The relaxed p-energy of manifold constrained mappings

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Abstract. *The p-energy of Sobolev mappings between Riemannian manifolds is studied, for each integer p greater than two. We analyse the lower semicontinuous extension of the energy to currents. We then restrict to mappings with values into the p-sphere, by giving an explicit relaxed p-energy formula, whose proof depends on a strong density result. Finally, a related coarea formula is obtained.*

Keywords: Relaxed energy, Sobolev mappings, Riemannian manifolds, currents, coarea formula.

1 Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be two smooth, compact, connected, oriented Riemannian manifolds, where $\mathcal{X}$ is possibly with a non-empty boundary $\partial\mathcal{X}$, but $\mathcal{Y}$ is closed.

The *Dirichlet energy*, or *action* in Physics, of a smooth map $U:\mathcal{X}\to\mathcal{Y}$ is defined as the integral of the square of the derivative $\mathrm{d}U$, so that

$$\frac{1}{2}\int_{\mathcal{X}}|dU_x|^2\,\mathrm{dvol}_{\mathcal{X}}\,,\quad |\mathrm{d}U_x|^2=\mathrm{tr}\,[(\mathrm{d}U_x)^*\mathrm{d}U_x]\,, \tag{1.1}$$

where $A\mapsto \mathrm{tr}\,A$ and $G\mapsto G^*$ denote the trace and adjoint operator, respectively, compare [13, Sec. 4.1].

By Nash embedding theorem, we assume that the target manifold $\mathcal{Y}$ is isometrically embedded, as a submanifold, in some Euclidean space $\mathbb{R}^N$. Therefore, since the Riemannian metric on $\mathcal{Y}$ is induced by the standard metric on $\mathbb{R}^N$, the inner product of two tangent vectors to $\mathcal{Y}$ at a point $y\in\mathcal{Y}$ is simply their inner product, and the metric tensor on $\mathcal{Y}$ is given by the Kronecker symbols $\gamma_{ij}=\delta_{ij}$, where $i,j=1,\dots,N$. We thus consider maps $U:\mathcal{X}\to\mathbb{R}^N$ that are constrained to take values into the submanifold $\mathcal{Y}$.

Let $n=\dim\mathcal{X}$. Given a local parameterization $\phi:\Omega\to\mathcal{X}$, where $\Omega\subset\mathbb{R}^n$ is a bounded open set, the metric tensor $g=(g_{\alpha\beta})$ at $x\in\Omega$ is given by $g_{\alpha\beta}(x)=\partial_{x_\alpha}\phi(x)\cdot\partial_{x_\beta}\phi(x)$, for $\alpha,\beta=1,\dots,n$. Therefore, by compactness, there exists a positive real constant C, depending on Ω, such that for every $x\in\Omega$ and $\tau\in\mathbb{R}^n$

$$C|\tau|^2\le|\tau|^2_{g(x)}\le\frac{1}{C}\,|\tau|^2\,,\qquad |\tau|^2_{g(x)}:=\tau^\top g(x)\,\tau\,, \tag{1.2}$$

where $G\mapsto G^\top$ denotes the transposition operator on matrices and column vectors.

If $(g^{\alpha\beta})=(g_{\alpha\beta})^{-1}$ denotes the inverse of the metric tensor, with $u=U\circ\phi$ one computes

$$\mathrm{tr}\,[(dU_x)^*dU_x]=\sum_{\alpha,\beta=1}^n\sum_{i,j=1}^N g^{\alpha\beta}(x)\delta_{ij}\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}\,,\quad x\in\Omega\,,$$

where $u=(u^1,\dots,u^N)$. Therefore, the volume element $\mathrm{dvol}_{\mathcal{X}}$ being equal to $\sqrt{\det g}\,\mathrm{d}x$, the Dirichlet energy in local coordinates takes the form

$$\frac{1}{2}\int_\Omega\sum_{\alpha,\beta=1}^n\sum_{i,j=1}^N g^{\alpha\beta}(x)\delta_{ij}\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}\,\sqrt{\det g(x)}\,\mathrm{d}x\,. \tag{1.3}$$

The Dirichlet energy is a conformally invariant functional in dimension $n=2$. In case $\mathcal{X}=B^n$, the unit ball in $\mathbb{R}^n$, and $g_{\alpha\beta}(x)\equiv\delta_{\alpha\beta}$, one recovers the standard Dirichlet integral. In a similar way, for any integer $\mathfrak{p}\ge 2$, the functional

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}}\int_{B^n}|Du(x)|^{\mathfrak{p}}\,\mathrm{d}x\,,\quad n\ge\mathfrak{p} \tag{1.4}$$

is conformally invariant in the critical dimension $n = \mathfrak{p}$. It will be called *Euclidean* $\mathfrak{p}$*-energy functional*.

In this paper, we deal with the $\mathfrak{p}$-*energy* of maps $U : \mathcal{X} \to \mathcal{Y}$. Therefore, we assume $n \geq \mathfrak{p} \geq 2$ integer, and $\dim \mathcal{Y} \geq \mathfrak{p}$, so that $N > \mathfrak{p}$.

For the sake of simplicity, in the sequel we only consider local arguments, and hence we assume $\mathcal{X} = (B^n, g)$. Therefore, without loss of generality we can find an absolute constant $C > 0$ such that the bound (1.2) holds for every $x \in B^n$ and $\tau \in \mathbb{R}^n$. Global results can be obtained by using arguments as e.g. in [9], where the case $\mathfrak{p} = 2$ was analysed.

When $\mathcal{X} = (B^n, g)$, the $\mathfrak{p}$-*energy* of smooth maps $u : B^n \to \mathcal{Y} \subset \mathbb{R}^N$ is given by

$$\mathbf{D}^{\mathfrak{p}}_g(u, B^n) := \int_{B^n} e^{\mathfrak{p}}_g(x, Du(x))\, \mathrm{d}x \,, \tag{1.5}$$

where the $\mathfrak{p}$-*energy density* is defined for any $x \in B^n$ and any real valued $N \times n$-matrix G in $M(N, n)$ as:

$$e^{\mathfrak{p}}_g(x, G) := \Big(\frac{1}{\mathfrak{p}} \sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} g^{\alpha\beta}(x) \delta_{ij}\, G^i_\alpha\, G^j_\beta \cdot (\det g(x))^{1/\mathfrak{p}} \Big)^{\mathfrak{p}/2} . \tag{1.6}$$

From another viewpoint, one may be interested in studying the energy

$$\int_{B^n} f_{\mathfrak{p}}(x, Du)\, \mathrm{d}x \tag{1.7}$$

of mappings $u : B^n \to \mathcal{Y} \subset \mathbb{R}^N$, where the integrand $f_{\mathfrak{p}} : B^n \times M(N, n) \to \mathbb{R}^+$ is defined by

$$f_{\mathfrak{p}}(x, G) := \Big(\frac{1}{\mathfrak{p}} \operatorname{tr} (G\, A(x)\, G^\top) \Big)^{\mathfrak{p}/2} , \quad x \in B^n , \quad G \in M(N, n) , \tag{1.8}$$

$x \mapsto A(x)$ being a continuous map from B^n to the space of positive definite matrices in $M(n, n)$.

