EMERGENCE OF TOPOLOGICAL AND GEOMETRIC DEFECTS IN THE Γ-LIMIT OF DISCRETE ENERGIES

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Abstract. We introduce and analyse a new variant of the two-dimensional XY-model energy which is suited to detect both topological defects and geometric defects in form of fractional vortices and domain walls, respectively. In contrast to previously introduced variants, the energies we consider here are defined without using an angular lifting of the $\mathbb{S}^1$-valued spin variables. Moreover, they combine in an explicit way the features of the XY-model energy on the one hand and weak-membrane energies on the other hand. This leads to simplified proofs of compactness and lower bound in the Γ-convergence analysis.

1. Introduction

The macroscopic behaviour of crystalline materials is highly influenced by the presence and interaction of material defects, that can roughly be described as local deviations from the otherwise regular crystalline structure. Such defects can be of different (co-)dimension ranging from point and line to planar and volume defects (see, *e.g.*, [26, Chapter 1.3]), and the derivation and analysis of corresponding mathematical models has constantly given rise to interesting challenges and problems. One challenge that has drawn the attention of the mathematical community in recent years consists in obtaining continuum-mechanical models for material defects from more elementary discrete models via a rigorous coarse-graining procedure by means of Γ-convergence. Such a discrete-to-continuum variational analysis has been applied to models for defects of different (co-)dimension including, among others, models for volume defects such as voids [22], for planar (or co-dimension 1) defects such as grain boundaries [21], and for co-dimension 2 defects such as dislocations [28, 2, 3, 4, 5, 19], to name just a few. It is also worth mentioning that in a reduced two-dimensional setting discrete energies for so-called screw dislocations have been shown to be equivalent to the XY-model energy used in micromagnetism (see [2]).

Here we are interested in further investigating the relation between the screw-dislocation and the (ferromagnetic) XY-spin model when both co-dimension 2 and co-dimension 1 defects are taken into account as in the recent contributions [11] and [8]. Those defects are typically referred to as topological and geometric defects, and they are observed in nature as (partial) dislocations and stacking faults in the context of crystal plasticity [25, 26] or as (fractional) vortices and domain walls in the context of micromagnetism [31]. In the reduced two-dimensional setting we will consider here, they are point and line defects. To motivate ideas we start recalling the definition of the screw-dislocation and the XY-model energy. The latter is defined on spin fields $v : \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathbb{S}^1$ mapping from the portion of an ε-spaced square lattice contained in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ to the unit vectors, and it assigns to any such spin field the energy

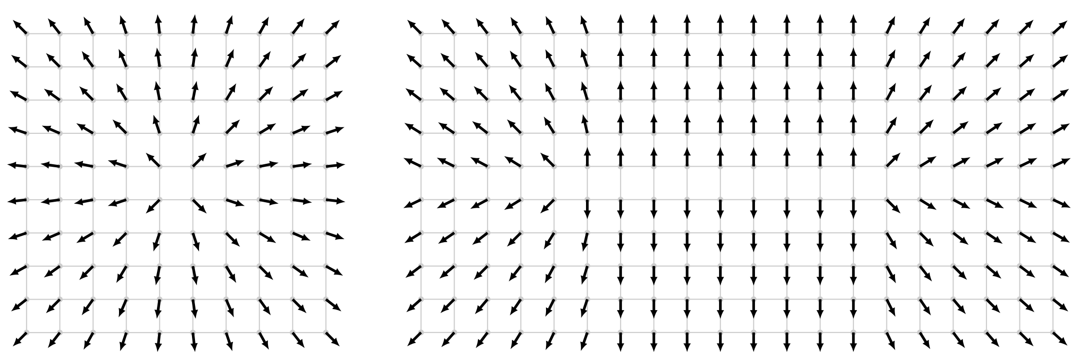
$$XY_\varepsilon(v) = \frac{1}{2} \sum_{(i,j)\ \text{n.n.}} |v(j) - v(i)|^2 . \tag{1.1}$$

In (1.1) the sum is taken over all pairs of nearest neighbours $i, j \in \varepsilon\mathbb{Z}^2 \cap \Omega$. The XY-model energy is well-suited to detect topological singularities at the leading order logarithmic scaling. Roughly speaking, configurations v_ε satisfying a uniform bound $XY_\varepsilon(v_\varepsilon) \leq C|\log\varepsilon|$ develop in the limit as $\varepsilon \to 0$ finitely many point singularities (interpreted as vortices) as those in Figure 1a. Around these singularities an energetic contribution of order $|\log\varepsilon|$ is stored (see [1, 3]).

Upon choosing an angular lifting $u : \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathbb{R}$ of v satisfying $v = \exp(2\pi\imath u)$ we have that $\frac{1}{2}|v(j) - v(i)|^2 = 1 - \cos\big(2\pi(u(j) - u(i)\big)$ for i, j. This allows to rewrite $XY_\varepsilon(v)$ in terms of the angular variable u, which in turn makes it possible to relate the XY-model energy to the screw-dislocation energy. Indeed, the latter associates to any scalar configuration $u : \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathbb{R}$ (interpreted as a lattice displacement in the context of plasticity) an energy of the form

$$SD_\varepsilon(u) = 2\pi^2 \sum_{(i,j)\text{ n.n.}} \operatorname{dist}^2\big(u(j) - u(i); \mathbb{Z}\big)\,. \tag{1.2}$$

Since $2\pi^2 \operatorname{dist}^2(t; \mathbb{Z}) \approx 1 - \cos(2\pi t)$ for t close to $\mathbb{Z}$, one can show that XY_ε and SD_ε have the same asymptotic behaviour at the leading order logarithmic scaling (see [2] and [3]). Moreover, rewriting XY_ε in terms of the angular variable is the starting point in [11] for introducing a class of discrete energies that is suited to detect the formation of both topological and geometric defects which in the spin variable correspond to fractional vortices and domain walls as sketched in Figure 1b.



(a) Vortex singularity **(b)** Fractional vortices and string defect

Figure 1. Schematic pictures of a topological singularity in form of a vortex (left) and of two topological singularities in form of fractional vortices that are connected by a geometric singularity (right).

Here, we follow the approach in [11] to obtain a model for partial screw dislocations and stacking faults, which we use as a point of reference (see Section 2.4 for a precise definition). Instead, for the XY-model energy we take a different approach and define discrete energies directly on the spin field v. This is done by taking inspiration from a different class of discrete energies which is commonly used to detect line singularities. Those are discrete energies that approximate free-discontinuity problems as considered, *e.g.*, in [16, 17, 30, 7]. A prototypical energy of this type is the so-called weak-membrane energy which assigns to any configuration $w : \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathbb{R}^2$ an energy of the form

$$WM_\varepsilon(w) = \sum_{i,j\text{ n.n.}} \min\big\{|w(j) - w(i)|^2, \varepsilon\big\}\,. \tag{1.3}$$

It is known that the weak-membrane energies Γ-converge as $\varepsilon \to 0$ to a free-discontinuity functional, *i.e.*, a functional consisting of a bulk and a surface contribution. In particular, the energies are suited to detect line defects.

Energies as in (1.3) were already exploited in [8] to detect line defects in a model for partial edge dislocations and stacking faults. Although these energies itself are not suited to detect topological defects (see Example 4.3), we use them to define a variant of the XY-model energy which combines the features of both energies and is thus indeed able to detect topological and geometric singularities. Specifically, for fixed $\mathbf{n} \in \mathbb{N}$ we consider energies of the form

$$XY_\varepsilon^{\mathrm{frac}}(v) = \sum_{i,j \text{ n.n.}} \max\left\{ \tfrac{1}{\mathbf{n}^2} |v^{\mathbf{n}}(j) - v^{\mathbf{n}}(i)|^2, \min\left\{ |v(j) - v(i)|^2, \varepsilon \right\} \right\}. \tag{1.4}$$

Here, $v^{\mathbf{n}}$ is the complex $\mathbf{n}$-th power of $v : \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathbb{S}^1$ which allows for fractional-integer windings of the spin field. Moreover, for small values of $|v(j) - v(i)|$ we have that $|v(j) - v(i)|^2 \approx \frac{1}{\mathbf{n}^2}|v^{\mathbf{n}}(j) - v^{\mathbf{n}}(i)|^2$. This suggests that far from the singularities the two terms should give the same energetic contribution. Finally, for specific jumps of v, namely jumps by $\frac{2\pi}{\mathbf{n}}$ (as in Figure 1b for $\mathbf{n} = 2$), the field $v^{\mathbf{n}}$ will not jump and thus the value of $|v^{\mathbf{n}}(j) - v^{\mathbf{n}}(i)|$ will be of order ε^2, while $|v(j) - v(i)|$ will be of order one. Thus, for those terms the dominating term will be ε, which makes the energetic contribution of macroscopic line defects across which the spin field jumps by $\frac{2\pi}{\mathbf{n}}$ of order one.

The above heuristics are made rigorous in Theorem 4.1 where we show that after removing the logarithmic contribution of a fixed number of limiting topological defects the energies $XY_\varepsilon^{\mathrm{frac}}$ converge as $\varepsilon \to 0$ to a continuum energy consisting of three terms. More specifically, Theorem 4.1 implies that for fixed $M \in \mathbb{N}$ the Γ-limit of the excess energies $XY_\varepsilon^{\mathrm{frac}}(v) - \frac{M\pi}{\mathbf{n}^2}|\log\varepsilon|$ is given by

$$E^{\mathrm{frac}}(w) = \frac{M}{\mathbf{n}^2}\gamma + \frac{1}{\mathbf{n}^2}\mathcal{W}(w^{\mathbf{n}}) + \int_{S_w} |\nu|_1 \, \mathrm{d}\mathcal{H}^1, \quad w \in \mathcal{D}_M^{\frac{1}{\mathbf{n}}}(\Omega). \tag{1.5}$$

Here, $\mathcal{D}_M^{\frac{1}{\mathbf{n}}}(\Omega)$ is a suitable subclass of $\mathbb{S}^1$-valued SBV-functions w whose jumpset S_w has finite length, their complex $\mathbf{n}$-th power $w^{\mathbf{n}}$ is a Sobolev function, and the jacobian $J(w^{\mathbf{n}})$ is a sum of signed dirac measures supported on the limiting topological defects (see 4.4 for the precise definition). The function w in (1.5) is obtained as weak H^1_{loc}-limit of the sequence of spin fields v_ε away from the limiting point defects. Moreover, γ is the core energy of the XY-model energy, a fixed quantity concentrated around each limiting topological defect. The far-field energy $\mathcal{W}$ instead accounts for the interaction of the topological defects and coincides again with the far-field energy of the XY-model energy. Finally, the surface contribution is concentrated on the limiting line defects and coincides with the surface contribution of the weak-membrane energies. In this way, we provide here a class of discrete energies which have the same asymptotic behaviour in terms of Γ-convergence as the discrete energies defined in [11]. Re-interpreting the latter energies as a model for partial screw dislocations and stacking faults thus allows to compare those models with XY-spin models in a framework that takes into account both topological and geometric defects. At the same time, defining $XY_\varepsilon^{\mathrm{frac}}$ directly in terms of the spin variable v makes it more evident how to compare those energies with the weak-membrane energies, which in turn leads to a simplified proof of the lower-bound inequality and the corresponding compactness result. Finally, a crucial step in obtaining the upper-bound inequality consists in showing that the core energy concentrated around each limiting point singularity is the same (up to a factor) for the fractional XY-model energy (1.4) and the original XY-model energy (1.1). This can be shown in a concise way using the recent result [23, Theorem 2.4] which allows us to choose competitors for the minimization problem defining γ (see (4.12) for the precise definition) that are suitably regular in the sense that they do not contain so-called short dipoles.

We finally observe that the limiting energies we obtain here also appear as Γ-limits of the continuum phase-field functionals introduced in [24] which are defined by coupling suitable variants of the Ginzburg-Landau functionals and the Ambrosio-Tortorelli functionals. Building upon the analysis carried out in the present paper and using the similarities between weak-membrane energies and discretizations of the Ambrosio-Tortorelli functionals (see [10, Section 3.3]), it will thus be a subject of further investigation to provide a discrete counterpart of those phase-field functionals.

2. Notation and Setting of the problem

In this section we fix some basic notation and introduce the discrete energies considered in this paper.

2.1. **Basic notation.** We start fixing some notation employed throughout. Given two unit vectors $a, b \in \mathbb{S}^1$ we let $\mathrm{d}_{\mathbb{S}^1}(a,b) := 2\arcsin(\frac{1}{2}|a-b|)$ denote the geodesic distance between them. We recall that

$$|a-b| \leq \mathrm{d}_{\mathbb{S}^1}(a,b) \leq \frac{\pi}{2}|a-b|\,, \tag{2.1}$$

where $|\cdot|$ denotes the euclidian distance in $\mathbb{R}^2$. For any $R > r > 0$ we set $B_r := \{x \in \mathbb{R}^2 \colon |x| < r\}$ and $A_{r,R} := B_R \setminus \overline{B}_r$. Finally, for any $x \in \mathbb{R}^2$ we set $B_r(x) := B_r + x$ and $A_{r,R}(x) := A_{r,R} + x$. Moreover, the closed segment joining two vectors $x, y \in \mathbb{R}^2$ is denoted by $[x,y]$.

Throughout the note, $\varepsilon > 0$ is a positive parameter varying in strictly decreasing sequence converging to zero.

2.2. **The discrete lattice.** Throughout we will consider the ε-spaced square lattice $\varepsilon\mathbb{Z}^2$. For any $\varepsilon > 0$, $k \in \{1,2\}$, and any Borel subset $A \subset \mathbb{R}^2$ we let

$$\mathbb{Z}_\varepsilon^{e_k}(A) := \big\{i \in \varepsilon\mathbb{Z}^2 \cap A \colon [i, i+\varepsilon e_k] \subset A\big\} \tag{2.2}$$

denote the portion of lattice points contained in A for which the closed segment joining i and its nearest neighbour $i + \varepsilon e_k$ is contained in A as well. Moreover, we define the collection of closed cubes subordinated to the lattice $\varepsilon\mathbb{Z}^2$ via

$$\mathcal{Q}_\varepsilon := \big\{Q_\varepsilon = Q_\varepsilon(i) = i + [0,\varepsilon]^2 \colon i \in \varepsilon\mathbb{Z}^2\big\}\,, \tag{2.3}$$

and for any cube $Q_\varepsilon = i + [0,\varepsilon]^2 \in \mathcal{Q}_\varepsilon$ we write $b(Q_\varepsilon) := i + \frac{\varepsilon}{2}(e_1 + e_2)$ for its barycentre. For any Borel set $A \subset \mathbb{R}^2$ we let

$$\mathcal{Q}_\varepsilon^{\mathrm{int}}(A) := \{Q_\varepsilon \in \mathcal{Q}_\varepsilon \colon Q_\varepsilon \subset A\} \quad \text{and} \quad \mathcal{Q}_\varepsilon^{\mathrm{ext}}(A) := \{Q_\varepsilon \in \mathcal{Q}_\varepsilon \colon Q_\varepsilon \cap A \neq \emptyset\}$$

denote the subclasses of lattice cubes contained in A and intersecting A, respectively. Accordingly we set

$$A_\varepsilon^{\mathrm{int}} := \bigcup_{Q_\varepsilon \in \mathcal{Q}_\varepsilon^{\mathrm{int}}(A)} Q_\varepsilon \quad \text{and} \quad A_\varepsilon^{\mathrm{ext}} := \bigcup_{Q_\varepsilon \in \mathcal{Q}_\varepsilon^{\mathrm{ext}}(A)} Q_\varepsilon\,.$$

Finally $\partial_\varepsilon A := \varepsilon\mathbb{Z}^2 \cap \partial A_\varepsilon^{\mathrm{int}}$ denotes the discrete boundary of A. It is also convenient to fix a triangulation $\mathcal{T}_\varepsilon$ of $\mathbb{R}^2$ subordinated to $\varepsilon\mathbb{Z}^2$ by setting

$$\mathcal{T}_\varepsilon := \Big\{T_\varepsilon^+ = \mathrm{conv}(i, i+\varepsilon e_2, i+\varepsilon(e_1+e_2))\,,\ T_\varepsilon^- = \mathrm{conv}(i, i+\varepsilon e_1, i+\varepsilon(e_1+e_2)) \colon i \in \varepsilon\mathbb{Z}^2\Big\}\,. \tag{2.4}$$

Here conv denotes the closed and convex hull.

Next we introduce the sets of discrete variables taking values in the real numbers and the unit sphere, denoted respectively by

$$\mathcal{AD}_\varepsilon := \{u : \varepsilon\mathbb{Z}^2 \to \mathbb{R}\} \quad \text{and} \quad \mathcal{SF}_\varepsilon := \{v : \varepsilon\mathbb{Z}^2 \to \mathbb{S}^1\}\,.$$

We will often refer to variables in $\mathcal{AD}_\varepsilon$ and $\mathcal{SF}_\varepsilon$ as admissible displacements and spin fields, respectively. For $u, \widetilde{u} \in \mathcal{AD}_\varepsilon$ we write

$$u \overset{\mathbb{Z}}{\equiv} \widetilde{u} \quad \text{if} \quad u(i) - \widetilde{u}(i) \in \mathbb{Z} \text{ for every } i \in \varepsilon\mathbb{Z}^2\,.$$

Upon identifying $\mathbb{R}^2$ with $\mathbb{C}$ we can associate to any $u \in \mathcal{AD}_\varepsilon$ a spin field $v \in \mathcal{SF}_\varepsilon$ by considering the complex exponential

$$v = \exp(2\pi\iota u) \tag{2.5}$$

with ι being the imaginary unit. Vice versa, any $v \in \mathcal{SF}_\varepsilon$ can be written in the form (2.5) for some $u \in \mathcal{AD}_\varepsilon$ (not unique). In this case, we will refer to u as an angular lifting of v. We shall often interpret points on $\mathbb{S}^1$ as complex numbers and implicitly use complex products and complex powers.

