L^∞ energies on discontinuous functions

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Abstract

We study necessary and sufficient conditions for the lower-semicontinuity of one-dimensional energies defined on (BV and) SBV of the model form $F(u) = \sup f(u') \vee \sup g([u])$, and prove a relaxation theorem. We apply these results to the study of problems with Dirichlet boundary conditions, highlighting a complex behaviour of solutions. We draw a comparison with the parallel theory for integral energies on SBV.

1 Introduction

Many interesting phenomena of a variational nature present a complex dependence on boundary conditions or forcing terms, highlighting the relevant effect of non-convex energies. Among the many examples, in the framework of the so-called free-discontinuity problems we point out variational theories of softening and fracture (see e.g. [10] and [13]) and models in Computer Vision (see e.g. [16]). The study of problems involving such types of energies has been fruitfully addressed in the framework of the direct methods of the Calculus of Variations by characterizing some classes of non-convex functionals that are lower semicontinuous on spaces of discontinuous functions; namely, BV spaces or Ambrosio and De Giorgi's SBV spaces (see [1]). In one space dimension we can think of SBV functions as piecewise-Sobolev functions; if we denote by $S(u)$ the set of discontinuity points of an SBV function u, then the typical shape of such functionals is

$$\int_\Omega f(u')\,dt + \sum_{S(u)\cap\Omega} g([u]), \tag{1.1}$$

where $[u](t) = u(t+) - u(t-)$ is the jump of u across the point $t \in S(u)$ and u' is the derivative of u, that is defined almost everywhere on Ω. In order for such energies (or more precisely their extensions on BV) to be lower-semicontinuous in an appropriate topology, the necessary and sufficient conditions are of two types: (i) *structure conditions* on f and g (namely, that f be convex and g be subadditive; i.e. that $g(a+b) \leq g(a) + g(b)$); (ii) *compatibility conditions* between the growth of g at 0 and of f at infinity; i.e.,

$$\lim_{z\to\pm\infty} \frac{f(z)}{z} = \lim_{z\to 0\pm} \frac{g(z)}{z}.$$

Note that these two conditions do not imply that the energy (1.1) be convex, and in particular they allow for complex non-monotone behaviour with respect to the boundary data. We refer, for example, to [6] and [18] for an analysis of the behaviour of (local) minima with Dirichlet boundary conditions and their interpretation in terms of softening and fracture. Note that similar behaviours can be obtained with 'asymptotically equivalent' energies of different shapes involving scale parameters (see [4] for a survey on this subject).

Recently, other types of functionals of a different nature that allow for a non-convex behaviour have been studied by considering energies of the sup norm. The simplest type of such energies is defined on $W^{1,\infty}(\Omega)$ and takes the form

$$\sup_{x\in\Omega} f(\nabla u(x)) \tag{1.2}$$

(by sup we mean the essential supremum). This type of functionals with $f(z) = |z|$ arises naturally, for example, when looking for the best Lipschitz extension of a function defined on $\partial\Omega$ (see e.g. [3, 7]).

Necessary and sufficient conditions for the lower semicontinuity of functionals of this type have been studied in [2, 7], where it is shown that a necessary condition is that the sub-level sets of f be convex or, equivalently, that f satisfy

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \tag{1.3}$$

for all $t \in [0,1]$. This property is called the *level convexity* or *quasi-convexity* of f. Note that level convexity is implied by convexity. In one space dimension this property is equivalent to being either monotone or decreasing-increasing; in particular this class of f contains many more functions than just convex functions.

An interesting remark, observed in [14] for convex f and subsequently proved in [11] for f superlinear at infinity, is that the functional in (1.2) is the Γ-limit of the power-law integral energies (for simplicity f is considered positive)

$$F_p(u) = \Big(\int_\Omega (f(\nabla u(x)))^p\,dx\Big)^{1/p}, \tag{1.4}$$

and hence it can be used to deduce 'approximate variational principles' for F_p for p large. This observation extends to more complex integrands and in itself is a justification of the study of the energies in (1.2). As an example of an application to dielectic breakdown following this approach we refer to [14].

In this work, we analyze the structure of minimum problems of the sup norm for discontinuous functions, in the simple case of one space dimension. The goal of the paper is to show that some energies of the sup norm are meaningful also in BV and SBV spaces, that necessary and sufficient conditions for lower semicontinuity can be easily described and compared with the corresponding condition for integral functionals, and that the solutions of general minimum problems can be described by relaxation showing interesting new behaviours.

We first extend the definition in (1.2) to functions in SBV in parallel with (1.1), by considering

$$F(u) = \max\Big\{\sup_{t\in\Omega} f(u'(t)),\ \sup_{t\in S(u)\cap\Omega} g([u](t))\Big\}. \tag{1.5}$$

A technical issue must be mentioned at this point: since the lower-semicontinuity properties of L^∞ energies are invariant under composition of the energy densities with bijective increasing functions, it is not restrictive to impose a one-sided growth condition on f (namely, that $f(z) \to +\infty$ as $z \to -\infty$), that automatically also implies that $g(z) = +\infty$ for $z < 0$. Our model energy densities f are then Lennard-Jones type potentials (the case of monotone f being less interesting).

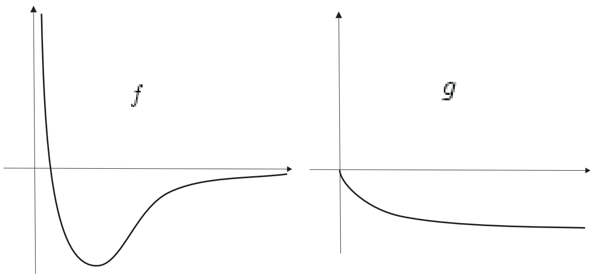

Figure 1: The functions f and g

In parallel with the theory for the energies in (1.1), we then show that necessary and sufficient conditions for the lower semicontinuity of F are again of two types:
(i) *structure conditions* on f and g. Namely, that f be level convex and g be *sub-maximal*; i.e. that

$$g(a+b) \leq \max\{g(a), g(b)\}; \tag{1.6}$$

(ii) *compatibility conditions* between the growth of g at 0 and of f at infinity:

$$\lim_{z\to 0+} g(z) = \lim_{z\to +\infty} f(z). \tag{1.7}$$

It must be remarked that while level convexity is implied by convexity, the new condition of 'sub-maximality' of g implies its sub-additivity. A simple class of sub-maximal g are decreasing functions, but much more complex shapes can be exhibited. In the case of f of Lennard-Jones type and g decreasing as in Figure 1 we can plot as an example the minimal energy $m(d)$ of the problems

$$m(d) = \min\{F(u) : u(0) = 0,\ u(1) = d\}, \tag{1.8}$$

obtaining a graph as in Figure 2.

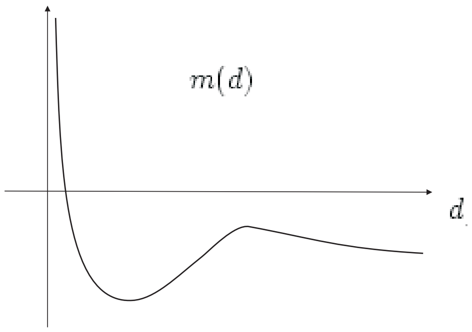

Figure 2: Non-monotone dependence on boundary conditions

In order to study the structure of solutions of minimum problems for general non-lower semicontinuous F, we prove a relaxation theorem showing that the L^1-lower semicontinuous envelope of such F is a functional of the same form with f and g substituted by the suitably defined level-convex and sub-maximal envelopes, respectively. A simple formula is obtained when g is a level-convex function itself, in which case the sub-maximal envelope is simply $\inf_k g(x/k)$.

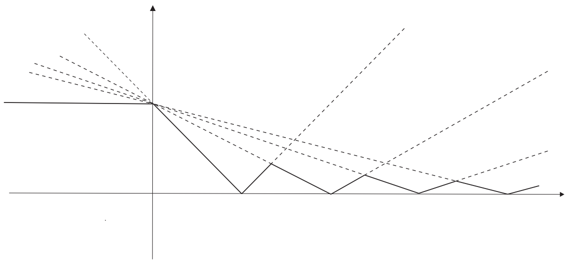

Figure 3: A non-trivial sub-maximal envelope

By plotting the 'stress-strain' curve relating the bulk gradient of the solutions to the boundary datum we highlight a 'multiple cracking' phenomenon analogous to that observed for non-subadditive free-discontinuity integral energies (see the concluding section).

2 Preliminaries

The integer part of t will be denoted by $[t]$. We will sometimes write $a\vee b = \max\{a, b\}$, $a\wedge b = \min\{a, b\}$, $\bigvee_i a_i = \sup\{a_i\}$, and so on. We will use standard notation for Lebesgue and Sobolev spaces. If μ is a measure and f a real function μ-sup f denotes the essential supremum of the values of f with respect to the measure μ. With a little abuse of notation, we simply write $\sup f$ if μ is the Lebesgue measure. We write l.s.c. as a shorthand for 'lower semicontinuous'.

We first recall the notion of level convexity (also referred to as *quasiconvexity* – not to be confused with Morrey's quasiconvexity – in part of the literature). A Borel function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *level convex* if for any $t \in \overline{\mathbb{R}}$ the sub-level set $\{x \in \mathbb{R}^n : \ f(x) \leq t\}$ is convex. We remark that level convexity can be equivalently stated by requiring the condition:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq f(x_1) \vee f(x_2) \quad \forall \lambda \in (0,1),\ x_1, x_2 \in \mathbb{R}^n.$$

In the one-dimensional case $n = 1$, this condition is equivalent to f being either monotone or decreasing in $(-\infty, t_0]$ and increasing in $(t_0, +\infty)$ for some $t_0 \in \mathbb{R}$ (we label this situation as a decreasing/increasing f, for short). Moreover we recall the analogue of Jensen's inequality for level convex functions (see [17]).

Theorem 2.1 *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be a lower-semicontinuous and level-convex function and let μ be a probability measure on $\mathbb{R}$. Then for every function $u \in L^1_\mu(\mathbb{R})$*

$$f\Big(\int_{\mathbb{R}} u \ d\mu\Big) \leq \mu\text{-}\sup(f \circ u).$$

In what follows $I = (a, b)$ is a bounded open interval in $\mathbb{R}$. Let $F : L^\infty(I) \to \overline{\mathbb{R}}$ be defined as follows:

$$F(u) = \sup_{t \in I} f(u(t)). \tag{2.9}$$

The following theorems, concerning the lower-semicontinuity and relaxation of L^∞-functionals on $L^\infty(I)$, have been proved in [17].