Setting $c(\mathfrak{p}, \mathfrak{p}) = 0$, and $c(n, \mathfrak{p}) = 1/(n - \mathfrak{p})$ if $n > \mathfrak{p}$, we get for any $n \geq \mathfrak{p}$ the equivalence:

$$g(x) := (\det A(x))^{c(n,\mathfrak{p})}\, A(x)^{-1} \iff A_{\alpha\beta}(x) := \big(\det g(x)\big)^{1/\mathfrak{p}}\, g^{\alpha\beta}(x) , \quad \forall\, x \in B^n . \tag{1.9}$$

Therefore, it turns out that the integral (1.7) agrees with the $\mathfrak{p}$-energy (1.5) of mappings $u : \mathcal{X} \to \mathcal{Y} \subset \mathbb{R}^N$, where $\mathcal{X} = (B^n, g)$, i.e., on account of the equivalence in (1.9), we have:

$$f_{\mathfrak{p}}(x, G) = e^{\mathfrak{p}}_g(x, G) \qquad \forall\, (x, G) \in B^n \times M(N, n) . \tag{1.10}$$

Note that for $n = \mathfrak{p} + 1$, in (1.9) we have $g(x) = \operatorname{cof} A(x)$. Therefore, since $|\tau|^2_{g(x)} = \tau^\top g(x)\, \tau$, we have

$$|\tau|^2_{g(x)} = \tau^\top (\operatorname{cof} A(x))\, \tau , \quad \forall\, \tau \in \mathbb{R}^{\mathfrak{p}+1}. \tag{1.11}$$

We wish to analyse the *relaxed* $\mathfrak{p}$-*energy* (1.12) of non-smooth maps $u : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X} = (B^n, g)$ as above. For the sake of simplicity, we only discuss the easier case when the target manifold $\mathcal{Y}$ is equal to the unit $\mathfrak{p}$-sphere

$$\mathbb{S}^{\mathfrak{p}} := \big\{ y \in \mathbb{R}^{\mathfrak{p}+1} \,:\, |y| = 1 \big\} , \quad \mathfrak{p} \geq 2 .$$

Some of the new results contained in this paper extend the ones obtained in [9] when $\mathfrak{p} = 2$. Moreover, similar problems concerning energies with a non-negative measurable (or continuous) weight

$$\int_{B^n} a(x)\, |Du|^{\mathfrak{p}}\, \mathrm{d}x$$

of non-smooth maps $u : B^n \to \mathbb{S}^{\mathfrak{p}}$ have been studied. We refer to [19] and [19] for the case $n = 3$ and $\mathfrak{p} = 2$, and to [22] for the case $n \geq \mathfrak{p} + 1 \geq 3$. In addition, $H^{1/2}$-maps with measurable weights and taking values into the circle have been thoroughly analysed in [21].

1.1 Main results

In the same spirit as for Lebesgue's relaxed area, for every map $u \in L^{\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ we define:

$$\widetilde{\mathbf{D}}^{\mathfrak{p}}_g(u, B^n) := \inf\Big\{ \liminf_{k\to\infty} \mathbf{D}^{\mathfrak{p}}_g(u_k, B^n) \quad | \quad \{u_k\} \subset C^\infty(B^n, \mathbb{S}^{\mathfrak{p}}),\ u_k \to u \text{ strongly in } L^{\mathfrak{p}}(B^n, \mathbb{R}^{\mathfrak{p}+1}) \Big\}, \tag{1.12}$$

where, for $\mathcal{F} = L^p$, $W^{1,p}$, or C^∞, we denote

$$\mathcal{F}(B^n, \mathbb{S}^{\mathfrak{p}}) := \{u \in \mathcal{F}(B^n, \mathbb{R}^{\mathfrak{p}+1}) \,:\, |u(x)| = 1 \text{ for a.e. } x \in B^n\}.$$

Due to the bound (1.2) on the metric g, a map $u \in L^{\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ with finite relaxed $\mathfrak{p}$-energy belongs to the Sobolev class $W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$. In that case, moreover, one can replace the strong $L^{\mathfrak{p}}$-convergence with the sequential weak $W^{1,\mathfrak{p}}(B^n, \mathbb{R}^{\mathfrak{p}+1})$ convergence $u_k \rightharpoonup u$, without affecting the relaxed functional. In addition, by the convexity of the $\mathfrak{p}$-energy functional, the $\mathfrak{p}$-*energy gap*

$$\mathbf{G}^p_g(u, B^n) := \widetilde{\mathbf{D}}^{\mathfrak{p}}_g(u, B^n) - \mathbf{D}^{\mathfrak{p}}_g(u, B^n), \quad u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}}), \tag{1.13}$$

is always non-negative. Furthermore, in low dimension $n = \mathfrak{p}$, by Schoen-Uhlenbeck density theorem [25], and by dominated convergence, one has

$$\widetilde{\mathbf{D}}^{\mathfrak{p}}_g(u, B^{\mathfrak{p}}) = \mathbf{D}^{\mathfrak{p}}_g(u, B^{\mathfrak{p}}) \qquad \forall\, u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}}). \tag{1.14}$$

This property follows essentially thanks to the embedding of the Sobolev space $W^{1,\mathfrak{p}}(B^{\mathfrak{p}})$ in the class VMO of functions with vanishing mean oscillation.

Therefore, we now assume $n \geq \mathfrak{p}+1$. In the Euclidean case, so that the energy $\mathbf{D}^{\mathfrak{p}}_g(u, B^n)$ of smooth maps is equal to the integral in (1.4), the explicit formula for the relaxed energy is well-known, see Theorem 2.12 below. In case $\mathfrak{p} = 2$, it was first proved in [10] and independently (see Eq. (2.28) below) in [3], in low dimension $n = 3$, and then extended to any high dimension n in [26].

In this paper, we show that the relaxed $\mathfrak{p}$-energy (1.12) of Sobolev maps is always finite, so that

$$\widetilde{\mathbf{D}}^{\mathfrak{p}}_g(u, B^n) < \infty \iff u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}}), \quad \forall\, n \geq \mathfrak{p}+1. \tag{1.15}$$

In addition, we find an explicit formula for the $\mathfrak{p}$-energy gap (1.13). As in the Euclidean case, it depends on the size of the *minimal connection* of the *singularities* of u.

Precisely, using homological tools from Geometric Measure Theory, it is well-known [13] that the relevant singularities of a map u in $W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ are described by an $(n-\mathfrak{p}-1)$-dimensional *current* $\mathbb{P}(u)$, that turns out to be an *integral flat chain*. Referring to Sec. 2 for the notation adopted here, the latter property means that there exists an i.m. rectifiable current $L \in \mathcal{R}_{n-\mathfrak{p}}(B^n)$, with *finite mass*, $\mathbf{M}(L) < \infty$, that bounds the singularities of u, i.e., such that equation $L(\mathrm{d}\eta) = \mathbb{P}(u)(\eta)$ holds for every compactly supported smooth $(n-\mathfrak{p}-1)$-form η in B^n, where $\mathrm{d}\eta$ is the differential of η. In this case, we write shortly

$$(\partial L) \llcorner B^n = \mathbb{P}(u).$$

In dimension $n = \mathfrak{p}+1$, if e.g. $u_V(x) = x/|x|$, the vortex map, we find that $\mathbb{P}(u_V) = -\delta_{\mathbf{0}}$, and a 1-current L as above is obtained by integration of 1-forms along any oriented segment connecting a boundary point of $B^{\mathfrak{p}+1}$ to the origin $\mathbf{0}$.