2.3. **Discrete gradients and discrete topological singularities.** For $u \in \mathcal{AD}_\varepsilon$, $v \in \mathcal{SF}_\varepsilon$ and $i, j \in \varepsilon\mathbb{Z}^2$ with $|i-j| = \varepsilon$ we consider the directional discrete derivatives

$$\mathrm{d}u(i,j) := u(j) - u(i) \qquad \mathrm{d}v(i,j) := v(j) - v(i)\,. \tag{2.6}$$

Moreover, to any $u \in \mathcal{AD}_\varepsilon$ we associate a discrete vorticity measure as follows. For any $t \in \mathbb{R}$ let

$$P_{\mathbb{Z}}(t) := \operatorname{argmin}\{|t-z| \colon z \in \mathbb{Z}\} \tag{2.7}$$

denote its projection onto $\mathbb{Z}$ (with the convention of taking the minimal argmin in (2.7) if it is not unique). In other words, $P_{\mathbb{Z}}(t) = \lceil t - \frac{1}{2} \rceil$. In this way,

$$P_{\mathbb{Z}}(t+z) = P_{\mathbb{Z}}(t) + z \quad \text{for every } t \in \mathbb{R} \text{ and every } z \in \mathbb{Z}. \tag{2.8}$$

For $i, j \in \varepsilon\mathbb{Z}^2$ with $|i-j| = \varepsilon$ we now define the elastic part of $\mathrm{d}u(i,j)$ by

$$\mathrm{d}^e u(i,j) := \begin{cases} \mathrm{d}u(i,j) - P_{\mathbb{Z}}\big(\mathrm{d}u(i,j)\big) & \text{if } i \leq j\,, \\ \mathrm{d}u(i,j) + P_{\mathbb{Z}}\big(\mathrm{d}u(j,i)\big) & \text{if } j \leq i\,, \end{cases} \tag{2.9}$$

where $i = (i_1, i_2) \leq j = (j_1, j_2)$ means that $i_1 \leq j_1$ and $i_2 \leq j_2$. Note that $\mathrm{d}^e u(i,j) = -\mathrm{d}^e u(j,i)$. Moreover

$$|\mathrm{d}^e u(i,j)| = \operatorname{dist}\big(\mathrm{d}u(i,j); \mathbb{Z}\big)\,. \tag{2.10}$$

Using $\mathrm{d}^e u$ we can associate a discrete circulation measure μ_u to u as follows. For any cube $Q_\varepsilon(i) \in \mathcal{Q}_\varepsilon$ with lower left corner i and vertices $\{i, j, k, \ell\}$ ordered counter clockwise we define $\mu_u(Q_\varepsilon(i))$ by setting

$$\mu_u(Q_\varepsilon(i)) := \mathrm{d}^e u(i,j) + \mathrm{d}^e u(j,k) + \mathrm{d}^e u(k,\ell) + \mathrm{d}^e u(\ell,i)\,. \tag{2.11}$$

By construction, $\mu_u(Q_\varepsilon) \in \{-1, 0, 1\}$. We then define the measure μ_u as

$$\mu_u := \sum_{Q_\varepsilon \in \mathcal{Q}_\varepsilon} \mu_u(Q_\varepsilon) \delta_{b(Q_\varepsilon)}\,. \tag{2.12}$$

If $v \in \mathcal{SF}_\varepsilon$ we write $v = \exp(2\pi\iota u)$ for some $u \in \mathcal{AD}_\varepsilon$ and set

$$\mu_v(Q_\varepsilon) := \mu_u(Q_\varepsilon) \quad \text{and} \quad \mu_v := \mu_u\,. \tag{2.13}$$

The measure μ_v is well-defined, since it does not depend on the choice of the angular lifting u. Indeed, if $u, \widetilde{u} \in \mathcal{AD}_\varepsilon$ with $u \overset{\mathbb{Z}}{\equiv} \widetilde{u}$ one can use (2.8) to verify that $\mathrm{d}^e u(i,j) = \mathrm{d}^e \widetilde{u}(i,j)$ for every $i, j \in \varepsilon\mathbb{Z}^2$, $|i-j| = \varepsilon$. Thus, $\mu_u = \mu_{\widetilde{u}}$.

2.4. **Definition of the discrete energies.** This section collects all discrete energies we will consider. To define them, it is convenient to first introduce the following pairwise interaction-energy densities. For every $n \in \mathbb{N} \setminus \{0\}$ we define $f_{\frac{1}{n}} : \mathbb{R} \to [0, +\infty)$ by

$$f_{\frac{1}{n}}(t) := 2\pi^2 \operatorname{dist}^2\left(t, \frac{1}{n}\mathbb{Z}\right).$$

In this way, we have that

$$f_{\frac{1}{n}}(t) = \frac{1}{n^2} f_1(nt) \quad \text{for every } t \in \mathbb{R} \tag{2.14}$$

and

$$f_1(t) = f_{\frac{1}{n}}(t) \quad \iff \quad \operatorname{dist}(t, \mathbb{Z}) \leq \frac{1}{2n}. \tag{2.15}$$

Moreover, for any $\varepsilon, \tau > 0$ we define $f^{\tau}_{\varepsilon, \frac{1}{n}}, g^{\tau}_{\varepsilon} : \mathbb{R} \to [0, +\infty)$ by

$$f^{\tau}_{\varepsilon, \frac{1}{n}}(t) := \max\left\{ f_{\frac{1}{n}}(t), \varepsilon\tau \mathbb{1}_{\{x:\ \operatorname{dist}(x,\mathbb{Z}) > \frac{1}{2\mathbf{n}}\}}(t) \right\} \qquad \text{and} \qquad g^{\tau}_{\varepsilon}(t) := \min\left\{ \frac{t^2}{2}, \varepsilon\tau \right\}.$$

We now introduce the two main families of discrete energies under consideration. Throughout we fix $\tau_1, \tau_2 > 0$ and $\mathbf{n} \in \mathbb{N}$. For any $u \in \mathcal{AD}_\varepsilon$, $w \in \mathcal{SF}_\varepsilon$, and $A \subset \mathbb{R}^2$ Borel we define

$$SD^{\mathrm{part}}_{\varepsilon, \frac{1}{\mathbf{n}}}(u, A) := \sum_{i \in \mathbb{Z}^{e_1}_\varepsilon(A)} f^{\tau_1}_{\varepsilon, \frac{1}{\mathbf{n}}}\big(\mathrm{d}u(i, i + \varepsilon e_1)\big) + \sum_{i \in \mathbb{Z}^{e_2}_\varepsilon(A)} f^{\tau_2}_{\varepsilon, \frac{1}{\mathbf{n}}}\big(\mathrm{d}u(i, i + \varepsilon e_2)\big) \tag{2.16}$$

and

$$\begin{aligned} XY^{\mathrm{frac}}_{\varepsilon, \frac{1}{\mathbf{n}}}(w, A) := &\sum_{i \in \mathbb{Z}^{e_1}_\varepsilon(A)} \max\left\{ \frac{1}{2\mathbf{n}^2} |\mathrm{d}w^{\mathbf{n}}(i, i + \varepsilon e_1)|^2, g^{\tau_1}_\varepsilon\left(\mathrm{d}w(i, i + \varepsilon e_1)\right) \right\} \\ &+ \sum_{i \in \mathbb{Z}^{e_2}_\varepsilon(A)} \max\left\{ \frac{1}{2\mathbf{n}^2} |\mathrm{d}w^{\mathbf{n}}(i, i + \varepsilon e_2)|^2, g^{\tau_2}_\varepsilon\left(\mathrm{d}w(i, i + \varepsilon e_2)\right) \right\}. \end{aligned} \tag{2.17}$$

We also recall the definition of the screw dislocation energy and the XY-model energy studied in [3] and [1], respectively. For every $A \subset \mathbb{R}^2$ Borel, $u \in \mathcal{AD}_\varepsilon$, and $v \in \mathcal{SF}_\varepsilon$ they are given by

$$SD_\varepsilon(u, A) := \sum_{k=1}^{2} \sum_{i \in \mathbb{Z}^{e_k}_\varepsilon(A)} f_1\big(\mathrm{d}u(i, i + \varepsilon e_k)\big) \tag{2.18}$$

and

$$XY_\varepsilon(v, A) := \frac{1}{2} \sum_{k=1}^{2} \sum_{i \in \mathbb{Z}^{e_k}_\varepsilon(A)} |\mathrm{d}v(i, i + \varepsilon e_k)|^2, \tag{2.19}$$

respectively. Finally, we will make use of an anisotropic variant of the weak-membrane energies. In our setting, they will depend on the parameters $\tau_1, \tau_2 > 0$; specifically, we set

$$WM_\varepsilon(w, A) := \sum_{i \in \mathbb{Z}^{e_1}_\varepsilon(A)} g^{\tau_1}_\varepsilon(\mathrm{d}w(i, i + \varepsilon e_1)) + \sum_{i \in \mathbb{Z}^{e_2}_\varepsilon(A)} g^{\tau_2}_\varepsilon(\mathrm{d}w(i, i + \varepsilon e_2)) \tag{2.20}$$

for any $A \subset \mathbb{R}^2$ Borel and $w : \varepsilon\mathbb{Z}^2 \to \mathbb{R}^2$. It is well known that the discrete energies SD_ε and XY_ε (when scaled properly) account for topological defects in the continuum Γ-limit (see [1] and [3]). Those topological defects are then identified with screw dislocations and vortices, respectively. Instead, the macroscopic Γ-limit of the weak-membrane energies is a free-discontinuity functional consisting of a volume and a surface term (see [17] and [30]). In particular, in our two-dimensional setting it accounts for line defects. Based on those two results we will show that the macroscopic

Γ-limit of suitably scaled versions of $SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and $XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ accounts for both topological defects and line defects.

3. Preliminary results

In this section we collect some preliminary results that allow to compare the discrete energies introduced in Section 2.4 with each other. Moreover, we recall some useful interpolation results.

3.1. **Comparison between the discrete models.** We start comparing the discrete energies introduced in Section 2.4. This comparison will be useful to derive compactness properties and lower bounds for the energies $SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and $XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$. A first observation is that the discrete models $SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and $XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ accounting for partial dislocations and fractional vortices, respectively, are lower bounded by the corresponding models accounting for full dislocations and vortices.

Remark 3.1. Let $u \in \mathcal{AD}_\varepsilon$ and let $A \subset \mathbb{R}^2$ be a Borel set. The definition of $SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ together with (2.14) implies that

$$SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(u,A) \geq \frac{1}{\mathbf{n}^2} SD_\varepsilon(\mathbf{n}u, A)\,. \tag{3.1}$$

Moreover, for any $w \in \mathcal{SF}_\varepsilon$ by definition we have that

$$XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}(w,A) \geq \frac{1}{\mathbf{n}^2} XY_\varepsilon(w^{\mathbf{n}}, A)\,. \tag{3.2}$$

A second observation is the following elementary comparison between SD_ε and XY_ε.

Remark 3.2 (Comparison between XY_ε and SD_ε). Let $v \in \mathcal{SF}_\varepsilon$ and let $u \in \mathcal{AD}_\varepsilon$ be any angular lifting of v. Then the estimate

$$SD_\varepsilon(u,A) \geq XY_\varepsilon(v,A) \tag{3.3}$$

holds for any Borel set $A \subset \mathbb{R}^2$. This is an immediate consequence of the identity

$$2\pi^2 \operatorname{dist}^2\big(\mathrm{d}u(i,j);\mathbb{Z}\big) = \frac{1}{2}\mathrm{d}^2_{\mathbb{S}^1}\big(v(i),v(j)\big) \tag{3.4}$$

for any $i,j \in \varepsilon\mathbb{Z}^2$ with $|i-j| = \varepsilon$. Indeed, (2.1) together with (3.4) implies that

$$f_1\big(\mathrm{d}u(i,j)\big) \geq \frac{1}{2}|\mathrm{d}v(i,j)|^2 \tag{3.5}$$

for any $i,j \in \varepsilon\mathbb{Z}^2$ with $|i-j| = \varepsilon$, which in turn implies (3.3).

Remark 3.2 allows us to lower bound the screw dislocation energies by the XY-model energies. The next lemma shows that an analogue estimate to (3.3) holds for the corresponding models for partial dislocations and fractional vortices.

Lemma 3.3 (Comparison between $SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and $XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$)**.** *Let $\varepsilon > 0$, let $w \in \mathcal{SF}_\varepsilon$, and let $u \in \mathcal{AD}_\varepsilon$ be an angular lifting of w. Then the estimate*

$$SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(u,A) \geq XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}(w,A) \tag{3.6}$$

holds for any Borel subset $A \subset \mathbb{R}^2$.

Proof. Let $w \in \mathcal{SF}_\varepsilon$ and $u \in \mathcal{AD}_\varepsilon$ be as in the statement. Moreover, set $v := w^{\mathbf{n}}$. To obtain (3.6), we fix $\tau > 0$ and we show that for any $i,j \in \varepsilon\mathbb{Z}^2$ with $|i-j| = \varepsilon$ we have

$$f^\tau_{\varepsilon,\frac{1}{\mathbf{n}}}\big(\mathrm{d}u(i,j)\big) \geq \max\left\{\frac{1}{2\mathbf{n}^2}|\mathrm{d}v(i,j)|^2, g^\tau_\varepsilon\big(\mathrm{d}w(i,j)\big)\right\}. \tag{3.7}$$

Then (3.6) follows by applying (3.7) with $j = i + \varepsilon e_1$, $\tau = \tau_1$ and $j = j + \varepsilon e_2$, $\tau = \tau_2$, respectively, and summing up over all i.

To establish (3.7) we fix $i, j \in \varepsilon\mathbb{Z}^2$ with $|i - j| = \varepsilon$, and we start with the following preliminary observation. Since $v = w^{\mathbf{n}}$, $\mathbf{n}u$ is an angular lifting of v. Applying (3.5) with v and $\mathbf{n}u$ and using (2.14) thus yields

$$\frac{1}{2\mathbf{n}^2}|\mathrm{d}v(i,j)|^2 \leq \frac{1}{\mathbf{n}^2} f_1\big(\mathbf{n}\mathrm{d}u(i,j)\big) = f_{\frac{1}{\mathbf{n}}}\big(\mathrm{d}u(i,j)\big) \leq f^\tau_{\varepsilon,\frac{1}{\mathbf{n}}}\big(\mathrm{d}u(i,j)\big)\,. \tag{3.8}$$

Hence, (3.7) follows if we can show that also

$$g^\tau_\varepsilon\big(\mathrm{d}w(i,j)\big) \leq f^\tau_{\varepsilon,\frac{1}{\mathbf{n}}}\big(\mathrm{d}u(i,j)\big)\,.$$

Suppose first that $\mathrm{dist}(\mathrm{d}u(i,j);\mathbb{Z})) > \frac{1}{2\mathbf{n}}$. Then we have by definition

$$f^\tau_{\varepsilon,\frac{1}{\mathbf{n}}}(\mathrm{d}u(i,j)) \geq \varepsilon\tau \geq g^\tau_\varepsilon(\mathrm{d}w(i,j))\,.$$

If on the contrary $\mathrm{dist}(\mathrm{d}u(i,j);\mathbb{Z}) \leq \frac{1}{2\mathbf{n}}$, then (2.15) ensures that

$$f_1\big(\mathrm{d}u(i,j)\big) = f_{\frac{1}{\mathbf{n}}}\big(\mathrm{d}u(i,j)\big)\,. \tag{3.9}$$

Thus, applying (3.5) now with w and u yields

$$g^\tau_\varepsilon(\mathrm{d}w(i,j)) \leq \frac{1}{2}|\mathrm{d}w(i,j)|^2 \leq f_1\big(\mathrm{d}u(i,j)\big) = f_{\frac{1}{\mathbf{n}}}\big(\mathrm{d}u(i,j)\big) \leq f^\tau_{\varepsilon,\frac{1}{\mathbf{n}}}\big(\mathrm{d}u(i,j)\big)\,,$$

which concludes the proof. □

Remark 3.4 (Comparison with WM_ε). Let $\varepsilon > 0$, $w \in \mathcal{SF}_\varepsilon$, and $A \subset \mathbb{R}^2$ a Borel set. By definition, we clearly have $XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}(w, A) \geq WM_\varepsilon(w, A)$. Together with Lemma 3.3 we thus obtain the chain of inequalities

$$SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(u, A) \geq XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}(w, A) \geq WM_\varepsilon(w, A) \tag{3.10}$$

for any angular lifting $u \in \mathcal{AD}_\varepsilon$ of w.

The estimates collected in Remarks 3.1–3.2 and Lemma 3.3 allow us to lower bound on the one hand dislocation energies by vortex energies and on the other hand energies accounting for partial dislocations and fractional vortices by the corresponding energies for full dislocations and vortices. Using a Taylor expansion we obtain approximate reverse estimates far from the singularities.

