsup_{h\to\infty} \frac{P(E_h, B_1)}{\lambda_h^2} = 0. \tag{3.9}$$

Therefore, by the relative isoperimetric inequality,

$$\lim_{h\to\infty} \min\left\{\frac{|E_h^*|}{\lambda_h^2}, \frac{|B_1 \setminus E_h|}{\lambda_h^2}\right\} \leq c(n) \lim_{h\to\infty} \frac{(P(E_h, B_1))^{\frac{n}{n-1}}}{\lambda_h^2} = 0. \tag{3.10}$$

In the sequel the proof will proceed differently depending on

$$\min\{|E_h^*|, |B_1 \setminus E_h|\} = |E_h^*| \qquad \text{or} \qquad \min\{|E_h^*|, |B_1 \setminus E_h|\} = |B_1 \setminus E_h|.$$

The first case is easier to handle. To understand the reason, let us introduce the expansions of F and G around A_h as follows:

$$\begin{aligned} F_h(\xi) &:= \frac{F(A_h + \lambda_h \xi) - F(A_h) - DF(A_h)\lambda_h \xi}{\lambda_h^2}, \\ G_h(\xi) &:= \frac{G(A_h + \lambda_h \xi) - G(A_h) - DG(A_h)\lambda_h \xi}{\lambda_h^2}, \end{aligned} \tag{3.11}$$

for any $\xi \in \mathbb{R}^{n \times N}$. In the first case the suitable rescaled functional to consider in the blow-up procedure is the following

$$\mathcal{I}_h(w) := \int_{B_1} \left[F_h(Dw)dy + \mathbb{1}_{E_h^*} G_h(Dw)\right] dy. \tag{3.12}$$

We claim that v_h satisfies the minimality inequality

$$\mathcal{I}_h(v_h) \leq \mathcal{I}_h(v_h + \psi) + \frac{1}{\lambda_h} \int_{B_1} \mathbb{1}_{E_h^*} DG(A_h) D\psi(y)\, dy, \tag{3.13}$$

for any $\psi \in W_0^{1,p}(B_1;\mathbb{R}^N)$. Indeed, using the minimality of (u,E) with respect to $(u+\varphi,E)$, for $\varphi \in W_0^{1,p}(B_{r_h}(x_h);\mathbb{R}^N)$, the change of variable $x = x_h + r_h y$, setting $\psi(y) := \frac{\varphi(x_h+r_hy)}{\lambda_h r_h}$, it holds that

$$\begin{aligned}&\int_{B_1} \big[(F_h(Dv_h(y)) + \mathbb{1}_{E_h^*}G_h(Dv_h(y))\big]\,dy\\&\le \int_{B_1} \big[F_h(Dv_h(y)+D\psi(y)) + \mathbb{1}_{E_h^*}G_h(Dv_h(y)+D\psi(y))\big]\,dy + \frac{1}{\lambda_h}\int_{B_1}\mathbb{1}_{E_h^*}DG(A_h)D\psi(y)\,dy,\end{aligned}$$

and (3.13) follows by the definition of $\mathcal{I}_h$ in (3.12).
In the second case, the suitable rescaled functional to consider in the blow-up procedure is

$$\mathcal{H}_h(w) := \int_{B_1}\big[F_h(Dw)+G_h(Dw)\big]\,dy.$$

We claim that

$$\mathcal{H}_h(v_h) \le \mathcal{H}_h(v_h+\psi) + \frac{L_2}{\lambda_h^2}\int_{(B_1\setminus E_h)\cap \mathrm{supp}\psi}(\mu^2+|A_h+\lambda_h Dv_h|^2)^{\frac{p}{2}}\,dy, \tag{3.14}$$

for all $\psi \in W_0^{1,p}(B_1;\mathbb{R}^N)$. Indeed, the minimality of (u,E) with respect to $(u+\varphi,E)$, for $\varphi \in W_0^{1,p}(B_{r_h}(x_h);\mathbb{R}^N)$, implies that

$$\begin{aligned}\int_{B_{r_h}(x_h)}(F+G)(Du)\,dx &= \int_{B_{r_h}(x_h)}\big[F(Du)+\mathbb{1}_E G(Du)\big]\,dx + \int_{B_{r_h}(x_h)\setminus E} G(Du)dx\\&\le \int_{B_{r_h}(x_h)}\big[F(Du+D\varphi)+\mathbb{1}_E G(Du+D\varphi)\big]\,dx + \int_{B_{r_h}(x_h)\setminus E} G(Du)dx\\&= \int_{B_{r_h}(x_h)}(F+G)(Du+D\varphi)dx + \int_{B_{r_h}(x_h)\setminus E}\big[G(Du)-G(Du+D\varphi)\big]\,dx\\&\le \int_{B_{r_h}(x_h)}(F+G)(Du+D\varphi)dx + \int_{(B_{r_h}(x_h)\setminus E)\cap\mathrm{supp}\varphi} G(Du)dx,\end{aligned}$$

where we used that the last integral vanishes outside the support of φ and that $G \ge 0$. Using the change of variable $x = x_h + r_h y$ in the previous formula, we get

$$\begin{aligned}\int_{B_1}(F+G)(Du(x_h+r_hy))dy &\le \int_{B_1}(F+G)(Du(x_h+r_hy)+D\varphi(x_h+r_hy))\,dy\\&\quad+\int_{(B_1\setminus E_h)\cap\mathrm{supp}\psi} G(Du(x_h+r_hy))dy,\end{aligned}$$

or, equivalently, using the definitions of v_h,

$$\begin{aligned}\int_{B_1}(F+G)(A_h+\lambda_h Dv_h)dy &\le \int_{B_1}(F+G)(A_h+\lambda_h(Dv_h+D\psi))\,dy\\&\quad+\int_{(B_1\setminus E_h)\cap\mathrm{supp}\psi} G(A_h+\lambda_h Dv_h)dy\end{aligned}$$

where $\psi(y) := \frac{\varphi(x_h + r_h y)}{\lambda_h r_h}$, for $y \in B_1$. Therefore, setting

$$H_h := F_h + G_h,$$

by the definitions of F_h and G_h in (3.11) and using the assumption $(G1)$, we have that

$$\begin{aligned}\int_{B_1} H_h(Dv_h)dy &\le \int_{B_1} H_h(Dv_h + D\psi)dy + \frac{1}{\lambda_h^2}\int_{(B_1\setminus E_h)\cap \operatorname{supp}\psi} G(A_h + \lambda_h Dv_h)\,dy\\ &\le \int_{B_1} H_h(Dv_h + D\psi)\,dy + \frac{L_2}{\lambda_h^2}\int_{(B_1\setminus E_h)\cap \operatorname{supp}\psi}\big(1 + |A_h + \lambda_h Dv_h|^2\big)^{\frac{p}{2}}\,dy, \end{aligned}\tag{3.15}$$

i.e. (3.14).

Step 2. *A Caccioppoli type inequality.*

We claim that there exists a constant $c = c(n, p, \ell_1, \ell_2, L_1, L_2, M) > 0$ such that for every $0 < \rho < 1$ there exists $h_0 = h_0(n, p, M, \rho) \in \mathbb{N}$ such that

$$\begin{aligned}&\int_{B_{\frac{\rho}{2}}} \big|V\big(\lambda_h(Dv_h - (Dv_h)_{\frac{\rho}{2}}\big)\big|^2\,dy \\ &\le c\left[\int_{B_\rho}\left|V\left(\frac{\lambda_h\big(v_h - (v_h)_\rho - (Dv_h)_{\frac{\rho}{2}}\,y\big)}{\rho}\right)\right|^2 dy + P(E_h, B_1)^{\frac{n}{n-1}}\right],\end{aligned}\tag{3.16}$$

for all $h > h_0$. We divide the proof into two steps.

Substep 2.a *The case* $\min\{|E_h^*|, |B_1 \setminus E_h|\} = |E_h^*|$.

We consider $0 < \frac{\rho}{2} < s < t < \rho < 1$ and let $\eta \in C_0^\infty(B_t)$ be a cut off function between B_s and B_t, i.e. $0 \le \eta \le 1$, $\eta \equiv 1$ on B_s and $|\nabla\eta| \le \frac{c}{t-s}$. Set $b_h := (v_h)_{B_\rho}$, $B_h := (Dv_h)_{B_{\frac{\rho}{2}}}$, and set

$$w_h(y) := v_h(y) - b_h - B_h y,\tag{3.17}$$

for any $y \in B_1$. Proceeding similarly as in (3.6), we rescale F and G around $A_h + \lambda_h B_h$,

$$\begin{aligned}\widetilde{F}_h(\xi) &:= \frac{F(A_h + \lambda_h B_h + \lambda_h\xi) - F(A_h + \lambda_h B_h) - DF(A_h + \lambda_h B_h)\lambda_h\xi}{\lambda_h^2},\\ \widetilde{G}_h(\xi) &:= \frac{G(A_h + \lambda_h B_h + \lambda_h\xi) - G(A_h + \lambda_h B_h) - DG(A_h + \lambda_h B_h)\lambda_h\xi}{\lambda_h^2},\end{aligned}\tag{3.18}$$

for any $\xi \in \mathbb{R}^{n\times N}$. By Lemma 2.5, two growth estimates on $\widetilde{F}_h$, $\widetilde{G}_h$ and their gradients hold with some constants that depend on p, L_1, L_2, M (see (3.3)) and could also depend on ρ through $|\lambda_h B_h|$. However, given ρ, we may choose $h_0 = h_0(n, p, M, \rho)$ large enough to have

$$|\lambda_h B_h| < \frac{c(n, p, M)\lambda_h}{\rho^{\frac{n}{p}}} < 1,$$

for any $h \ge h_0$. Indeed, by (3.7) the sequence $\{Dv_h\}_h$ is equibounded in $L^p(B_1)$, then we have

$$|B_h| \le \frac{2^n}{\omega_n\rho^{\frac{n}{p}}}\left[\int_{B_{\frac{\rho}{2}}\cap\{|Dv_h|\le 1\}} |Dv_h|\,dy + \int_{B_{\frac{\rho}{2}}\cap\{|Dv_h|>1\}} |Dv_h|\,dy\right]$$

$$\leq \frac{2^n}{\omega_n \rho^{\frac{n}{p}}} \left[\left(\int_{B_{\frac{\rho}{2}}} |V(Dv_h)|^2 \, dy \right)^{\frac{1}{2}} + \left(\int_{B_{\frac{\rho}{2}}} |V(Dv_h)|^2 \, dy \right)^{\frac{1}{p}} \right] \leq \frac{c(n,p,M)}{\rho^{\frac{n}{p}}},$$

and so the constant in (2.3) can be taken independently of ρ.