Theorem 2.2 *F is L^∞ weakly*-lower semicontinuous if and only if f is lower semicontinuous and level convex.*

Theorem 2.3 *Let F be as in* (2.9)*. Then the lower-semicontinuous envelope of F with respect to the weak*-topology of $L^\infty(I)$ is given by*

$$\overline{F}(u) = \sup_{t \in I} f^{\mathrm{lc}}(u(t))$$

where $f^{\mathrm{lc}}(z) = \sup\{g(z) : \ g$ is lower semicontinuous and level convex, $g \leq f\}$ is the lower-semicontinuous and level-convex envelope of f.

We finally recall an extension of L^∞ energies on gradients in a BV setting (see Section 4 for definitions).

Remark 2.4 Let f be a lower-semicontinuous level-convex function. Upon changing $f(z)$ into $f(-z)$, we can suppose that

$$\lim_{z \to -\infty} f(z) = \sup f.$$

Let $K \in \overline{\mathbb{R}}$, and let $F_\infty : L^1_{\mathrm{loc}}(I) \to \overline{\mathbb{R}}$ be defined as

$$F_\infty(u) = \begin{cases} \sup\limits_{t \in I} f(u'(t)) & \text{if } u \in W^{1,1}_{\mathrm{loc}}(I) \\ \max\{\sup\limits_{t \in I} f(u'(t)), C_\infty\} & \text{if } u \in BV^+_{\mathrm{loc}}(I) \setminus W^{1,1}_{\mathrm{loc}}(I) \\ K & \text{otherwise.} \end{cases} \tag{2.10}$$

Thanks to the results in [15], F_∞ is l.s.c. with respect to $L^1_{\mathrm{loc}}(I)$ convergence if and only if $K = \sup f$ and $C_\infty = \lim\limits_{z \to +\infty} f(z)$.

3 Sub-maximality

In this section we define the notion of sub-maximality which is related to the lower semicontinuity of L^∞ energies defined on piecewise-constant functions.

We say that a function $u : (a,b) \to \mathbb{R}$ is *piecewise constant* on (a,b) if there exist points $a = t_0 < t_1 < \ldots < t_N < t_{N+1} = b$ such that

$$u(t) \text{ is constant a.e. on } (t_{i-1}, t_i) \text{ for all } i = 1, 2, \ldots, N+1. \tag{3.11}$$

We define $S(u)$ as the minimal set $\{t_1, t_2, \ldots, t_N\} \subset (a,b)$ such that (3.11) holds. The subspace of $L^\infty(a,b)$ of all such u is denoted by $PC(a,b)$.

In this section, we will study functionals $H(u) : PC(I) \to \overline{\mathbb{R}}$ of the form

$$H(u) = \sup_{t \in S(u)} g([u](t)), \tag{3.12}$$

where $g : \mathbb{R} \setminus \{0\} \to \overline{\mathbb{R}}$.

We say that a function $g : \mathbb{R} \setminus \{0\} \to \overline{\mathbb{R}}$ is *sub-maximal* if the following inequality holds true for all $x_1, x_2 \in \mathbb{R} \setminus \{0\}$ such that $x_1 \neq -x_2$:

$$g(x_1 + x_2) \leq \max\{g(x_1), g(x_2)\}. \tag{3.13}$$

We will also consider functions $g : (0, +\infty) \to \overline{\mathbb{R}}$ or $g : (-\infty, 0) \to \overline{\mathbb{R}}$ for which the definition extends likewise.

In the sequel we sometimes prefer to consider g as defined on the whole $\mathbb{R}$, even though its value in 0 is never taken into account, by setting $g(0) = \inf g$. Note that g is sub-maximal and lower semicontinuous in $\mathbb{R} \setminus \{0\}$ if and only if such its extension is sub-maximal and lower semicontinuous.

The following theorem holds true.

Theorem 3.1 (**Semicontinuity**) *H is lower semicontinuous with respect to the $L^1(I)$-convergence if and only if g is lower semicontinuous and sub-maximal.*

Proof. First suppose H to be semicontinuous. To prove the lower semicontinuity of g, let (w_n) be a sequence of real numbers converging to w. Let $u_h(t)$ be defined as

$$u_h(t) = \begin{cases} z & \text{if } t \leq t_0 \\ z + w_h & \text{if } t > t_0. \end{cases}$$

As u_h converges to

$$u(t) = \begin{cases} z & \text{if } t \leq t_0 \\ z + w & \text{if } t > t_0 \end{cases}$$

in $L^\infty(I)$, we get

$$g(w) \leq \liminf_{h \to +\infty} g(w_h). \tag{3.14}$$

Consider now u_h be defined as

$$u_h(t) = \begin{cases} z & \text{if } t \leq t_0 \\ z + w_1 & \text{if } t_0 < t \leq t_0 + \frac{1}{h} \\ z + w_1 + w_2 & \text{if } t > t_0 + \frac{1}{h}. \end{cases}$$

As u_h converges to

$$u(t) = \begin{cases} z & \text{if } t \leq t_0 \\ z + w_1 + w_2 & \text{if } t > t_0 \end{cases}$$

in $L^1(I)$, we obtain the sub-maximality of g.

Conversely suppose g to be lower semicontinuous and sub-maximal. Let (u_h) be a sequence of functions such that $u_h \to u$ in $L^1_{\text{loc}}(I)$. Up to subsequences we can suppose that $u_h \to u$ a.e.. As

$u \in PC(I)$, there exists $\varepsilon > 0$ such that $\varepsilon < \inf\{|t-s| \ : \ t,s \in S(u),\ t \neq s\}$. Fix $t \in S(u)$, we can suppose that

$$u_h(t \pm \varepsilon) \to u(t \pm \varepsilon) = u(t\pm)$$

and that, for all h, $(t \pm \varepsilon) \notin S(u_h)$. Thanks to the hypotheses on g, we have

$$g([u](t)) \leq \liminf_{h\to+\infty} \sup_{s\in S(u_h)\cap(t-\varepsilon,t+\varepsilon)} g([u_h](s)) \leq \liminf_{h\to+\infty} \sup_{t\in S(u_h)} g([u_h](t)).$$

Thanks to the arbitrariness of $t \in S(u)$ we finally get

$$\sup_{t\in S(u)} g([u](t)) \leq \liminf_{h\to+\infty} H(u_h)$$

as desired. $\square$

Remark 3.2 (i) If $g : (0,+\infty) \to \mathbb{R}$ is decreasing then it is sub-maximal. Analogously, if $g : (-\infty,0) \to \mathbb{R}$ is increasing then it is sub-maximal.

(ii) If $g : (0,+\infty) \to \mathbb{R}$ ($g : (-\infty,0) \to \mathbb{R}$) is sub-maximal, a sub-maximal extension $\tilde{g}$ of g defined on $\mathbb{R} \setminus \{0\}$ is given by

$$\tilde{g}(w) = \begin{cases} C & \text{if } w < 0 \ (w > 0) \\ g(w) & \text{if } w > 0 (w < 0), \end{cases}$$

where $C \geq \sup_{(0,+\infty)} g$.

Example 3.3 The function $g(w) = \dfrac{|\sin(w)|}{w}$ is sub-maximal on $(0,+\infty)$. More generally, if $f : (0,+\infty) \to [0,+\infty)$ is sub-additive and $h : (0,+\infty) \to (0,+\infty)$ is super-additive, then the function $g := \dfrac{f}{h}$ is sub-maximal. In fact, given $w_1,\ w_2 > 0$ there holds

$$\begin{aligned} g(w_1+w_2) &\leq \frac{1}{h(w_1+w_2)}(f(w_1)+f(w_2)) \\ &= \frac{h(w_1)}{h(w_1+w_2)}\frac{f(w_1)}{h(w_1)} + \frac{h(w_2)}{h(w_1+w_2)}\frac{f(w_2)}{h(w_2)} \\ &\leq \left(\frac{h(w_1)}{h(w_1+w_2)} + \frac{h(w_2)}{h(w_1+w_2)}\right)\Big(g(w_1) \vee g(w_2)\Big) \\ &\leq g(w_1) \vee g(w_2). \end{aligned}$$

Remark 3.4 (properties of sub-maximal functions) Let g be sub-maximal.

(i) If g is positive then g is sub-additive;
(ii) If $k \in \mathbb{N}$ then $g(kw) \leq g(w)$ for all $w \in \mathbb{R}$;
(iii) $g + c$ is sub-maximal (c a constant).

Furthermore, if (g_i) is a family of sub-maximal functions then $g = \sup_i g_i$ is sub-maximal.

We introduce the *lower-semicontinuous and sub-maximal envelope* of g as

$$g^{\rm sm}(w) = \sup\{f(w) : f \text{ lower semicontinuous and sub-maximal, } f \leq g\}.$$

Following the analogous argument for sub-additive functions (see e.g. [5]) we easily check that

$$g^{\rm sm}(w) = \inf\Big\{\liminf_{j\to+\infty} \bigvee_{i=1}^{N_j} g(w_j^i) \ : \sum_{i=1}^{N_j} w_j^i \to w\Big\}. \tag{3.15}$$

Theorem 3.5 (relaxation) *Let $H(u) : PC(I) \to \mathbb{R}$ be the functional defined in* (3.12)*. Then the lower-semicontinuous envelope of H with respect to the $L^1(I)$-topology is given on $PC(I)$ by the functional G defined as*

$$G(u) = \sup_{t \in S(u)} g^{\mathrm{sm}}([u](t)). \tag{3.16}$$

PROOF. Since g^{sm} is l.s.c., sub-maximal and $g^{\mathrm{sm}} \le g$, thanks to Theorem 3.1 $G(u) \le \overline{F}(u)$. We now prove that $\overline{F}(u) \le G(u)$. For simplicity we can suppose that

$$u(t) = \begin{cases} \alpha & \text{if } t \in (a, t_0) \\ \alpha + w & \text{if } t \in [t_0, b). \end{cases}$$

Then, for all $\varepsilon > 0$, thanks to (3.15) there exist w_j^i and N_j such that

$$g^{\mathrm{sm}}(w) \ge \bigvee_{i=1}^{N_j} g(w_j^i) - \varepsilon \qquad \text{and} \qquad \Big|\sum_{i=1}^{N_j} w_j^i - w\Big| < \frac{1}{j}.$$

Let $M_j \in \mathbb{N}$ be such that $\sup_i \left(\dfrac{N_j^2 |w_j^i|}{M_j}\right) < \dfrac{1}{j}$ and let u_j be defined as follows

$$u_j(t) = \begin{cases} \alpha & a < t \le t_0 \\ \alpha + \sum_{i=1}^{k} w_j^i & t_0 + \frac{k-1}{M_j} < t \le t_0 + \frac{k}{M_j} \\ & k = 1, 2, \ldots, N_j - 1 \\ \alpha + \sum_{i=1}^{N_j} w_j^i & t_0 + \frac{N_j - 1}{M_j} < t < b. \end{cases}$$

We have that $u_j \to u$ in $L^1(I)$ and that

$$F(u_j) = \bigvee_{i=1}^{N_j} g(w_j^i) \le g^{\mathrm{sm}}(w) + \varepsilon = G(u) + \varepsilon.$$

Letting $j \to +\infty$ we get

$$F(u) \le G(u) + \varepsilon$$

and the conclusion follows by the arbitrariness of ε. □

The following simple formula for the sub-maximal envelope of a level-convex function holds.