Remark 3.5 (Comparison via Taylor expansion). Let $v \in \mathcal{SF}_\varepsilon$ and let $u \in \mathcal{AD}_\varepsilon$ be an angular lifting of v. Then we have that

$$\frac{1}{2}|\mathrm{d}v(i,j)|^2 = 1 - \cos(2\pi\mathrm{d}u(i,j))\,.$$

Expanding the cosine around $2\pi\mathbb{Z}$ thus yields the existence of a constant $C > 0$ such that

$$f_1\big(\mathrm{d}u(i,j)\big) \leq \frac{1}{2}|\mathrm{d}v(i,j)|^2 + C\,\mathrm{dist}^4(\mathrm{d}u(i,j);\mathbb{Z}) \leq \frac{1}{2}|\mathrm{d}v(i,j)|^2 + C|\mathrm{d}v(i,j)|^4\,, \tag{3.11}$$

where the last inequality follows from the identity (3.4) together with the second estimate in (2.1).

Suppose now that $w \in \mathcal{SF}_\varepsilon$ is given by $w := \exp\big(\frac{2\pi}{\mathbf{n}}\iota u\big)$. Similar to Remark 3.2 we obtain that

$$\frac{1}{2}|\mathrm{d}w(i,j)|^2 \leq f_1\big(\tfrac{\mathrm{d}u(i,j)}{\mathbf{n}}\big)\,. \tag{3.12}$$

In particular, if $\mathrm{dist}\big(\frac{\mathrm{d}u(i,j)}{\mathbf{n}};\mathbb{Z}\big) \leq \frac{1}{2\mathbf{n}}$, then (2.15) together with (2.14) ensures that

$$f_1\big(\tfrac{\mathrm{d}u(i,j)}{\mathbf{n}}\big) = f_{\frac{1}{\mathbf{n}}}\big(\tfrac{\mathrm{d}u(i,j)}{\mathbf{n}}\big) = \frac{1}{\mathbf{n}^2} f_1\big(\mathrm{d}u(i,j)\big)\,.$$

In combination with (3.11) and (3.12) this yields that

$$\frac{1}{2}|\mathrm{d}w(i,j)|^2 \le \frac{1}{2\mathbf{n}^2}|\mathrm{d}v(i,j)|^2 + C|\mathrm{d}v(i,j)|^4 \quad \text{if } \operatorname{dist}\big(\tfrac{\mathrm{d}u(i,j)}{\mathbf{n}};\mathbb{Z}\big) \le \frac{1}{2\mathbf{n}}\,. \tag{3.13}$$

3.2. **Interpolation of discrete functions.** We conclude this section by introducing useful interpolations of discrete functions.

Piecewise constant interpolations. Throughout the paper, discrete functions $u_\varepsilon\colon \varepsilon\mathbb{Z}^2 \to \mathbb{R}^m$ will be tacitly identified with their piecewise constant interpolations taking values $u_\varepsilon(i)$ on every cube $Q_\varepsilon(i) \in \mathcal{Q}_\varepsilon$.

Piecewise affine interpolations. For $\varepsilon > 0$ let $\mathcal{T}_\varepsilon$ be the triangulation defined in (2.4); for any $v_\varepsilon \in \mathcal{SF}_\varepsilon$ we let $\hat{v}_\varepsilon \in H^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ denote the function satisfying $\hat{v}_\varepsilon(i) = v_\varepsilon(i)$ for every $i \in \varepsilon\mathbb{Z}^2$ and being affine on every triangle $T^+_\varepsilon, T^-_\varepsilon \in \mathcal{T}_\varepsilon$. In this way, on every cube $Q_\varepsilon \in \mathcal{Q}_\varepsilon$ the identity

$$XY_\varepsilon(v_\varepsilon, Q_\varepsilon) = \int_{Q_\varepsilon} |\nabla \hat{v}_\varepsilon|^2 \,\mathrm{d}x\,. \tag{3.14}$$

holds. Combining this identity with well-known interpolation estimates leads to suitable continuum upper bounds for XY_ε and thanks to (3.11) also for SD_ε.

Remark 3.6 (Interpolation estimate for XY_ε and SD_ε)*.* Let $U, U' \subset \mathbb{R}^2$ be open, bounded, and connected, with $U \subset\subset U'$. Let $v \in C^\infty(U';\mathbb{S}^1) \cap H^2(U';\mathbb{S}^1)$ and suppose that $v_\varepsilon \in \mathcal{SF}_\varepsilon$ is such that $v_\varepsilon(i) = v(i)$ for every $i \in \varepsilon\mathbb{Z}^2 \cap U'$. Then elliptic interpolation estimates (see *e.g.*, [29, Theorem 3.4.1]) provide us with a constant $C > 0$ such that

$$\|\nabla v - \nabla \hat{v}_\varepsilon\|^2_{L^2(Q_\varepsilon)} \le C\varepsilon^2 \|\nabla^2 v\|^2_{L^2(Q_\varepsilon)}$$

for every $Q_\varepsilon \in \mathcal{Q}^{\mathrm{int}}_\varepsilon(U')$. Together with Young inequality this leads to

$$\|\nabla \hat{v}_\varepsilon\|^2_{L^2(Q_\varepsilon)} \le (1+\varepsilon^\alpha)\|\nabla v\|^2_{L^2(Q_\varepsilon)} + C\varepsilon^2(1+\varepsilon^{-\alpha})\|\nabla^2 v\|^2_{L^2(Q_\varepsilon)} \quad \text{for any } \alpha > 0\,. \tag{3.15}$$

Combining (3.14) and (3.15) finally gives

$$XY_\varepsilon(v_\varepsilon, U) \le \frac{1}{2}\sum_{Q_\varepsilon \in \mathcal{Q}^{\mathrm{ext}}_\varepsilon(U)} XY_\varepsilon(v_\varepsilon, Q_\varepsilon) \le \frac{1+\varepsilon^\alpha}{2}\int_{U^{\mathrm{ext}}_\varepsilon} |\nabla v|^2 \,\mathrm{d}x + C\varepsilon^{2-\alpha}\int_{U^{\mathrm{ext}}_\varepsilon} |\nabla^2 v|^2 \,\mathrm{d}x \tag{3.16}$$

for any $\alpha > 0$ and for $\varepsilon > 0$ sufficiently small such that $U^{\mathrm{ext}}_\varepsilon \subset U'$.

Suppose now that u is an angular lifting of v and for every $i \in \varepsilon\mathbb{Z}^2$ let $u_\varepsilon(i) := u(i)$.[1] Using (3.11) and applying the mean-value inequality leads to

$$f_1\big(\mathrm{d}u_\varepsilon(i,j)\big) \le \frac{1}{2}|\mathrm{d}v_\varepsilon(i,j)|^2 + C\varepsilon^2\|\nabla v\|^2_{L^\infty(U')}|\mathrm{d}v_\varepsilon(i,j)|^2$$

for every $i, j \in \varepsilon\mathbb{Z}^2 \cap U$ with $|i-j| = \varepsilon$. Together with (3.16) this yields

$$\begin{aligned} SD_\varepsilon(u_\varepsilon, U) &\le \Big(1 + C\varepsilon^2\|\nabla v\|^2_{L^\infty(U')}\Big) XY_\varepsilon(v_\varepsilon, U) \\ &\le \Big(1 + C\varepsilon^2\|\nabla v\|^2_{L^\infty(U')}\Big)\left(\frac{1+\varepsilon^\alpha}{2}\int_{U^{\mathrm{ext}}_\varepsilon} |\nabla v|^2 \,\mathrm{d}x + C\varepsilon^{2-\alpha}\int_{U^{\mathrm{ext}}_\varepsilon} |\nabla^2 v|^2 \,\mathrm{d}x\right). \end{aligned} \tag{3.17}$$

[1]If U is not simply connected, u might not be smooth anymore. However, by introducing suitable cuts in U not intersecting $\varepsilon\mathbb{Z}^2$ we can assume that u is piecewise smooth and can be evaluated in $\varepsilon\mathbb{Z}^2$.

$\mathbb{S}^1$-valued interpolations. The piecewise affine interpolation $\hat{v}_\varepsilon$ of a spin field $v_\varepsilon \in \mathcal{SF}_\varepsilon$ in general does not take values in $\mathbb{S}^1$ any more. The following result instead provides an $\mathbb{S}^1$-valued interpolation of v_ε. It has been stated and proved in [9, Remark 3.2] on a triangular lattice. Since the proof on the square lattice is analogous to the one in [9, Remark 3.2], we do not repeat it here.

Lemma 3.7 ($\mathbb{S}^1$-valued interpolation)**.** *Let $\varepsilon > 0$, let $v_\varepsilon \in \mathcal{SF}_\varepsilon$, and let $u_\varepsilon \in \mathcal{AD}_\varepsilon$ be an arbitrary angular lifting of v_ε. Then there exists $\widehat{v}_\varepsilon \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{S}^1) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \operatorname{supp}\mu_{u_\varepsilon};\mathbb{S}^1)$ satisfying the following properties:*

(1) $\widehat{v}_\varepsilon(i) = v_\varepsilon(i) = \exp\big(2\pi\iota u_\varepsilon(i)\big)$ for any $i \in \varepsilon\mathbb{Z}^2$;
(2) $J(\widehat{v}_\varepsilon) = \pi\mu_{u_\varepsilon}$ with $J(\widehat{v}_\varepsilon)$ being the distributional jacobian (see Section 4.1 for the definition);
(3) $\int_{Q_\varepsilon} |\nabla\widehat{v}_\varepsilon|^2 \,\mathrm{d}x = F^{\mathrm{screw}}_\varepsilon(u_\varepsilon, Q_\varepsilon)$ whenever $\mu_{u_\varepsilon}(Q_\varepsilon) = 0$.

Remark 3.8 (Lifting of $\widehat{v}_\varepsilon$). The $\mathbb{S}^1$-valued interpolation introduced in Lemma 3.7 is particularly useful to provide a "smooth" version of the displacement variable u_ε in Lemma 3.7 by using well-known lifting results in the continuum setting for $\widehat{v}_\varepsilon$. Indeed, if we suppose that $U \subset \mathbb{R}^2$ is an open, bounded and simply connected set with $\operatorname{supp}\mu_{u_\varepsilon} \cap U = \emptyset$, then $\widehat{v}_\varepsilon \in W^{1,\infty}_{\mathrm{loc}}(U;\mathbb{S}^1)$ admits a lifting $\phi_\varepsilon \in W^{1,\infty}_{\mathrm{loc}}(U)$ satisfying $\widehat{v}_\varepsilon(x) = \exp\big(2\pi\iota\phi_\varepsilon(x)\big)$ for every $x \in U$ and $2\pi|\nabla\phi_\varepsilon(x)| = |\nabla\widehat{v}_\varepsilon(x)|$ for a.e. $x \in U$ (see [15, Theorem 1.1]). By construction, ϕ_ε coincides (modulo $\mathbb{Z}$) with u_ε on $\varepsilon\mathbb{Z}^2$ and it satisfies

$$|\mathrm{d}\phi_\varepsilon(i, i+\varepsilon e_k)| = |\mathrm{d}^e u_\varepsilon(i, i+\varepsilon e_k)| \tag{3.18}$$

for every $k \in \{1,2\}$ and every $i \in \mathbb{Z}^{e_k}_\varepsilon(U)$ (see [9, Remark 3.4]). In this sense, ϕ_ε can be seen as a "smooth" mod $\mathbb{Z}$ representative of u_ε.

4. Statement of the main results

In this section we state a compactness and Γ-convergence result for a suitable rescaling of the energies $SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and $XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ using the lower bounds established in Section 3.1. We start by introducing the space of limiting fields.

4.1. Space of limiting fields.

For any $U \subset \mathbb{R}^2$ open and $M \in \mathbb{N}$ we consider the families of measures

$$X(U) := \Big\{\mu = \sum_{h=1}^{N} d_h\delta_{x_h} \ \text{ with } \ N \in \mathbb{N}\,,\ d_h \in \mathbb{Z}\setminus\{0\}\,,\ x_h \in U\,,\ x_h \neq x_{h'} \ \text{ for } \ h \neq h'\Big\} \tag{4.1}$$

and

$$X_M(U) := \Big\{\mu = \sum_{h=1}^{M} d_h\delta_{x_h} \in X(U) \ \text{ with } \ d_h \in \{-1,1\}\Big\}. \tag{4.2}$$

It will be convenient to equip $X(U)$ with the convergence induced by the flat topology. Namely, for any distribution $T \in \mathcal{D}'(U)$ we let

$$\|T\|_{\mathrm{flat}} := \sup\big\{\langle T,\psi\rangle \colon \psi \in C^\infty_c(U)\,,\ \|\psi\|_{L^\infty(U)} \le 1\,,\ \|\nabla\psi\|_{L^\infty(U)} \le 1\big\}$$

be its flat norm. We say that a sequence $(\mu_n)_n \subset X(U)$ converges flat to some $\mu \in X(U)$ and we write $\mu_n \overset{\mathrm{flat}}{\to} \mu$, if $\|\mu_n - \mu\|_{\mathrm{flat}} \to 0$ as $n \to +\infty$.

From now on, if not specified otherwise, $\Omega \subset \mathbb{R}^2$ is an open, bounded, and simply connected subset of $\mathbb{R}^2$ with Lipschitz boundary and $M \in \mathbb{N}$ is a fixed integer. The space of limiting fields will be a certain class of special functions of bounded variation that can be related to a measure $\mu \in X_M(\Omega)$. Specifically, we set

$$\mathcal{D}_M(\Omega) := \Big\{ v \in W^{1,1}(\Omega;\mathbb{S}^1) \colon J(v) = \pi\mu \ \text{ for some } \ \mu \in X_M(\Omega) \ \text{ and } \ v \in H^1_{\mathrm{loc}}\big(\Omega \setminus \operatorname{supp}\mu;\mathbb{S}^1\big) \Big\}\,, \tag{4.3}$$

where for any $v = (v_1, v_2) \in W^{1,1}(\Omega;\mathbb{R}^2) \cap W^{1,\infty}(\Omega;\mathbb{R}^2)$ the jacobian $J(v)$ is defined in a distributional sense as $J(v) := \operatorname{curl} j(v)$. Here

$$j(v) := \frac{1}{2}\big(v_1 \nabla v_2 - v_2 \nabla v_1\big)$$

is the so-called *current* or *pre-jacobian* (see [9, Section 3] for more details on the pre-jacobian and its relation to the degree of v). Moreover, we consider the family of functions

$$\mathcal{D}^{\frac{1}{\mathbf{n}}}_M(\Omega) := \Big\{ w \in SBV(\Omega;\mathbb{S}^1) \colon w^{\mathbf{n}} \in \mathcal{D}_M(\Omega)\,,\ w \in SBV^2_{\mathrm{loc}}(\Omega \setminus \operatorname{supp}\mu;\mathbb{S}^1)\,,\ \mathcal{H}^1(S_w \cap \Omega) < +\infty \Big\}\,, \tag{4.4}$$

where $J(w^{\mathbf{n}}) = \pi\mu$ according to (4.3).

4.2. **Renormalised energy and core energies.** For any $\mu = \sum_{h=1}^M d_h \delta_{x_h} \in X_M(\Omega)$ and $\sigma > 0$ sufficiently small such that

$$B_\sigma(x_h) \subset \Omega \quad \text{and} \quad B_\sigma(x_h) \cap B_\sigma(x'_h) = \emptyset \ \text{ for } h, h' \in \{1, \dots, M\}\,,\ h \neq h' \tag{4.5}$$

we set

$$\Omega^\sigma(\mu) := \Omega \setminus \bigcup_{h=1}^M B_\sigma(x_h)\,, \tag{4.6}$$

and to any $v \in \mathcal{D}_M(\Omega)$ with $J(v) = \pi\mu$ we associate the quantity

$$\mathcal{W}(v,\Omega) := \lim_{\sigma \to 0} \left(\frac{1}{2} \int_{\Omega^\sigma(\mu)} |\nabla v|^2 \,\mathrm{d}x - M\pi |\log \sigma| \right) \in \mathbb{R} \cup \{+\infty\}\,. \tag{4.7}$$

The quantity $\mathcal{W}(v,\Omega)$ is well defined thanks to [3, Section 4.4] (see also [8, Remark 3.2]). Moreover, if $v \in \mathcal{D}_M(\Omega)$ with $J(v) = \pi \sum_{h=1}^M d_h \delta_{x_h}$ is such that $\mathcal{W}(v,\Omega) < +\infty$, then

$$\lim_{\sigma \to 0} \int_{A_{\frac{\sigma}{2},\sigma}(x_h)} |\nabla v|^2 \,\mathrm{d}x = \pi \log 2 \tag{4.8}$$

for every $h \in \{1, \dots, M\}$ (cf. [8, Remark 3.2]).

We finally recall the characterisation of the so-called core energy for the screw-dislocation model and the XY-model, respectively. For $\varepsilon > 0$, $\sigma > 2\varepsilon$, and $x_0 \in \mathbb{R}^2$ let

$$\gamma^{SD}_\varepsilon\big(B_\sigma(x_0)\big) := \min \left\{ SD_\varepsilon(u, B_\sigma(x_0)) \colon \exp(2\pi\iota u(i)) = \frac{i - x_0}{|i - x_0|} \ \text{for all } i \in \partial_\varepsilon B_\sigma(x_0) \right\} \tag{4.9}$$

and

$$\gamma^{XY}_\varepsilon\big(B_\sigma(x_0)\big) := \min \left\{ XY_\varepsilon(v, B_\sigma(x_0)) \colon v(i) = \frac{i - x_0}{|i - x_0|} \ \text{for all } i \in \partial_\varepsilon B_\sigma(x_0) \right\}. \tag{4.10}$$

Thanks to [3, Theorem 4.1] and [18, Lemma 7.2] the limits

$$\gamma^{SD} := \lim_{\varepsilon \to 0} \left(\gamma^{SD}_\varepsilon\big(B_\sigma(x_0)\big) - \pi \log \frac{\sigma}{\varepsilon} \right) \tag{4.11}$$

and

$$\gamma^{XY} := \lim_{\varepsilon \to 0} \left(\gamma^{XY}_\varepsilon\big(B_\sigma(x_0)\big) - \pi \log \frac{\sigma}{\varepsilon} \right) \tag{4.12}$$

exist and are independent of $x_0 \in \mathbb{R}^2$ and $\sigma > 0$.