Set

$$\psi_{1,h} := \eta w_h \quad \text{and} \quad \psi_{2,h} := (1-\eta) w_h.$$

By the uniformly strict quasiconvexity of $\widetilde{F}_h$ we have

$$\begin{aligned}
&\frac{\ell_1}{\lambda_h^2} \int_{B_s} |V(\lambda_h Dw_h)|^2 \, dy \\
&\leq \ell_1 \int_{B_t} \left(1 + |\lambda_h D\psi_{1,h}|^2\right)^{\frac{p-2}{2}} |D\psi_{1,h}|^2 \, dy \leq \int_{B_t} \widetilde{F}_h(D\psi_{1,h}) \, dy \\
&= \int_{B_t} \widetilde{F}_h(Dw_h) \, dy + \int_{B_t} \widetilde{F}_h(Dw_h - D\psi_{2,h}) \, dy - \int_{B_t} \widetilde{F}_h(Dw_h) \, dy \\
&= \int_{B_t} \widetilde{F}_h(Dw_h) \, dy - \int_{B_t} \int_0^1 D\widetilde{F}_h(Dw_h - \theta D\psi_{2,h}) D\psi_{2,h} \, d\theta \, dy.
\end{aligned} \tag{3.19}$$

We estimate separately the two addends in the right-hand side of the previous chain of inequalities. We deal with the first addend by means of a rescaling of the minimality condition of (u, E). Using the change of variable $x = x_h + r_h y$, the fact that $G \geq 0$ and the minimality of (u, E) with respect to $(u + \varphi, E)$ for $\varphi \in W_0^{1,p}(B_{r_h}(x_h); \mathbb{R}^N)$, we have

$$\begin{aligned}
&\int_{B_1} F(Du(x_h + r_h y)) dy \leq \int_{B_1} \left[F(Du(x_h + r_h y)) + \mathbb{1}_{E_h^*} G(Du(x_h + r_h y)) \right] dy \\
&\leq \int_{B_1} \left[F(Du(x_h + r_h y) + D\varphi(x_h + r_h y)) + \mathbb{1}_{E_h^*} G(Du(x_h + r_h y) + D\varphi(x_h + r_h y)) \right] dy,
\end{aligned}$$

i.e., by the definitions of v_h and w_h, (3.5) and (3.17) respectively,

$$\begin{aligned}
&\int_{B_1} F(A_h + \lambda_h B_h + \lambda_h Dw_h) dy \\
&\leq \int_{B_1} \left[F(A_h + \lambda_h B_h + \lambda_h (Dw_h + D\psi)) + \mathbb{1}_{E_h^*} G(A_h + \lambda_h B_h + \lambda_h (Dw_h + D\psi)) \right) dy,
\end{aligned}$$

for $\psi := \frac{\varphi(x_h + r_h y)}{\lambda_h r_h} \in W_0^{1,p}(B_1; \mathbb{R}^N)$. Therefore, recalling the definitions of $\widetilde{F}_h$ and $\widetilde{G}_h$ in (3.18), we have that

$$\begin{aligned}
&\int_{B_1} \widetilde{F}_h(Dw_h) dy \leq \int_{B_1} \left[\widetilde{F}_h(Dw_h + D\psi) + \mathbb{1}_{E_h^*} \widetilde{G}_h(Dw_h + D\psi) \right] dy \\
&+ \frac{1}{\lambda_h^2} \int_{B_1} \mathbb{1}_{E_h^*} \left[G(A_h + \lambda_h B_h) + DG(A_h + \lambda_h B_h) \lambda_h (Dw_h + D\psi) \right] dy.
\end{aligned}$$

Choosing φ such that $\psi = -\psi_{1,h}$, the previous inequality becomes

$$\int_{B_t} \widetilde{F}_h(Dw_h) \, dy \leq \int_{B_t} \left[\widetilde{F}_h\big(Dw_h - D\psi_{1,h}\big) + \mathbb{1}_{E_h^*} \widetilde{G}_h(Dw_h - D\psi_{1,h}) \right] dy \tag{3.20}$$

$$
\begin{aligned}
&+\frac{1}{\lambda_h^2}\int_{B_1}\mathbb{1}_{E_h^*}\big[G(A_h+\lambda_h B_h)+DG(A_h+\lambda_h B_h)\lambda_h(Dw_h-D\psi_{1,h})\big]\,dy\\
&=\int_{B_t\setminus B_s}\big[\widetilde{F}_h(D\psi_{2,h})+\mathbb{1}_{E_h^*}\widetilde{G}_h(D\psi_{2,h})\big]\,dy\\
&+\frac{1}{\lambda_h^2}\int_{B_1}\mathbb{1}_{E_h^*}\big[G(A_h+\lambda_h B_h)+DG(A_h+\lambda_h B_h)\lambda_h D\psi_{2,h}\big]\,dy\\
&\le \frac{c(p,L_1,L_2,M)}{\lambda_h^2}\int_{B_t\setminus B_s}|V(\lambda_h D\psi_{2,h})|^2\,dy + c(n,p,L_2,M)\left[\frac{|E_h^*|}{\lambda_h^2}+\frac{1}{\lambda_h}\int_{E_h^*}|D\psi_{2,h}|\,dy\right],
\end{aligned}
$$

where we have used Lemma 2.5, the second estimate in (2.1), and the fact that $|A_h+\lambda_h B_h|\le M+1$. By applying Hölder's and Young's inequalities, we get

$$
\begin{aligned}
\frac{1}{\lambda_h}\int_{E_h^*}|D\psi_{2,h}|\,dy &\le \frac{|E_h^*|^{\frac{p-1}{p}}}{\lambda_h^2}\left(\int_{E_h^*\cap(B_t\setminus B_s)}|\lambda_h D\psi_{2,h}|^p\,dy\right)^{\frac{1}{p}}\\
&\le \frac{1}{\lambda_h^2}\left[|E_h^*|+\int_{E_h^*\cap(B_t\setminus B_s)}|\lambda_h D\psi_{2,h}|^p\,dy\right]\\
&\le \frac{1}{\lambda_h^2}\left[2|E_h^*|+\int_{E_h^*\cap(B_t\setminus B_s)\cap\{|\lambda_h D\psi_{2,h}|>1\}}|\lambda D\psi_{2,h}|^p\,dy\right]\\
&\le \frac{1}{\lambda_h^2}\left[2|E_h^*|+\int_{B_t\setminus B_s}|V(\lambda_h D\psi_{2,h}))|^2\,dy\right].
\end{aligned}
$$

The previous chain of inequalities combined with (3.20) yields

$$
\int_{B_1}\widetilde{F}_h(Dw_h)dy \le \frac{c(n,p,L_1,L_2,M)}{\lambda_h^2}\left[\int_{B_t\setminus B_s}|V(\lambda_h D\psi_{2,h})|^2\,dy+|E_h^*|\right]. \tag{3.21}
$$

Now we estimate the second addend in the right-hand side of (3.19). Using the upper bound on $D\widetilde{F}_h$ in Lemma 2.5,

$$
\begin{aligned}
&\int_{B_t}\int_0^1 D\widetilde{F}_h(Dw_h-\theta D\psi_{2,h})D\psi_{2,h}\,d\theta dy \qquad (3.22)\\
&\le c(p,L_1,M)\int_{B_t\setminus B_s}\int_0^1\big(1+\lambda_h^2|Dw_h-\theta D\psi_{2,h}|^2\big)^{\frac{p-2}{2}}|Dw_h-\theta D\psi_{2,h}||D\psi_{2,h}|d\theta dy.
\end{aligned}
$$

Regarding the integrand in the latest estimate, we distinguish two cases:

Case 1: $|D\psi_{2,h}|\le|Dw_h-\theta D\psi_{2,h}|$.

By the definition of V, we have

$$
\big(1+\lambda_h^2|Dw_h-\theta D\psi_{2,h}|^2\big)^{\frac{p-2}{2}}|Dw_h-\theta D\psi_{2,h}||D\psi_{2,h}|\le\lambda_h^{-2}|V(\lambda_h(Dw_h-\theta D\psi_{2,h})|^2.
$$

Case 2: $|Dw_h-\theta D\psi_{2,h}|<|D\psi_{2,h}|$.

If $|D\psi_{2,h}|<1/\lambda_h$, using *(i)* of Lemma 2.2 we get

$$
\big(1+\lambda_h^2|Dw_h-\theta D\psi_{2,h}|^2\big)^{\frac{p-2}{2}}|Dw_h-\theta D\psi_{2,h}||D\psi_{2,h}|\le|D\psi_{2,h}|^2\le\lambda_h^{-2}|V(\lambda_h D\psi_{2,h})|^2.
$$

If $|D\psi_{2,h}| \geq 1/\lambda_h$, using again (i) of Lemma 2.2 we deduce that

$$\begin{aligned}&\left(1+\lambda_h^2|Dw_h-\theta D\psi_{2,h}|^2\right)^{\frac{p-2}{2}}|Dw_h-\theta D\psi_{2,h}||D\psi_{2,h}| \leq \\ &\leq \lambda_h^{p-2}|Dw_h-\theta D\psi_{2,h}|^{p-1}|D\psi_{2,h}| \leq \lambda_h^{-2}|\lambda_h D\psi_{2,h}|^p \leq \lambda_h^{-2}|V(\lambda_h D\psi_{2,h})|^2.\end{aligned}$$

By combining the two previous cases, we can proceed in the estimate (3.22) as follows:

$$\begin{aligned}&\int_{B_t}\int_0^1 D\widetilde{F}_h(Dw_h-\theta D\psi_{2,h})D\psi_{2,h}\,d\theta\,dy \\ &\leq \frac{c(p,L_1,M)}{\lambda_h^2}\int_{B_t\setminus B_s}\left(|V(\lambda_h(Dw_h-\theta D\psi_{2,h})|^2+|V(\lambda_h D\psi_{2,h})|^2\right)dy \\ &\leq \frac{c(p,L_1,M)}{\lambda_h^2}\int_{B_t\setminus B_s}\left(|V(\lambda_h Dw_h)|^2+|V(\lambda_h D\psi_{2,h})|^2\right)dy\end{aligned} \tag{3.23}$$

Hence, combining (3.19) with (3.21) and (3.23), we obtain

$$\begin{aligned}&\frac{\ell_1}{\lambda_h^2}\int_{B_s}|V(\lambda_h Dw_h)|^2\,dy \\ &\leq \frac{c(n,p,L_1,L_2,M)}{\lambda_h^2}\left[\int_{B_t\setminus B_s}\left(|V(\lambda_h Dw_h)|^2+|V(\lambda_h D\psi_{2,h})|^2\right)\,dy+|E_h^*|\right]\end{aligned}$$

By the definition of $\psi_{2,h}$ and *(iii)* of Lemma 2.2, we infer that

$$\begin{aligned}&\ell_1\int_{B_s}|V(\lambda_h Dw_h)|^2\,dy \\ &\leq \tilde{C}\left[\int_{B_t\setminus B_s}\left(|V(\lambda_h Dw_h)|^2+\left|V\left(\lambda_h\frac{w_h}{t-s}\right)\right|^2\right)dy+|E_h^*|\right],\end{aligned}$$

for some $\tilde{C}=\tilde{C}(n,p,L_1,L_2,M)$

By adding $\tilde{C}\int_{B_s}|V(\lambda_h Dw_h)|^2\,dy$ to both sides of the previous estimate, dividing by $\ell_1+\tilde{C}$ and thanks to Lemma 2.4, we deduce that

$$\int_{B_{\frac{\rho}{2}}}|V(\lambda_h Dw_h)|^2\,dy \leq c(n,p,\ell_1,L_1,L_2,M)\left(\int_{B_\rho}\left|V\left(\lambda_h\frac{w_h}{\rho}\right)\right|^2 dy+|E_h^*|\right).$$

Therefore, by the definition of w_h, we conclude that

$$\begin{aligned}&\int_{B_{\frac{\rho}{2}}}\left|V(\lambda_h(Dv_h-(Dv_h)_{\frac{\rho}{2}})\right|^2 dy \\ &\leq c(n,p,\ell_1,L_1,L_2,M)\left[\int_{B_\rho}\left|V\left(\frac{\lambda_h\left(v_h-(v_h)_\rho-(Dv_h)_{\frac{\rho}{2}}\,y\right)}{\rho}\right)\right|^2 dy+|E_h^*|\right]\end{aligned}$$

which, by the relative isoperimetric inequality and the hypothesis of this substep, i.e. $\min\{|E_h^*|,|B_1\setminus E_h|\}=|E_h^*|$, yields the estimate (3.16).