Proposition 3.6 *If g is lower semicontinuous, level convex and continuous in zero then*

$$g^{\mathrm{sm}}(w) = \bigwedge_{j=1}^{+\infty} g\Big(\frac{w}{j}\Big). \tag{3.17}$$

PROOF. Set $\phi(w) = \sup\{f(w) : \ f \text{ sub-maximal}, \ f \le g\}$. It can be easily checked that $\phi(w) = \inf\Big\{ \bigvee_{i=1}^{N} g(w_j) \ : \sum_{j=1}^{N} w_j = w \Big\}$. Note that $g^{\mathrm{sm}} \le \phi \le g$. We will prove (3.6) in two steps.

Step 1. We want to show that $\phi(w) = \bigwedge_{j=1}^{+\infty} g(\frac{w}{j})$. For all j, choosing $w_j = \frac{w}{j}$ in the minimum problem defining $\phi(w)$ we get that $\phi(w) \le g(\frac{w}{j})$ and so

$$\phi(w) \le \bigwedge_{j=1}^{+\infty} g(\frac{w}{j}).$$

To prove the opposite inequality, thanks to the level convexity of g, we have that $g(\sum_{i=1}^{j} \frac{w_i}{j}) \leq \bigvee_{i=1}^{j} g(w_i)$, hence

$$\phi(w) \geq \inf\Big\{g(\sum_{i=1}^{j} \frac{w_i}{j}),\ \sum_{i=1}^{j} w_i = w\Big\} = \bigwedge_{j=1}^{+\infty} g(\frac{w}{j}).$$

Step 2. To prove (3.6) ot suffices to prove that $\phi(w)$ is l.s.c. Let $w_n \to w$. Fixed $\varepsilon > 0$, thanks to the continuity of g in zero, there exists $k \in \mathbb{N}$ such that

$$\bigwedge_{j=k+1}^{+\infty} g(\frac{w_n}{j}) > \bigwedge_{j=k+1}^{+\infty} g(\frac{w}{j}) - \varepsilon \quad \forall n \in \mathbb{N}. \tag{3.18}$$

Then, thanks to the l.s.c. of g and to (3.18), we have

$$\begin{aligned}
\liminf_n \phi(w_n) &= \liminf_n \bigwedge_{j=1}^{k} g(\frac{w_n}{j}) \wedge \liminf_n \bigwedge_{j=k+1}^{+\infty} g(\frac{w_n}{j}) \\
&= \bigwedge_{j=1}^{k} \liminf_n g(\frac{w_n}{j}) \wedge \liminf_n \bigwedge_{j=k+1}^{+\infty} g(\frac{w_n}{j}) \\
&\geq \bigwedge_{j=1}^{k} g(\frac{w}{j}) \wedge \bigwedge_{j=k+1}^{+\infty} g(\frac{w}{j}) - \varepsilon = \phi(w) - \varepsilon.
\end{aligned}$$

Step 2 is completed by letting ε tend to zero. □

Remark 3.7 If $g : \mathbb{R} \to \overline{\mathbb{R}}$ is such that $g(w) = +\infty$ for all $w < 0$, the previous proposition still holds if the continuity hypothesis is replaced by the continuity from the right in zero.

Example 3.8 Let $g : (0, +\infty) \to \mathbb{R}$ be of quadratic type and such that $g(0+) = 0$, that is

$$g(w) = b(w^2 - 2ww_0)$$

for some positive constants b and w_0, and set

$$K := \min g = g(w_0) = -bw_0^2.$$

By Proposition 3.6 and Remark 3.7, we have that the lower semicontinuous and sub-maximal envelope of g on $(0, +\infty)$ is given by formula (3.17). Note that $g^{\rm sm}(kw_0) = K$ for any $k \in \mathbb{N}$. Moreover a simple calculation shows that

$$g^{\rm sm}(w) = g(w/k) \text{ if } w \in [w_{k-1}, w_k]$$

where $w_k := 2w_0k(k+1)/(2k+1)$ are relative maximum points for $g^{\rm sm}$. Note that $kw_0 < w_k < (k+1)w_0$ and $g^{\rm sm}(w_k)$ converge decreasingly to K as $k \to \infty$.

Remark 3.9 Let $g : \mathbb{R} \to \overline{\mathbb{R}}$. Then

$$\inf g(w) \leq g^{\rm sm}(w) \leq \limsup_{w \to 0^+} g(w) \quad \forall w > 0,$$

$$\inf g(w) \leq g^{\rm sm}(w) \leq \limsup_{w \to 0^-} g(w) \quad \forall w < 0.$$

The lower bound inequality is trivial. We prove the upper bound for $w > 0$ only, the proof in the other case being analogous. For any $\varepsilon \in (0, w)$ we can write

$$\begin{aligned}
g^{\rm sm}(w) &= g^{\rm sm}\Big(\varepsilon\Big[\frac{w}{\varepsilon}\Big] + \Big(w - \varepsilon\Big[\frac{w}{\varepsilon}\Big]\Big)\Big) \leq g^{\rm sm}\Big(\varepsilon\Big[\frac{w}{\varepsilon}\Big]\Big) \vee g^{\rm sm}\Big(w - \varepsilon\Big[\frac{w}{\varepsilon}\Big]\Big) \\
&\leq g^{\rm sm}(\varepsilon) \vee g^{\rm sm}v\Big(w - \varepsilon\Big[\frac{w}{\varepsilon}\Big]\Big) \leq \sup_{(0,\varepsilon]} g(w),
\end{aligned}$$

and we conclude by letting ε go to zero.

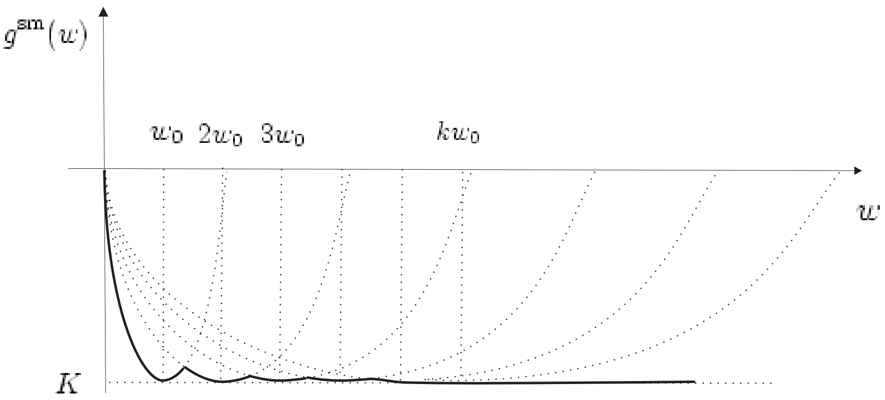

Figure 4: the sub-maximal envelope of g in Example 3.8

4 Semicontinuity and relaxation

In this section we study lower-semicontinuity properties for functionals whose natural framework is that of functions with bounded variation.

Let $I \subset \mathbb{R}$ be a bounded open set. We recall that a function $u \in L^1(I)$ is a function of bounded variation if its distributional derivative Du is a measure on I. In this case u is approximately differentiable almost everywhere and its a.e. derivative is denoted by u'; moreover the set of approximate discontinuity points (or jump set) of u, denoted by $S(u)$, is at most countable and at each point $t \in S(u)$ there exist the right-hand and left-hand approximate limits, denoted by $u(t\pm)$. With a slight abuse of notation we will also denote $[u](t) = u(t+) - u(t-)$ the *jump* of u at t. With this notation, the distributional derivative Du admits the decomposition

$$Du = u'\mathcal{L}^1 + D^s u = u'\mathcal{L}^1 + \sum_{t \in S(u)} (u(t+) - u(t-))\delta_t + D^c u,$$

where $\mathcal{L}^1$ stands for the one-dimensional Lebesgue measure, δ_t is the Dirac delta at t, and $D^c u$, the *Cantor part* of Du, is a non-atomic measure which is orthogonal to the Lebesgue measure. The notation $D^s u$ stands for the *singular part* of the measure Du (with respect to the Lebesgue measure). The space of functions of bounded variation on I is denoted by $BV(I)$.

The general (homogeneous and translation-invariant) local functional on $BV(I)$ is of the form

$$\int_I f(u')\,dt + \sum_{t \in S(u)} g([u]) + \int_I h\Big(\frac{D^c u}{|D^c u|}\Big) d|D^c u|. \tag{4.19}$$

Lower-semicontinuity properties for F are translated into the convexity of f, the subadditivity of g and a compatibility condition between the three energy densities (see e.g. [8] for details).

A function $u \in BV(I)$ is said to be a *special function of bounded variation* if $D^c u$ is the null measure. We will write $u \in SBV(I)$. In particular, piecewise-Sobolev functions belong to $SBV(I)$. Integral functionals on $SBV(I)$ take the form (4.19) without the integral depending on the Cantor part, or, equivalently, can be see as particular functionals on $BV(I)$ with $h(0) = 0$ and $h(z) = +\infty$ otherwise.

We refer to [1] for a thorough introduction to (special) functions of bounded variation; a quick overview is contained in [4].

In this section we study lower-semicontinuity properties of L^∞-functionals whose natural domain is $BV_{\rm loc}(I)$. In parallel with (4.19) these energies take the form

$$F(u) = \max\Big\{\sup_{t \in I} f(u'(t)),\ \sup_{t \in S(u)\cap I} g([u](t)),\ |D^c u|\text{-}\sup_{t \in I} h\Big(\frac{D^c u}{|D^c u|}(t)\Big)\Big\}, \tag{4.20}$$

with $f:\mathbb{R}\to\overline{\mathbb{R}}$, $g:\mathbb{R}\setminus\{0\}\to\overline{\mathbb{R}}$ and $h:\{-1,1\}\to\overline{\mathbb{R}}$ Borel functions. Note that we denote by $|D^c u|$-sup the essential supremum with respect to the measure $|D^c u|$. A particular case is when we take $h(z)=+\infty$ if $z\neq 0$, in which case the domain of F is $SBV(I)$. We can also equivalently consider F as defined on $L^1_{\rm loc}(I)$, by setting

$$F(u)=\begin{cases}\max\Big\{\sup\limits_{t\in I} f(u'),\ \sup\limits_{t\in S(u)\cap I} g([u]),\ |D^c u|\text{-}\sup\limits_{t\in I} h\Big(\frac{D^c u}{|D^c u|}\Big)\Big\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u\in BV_{\rm loc}(I) & \\ K & \text{otherwise,}\end{cases}\tag{4.21}$$

with $K\in\overline{\mathbb{R}}$ a constant (in particular we can take $K=+\infty$, in which case the functional defined in $L^1_{\rm loc}(I)$ is equivalent to that defined on $BV_{\rm loc}(I)$).