4.3. **Γ-convergence results for $XY^{\rm frac}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and $SD^{\rm part}_{\varepsilon,\frac{1}{\mathbf{n}}}$.** We are now in a position to state the main compactness and Γ-convergence results of this note. We start defining the candidate limiting energies. Specifically, for every $w \in \mathcal{D}^{\frac{1}{\mathbf{n}}}_M(\Omega)$ we set

$$F^{\rm part}_{\frac{1}{\mathbf{n}}}(w,\Omega) := \frac{1}{\mathbf{n}^2}\Big(M\gamma^{SD} + \mathcal{W}(w^{\mathbf{n}},\Omega)\Big) + \int_{S_w}\big(\tau_1|\nu_w\cdot e_1| + \tau_2|\nu_w\cdot e_2|\big)\,\mathrm{d}\mathcal{H}^1 \tag{4.13}$$

and

$$E^{\rm frac}_{\frac{1}{\mathbf{n}}}(w,\Omega) := \frac{1}{\mathbf{n}^2}\Big(M\gamma^{XY} + \mathcal{W}(w^{\mathbf{n}},\Omega)\Big) + \int_{S_w}\big(\tau_1|\nu_w\cdot e_1| + \tau_2|\nu_w\cdot e_2|\big)\,\mathrm{d}\mathcal{H}^1\,. \tag{4.14}$$

The following results show that $F^{\rm part}_{\frac{1}{\mathbf{n}}}$ and $E^{\rm frac}_{\frac{1}{\mathbf{n}}}$ capture the asymptotic behaviour in terms of Γ-convergence of $SD^{\rm part}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and $XY^{\rm frac}_{\varepsilon,\frac{1}{\mathbf{n}}}$, respectively. This shows in particular that the asymptotic behaviour of the discrete energies accounting for partial dislocations and stacking faults and the energies accounting for fractional vortices and string defects is the same far from the limiting point singularities. This reflects the feature that also the corresponding models for full dislocations and vortices share the same asymptotic behaviour far from the limiting singularities (see [3] and [2]). Moreover, close to the limiting point singularities the models for partial dislocations and for fractional vortices concentrate the same energetic contribution as their counterparts for full dislocations and vortices, but lowered by a factor $\frac{1}{\mathbf{n}^2}$.

Theorem 4.1. *Let $XY^{\rm frac}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and $E^{\rm frac}_{\frac{1}{\mathbf{n}}}$ be as in* (2.16) *and* (4.14)*, respectively. The following holds true.*

(i) (Compactness) Let (w_ε) be a sequence of spin fields $w_\varepsilon \in \mathcal{SF}_\varepsilon$ satisfying

$$\sup_{\varepsilon>0}\Big(XY^{\rm frac}_{\varepsilon,\frac{1}{\mathbf{n}}}(w_\varepsilon,\Omega) - \frac{M\pi}{\mathbf{n}^2}|\log\varepsilon|\Big) < +\infty \tag{4.15}$$

and let $u_\varepsilon \in \mathcal{AD}_\varepsilon$ be an arbitrary angular lifting of w_ε. Then up to a subsequence (not relabeled) $\mu_{\mathbf{n}u_\varepsilon}\llcorner\Omega \overset{\rm flat}{\to} \mu$ for some $\mu = \sum_{h=1}^N d_h\delta_{x_h} \in X(\Omega)$ with $|\mu|(\Omega) \leq M$. Moreover, if $|\mu|(\Omega) = M$, then $N = M$ and $|d_h| = 1$ for every $h \in \{1,\dots,N\}$ (i.e., $\mu \in X_M(\Omega)$) and there exists $w \in \mathcal{D}^{\frac{1}{\mathbf{n}}}_M(\Omega)$ with $J(w^{\mathbf{n}}) = \pi\mu$ such that (up to a further subsequence) $w_\varepsilon \to w$ in $L^1(\Omega;\mathbb{R}^2)$.

(ii) (Lower bound) Let $w \in \mathcal{D}^{\frac{1}{\mathbf{n}}}_M(\Omega)$ and let $u_\varepsilon \in \mathcal{AD}_\varepsilon$, $w_\varepsilon = \exp(2\pi\imath u_\varepsilon) \in \mathcal{SF}_\varepsilon$ be such that $\pi\mu_{\mathbf{n}u_\varepsilon}\llcorner\Omega \overset{\rm flat}{\to} J(w^n)$, $w_\varepsilon \to w$ in $L^1(\Omega;\mathbb{R}^2)$. Then

$$\liminf_{\varepsilon\to0}\Big(XY^{\rm frac}_{\varepsilon,\frac{1}{\mathbf{n}}}(w_\varepsilon,\Omega) - \frac{M\pi}{\mathbf{n}^2}|\log\varepsilon|\Big) \geq E^{\rm frac}_{\frac{1}{\mathbf{n}}}(w,\Omega)\,. \tag{4.16}$$

(iii) (Upper bound) For every $w \in \mathcal{D}^{\frac{1}{\mathbf{n}}}_M(\Omega)$ there exist $u_\varepsilon \in \mathcal{AD}_\varepsilon$ such that $\pi\mu_{\mathbf{n}u_\varepsilon}\llcorner\Omega \overset{\rm flat}{\to} J(w^{\mathbf{n}})$ and the sequence of spin fields $w_\varepsilon := \exp(2\pi\imath u_\varepsilon) \in \mathcal{SF}_\varepsilon$ satisfies $w_\varepsilon \to w$ in $L^1\big(\Omega;\mathbb{R}^2\big)$ and

$$\limsup_{\varepsilon\to0}\Big(XY^{\rm frac}_{\varepsilon,\frac{1}{\mathbf{n}}}(w_\varepsilon,\Omega) - \frac{M\pi}{\mathbf{n}^2}|\log\varepsilon|\Big) \leq E^{\rm frac}_{\frac{1}{\mathbf{n}}}(w,\Omega)\,. \tag{4.17}$$

The above theorem characterizes the asymptotic behaviour of the fractional XY-model energies and is the main result of this paper. An analogue result holds for the partial dislocation energies $SD^{\rm part}_{\varepsilon,\frac{1}{\mathbf{n}}}$. In the case $\tau_1 = \tau_2 = 1$ the result corresponds to the one already established in [11]. However, the proof of the compactness and the lower bound can be simplified by using Theorem 4.1.

Theorem 4.2. *Let $SD^{\rm part}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and $F^{\rm part}_{\frac{1}{\mathbf{n}}}$ be as in* (2.16) *and* (4.14)*, respectively. The following holds true.*

(i) *(Compactness) Let* (u_ε) *be a sequence of displacements* $u_\varepsilon \in \mathcal{AD}_\varepsilon$ *satisfying*

$$\sup_{\varepsilon>0}\Big(SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(u_\varepsilon,\Omega)-\frac{M\pi}{\mathbf{n}^2}|\log\varepsilon|\Big)<+\infty \tag{4.18}$$

and let $w_\varepsilon \in \mathcal{SF}_\varepsilon$ *be given by* $w_\varepsilon := \exp(2\pi\iota u_\varepsilon)$*. Then up to a subsequence (not relabeled)* $\mu_{\mathbf{n}u_\varepsilon} \llcorner \Omega \overset{\mathrm{flat}}{\to} \mu$ *for some* $\mu = \sum_{h=1}^N d_h\delta_{x_h} \in X(\Omega)$ *with* $|\mu|(\Omega) \le M$*. Moreover, if* $|\mu|(\Omega) = M$*, then* $N = M$ *and* $|d_h| = 1$ *for every* $h \in \{1,\dots,N\}$ *(i.e.,* $\mu \in X_M(\Omega)$*) and there exists* $w \in \mathcal{D}^{\frac{1}{\mathbf{n}}}_M(\Omega)$ *with* $J(w^{\mathbf{n}}) = \pi\mu$ *such that (up to a further subsequence)* $w_\varepsilon \to w$ *in* $L^1(\Omega;\mathbb{R}^2)$*.*

(ii) *(Lower bound) Let* $w \in \mathcal{D}^{\frac{1}{\mathbf{n}}}_M(\Omega)$ *and let* $u_\varepsilon \in \mathcal{AD}_\varepsilon$*,* $w_\varepsilon = \exp(2\pi\iota u_\varepsilon) \in \mathcal{SF}_\varepsilon$ *be such that* $\pi\mu_{\mathbf{n}u_\varepsilon} \llcorner \Omega \overset{\mathrm{flat}}{\to} J(w^n)$*,* $w_\varepsilon \to w$ *in* $L^1(\Omega;\mathbb{R}^2)$*. Then*

$$\liminf_{\varepsilon\to0}\Big(SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(u_\varepsilon,\Omega)-\frac{M\pi}{\mathbf{n}^2}|\log\varepsilon|\Big)\ge F^{\mathrm{part}}_{\frac{1}{\mathbf{n}}}(w,\Omega)\,. \tag{4.19}$$

(iii) *(Upper bound) For every* $w \in \mathcal{D}^{\frac{1}{\mathbf{n}}}_M(\Omega)$ *there exist* $u_\varepsilon \in \mathcal{AD}_\varepsilon$ *such that* $\pi\mu_{\mathbf{n}u_\varepsilon} \llcorner \Omega \overset{\mathrm{flat}}{\to} J(w^{\mathbf{n}})$ *and the sequence of spin fields* $w_\varepsilon := \exp(2\pi\iota u_\varepsilon) \in \mathcal{SF}_\varepsilon$ *satisfies* $w_\varepsilon \to w$ *in* $L^1\big(\Omega;\mathbb{R}^2\big)$ *and*

$$\limsup_{\varepsilon\to0}\Big(SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(u_\varepsilon,\Omega)-\frac{M\pi}{\mathbf{n}^2}|\log\varepsilon|\Big)\le F^{\mathrm{part}}_{\frac{1}{\mathbf{n}}}(w,\Omega)\,. \tag{4.20}$$

We conclude this section by providing an example of a sequence of spinfields $w_\varepsilon \in \mathcal{SF}_\varepsilon$ along which WM_ε is uniformly bounded in the unit ball, but $XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ blows up. This shows in particular that the weak-membrane energies in general do not provide an upper bound for the discrete energies considered in this paper, not even asymptotically. In fact, the example below highlights that the weak-membrane energies are not suited to detect limiting topological singularities.

Example 4.3. We define $u_\varepsilon \in \mathcal{AD}_\varepsilon$ by setting

$$u_\varepsilon(i) := \begin{cases} 0 & \text{if } i\cdot e_1 \ge 0 \text{ and } i\cdot e_2 \ge 0\,,\\ \frac18 & \text{if } i\cdot e_1 < 0 \text{ and } i\cdot e_2 \ge 0\,,\\ \frac14 & \text{if } i\cdot e_1 < 0 \text{ and } i\cdot e_2 < 0\,,\\ \frac38 & \text{if } i\cdot e_1 > 0 \text{ and } i\cdot e_2 \le 0\,,\end{cases}$$

We then set $w_\varepsilon(i) := \exp\big(2\pi\iota u_\varepsilon(i)\big)$ for every $i \in \varepsilon\mathbb{Z}^2$. In this way we have that

$$WM_\varepsilon(w_\varepsilon,B_1) \le \varepsilon\sum_{k=1}^2 \tau_k\#\big\{i\in\mathbb{Z}^{e_k}_\varepsilon(B_1)\colon [i,i+\varepsilon e_k]\cap\Pi^{e_k}\neq\emptyset\big\} \le C\,, \tag{4.21}$$

where $\Pi^{e_k} := \{x\in\mathbb{R}^2\colon x\cdot e_k = 0\}$. Instead, for $\mathbf{n}=2$ and $v_\varepsilon := w_\varepsilon^2 = \exp(4\pi\iota u_\varepsilon)$ we find that

$$\frac12|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_1)|^2 = 1-\cos\big(4\pi\mathrm{d}u_\varepsilon(i,i+\varepsilon e_1)\big) = 1-\cos(\pi/2) = 1$$

for any $i\in\varepsilon\mathbb{Z}^2$ with $(i,i+\varepsilon e_1]\cap\Pi^{e_1}\neq\emptyset$. Similarly, $\frac12|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_2)|^2 = 1$ for $i\in\varepsilon\mathbb{Z}^2$ with $(i,i+\varepsilon e_2]\cap\Pi^{e_2}\neq\emptyset$. In view of (3.2) this implies that

$$\begin{aligned} SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(u_\varepsilon,B_1) &\ge XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}(w_\varepsilon,B_1) \ge \frac14 XY_\varepsilon(v_\varepsilon,B_1)\\ &\ge C\Big(\#\{i\in\varepsilon\mathbb{Z}^2\cap B_1\cap\Pi^{e_1}\}+\#\{i\in\varepsilon\mathbb{Z}^2\cap B_1\cap\Pi^{e_2}\}\Big)\ge\frac{C}{\varepsilon}\,.\end{aligned}$$

Thus, $SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(u_\varepsilon,B_1)$ and $XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}(w_\varepsilon,B_1)$ diverge as $\varepsilon\to0$, while (4.21) shows that $WM_\varepsilon(w_\varepsilon,B_1)$ is uniformly bounded. The latter also implies that $WM_\varepsilon(w_\varepsilon,B_1)\ll|\log\varepsilon|$. Finally, we observe that $\mu_{2u_\varepsilon} \overset{\text{flat}}{\to} \delta_0$ as $\varepsilon \to 0$, but since $WM_\varepsilon(w_\varepsilon, B_1) \ll |\log\varepsilon|$, WM_ε will not detect the logarithmic contribution of the limiting point singularity as $\varepsilon \to 0$.

5. Proof of compactness and lower bound

In this section we prove Theorem 4.2 (i)–(ii) and Theorem 4.1 (i)–(ii). We start establishing Theorem 4.1 (i)–(ii), then Theorem 4.2 (i)–(ii) will follow from Lemma 3.3.

Proof of Theorem 4.1 (i)–(ii). The proof is divided into several steps establishing separately the compactness of $\mu_{\mathbf{n}u_\varepsilon}$ and w_ε. The liminf inequality will essentially be established in parallel.

Step 1: Compactness of $\mu_{\mathbf{n}u_\varepsilon}$. Suppose that $w_\varepsilon \in \mathcal{SF}_\varepsilon$ satisfy (4.15). Defining $v_\varepsilon \in \mathcal{SF}_\varepsilon$ pointwise via $v_\varepsilon(i) := w_\varepsilon^{\mathbf{n}}(i)$ for every $i \in \varepsilon\mathbb{Z}^2$, we deduce from (3.2) that

$$\sup_{\varepsilon>0}\big(XY_\varepsilon(v_\varepsilon,\Omega) - M\pi|\log\varepsilon|\big) < +\infty\,. \tag{5.1}$$

Since $\mathbf{n}u_\varepsilon$ is an angular lifting of v_ε, [3, Theorem 4.2 (i)] implies that up to a subsequence (not relabeled) $\mu_{\mathbf{n}u_\varepsilon} \llcorner \Omega \overset{\text{flat}}{\to} \mu$ for some $\mu = \sum_{h=1}^N d_h x_h \in X(\Omega)$ with $|\mu|(\Omega) \leq M$. Suppose now that $|\mu|(\Omega) = M$; then [3, Theorem 4.2 (i)] ensures the following. We have that $N = M$ and $|d_h| = 1$ for every $h \in \{1, \dots, N\}$. Moreover, up to taking another non-relabeled subsequence $\hat{v}_\varepsilon$ converges weakly in $H^1_{\mathrm{loc}}(\Omega \setminus \operatorname{supp}\mu; \mathbb{R}^2)$ to some $v \in \mathcal{D}_M(\Omega)$ with $J(v) = \pi\mu$ (*cf.*also [11, Remark 3.4]). We will use this below to establish the required convergence of w_ε.