In high dimension $n > \mathfrak{p}+1$, if $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ is defined by

$$u(x) = \frac{\widetilde{x}}{|\widetilde{x}|}, \qquad x = (\widetilde{x}, \widehat{x}) \in \mathbb{R}^{\mathfrak{p}+1} \times \mathbb{R}^{n-\mathfrak{p}-1}, \tag{1.16}$$

the singular set of u is the $(n-\mathfrak{p}-1)$-dimensional disk

$$\Delta^{n-\mathfrak{p}-1} := \{(0_{\mathbb{R}^{\mathfrak{p}+1}}, \widehat{x}) \in \mathbb{R}^n \,:\, |\widehat{x}| \leq 1\}.$$

Moreover, denoting by $[\![\Delta^{n-\mathfrak{p}-1}]\!]$ the current obtained by integration of $(n-\mathfrak{p}-1)$-forms on the disk $\Delta^{n-\mathfrak{p}-1}$, equipped with the natural orientation induced by the canonical basis in $\mathbb{R}^n$, we have:

$$\mathbb{P}(u) = (-1)^{n-\mathfrak{p}}[\![\Delta^{n-\mathfrak{p}-1}]\!] . \tag{1.17}$$

The minimal connection among currents L as above, is computed with respect to the *g-mass* $\mathbf{M}_g(L)$. In the Euclidean case, it agrees with the usual mass. If e.g. $n = \mathfrak{p}+1$, so that $L \in \mathcal{R}_1(B^{\mathfrak{p}+1})$, the g-mass of L depends on formula (1.11). More precisely, in case L is obtained by integration along a smooth and oriented curve $\gamma : [a,b] \to \overline{B}^{\mathfrak{p}+1}$, with $\gamma(t) \in B^{\mathfrak{p}+1}$ for each $t \in]a,b[$, then

$$\mathbf{M}_g(L) = \int_a^b |\gamma'(t)|_{g(\gamma(t))}\, \mathrm{d}t = \int_a^b \big(\gamma'(t)^\top (\mathrm{cof}\, A(\gamma(t)))\, \gamma'(t)\big)^{1/2}\, \mathrm{d}t\,,$$

where $A(x)$ is given by formula (1.9). Therefore, the g-mass of L agrees with the length of γ in the Riemannian manifold $\mathcal{X} = (B^{\mathfrak{p}+1}, g)$. More generally, if $T \in \mathcal{R}_k(B^n)$ is the k-current integration on a smooth, embedded, and oriented k-surface $\mathcal{M}$ of B^n, say $L = [\![\mathcal{M}]\!]$, then for every integer $1 \le k \le n$ we have

$$\mathbf{M}_g(L) = \mathbf{M}_g([\![\mathcal{M}]\!]) = \mathcal{H}^k_g(\mathcal{M})\,,$$

where $\mathcal{H}^k_g$ denotes the k-dimensional Hausdorff measure in B^n with respect to the distance induced by the metric tensor g, compare e.g. [24].

The *integral g-mass* of $\mathbb{P}(u)$ is defined for every $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ by

$$m^g_{i,B^n}(\mathbb{P}(u)) := \inf\big\{\mathbf{M}_g(L) \mid L \in \mathcal{R}_{n-\mathfrak{p}}(B^n)\,, \quad (\partial L) \llcorner B^n = \mathbb{P}(u)\big\}$$

and the minimum is always attained. In case e.g. of the vortex map $u_V(x) = x/|x|$, it is equal to the minimal length in $\mathcal{X} = (B^{\mathfrak{p}+1}, g)$ among the smooth geodesic arcs between a boundary point of $B^{\mathfrak{p}+1}$ and the origin.

In this paper, we show that in any dimension $n \ge \mathfrak{p}+1$

$$\mathbf{G}^{\mathfrak{p}}_g(u, B^n) = \alpha_{\mathfrak{p}} \cdot m^g_{i,B^n}(\mathbb{P}(u)) < \infty\,, \quad \forall\, u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})\,, \tag{1.18}$$

where $\alpha_{\mathfrak{p}} := \mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})$ is the $\mathfrak{p}$-dimensional area of the unit $\mathfrak{p}$-sphere, see Sec. 4.2.

Note that in dimension $n = \mathfrak{p}+1$, we equivalently have:

$$\mathbf{G}^{\mathfrak{p}}_g(u, B^{\mathfrak{p}+1}) = \alpha_{\mathfrak{p}} \cdot \mathbf{L}_g(u, B^{\mathfrak{p}+1})\,, \quad \forall\, u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}})\,, \tag{1.19}$$

where the *flat g-norm* of a function $u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}})$ is defined in the sense of Brezis-Coron-Lieb [5] by

$$\mathbf{L}_g(u, B^{\mathfrak{p}+1}) := \frac{1}{\alpha_{\mathfrak{p}}} \sup\Big\{\int_{B^{\mathfrak{p}+1}} D(u)\cdot D\phi\, \mathrm{d}x \mid \phi \in C^\infty_c(B^{\mathfrak{p}+1})\,,\ |D\phi|_{g(x)} \le 1\ \forall\, x \in B^{\mathfrak{p}+1}\Big\}\,, \tag{1.20}$$

with $D(u) \in L^1(B^{\mathfrak{p}+1}, \mathbb{R}^{\mathfrak{p}+1})$ the D-field $D(u) = (D^1(u), \dots, D^{\mathfrak{p}+1}(u))$, with components given by (2.10).

We also obtain an estimate concerning the energy gap. In fact, extending the *coarea formula* by Almgren-Browder-Lieb [1], in Theorem 4.4 we prove for every smooth map $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ the $\mathfrak{p}$-energy lower bound

$$\mathbf{D}^{\mathfrak{p}}_g(u, B^n) \ge \int_{B^n} J^g_u(x)\, \mathrm{d}\mathcal{H}^n_g(x) = \int_{\mathbb{S}^{\mathfrak{p}}} \mathcal{H}^{n-\mathfrak{p}}_g(u^{-1}(y))\, \mathrm{d}\mathcal{H}^{\mathfrak{p}}(y)\,,$$

where J^g_u is the Jacobian of u with respect to the metric g on B^n.

Moreover, the latter formula extends to the class $R^\infty_{\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$, given by the Sobolev maps $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ that are smooth outside a "smooth" singular set of dimension $(n-\mathfrak{p}-1)$. This is e.g. the case of the vortex map $u_V(x) = x/|x|$, in dimension $n = \mathfrak{p}+1$, or the map in (1.16), in higher dimension. The above class was introduced by Bethuel [2], who showed that it is strongly dense in $W^{1,p}(B^n, \mathbb{S}^{\mathfrak{p}})$. As a consequence, we readily obtain in any dimension $n \ge \mathfrak{p}+1$ the energy gap estimate

$$\mathbf{G}^{\mathfrak{p}}_g(u, B^n) \le \mathbf{D}^{\mathfrak{p}}_g(u, B^n)\,, \quad \forall\, u \in R^\infty_{\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})\,. \tag{1.21}$$

Finally, by Federer's theorem [6] on 0-dimensional integral flat chains, in dimension $n = \mathfrak{p}+1$ we are able to extend the gap estimate (1.21) to the whole class $W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^{\mathfrak{p}})$, see Corollary 4.5.

1.2 Content of the paper

As in the case $\mathfrak{p} = 2$ treated in [9], following arguments by Giaquinta-Modica-Souček, the relaxed energy formula (1.18) stems from the theory of *Cartesian currents* in $B^n \times \mathbb{S}^{\mathfrak{p}}$ with finite $\mathfrak{p}$-energy.

For that reason, in Sec. 2 we introduce the basic GMT tools, and report the analysis of the class $\mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$, showing how the formula of the relaxation of the Euclidean $\mathfrak{p}$-energy (1.4) is obtained.

In Sec. 3, we analyze the *parametric polyconvex l.s.c. envelop* of the $\mathfrak{p}$-energy density integrand (1.6). We then write an explicit formula for the corresponding energy functional $T \mapsto \mathbf{D}^{\mathfrak{p}}_g(T)$ on currents in $\mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$. The results there obtained are new, and extend the case $\mathfrak{p} = 2$ analyzed in [9].

In Sec. 4, we prove the explicit formula (1.18) for the relaxed $\mathfrak{p}$-energy, and the coarea formula, Theorem 4.4, yielding to the energy gap estimate (1.21). Inequality "$\geq$" in formula (1.18) follows from the lower semicontinuity of the $\mathfrak{p}$-energy functional $T \mapsto \mathbf{D}^{\mathfrak{p}}_g(T)$. Inequality "$\leq$", instead, is a consequence of the validity of a suitable strong density result for Cartesian currents.