4.1 Lower semicontinuity. Statements of the results

We first analyze necessary and sufficient conditions for F as above to be lower semicontinuous with respect to the $L^1(I)$-convergence.

Theorem 4.1 (necessary conditions) *Let $F: L^1_{\rm loc}(I)\to\overline{\mathbb{R}}$ be defined in* (4.21). *Then necessary conditions for F to be lower semicontinuous with respect to the $L^1_{\rm loc}(I)$-convergence are that*

(a) *f is lower semicontinuous and level convex;*

(b) *$g\vee\inf f$ is lower semicontinuous and sub-maximal;*

and that the following compatibility conditions *hold:*

(c) $h(\pm1)\vee\inf f=\lim\limits_{z\to\pm\infty} f(z)=\lim\limits_{w\to 0\pm} g(w)\vee\inf f=\sup\limits_{\pm w>0} g(w)\vee\inf f;$

(d) $K=\sup f;$

(e) *either*

$$g(w)\vee\inf f=K\qquad \textit{for all } w<0\tag{4.22}$$

or $g(w)\vee\inf f=K$ for all $w>0$.

Remark 4.2 (necessary conditions under one-sided growth conditions) *Note that, due to its level-convexity, f is either monotone or decreasing/increasing. In both cases, the two limits $f(\pm\infty):=\lim_{z\to\pm\infty} f(z)$ exist and it is not restrictive to suppose that $f(-\infty)=\sup f$. Necessary conditions can be further simplified if the two limits are not equal, for example if*

$$\lim_{z\to-\infty} f(z)>\lim_{z\to+\infty} f(z).\tag{4.23}$$

We will assume that this condition holds whenever the limits are different; if the converse inequality holds, clearly the correct statements must be obtained by changing signs.

Then, if $\inf f<\sup f$, the necessary conditions (e) leads to

$$g(w)=h(-1)=K=\lim_{z\to-\infty} f(z)=\sup f\qquad \textit{for all } w<0,\tag{4.24}$$

and the compatibility conditions become

$$h(1)\vee\inf f=\lim_{z\to+\infty} f(z)=\lim_{w\to 0^+} g(w)\vee\inf f=\sup_{w>0} g(w)\vee\inf f.\tag{4.25}$$

The next proposition shows that (4.24) is a sufficient condition for coerciveness in $BV_{\rm loc}(I)$.

Proposition 4.3 (coerciveness) *Let* (4.24) *hold. Then from every bounded sequence (u_j) in $L^1(I)$ such that $\sup_j F(u_j)<K$ we can extract a subsequence weakly* converging in $BV_{\rm loc}(I)$.*

Upon changing sign to variables, we have that (a), (b), (4.24) and (4.25) are necessary for combined lower-semicontinuity and coerciveness of F. The following theorem shows that they are also sufficient.

Theorem 4.4 (sufficient conditions) *Let conditions* (a) *and* (b) *of Theorem* 4.1 *and conditions* (4.24) *and* (4.25) *of Remark* 4.2 *be satisfied. Then* $F : L^1_{\rm loc}(I) \to \overline{\mathbb{R}}$ *defined in* (4.21) *is lower semicontinuous with respect to the* $L^1_{\rm loc}(I)$*-convergence.*

Note that in Theorem 4.4 we do not require that (4.23) holds. Moreover, clearly, conditions (4.24) and (4.25) can be replaced by symmetric conditions changing z into $-z$, etc.

We summarize the results above in the following theorem.

Theorem 4.5 (necessary and sufficient conditions) *Conditions* (a)–(e) *of Theorem* 4.1 *are necessary and sufficient for the lower semicontinuity of* F *with respect to the* $L^1_{\rm loc}(I)$*-convergence. Moreover, such* F *is coercive with respect to the weak** BV*-convergence on the set* $\{u : F(u) < K\}$.

Before entering into the details of the proof of the results above, we note in the following remark that it is not restrictive to require that $K = +\infty$ and that one-sided growth conditions be imposed on f and g, due to the necessary conditions.

Remark 4.6 (modification of growth conditions by composition) We preliminarily note that by definition the value $F(u)$ does not change if we substitute g and h by $g \vee \inf f$ and $h \vee \inf f$, respectively, or, equivalently it is not restrictive to suppose that $g \geq \inf f$ and $h \geq \inf f$.

If F is lower semicontinuous then by Theorem 4.1 it is not restrictive to suppose that

$$\inf f \leq g \leq \sup f, \qquad \inf f \leq h \leq \sup f, \qquad K = \sup f = f(-\infty), \tag{4.26}$$

and that f is level convex, in particular the limits $f(\pm\infty)$ exist. Let now $H : [\inf f, \sup f] \to \overline{\mathbb{R}}$ be a strictly-increasing function, and consider the functional

$$G(u) = \begin{cases} \max\Big\{\sup\limits_{t\in I} H(f(u')),\ \sup\limits_{t\in S(u)\cap I} H(g([u])),\ |D^c u|\text{-}\sup\limits_{t\in I} H\Big(h\Big(\frac{D^c u}{|D^c u|}\Big)\Big)\Big\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in BV_{\rm loc}(I) & \\ H(K) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{otherwise.} \end{cases} \tag{4.27}$$

Then $G(u) = H(F(u))$ so that G is lower semicontinuous with respect to the $L^1_{\rm loc}$-convergence if and only if such is F. Note moreover that all notions involved in the statement of the results above are invariant with respect to the composition with H.

We are now free to choose H such that $H(\sup f) = +\infty$ (and, if needed, $H(\inf f) = 0$), so that by (4.26) it is not restrictive to suppose that a 'one-sided growth condition' on f holds:

$$\lim_{z\to-\infty} f(z) = +\infty. \tag{4.28}$$

Note that condition (4.24) translates into

$$g(w) = +\infty \quad \text{if } w < 0, \quad h(-1) = +\infty. \tag{4.29}$$

In conclusion, if (4.24) holds, it is not restrictive to assume that F is of the type

$$F(u) = \begin{cases} \sup\limits_{t\in I} f(u'(t)) & \text{if } u \in W^{1,1}_{\rm loc}(I) \\ \max\{\sup\limits_{t\in I} f(u'(t)),\ \sup\limits_{t\in S(u)\cap I} g([u](t))\} & \text{if } u \in SBV^+_{\rm loc}(I) \setminus W^{1,1}_{\rm loc}(I) \\ \max\{\sup\limits_{t\in I} f(u'(t)),\ \sup\limits_{t\in S(u)\cap I} g([u](t)), C_\infty\} & \text{if } u \in BV^+_{\rm loc}(I) \setminus SBV^+_{\rm loc}(I) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.30}$$

where $C_\infty \in \overline{\mathbb{R}}$ and

$$\begin{aligned} BV^+_{\rm loc}(I) &= \{u \in BV_{\rm loc}(I) :\ D_s u \text{ is a positive measure}\}, \\ SBV^+_{\rm loc}(I) &= SBV_{\rm loc}(I) \cap BV^+_{\rm loc}(I). \end{aligned}$$

Note that the condition for a function $u \in SBV_{\rm loc}(I)$ to belong to $SBV^+_{\rm loc}(I)$ is simply that all jumps be positive; i.e. $u(t+) > u(t-)$ on $S(u)$.

4.2 Lower semicontinuity. Proofs of the results

We now prove the results stated in the previous section.

PROOF OF THEOREM 4.1. Without loss of generality we assume that $g \geq \inf f$ and $h \geq \inf f$ (see Remark 4.6).

(a) Suppose that F is l.s.c. with respect to the $L^1_{\rm loc}(I)$ convergence. Since, in particular, the restriction of F to $W^{1,\infty}(I)$ is l.s.c. with respect to the $L^\infty(I)$-convergence, thanks to [17] f is l.s.c. and level convex.

(b) We now check that g is lower semicontinuous. Let (w_n) be a sequence of real numbers converging to w. For any $z \in \mathbb{R}$ let $u_n(t)$ be defined as

$$u_n(t) = \begin{cases} zt & \text{if } t \leq t_0 \\ zt + w_n & \text{if } t > t_0. \end{cases}$$

As u_n converges to

$$u(t) = \begin{cases} zt & \text{if } t \leq t_0 \\ zt + w & \text{if } t > t_0 \end{cases}$$

in $L^\infty(I)$, we get

$$f(z) \vee g(w) \leq f(z) \vee \liminf_{n\to+\infty} g(w_n). \tag{4.31}$$

Let (z_k) be a sequence such that $f(z_k) \to \inf f$, then passing to the limit as $k \to +\infty$ in (4.31) with $z = z_k$ we get the conclusion. For the sake of simplicity, from now on we suppose that there exists z_0 such that $f(z_0) = \inf f$. In order to prove that g is sub-maximal, take w_1, w_2 and let u_n be defined as

$$u_n(t) = \begin{cases} z_0 t & \text{if } t \leq t_0 \\ z_0 t + w_1 & \text{if } t_0 < t \leq t_0 + \frac{1}{n} \\ z_0 t + w_1 + w_2 & \text{if } t > t_0 + \frac{1}{n}. \end{cases}$$

As u_n converges to

$$u(t) = \begin{cases} z_0 t & \text{if } t \leq t_0 \\ zt + w_1 + w_2 & \text{if } t > t_0 \end{cases}$$

in $L^1(I)$, we obtain the sub-maximality of g.