Step 2: Compactness of w_ε. In this step we show that up to a further subsequence we have that $w_\varepsilon \to w$ in $L^1(\Omega;\mathbb{R}^2)$ for some $w \in L^1(\Omega;\mathbb{R}^2) \cap SBV^2_{\mathrm{loc}}(\Omega \setminus \operatorname{supp}\mu;\mathbb{R}^2)$. Since $|w_\varepsilon| = 1$, the sequence (w_ε) is uniformly bounded in $L^\infty(\Omega;\mathbb{R}^2)$ and thus there exists $w \in L^\infty(\Omega;\mathbb{R}^2)$ such that up to subsequences $w_\varepsilon \overset{*}{\rightharpoonup} w$ in $L^\infty(\Omega;\mathbb{R}^2)$. We now upgrade this convergence. To this end, let $\sigma > 0$ be sufficiently small such that (4.5) is satisfied. Applying (3.2) on the balls $B_\sigma(x_h)$ and (3.10) on $\Omega^\sigma(\mu)$ we find that

$$XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}(w_\varepsilon,\Omega) - \frac{M\pi}{\mathbf{n}^2}|\log\varepsilon| \geq \frac{1}{\mathbf{n}^2}\sum_{h=1}^M\Big(XY_\varepsilon(v_\varepsilon, B_\sigma(x_h)) - \pi|\log\varepsilon|\Big) + WM_\varepsilon(w_\varepsilon,\Omega^\sigma(\mu))\,. \tag{5.2}$$

Let $h \in \{1,\dots,M\}$; from Step 1 we deduce that $\mu_{\mathbf{n}u_\varepsilon}\llcorner B_\sigma(x_h) \overset{\text{flat}}{\to} d_h\delta x_h$. Thus, a local application of [3, Theorem 4.2 (ii)] on $B_\sigma(x_h)$ yields

$$\liminf_{\varepsilon\to0}\Big(XY_\varepsilon(v_\varepsilon, B_\sigma(x_h)) - \pi|\log\varepsilon|\Big) \geq \gamma^{XY} + \mathbb{W}(d_h\delta_{x_h}, B_\sigma(x_h)) = \gamma^{XY} - \pi|\log\sigma|\,, \tag{5.3}$$

where $\mathbb{W}$ is the renormalised energy introduced in [12]. It satisfies $\mathbb{W}(d_h\delta_{x_h}, B_\sigma(x_h)) = \pi\log\sigma$. Together with (5.2) and (4.15) this implies that there exists $C > 0$ such that

$$WM_\varepsilon(w_\varepsilon,\Omega^\sigma(\mu)) \leq C|\log\sigma|$$

for every $\varepsilon > 0$. Since in addition (w_ε) is uniformly bounded by one in $L^\infty(\Omega;\mathbb{R}^2)$, an application of [30, Lemma 5.6] yields that $w_\varepsilon \to w$ in $L^1(\Omega^\sigma(\mu);\mathbb{R}^2)$ and $w \in SBV^2(\Omega^\sigma(\mu);\mathbb{R}^2)$. By the arbitrariness of $\sigma > 0$ we conclude that $w_\varepsilon \to w$ in $L^1(\Omega;\mathbb{R}^2)$ and $w \in SBV^2_{\mathrm{loc}}(\Omega\setminus\operatorname{supp}\mu;\mathbb{R}^2)$.

Step 3: Identification of w. In this step we show that $w^{\mathbf{n}} = v$, where v is the limit of $\hat{v}_\varepsilon$ obtained in Step 1. To prove this fact, we compare the piecewise affine functions $\hat{v}_\varepsilon$ and the piecewise constant functions $w_\varepsilon^{\mathbf{n}}$. More precisely, we let $\Omega' \subset\subset \Omega$ be arbitrarily fixed and we show that

$$\|\hat{v}_\varepsilon - w_\varepsilon^{\mathbf{n}}\|^2_{L^2(\Omega')} \leq C\varepsilon^2|\log\varepsilon| \tag{5.4}$$

for some constant $C > 0$ independent of Ω'. To this end, let $i \in \varepsilon\mathbb{Z}^2$ be such that $Q_\varepsilon(i) \in \mathcal{Q}_\varepsilon^{\rm ext}(\Omega')$. Then we have that

$$\begin{aligned}\|\hat{v}_\varepsilon - w_\varepsilon^{\mathbf{n}}\|^2_{L^2(Q_\varepsilon(i))} \leq C\varepsilon^2 \Big(& |v_\varepsilon(i+\varepsilon e_1) - v_\varepsilon(i)|^2 + |v_\varepsilon(i+\varepsilon(e_1+e_2)) - v_\varepsilon(i+\varepsilon e_1)|^2 \\ & + |v_\varepsilon(i+\varepsilon e_2) - v_\varepsilon(i)|^2 + |v_\varepsilon(i+\varepsilon(e_1+e_2)) - v_\varepsilon(i+\varepsilon e_2)|^2 \Big) .\end{aligned}$$

Since for $\varepsilon > 0$ sufficiently small we have $Q_\varepsilon(i) \subset \Omega$, by summing up the above estimate we obtain that

$$\|\hat{v}_\varepsilon - w_\varepsilon^{\mathbf{n}}\|^2_{L^2(\Omega')} \leq \sum_{Q_\varepsilon \in \mathcal{Q}_\varepsilon^{\rm ext}(\Omega')} \|\hat{v}_\varepsilon - w_\varepsilon^{\mathbf{n}}\|^2_{L^2(Q_\varepsilon)} \leq C\varepsilon^2 XY_\varepsilon(v_\varepsilon, \Omega)$$

for ε small enough. Thus, (5.4) follows from (5.1). By the arbitrariness of Ω' we conclude that $(\hat{v}_\varepsilon - w_\varepsilon^{\mathbf{n}}) \to 0$ in $L^2(\Omega;\mathbb{R}^2)$. Since Step 1 and Step 2 imply that $\hat{v}_\varepsilon \to v$ in $L^2(\Omega;\mathbb{R}^2)$ and $w_\varepsilon^{\mathbf{n}} \to w^{\mathbf{n}}$ in $L^1(\Omega;\mathbb{R}^2)$ respectively, we finally obtain that $w^{\mathbf{n}} = v$.

Step 4: w belongs to $\mathcal{D}_M^{\frac{1}{\mathbf{n}}}(\Omega)$. We deduce from Step 3 that $w \in SBV^2_{\rm loc}(\Omega \setminus \operatorname{supp}\mu;\mathbb{S}^1)$ and $w^{\mathbf{n}} = v \in \mathcal{D}_M(\Omega)$. Thus, to conclude that $w \in \mathcal{D}_M^{\frac{1}{\mathbf{n}}}(\Omega)$, it remains to show that $\mathcal{H}^1(S_w \cap \Omega) < +\infty$. In doing so, we essentially prove the liminf inequality. Let $\sigma > 0$ be fixed such that (4.5) is satisfied. Since $w_\varepsilon \to w$ in $L^1(\Omega;\mathbb{R}^2)$, applying the Γ-convergence result [17, Theorem 1] (see also [30, Theorem 3.5]) yields

$$\begin{aligned}\liminf_{\varepsilon\to 0} WM_\varepsilon(w_\varepsilon, \Omega^\sigma(\mu)) &\geq \frac{1}{2}\int_{\Omega^\sigma(\mu)} |\nabla w|^2 \,\mathrm{d}x + \int_{S_w \cap \Omega^\sigma(\mu)} \big(\tau_1|\nu_w \cdot e_1| + \tau_2|\nu_w \cdot e_2|\big)\,\mathrm{d}\mathcal{H}^1 \\ &= \frac{1}{2\mathbf{n}^2}\int_{\Omega^\sigma(\mu)} |\nabla v|^2 \,\mathrm{d}x + \int_{S_w \cap \Omega^\sigma(\mu)} \big(\tau_1|\nu_w \cdot e_1| + \tau_2|\nu_w \cdot e_2|\big)\,\mathrm{d}\mathcal{H}^1 ,\end{aligned} \tag{5.5}$$

where the second equality follows from the fact that $w^{\mathbf{n}} = v$ and hence $|\nabla w| = \frac{1}{\mathbf{n}}|\nabla v|$. Combining (5.5) with (5.2)–(5.3) and using the definition of $\mathcal{W}(v,\Omega)$ we deduce that

$$\begin{aligned}\liminf_{\varepsilon\to 0} \Big(XY^{\rm frac}_{\varepsilon,\frac{1}{\mathbf{n}}}(w_\varepsilon,\Omega) - \frac{M\pi}{\mathbf{n}^2}|\log\varepsilon| \Big) &\geq \frac{\gamma^{XY}}{\mathbf{n}^2} + \frac{1}{\mathbf{n}^2}\Big(\frac{1}{2}\int_{\Omega^\sigma(\mu)} |\nabla v|^2\,\mathrm{d}x - M\pi|\log\sigma|\Big) \\ &\quad + \int_{S_w\cap\Omega^\sigma(\mu)} \big(\tau_1|\nu_w \cdot e_1| + \tau_2|\nu_w \cdot e_2|\big)\,\mathrm{d}\mathcal{H}^1 \\ &\geq \frac{1}{\mathbf{n}^2}\Big(\gamma^{XY} + \mathcal{W}(v,\Omega)\Big) + r(\sigma) \\ &\quad + \int_{S_w\cap\Omega^\sigma(\mu)} \big(\tau_1|\nu_w \cdot e_1| + \tau_2|\nu_w \cdot e_2|\big)\,\mathrm{d}\mathcal{H}^1 ,\end{aligned} \tag{5.6}$$

with $r(\sigma) \to 0$ as $\sigma \to 0$. In view of (4.18) this implies that

$$\mathcal{H}^1(S_w \cap \Omega) = \bigcup_{\sigma>0} \mathcal{H}^1(S_w \cap \Omega^\sigma(\mu)) < +\infty$$

and hence $w \in \mathcal{D}_M^{\frac{1}{\mathbf{n}}}(\Omega)$.

Step 5: Liminf inequality. We finally obtain (4.16) from (5.6) by letting $\sigma \to 0$. □

Proof of Theorem 4.2 (i)–(ii). Theorem 4.2 (i) is a direct consequence of Theorem 4.1 (i) and (3.10). Indeed, if $u_\varepsilon \in \mathcal{AD}_\varepsilon$ satisfy (4.18), then (3.10) implies that the spin fields $w_\varepsilon \in \mathcal{SF}_\varepsilon$ defined as $w_\varepsilon := \exp(2\pi\iota u_\varepsilon)$ satisfy (4.15) and thus the required compactness of $\mu_{\mathbf{n}u_\varepsilon}$ and w_ε follows from

Theorem 4.1 (i). Finally, if $w_\varepsilon \to w$ in $L^1(\Omega;\mathbb{R}^2)$ and $\pi\mu_{\mathbf{n}u_\varepsilon} \llcorner \Omega \overset{\text{flat}}{\to} J(w^{\mathbf{n}})$ with $w \in \mathcal{D}^{\frac{1}{\mathbf{n}}}_M(\Omega)$, then (3.1) together with (3.10) implies that

$$SD^{\text{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(u_\varepsilon,\Omega) - \frac{M\pi}{\mathbf{n}^2}|\log\varepsilon| \geq WM_\varepsilon(w_\varepsilon,\Omega^\sigma(\mu)) - \frac{1}{\mathbf{n}^2}\sum_{h+1}^{M}\Big(SD_\varepsilon(\mathbf{n}u_\varepsilon, B_\sigma(x_h)) - \pi|\log\varepsilon|\Big). \quad (5.7)$$

Thus, the liminf inequality follows by repeating the estimates in (5.2), (5.3), and (5.6) with $XY^{\text{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and XY_ε replaced by $SD^{\text{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and SD_ε, respectively. □

6. Proof of the upper bound

In this section we prove Theorem 4.2 (iii) and Theorem 4.1 (iii).

6.1. **The core energy.** As a first step we establish an alternative characterisation of γ^{SD} and γ^{XY} defined in (4.11)–(4.12) in terms of minimisation problems involving the energies $SD^{\text{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and $XY^{\text{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$, respectively. More precisely, for every $\varepsilon > 0$, $\sigma > 2\varepsilon$, and $x_0 \in \mathbb{R}^2$ we set

$$\gamma^{\text{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_\sigma(x_0)\big) := \min\left\{SD^{\text{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(u, B_\sigma(x_0))\colon \exp(2\pi\iota\mathbf{n}u(i)) = \frac{i-x_0}{|i-x_0|} \text{ for all } i \in \partial_\varepsilon B_\sigma(x_0)\right\} \quad (6.1)$$

and

$$\gamma^{\text{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_\sigma(x_0)\big) := \min\left\{XY^{\text{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}(w, B_\sigma(x_0))\colon w^{\mathbf{n}}(i) = \frac{i-x_0}{|i-x_0|} \text{ for all } i \in \partial_\varepsilon B_\sigma(x_0)\right\}. \quad (6.2)$$

Remark 6.1 (Invariance under rotations). Since $SD^{\text{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ is invariant under translations and $XY^{\text{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ is invariant under rotations, we can replace the boundary conditions $\frac{i-x_0}{|i-x_0|}$ in (6.1) and (6.2) by a rotated version $\alpha\frac{i-x_0}{|i-x_0|}$ with $\alpha \in \mathbb{C}$, $|\alpha| = 1$ without affecting the value of $\gamma^{\text{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_\sigma(x_0)\big)$ and $\gamma^{\text{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_\sigma(x_0)\big)$, respectively.

Below we show that after correctly weighting the logarithmic correction term the quantities γ^{SD} and γ^{XY} can be characterised by replacing γ^{SD}_ε and γ^{XY}_ε in (4.11) and (4.12) with $\gamma^{\text{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and $\gamma^{\text{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$, respectively, and letting the radii of the balls $B_\sigma(x_0)$ tend to zero. For γ^{SD} such a result is already contained in [11, Lemma 4.1]. We still include it here, giving a simplified proof based on the recent result [23, Theorem 2.4].

Proposition 6.2 (Core energy involving $SD^{\text{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and $XY^{\text{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$)**.** *Let γ^{SD} and γ^{XY} be as in* (4.11)–(4.12) *and $\gamma^{\text{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and $\gamma^{\text{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ as in* (6.1)–(6.2)*, respectively. Then we have that*

$$\gamma^{SD} = \lim_{\sigma\to0}\limsup_{\varepsilon\to0}\Big(\mathbf{n}^2\gamma^{\text{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_\sigma(x_0)\big) - \pi\log\frac{\sigma}{\varepsilon}\Big) = \lim_{\sigma\to0}\liminf_{\varepsilon\to0}\Big(\mathbf{n}^2\gamma^{\text{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_{\sigma_\varepsilon}(x_0)\big) - \pi\log\frac{\sigma}{\varepsilon}\Big),$$

$$\gamma^{XY} = \lim_{\sigma\to0}\limsup_{\varepsilon\to0}\Big(\mathbf{n}^2\gamma^{\text{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_\sigma(x_0)\big) - \pi\log\frac{\sigma}{\varepsilon}\Big) = \lim_{\sigma\to0}\limsup_{\varepsilon\to0}\Big(\mathbf{n}^2\gamma^{\text{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_\sigma(x_0)\big) - \pi\log\frac{\sigma}{\varepsilon}\Big)$$

for every $x_0 \in \mathbb{R}^2$.

Proof. Let $x_0 \in \mathbb{R}^2$ be fixed. Thanks to (4.11) and (4.12) an application of (3.1) and (3.2) leads to

$$\liminf_{\varepsilon\to0}\Big(\mathbf{n}^2\gamma^{\text{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_\sigma(x_0)\big) - \pi\log\frac{\sigma}{\varepsilon}\Big) \geq \gamma^{SD},$$
$$\liminf_{\varepsilon\to0}\Big(\mathbf{n}^2\gamma^{\text{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_\sigma(x_0)\big) - \pi\log\frac{\sigma}{\varepsilon}\Big) \geq \gamma^{XY}$$

for every $\sigma > 0$. Thus, it remains to show that

$$\limsup_{\sigma\to 0}\limsup_{\varepsilon\to 0}\Big(\mathbf{n}^2\gamma^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_\sigma(x_0)\big) - \pi\log\frac{\sigma}{\varepsilon}\Big) \leq \gamma^{SD}\,. \tag{6.3}$$

and

$$\limsup_{\sigma\to 0}\limsup_{\varepsilon\to 0}\Big(\mathbf{n}^2\gamma^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_\sigma(x_0)\big) - \pi\log\frac{\sigma}{\varepsilon}\Big) \leq \gamma^{XY}\,. \tag{6.4}$$

Step 1: *Proof of* (6.3). Let $u_\varepsilon \in \mathcal{AD}_\varepsilon$ be a solution to the minimisation problem defining $\gamma^{SD}_\varepsilon(B_\sigma(x_0))$. Thanks to [23, Theorem 2.4] we can assume that $\mu_{u_\varepsilon} = \delta_{x_\varepsilon}$ with $x_\varepsilon \in B_\sigma(x_0)$ being the barycenter of a cube $Q_\varepsilon \in \mathcal{Q}^{\mathrm{int}}_\varepsilon\big(B_\sigma(x_0)\big)$. We now use Remark 3.8 to remove unnecessary jumps from u_ε. In this way, we will obtain a suitable competitor for $\gamma^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(B_\sigma(x_0))$. We start by setting $v_\varepsilon := \exp(2\pi\iota u_\varepsilon)$ and by letting S be a segment joining x_ε and $\partial B_\sigma(x_0)$ not intersection $\varepsilon\mathbb{Z}^2$ (this possible by choosing for a example a horizontal or vertical segment). Since $U := B_\sigma(x_0)\setminus S$ is simply connected with $\operatorname{supp}\mu_{v_\varepsilon}\cap U = \operatorname{supp}\mu_{u_\varepsilon}\cap U = \emptyset$, the $\mathbb{S}^1$-valued interpolation $\widehat{v}_\varepsilon$ admits a lifting $\phi_\varepsilon \in W^{1,\infty}_{\mathrm{loc}}(U)$ satisfying

$$\phi_\varepsilon \overset{\mathbb{Z}}{\equiv} u_\varepsilon \text{ on } U \quad\text{and}\quad |\mathrm{d}\phi_\varepsilon(i,i+\varepsilon e_k)| = |\mathrm{d}^e u_\varepsilon(i,i+\varepsilon e_k)| \tag{6.5}$$

for $k \in \{1,2\}$ and every $i \in \mathbb{Z}^{e_k}_\varepsilon(U)$ (see Remark 3.8). We then extend ϕ_ε by u_ε to $\varepsilon\mathbb{Z}^2\setminus B_\sigma(x_0)$ and we define $\widetilde{u}_\varepsilon \in \mathcal{AD}_\varepsilon$ by setting $\widetilde{u}_\varepsilon(i) := \frac{1}{\mathbf{n}}\phi_\varepsilon(i)$ for every $i \in \varepsilon\mathbb{Z}^2$. Thanks to the first condition in (6.5) we know that $\widetilde{u}_\varepsilon$ satisfies the required boundary conditions for $\gamma^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(B_\sigma(x_0))$, so that

$$\gamma^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(B_\sigma(x_0)) \leq SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(\widetilde{u}_\varepsilon, B_\sigma(x_0))\,. \tag{6.6}$$