Namely, in Theorem 4.1 we show that *for every $T \in \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$, there exists a sequence of smooth maps $\{u_k\} \subset C^\infty(B^n, \mathbb{S}^{\mathfrak{p}})$ such that the corresponding graph currents weakly converge to T in $\mathcal{D}_n(B^n \times \mathbb{S}^{\mathfrak{p}})$, and $\mathbf{D}^{\mathfrak{p}}_g(u_k, B^n) \to \mathbf{D}^{\mathfrak{p}}_g(T)$.*

Following an idea by M. Giaquinta, the latter strong density result is a consequence of the approximation argument contained in Theorem 4.2, whose technical proof is postponed to Sec. 5. It is based essentially on adaptations of the one in case $\mathfrak{p} = 2$ obtained in [9], but in the easier situation when $\mathcal{X} = (B^n, g)$ and $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}}$. Therefore, it relies on arguments taken from the density theorems in [16] and [18], see also [17].

As a consequence of the previous results, we also discuss the relaxed p-energy of mappings in $W^{1,p}(B^n, \mathbb{S}^{\mathfrak{p}})$, where $p > 2$ is a non-integer exponent and $\mathfrak{p}$ is the integer part of p. Roughly speaking, since $p > \mathfrak{p}$, in the relaxation process, concentration along $(n - \mathfrak{p})$-dimensional sets cannot be obtained with a finite amount of p-energy, see Theorem 4.6.

We finally remark that the relaxed $\mathfrak{p}$-energy of mappings satisfying a suitable *Dirichlet-type* boundary conditions can be tackled in a similar way, following arguments taken e.g. from [13, Sec. 4.2.5]. In another direction, similar results concerning more general target manifolds[1] can be treated using arguments taken from [9] for the case $\mathfrak{p} = 2$. However, for the sake of brevity, neither of the latter items is reported here.

2 Notation and background material

We refer to [12], [13], and [17] for further details concerning the following notation.

2.1 Multivectors and linear mappings

Denote by $I(k, m)$ the class of ordered multi-indices α in $\{1, \dots, m\}$ of length $|\alpha|$ equal to k, i.e., $\alpha = (\alpha_1, \dots, \alpha_k)$ where $1 \leq \alpha_1 < \cdots < \alpha_k \leq m$, and, for convenience, $I(0, m) := \{0\}$. Moreover, let $\overline{\alpha}$ be the element in $I(m - k, m)$ which complements α, and $\sigma(\alpha, \overline{\alpha})$ the sign of the permutation that reorders the multi-index $(\alpha, \overline{\alpha})$ in the natural way.

Let $(e_i)_{i=1}^n$ and $(\varepsilon_j)_{j=1}^N$ be the canonical bases in $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively. The dual bases of covector are denoted by $(\mathrm{d}x^i)_{i=1}^n$ and $(\mathrm{d}y^j)_{j=1}^N$. Also, for $\alpha \in I(k, n)$ and $\beta \in I(h, N)$, the corresponding unit *simple* multi-vectors are $e_\alpha := e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k}$ and $\varepsilon_\beta := \varepsilon_{\beta_1} \wedge \cdots \wedge \varepsilon_{\beta_h}$. Moreover, $\wedge_n \mathbb{R}^{n+N}$ is the space of n-vectors ξ in $\mathbb{R}^{n+N}$, so that every $\xi \in \wedge_n \mathbb{R}^{n+N}$ can be written as

$$\xi = \sum_{|\alpha| + |\beta| = n} \xi^{\alpha\beta} e_\alpha \wedge \varepsilon_\beta \,, \qquad \xi^{\alpha\beta} \in \mathbb{R} \,.$$

If $G : \mathbb{R}^n \to \mathbb{R}^N$ is a linear map, we also denote by G the $N \times n$-matrix in $M(N, n)$ associated to G with respect to the standard bases. For multi-indices $\alpha \in I(k, n - k)$ and $\beta \in I(N, k)$, where $1 \leq k \leq \min\{n, N\}$,

[1]Let $\mathcal{Y}$ be $(\mathfrak{p} - 1)$-connected, so that by the Hurewicz theorem, the $\mathfrak{p}$-th homotopy group $\pi_{\mathfrak{p}}(\mathcal{Y})$ and the $\mathfrak{p}$-th homology group with integer coefficients $H_{\mathfrak{p}}(\mathcal{Y})$ are isomorphic. Moreover, denoting by $H^{sph}_{\mathfrak{p}}(\mathcal{Y})$ the spherical subgroup of $H_{\mathfrak{p}}(\mathcal{Y})$, assume that the quotient $H_{\mathfrak{p}}(\mathcal{Y})/H^{sph}_{\mathfrak{p}}(\mathcal{Y})$ is torsion-free, compare [12, Sec. 5.4.1]. Then, it turns out that the relevant singularities of non-smooth maps can be treated through homological arguments, as in the easier case $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}}$.

we denote by $G^{\beta}_{\overline{\alpha}}$ the minor of order k of G with rows β and columns $\overline{\alpha}$, and by $M^{\beta}_{\overline{\alpha}}(G) := \det G^{\beta}_{\overline{\alpha}}$ its determinant, where by definition we set $M^0_0(G) := 1$.

For $G \in M(N,n)$, the vectors $e_i + Ge_i \in \mathbb{R}^{n+N}$, $i = 1,\dots,n$, yield a basis of the tangent n-plane to the graph of the linear map G in $\mathbb{R}^{n+N}$. The *simple* n-vector

$$M(G) := (e_1 + Ge_1) \wedge \cdots \wedge (e_n + Ge_n) \in \Lambda_n \mathbb{R}^{n+N} \tag{2.1}$$

identifies the graph of G, and the unit n-vector $\xi_G := \frac{M(G)}{|M(G)|}$ in fact orients such an n-plane. Note that the map $G \mapsto M(G)$ from $M(N,n)$ to $\wedge_n \mathbb{R}^{n+N}$ is injective, as

$$M(G) = \sum_{|\alpha|+|\beta|=n} \sigma(\alpha,\overline{\alpha})\, M^{\beta}_{\overline{\alpha}}(G)\, e_\alpha \wedge \varepsilon_\beta\,. \tag{2.2}$$

Let $L : V \to W$ be a linear map between finite dimensional vector spaces V and W. The *induced linear transformation* $\wedge_k L : \wedge_k V \to \wedge_k W$ is defined on simple k-vectors of V by

$$\wedge_k L(v_1 \wedge \cdots \wedge v_k) := Lv_1 \wedge \cdots \wedge Lv_k\,.$$

Denoting by $(\mathrm{Id} \bowtie G) : \mathbb{R}^n \to \mathbb{R}^{n+N}$ the graph map $(\mathrm{Id} \bowtie G)(x) := (x, Gx)$, we have

$$M(G) = \wedge_n (\mathrm{Id} \bowtie G)(e_1 \wedge \cdots \wedge e_n) \qquad \forall\, G \in M(N,n)\,.$$

Moreover, the following Laplace's formulas hold true (cf. [9, Lemma 2.1]):

Lemma 2.1 *Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a non-singular linear map. Then for any $0 \le |\alpha| = |\gamma| \le n$*

$$\sigma(\gamma,\overline{\gamma})\,\sigma(\alpha,\overline{\alpha})\, M^{\overline{\alpha}}_{\overline{\gamma}}(L) = (\det L)\, M^{\gamma}_{\alpha}(L^{-1})\,.$$