(c) We only deal the case of positive variables. We first prove that $\lim_{z\to+\infty} f(z) = \lim_{w\to0^+} g(w) = \sup_{w>0} g$. Consider, for $z > z_0$, $u(t) = zt$ and let (w_n) be a decreasing sequence of positive real numbers converging to 0 such that $\lim_n g(w_n) = \liminf\limits_{w\to0^+} g(w)$. Let u_n be defined as

$$u_n(t) = \begin{cases} z_0 t + j w_n & \text{if } a + j(\frac{b-a}{k_n}) < t < a + (j+1)(\frac{b-a}{k_n}) \\ & \text{with } j \in \{1, 2, \ldots, [k_n] - 1\} \\ & \\ z_0 t + [k_n] w_n & \text{if } a + [k_n](\frac{b-a}{k_n}) < t < b \end{cases}$$

where $k_n = (z - z_0)(\frac{b-a}{w_n})$. With this approximation we obtain that $f(z) \leq \liminf\limits_{w\to0^+} g(w)$. This inequality gives

$$\lim_{z\to+\infty} f(z) \leq \liminf_{w\to0^+} g(w). \tag{4.32}$$

Consider now, for all $w > 0$, u_n defined as

$$u_n(t) = \begin{cases} z_0 t & \text{if } a < t < t_0 \\ nwt + (z_0 - nw)t_0 & \text{if } t_0 \leq t < t_0 + \frac{1}{n} \\ z_0 t + w + z_0 t_0 & \text{if } t_0 + \frac{1}{n} \leq t < b. \end{cases}$$

As u_n converges to

$$u(t) = \begin{cases} z_0 t & \text{if } a < t < t_0 \\ z_0 t + w & \text{if } t_0 \leq t < b \end{cases}$$

in $L^1(I)$ we have that

$$g(w) \leq \liminf_{z\to+\infty} f(z). \tag{4.33}$$

Letting $w \to 0^+$, from inequalities (4.32) and (4.33) we get

$$\limsup_{w\to 0^+} g(w) \leq \sup_{w>0} g(w) \leq \lim_{z\to+\infty} f(z) \leq \liminf_{w\to 0^+} g(w). \tag{4.34}$$

Let now $\gamma : [0,1] \to [0,1]$ be the Cantor-Vitali function and let $\gamma_n : [0,1] \to [0,1]$ be its standard piecewise affine approximations with maximum slope $z_n = (\frac{3}{2})^n$. Set $u(t) = \gamma(t) + z_0 t$ we have that $u_n(t) = \gamma_n(t) + z_0 t$ converges to u and that

$$\lim_{n\to+\infty} f(z_n) = \lim_{z\to+\infty} f(z) \geq h(1). \tag{4.35}$$

To get the opposite inequality, for all $w > 0$ consider the sequence

$$u_n(t) = \begin{cases} z_0 t & \text{if } a < t < t_0 \\ w\gamma(n(t-t_0)) + z_0 t & \text{if } t_0 \leq t < t_0 + \frac{1}{n} \\ z_0 t + w & \text{if } t_0 + \frac{1}{n} \leq t < b. \end{cases}$$

Then u_n converges in L^1 to

$$u(t) = \begin{cases} z_0 t & \text{if } a < t < t_0 \\ w + z_0 t_0 & \text{if } t_0 \leq t < b \end{cases}$$

and so

$$h(1) \geq g(w) \quad \forall w > 0.$$

Passing to the limsup as $w \to 0^+$ in the right-hand side of the previous estimate, thanks to (4.35) and to (4.34) we conclude the proof.

(d) Take $u \in L^1(I) \setminus BV_{\rm loc}(I)$, and let $u_j \to u$ in $L^1(I)$ with $u_j \in W^{1,\infty}(I)$. We then obtain $K = F(u) \leq \liminf_j F(u_j) \leq \sup f$. Conversely, let $z \in \mathbb{R}$, $I = (0,1)$, and let

$$u_j(t) = \begin{cases} \frac{1}{j}u(jt) & \text{if } t \in (0, 1/j), \\ zt & \text{otherwise} \end{cases}$$

(u as above). We then have $u_j \to zt$ so that $f(z) = F(zt) \leq \liminf_j F(u_j) = K$. By the arbitrariness of z we conclude the proof.

(e) Suppose (e) is not satisfied; i.e., $w_1 > 0$ and $w_2 < 0$ exist such that $\max\{g(w_1), g(w_2)\} < K$ (and $\inf f < \sup f = K$ by (d)). Now, note that from the sub-maximality of g we have

$$g(k_1 w_1 + k_2 w_2) \leq C := \max\{g(w_1), g(w_2)\} < K \qquad \text{for all } k_1, k_2 \in \mathbb{N}. \tag{4.36}$$

If w_1 and w_2 are linearly independent in $\mathbb{Z}$, then $\mathbb{N}w_1 + \mathbb{N}w_2$ is dense in $\mathbb{R}$. Since by (b) g is also lower semicontinuous, from (4.36) we deduce that $\sup g \leq C < K$, that contradicts condition (c) (taking (d) into account).

If otherwise w_1 and w_2 are linearly dependent in $\mathbb{Z}$ then let $\pm w_0 \in \mathbb{N}w_1 + \mathbb{N}w_2$, let

$$u_h(t) = z_0 t + w_0 \sum_{j=1}^{h} (-1)^j \chi_{(0,1/j)},$$

and u the limit of u_h. Note that $u \notin BV_{\rm loc}(-1,1)$, so that (taking $I = (-1,1)$) we obtain $K = F(u) \leq \liminf_h F(u_h) = \max\{g(w_0), g(-w_0)\} \vee \inf f$, that contradicts (4.36). $\square$

We now prove the remaining results under the simplifying non-restrictive hypotheses derived from Remark 4.6.

Proof of Proposition 4.3. Consider a bounded sequence (u_n) in $L^1(I)$ with $F(u_n)$ equibounded. The sequence then belongs to $BV^+_{\rm loc}(I)$ and by (4.28) there exists $\overline{z}$ such that $u'_n \geq \overline{z}$ a.e. for all n. Upon extracting a subsequence, for each fixed $\eta > 0$ we can find $a_\eta \in (a, a+\eta)$ and $b_\eta \in (b-\eta, b)$ such that $(u_n(a_\eta))$ and $(u_n(b_\eta))$ converge. We then have

$$\begin{aligned} |Du_n|(a_\eta, b_\eta) &= D^s u_n(a_\eta, b_\eta) + \int_{a_\eta}^{b_\eta} |u'_n|\, dt \\ &\leq u_n(b_\eta) - u_n(a_\eta) + 2(b-a)|\overline{z}|, \end{aligned}$$

so that the sequence (u_n) is bounded in $BV(a_\eta, b_\eta)$ and hence we can extract from it a further converging subsequence. From this argument, we easily obtain the thesis. $\square$

Proof of Theorem 4.4. Suppose now that f is lower semicontinuous and level convex, g is lower semicontinuous and sub-maximal and (4.25) holds true. Let (u_h) be a sequence of functions such that $u_h \to u$ in $L^1_{\rm loc}(I)$. Up to subsequences we can suppose that also $u_h \to u$ a.e. and that there exists

$$\lim_{h\to+\infty} F(u_h) < +\infty.$$

Thanks to (4.28), $u_h \in BV^+_{\rm loc}(I)$ and there exists $k > 0$ such that

$$u'_h(t) \geq -k \ \text{ a.e. } t \in I, \ \forall h \in \mathbb{N}.$$

If we set $v_h(t) = u_h(t) + kt$, then (v_h) is a sequence of non decreasing functions in $BV^+_{\rm loc}(I)$ converging in $L^1_{\rm loc}$ to $v(t) = u(t) + kt$. It implies that u belongs to $BV^+_{\rm loc}(I)$. Note that the lower-semicontinuity inequality along a sequence (u_h) can be easily checked in the following three cases:

1. $u_h \in BV^+_{\rm loc}(I) \setminus SBV^+_{\rm loc}(I) \cup W^{1,1}_{\rm loc}(I)$,
2. $\liminf\limits_{h\to+\infty} \inf\limits_{t\in S(u_h)} ([u_h](t)) = 0$,
3. $\limsup\limits_{h\to+\infty} \sup_{t\in I} f(u'_h(t)) = +\infty$.

In fact, thanks to hypothesis (4.25), $F(u) \leq F_\infty(u)$ and $F(u) = F_\infty(u)$ $\forall u \in W^{1,1}_{\rm loc}(I) \cup \big(BV^+_{\rm loc}(I) \setminus SBV^+_{\rm loc}(I)\big)$. Then, by Remark 2.4, it holds

$$\lim_{h\to+\infty} F(u_h) = \lim_{h\to+\infty} F_\infty(u_h) \geq F_\infty(u) \geq F(u).$$

It remains to prove the lower semicontinuity when none of the above cases is satisfied; i.e. when $(u_h) \in SBV^+_{\rm loc}(I) \setminus W^{1,1}_{\rm loc}(I)$ and is such that

$$\exists k > 0,\ \delta > 0 \qquad \text{such that} \quad |u'_h(t)| \leq k \quad \forall h \in \mathbb{N}, \ \text{ for a.e. } t \in I \tag{4.37}$$

$$\inf_{t\in S(u_h)} [u_h](t) \geq \delta. \tag{4.38}$$

Let $\eta > 0$ be such that $a+\eta,\ b-\eta \in (a,b) \setminus \bigcup_h S(u_h)$, $a+\eta < b-\eta$ and $u_h(a+\eta) \to u(a+\eta) < +\infty$, $u_h(b-\eta) \to u(b-\eta) < +\infty$. Set $I_\eta := (a+\eta, b-\eta)$. By (4.38) we can suppose that $\#(S(u_h)\cap I_\eta) < +\infty$ and that it is independent of h. If $\#(S(u_h) \cap I_\eta) = 0$ then the sequence is weakly* converging in $W^{1,\infty}_{\rm loc}(I)$ and the lower semicontinuity follows by the results in Sobolev spaces (see e.g. [15] and [17]). Hence, we can suppose that $S(u_h) = \{t^0_h, \dots, t^N_h\}$, with $t^{j-1}_h < t^j_h$, and that $t^j_h \to t^j$ as $h \to +\infty$. Let $S = \{t^0, \dots, t^N\}$, and for each $\delta > 0$ let $S_\delta = \{t \in I \ : \ \inf_j |t - t^j| \leq \delta\}$. As before, $u_h \rightharpoonup u$ weakly* in $W^{1,\infty}(I_\eta \setminus S_\delta)$ so that

$$\liminf_{h\to+\infty} F(u_h) \geq \liminf_{h\to+\infty} \sup_{t\in I_\eta\setminus S_\delta} f(u'_h(t)) \geq \sup_{t\in I_\eta\setminus S_\delta} f(u'(t)).$$

and, as η and δ go to zero, we have

$$\liminf_{h\to+\infty} F(u_h) \geq \sup_{t\in I} f(u'(t)). \tag{4.39}$$

If $S \cap I = \emptyset$, the proof is completed . Suppose otherwise that there exists $t \in S \cap I$ and let $j = j_0, j_0+1, \ldots, j_0+M$ be the indices such that $t = t^j$. As $u_h \to u$ uniformly on $L^\infty(I_\eta \setminus S_\delta)$ for all $\delta > 0$ and u_h are equi-uniformly continuous on each interval (t_h^{j-1}, t_h^j), we get that $u_h(t_h^{j_0}-) \to u(t-)$, $u_h(t_h^{j_0+M}+) \to u(t+)$ and that $(u_h(t_h^{j-1}+) - u_h(t_h^j-)) \to 0 \ \forall j \in \{j_0+1, j_0+2, \ldots, j_0+M\}$. Thus we have

$$\sum_{j=j_0}^{j_0+M} (u_h(t_h^j+) - u_h(t_h^j-)) \to (u(t+) - u(t-)).$$

By the sub-maximality and lower-semicontinuity of g we obtain

$$\begin{aligned} g(u(t+) - u(t-)) &\leq \liminf_{h\to+\infty} g\Big(\sum_{j=j_0}^{j_0+M} (u_h(t_h^j+) - u_h(t_h^j-))\Big) \\ &\leq \liminf_h \sup_{j\in\{j_0, j_0+1, \ldots, j_0+M\}} g(u_h(t_h^j+) - u_h(t_h^j-)) \\ &\leq \liminf_{h\to+\infty} F(u_h). \end{aligned}$$

Thanks to the arbitrariness of $t \in S \cap I_\eta$, when $\eta \to 0$, as $S(u) \subset S \cap I$ we have

$$\sup_{t\in S(u)} g(u(t+) - u(t-)) \leq \liminf_{h\to+\infty} F(u_h). \tag{4.40}$$

The thesis follows from the inequalities in (4.39) and in (4.40). □

Remark 4.7 Following the outline of the previous proof, one can show that, were F of the type (4.21), then the lower semicontinuity of F implies that the following three conditions are equivalent

$$\text{(a)} \quad \lim_{z\to-\infty} f(z) = +\infty, \qquad \text{(b)} \quad \lim_{w\to 0^-} g(w) = +\infty, \qquad \text{(c)} \quad h(-1) = +\infty.$$

Proof of Theorem 4.5. The proof immediately follows from Theorems 4.1 and 4.4 since it is not restrictive to suppose that the hypotheses of Theorem 4.4 hold. □

4.3 Relaxation

We now prove a relaxation theorem under the simplifying hypotheses derived from Remark 4.6. For a general introduction to relaxation we refer to [9]. In what follows, for short, we set $f(+\infty) := \lim_{z\to+\infty} f(z)$ and $\underline{g}(0^+) := \liminf_{w\to 0^+} g(w)$.