Substep 2.b *The case* $\min\{|E_h^*|,|B_1\setminus E_h|\}=|B_1\setminus E_h|$.

As in the previous substep, we fix $0 < \frac{\rho}{2} < s < t < \rho < 1$ and let $\eta \in C_0^\infty(B_t)$ be a cut off function between B_s and B_t, i.e., $0 \le \eta \le 1$, $\eta \equiv 1$ on B_s and $|\nabla \eta| \le \frac{c}{t-s}$. Also, we set $b_h := (v_h)_{B_\rho}$, $B_h := (Dv_h)_{B_{\frac{\rho}{2}}}$ and define

$$w_h(y) := v_h(y) - b_h - B_h y, \quad \forall y \in B_1,$$

and

$$\widetilde{H}_h := \widetilde{F}_h + \widetilde{G}_h.$$

We remark that Lemma 2.5 can be applied to $\widetilde{H}_h$, that is

$$|\widetilde{H}_h(\xi)| \le c(p, L_1, L_2, M)\big(1 + |\lambda_h \xi|^2\big)^{\frac{p-2}{2}} |\xi|^2, \quad \forall \xi \in \mathbb{R}^{n\times N},$$

and, by the uniformly strict quasiconvexity conditions $(F2)$ and $(G2)$,

$$\int_{B_1} \widetilde{H}_h(\xi + D\psi)\, dx \ge \int_{B_t} \big[\widetilde{H}_h(\xi) + \tilde{\ell}\big(1 + |\lambda_h D\psi|^2\big)^{\frac{p-2}{2}} |D\psi|^2\big]\, dy, \quad \forall \psi \in W_0^{1,p}(B_1; \mathbb{R}^N), \tag{3.24}$$

where we have set

$$\widetilde{\ell} = \ell_1 + \ell_2.$$

We set again

$$\psi_{1,h} := \eta w_h \quad \text{and} \quad \psi_{2,h} := (1-\eta) w_h.$$

By the quasiconvexity condition (3.24) and since $\widetilde{H}_h(0) = 0$, we have

$$\begin{aligned}
\frac{\widetilde{\ell}}{\lambda_h^2} \int_{B_s} |V(\lambda_h Dw_h)|^2\, dy &\le \widetilde{\ell} \int_{B_s} \big(1 + |\lambda_h Dw_h|^2\big)^{\frac{p-2}{2}} |Dw_h|^2\, dy \\
&\le \widetilde{\ell} \int_{B_t} \big(1 + |\lambda_h D\psi_{1,h}|^2\big)^{\frac{p-2}{2}} |D\psi_{1,h}|^2\, dy \le \int_{B_t} \widetilde{H}_h(D\psi_{1,h})\, dy = \int_{B_t} \widetilde{H}_h(Dw_h - D\psi_{2,h})\, dy \\
&= \int_{B_t} \widetilde{H}_h(Dw_h)\, dy + \int_{B_t} \widetilde{H}_h(Dw_h - D\psi_{2,h})\, dy - \int_{B_t} \widetilde{H}_h(Dw_h)\, dy \\
&= \int_{B_t} \widetilde{H}_h(Dw_h)\, dy - \int_{B_t} \int_0^1 D\widetilde{H}_h(Dw_h - \theta D\psi_{2,h}) D\psi_{2,h}\, d\theta\, dy.
\end{aligned} \tag{3.25}$$

Similarly to the previous case, we estimate separately the two addends in the right-hand side of the previous chain of inequalities. Using the minimality condition (3.15) for the rescaled functions v_h and recalling the definition of $\widetilde{H}_h$, since $Dv_h = Dw_h + B_h$, we get

$$\begin{aligned}
\int_{B_1} \widetilde{H}_h(Dw_h) dy \le& \int_{B_1} \widetilde{H}_h(Dw_h + D\psi)\, dy \\
&+ \frac{L_2}{\lambda_h^2} \int_{(B_1 \setminus E_h)\cap \operatorname{supp}\psi} \big(1 + |A_h + \lambda_h B_h + \lambda_h Dw_h|^2\big)^{\frac{p}{2}}\, dy.
\end{aligned} \tag{3.26}$$

Choosing $\psi = -\psi_{1,h}$ as test function in (3.26) and using the fact that $\widetilde{H}_h(0) = 0$, we estimate

$$\int_{B_t} \widetilde{H}_h(Dw_h)\, dy$$

$$
\begin{aligned}
&\le \int_{B_t} \widetilde{H}_h(Dw_h - D\psi_{1,h})\, dy + \frac{L_2}{\lambda_h^2}\int_{B_t\setminus E_h} \big(1+|A_h+\lambda_h B_h+\lambda_h Dw_h|^2\big)^{\frac{p}{2}}\, dy\\
&= \int_{B_t\setminus B_s} \widetilde{H}_h\big(D\psi_{2,h}\big)\, dy + \frac{L_2}{\lambda_h^2}\int_{B_t\setminus E_h} \big(1+|A_h+\lambda_h B_h+\lambda_h Dw_h|^2\big)^{\frac{p}{2}}\, dy\\
&\le \frac{c(p,L_1,L_2,M)}{\lambda_h^2}\int_{B_t\setminus B_s} \big|V\big(\lambda_h D\psi_{2,h}\big)\big|^2\, dy + \frac{L_2}{\lambda_h^2}\int_{B_t\setminus E_h} \big(1+|A_h+\lambda_h B_h+\lambda_h Dw_h|^2\big)^{\frac{p}{2}}\, dy.
\end{aligned}
$$

We note that, since $|A_h+\lambda_h B_h|\le c(M)$, for every fixed $\varepsilon>0$ there exists a constant $C=C(p,\varepsilon)$ such that

$$
\big(1+|A_h+\lambda_h B_h+\lambda_h Dw_h|^2\big)^{\frac{p}{2}} \le C(p,\varepsilon)c(M)^p + (1+\varepsilon)\lambda_h^p|Dw_h|^p.
$$

Summarizing, we get

$$
\begin{aligned}
&\int_{B_t} \widetilde{H}_h(Dw_h)\, dy \le \frac{c(p,L_1,L_2,M)}{\lambda_h^2}\int_{B_t\setminus B_s} \big|V\big(\lambda_h D\psi_{2,h}\big)\big|^2\, dy \\
&+ (1+\varepsilon)\frac{L_2}{\lambda_h^2}\int_{B_t} \mathbb{1}_{\{|\lambda_h Dw_h|\ge 1\}}|\lambda_h Dw_h|^p\, dy + c(p,L_2,M,\varepsilon)\frac{|B_1\setminus E_h|}{\lambda_h^2}.
\end{aligned}
\tag{3.27}
$$

Now we estimate the second addend in the right-hand side of (3.25). Using the upper bound on $D\widetilde{H}_h$ in Lemma 2.5, we obtain

$$
\begin{aligned}
&\int_{B_t}\int_0^1 D\widetilde{H}_h(Dw_h-\theta D\psi_{2,h})D\psi_{2,h}\, d\theta\, dy\\
&\le c(p,L_1,L_2,M)\int_{B_t\setminus B_s}\int_0^1 \big(1+\lambda_h^2|Dw_h-\theta D\psi_{2,h}|^2\big)^{\frac{p-2}{2}}|Dw_h-\theta D\psi_{2,h}||D\psi_{2,h}|\, dy.
\end{aligned}
$$

Proceeding exactly as in the estimate (3.23) of the step $2.a$, we obtain

$$
\begin{aligned}
&\int_{B_t}\int_0^1 D\widetilde{H}_h(Dw_h-\theta D\psi_{2,h})D\psi_{2,h}\, d\theta\, dy\\
&\le \frac{c(p,L_1,L_2,M)}{\lambda_h^2}\int_{B_t\setminus B_s} \big(|V(\lambda_h Dw_h)|^2 + |V(\lambda_h D\psi_{2,h})|^2\big)\, dy.
\end{aligned}
\tag{3.28}
$$

Inserting (3.27) and (3.28) in (3.25), we infer that

$$
\begin{aligned}
&\frac{\widetilde{\ell}}{\lambda_h^2}\int_{B_s}|V(\lambda_h Dw_h)|^2\, dy\\
&\quad\le \frac{c(p,L_1,L_2,M)}{\lambda_h^2}\int_{B_t\setminus B_s} \big(|V(\lambda_h Dw_h)|^2 + |V(\lambda_h D\psi_{2,h})|^2\big)\, dy\\
&\quad+ (1+\varepsilon)\frac{L_2}{\lambda_h^2}\int_{B_t} \mathbb{1}_{\{|\lambda_h Dw_h|\ge 1\}}|\lambda_h Dw_h|^p\, dy + c(p,L_2,M,\varepsilon)\frac{|B_1\setminus E_h|}{\lambda_h^2}\\
&\quad\le \frac{c(p,L_1,L_2,M)}{\lambda_h^2}\int_{B_t\setminus B_s}|V(\lambda_h Dw_h)|^2\, dy + \frac{c(p,M,L_1,L_2)}{\lambda_h^2}\int_{B_t\setminus B_s}\left|V\left(\lambda_h\frac{w_h}{t-s}\right)\right|^2 dy\\
&\quad+ (1+\varepsilon)\frac{L_2}{\lambda_h^2}\int_{B_t}|V(\lambda_h Dw_h)|^2\, dy + c(p,L_2,M,\varepsilon)\frac{|B_1\setminus E_h|}{\lambda_h^2}.
\end{aligned}
$$

Taking advantage of the hole filling technique as in the previous case, we obtain

$$\begin{aligned}&\int_{B_t\setminus B_s}|V(\lambda_h Dw_h)|^2\,dy\\&\le\frac{(c(p,L_1,L_2,M)+(1+\varepsilon)L_2)}{(c(p,M,L_1,L_2)+\widetilde{\ell})}\int_{B_t}|V(\lambda_h Dw_h)|^2\,dy\\&+c(p,M,L_1,L_2)\int_{B_t\setminus B_s}\left|V\left(\lambda_h\frac{w_h}{t-s}\right)\right|^2dy+c(p,L_2,M,\varepsilon)\frac{|B_1\setminus E_h|}{\lambda_h^2}.\end{aligned}$$

The assumption (H) implies that there exists $\varepsilon=\varepsilon(p,\ell_1,\ell_2,L_2)>0$ such that $\frac{(1+\varepsilon)L_2}{\ell_1+\ell_2}<1$. Therefore we have

$$\frac{c+(1+\varepsilon)L_2}{c+\widetilde{\ell}}=\frac{c+(1+\varepsilon)L_2}{c+\ell_1+\ell_2}<1.$$

So, by virtue of Lemma 2.4, from the previous estimate we deduce that

$$\int_{B_{\frac{\rho}{2}}}|V(\lambda_h Dw_h)|^2\,dy\le c(n,p,\ell_1,\ell_2,L_1L_2,M)\bigg(\int_{B_\rho}\left|V\left(\lambda_h\frac{w_h}{\rho}\right)\right|^2dy+|B_1\setminus E_h|\bigg).$$

By definition of w_h and the relative isoperimetric inequality, since $|B_1\setminus E_h|=\min\{|E_h^*|,|B_1\setminus E_h|\}$, we get the estimate (3.16).