For any square matrix $L \in M(n,n)$, let $\mathcal{L}_L : \wedge_n \mathbb{R}^{n+N} \to \wedge_n \mathbb{R}^{n+N}$ be the linear map defined by

$$\mathcal{L}_L(\xi) := \sum_{|\alpha|+|\beta|=n} \sigma(\alpha,\overline{\alpha})\, \xi^{\alpha\beta}_L e_\alpha \wedge \varepsilon_\beta\,, \qquad \xi^{\alpha\beta}_L := \sum_{|\gamma|=|\alpha|} \sigma(\gamma,\overline{\gamma})\, \xi^{\gamma\beta}\, M^{\overline{\gamma}}_{\overline{\alpha}}(L)\,, \tag{2.3}$$

if $\xi = \sum_{|\gamma|+|\beta|=n} \xi^{\gamma\beta} e_\gamma \wedge \varepsilon_\beta \in \Lambda_n \mathbb{R}^{n+N}$. It turns out (cf. [9, Lemma 2.3]) that $\mathcal{L}_L(M(G)) = M(GL)$ for any $G \in M(N,n)$. Moreover, if $\det L \neq 0$, then $\mathcal{L}_L$ is bijective and

$${\mathcal{L}_L}^{-1} = \mathcal{L}_{L^{-1}}\,. \tag{2.4}$$

2.2 Integer rectifiable currents

Let $U \subset \mathbb{R}^m$ be open and k be integer, with $0 \le k \le m$. We denote by $\mathcal{D}_k(U)$ the strong dual of the space $\mathcal{D}^k(U)$ of compactly supported smooth k-forms, whence $\mathcal{D}_0(U)$ is the class of distributions in U. For any k-*current* $T \in \mathcal{D}_k(U)$, we define its *mass* $\mathbf{M}(T)$ as

$$\mathbf{M}(T) := \sup\{T(\omega) \mid \omega \in \mathcal{D}^k(U)\,,\ \|\omega\| \le 1\}\,,$$

where $\|\omega\|$ is the *comass* of ω, see (3.6) below, and its *support* $\operatorname{spt} T$ is defined in a way similar to the case of distributions in $\mathcal{D}_0(U)$. For $k \ge 1$, the *boundary* of T is the $(k-1)$-current ∂T defined by the relation

$$\partial T(\eta) := T(\mathrm{d}\eta)\,, \qquad \forall \eta \in \mathcal{D}^{k-1}(U)\,,$$

where $\mathrm{d}\eta$ is the differential of η. The *weak convergence* $T_h \rightharpoonup T$ in the sense of currents in $\mathcal{D}_k(U)$ is defined through the formula

$$\lim_{h\to\infty} T_h(\omega) = T(\omega)\,, \qquad \forall\, \omega \in \mathcal{D}^k(U)\,.$$

If $T_h \rightharpoonup T$, by lower semicontinuity one has

$$\mathbf{M}(T) \le \liminf_{h\to\infty} \mathbf{M}(T_h)\,.$$

If $T \in \mathcal{D}_k(U)$ has finite mass, there exists a Borel regular and finite measure $\|T\|$ in U, and a $\|T\|$-measurable map $\overrightarrow{T} : U \to \wedge_k \mathbb{R}^m$, with $|\overrightarrow{T}| = 1$ for $\|T\|$-almost every $x \in U$, such that

$$T(\omega) = \int_U \langle \omega, \overrightarrow{T} \rangle \, \mathrm{d}\|T\| \quad \forall \, \omega \in \mathcal{D}^k(U) \,, \tag{2.5}$$

where $\overrightarrow{T}$ is the *Radon-Nikodym derivative* of T with respect to $\|T\|$. Therefore, one has $T = \overrightarrow{T} \llcorner \|T\|$.

Denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure in $\mathbb{R}^m$. For $k \geq 1$, a current $T \in \mathcal{D}_k(U)$ is said to be of the type $(\mathcal{M}, \theta, \overrightarrow{\xi})$, say $T = [\![\mathcal{M}, \theta, \overrightarrow{\xi}]\!]$, if the action of T is given by

$$T(\omega) = \int_{\mathcal{M}} \langle \omega(z), \overrightarrow{\xi}(z) \rangle \, \theta(z) \, \mathrm{d}\mathcal{H}^k(z) \qquad \forall \, \omega \in \mathcal{D}^k(U) \,, \tag{2.6}$$

where $\mathcal{M} \subset U$ is countably $\mathcal{H}^k$-rectifiable, the multiplicity $\theta : \mathcal{M} \to]0, +\infty]$ is $\mathcal{H}^k$-measurable and locally $(\mathcal{H}^k \llcorner \mathcal{M})$-summable, and $\overrightarrow{\xi} : \mathcal{M} \to \wedge_k \mathbb{R}^m$ is $\mathcal{H}^k$-measurable with $|\overrightarrow{\xi}| = 1$ $(\mathcal{H}^k \llcorner \mathcal{M})$-a.e. Furthermore, T is said to be an *integer multiplicity* (i.m) *rectifiable current*, $T \in \mathcal{R}_k(U)$, if in addition T has finite mass, the density θ takes integer values, and for $\mathcal{H}^k$-almost every $z \in \mathcal{M}$ the unit k-vector $\overrightarrow{\xi}(z) \in \Lambda_k \mathbb{R}^m$ provides an orientation to the approximate tangent space to $\mathcal{M}$ at z. In that case, $\mathbf{M}(T) = \int_{\mathcal{M}} \theta \, \mathrm{d}\mathcal{H}^k < \infty$.

If e.g. $\mathcal{M}$ is a smooth, embedded and oriented k-manifold in U, with $\mathcal{H}^k(\mathcal{M}) < \infty$, a current $[\![\mathcal{M}]\!]$ in $\mathcal{R}_k(U)$ is naturally associated to $\mathcal{M}$, its action on k-forms being given in the sense of Differential Geometry:

$$[\![\mathcal{M}]\!](\omega) := \int_{\mathcal{M}} \omega \,, \quad \forall \, \omega \in \mathcal{D}^k(U) \,.$$

Finally, when $k = 0$, a current $T \in \mathcal{R}_0(U)$ is given by a finite sum $T = \sum_{j=1}^m \Delta_j \, \delta_{a_j}$, where $\Delta_j \in \mathbb{Z}$ and δ_{a_j} is the unit *Dirac mass* at a point $a_j \in U$.

2.3 Sobolev maps into the p-sphere

Let $n \geq \mathfrak{p} \geq 2$ be integer. For every $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ and for any Borel set $B \subset B^n$, we denote by

$$\mathbf{D}^{\mathfrak{p}}(u, B) := \int_B \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} |Du|^{\mathfrak{p}} \, dx \,, \qquad \mathbf{D}^{\mathfrak{p}}(u) := \mathbf{D}^{\mathfrak{p}}(u, B^n) \,,$$

the *Euclidean* $\mathfrak{p}$*-energy functional*. It is scale invariant in the critical dimension $n = \mathfrak{p}$, where the so called *bubbling-off* phenomenon occurs, see Example 2.9.