Theorem 4.8 (Relaxation) *Let f be such that condition (4.28) holds and let $F : L^1_{\rm loc}(I) \to [0, +\infty]$ be the functional defined in (4.30). Then the lower-semicontinuous envelope of F with respect to the $L^1_{\rm loc}(I)$-topology is given by the functional $G : L^1_{\rm loc}(I) \to \overline{\mathbb{R}}$ defined as*

$$G(u) = \begin{cases} \sup_{t\in I} \hat{f}(u'(t)) & \text{if } u \in W^{1,1}_{\rm loc}(I) \\ \max\{\sup_{t\in I} \hat{f}(u'(t)), \ \sup_{t\in S(u)\cap I} \hat{g}([u](t))\} & \text{if } u \in SBV^+_{\rm loc}(I) \setminus W^{1,1}_{\rm loc}(I) \\ \max\{\sup_{t\in I} f(u'(t)), \hat{C}_\infty\} & \text{if } u \in BV^+_{\rm loc}(I) \setminus SBV^+_{\rm loc}(I) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.41}$$

where

$$\begin{aligned} &\hat{f}(z) = f^{\rm lc}(z) \wedge \tilde{f}(z) \wedge C_\infty \ \text{ with } \tilde{f}(z) = \inf\{f^{\rm lc}(y), \ y \leq z\} \vee g^{\rm sm}(0^+), \\ &\hat{g}(w) = g^{\rm sm}(w) \wedge f^{\rm lc}(+\infty) \wedge g^{\rm sm}(0^+) \wedge C_\infty, \\ &\hat{C}_\infty = C_\infty \wedge f^{\rm lc}(+\infty) \wedge g^{\rm sm}(0^+). \end{aligned}$$

PROOF. By Theorems 2.3 and 3.5 the functional H defined by substituting f with $f^{\rm lc}$ and g with $g^{\rm sm}$ in formula (4.30) is such that $\overline{H} = \overline{F}$. Thus we may suppose f and g to be l.s.c., and level-convex and sub-maximal, respectively, (see Sections 2 and 3). We want to prove that $\overline{F}(u) = G(u)$. Since $G(u)$ is such that $G(u) \leq F(u)$ by definition, and it satisfies all the hypotheses of the lower semicontinuity Theorem 4.5, we have that $\overline{F}(u) \geq G(u)$. Now we prove that $\overline{F}(u) \leq G(u)$ in the case f is decreasing-increasing, the proof in the other case being analogous. The following two cases are possible:

1. $G(u) < f(+\infty) \wedge \underline{g}(0^+) \wedge C_\infty$,
2. $G(u) = f(+\infty) \wedge \underline{g}(0^+) \wedge C_\infty$.

In the first case there is nothing to prove since $G(u) = F(u)$. The second possibility can be further subdivided into three sub-cases:

2a. $G(u) = f(+\infty)$, 2b. $G(u) = \underline{g}(0^+)$, 2c. $G(u) = C_\infty$.

Case 2a. In this case, set

$$\tilde{z} = \min\{z : \ f(z) \leq f(+\infty)\},$$

we have that $u'(t) \geq \tilde{z}$ for a.e. $t \in I$. Consider the function $v(t) = u(t) - \tilde{z}t$, and let (v_n) be a sequence of increasing piecewise-constant functions converging to v in $L^1_{\rm loc}(I)$. Fixed n, suppose for simplicity, the construction being analogous in the general case, that $S(v_n) = \{t_0\}$. Let $v_{n,j}$ be defined as follows

$$v_{n,j}(t) = \begin{cases} v_n(t) & \text{if } t \in (a, t_0) \cup (t_0 + \frac{1}{j}, b) \\ v_n(t_0-) + j[v_n](t_0)(t - t_0) & \text{if } t \in [t_0, t_0 + \frac{1}{j}] \end{cases}$$

Using a diagonalization argument there exists $n(j)$ such that $u_{n(j),j}(t) = v_{n(j),j}(t) + \tilde{z}t$ satisfies $u'_{n(j),j} \geq \tilde{z}$ and $D^s(u_{n(j),j}) = 0$. Moreover, it converges to u in $L^\infty(I)$ and

$$\lim_{j\to+\infty} F(u_{n(j),j}) = \lim_{j\to+\infty} \sup_{t\in I} f(u'_{n(j),j}(t)) = f(+\infty) = G(u).$$

Case 2b. In this case, denoted $\tilde{z} := \min\{z : \ f(z) \leq \underline{g}(0+)\}$ we have $u'(t) \geq \tilde{z}$ for a.e. $t \in I$. Consider the function $v(t) = u(t) - \tilde{z}t$, and let v_n be a sequence of increasing piecewise-constant functions converging to v in $L^1_{\rm loc}(I)$. With fixed n, suppose for simplicity that $S(v_n) = \{t_0\}$, the construction being analogous in the general case. Let (w_j) be a sequence such that $\lim_j g(w_j) = \liminf_{w\to 0^+} g(w) = \underline{g}(0^+)$ and let $v_{n,j}$ be defined as follows

$$v_{n,j}(t) = \begin{cases} v_n(t) & \text{if } a < t \leq t_0 \\ v_n(t_0-) + hw_j & \text{if } t_0 + (h-1)M_j < t \leq t_0 + hM_j \\ & \quad \text{and } h \in \left\{1, 2, \dots \left[\frac{[v_n](t_0)}{w_j}\right]\right\} \\ v_n(t_0-) + \left[\frac{[v_n](t_0)}{w_j}\right] w_j & \text{if } \left[\frac{[v_n](t_0)}{w_j}\right] M_j < t < b \end{cases}$$

where $M_j := \left[\frac{w_j}{jw}\right]$. Using a diagonalization argument we obtain that the sequence $u_{n(j),j}(t) = v_{n(j),j}(t) + \tilde{z}t$ converges to u with $u'_{n(j),j} \geq \tilde{z}$, $D^c(u_{n(j),j}) = 0$ and is such that,

$$\begin{aligned} \lim_{j\to+\infty} F(u_{n(j),j}) &= \max\{\sup_{t\in I} f(u'_{n(j),j}(t)), \sup_{t\in S(u_{n(j),j})\cap I} g([u_{n(j),j}](t))\} \\ &= \underline{g}(0^+) = G(u). \end{aligned}$$

Case 2c. Set $\tilde{z} := \min\{z : \ f(z) \leq C_\infty\}$. We have that $u'(t) \geq \tilde{z}$ for a.e. $t \in I$. We can construct v, v_n and $v_{n,j}$ as before. In particular, supposing for simplicity that, with n fixed, $S(v_n) = \{t_0\}$ and denoting by $\gamma : [0,1] \to [0,1]$ the Cantor-Vitali function, $v_{n,j}$ will be defined as follows

$$v_{n,j}(t) = \begin{cases} v_n(t) & \text{if } t \in (a, t_0) \cup (t_0 + \frac{1}{j}, b) \\ v_n(t_0-) + [v_n](t_0)\gamma(j(t - t_0)) & \text{if } t_0 \leq t \leq t_0 + \frac{1}{j}. \end{cases}$$

Thanks to a diagonalization argument, the sequence $u_{n(j),j}(t) = v_{n(j),j}(t) + \tilde{z}t$ converges to u with $u'_{n(j),j} \geq \tilde{z}$, $D^j(u_{n(j),j}) = 0$ and is such that

$$\lim_{j\to+\infty} F(u_{n(j),j}) = \max\{\sup_{t\in I} f(u'_{n(j),j}(t)), C_\infty\} = C_\infty = G(u),$$

as desired. □

Example 4.9 Let f and g satisfy the hypotheses of Theorem 4.8 and let $C_\infty = +\infty$. Suppose in addition that f be continuous and level convex, decreasing on $(-\infty, 0)$ and increasing on $(0, +\infty)$, and g be continuous and decreasing on $(0, +\infty)$. Then we simply have

$$\hat{f}(z) = \begin{cases} f(z) & \text{if } z \leq 0 \\ f(z) \wedge (g(0^+) \vee \inf f) & \text{if } z > 0 \end{cases}, \qquad \hat{g}(w) = g(w) \wedge f(+\infty), \qquad \hat{C}_\infty = f(+\infty) \wedge g(0^+).$$

5 Minimum Problems

Let F be an L^∞-functional defined as in (4.30) with f satisfying (4.28). In what follows, for simplicity, we suppose that $g \geq \inf f$, $C_\infty \geq \inf f$ and $I = (0,1)$. We will consider minimum problems with Dirichlet boundary conditions

$$m(d) = \inf\Big\{F(u) : u(0) = 0,\ u(1) = d\}.$$

As usual in problems on BV spaces, it is convenient to 'relax' the boundary conditions. In place of the definition above, for all $d \geq 0$ set

$$m(d) := \inf\{F(u) \vee \tilde{g}(u(0+)) \vee \tilde{g}(d - u(1-))\}$$

where $\tilde{g} : \mathbb{R} \to \overline{\mathbb{R}}$ is an extension of g such that $\tilde{g}(0) = \inf f$ and $\tilde{g}(w) = +\infty$ for all $w < 0$. We will show *a posteriori* that the two definitions of $m(d)$ are the same.