Step 3. *v solves a linear system in B_1.*

Let us divide the proof into two cases, depending on which one is the smallest between $|E_h^*|$ and $|B_1\setminus E_h|$.

We divide the proof in two substeps.

Substep 3.a *The case* $\min\{|E_h^*|,|B_1\setminus E_h|\}=|E_h^*|$. We claim that v solves the linear system

$$\int_{B_1}D^2F(A)DvD\psi\,dy=0,$$

for all $\psi\in C_0^1(B_1;\mathbb{R}^N)$. Since v_h satisfies (3.13), we have that

$$0\le\mathcal{I}_h(v_h+s\psi)-\mathcal{I}_h(v_h)+\frac{1}{\lambda_h}\int_{B_1}\mathbb{1}_{E_h^*}DG(A_h)sD\psi\,dy,$$

for every $\psi\in C_0^1(B_1;\mathbb{R}^N)$ and $s\in(0,1)$. Dividing by s and passing to the limit as $s\to0$, by the definition of $\mathcal{I}_h$, we get (see [9] or [11])

$$\begin{aligned}0\le&\frac{1}{\lambda_h}\int_{B_1}(DF(A_h+\lambda_h Dv_h)-DF(A_h))D\psi\,dy\\&+\frac{1}{\lambda_h}\int_{B_1}\mathbb{1}_{E_h^*}DG(A_h+\lambda_h Dv_h)D\psi\,dy.\end{aligned}\tag{3.29}$$

We partition the unit ball as follows:

$$B_1=\mathbf{B}_h^+\cup\mathbf{B}_h^-=\{y\in B_1:\lambda_h|Dv_h|>1\}\cup\{y\in B_1:\lambda_h|Dv_h|\le1\}.$$

By (3.7), we get

$$|\mathbf{B}_h^+| \le \int_{\mathbf{B}_h^+} \lambda_h^p |Dv_h|^p \, dy \le \lambda_h^p \int_{B_1} |Dv_h|^p \, dy \le c(n,p,M)\lambda_h^p. \tag{3.30}$$

We rewrite (3.29) as follows:

$$\begin{aligned} 0 \le & \frac{1}{\lambda_h} \int_{\mathbf{B}_h^+} (DF(A_h + \lambda_h Dv_h) - DF(A_h)) D\psi \, dy \\ & + \int_{\mathbf{B}_h^-} \int_0^1 \big(D^2F(A_h + t\lambda_h Dv_h) - D^2F(A)\big) \, dt Dv_h D\psi \, dy \\ & + \int_{\mathbf{B}_h^-} D^2F(A) Dv_h D\psi \, dy + \frac{1}{\lambda_h} \int_{B_1} \mathbb{1}_{E_h^*} DG(A_h + \lambda_h Dv_h) D\psi \, dy. \end{aligned} \tag{3.31}$$

By growth condition in (2.1) and Hölder's inequality, we get

$$\begin{aligned} & \frac{1}{\lambda_h} \left| \int_{\mathbf{B}_h^+} (DF(A_h + \lambda_h Dv_h) - DF(A_h)) D\psi \, dy \right| \\ & \le c(p, L_1, M, D\psi) \left[\frac{|\mathbf{B}_h^+|}{\lambda_h} + \lambda_h^{p-2} \int_{\mathbf{B}_h^+} |Dv_h|^{p-1} \, dy \right] \\ & \le c(n,p,L_1,M,D\psi) \left[\lambda_h^{p-1} + \lambda_h^{p-1} \left(\int_{\mathbf{B}_1} |Dv_h|^p \, dy \right)^{\frac{p-1}{p}} \left(\frac{|\mathbf{B}_h^+|}{\lambda_h^p} \right)^{\frac{1}{p}} \right] \le c(n,p,L_1,M,D\psi)\lambda_h^{p-1}, \end{aligned}$$

thanks to (3.3), (3.7) and (3.30). Thus

$$\lim_{h\to\infty} \frac{1}{\lambda_h} \left| \int_{\mathbf{B}_h^+} (DF(A_h + \lambda_h Dv_h) - DF(A_h)) D\psi \, dy \right| = 0. \tag{3.32}$$

By (3.3) and the definition of $\mathbf{B}_h^-$ we have that $|A_h + \lambda_h Dv_h| \le M+1$ on $\mathbf{B}_h^-$. Hence we estimate

$$\begin{aligned} & \left| \int_{\mathbf{B}_h^-} \int_0^1 \big(D^2F(A_h + t\lambda_h Dv_h) - D^2F(A)\big) \, dt Dv_h D\psi \, dy \right| \\ & \le \int_{\mathbf{B}_h^-} \left| \int_0^1 \big(D^2F(A_h + t\lambda_h Dv_h) - D^2F(A)\big) \, dt \right| |Dv_h||D\psi| \, dy \\ & \le \left(\int_{\mathbf{B}_h^-} \left| \int_0^1 \big(D^2F(A_h + t\lambda_h Dv_h) - D^2F(A)\big) \, dt \right|^{\frac{p}{p-1}} dy \right)^{\frac{p-1}{p}} \|Dv_h\|_{L^p(B_1)} \|D\psi\|_{L^\infty(B_1)} \\ & \le c(n,p,M,D\psi) \left(\int_{\mathbf{B}_h^-} \left| \int_0^1 \big(D^2F(A_h + t\lambda_h Dv_h) - D^2F(A)\big) \, dt \right|^{\frac{p}{p-1}} dy \right)^{\frac{p-1}{p}}, \end{aligned}$$

where we have used (3.7). Since, by (3.8), $\lambda_h Dv_h \to 0$ a.e. in B_1, the uniform continuity of D^2F on bounded sets implies that

$$\lim_{h\to\infty} \left| \int_{\mathbf{B}_h^-} \int_0^1 \big(D^2F(A_h + t\lambda_h Dv_h) - D^2F(A)\big) \, dt Dv_h D\psi \, dy \right| = 0. \tag{3.33}$$

Note that (3.30) yields that $\mathbb{1}_{\mathbf{B}_h^-} \to \mathbb{1}_{B_1}$ in $L^r(B_1)$, for every $r < \infty$. Therefore, by the weak convergence of Dv_h to Dv in $L^p(B_1)$, it follows that

$$\lim_{h\to\infty} \int_{\mathbf{B}_h^-} D^2F(A)Dv_h D\psi\, dy = \int_{B_1} D^2F(A)DvD\psi\, dy. \tag{3.34}$$

By growth condition (2.1), we deduce

$$\begin{aligned}
&\frac{1}{\lambda_h}\left|\int_{B_1} \mathbb{1}_{E_h^*}[D_\xi G(A_h+\lambda_h Dv_h)D\psi\, dy\right| \le \frac{c(p,L_2)}{\lambda_h}\int_{B_1} \mathbb{1}_{E_h^*}\big(1+|A_h+\lambda_h Dv_h|^2\big)^{\frac{p-1}{2}}|D\psi|\, dy\\
&\le c(p,L_2,M,D\psi)\left[\frac{1}{\lambda_h}|E_h^*| + \lambda_h^{p-2}\int_{E_h^*}|Dv_h|^{p-1}\, dy\right]\\
&\le c(p,L_2,M,D\psi)\left[\frac{1}{\lambda_h}|E_h^*| + \lambda_h^{p-2+\frac{2}{p}}\left(\int_{B_1}|Dv_h|^p\, dy\right)^{\frac{p-1}{p}}\left(\frac{|E_h^*|}{\lambda_h^2}\right)^{\frac{1}{p}}\right]\\
&\le c(n,p,L_2,M,D\psi)\left[\frac{1}{\lambda_h}|E_h^*| + \lambda_h^{p-2-\frac{2}{p}}\left(\frac{|E_h^*|}{\lambda_h^2}\right)^{\frac{1}{p}}\right],
\end{aligned}$$

where we have used (3.3) and (3.7). Since $\min\{|E_h^*|, |B_1\setminus E_h|\} = |E_h^*|$, by (3.10), we have

$$\lim_{h\to\infty}\frac{|E_h^*|}{\lambda_h^2} = 0,$$

and so

$$\lim_{h\to\infty}\frac{1}{\lambda_h}\int_{B_1}\mathbb{1}_{E_h^*}DG(A_h+\lambda_h Dv_h)D\psi\, dy = 0. \tag{3.35}$$

By (3.32), (3.33), (3.34) and (3.35), passing to the limit as $h\to\infty$ in (3.31), we get

$$\int_{B_1} DF(A)DvD\psi\, dy \ge 0.$$

Furthermore, plugging $-\psi$ in place of ψ, we get

$$\int_{B_1} DF(A)DvD\psi\, dy = 0,$$

i.e. v solves a linear system with constant coefficients.

Substep 3.b *The case* $\min\{|E_h^*|, |B_1\setminus E_h|\} = |B_1\setminus E_h|$.

We claim that v solves the linear system

$$\int_{B_1} D^2(F+G)(A)DvD\psi\, dy = 0,$$

for all $\psi\in C_0^1(B_1;\mathbb{R}^N)$. Dividing by s and passing to the limit as $s\to 0$, by the definition of $\mathcal{H}_h$ we get (see [9] or [11])

$$0 \le \frac{1}{\lambda_h}\int_{B_1}\big[D(F+G)(A_h+\lambda_h Dv_h)D\psi - D(F+G)(A_h)D\psi\big]dy \tag{3.36}$$

$$+ c(p, L_2, M)\left[\frac{1}{\lambda_h}\int_{B_1\setminus E_h}|D\psi|dy + \int_{B_1\setminus E_h}\lambda_h^{p-2}|Dv_h|^{p-1}|D\psi|\,dy\right].$$

As before, we partition B_1 as follows:

$$B_1 = \mathbf{B}_h^+ \cup \mathbf{B}_h^- = \{y \in B_1 :\, \lambda_h|Dv_h| > 1\} \cup \{y \in B_1 :\, \lambda_h|Dv_h| \le 1\}.$$

We rewrite (3.36) as

$$\begin{aligned}
0 \le\ & \frac{1}{\lambda_h}\int_{\mathbf{B}_h^+}(D(F+G)(A_h+\lambda_h Dv_h) - D(F+G)(A_h))D\psi\,dy \\
&+ \frac{1}{\lambda_h}\int_{\mathbf{B}_h^-}(D(F+G)(A_h+\lambda_h Dv_h) - D(F+G)(A_h))D\psi\,dy \\
&+ c(p, L_2, M)\left[\frac{1}{\lambda_h}\int_{B_1\setminus E_h}|D\psi|dy + \int_{B_1\setminus E_h}\lambda_h^{p-2}|Dv_h|^{p-1}|D\psi|\,dy\right].
\end{aligned} \tag{3.37}$$

Arguing as in (3.32), we obtain that

$$\lim_{h\to\infty}\frac{1}{\lambda_h}\left|\int_{\mathbf{B}_h^+}(D(F+G)(A_h+\lambda_h Dv_h) - D(F+G)(A_h))D\psi\,dy\right| = 0, \tag{3.38}$$

and, as in (3.33) and (3.34),

$$\begin{aligned}
&\lim_{h\to\infty}\frac{1}{\lambda_h}\int_{\mathbf{B}_h^-}[D(F+G)(A_h+\lambda_h Dv_h) - D(F+G)(A_h)]D\psi\,dy \\
&= \int_{B_1} D(F+G)(A)DvD\psi\,dy.
\end{aligned} \tag{3.39}$$