The stereographic projection σ of the unit $\mathfrak{p}$-sphere $\mathbb{S}^{\mathfrak{p}}$ onto $\mathbb{R}^{\mathfrak{p}}$, from the south pole $P_S := (0_{\mathbb{R}^{\mathfrak{p}}}, -1)$, maps $(y, z) \in \mathbb{S}^{\mathfrak{p}} \subset \mathbb{R}^{\mathfrak{p}} \times \mathbb{R}$, with $|y|^2 + z^2 = 1$, to $y/(1+z) \in \mathbb{R}^{\mathfrak{p}}$. Its inverse $\sigma^{-1} : \mathbb{R}^{\mathfrak{p}} \to \mathbb{S}^{\mathfrak{p}}$ is given by

$$\sigma^{-1}(x) := \left(\frac{2}{1+|x|^2} \, x, \, \frac{1-|x|^2}{1+|x|^2} \right), \quad \forall \, x \in \mathbb{R}^{\mathfrak{p}} \,.$$

Since the Jacobian $J_{\sigma^{-1}}$ is equal to $\mathfrak{p}^{-\mathfrak{p}/2} \, |D\sigma^{-1}|^{\mathfrak{p}}$, by the area formula we have

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} \int_{\mathbb{R}^{\mathfrak{p}}} |D\sigma^{-1}|^{\mathfrak{p}} \, dx = \int_{\mathbb{R}^{\mathfrak{p}}} J_{\sigma^{-1}} \, dx = \alpha_{\mathfrak{p}} \,,$$

where here and in the sequel we denote

$$\alpha_{\mathfrak{p}} := \mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}) \,.$$

The map $(-1)^{\mathfrak{p}} \, \sigma^{-1}$ is an orientation preserving conformal diffeomorphism from $\mathbb{R}^{\mathfrak{p}}$ into $\mathbb{S}^{\mathfrak{p}} \setminus \{P_S\}$, where $\mathbb{S}^{\mathfrak{p}}$ is equipped with the natural orientation induced from the outward unit normal; in particular,

$$(-1)^{\mathfrak{p}} \, \sigma^{-1}_{\#} [\![\mathbb{R}^{\mathfrak{p}}]\!] = [\![\mathbb{S}^{\mathfrak{p}}]\!] \,.$$

We modify σ^{-1} as follows. We first write

$$\sigma^{-1}(x) = \Big(\frac{x}{|x|} \, \sin\theta(|x|), -\cos\theta(|x|) \Big) \,, \qquad x \in \mathbb{R}^{\mathfrak{p}} \,,$$

where $\theta(r)$, for $r > 0$, is the angular distance of $\sigma^{-1}(\partial B^{\mathfrak{p}}_r)$ from the south pole P_S. For $\varepsilon > 0$ small, we set

$$\theta_\varepsilon(r) := \begin{cases} \theta(r) & \text{if} \quad r < R_\varepsilon \\ \varepsilon\,(2R_\varepsilon - r)/R_\varepsilon & \text{if} \quad R_\varepsilon \le r \le 2R_\varepsilon \\ 0 & \text{if} \quad r > 2R_\varepsilon\,, \end{cases}$$

where $R_\varepsilon := \theta^{-1}(\varepsilon)$, and define $\varphi_\varepsilon : \mathbb{R}^{\mathfrak{p}} \to \mathbb{S}^{\mathfrak{p}}$ by

$$\varphi_\varepsilon(x) := (-1)^{\mathfrak{p}} \Big(\frac{x}{|x|} \sin\theta_\varepsilon(|x|), -\cos\theta_\varepsilon(|x|) \Big)\,, \qquad x \in \mathbb{R}^{\mathfrak{p}}\,.$$

Clearly, φ_ε is Lipschitz-continuous, with $\varphi_\varepsilon(x) = (-1)^{\mathfrak{p}}\sigma^{-1}(x)$ for $|x| < R_\varepsilon$ and $\varphi_\varepsilon(x) \equiv (-1)^{\mathfrak{p}}\, P_S$ for $|x| > 2R_\varepsilon$. Moreover, see [13, Sec. 4.1.1], it can be shown that its Euclidean $\mathfrak{p}$-energy satisfies

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} \int_{\mathbb{R}^{\mathfrak{p}}} |D\varphi_\varepsilon|^{\mathfrak{p}}\, dx \le \alpha_{\mathfrak{p}} + c\,\varepsilon\,,$$

where $c > 0$ is an absolute constant, and that the image current

$$\varphi_{\varepsilon\#}[\![\, \mathbb{R}^{\mathfrak{p}} \,]\!] = \varphi_{\varepsilon\#}[\![\, B^{\mathfrak{p}}_{2R_\varepsilon} \,]\!] = [\![\, \mathbb{S}^{\mathfrak{p}} \,]\!]\,.$$

Finally, by considering the mapping $\varphi_{\varepsilon,\delta}(x) := \varphi_\varepsilon(R_\varepsilon x/\delta)$, where the positive parameter δ can be chosen independently of ε, one can even shrink the set $\{x \in \mathbb{R}^{\mathfrak{p}} \mid \varphi_\varepsilon(x) \neq (-1)^{\mathfrak{p}}\, P_S\}$ to $\{0_{\mathbb{R}^{\mathfrak{p}}}\}$, without affecting the Euclidean $\mathfrak{p}$-energy, and state the following

Proposition 2.2 *For any $\varepsilon, \delta > 0$, there exists a smooth map $\varphi_{\varepsilon,\delta} : \mathbb{R}^{\mathfrak{p}} \to \mathbb{S}^{\mathfrak{p}}$ such that $\varphi_{\varepsilon,\delta}$ is conformal on $B^{\mathfrak{p}}_{\delta/2}$, $\varphi_{\varepsilon,\delta} \equiv (-1)^{\mathfrak{p}}\, P_S$ outside $B^{\mathfrak{p}}_\delta$, $\varphi_{\varepsilon,\delta\#}[\![\, \mathbb{R}^{\mathfrak{p}} \,]\!] = [\![\, \mathbb{S}^{\mathfrak{p}} \,]\!]$, and*

$$\alpha_{\mathfrak{p}} \le \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} \int_{\mathbb{R}^{\mathfrak{p}}} |D\varphi_{\varepsilon,\delta}|^{\mathfrak{p}}\, dx \le \alpha_{\mathfrak{p}} + c\,\varepsilon\,.$$

In dimension $n = \mathfrak{p}$, Schoen-Uhlenbeck density theorem [25] yields that the class of smooth maps in $W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$ is strongly dense in $W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$. However, strong density of smooth maps is false in higher dimension, a counterexample in case $n = \mathfrak{p}+1$ being given by the vortex map $u_V(x) = x/|x|$. For that reason, Bethuel [2] introduced the relevant class $R^\infty_{\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ of maps $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ that are smooth outside a "smooth" closed singular subset $\Sigma(u)$ of B^n of dimension $(n - \mathfrak{p} - 1)$, e.g., a discrete set for $n = \mathfrak{p} + 1$. In fact, he proved:

Theorem 2.3 *For any $n \ge \mathfrak{p} + 1$, the class $R^\infty_{\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ is strongly dense in $W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$.*

2.4 Singularities and degree

Let $n \ge \mathfrak{p} + 1$, where $\mathfrak{p} \ge 2$. Let $\omega_{\mathbb{S}^{\mathfrak{p}}}$ denote the *volume* $\mathfrak{p}$*-form on* $\mathbb{S}^{\mathfrak{p}}$,

$$\omega_{\mathbb{S}^{\mathfrak{p}}} := \sum_{j=1}^{\mathfrak{p}+1} (-1)^{j-1} y^j\, \widehat{dy^j}\,, \qquad y = (y^1, \ldots, y^{\mathfrak{p}+1})$$

where $\widehat{\mathrm{d}y^j} := \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^{j-1} \wedge \mathrm{d}y^{j+1} \wedge \cdots \wedge \mathrm{d}y^{\mathfrak{p}+1}$, so that $\mathrm{d}\omega_{\mathbb{S}^{\mathfrak{p}}} = (\mathfrak{p}+1) \cdot \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^{\mathfrak{p}+1}$ and

$$[\![\, \mathbb{S}^{\mathfrak{p}} \,]\!](\omega_{\mathbb{S}^{\mathfrak{p}}}) = \int_{\mathbb{S}^{\mathfrak{p}}} \omega_{\mathbb{S}^{\mathfrak{p}}} = \mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}) =: \alpha_{\mathfrak{p}}\,.$$

To every Sobolev map $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$, we associate an $(n - \mathfrak{p} - 1)$-current $\mathbb{P}(u)$ in B^n given by

$$\mathbb{P}(u)(\phi) := \frac{1}{\alpha_{\mathfrak{p}}} \int_{B^n} \mathrm{d}\phi \wedge u^{\#}\omega_{\mathbb{S}^{\mathfrak{p}}}\,, \quad \forall\, \phi \in \mathcal{D}^{n-\mathfrak{p}-1}(B^n)\,. \tag{2.7}$$

The current $\mathbb{P}(u)$ in (2.7) describes the relevant *singularities* of maps $u \in W^{1,\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})$, compare [13]. If e.g. $n = \mathfrak{p}+1$ and u is in $R^\infty_{\mathfrak{p}}(B^{\mathfrak{p}+1},\mathbb{S}^{\mathfrak{p}})$, with singular set $\Sigma(u) = \{a_j \mid j = 1,\dots m\}$, we have

$$\mathbb{P}(u) = -\sum_{j=1}^{m} \Delta_j\, \delta_{a_j}\,, \tag{2.8}$$

where $\Delta_j \in \mathbb{Z}$ is the *degree* of u around the point a_j, see (2.12) below. For e.g. the vortex map $u_V(x) = x/|x|$, we get $\mathbb{P}(u_V) = -\delta_{\mathbf{0}}$. In high dimension $n \geq \mathfrak{p}+2$, for e.g. the map in (1.16) we get to Eq. (1.17).