Remark 5.1 If F is lower semicontinuous, then it can be easily checked that, for any $d \geq 0$, the functional F^d defined as

$$F^d(u) = F(u) \vee \tilde{g}(u(0+)) \vee \tilde{g}(d - u(1-))$$

is also lower semicontinuous (the functional F^d can be viewed e.g. as obtained by extending each function u to the affine functions $z_0 t$ for $t \leq 0$ and $d + z_0(t-1)$ for $t \geq 1$, where $f(z_0) = \min f$). Moreover F^d is coercive in $L^1(I)$. In fact, let $u_n \in BV^+_{\mathrm{loc}}(I)$ be such that $\sup_n F(u_n) < +\infty$. Then, for any n, $u_n(0+) \geq 0$, $u_n(1-) \leq d$ and there exists $k > 0$ such that $\inf_{t\in I} u'_n(t) \geq -k$. Then

$$|Du_n|(0,1) \leq Du_n(0,1) + 2k = u_n(1-) - u_n(0+) + 2k \leq d + 2k.$$

Since the previous inequality implies that u_n is bounded in $L^\infty(I)$, we get that (u_n) is relatively compact in $L^1(I)$. If F is lower semicontinuous then it follows that $m(d)$ is actually a minimum.

Proposition 5.2 *If F is lower semicontinuos, then for any $d > 0$, we have*

$$m(d) = f(d) \wedge \inf_{z<d}\{f(z) \vee g(d-z)\}, \tag{5.42}$$

and, for any $\overline{t} \in I$ the function

$$u^d(t) := \begin{cases} dt & \text{if } m(d) = f(d) \\ z_d t + (d - z_d)\chi_{[\overline{t},+\infty)}(t) & \text{if } m(d) = f(z_d) \vee g(d - z_d) \end{cases} \tag{5.43}$$

is a minimizer for $m(d)$.

Proof. We can restrict to $SBV^+(I)$ in the definition of $m(d)$. In fact, let $u \in BV^+(I)$ be such that $u(0) = 0$ and $u(1) = d$ and set

$$v(t) := u(t) - D^c u(0,t) + D^c u(I)\chi_{[\bar{t},+\infty)}(t).$$

Then $v \in SBV^+(I)$, $v(0) = 0$, $v(1) = d$ and, by (4.25), $F(v) \leq F(u)$.

Moreover, we may assume that $\#S(u) \leq 1$. In fact, if $u \in SBV^+(I)$ and $S(u) = \{t_i : i \in \mathbb{N}\}$, $z_i := [u](t_i)$, the function

$$v(t) := u(t) - \sum_{i=1}^{+\infty} z_i \chi_{[t_i,+\infty)}(t) + \sum_{i=1}^{+\infty} z_i \chi_{[\bar{t},+\infty)}(t)$$

satisfies the same boundary condition as u, $S(v) = \{\bar{t}\}$, and, by the sub-maximality of g, $F(v) \leq F(u)$.

Eventually, we may suppose $u'(t)$ constant. In fact, if $u \in SBV^+(I)$ and $z := \int_0^1 u'(t)\,dt$, the function

$$v(t) = zt + (d - z)\chi_{[\bar{t},+\infty)}(t)$$

satisfies the same boundary condition of u and $F(v) \leq F(u)$, since, by the level-convexity of f, we get

$$f(z) = f\Big(\int_0^1 u'(t)\,dt\Big) \leq \sup_{t\in I} f(u'(t)),$$

and it is easy to get the conclusion. □

In the following examples we give a detailed description of $m(d)$, under particular choices of f and g.

Example 5.3 Let f be a Lennard-Jones type potential; that is, a continuous decreasing-increasing function such that $f(z) = +\infty$ if $z \leq 0$, $\lim_{z\to 0+} f(z) = +\infty$ and $\lim_{z\to+\infty} f(z) = 0$. In particular f is a level-convex function. Set

$$f(\hat{z}) = 0, \qquad f(z_m) = m := \min_{\mathbb{R}} f.$$

Let $g : (0, +\infty) \to \mathbb{R}$ be any decreasing and continuous function such that $g(0+) = 0$ and

$$K := \lim_{z\to+\infty} g(z) > m.$$

In particular g is sub-maximal. By Theorem 4.5, the functional F, defined as in (4.30), with this choice of f and g and with $C_\infty = 0$ is lower semicontinuous. We study $m(d)$ in this case. Set

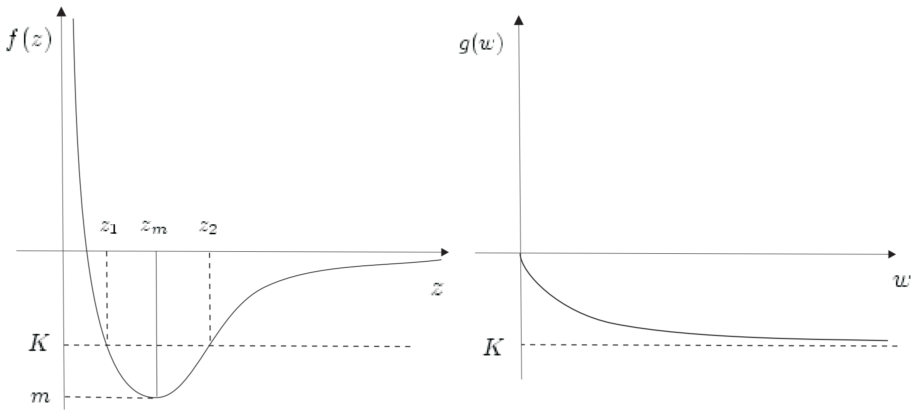

Figure 5: f and g in Example 5.3

$$H(z,d) = f(z) \vee g(d-z), \quad h(d) = \min_{0<z<d} H(z,d),$$

and let $z_1 < z_2$ be such that

$$f^{-1}(K) = \{z_1, z_2\}.$$

Note that, for $z \in [z_1, z_2]$, $f(z) \le K$ and so $H(z,d) = g(d-z)$.

We have that $m(d) = f(d)$ for $d \le z_2$. In fact, if $d \le z_m$ then f is decreasing and so $f(d) \le h(d)$. If $z_m < d \le z_2$ then $f(d) \le \min g \le h(d)$ and so $m(d) = f(d)$.

The function $z_0 : (\hat{z}, +\infty) \to (\hat{z}, z_1]$ defined as

$$z_0(d) = \min\{z > \hat{z} : f(z) = g(d-z)\}$$

is continuous increasing and such that

$$h(d) = f(z_0(d)) = g(d - z_0(d)), \qquad \text{and} \qquad \lim_{d\to+\infty} z_0(d) = z_1.$$

In particular, h is decreasing in $(\hat{z}, +\infty)$ and $\lim_{d\to+\infty} h(d) = K$. Then there exists a unique $d_0 > z_2$ such that $h(d_0) = f(d_0)$ and

$$m(d) = \begin{cases} f(d) & \text{if } d \in (0, d_0] \\ f(z_0(d)) & \text{if } d \in (d_0, +\infty). \end{cases}$$

Moreover, a minimizer for $m(d)$ is given by the following function

$$u^d(t) = \begin{cases} dt & \text{if } d \in (0, d_0] \\ z_0(d)t + (d - z_0(d))\chi_{[\overline{t},+\infty)}(t) & \text{if } d \in (d_0, +\infty). \end{cases}$$

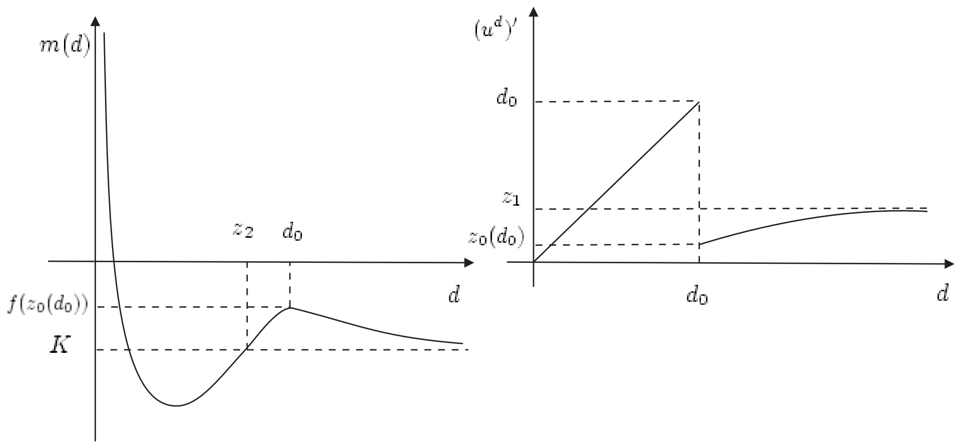

Figure 6: $m(d)$ and $(u^d)'$ in Example 5.3

Example 5.4 Let f be as in Example 5.3 and let g be as in Example 3.8 with $K > m$. We study $m(d)$ with this choice of f and g and with $C_\infty = 0$. By Theorem 4.8 and Proposition 5.2, we get that

$$m(d) = f(d) \wedge \min_{0<z<d} \{f(z) \vee g^{\mathrm{sm}}(d-z)\}.$$

Set

$$H(z,d) = f(z) \vee g^{\mathrm{sm}}(d-z), \quad h(d) = \min_{0<z<d} H(z,d),$$

and let $z_1 < z_2$ be such that

$$f^{-1}(K) = \{z_1, z_2\}.$$

As in Example 5.3, we easily infer that $m(d) = f(d)$ for $d \leq z_2$. For $d > z_2$ we distinguish the two cases $w_0 \leq z_2 - z_1$ and $w_0 > z_2 - z_1$.

Case 1: $w_0 \leq z_2 - z_1$. In this case, if $d \in [z_1 + kw_0, z_2 + kw_0]$, $k \in \mathbb{N}$, then $m(d) = H(d - kw_0, d) = g^{\rm sm}(kw_0) = K$. Thus, we have

$$m(d) = \begin{cases} f(d) & \text{if } d \in (0, z_2] \\ K & \text{if } d \in (z_2, +\infty) \end{cases}$$

and a minimizer for $m(d)$ is given by the following function

$$u^d(t) = \begin{cases} dt & \text{if } d \in (0, z_2] \\ (d - kw_0)t + w_0 \sum_{i=1}^k \chi_{[t_i,+\infty)}(t) & \\ \qquad\qquad \text{if } d \in (z_2 + (k-1)w_0, z_2 + kw_0],\ k \in \mathbb{N}, \end{cases}$$

where $(t_i)_{i\in\mathbb{N}}$ is any increasing sequence of points in $(0, 1)$.