Moreover, we have that

$$\begin{aligned}
&\frac{1}{\lambda_h}\int_{B_1\setminus E_h}|D\psi|dy + \int_{B_1\setminus E_h}\lambda_h^{p-2}|Dv_h|^{p-1}|D\psi|\,dy \\
&\le c(p, D\psi)\left[\frac{|B_1\setminus E_h|}{\lambda_h} + \lambda_h^{p-2+\frac{2}{p}}\left(\int_{\mathbf{B}_1}|Dv_h|^p\,dy\right)^{\frac{p-1}{p}}\left(\frac{|B_1\setminus E_h|}{\lambda_h^2}\right)^{\frac{1}{p}}\right] \\
&\le c(n, p, D\psi)\left[\frac{|B_1\setminus E_h|}{\lambda_h} + \lambda_h^{p-2+\frac{2}{p}}\left(\frac{|B_1\setminus E_h|}{\lambda_h^2}\right)^{\frac{1}{p}}\right],
\end{aligned}$$

where we used (3.7). Since $\min\{|E_h^*|, |B_1\setminus E_h|\} = |B_1\setminus E_h|$, by (3.10), we have

$$\lim_{h\to\infty}\frac{|B_1\setminus E_h|}{\lambda_h^2} = 0,$$

and we obtain

$$\lim_{h\to\infty}\left[\frac{1}{\lambda_h}\int_{B_1\setminus E_h}|D\psi|dy + \int_{B_1\setminus E_h}\lambda_h^{p-2}|Dv_h|^{p-1}|D\psi|\,dy\right] = 0. \tag{3.40}$$

By (3.38), (3.39) and (3.40), passing to the limit as $h \to \infty$ in (3.37) we conclude that

$$\int_{B_1} D^2(F+G)(A)DvD\psi\, dy \geq 0$$

and, with $-\psi$ in place of ψ, we finally get

$$\int_{B_1} D^2(F+G)(A)DvD\psi\, dy = 0,$$

as claimed.

By Proposition 2.1 and the theory of linear systems (see [29, Theorem 2.1 and Chapter 3]), we deduce in both cases that $v \in C^\infty$ and there exists a constant $\tilde{c} = \tilde{c}(n, N, p, \ell_1, \ell_2, L_1, L_2) > 0$ such that

$$\fint_{B_\tau} |Dv - (Dv)_\tau|^2 \leq \tilde{c}\tau^2 \fint_{B_{\frac{1}{2}}} |Dv - (Dv)_{\frac{1}{2}}|^2\, dx,$$

for any $\tau \in \left(0, \frac{1}{2}\right)$. Moreover, by Proposition 2.1 again,

$$\fint_{B_{\frac{1}{2}}} |Dv - (Dv)_{\frac{1}{2}}|^2\, dx \leq \sup_{B_{\frac{1}{2}}} |Dv|^2 \leq \tilde{c}\left(\fint_{B_1} |Dv|^p\, dx\right)^{2/p}.$$

Observing that

$$\|Dv\|_{L^p(B_1)} \leq \limsup_h \|Dv_h\|_{L^p(B_1)} \leq c(n,p),$$

it follows that

$$\fint_{B_\tau} |Dv - (Dv)_\tau|^2 \leq \overline{C}\tau^2, \tag{3.41}$$

for some fixed $\overline{C} = \overline{C}(n, N, p, \ell_1, \ell_2, L_1, L_2)$.

Step 4. *An estimate for the perimeters.*

Our aim is to show that there exists a constant $c = c(n, p, L_2, \Lambda, M) > 0$ such that

$$P(E_h, B_\tau) \leq c\left[\frac{1}{\tau} P(E_h, B_1)^{\frac{n}{n-1}} + r_h\tau^n + r_h\lambda_h^p\right]. \tag{3.42}$$

By the minimality of (u, E) with respect to $(u, \widetilde{E})$, where $\widetilde{E}$ is a set of finite perimeter such that $\widetilde{E}\Delta E \Subset B_{r_h}(x_h)$ and $B_{r_h}(x_h)$ are the balls of the contradiction argument, we get

$$\int_{B_{r_h}(x_h)} \mathbb{1}_E G(Du) + \mathbf{\Phi}(E; B_{r_h}(x_h)) \leq \int_{B_{r_h}(x_h)} \mathbb{1}_{\widetilde{E}} G(Du) + \mathbf{\Phi}(\widetilde{E}; B_{r_h}(x_h)).$$

Using the change of variable $x = x_h + r_h y$ and dividing by r_h^{n-1}, we have

$$r_h \int_{B_1} \mathbb{1}_{E_h} G(A_h + \lambda_h Dv_h) dy + \mathbf{\Phi}_h(E_h; B_1) \leq r_h \int_{B_1} \mathbb{1}_{\widetilde{E}_h} G(A_h + \lambda_h Dv_h) dy + \mathbf{\Phi}_h(\widetilde{E}_h; B_1),$$

where

$$\mathbf{\Phi}_h(E_h;V) := \int_{V\cap\partial^* E_h} \Phi(x_h + r_h y, \nu_{E_h}(y))\, d\mathcal{H}^{n-1}(y),$$

for every Borel set $V \subset \Omega$. Assume first that $\min\{|B_1 \setminus E_h|, |E_h^*|\} = |B_1 \setminus E_h|$. Choosing $\widetilde{E}_h := E_h \cup B_\rho$, we get

$$\mathbf{\Phi}_h(E_h;B_1) \le r_h \int_{B_1} \mathbb{1}_{B_\rho} G(A_h + \lambda_h Dv_h)dy + \mathbf{\Phi}_h(\widetilde{E}_h;B_1). \tag{3.43}$$

By the coarea formula, the relative isoperimetric inequality, the choice of the representative $E_h^{(1)}$ of E_h, which is a Borel set, we get

$$\int_\tau^{2\tau} \mathcal{H}^{n-1}(\partial B_\rho \setminus E_h)\, d\rho \le |B_1 \setminus E_h| \le c(n) P(E_h, B_1)^{\frac{n}{n-1}}.$$

Therefore, thanks to Chebyshev's inequality, we may choose $\rho \in (\tau, 2\tau)$, independent of n, such that, up to subsequences, it holds

$$\mathcal{H}^{n-1}(\partial^* E_h \cap \partial B_\rho) = 0 \quad \text{and} \quad \mathcal{H}^{n-1}(\partial B_\rho \setminus E_h) \le \frac{c(n)}{\tau} P(E_h, B_1)^{\frac{n}{n-1}}. \tag{3.44}$$

We remark that Proposition 2.9 holds also for $\mathbf{\Phi}_h$. Thus, thanks to the choice of ρ, being $\mathcal{H}^{n-1}(\partial^* E_h \cap \partial B_\rho) = 0$, we have that

$$\begin{aligned} \mathbf{\Phi}_h(\widetilde{E}_h;B_1) &= \mathbf{\Phi}_h(E_h;B_\rho^{(0)}) + \mathbf{\Phi}_h(B_\rho;E_h^{(0)}) + \mathbf{\Phi}_h(E_h;\{\nu_{E_h} = \nu_{B_\rho}\}) \\ &= \mathbf{\Phi}_h(E_h;B_1 \setminus \overline{B_\rho}) + \mathbf{\Phi}_h(B_\rho;E_h^{(0)}). \end{aligned}$$

By the choice of the representative of E_h (see Remark 2.7), taking into account (2.8) and the inequality in (3.44), it follows that

$$\begin{aligned} \mathbf{\Phi}_h(\widetilde{E}_h;B_1) &\le \mathbf{\Phi}_h(E_h;B_1 \setminus \overline{B_\rho}) + \Lambda \mathcal{H}^{n-1}(\partial B_\rho \cap E_h^{(0)}) \\ &\le \mathbf{\Phi}_h(E_h;B_1 \setminus \overline{B_\rho}) + \Lambda \mathcal{H}^{n-1}(\partial B_\rho \setminus E_h). \\ &\le \mathbf{\Phi}_h(E_h;B_1 \setminus \overline{B_\rho}) + \Lambda \frac{c(n)}{\tau} P(E_h, B_1)^{\frac{n}{n-1}}. \end{aligned} \tag{3.45}$$

On the other hand, by (2.8) and the additivity of the measure $\mathbf{\Phi}_h(E_h, \cdot)$ it holds that

$$\frac{1}{\Lambda} P(E_h, B_\tau) \le \mathbf{\Phi}_h(E_h;B_\tau) \le \mathbf{\Phi}_h(E_h;B_1) - \mathbf{\Phi}_h(E_h;B_1 \setminus \overline{B}_\rho), \tag{3.46}$$

since $\rho > \tau$. Combining (3.43), (3.45) and (3.46), we obtain

$$\begin{aligned} \frac{1}{\Lambda} P(E_h, B_\tau) &\le \mathbf{\Phi}_h(E_h;B_1) - \mathbf{\Phi}_h(E_h;B_1 \setminus \overline{B}_\rho) \\ &\le \mathbf{\Phi}_h(\widetilde{E}_h;B_1) + r_h \int_{B_1} \mathbb{1}_{B_\rho} G(A_h + \lambda_h Dv_h)dy - \mathbf{\Phi}_h(E_h;B_1 \setminus \overline{B}_\rho) \\ &\le \Lambda \frac{c(n)}{\tau} P(E_h, B_1)^{\frac{n}{n-1}} + r_h \int_{B_1} \mathbb{1}_{B_\rho} G(A_h + \lambda_h Dv_h)dy \end{aligned} \tag{3.47}$$

$$
\begin{aligned}
&\le \Lambda\frac{c(n)}{\tau}P(E_h,B_1)^{\frac{n}{n-1}} + c(p,L_2)r_h\int_{B_{2\tau}}(1+|A_h+\lambda_h Dv_h|^2)^{\frac{p}{2}}\,dy\\
&\le \Lambda\frac{c(n)}{\tau}P(E_h,B_1)^{\frac{n}{n-1}} + c(n,p,L_2,M)r_h\tau^n + c(p,L_2)r_h\lambda_h^p\int_{B_{2\tau}}|Dv_h|^p\,dy\\
&\le \Lambda\frac{c(n)}{\tau}P(E_h,B_1)^{\frac{n}{n-1}} + c(n,p,L_2,M)r_h\tau^n + c(n,p,L_2)r_h\lambda_h^p,
\end{aligned}
$$

where we used (3.7). The previous estimate leads to (3.42). We reach the same conclusion if $\min\{|B_1\setminus E_h|,|E_h^*|\} = |E_h^*|$, choosing $\widetilde{E}_h = E_h\setminus B_\rho$ as a competitor set.