Finally, the second condition in (6.5) together with (2.10) implies that for $k \in \{1,2\}$ we have

$$|\mathrm{d}\widetilde{u}_\varepsilon(i,i+\varepsilon e_k)| = \frac{1}{\mathbf{n}}|\mathrm{d}^e u_\varepsilon(i,i+\varepsilon e_k)| \leq \frac{1}{2\mathbf{n}} \text{ for every } i \in \mathbb{Z}^{e_k}_\varepsilon(U)\,. \tag{6.7}$$

By definition, this implies that $f^{\tau_k}_{\varepsilon,\frac{1}{\mathbf{n}}}(\mathrm{d}\widetilde{u}_\varepsilon(i,i+\varepsilon e_k) = f_{\frac{1}{\mathbf{n}}}(\mathrm{d}\widetilde{u}_\varepsilon(i,i+\varepsilon e_k))$ for all such i and k. Since in addition $f_{\frac{1}{\mathbf{n}}}(\mathrm{d}\widetilde{u}_\varepsilon(i,i+\varepsilon e_k)) = \frac{1}{\mathbf{n}^2}f_1(\mathrm{d}\phi_\varepsilon(i,i+\varepsilon e_k))$ thanks to (2.14), we deduce that

$$\begin{aligned} SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(\widetilde{u}_\varepsilon, B_\sigma(x_0)) &= \mathbf{n}^2 SD_\varepsilon(\phi_\varepsilon, U) + \sum_{k=1}^2 \sum_{\substack{i\in\mathbb{Z}^{e_k}_\varepsilon(B_\sigma(x_0))\\ [i,i+\varepsilon e_k]\cap S\neq\emptyset}} f^{\tau_k}_{\varepsilon,\frac{1}{\mathbf{n}}}(\mathrm{d}\widetilde{u}_\varepsilon(i,i+\varepsilon e_k)) \\ &\leq \mathbf{n}^2 SD_\varepsilon(u_\varepsilon, B_\sigma(x_0)) + \varepsilon\sum_{k=1}^2 \tau_k \#\big\{i \in \mathbb{Z}^{e_k}_\varepsilon(B_\sigma(x_0))\colon [i,i+\varepsilon e_k]\cap S \neq \emptyset\big\} \\ &\leq \mathbf{n}^2\gamma^{SD}_\varepsilon(B_\sigma(x_0)) + C\sigma\,, \end{aligned} \tag{6.8}$$

where the last estimate follows from the choice of u_ε and the fact that $\mathcal{H}^1(S) \leq \operatorname{diam} B_\sigma(x_0)$. Combining (6.6) and (6.8) and using the characterisation of γ^{SD} in (4.11) we finally obtain (6.3).

Step 2: *Proof of* (6.4). Similar to Step 1 we let $v_\varepsilon \in \mathcal{SF}_\varepsilon$ be a solution to the minimisation problem defining $\gamma^{XY}_\varepsilon\big(B_\sigma(x_0)\big)$ and we apply [23, Theorem 2.4] to argue that $\mu_{v_\varepsilon} = \delta_{x_\varepsilon}$ for some $x_\varepsilon \in B_\sigma(x_0)$ (not necessarily the same as in Step 1). Following the lines of Step 1 we construct a competitor w_ε for the minimisation problem defining $\gamma^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ satisfying

$$XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}(w_\varepsilon, B_\sigma(x_0)) \leq \frac{1}{\mathbf{n}^2}XY_\varepsilon(v_\varepsilon, B_\sigma(x_0)) + r(\varepsilon,\sigma) \tag{6.9}$$

with $r(\varepsilon,\sigma)\to 0$ as $\varepsilon\to 0$ and $\sigma\to 0$. Specifically, we let S and U be as in Step 1, $\widehat{v}_\varepsilon$ the $\mathbb{S}^1$-valued interpolation of v_ε and $\phi_\varepsilon\in W^{1,\infty}_{\mathrm{loc}}(U)$ the lifting of $\widehat{v}_\varepsilon$ provided by Remark 3.8. We then define $w_\varepsilon\in\mathcal{SF}_\varepsilon$ by setting $w_\varepsilon(i):=\exp\big(\frac{2\pi}{\mathbf{n}}\iota\phi_\varepsilon(i)\big)$ for every $i\in\varepsilon\mathbb{Z}^2$. In this way, we have that $w_\varepsilon^{\mathbf{n}}=v_\varepsilon$, hence w_ε satisfies the required boundary conditions for the minimisation problem defining $\gamma^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_\sigma(x_0)\big)$.

It remains to show that w_ε satisfies (6.9). To this end, for $k\in\{1,2\}$ we define

$$\mathcal{I}^{\mathrm{good}}_{\varepsilon,k}:=\Big\{i\in\mathbb{Z}^{e_k}_\varepsilon\big(B_\sigma(x_0)\big)\colon g^{\tau_k}_\varepsilon\big(\mathrm{d}w_\varepsilon(i,i+\varepsilon e_k)\big)\leq\frac{1}{2\mathbf{n}^2}\big|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_k)\big|^2\Big\}\,,$$
$$\mathcal{I}^{\mathrm{bad}}_{\varepsilon,k}:=\Big\{i\in\mathbb{Z}^{e_k}_\varepsilon\big(B_\sigma(x_0)\big)\colon g^{\tau_k}_\varepsilon\big(\mathrm{d}w_\varepsilon(i,i+\varepsilon e_k)\big)>\frac{1}{2\mathbf{n}^2}\big|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_k)\big|^2\Big\}\,.$$

In this way, we can write

$$XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(w_\varepsilon,B_\sigma(x_0)\big)=\sum_{k=1}^{2}\bigg(\sum_{i\in\mathcal{I}^{\mathrm{good}}_{\varepsilon,k}}\frac{1}{2\mathbf{n}^2}\big|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_k)\big|^2+\sum_{i\in\mathcal{I}^{\mathrm{bad}}_{\varepsilon,k}}g^{\tau_k}_\varepsilon\big(\mathrm{d}w_\varepsilon(i,i+\varepsilon e_k)\big)\bigg)\,. \tag{6.10}$$

To obtain (6.9) it suffices to estimate the last term in (6.10). To this end, we fix $k\in\{1,2\}$ and we distinguish between the following three exhaustive cases.

1) $i\in\mathcal{I}^{\mathrm{bad}}_{\varepsilon,k}$ with $[i,i+\varepsilon e_k]\cap S\neq\emptyset$;
2) $i\in\mathcal{I}^{\mathrm{bad}}_{\varepsilon,k}\cap\mathbb{Z}^{e_k}_\varepsilon(U)$ with $\big|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_k)\big|>\varepsilon^{\frac{1}{3}}$;
3) $i\in\mathcal{I}^{\mathrm{bad}}_{\varepsilon,k}\cap\mathbb{Z}^{e_k}_\varepsilon(U)$ with $\big|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_k)\big|\leq\varepsilon^{\frac{1}{3}}$.

Case 1. Suppose first that $i\in\mathcal{I}^{\mathrm{bad}}_{\varepsilon,k}$ is such that $[i,i+\varepsilon e_k]\cap S\neq\emptyset$. Then we use that $g^{\tau_k}_\varepsilon\big(\mathrm{d}w_\varepsilon(i,i+\varepsilon e_k)\big)\leq\varepsilon\tau_k$ and we estimate the number of such i. Since S is a segment connecting x_ε with $\partial B_\sigma(x_0)$ we have that

$$\#\big\{i\in\mathcal{I}^{\mathrm{bad}}_{\varepsilon,k}\colon[i,i+\varepsilon e_k]\cap S\neq\emptyset\big\}\leq\frac{C}{\varepsilon}\mathcal{H}^1(S)\leq\frac{C\sigma}{\varepsilon}\,.$$

Thus,

$$\sum_{\substack{i\in\mathcal{I}^{\mathrm{bad}}_{\varepsilon,k}\\ [i,i+\varepsilon e_k]\cap S\neq\emptyset}}g^{\tau_k}_\varepsilon\big(\mathrm{d}w_\varepsilon(i,i+\varepsilon e_k)\big)\leq C\sigma\,. \tag{6.11}$$

Case 2. We start observing that

$$XY_\varepsilon\big(v_\varepsilon,B_\sigma(x_0)\big)\geq 2\varepsilon^{\frac{2}{3}}\#\big\{i\in\mathcal{I}^{\mathrm{bad}}_{\varepsilon,k}\cap\mathbb{Z}^{e_k}_\varepsilon(U)\colon|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_k)|>\varepsilon^{\frac{1}{3}}\big\}\,.$$

Moreover, since v_ε is a solution to the minimisation problem defining $\gamma^{XY}_\varepsilon(B_\sigma(x_0)$ we deduce from (4.12) that $XY_\varepsilon(v_\varepsilon,B_\sigma(x_0))\leq C\log\frac{\sigma}{\varepsilon}$. Thus, using again that $g^{\tau_k}_\varepsilon\leq\varepsilon\tau_k$, we infer

$$\begin{aligned}\sum_{\substack{i\in\mathcal{I}^{\mathrm{bad}}_{\varepsilon,k}\cap\mathbb{Z}^{e_k}_\varepsilon(U)\\ |\mathrm{d}v_\varepsilon(i,i+\varepsilon e_k)|>\varepsilon^{\frac{1}{3}}}}g^{\tau_k}_\varepsilon\big(\mathrm{d}w_\varepsilon(i,i+\varepsilon e_k)\big)&\leq\varepsilon\tau_k\#\big\{i\in\mathcal{I}^{\mathrm{bad}}_{\varepsilon,k}\cap\mathbb{Z}^{e_k}_\varepsilon(U)\colon|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_k)|>\varepsilon^{\frac{1}{3}}\big\}\\ &\leq C\varepsilon^{\frac{1}{3}}XY_\varepsilon(v_\varepsilon,B_{\sigma_\varepsilon}(x_0))\leq C\varepsilon^{\frac{1}{3}}\log\frac{\sigma}{\varepsilon}\;\to 0\;\text{ as }\varepsilon\to 0\,.\end{aligned} \tag{6.12}$$

Case 3. Suppose finally that $i\in\mathcal{I}^{\mathrm{bad}}_{\varepsilon,k}\cap\mathbb{Z}^{e_k}_\varepsilon(U)$ satisfies $|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_k)|\leq\varepsilon^{\frac{1}{3}}$. Thanks to (6.7) an application of (3.13) in Remark 3.5 ensures that

$$|\mathrm{d}w_\varepsilon(i,i+\varepsilon e_k)|^2\leq\frac{1}{\mathbf{n}^2}|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_k)|^2+C|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_k)|^4\leq\frac{1+C\varepsilon^{\frac{2}{3}}}{\mathbf{n}^2}|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_k)|^2\,.$$

Summing up over all such i we thus obtain

$$\begin{aligned}\sum_{\substack{i\in\mathcal{I}^{\rm bad}_{\varepsilon,k}\cap\mathbb{Z}^{e_k}_\varepsilon(U)\\|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_k)|\leq\varepsilon^{\frac13}}} g^{\tau_k}_\varepsilon\big(\mathrm{d}w_\varepsilon(i,i+\varepsilon e_k)\big) &\leq \frac12 \sum_{\substack{i\in\mathcal{I}^{\rm bad}_{\varepsilon,k}\cap\mathbb{Z}^{e_k}_\varepsilon(U)\\|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_k)|\leq\varepsilon^{\frac13}}} |\mathrm{d}w_\varepsilon(i,i+\varepsilon e_k)|^2\\ &\leq \frac{1}{2\mathbf{n}^2}\sum_{i\in\mathcal{I}^{\rm bad}_{\varepsilon,k}}|\mathrm{d}v_\varepsilon(i,i+\varepsilon e_k)|^2 + C\varepsilon^{\frac23}XY_\varepsilon\big(v_\varepsilon,B_\sigma(x_0)\big)\,.\end{aligned}\tag{6.13}$$

Using once again that $XY_\varepsilon\big(v_\varepsilon,B_\sigma(x_0)\big)\leq C\log\frac{\sigma}{\varepsilon}$, a combination of (6.10)–(6.13) yields (6.9) with $r(\varepsilon,\sigma)=C\big(\varepsilon^{\frac13}\log\frac{\sigma}{\varepsilon}+\sigma\big)$. □

6.2. **Proof of Theorems 4.2(iii) and 4.1(iii).** The proof of Theorems 4.2(iii) and 4.1(iii) is based on Proposition 6.2 and the density result [11, Lemma 4.3] that we recall below for the readers' convenience.

Lemma 6.3. *Let $\mu=\sum_{h=1}^M d_h\delta_{x_h}\in X_M(\Omega)$ and $w\in\mathcal{D}^{\frac{1}{\mathbf{n}}}_M(\Omega)$ with $J(w^{\mathbf{n}})=\pi\mu$. Let moreover $\Gamma_1,\dots\Gamma_M$ be pairwise disjoint segments connecting $x_1,\dots,x_M$ to $\partial\Omega$ and satisfying $\mathcal{H}^1(\Gamma_h\cap S_w)=0$ for every $h\in\{1,\dots,M\}$. Let $\sigma>0$ be fixed such that (4.5) is satisfied. Then there exist $\phi\in H^1(\Omega^\sigma(\mu)\setminus\bigcup_h\Gamma_h)$ such that $w^{\mathbf{n}}=\exp(2\pi\iota\phi)$ a.e. in $\Omega^\sigma(\mu)$ and a partition function $\chi\in SBV(\Omega^\sigma(\mu);\{0,\dots,\mathbf{n}\})$ such that $\psi:=\frac{\phi+\chi}{\mathbf{n}}$ satisfies $w=\exp(2\pi\iota\psi)$ a.e. in $\Omega^\sigma(\mu)$ and*

$$[\psi]_{|\Gamma_h}\in\mathbb{Z}\quad\text{for every } h\in\{1,\dots,M\}\,.\tag{6.14}$$

Moreover, there exist two sequences $(\phi_n)\subset C^\infty(\Omega^\sigma(\mu)\setminus\bigcup_h\Gamma_h)$ and $(\chi_n)\subset SBV\big(\Omega^\sigma(\mu);\{0,\dots,\mathbf{n}\}\big)$ with $[\phi_n]=[\phi]$ and S_{χ_n} polyhedral such that setting $\psi_n:=\frac{\phi_n+\chi_n}{\mathbf{n}}$ and $v_n:=\exp(2\pi\iota\phi_n)$, $w_n:=\exp(2\pi\iota\psi_n)$ the following are satisfied

(i) $(v_n)\subset C^\infty\big(\Omega^\sigma(\mu);\mathbb{S}^1\big)$ *with* $\deg\big(v_n,\partial B_\rho(x_h)\big)=d_h$ *for all* $h\in\{1,\dots,M\}$ *and* $\rho>\sigma$ *satisfying* (4.5);

(ii) $w_n\subset C^\infty\big(\Omega^\sigma(\mu)\setminus(\bigcup_h\Gamma_h\cup S_{\chi_n});\mathbb{S}^1\big)$ *with*

$$S_{w_n}=\Big\{x\in S_{\psi_n}:[\psi_n](x)\in\frac{1}{\mathbf{n}}\mathbb{Z}\setminus\mathbb{Z}\Big\}\tag{6.15}$$

up to $\mathcal{H}^1$-negligible sets;

(iii) $\phi_n\to\phi$ *in* $H^1(\Omega^\sigma(\mu)\setminus\bigcup_h\Gamma_h)$ *and* $v_n\to v$ *in* $H^1(\Omega^\sigma(\mu);\mathbb{R}^2)$ *as* $n\to\infty$;

(iv) $\psi_n\overset{*}{\rightharpoonup}\psi$ *and* $w_n\overset{*}{\rightharpoonup}w$ *as* $n\to\infty$;

(v) For any bounded and continuous function $g:\mathbb{R}\times\mathbb{R}\times\mathbb{S}^1\to[0,+\infty)$ satisfying $g(a,b,\nu)=g(b,a,-\nu)$ we have that

$$\lim_{n\to+\infty}\int_{S_{\psi_n}\cap\Omega^\sigma(\mu)}g(\psi_n^+,\psi_n^-,\nu_{\psi_n})\,\mathrm{d}\mathcal{H}^1=\int_{S_\psi\cap\Omega^\sigma(\mu)}g(\psi^+,\psi^-,\nu_\psi)\,\mathrm{d}\mathcal{H}^1\,.$$

Remark 6.4. We briefly comment on Properties (i)–(ii) in Lemma 6.3, since they are not explicitely stated in [11, Lemma 4.3]. The fact that $(v_n)\subset C^\infty\big(\Omega^\sigma(\mu);\mathbb{S}^1\big)$ follows from Step 2 in the proof of [11, Lemma 4.3]. In fact, ϕ_n is constructed by approximating $v\in H^1\big(\Omega^\sigma(\mu);\mathbb{S}^1\big)$ with a sequence $(v_n)\subset C^\infty\big(\Omega^\sigma(\mu);\mathbb{S}^1\big)$ approximating v in H^1-norm, which is possible thanks to [32, Section 4]. Then ϕ_n is chosen to be smooth angular lifting of v_n in the cut domain $\Omega\setminus\bigcup_h\Gamma_h$. The continuity of the degree then ensures that for n sufficiently large $\deg\big(v_n,\partial B_\rho(x_h)\big)=\deg\big(v,\partial B_\rho(x_h)\big)=d_h$ for every $h\in\{1,\dots,M\}$ and correspondingly $[\phi_n]=[\phi]$ on $\bigcup_h\Gamma_h$.