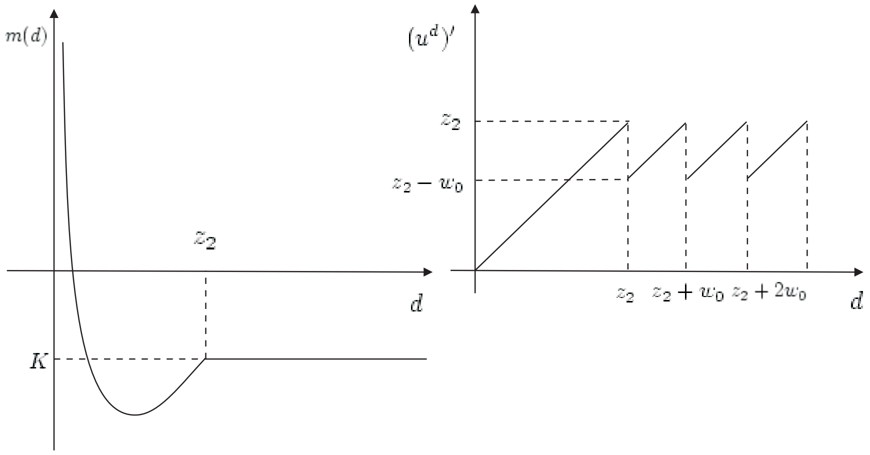

Figure 7: $m(d)$ and $(u^d)'$ in Example 5.4 - Case 1

Case 2: $w_0 > z_2 - z_1$. Analogously to Example 5.3, the function $z_0 : (\hat{z}, w_0 + z_1] \to (\hat{z}, z_1]$ defined as

$$z_0(d) = \min\{z > \hat{z} : f(z) = g(d - z)\}$$

is continuous, increasing, and such that

$$h(d) = f(z_0(d)) = g(d - z_0(d)) \quad \text{ and } \quad z_0(w_0 + z_1) = z_1.$$

In particular, h is continuous and decreasing in $(\hat{z}, w_0 + z_1]$ and $h(w_0 + z_1) = K$. Then there exists a unique $d_0 \in (z_2, w_0 + z_1]$ such that $h(d_0) = f(d_0)$ and

$$m(d) = \begin{cases} f(d) & \text{if } d \in (0, d_0] \\ f(z_0(d)) & \text{if } d \in (d_0, w_0 + z_1]. \end{cases}$$

If $k \in \mathbb{N}$ and $d \in [kw_0 + z_1, kw_0 + z_2]$, then $m(d) = h(d) = H(d - kw_0, d) = g^{\rm sm}(kw_0) = K$.

If $d \in [kw_0 + z_2, (k+1)w_0 + z_1]$, then it can be easily shown that there exist two increasing and continuous functions $z_k^1 : [kw_0+z_2, (k+1)w_0+z_1] \to [\hat{z}, z_1]$, $z_k^2 : [kw_0+z_2, (k+1)w_0+z_1] \to [z_2, w_0+z_1]$ such that

$$z_k^1((k+1)w_0 + z_1) = z_1, \quad z_k^2(kw_0 + z_2) = z_2,$$

$$f(z_k^1(d)) = g^{\rm sm}(d - z_k^1(d)) = g\Big(\frac{d - z_k^1(d)}{k+1}\Big),$$

$$f(z_k^2(d)) = g^{\mathrm{sm}}(d - z_k^2(d)) = g\big(\frac{d - z_k^2(d)}{k}\big),$$

and

$$m(d) = f(z_k^1(d)) \wedge f(z_k^2(d)).$$

Set

$$h_k^1(d) := f(z_k^1(d)), \quad h_k^2(d) := f(z_k^2(d)).$$

Since h_k^1 is continuous and decreasing, h_k^2 is continuous and increasing, and

$$\begin{array}{l} h_k^1((k+1)w_0 + z_1) = K < h_k^2((k+1)w_0 + z_1), \\ h_k^2(kw_0 + z_2) = K < h_k^1(kw_0 + z_2), \end{array}$$

there exist a unique $d_k \in (kw_0 + z_2, (k+1)w_0 + z_1)$ such that $h_k^1(d_k) = h_k^2(d_k)$ and

$$m(d) = \begin{cases} h_k^2(d) & \text{if } d \in [kw_0 + z_2, d_k] \\ h_k^1(d) & \text{if } d \in (d_k, (k+1)w_0 + z_1]. \end{cases}$$

Note that $m(d_k) \searrow K$ as $k \to +\infty$.

Eventually, we can write

$$m(d) = \begin{cases} f(d) & \text{if } d \in (0, d_0] \\ f(z_0(d)) & \text{if } d \in (d_0, w_0 + z_1] \\ K & \text{if } d \in (kw_0 + z_1, kw_0 + z_2],\ k \in \mathbb{N} \\ f(z_k^2(d)) & \text{if } d \in (kw_0 + z_2, d_k],\ k \in \mathbb{N} \\ f(z_k^1(d)) & \text{if } d \in (d_k, (k+1)w_0 + z_1],\ k \in \mathbb{N}, \end{cases}$$

and a minimizer for $m(d)$ is given by the following function

$$u^d(t) = \begin{cases} dt & \text{if } d \in (0, d_0] \\ z_0(d)t + (d - z_0(d))\chi_{[\bar{t}, +\infty)}(t) & \text{if } d \in (d_0, w_0 + z_1] \\ (d - kw_0)t + w_0 \sum_{i=1}^{k} \chi_{[t_i, +\infty)}(t) & \text{if } d \in (kw_0 + z_1, kw_0 + z_2],\ k \in \mathbb{N} \\ z_k^2(d)t + \sum_{i=1}^{k} \big(\frac{d - z_k^2(d)}{k}\big)\chi_{[t_i, +\infty)}(t) & \text{if } d \in (kw_0 + z_2, d_k],\ k \in \mathbb{N} \\ z_k^2(d)t + \sum_{i=1}^{k+1} \big(\frac{d - z_k^1(d)}{k+1}\big)\chi_{[t_i, +\infty)}(t) & \text{if } d \in (d_k, (k+1)w_0 + z_1],\ k \in \mathbb{N}, \end{cases}$$

where $(t_i)_{i \in \mathbb{N}}$ is any increasing sequence of points in $(0, 1)$.

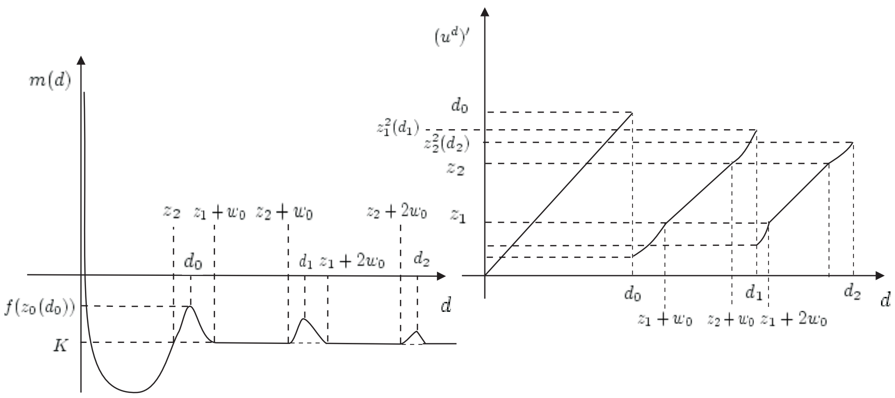

Figure 8: $m(d)$ and $(u^d)'$ in Example 5.4 - Case 2

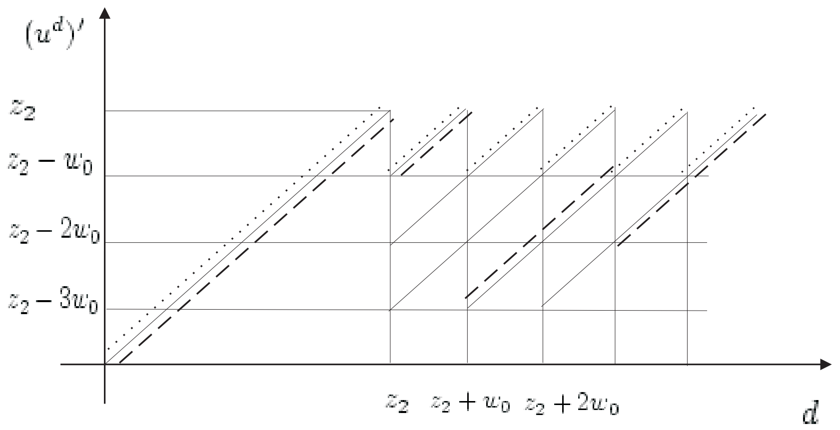

Figure 9: Non uniqueness: the dashed and the pointed $(u^d)'$ are equi-energetic.

Remark 5.5 Note that, in general, many equi-energetic states are possible. This feature is highlighted in Fig. 9 where a non unique multiple crack behavior is shown in the particular case $[\frac{z_2 - z_1}{w_0}] = 3$.

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References

[1] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford University Press, Oxford, 2000.

[2] E. Acerbi, G. Buttazzo and F. Prinari, The class of functionals which can be represented by a supremum. *J. Convex Anal.* **9** (2002), 225–236.

[3] G. Aronsson, Extension of functions satisfying Lipschitz conditions. *Ark. Mat.* **6** (1967), 551–561.

[4] A. Braides, *Approximation of Free-Discontinuity Problems*, Lecture Notes in Mathematics, Springer Verlag, Berlin, 1998.

[5] A. Braides, *Γ-convergence for Beginners*, Oxford University Press, Oxford, 2002.

[6] A. Braides, G. Dal Maso and A. Garroni. Variational formulation of softening phenomena in fracture mechanics: the one-dimensional case. *Arch. Rational Mech. Anal.* **146** (1999), 23–58.

[7] E. N. Barron, R. R. Jensen and C. Y. Wang, Lower semicontinuity of L^∞ functionals. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **18** (2001), 495–517.

[8] G. Bouchitté and G. Buttazzo. Integral representation of nonconvex functionals defined on measures. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **9** (1992), 101–117.

[9] G. Buttazzo. *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations.* Pitman, London, 1989.

[10] A. Carpinteri, Cusp catastrophe interpretation of fracture instability. *J. Mech. Phys. Solids* **37** (1989), 567–582.

[11] T. Champion, L. De Pascale and F. Prinari, Semicontinuity ad absolute minimizers for supremal functionals, to appear.

[12] G. Dal Maso, *An Introduction to Γ-convergence*, Birkhäuser, Boston, 1993.

[13] G.A. Francfort and J.J.Marigo, Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids* **46** (1998), 1319–1342.

[14] A. Garroni, V. Nesi and M. Ponsiglione, Dielectric breakdown: optimal bounds. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **457** (2001), 2317–2335.

[15] M. Gori, On the definition of a supremal functional on BV. Preprint Università di Pisa, 2002.

[16] J-M. Morel and S. Solimini, *Variational methods in image segmentation. With seven image processing experiments.* Progress in Nonlinear Differential Equations and their Applications **14**. Birkhäuser Boston, Inc., Boston, 1995.

[17] F. Prinari, Relaxation and Gamma-convergence of supremal functionals. Preprint Università di Pisa, 2002.

[18] L. Truskinovsky, Fracture as a phase transition. *Contemporary Research in the Mechanics and Mathematics of Materials* (R.C. Batra and M.F. Beatty eds.), 322–332, CIMNE, Barcelona, 1996.

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