Step 5. *Higher integrability of v_h.*
We need to prove that there exist two positive constants C and δ depending on $n,p,\ell_1,\ell_2,L_1,L_2$ such that for every $B_r\subset B_1$ it holds

$$
⨍_{B_{\frac{r}{2}}}|V(\lambda_h Dv_h)|^{2(1+\delta)}\,dy \le C\left[\left(⨍_{B_1}|V(\lambda_h Dv)|^2\,dy\right)^{1+\delta}+1\right].
$$

We remark that

$$
|F_h(\xi)|+|G_h(\xi)| \le \frac{c(p,L_1,L_2,M)}{\lambda_h^2}|V(\lambda_h\xi)|^2,\quad \forall\xi\in\mathbb{R}^{n\times N},
$$

and

$$
\int_{B_1}F_h(D\phi)\,dy \ge \frac{\ell_1}{\lambda_h^2}\int_{B_1}|V(\lambda_h D\phi)|^2\,dy,\quad \forall\phi\in C_c^1(B_1,\mathbb{R}^N)
$$

Let $r>0$ be such that $B_{3r}\subset B_1$, $\frac{r}{2}<s<t<r$ and $\eta\in C_c^1(B_t)$ be such that $0\le\eta\le 1$, $\eta=1$ on B_s, $|D\eta|\le\frac{c}{t-s}$, for some positive constant c. We define

$$
\phi_1 := [v_h-(v_h)_r]\eta,\quad \phi_2 := [v_h-(v_h)_r](1-\eta).
$$

We deal with the case $\min\{|E_h^*|,|B_1\setminus E_h|\} = |E_h^*|$, the other one is similar. Using the minimality relation (3.13) and the usual growth conditions, we get

$$
\begin{aligned}
&\frac{\ell_1}{\lambda_h^2}\int_{B_t}|V(\lambda_h D\phi_1)|^2\,dy \le \int_{B_t}F_h(D\phi_1)\,dy\\
&= \int_{B_t}F_h(Dv_h)\,dy + \int_{B_t\setminus B_s}[F_h(Dv_h-D\phi_2)-F_h(Dv_h)]\,dy\\
&\le \mathcal{I}_h(v_h) + \int_{B_t\setminus B_s}[F_h(Dv_h-D\phi_2)-F_h(Dv_h)]\,dy\\
&\le \mathcal{I}_h(\phi_2) + \int_{B_t\setminus B_s}[F_h(Dv_h-D\phi_2)-F_h(Dv_h)]\,dy + \frac{1}{\lambda_h^2}\int_{B_t\cap E_h^*}DG(A_h)|D\phi_1|\,dy\\
&\le \frac{c(p,L_1,L_2,M)}{\lambda_h^2}\left[\int_{B_t\setminus B_s}\left[|V(\lambda_h D\phi_2)|^2+|V(\lambda_h D\phi_1)|^2+|V(\lambda_h Dv_h)|^2\right]dy\right.\\
&\left.+\frac{1}{\lambda_h}\int_{B_t\cap E_h^*}|D\phi_1|\,dy\right].
\end{aligned}
$$

By the properties of V, it holds that

$$|\xi| \leq C(p)\big(1 + |V(\xi)|^{\frac{2}{p}}\big), \quad \forall \xi \in \mathbb{R}^{n\times N}.$$

Thus it follows

$$\begin{aligned}
\frac{1}{\lambda_h^2}\int_{B_t\cap E_h^*} |\lambda_h D\phi_1|\,dy &\leq \frac{c(p)}{\lambda_h^2}\left[|E_h^*\cap B_t| + \int_{B_t\cap E_h^*} V(|\lambda_h D\phi_1|)^{\frac{2}{p}}\,dy\right]\\
&\leq \frac{c(p)}{\lambda_h^2}\left[c(\varepsilon)|E_h^*\cap B_t| + \varepsilon\int_{B_t\cap E_h^*} |V(\lambda_h D\phi_1)|^2\,dy\right].
\end{aligned}$$

Combining the previous two chains of inequalities, we deduce that

$$\begin{aligned}
&\frac{\ell_1}{\lambda_h^2}\int_{B_t} |V(\lambda_h D\phi_1)|^2\,dy\\
&\leq \frac{c(p,L_1,L_2,M)}{\lambda_h^2}\left[\int_{B_t\setminus B_s}\big[|V(\lambda_h D\phi_2)|^2 + |V(\lambda_h D\phi_1)|^2 + |V(\lambda_h Dv_h)|^2\big]\,dy\right.\\
&\left.+\, c(\varepsilon)|E_h^*\cap B_t| + \varepsilon\int_{B_t\cap E_h^*} |V(\lambda_h D\phi_1)|^2\,dy\right].
\end{aligned}$$

Choosing ε sufficiently small, we absorb the last integral to the left-hand side

$$\begin{aligned}
&\frac{1}{\lambda_h^2}\int_{B_t} |V(\lambda_h D\phi_1)|^2\,dy\\
&\leq \frac{c(p,\ell_1,L_1,L_2,M)}{\lambda_h^2}\left[\int_{B_t\setminus B_s}\big[|V(\lambda_h D\phi_2)|^2 + |V(\lambda_h D\phi_1)|^2 + |V(\lambda_h Dv_h)|^2\big]\,dy + |E_h^*\cap B_t|\right].
\end{aligned}$$

By *(ii)* of Lemma 2.2, it follows

$$\begin{aligned}
&\int_{B_s} |V(\lambda_h Dv_h)|^2\,dy\\
&\leq c(p,\ell_1,L_1,L_2,M)\left[\int_{B_t\setminus B_s} |V(\lambda_h Dv_h)|^2\,dy + \int_{B_t\setminus B_s}\left|V\left(\lambda_h\frac{v_h-(v_h)_r}{t-s}\right)\right|^2 dy + |E_h^*\cap B_t|\right].
\end{aligned}$$

By applying the hole-filling technique, we add $c(p,\ell_1,L_1,L_2,M)\int_{B_s}|V(\lambda_h Dv_h)|^2\,dy$, and we get

$$\begin{aligned}
&\int_{B_s} |V(\lambda_h Dv_h)|^2\,dy\\
&\leq \frac{c(p,\ell_1,L_1,L_2,M)}{c(p,\ell_1,L_1,L_2,M)+1}\left[\int_{B_t} |V(\lambda_h Dv_h)|^2\,dy + \int_{B_t\setminus B_s}\left|V\left(\lambda_h\frac{v_h-(v_h)_r}{t-s}\right)\right|^2 dy + |E_h^*\cap B_t|\right].
\end{aligned}$$

Now we can apply Lemma 2.4 and derive

$$\int_{B_{r/2}} |V(\lambda_h Dv_h)|^2\,dy \leq c(p,\ell_1,L_1,L_2,M)\left[\int_{B_r}\left|V\left(\lambda_h\frac{v_h-(v_h)_r}{r}\right)\right|^2 dy + |E_h^*\cap B_r|\right].$$

Finally, by Hölder's inequality and Theorem 2.6 we gain

$$\begin{aligned}\fint_{B_{r/2}} |V(\lambda_h Dv_h)|^2\,dy &\le c(p,\ell_1,L_1,L_2,M)\left\{\left[\int_{B_r}\left|V\left(\lambda_h\frac{v_h-(v_h)_r}{r}\right)\right|^{2(1+\sigma)}dy\right]^{\frac{1}{1+\sigma}}+|B_r|\right\}\\ &\le c(p,\ell_1,L_1,L_2,M)\left\{\left[\fint_{B_{3r}}|V(\lambda_h Dv_h)|^{\alpha}\,dy\right]^{\frac{1}{2\alpha}}+|B_r|\right\}.\end{aligned}$$

We conclude the proof by applying Theorem 6.6 in [31].

Step 6. *Conclusion.* By the change of variable $x = x_h + r_h y$, *(v)* of Lemma 2.2 and the Caccioppoli inequality in (3.16), for every $0<\tau<\frac{1}{4}$ we have

$$\begin{aligned}&\limsup_{h\to\infty}\frac{U_*(x_h,\tau r_h)}{\lambda_h^2}\\ &\le \limsup_{h\to\infty}\fint_{B_{\tau r_h}(x_0)}\big|V(Du)-V\big((Du)_{x_0,r}\big)\big|^2\,dx+\limsup_{h\to\infty}\frac{P(E,B_{\tau r_h}(x_h))}{\lambda_h^2\tau^{n-1}r_h^{n-1}}+\limsup_{h\to\infty}\frac{\tau r_h}{\lambda_h^2}\\ &\le \limsup_{h\to\infty}\frac{1}{\lambda_h^2}\fint_{B_\tau}\big|V(\lambda_h Dv_h+A_h)-V\big(A_h+\lambda_h(Dv_h)_\tau\big)\big|^2\,dy+\limsup_{h\to\infty}\frac{P(E_h,B_\tau)}{\lambda_h^2\tau^{n-1}}+\tau\\ &\le \limsup_{h\to\infty}\frac{c(n,p)}{\lambda_h^2}\fint_{B_\tau}\big|V(\lambda_h\big(Dv_h-(Dv_h)_\tau\big)\big|^2\,dy+\limsup_{h\to\infty}\frac{P(E_h,B_\tau)}{\lambda_h^2\tau^{n-1}}+\tau\\ &\le c(n,p,\ell_1,\ell_2,L_1,L_2,\Lambda,M)\left\{\limsup_{h\to\infty}\frac{1}{\lambda_h^2}\fint_{B_{2\tau}}\left|V\left(\frac{\lambda_h\big(v_h-(v_h)_{2\tau}-(Dv_h)_\tau\,y\big)}{2\tau}\right)\right|^2dy\right.\\ &\left.+\frac{1}{\tau^n}\limsup_{h\to\infty}\frac{P(E_h,B_1)^{\frac{n}{n-1}}}{\lambda_h^2}+\frac{1}{\tau^{n-1}}\limsup_{h\to\infty}\left(\frac{r_h\tau^n}{\lambda_h^2}+\frac{r_h}{\lambda_h^2}\lambda_h^p\right)+\tau\right\},\end{aligned}$$

where we have used (3.7) and estimate (3.47).

Now we want to prove that

$$\begin{aligned}&\limsup_{h\to\infty}\frac{1}{\lambda_h^2}\int_{B_{2\tau}}\left|V\left(\frac{\lambda_h\big(v_h-(v_h)_{2\tau}-(Dv_h)_\tau\,y\big)}{2\tau}\right)\right|^2dy\\ &=\limsup_{h\to\infty}\frac{1}{\lambda_h^2}\int_{B_{2\tau}}\left|V\left(\frac{\lambda_h\big(v-(v)_{2\tau}-(Dv)_\tau\,y\big)}{2\tau}\right)\right|^2dy\le\int_{B_{2\tau}}\frac{|v-(v)_{2\tau}-(Dv)_\tau y|^2}{\tau^2}\,dy,\end{aligned}$$

where we have used that v and Dv are bounded, $\lambda_h\to 0$ and $|V(\xi)|\le|\xi|$ for $|\xi|\le 1$.

In view of this aim it is enough to prove that

$$I:=\lim_{h\to\infty}\frac{1}{\lambda_h^2}\fint_{B_{2\tau}}\left|V\left(\frac{\lambda_h\big((v_h-v)-(v_h-v)_{2\tau}-(Dv_h-Dv)_\tau\,y\big)}{2\tau}\right)\right|^2dy=0.$$

In the sequel σ will denote the exponent given in the Sobolev-Poincaré type inequality of the Theorem 2.6. We can assume that the higher integrability exponent δ given in the step 5 is such that $\delta<\sigma$.