In [13], the authors also defined the $(n-\mathfrak{p})$-current $\mathbb{D}(u) \in \mathcal{D}_{n-\mathfrak{p}}(B^n)$ given by

$$\mathbb{D}(u)(\gamma) := \frac{1}{\alpha_{\mathfrak{p}}} \int_\Omega \gamma \wedge u^{\#}\omega_{\mathbb{S}^{\mathfrak{p}}}$$

for every $\gamma \in \mathcal{D}^{n-\mathfrak{p}}(B^n)$, so that clearly

$$\mathbb{P}(u) = \partial\, \mathbb{D}(u) \qquad \text{on} \quad \mathcal{D}^{n-\mathfrak{p}-1}(B^n)\,. \tag{2.9}$$

In dimension $n = \mathfrak{p}+1$, Eq. (2.9) can be re-written in terms of the so called *D-field* of Brezis-Coron-Lieb [5, App. B], that is by the vector field $D(u) \in L^1(B^{\mathfrak{p}+1},\mathbb{R}^{\mathfrak{p}+1})$ defined in components by $D(u) = (D^1(u),\dots,D^{\mathfrak{p}+1}(u))$, where

$$D^i(u) := \det\Big(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_{i-1}}, u, \frac{\partial u}{\partial x_{i+1}},\dots,\frac{\partial u}{\partial x_{\mathfrak{p}+1}}\Big)\,. \tag{2.10}$$

Therefore, property (2.9) implies the equivalence:

$$\mathbb{P}(u) = 0 \qquad \Longleftrightarrow \qquad \mathrm{Div}\, D(u) = 0 \quad \text{on} \;\; B^{\mathfrak{p}+1}\,,$$

where Div denotes the distributional divergence.

In higher dimension $n \geq \mathfrak{p}+2$, the $(n-\mathfrak{p})$-vector field $D(u)$ can be defined as the dual to $u^{\#}\omega_{\mathbb{S}^{\mathfrak{p}}}$,

$$\langle \eta, D(u)(x)\rangle\, \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n := \eta \wedge u^{\#}\omega_{\mathbb{S}^{\mathfrak{p}}}(x) \qquad \forall\, \eta \in \wedge^{n-\mathfrak{p}}(\mathbb{R}^n)\,,$$

see [13, Sec. 5.2.1], and hence it may be identified with $*\, u^{\#}\omega_{\mathbb{S}^{\mathfrak{p}}}$, where $*$ is the *Hodge operator*.

In general, for maps $u \in W^{1,\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})$, where $n \geq \mathfrak{p}+1$, we have

$$\mathbb{D}(u)(\gamma) = \frac{1}{\alpha_{\mathfrak{p}}} \int_{B^n} \langle \gamma, D(u)\rangle\, \mathrm{d}x \qquad \forall\, \gamma \in \mathcal{D}^{n-\mathfrak{p}}(B^n)\,. \tag{2.11}$$

If in particular u belongs to $R^\infty_{\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})$, for a.e. $x \in B^n$, the $(n-\mathfrak{p})$-vector $D(u)(x) \in \wedge_{n-\mathfrak{p}}\mathbb{R}^n$ is tangent to the naturally oriented level $(n-\mathfrak{p})$-surfaces $\big\{z \in B^n \mid u(z) = u(x)\big\}$.

In dimension $n = \mathfrak{p}+1$, the *degree of a map* $u \in R^\infty_{\mathfrak{p}}(B^{\mathfrak{p}+1},\mathbb{S}^{\mathfrak{p}})$ *at a singular point* $a_j \in \Sigma(u)$ is given by:

$$\deg_{\mathbb{S}^{\mathfrak{p}}}(u,a_j) := \frac{1}{\alpha_{\mathfrak{p}}} \int_{\partial B^{\mathfrak{p}+1}(a_j,r)} D(u)\cdot \nu_{a_j,r}\, \mathrm{d}\mathcal{H}^{\mathfrak{p}} \in \mathbb{Z}\,, \tag{2.12}$$

where $D(u)$ is the D-field, $\nu_{a_j,r}$ is the outward unit normal to $\partial B^{\mathfrak{p}+1}(a_j,r)$, and the radius $r > 0$ is smaller than the distance of a_j from $\Sigma(v) \setminus \{a_j\}$ and from the boundary of $B^{\mathfrak{p}+1}$. The definition does not depend on the choice of r small. Moreover, if the current of the singularities $\mathbb{P}(u)$ satisfies (2.8), one has

$$\deg_{\mathbb{S}^{\mathfrak{p}}}(v,a_j) = \Delta_j \qquad \forall\, j = 1,\dots,m\,. \tag{2.13}$$

Finally, if u has zero degree at a_j, then the singularity at a_j can be removed by paying an arbitrary small amount of energy. More precisely, Bethuel-Zheng [4] proved the following:

Proposition 2.4 *Let $u \in R^\infty_{\mathfrak{p}}(B^{\mathfrak{p}+1},\mathbb{S}^{\mathfrak{p}})$ with degree Δ_j at a singular point $a_j \in \Sigma(u)$. For every $\varepsilon > 0$, there exists a map $u_\varepsilon \in R^\infty_{\mathfrak{p}}(B^{\mathfrak{p}+1},\mathbb{S}^{\mathfrak{p}})$, smooth in $B^{\mathfrak{p}+1}(a_j,r)$ for some $r = r(\varepsilon) > 0$ small, and equal to u outside $B^{\mathfrak{p}+1}(a_j,r)$, such that*

$$\mathbf{D}^{\mathfrak{p}}(u_\varepsilon, B^{\mathfrak{p}+1}) \leq \mathbf{D}^{\mathfrak{p}}(u, B^{\mathfrak{p}+1}) + |\Delta_j|\, \alpha_{\mathfrak{p}} + \varepsilon\,, \qquad \alpha_{\mathfrak{p}} := \mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})\,.$$

2.5 Minimal connection of the singularities

Let $n \geq \mathfrak{p}+1$. Given a current $\mathbb{P} \in \mathcal{D}_{n-\mathfrak{p}-1}(B^n)$, we denote by

$$m_{i,B^n}(\mathbb{P}) := \inf\big\{\mathbf{M}(L) \mid L \in \mathcal{R}_{n-\mathfrak{p}}(B^n)\,, \quad (\partial L) \llcorner B^n = \mathbb{P}\big\} \tag{2.14}$$

the *integral mass* of $\mathbb{P}$ *relative to* B^n. The current $\mathbb{P}$ is said to be an *integral flat chain* if there exists an i.m. rectifiable current $L \in \mathcal{R}_{n-\mathfrak{p}}(B^n)$ such that $(\partial L) \llcorner B^n = \mathbb{P}$ or, equivalently, if $m_{i,B^n}(\mathbb{P}) < \infty$. In that case, moreover, Federer-Fleming's closure theorem [7] yields that the minimum is always attained. Therefore, an i.m. rectifiable current $L \in \mathcal{R}_{n-\mathfrak{p}}(B^n)$ is called an *integral minimal connection* of $\mathbb{P}$ *allowing connections to the boundary* of B^n if $(\partial L) \llcorner B^n = \mathbb{P}$ and $\mathbf{M}(L) = m_{i,B^n}(\mathbb{P})$, see [13, Sec. 4.2.6].