To obtain (ii), it suffices to observe that by definition $S_{\psi_n}\subset\bigcup_h\Gamma_h\cup S_{\chi_n}$, which in turn implies that $w_n\in C^\infty\big(\Omega^\sigma(\mu)\setminus(\bigcup_h\Gamma_h\cup S_{\chi_n});\mathbb{S}^1\big)$ for every $n\in\mathbb{N}$. Moreover, (6.15) is a consequene of the chain rule for BV-functions [6, Theorem 3.96].

Remark 6.5. Note that (6.15) also holds with w_n, ψ_n replaced by w, ψ, respectively. Thus, Property (v) of Lemma 6.3 ensures that

$$\lim_{n\to+\infty}\int_{S_{w_n}\cap\Omega^\sigma(\mu)} h(\nu_{w_n})\,\mathrm{d}\mathcal{H}^1 = \int_{S_w\cap\Omega^\sigma(\mu)} h(\nu_w)\,\mathrm{d}\mathcal{H}^1 \tag{6.16}$$

for any continuous and bounded function $h:\mathbb{S}^1\to[0,+\infty)$ satisfying $h(\nu)=h(-\nu)$ for every $\nu\in\mathbb{S}^1$. To see this, it suffices to consider a smooth and symmetric function $\eta:\mathbb{R}\to[0,1]$ with

$$\eta\equiv 1 \ \text{ on } \ \big\{t\in\mathbb{R}:\ \mathrm{dist}(t,\mathbb{Z})>\tfrac{1}{2\mathbf{n}}\big\} \qquad\text{and}\qquad \eta\equiv 0 \ \text{ on } \ \big\{t\in\mathbb{R}:\ \mathrm{dist}(t,\mathbb{Z})<\tfrac{1}{4\mathbf{n}}\big\}\,.$$

Then the function $g:\mathbb{R}\times\mathbb{R}\times\mathbb{S}^1\to[0,+\infty)$, $g(a,b,\nu):=\eta(b-a)h(\nu)$ is bounded and continuous with $g(a,b,\nu)=g(b,a,-\nu)$, and thanks to (6.15) we have that

$$\int_{S_{w_n}\cap\Omega^\sigma(\mu)} h(\nu_{w_n})\,\mathrm{d}\mathcal{H}^1 = \int_{S_{\psi_n}\cap\Omega^\sigma(\mu)} g(\psi_n^+,\psi_n^-,\nu_{\psi_n})\,\mathrm{d}\mathcal{H}^1\,,$$

and similarly

$$\int_{S_w\cap\Omega^\sigma(\mu)} h(\nu_w)\,\mathrm{d}\mathcal{H}^1 = \int_{S_\psi\cap\Omega^\sigma(\mu)} g(\psi^+,\psi^-,\nu_\psi)\,\mathrm{d}\mathcal{H}^1\,.$$

Together with Lemma 6.3 (v) this yields (6.16).

We are now in a position to prove Theorems 4.2 (iii) and 4.1 (iii).

Proof of Theorem 4.2 (iii). Let $w\in\mathcal{D}_M^{\frac{1}{\mathbf{n}}}(\Omega)$ and $v:=w^{\mathbf{n}}$, so that $J(v)=\pi\mu$ for some $\mu=\sum_{h=1}^M d_h\delta_{x_h}\in X_M(\Omega)$. It is not restrictive to assume that $\mathcal{W}(v,\Omega)<+\infty$. We now proceed in several steps.

Step 1: Construction of an approximating sequence. Let $\Gamma_1,\dots,\Gamma_M$ be segments as in Lemma 6.3, $\sigma>0$ such that 2σ satisfies (4.5), and let $(\phi_n^\sigma)\subset C^\infty(\Omega^{\frac{\sigma}{4}}(\mu)\backslash\bigcup_h\Gamma_h)$, $(\chi_n^\sigma)\subset SBV(\Omega^{\frac{\sigma}{4}}(\mu);\{0,\dots,\mathbf{n}\})$, and $(\psi_n^\sigma)\subset SBV(\Omega^{\frac{\sigma}{4}}(\mu))$ be the sequences provided by Lemma 6.3 with σ replaced by $\frac{\sigma}{4}$. We also set

$$v_n^\sigma:=\exp(2\pi\iota\phi_n^\sigma)\quad\text{and}\quad w_n^\sigma:=\exp(2\pi\iota\psi_n^\sigma)\,.$$

Let $h\in\{1,\dots,M\}$; since $\phi_n^\sigma\in C^\infty\big(A_{\frac{\sigma}{4},2\sigma}(x_h)\backslash\Gamma_h\big)$ and $\deg(v_n^\sigma,\partial B_\rho(x_h))=d_h$ for any $\rho\in(\frac{\sigma}{4},2\sigma)$, there exists a lifting $\theta_h\in C^\infty\big(B_{2\sigma}(x_h)\setminus\Gamma_h\big)$ of a rotation of $\frac{x-x_h}{|x-x_h|}$ such that

$$\phi_n^\sigma-d_h\theta_h\in C^\infty\big(A_{\frac{\sigma}{4},2\sigma}(x_h)\big)\quad\text{and}\quad \fint_{A_{\frac{\sigma}{2},\sigma}(x_h)}\big(\phi_n^\sigma-d_h\theta_h\big)\,\mathrm{d}x=0\,. \tag{6.17}$$

Let now $\eta\in C^\infty\big([0,+\infty);[0,1]\big)$ with $\eta\equiv 0$ on $[0,\frac{5}{8}]$ and $\eta\equiv 1$ on $[\frac{7}{8},+\infty)$ be a smooth cut-off function and for $h\in\{1,\dots,M\}$ set

$$\vartheta_n^{\sigma,h}(x):=d_h\theta_h(x)+\eta\Big(\frac{|x-x_h|}{\sigma}\Big)\big(\phi_n^\sigma(x)-d_h\theta_h(x)\big)\ \text{ for every } x\in B_{2\sigma}(x_h)\setminus\{x_h\}\,.$$

By the choice of θ_h we have that $\vartheta_n^{\sigma,h}\in C^\infty(B_{2\sigma}(x_h)\setminus\Gamma_h)$. Moreover,

$$\vartheta_n^{\sigma,h}\equiv d_h\theta_h\ \text{ on } B_{\frac{5\sigma}{8}}(x_h)\setminus\{x_h\}\quad\text{and}\quad \vartheta_n^{\sigma,h}\equiv\phi_n^\sigma\ \text{ on } A_{\frac{7\sigma}{8},2\sigma}(x_h)\,. \tag{6.18}$$

We also set

$$v_n^{\sigma,h}:=\exp(2\pi\iota\vartheta_n^{\sigma,h})\quad\text{and}\quad w_n^{\sigma,h}:=\exp\big(\tfrac{2\pi}{\mathbf{n}}\iota\vartheta_n^{\sigma,h}\big)\ \text{ on } B_{2\sigma}(x_h)\setminus\{x_h\}\,. \tag{6.19}$$

Then $w_n^{\sigma,h} \in C^\infty\big(B_{2\sigma}(x_h) \setminus \Gamma_h;\mathbb{S}^1\big)$. Moreover, the first condition in (6.17) implies that $v_n^{\sigma,h} \in C^\infty\big(B_{2\sigma}(x_h) \setminus \{x_h\};\mathbb{S}^1\big)$. Finally, for every $h \in \{1,\dots,M\}$ we let $u_\varepsilon^h \in \mathcal{AD}_\varepsilon$ be such that

$$SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(u_\varepsilon^h, B_{\frac{\sigma}{2}}(x_h)\big) = \gamma^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_{\frac{\sigma}{2}}(x_h)\big) \tag{6.20}$$

and $\exp(2\pi\mathbf{n}u_\varepsilon^h) = \exp(\frac{2\pi}{\mathbf{n}}\iota d_h\theta_h)$ on $\partial_\varepsilon B_{\frac{\sigma}{2}}(x_h)$. This is possible thanks to Remark 6.1. We define $u_\varepsilon^{n,\sigma}$ on $\varepsilon\mathbb{Z}^2 \cap \Omega$ as follows.

$$u_\varepsilon^{n,\sigma}(i) := \begin{cases} u_\varepsilon^h(i) & \text{if } i \in \varepsilon\mathbb{Z}^2 \cap B_{\frac{\sigma}{2}}(x_h) \text{ for some } h \in \{1,\dots,M\}\,,\\[2ex] \frac{\vartheta_n^{\sigma,h}(i)}{\mathbf{n}} & \text{if } i \in \varepsilon\mathbb{Z}^2 \cap B_\sigma(x_h)\setminus B_{\frac{\sigma}{2}}(x_h) \text{ for some } h \in \{1,\dots,M\}\,,\\[2ex] \psi_n^\sigma(i) & \text{if } i \in \varepsilon\mathbb{Z}^2 \cap \Omega^\sigma(\mu). \end{cases}$$

In the above definition we identify ϕ_n^σ, χ_n^σ, and θ_h with their one-sided traces on their respective jumpsets, which can be uniquely defined up to a choice of normal to the jumpset. In this way, the point evaluation of both θ_n^σ and ψ_n^σ on $\varepsilon\mathbb{Z}^2$ is well-defined. Moreover, since Ω has Lipschitz boundary, as in [1, Remark 2] we can extend $u_\varepsilon^{n,\sigma}$ to $\varepsilon\mathbb{Z}^2\setminus\Omega$ without affecting the convergence of $\mu_{\mathbf{n}u_\varepsilon^{n,\sigma}}$. It remains to show that

$$\begin{aligned}\limsup_{\sigma\to0}\limsup_{n\to\infty}\limsup_{\varepsilon\to0}&\big(\mathbf{n}^2 SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(u_\varepsilon^{n,\sigma},\Omega) - M\pi|\log\varepsilon|\big)\\ &\le M\gamma^{SD} + \mathcal{W}(w^{\mathbf{n}},\Omega) + \mathbf{n}^2\int_{S_w}\big(\tau_1|\nu_w\cdot e_2| + \tau_2|\nu_w\cdot e_1|\big)\,\mathrm{d}\mathcal{H}^1\end{aligned} \tag{6.21}$$

and that $w_\varepsilon^{n,\sigma} \in \mathcal{SF}_\varepsilon$ defined pointwise by $w_\varepsilon^{n,\sigma}(i) := \exp(2\pi u_\varepsilon^{n,\sigma}(i))$ satisfies

$$\limsup_{\sigma\to0}\limsup_{n\to\infty}\limsup_{\varepsilon\to0}\|w_\varepsilon^{n,\sigma} - w\|_{L^1(\Omega)} = 0\,, \tag{6.22}$$

then we conclude by a diagonal argument. We establish (6.21) and (6.22) in several steps estimating separately the energy contribution of $u^\sigma_{\varepsilon,n}$ close to the singularities $x_1,\dots,x_M$ and away from the singularities. Specifically, we split the energy contribution into

$$SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(u_\varepsilon^{n,\sigma},\Omega\big) \le \sum_{h=1}^M SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(u_\varepsilon^{n,\sigma},B_\sigma(x_h)\big) + SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(u_\varepsilon^{n,\sigma},\Omega^{\sigma-2\varepsilon}(\mu)\big)\,, \tag{6.23}$$

and we estimate the two terms on the right-hand side of (6.23) separately.

Step 2: *Energy estimate on $B_\sigma(x_h)$.* In this step we show that for every $h \in \{1,\dots,M\}$ we have

$$\limsup_{\sigma\to0}\limsup_{n\to+\infty}\limsup_{\varepsilon\to0}\Big(\mathbf{n}^2 SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(u_\varepsilon^{n,\sigma},B_\sigma(x_h)\big) - \pi\log\frac{\sigma}{\varepsilon}\Big) \le \gamma^{SD}\,. \tag{6.24}$$

Let $h \in \{1,\dots,M\}$ be fixed. The definition of $u_\varepsilon^{n,\sigma}$ together with the choice of u_ε^h ensures that

$$SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(u_\varepsilon^{n,\sigma},B_\sigma(x_h)\big) \le \gamma^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_{\frac{\sigma}{2}}(x_h)\big) + SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(\tfrac{\vartheta_n^{\sigma,h}}{\mathbf{n}},A_{\frac{\sigma}{2}-2\varepsilon,\sigma}(x_h)\big)\,. \tag{6.25}$$

In order to shorten notation we set $A_\varepsilon := A_{\frac{\sigma}{2}-2\varepsilon,\sigma}(x_h)$ and we show that

$$\limsup_{\sigma\to0}\limsup_{n\to+\infty}\limsup_{\varepsilon\to0}\mathbf{n}^2 SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(\tfrac{\vartheta_n^{\sigma,h}}{\mathbf{n}},A_\varepsilon\big) \le \pi\log 2\,. \tag{6.26}$$

A combination of (6.25)–(6.26) and Proposition 6.2 then gives (6.24).

We will obtain (6.26) by first comparing $SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(\frac{\vartheta_n^\sigma}{\mathbf{n}},A_\varepsilon)$ with $SD_\varepsilon(\vartheta_n^\sigma,A_\varepsilon)$ and subsequently using interpolation estimates. To this end, we start recalling that $\vartheta_n^{\sigma,h} \in C^\infty\big(B_{2\sigma}(x_h)\setminus\Gamma_h\big)$ with

$[\vartheta_n^{\sigma,h}]_{|\Gamma_h} = d_h$, and that we identified $\vartheta_n^{\sigma,h}$ on Γ_h with its one-sided trace. Upon suitably choosing a normal to Γ_h it is thus not restrictive to assume that for $k \in \{1,2\}$ and $i \in \mathbb{Z}_\varepsilon^{e_k}(A_\varepsilon)$ with $[i, i+\varepsilon e_k) \cap \Gamma_h = \emptyset$ the function $t \mapsto \vartheta_n^{\sigma,h}(i+te_k)$ is continuous on $[0,\varepsilon]$ and differentiable on $(0,\varepsilon)$. An application of the mean-value inequality then yields that

$$|\mathrm{d}\vartheta_n^{\sigma,h}(i, i+\varepsilon e_k)| \leq \varepsilon \|\nabla \vartheta_n^{\sigma,h}\|_{L^\infty(A_{\frac{\sigma}{4},2\sigma}(x_h))} < \frac{1}{2\mathbf{n}} \tag{6.27}$$

for ε sufficiently small. Similarly if $i \in \mathbb{Z}_\varepsilon^{e_k}(A_\varepsilon)$ is such that $[i, i+\varepsilon e_k) \cap \Gamma_h \neq \emptyset$, we find that

$$|\mathrm{d}\vartheta_n^{\sigma,h}(i, i+\varepsilon e_k) - d_h| \leq \varepsilon \|\nabla \vartheta_n^{\sigma,h}\|_{L^\infty(A)} < \frac{1}{2\mathbf{n}} \ \text{ for } \varepsilon \text{ sufficiently small}\,. \tag{6.28}$$

By definition of $f_{\varepsilon,\frac{1}{\mathbf{n}}}^{\tau_k}$ this in turn implies that

$$f_{\varepsilon,\frac{1}{\mathbf{n}}}^{\tau_k}\big(\tfrac{\mathrm{d}\vartheta_n^{\sigma,h}(i,i+\varepsilon e_k)}{\mathbf{n}}\big) = f_{\frac{1}{\mathbf{n}}}\big(\tfrac{\mathrm{d}\vartheta_n^{\sigma,h}(i,i+\varepsilon e_k)}{\mathbf{n}}\big) = \frac{1}{\mathbf{n}^2} f_1\big(\mathrm{d}\vartheta_n^{\sigma,h}(i, i+\varepsilon e_k)\big)\,,$$

where the second equality follows from (2.14). In particular, we have that

$$\mathbf{n}^2 SD_{\varepsilon,\frac{1}{\mathbf{n}}}^{\mathrm{part}}\big(\tfrac{\vartheta_n^{\sigma,h}}{\mathbf{n}}, A_\varepsilon\big) \leq SD_\varepsilon\big(\vartheta_n^{\sigma,h}, A_\varepsilon\big)\,.$$

Together with the interpolation estimate (3.17) this yields

$$\limsup_{\varepsilon\to 0} \mathbf{n}^2 SD_{\varepsilon,\frac{1}{\mathbf{n}}}^{\mathrm{part}}\big(\tfrac{\vartheta_n^{\sigma,h}}{\mathbf{n}}, A_\varepsilon\big) \leq \frac{1}{2}\int_{A_{\frac{\sigma}{2},\sigma}(x_h)} |\nabla v_n^{\sigma,h}(x)|^2 \,\mathrm{d}x = \frac{1}{2}\int_{A_{\frac{\sigma}{2},\sigma}(x_h)} |\nabla \vartheta_n^{\sigma,h}(x)|^2 \,\mathrm{d}x \tag{6.29}$$

Following now exactly the proof [8, Estimate (6.69)], we obtain that

$$\limsup_{\sigma\to 0}\limsup_{n\to\infty} \frac{1}{2}\int_{A_{\frac{\sigma}{2},\sigma}(x_h)} |\nabla \vartheta_{n,\sigma}^h|^2 \,\mathrm{d}x \leq \pi \log 2\,,$$

which together with (6.29) finally gives (6.26).