Let us choose $\theta\in(0,1)$ such that $2\theta+\frac{1-\theta}{1+\sigma}=1$. Applying Hölder's inequality, it holds that

$$0\le I\le\limsup_{h\to\infty}\frac{1}{\lambda_h^2}\left(\fint_{B_{2\tau}}\left|V\left(\frac{\lambda_h\big((v_h-v)-(v_h-v)_{2\tau}-(Dv_h-Dv)_\tau\,y\big)}{2\tau}\right)\right|dy\right)^{2\theta}$$

$$\cdot\left(\fint_{B_{2\tau}}\left|V\left(\frac{\lambda_h\big((v_h-v)-(v_h-v)_{2\tau}-(Dv_h-Dv)_\tau\, y\big)}{2\tau}\right)\right|^{2(1+\sigma)}dy\right)^{\frac{1-\theta}{1+\sigma}}.$$

Using the fact that $|V(\xi)|\le|\xi|$ and *(iii)* of Lemma 2.2, for the first factor in the previous product, and using also Theorem 2.6 applied to $(v_h-v)-(v_h-v)_{2\tau}-(Dv_h-Dv)_\tau\, y$, we deduce

$$\begin{aligned}0\le I\le \limsup_{h\to\infty}\frac{c}{\lambda_h^2}\left(\lambda_h\fint_{B_{2\tau}}\left(\left|\frac{v_h-v}{\tau}\right|+\left|\frac{(Dv_h-Dv)_\tau}{\tau}\right|\right)dy\right)^{2\theta}\\ \times\left(\fint_{B_{6\tau}}\big|V\big(\lambda_h(Dv_h-Dv)\big)\big|^{\alpha}dy\right)^{\frac{2(1-\theta)}{\alpha}}.\end{aligned}$$

In the last term we can increase choosing $\alpha=2$ and accordingly, observing that

$$\int_{B_1}\big|V\big(\lambda_h Dv_h\big)\big|^2dy\le c(n)\lambda_h^2,$$

we conclude that

$$\begin{aligned}0\le I&\le \lim_{h\to\infty}\frac{c}{\lambda_h^2}\lambda_h^{2\theta}\left(\fint_{B_{2\tau}}\left(\left|\frac{v_h-v}{\tau}\right|+\left|\frac{(Dv_h-Dv)_\tau}{\tau}\right|\right)dy\right)^{2\theta}\cdot c\lambda_h^{2(1-\theta)}\\ &=\lim_{h\to\infty}C\left(\fint_{B_{2\tau}}\big(|v_h-v|+|(Dv_h-Dv)_\tau|\big)\,dy\right)^{2\theta}=0.\end{aligned}$$

By virtue of (3.7), (3.9), (3.10), (3.41), the Poincaré-Wirtinger inequality and (3.41), we get

$$\begin{aligned}\limsup_{h\to\infty}\frac{U_*(x_h,\tau r_h)}{\lambda_h^2}&\le c(n,p,\ell_1,\ell_2,L_2,\Lambda,M)\left\{\fint_{B_{2\tau}}\frac{|v-(v)_{2\tau}-(Dv)_\tau y|^2}{\tau^2}\,dy+\tau\right\}\\ &\le c(n,p,\ell_1,\ell_2,L_2,\Lambda,M)\left\{\fint_{B_{2\tau}}|Dv-(Dv)_\tau|^2\,dy+\tau\right\}\\ &\le c(n,N,p,\ell_1,\ell_2,L_1,L_2,\Lambda,M)\big[\tau^2+\tau\big]\le C(n,N,p,\ell_1,\ell_2,L_1,L_2,\Lambda,M)\tau.\end{aligned}$$

The contradiction follows, by choosing C_* such that $C_*>C$, since, by (3.4),

$$\liminf_h\frac{U_*(x_h,\tau r_h)}{\lambda_h^2}\ge C_*\tau.$$

□

If assumption (H) is not taken into account, it is still possible to establish a decay result for the excess, analogous to the previous one. However, this requires employing a modified "hybrid" excess, defined as:

$$U_{**}(x_0,r):=U(x_0,r)+\left(\frac{P(E,B_r(x_0))}{r^{n-1}}\right)^{\frac{\delta}{1+\delta}}+r^{\beta},$$

where $U(x_0,r)$ is defined in (3.1), δ is the higher integrability exponent given in the Step 5 of Proposition 3.1 and $0<\beta<\frac{\delta}{1+\delta}$. The following result still holds true.

Proposition 3.2. *Let* (u, E) *be a local minimizer of* $\mathcal{I}$ *in* (1.2) *under the assumptions* $(F1)$, $(F2)$, $(G1)$ *and* $(G2)$. *For every* $M > 0$ *and* $0 < \tau < \frac{1}{4}$, *there exist two positive constants* $\varepsilon_0 = \varepsilon_0(\tau, M)$ *and* $c_{**} = c_{**}(n, p, \ell_1, \ell_2, L_1, L_2, \Lambda, \delta, M)$ *for which, whenever* $B_r(x_0) \Subset \Omega$ *verifies*

$$|(Du)_{x_0,r}| \leq M \quad \text{and} \quad U_{**}(x_0, r) \leq \epsilon_0,$$

then

$$U_{**}(x_0, \tau r) \leq c_{**}\, \tau^{\beta}\, U_{**}(x_0, r).$$

In order to avoid unnecessary repetition we do not include the proof here, as it is almost identical to the proof of the Proposition 3.1, with the obvious adjustments, see [9].

4 Proof of the Main Theorem

Here we give the proof of Theorem 1.2 through a suitable iteration procedure. It is easy to show the validity of the following lemma by arguing exactly in the same way as in [11, Lemma 6.1].

Lemma 4.1. *Let* (u, E) *be a minimizer of the functional* $\mathcal{I}$ *and let* c_* *the constant introduced in Proposition 3.1. For every* $\alpha \in (0, 1)$ *and* $M > 0$ *there exists* $\vartheta_0 = \vartheta_0(c_*, \alpha) < 1$ *such that for* $\vartheta \in (0, \vartheta_0)$ *there exists a positive constant* $\varepsilon_1 = \varepsilon_1(n, p, \ell_1, \ell_2, L_1, L_2, M, \vartheta)$ *such that, if* $B_r(x_0) \Subset \Omega$,

$$|Du|_{x_0,r} < M \quad \textit{and} \quad U_*(x_0, r) < \varepsilon_1,$$

then

$$|Du|_{x_0,\vartheta^h r} < 2M \quad \textit{and} \quad U_*(x_0, \vartheta^h r) \leq \vartheta^{h\alpha} U_*(x_0, r), \quad \forall h \in \mathbb{N}_0. \tag{4.1}$$

Proof. Let $M > 0$, $\alpha \in (0, 1)$ and $\vartheta \in (0, \vartheta_0)$, where $\vartheta_0 < 1$. Let $\varepsilon_1 < \varepsilon_0$, where ε_0 is the constant appearing in Proposition 3.1. We first prove by induction that

$$|Du|_{x_0,\vartheta^h r} < 2M, \quad \forall h \in \mathbb{N}_0. \tag{4.2}$$

If $h = 0$, the statement holds. Assuming that (4.2) holds for $h > 0$, applying properties *(i)* and *(vi)* of Lemma 2.2, we compute:

$$\begin{aligned}
|Du|_{x_0,\vartheta^{h+1} r} &\leq |Du|_{x_0,r} + \sum_{j=1}^{h+1} \big||Du|_{x_0,\vartheta^j r} - |Du|_{x_0,\vartheta^{j-1} r}\big| \\
&\leq M + \sum_{j=1}^{h+1} \fint_{B_{\vartheta^j r}} |Du - (Du)_{x_0,\vartheta^{j-1} r}|\, dx \\
&\leq M + \vartheta^{-n} \sum_{j=1}^{h+1} \Bigg[\frac{1}{|B_{\vartheta^{j-1} r}|} \int_{B_{\vartheta^{j-1} r} \cap \{|Du - (Du)_{x_0,\vartheta^{j-1} r}| \leq 1\}} |Du - (Du)_{x_0,\vartheta^{j-1} r}|\, dx \\
&\quad + \frac{1}{|B_{\vartheta^{j-1} r}|} \int_{B_{\vartheta^{j-1} r} \cap \{|Du - (Du)_{x_0,\vartheta^{j-1} r}| > 1\}} |Du - (Du)_{x_0,\vartheta^{j-1} r}|\, dx \Bigg] \\
&\leq M + \vartheta^{-n} \sum_{j=1}^{h+1} \Bigg[\left(\fint_{B_{\vartheta^{j-1} r}} |V(Du - (Du)_{x_0,\vartheta^{j-1} r})|^2\, dx \right)^{\frac{1}{2}}
\end{aligned}$$

$$+ \left(\fint_{B_{\vartheta^{j-1}r}} |V(Du - (Du)_{x_0,\vartheta^{j-1}r})|^2 \, dx \right)^{\frac{1}{p}} \Bigg]$$

$$\leq M + c(p, M)\vartheta^{-n} \sum_{j=1}^{h+1} \left[U_*(x_0, \vartheta^{j-1}r)^{\frac{1}{2}} + U_*(x_0, \vartheta^{j-1}r)^{\frac{1}{p}} \right]$$

$$\leq M + c(p, c_*, M)\varepsilon_1^{\frac{1}{2}} \vartheta^{-n} \sum_{j=1}^{h+1} \vartheta^{\frac{j-1}{2}} \leq M + c(p, c_*, M)\varepsilon_1^{\frac{1}{2}} \frac{\vartheta^{-n}}{1 - \vartheta^{\frac{1}{2}}} \leq 2M,$$

where we have chosen $\varepsilon_1 = \varepsilon_1(p, c_*, M, \vartheta) > 0$ sufficiently small. Now we prove the second inequality in (4.1). The statement is obvious for $h = 0$. If $h > 0$ and (4.1) holds, we have that

$$U_*(x_0, \vartheta^h r) \leq \vartheta^{h\alpha} U_*(x_0, r) < \varepsilon_1, \tag{4.3}$$

by our choice of ϑ and ε_1. Thus thanks to (4.2) we can apply Proposition 3.1 with $\vartheta^h r$ in place of r to deduce that

$$U_*(x_0, \vartheta^{h+1}r) \leq \vartheta^\alpha U_*(x_0, \vartheta^h r) \leq \vartheta^{(h+1)\alpha} U_*(x_0, r),$$

where we have chosen $\vartheta_0 = \vartheta_0(c_*, \alpha)$ sufficiently small and we have used (4.3). Therefore, the second inequality in (4.1) is also true for every $k \in \mathbb{N}$. □

Analogously, it is possible to prove an iteration lemma for U_{**}.

Lemma 4.2. *Let (u, E) be a minimizer of the functional $\mathcal{I}$ and let β be the exponent of Proposition 3.2. For every $M > 0$ and $\vartheta \in (0, \vartheta_0)$, with $\vartheta_0 < \min\left\{c_{**}, \frac{1}{4}\right\}$, there exist $\varepsilon_1 > 0$ and $R > 0$ such that, if $r < R$ and $x_0 \in \Omega$ satisfy*

$$B_r(x_0) \Subset \Omega, \quad |Du|_{x_0,r} < M \quad \textit{and} \quad U_{**}(x_0, r) < \varepsilon_1,$$

where c_{**} is the constant introduced in Proposition 3.2, then

$$|Du|_{x_0,\vartheta^h r} < 2M \quad \textit{and} \quad U_{**}(x_0, \vartheta^k r) \leq \vartheta^{k\beta} U_{**}(x_0, r), \quad \forall k \in \mathbb{N}.$$

Proof of Theorem 1.2. We consider the set

$$\Omega_1 := \left\{ x \in \Omega : \limsup_{\rho \to 0} |(Du)_{x,\rho}| < \infty \text{ and } \limsup_{\rho \to 0} U_*(x, \rho) = 0 \right\}$$

and let $x_0 \in \Omega_1$. For every $M > 0$ and for ε_1 determined in Lemma 4.1 there exists a radius $R_{M,\varepsilon_1} > 0$ such that

$$|Du|_{x_0,r} < M \quad \text{and} \quad U_*(x_0, r) < \varepsilon_1,$$

for every $0 < r < R_{M,\varepsilon_1}$. Let $0 < \rho < \vartheta r < R$ and $h \in \mathbb{N}$ be such that $\vartheta^{h+1} r < \rho < \vartheta^h r$, where $\vartheta = \frac{\vartheta_0}{2}$ and ϑ_0 is the same constant appearing in Lemma 4.1. By Lemma 4.1, we obtain

$$|Du|_{x_0,\rho} \leq \frac{1}{\vartheta^n} |Du|_{x_0,\vartheta^h r} \leq c(M, c_*, \alpha).$$

Using *(iv)* of Lemma 2.2 and reasoning as in the proof of Lemma 4.1, we estimate

$$\begin{aligned}|V((Du)_{x_0,\vartheta^h r}) - V((Du)_{x_0,\rho})|^2 &\leq c(n,p)|(Du)_{x_0,\vartheta^h r} - (Du)_{x_0,\rho}|^2 \\ &\leq c(n,p,c_*,M)\vartheta_0^{-2n}\vartheta^{h\alpha}U_*(x_0,r).\end{aligned}$$

Thus, taking the previous two inequalities into account, applying again Lemma 4.1, we estimate

$$\begin{aligned}U_*(x_0,\rho) &\leq 2\fint_{B_\rho(x_0)} |Du - (Du)_{x_0,\vartheta^h r}|^2\,dx + 2|(Du)_{x_0,\vartheta^h r} - (Du)_{x_0,\rho}|^2 + \frac{P(E,B_\rho(x_0))}{\rho^{n-1}} + \rho \\ &\leq c(n,p,M,c_*\vartheta_0)\left[\fint_{B_{\vartheta^h r}(x_0)} |Du - (Du)_{x_0,\vartheta^h r}|^2\,dx + \vartheta^{h\alpha}U_*(x_0,r) + \frac{P(E,B_{\vartheta^h r}(x_0))}{(\vartheta^h r)^{n-1}} + \vartheta^h r\right] \\ &\leq c(n,p,c_*,M,\vartheta_0)\big[U_*(x_0,\vartheta^h r) + \vartheta^{h\alpha}U_*(x_0,r)\big] \leq c(n,p,c_*,M,\vartheta_0)\left(\frac{\rho}{r}\right)^\alpha U_*(x_0,r).\end{aligned}$$

The previous estimate implies that

$$U(x_0,\rho) \leq C_*\left(\frac{\rho}{r}\right)^\alpha U_*(x_0,r),$$

where $C_* = C_*(n,p,c_*,M,\vartheta_0)$. Since $U_*(y,r)$ is continuous in y, we have that $U_*(y,r) < \varepsilon_1$ for every y in a suitable neighborhood I of x_0. Therefore, for every $y \in I$ we have that

$$U(y,\rho) \leq C_*\left(\frac{\rho}{r}\right)^\alpha U_*(y,r).$$

The last inequality implies, by the Campanato characterization of Hölder continuous functions (see [31, Theorem 2.9]), that u is $C^{1,\alpha}$ in I for every $0 < \alpha < \frac{1}{2}$, and we can conclude that the set Ω_1 is open and the function u has Hölder continuous derivatives in Ω_1.

When the assumption (H) is not enforced, the proof goes exactly in the same way provided we use Lemma 4.2 in place of Lemma 4.1, with

$$\Omega_0 := \left\{x \in \Omega : \limsup_{\rho\to 0} |(Du)_{x_0,\rho}| < \infty \text{ and } \limsup_{\rho\to 0} U_{**}(x_0,\rho) = 0\right\}.$$

□

Acknowledgments: The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and wish to acknowledge financial support from INdAM GNAMPA Project 2024 "Regolarità per problemi a frontiera libera e disuguaglianze funzionali in contesto finsleriano".

Conflict of interest: The authors state no conflict of interest.

References

[1] E. Acerbi and N. Fusco, An approximation lemma for $W^{1;p}$ functions, *Material instabilities in continuum mechanics* (Edinburgh, 19851986), Oxford Sci. Publ., Oxford Univ. Press, New York (1988), 1–5.

[2] E. Acerbi and N. Fusco, A regularity theorem for minimizers of quasiconvex integrals, *Arch. Ration. Mech. Anal.* **99** (1987), 261–281.

[3] F.J. Almgren, Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure, *Ann. Math.* **87** (1968), 321–391.

[4] L. Ambrosio and G. Buttazzo, An optimal design problem with perimeter penalization. *Calc. Var. Partial Differential Equations* **1** (1993), 55–69.

[5] A. Arroyo-Rabasa, Regularity for free interface variational problems in a general class of gradients, *Calc. Var. Partial Differential Equations* **55** (2016), 1–44.

[6] G. Bellettini, M. Novaga and M. Paolini, On a Crystalline Variational Problem, Part I: First Variation and Global L^∞ Regularity, *Arch. Rational Mech. Anal.* **157** (2001), 165–191.

[7] G. Bellettini, M. Novaga and M. Paolini, On a Crystalline Variational Problem, Part II: BV Regularity and Structure of Minimizers on Facets, *Arch. Rational Mech. Anal.* **157** (2001), 193–217.

[8] E. Bombieri, Regularity theory for almost minimal currents, *Arch. Ration. Mech. Anal.* **78** (1982), 99–130.

[9] M. Carozza, L. Esposito and L. Lamberti, Quasiconvex Bulk and Surface Energies: $C^{1,\alpha}$ Regularity, *Adv. Nonlinear Anal.* **13**(1) (2024), 20240021.

[10] M. Carozza, I. Fonseca and A. Passarelli Di Napoli, Regularity results for an optimal design problem with a volume constraint, *ESAIM Control Optim. Calc. Var.* **20** no. 2, (2014), 460–487.

[11] M. Carozza, I. Fonseca and A. Passarelli Di Napoli, Regularity results for an optimal design problem with quasiconvex bulk energies, *Calc. Var. Partial Differential Equations* **57** (2018), 68.

[12] M. Carozza, N. Fusco and G. Mingione, Partial Regularity of Minimizers of Quasiconvex Integrals with Subquadratic Growth, *Ann. Mat. Pura Appl.* **175** (1998), 141–164.

[13] M. Carozza and G. Mingione, Partial regularity of minimizers of quasiconvex integrals of subquadratic growth: the general case, *Ann. Pol. Math.* **77** (2001), 219–243.

[14] M. Carozza and A. Passarelli Di Napoli, A regularity theorem for minimisers of quasiconvex integrals: The case $1 < p < 2$, *Proc. Roy. Soc. Edinburgh Sect. A* **126** (1996), 1181–1200.

[15] M. Carozza and A. Passarelli Di Napoli, Partial regularity of local minimizers of quasiconvex integrals with sub-quadratic growth. *Proc. Roy. Soc. Edinburgh Sect. A* **133** (2003), 1249–1262.

[16] R. Choksi, R. Neumayer and I. Topaloglu, Anisotropic liquid drop models, *Adv. Calc. Var.* **15**, no. 1, (2022), 109–131.

[17] G. De Philippis and A. Figalli, A note on the dimension of the singular set in free interface problems, *Differ. Integral Equ.* **28** (2015), 523–536.

[18] G. De Philippis, N. Fusco and M. Morini, Regularity of capillarity droplets with obstacle, preprint (2022), http://arxiv.org/abs/2205.15272.

[19] G. De Philippis and F. Maggi, Dimensional estimates for singular sets in geometric variational problems with free boundaries, *J. Reine Angew. Math.* **725** (2017), 217–234.

[20] F.Duzaar and K.Steffen, Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals, *J. Reine Angew. Math.*, **546** (2002), 73–138.

[21] L. Esposito, Density lower bound estimate for local minimizer of free interface problem with volume constraint, *Ric. di Mat.* **68**, no. 2, (2019), 359–373.

[22] L. Esposito and N. Fusco, A remark on a free interface problem with volume constraint, *J. Convex Anal.* **18**, no. 2, (2011), 417–426.

[23] L. Esposito and L. Lamberti, Regularity Results for an Optimal Design Problem with lower order terms, *Adv. Calc. Var.* (2022).

[24] L. Esposito and L. Lamberti, Regularity results for a free interface problem with Hölder coefficients, *Calc. Var. Partial Differential Equations* **62** (2023), 156.

[25] L. Esposito, L. Lamberti and G. Pisante, Epsilon-regularity for almost-minimizers of anisotropic free interface problem with Hölder dependence on the position. *Interfaces Free Bound* (2024), published online first.

[26] A. Figalli, Regularity of codimension-1 minimizing currents under minimal assumptions on the integrand, *J. Differential Geom.* **106** (2017), 371–391.

[27] A. Figalli and F. Maggi, On the Shape of Liquid Drops and Crystals in the Small Mass Regime, *Arch. Rational Mech. Anal.* **201** (2011), 143–207.

[28] N. Fusco and V. Julin, On the regularity of critical and minimal sets of a free interface problem, *Interfaces Free Bound.* **17**, no.1, (2015), 117–142.

[29] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Annals of Mathematical Studies **105**, Princeton University Press, Princeton NJ, 1983.

[30] M. Giaquinta and G. Modica, *Partial regularity of minimizers of quasiconvex integrals.* Ann. Inst. H. Poincaré, Anal. Non Linéaire **3** (1986), 185–208.

[31] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific, 2003.

[32] M. Gurtin, On phase transitions with bulk, interfacial, and boundary energy, *Arch. Ration. Mech. Anal.* **96** (1986), 243–264.

[33] L. Lamberti, A regularity result for minimal configurations of a free interface problem, *Boll. Un. Mat. Ital.* **14** (2021), 521–539.

[34] F.H. Lin, Variational problems with free interfaces, *Calc. Var. Partial Differential Equations* **1** (1993), 149–168.

[35] F.H. Lin and R.V. Kohn, Partial regularity for optimal design problems involving both bulk and surface energies, *Chinese Ann. of Math. Ser. B* **20**, n.2, (1999), 137–158.

[36] F. Maggi, *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory*, Cambridge University Press, 2012.

[37] P. Marcellini, Approximation of quasiconvex functionals and lower semicontinuity of multiple integrals, *Manuscripta Math.* **51** (1985), 1–28.

[38] R. Schoen and L. Simon, A new proof of the regularity theorem for rectifiable currents which minimize parametric elliptic functionals, *Indiana Univ. Math. J.* **31** (1982), 415–434.

[39] D. A. Simmons, Regularity of almost-minimizers of Hölder-coefficient surface energies, *Discrete. Contin. Dyn. Syst.* **42**(7) (2022), 3233–3299.

Menita Carozza, Dipartimento di Ingegneria, Università del Sannio, Corso Garibaldi, Benevento 82100, Italy
carozza@unisannio.it

Luca Esposito, Dipartimento di Matematica, Università degli Studi di Salerno, Via Giovanni Paolo II 132, Fisciano 84084, Italy
luesposi@unisa.it

Lorenzo Lamberti, Dipartimento di Matematica, Università degli Studi di Salerno, Via Giovanni Paolo II 132, Fisciano 84084, Italy
llamberti@unisa.it