Step 3: *Energy estimate on $\Omega^\sigma(\mu)$.* In this step we show that

$$\begin{aligned}\limsup_{n\to\infty}\limsup_{\varepsilon\to 0} SD_{\varepsilon,\frac{1}{\mathbf{n}}}^{\mathrm{part}}\big(u_\varepsilon^{n,\sigma}, \Omega^{\sigma-2\varepsilon}(\mu)\big) \leq \frac{1}{2\mathbf{n}^2}&\int_{\Omega^\sigma(\mu)} |\nabla v|^2 \,\mathrm{d}x \\ &+ \int_{S_w} \big(\tau_1|\nu_w\cdot e_1| + \tau_2|\nu_w\cdot e_2|\big)\,\mathrm{d}\mathcal{H}^1 + C\sigma\end{aligned} \tag{6.30}$$

for some fixed constant $C>0$. By definition, we have that $u_\varepsilon^{n,\sigma}(i) = \psi_n^\sigma(i) = \frac{\phi_n^\sigma(i)+\chi_n^\sigma(i)}{\mathbf{n}}$ for every $i \in \varepsilon\mathbb{Z}^2 \cap \Omega^\sigma(\mu)$. If instead $i \in \varepsilon\mathbb{Z}^2 \cap \big(\Omega^{\sigma-2\varepsilon}(\mu)\setminus\Omega^\sigma(\mu)\big)$, then $i \in \varepsilon\mathbb{Z}^2 \cap A_{\sigma-2\varepsilon,\sigma}(x_h)$ for some $h \in \{1,\dots,M\}$. Thus, the definition of $u_\varepsilon^{n,\sigma}$ together with the boundary conditions (6.18) imply that $u_\varepsilon^{n,\sigma}(i) = \frac{\phi_n^\sigma(i)}{\mathbf{n}}$. Since χ_n^σ takes values in $\mathbb{Z}$, we deduce that

$$\mathbf{n}\mathrm{d}u_\varepsilon^{n,\sigma}(i, i+\varepsilon e_k) = \mathbf{n}\mathrm{d}\phi_n^\sigma(i, i+\varepsilon e_k) \mod \mathbb{Z}$$

for every $k \in \{1,2\}$ and $i \in \mathbb{Z}_\varepsilon^{e_k}\big(\Omega^{\sigma-2\varepsilon}(\mu)\big)$. Thus, estimating the maximum in the definition of $f_{\varepsilon,\frac{1}{\mathbf{n}}}^{\tau_k}$ with the sum and using (2.14) yields

$$f_{\varepsilon,\frac{1}{\mathbf{n}}}^{\tau_k}\big(\mathrm{d}u_\varepsilon^{n,\sigma}(i, i+\varepsilon e_k)\big) \leq \frac{1}{\mathbf{n}^2} f_1\big(\mathrm{d}\phi_n^\sigma(i, i+\varepsilon e_k)\big) + \varepsilon\tau_k \mathbb{1}_{\{\mathrm{dist}(t,\mathbb{Z})>\frac{1}{2\mathbf{n}}\}}\big(\mathrm{d}u_\varepsilon^{n,\sigma}(i, i+\varepsilon e_k)\big)\,. \tag{6.31}$$

Summing (6.31) over $k \in \{1,2\}$ and all $i \in \mathbb{Z}^{e_k}_\varepsilon\big(\Omega^{\sigma-2\varepsilon}(\mu)\big)$ we arrive at

$$\begin{aligned} SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(u^{n,\sigma}_\varepsilon,\Omega^{\sigma-2\varepsilon}(\mu)\big) \leq \frac{1}{\mathbf{n}^2} SD_\varepsilon\big(\phi^\sigma_n,\Omega^{\sigma-2\varepsilon}(\mu)\big) \\ + \varepsilon\sum_{k=1}^2 \tau_k \#\Big\{i\in\mathbb{Z}^{e_k}_\varepsilon\big(\Omega^{\sigma-2\varepsilon}(\mu)\big)\colon \operatorname{dist}\big(\mathrm{d}u^{n,\sigma}_\varepsilon(i,i+\varepsilon e_k),\mathbb{Z}\big) > \frac{1}{2\mathbf{n}}\Big\}\,. \end{aligned} \tag{6.32}$$

We now show that the second term on the right-hand side of (6.32) concentrates around the set $S_{w^\sigma_n}\cap\Omega^\sigma(\mu)$. To this end, we fix $k\in\{1,2\}$ and we assume that $i\in\mathbb{Z}^{e_k}_\varepsilon(\Omega^\sigma(\mu))$ is such that $[i,i+\varepsilon e_k)\cap S_{w^\sigma_n}=\emptyset$. Then (6.15) ensures that either $[i,i+\varepsilon e_k)\cap S_{\psi^\sigma_n}=\emptyset$ as well or there exists $x\in[i,i+\varepsilon e_k)\cap S_{\psi^\sigma_n}$ with $[\psi^\sigma_n](x)\in\mathbb{Z}$. As in (6.27)–(6.28) we deduce that

$$\operatorname{dist}\big(\mathrm{d}u^{n,\sigma}_\varepsilon(i,i+\varepsilon e_k),\mathbb{Z}\big) = \operatorname{dist}\big(\mathrm{d}\psi^\sigma_n(i,i+\varepsilon e_k),\mathbb{Z}\big) \leq \varepsilon\|\nabla\phi^\sigma_n\|_{L^\infty(\Omega^\sigma(\mu))} < \frac{1}{2\mathbf{n}}$$

for ε sufficiently small. We thus continue the estimate in (6.32) as follows.

$$\begin{aligned} SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(u^{n,\sigma}_\varepsilon,\Omega^{\sigma-2\varepsilon}(\mu)\big) \leq \frac{1}{\mathbf{n}^2} SD_\varepsilon\big(\phi^\sigma_n,\Omega^{\sigma-2\varepsilon}(\mu)\big) \\ + \varepsilon\sum_{k=1}^2 \tau_k\#\Big\{i\in\mathbb{Z}^{e_k}_\varepsilon\big(\Omega^\sigma(\mu)\big)\colon [i,i+\varepsilon e_k)\cap S_{w^\sigma_n}\neq\emptyset\Big\} \\ + \varepsilon(\tau_1+\tau_2)\#\Big\{i\in\varepsilon\mathbb{Z}^2\cap\big(\Omega^{\sigma-2\varepsilon}(\mu)\setminus\Omega^\sigma(\mu)\big)\Big\}\,. \end{aligned} \tag{6.33}$$

For the first term on the right-hand side of (6.32) we can use once again the interpolation estimate (3.17) to deduce that

$$\limsup_{\varepsilon\to0} SD_\varepsilon\big(\phi^\sigma_n,\Omega^{\sigma-2\varepsilon}(\mu)\big) \leq \frac12\int_{\Omega^\sigma(\mu)}|\nabla v^\sigma_n|^2\,\mathrm{d}x\,. \tag{6.34}$$

Moreover, Lemma 6.3 (ii) ensures that $S_{w^\sigma_n}$ is polyhedral. This implies that

$$\limsup_{\varepsilon\to0}\varepsilon\#\Big\{i\in\mathbb{Z}^{e_k}_\varepsilon\big(\Omega^\sigma(\mu)\big)\colon[i,i+\varepsilon e_k)\cap S_{w^\sigma_n}\neq\emptyset\Big\} \leq \int_{S_{w^\sigma_n}}|\nu_{w^\sigma_n}\cdot e_k|\,\mathrm{d}\mathcal{H}^1 \tag{6.35}$$

for any $k\in\{1,2\}$. Finally, we have that

$$\limsup_{\varepsilon\to0}\varepsilon\#\Big\{i\in\varepsilon\mathbb{Z}^2\cap\big(\Omega^{\sigma-2\varepsilon}(\mu)\setminus\Omega^\sigma(\mu)\big)\Big\} \leq C\mathcal{H}^1\bigg(\bigcup_{h=1}^M\partial B_\sigma(x_h)\bigg)\leq C\sigma\,,$$

which together with (6.33)–(6.35) yields

$$\limsup_{\varepsilon\to0} SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(u^{n,\sigma}_\varepsilon,\Omega^{\sigma-2\varepsilon}(\mu)\big) \leq \frac{1}{2\mathbf{n}^2}\int_{\Omega^\sigma(\mu)}|\nabla v^\sigma_n|^2\,\mathrm{d}x + \int_{S_{w^\sigma_n}} h(\nu_{w^\sigma_n})\,\mathrm{d}\mathcal{H}^1 + C\sigma$$

with $h(\nu):=\tau_1|\nu\cdot e_1|+\tau_2|\nu\cdot e_2|$. Thus, (6.30) follows from Lemma 6.3 (iii) together with (6.16).

<u>Step 4:</u> *Conclusion.* Combining (6.24) and (6.30) we deduce that

$$\begin{aligned} &\limsup_{n\to\infty}\limsup_{\varepsilon\to0}\Big(\mathbf{n}^2 SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}(u^{n,\sigma}_\varepsilon,\Omega) - M\pi|\log\varepsilon|\Big) \\ &\qquad \leq M\gamma^{SD} + \frac12\int_{\Omega^\sigma(\mu)}|\nabla v|^2\,\mathrm{d}x - M\pi|\log\sigma| + \mathbf{n}^2\int_{S_w}\big(\tau_1|\nu_w\cdot e_2|+\tau_2|\nu_w\cdot e_1|\big)\,\mathrm{d}\mathcal{H}^1 \end{aligned}$$

and thus (6.21) follows by letting $\sigma \to 0$ and using the definition of $\mathcal{W}(w^{\mathbf{n}},\Omega) = \mathcal{W}(v,\Omega)$. To conclude it thus suffices to show (6.22). To obtain (6.22) we start observing that

$$\begin{aligned}\|w^\sigma_{\varepsilon,n} - w\|_{L^1(\Omega)} &= \sum_{Q_\varepsilon(i)\in\mathcal{Q}_\varepsilon} \int_{\Omega\cap Q_\varepsilon(i)} |w^\sigma_{\varepsilon,k}(i) - w(x)|\,\mathrm{d}x \\ &\leq \sum_{Q_\varepsilon(i)\in\mathcal{Q}_\varepsilon(\Omega^\sigma(\mu)\setminus S_{w^\sigma_n})} \int_{Q_\varepsilon(i)} |w^\sigma_n(i) - w(x)|\,\mathrm{d}x + 2\sum_{\substack{Q_\varepsilon\in\mathcal{Q}_\varepsilon\\ Q_\varepsilon\cap\mathbb{R}^2\setminus\Omega^\sigma(\mu)\neq\emptyset}} |Q_\varepsilon\cap\Omega| \\ &\quad + 2\varepsilon^2\#\big\{Q_\varepsilon\in\mathcal{Q}_\varepsilon(\Omega^\sigma(\mu))\colon Q_\varepsilon\cap S_{w^\sigma_n}\neq\emptyset\big\}\,,\end{aligned} \tag{6.36}$$

where we have used that $|w^\sigma_{\varepsilon,n}|, |w| \leq 1$. For every $Q_\varepsilon\in\mathcal{Q}_\varepsilon$ with $Q_\varepsilon\cap\mathbb{R}^2\setminus\Omega^\sigma(\mu)\neq\emptyset$, the inclusion $Q_\varepsilon\cap\Omega\subset\{\mathrm{dist}(x,\partial\Omega)\leq\sqrt{2}\varepsilon\}\cup\bigcup_{h=1}^M B_{\sigma+\sqrt{2}\varepsilon}(x_h)$ holds. From this we deduce that

$$\sum_{\substack{Q_\varepsilon\in\mathcal{Q}_\varepsilon\\ Q_\varepsilon\cap\mathbb{R}^2\setminus\Omega^\sigma(\mu)\neq\emptyset}} |Q_\varepsilon\cap\Omega| \leq \big|\{x\in\Omega\colon\ \mathrm{dist}(x,\partial\Omega)\leq\sqrt{2}\varepsilon\}\big| + M\big|B_{\sigma+\sqrt{\varepsilon}}\big| \to 0 \ \text{ as } \varepsilon\to 0,\ \sigma\to 0, \tag{6.37}$$

where we have used that $\partial\Omega$ is Lipschitz and thus admits and $(n-1)$-dimensional Minkowsky content. Similarly, we find that

$$\varepsilon^2\#\big\{Q_\varepsilon\in\mathcal{Q}_\varepsilon(\Omega^\sigma(\mu))\colon Q_\varepsilon\cap S_{w^\sigma_n}\neq\emptyset\big\} \leq C\varepsilon\mathcal{H}^1(S_{w^\sigma_n}\cap\Omega^\sigma(\mu)) \to 0 \ \text{ as } \varepsilon\to 0\,. \tag{6.38}$$

The remaining term in (6.36) can be estimated by observing the following. For any $Q_\varepsilon(i)\in\mathcal{Q}_\varepsilon(\Omega^\sigma(\mu)\setminus S_{w^\sigma_n})$ and any $x\in Q_\varepsilon(i)$ we have that $|w^\sigma_n(i) - w^\sigma_n(x)| \leq \sqrt{2}\varepsilon\|\nabla w^\sigma_n\|_{L^\infty(\Omega^\sigma(\mu))}$. From this we infer that

$$\sum_{Q_\varepsilon(i)\in\mathcal{Q}_\varepsilon(\Omega^\sigma(\mu)\setminus S_{w^\sigma_n})} \int_{Q_\varepsilon(i)} |w^\sigma_n(i) - w(x)|\,\mathrm{d}x \leq \sqrt{2}\varepsilon\|\nabla w^\sigma_n\|_{L^\infty(\Omega^\sigma(\mu))}|\Omega^\sigma(\mu)| + \|w^\sigma_n - w\|_{L^1(\Omega^\sigma(\mu))}\,. \tag{6.39}$$

Finally, $\|w^\sigma_n - w\|_{L^1(\Omega^\sigma(\mu))}\to 0$ as $n\to\infty$ thanks to Lemma 6.3 (iv). Together with (6.36)–(6.39) this gives (6.22) and we conclude. □

Proof of Theorem 4.2 (iii). Theorem 4.1 (iii) is a direct consequence of Theorem 4.2 (iii), Proposition 6.2, and Lemma 3.3. In fact, to construct a recovery sequence for $XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ it suffices to take the sequence $u^{n,\sigma}_\varepsilon$ constructed in the proof of Theorem 4.2 (iii) and for every $h\in\{1,\dots,M\}$ a spin field $w^h_\varepsilon\in\mathcal{SF}_\varepsilon$ satisfying

$$XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(w^h_\varepsilon, B_{\frac{\sigma}{2}}(x_h)\big) = \gamma^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_{\frac{\sigma}{2}}(x_h)\big) \tag{6.40}$$

and $w^h_\varepsilon = \exp(\frac{2\pi}{\mathbf{n}}\iota d_h\theta_h)$ on $\partial_\varepsilon B_{\frac{\sigma}{2}}(x_h)$ with θ_h as in the proof of Theorem 4.2 (iii). We then define $w^{n,\sigma}_\varepsilon\in\mathcal{SF}_\varepsilon$ by setting

$$w^{n,\sigma}_\varepsilon(i) := \begin{cases} w^h_\varepsilon(i) & \text{if } i\in\varepsilon\mathbb{Z}^2\cap B_{\frac{\sigma}{2}}(x_h) \text{ for some } h\in\{1,\dots,M\}\,,\\[1em] \exp\big(2\pi\iota u^{n,\sigma}_\varepsilon(i)\big) & \text{otherwise in } \varepsilon\mathbb{Z}^2\,.\end{cases}$$

In this way, we deduce from Lemma 3.3 that

$$XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(w^{n,\sigma}_\varepsilon,\Omega\big) \leq \sum_{h=1}^M XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(w^{n,\sigma}_\varepsilon, B_\sigma(x_h)\big) + SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(u^{n,\sigma}_\varepsilon,\Omega^{\sigma-2\varepsilon}(\mu)\big)\,. \tag{6.41}$$

Moreover, (6.40) ensures that for every $h\in\{1,\dots,M\}$ we have

$$XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(w^{n,\sigma}_\varepsilon, B_\sigma(x_h)\big) \leq \gamma^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(B_{\frac{\sigma}{2}}(x_h)\big) + SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}\big(\tfrac{\vartheta^{\sigma,h}_n}{\mathbf{n}}, A_{\frac{\sigma}{2}-2\varepsilon,\sigma}(x_h)\big)\,. \tag{6.42}$$

In particular, Proposition 6.2 together with (6.26) implies that (6.24) holds with $SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and γ^{SD} replaced by $XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and γ^{XY}, respectively. Together with (6.41) and (6.30) this in turn yields (6.21) with $SD^{\mathrm{part}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and γ^{SD} replaced by $XY^{\mathrm{frac}}_{\varepsilon,\frac{1}{\mathbf{n}}}$ and γ^{XY}, respectively. We thus conclude by observing that $w^{n,\sigma}_{\varepsilon}$ still satisfies (6.22). □

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