This is the case of the current $\mathbb{P} = \mathbb{P}(u)$ of the singularities of a Sobolev map $u \in R^\infty_{\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$, see (2.7). More precisely, for future purpose we report the proof of the following estimate, that goes back to Almgren-Browder-Lieb [1]:

Theorem 2.5 *For every $u \in R^\infty_{\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$, we have:*

$$\alpha_{\mathfrak{p}} \cdot m_{i,B^n}(\mathbb{P}(u)) \leq \mathbf{D}^{\mathfrak{p}}(u, B^n)\,, \quad \alpha_{\mathfrak{p}} = \mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})\,.$$

Proof: By the parallelogram inequality and the *coarea formula* (cf. the proof of Theorem 4.4 below), one gets the energy lower bound

$$\mathbf{D}^{\mathfrak{p}}(u, B^n) \geq \int_{B^n} J_u \,\mathrm{d}x = \int_{\mathbb{S}^{\mathfrak{p}}} \mathcal{H}^{n-\mathfrak{p}}(u^{-1}(y))\,\mathrm{d}\mathcal{H}^{\mathfrak{p}}(y)\,.$$

Since $u \in R^\infty_{\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$, for $\mathcal{H}^{\mathfrak{p}}$-almost every $y \in \mathbb{S}^{\mathfrak{p}}$, the current

$$L_y := [\![\, u^{-1}(y), 1, \overrightarrow{L}_y \,]\!]\,, \qquad \overrightarrow{L}_y(x) := \frac{D(u(x))}{|D(u(x))|}\,, \quad x \in u^{-1}(y)\,,$$

acting on forms $\gamma \in \mathcal{D}^{n-\mathfrak{p}}(B^n)$ as $L_y(\gamma) = \int_{u^{-1}(y)} \langle \gamma, \overrightarrow{L}_y \rangle \,\mathrm{d}\mathcal{H}^{n-\mathfrak{p}}$, see (2.6), is i.m. rectifiable in $\mathcal{R}_{n-\mathfrak{p}}(B^n)$, with finite mass

$$\mathbf{M}(L_y) = \mathcal{H}^{n-\mathfrak{p}}(u^{-1}(y)) < \infty\,,$$

whereas by (2.9) and (2.11), it bounds the singularities of u, i.e., $(\partial L_y) \llcorner B^n = \mathbb{P}(u)$. Since one can choose $y_0 \in \mathbb{S}^{\mathfrak{p}}$ as above in such a way that

$$\alpha_{\mathfrak{p}} \cdot \mathbf{M}(L_{y_0}) \leq \int_{\mathbb{S}^{\mathfrak{p}}} \mathbf{M}(L_y)\,\mathrm{d}\mathcal{H}^{\mathfrak{p}}(y) \leq \mathbf{D}^{\mathfrak{p}}(u, B^n)\,,$$

the assertion follows from Def. (2.14). □

2.6 Cartesian currents

If $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ is smooth, the graph current G_u is defined by integrating n-forms over the naturally oriented graph manifold $\mathcal{G}_u$ of u, i.e., $G_u = [\![\, \mathcal{G}_u \,]\!]$. By Federer's support theorem, G_u is an i.m. rectifiable current in $\mathcal{R}_n(B^n \times \mathbb{S}^{\mathfrak{p}})$, and by a change of variables, we have:

$$G_u(\omega) = \int_{B^n} (\mathrm{Id} \bowtie u)^{\#}\omega \qquad \forall\, \omega \in \mathcal{D}^n(B^n \times \mathbb{S}^{\mathfrak{p}})\,, \tag{2.15}$$

where $(\mathrm{Id} \bowtie u)(x) := (x, u(x))$ is the graph map. Note that an unit n-vector orienting $\mathcal{G}_u$ at $(x, u(x))$ is given by $\frac{M(Du(x))}{|M(Du(x))|}$, where $M(G)$ is defined by (2.1).

More generally, to every Sobolev map u in $W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ we associate an i.m. rectifiable current $G_u \in \mathcal{R}_n(B^n \times \mathbb{S}^{\mathfrak{p}})$ by means of Def. (2.15), where this time the pull-back involves the distributional gradient of u. More precisely, G_u acts on forms in $\mathcal{D}^n(B^n \times \mathbb{S}^{\mathfrak{p}})$ by integration on the *rectifiable graph* $\mathcal{G}_v$ of v, and its mass agrees with the *area* of $\mathcal{G}_u$, i.e.,

$$\mathbf{M}(G_u) = \mathcal{H}^n(\mathcal{G}_u) \leq c\,\mathbf{D}^{\mathfrak{p}}(u, B^n) < \infty\,,$$

where $c = c(n, \mathfrak{p})$ is a positive constant not depending on u. The bound in terms of the Euclidean $\mathfrak{p}$-energy follows from the parallelogram and Hölder's inequalities, since (in case $n \geq \mathfrak{p}+1$) the minors of order $\mathfrak{p}+1$ of the gradient matrix $Du(x)$ have zero determinant for a.e. $x \in B^n$, by the area formula. As a consequence, we always have $G_u \in \mathcal{R}_n(B^n \times \mathbb{S}^{\mathfrak{p}})$, even if in general the boundary current ∂G_u is non-trivial.

When $n \geq \mathfrak{p}+1$, in fact, by (2.7) and (2.15) we find that for any $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$

$$\alpha_{\mathfrak{p}} \cdot \langle \mathbb{P}(u), \phi \rangle = G_u(\mathrm{d}\phi \wedge \omega_{\mathbb{S}^{\mathfrak{p}}}) = \partial G_u(\phi \wedge \omega_{\mathbb{S}^{\mathfrak{p}}}), \quad \forall \phi \in \mathcal{D}^{n-\mathfrak{p}-1}(B^n), \tag{2.16}$$

as $G_u(\phi \wedge \mathrm{d}\omega_{\mathbb{S}^{\mathfrak{p}}}) = 0$. More precisely, the proof of Proposition 2.8 below (cf. [23, Prop. 6.5]) gives that

$$\partial G_u = \mathbb{P}(u) \times [\![\mathbb{S}^{\mathfrak{p}}]\!] \qquad \text{on} \quad \mathcal{D}^{n-1}(B^n \times \mathbb{S}^{\mathfrak{p}}).$$

If e.g. $n = \mathfrak{p}+1$ and $u_V(x) = x/|x|$, then

$$(\partial G_{u_V}) \llcorner B^{\mathfrak{p}+1} \times \mathbb{S}^{\mathfrak{p}} = -\delta_{\mathbf{0}} \times [\![\mathbb{S}^{\mathfrak{p}}]\!].$$

In low dimension $n = \mathfrak{p}$, by a density argument we always have $(\partial G_u) \llcorner B^{\mathfrak{p}} \times \mathbb{S}^{\mathfrak{p}} = 0$.

The following result motivates Def. 2.7 below, that agrees with the one in [13], when $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}}$.

Theorem 2.6 (Giaquinta-Modica-Souček) *Let $\{u_k\}$ be a sequence of smooth maps in $W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ such that $\sup_k \mathbf{D}^{\mathfrak{p}}(u_k, B^n) < \infty$. Then, possibly passing to a subsequence, the graph currents G_{u_k} weakly converge in $\mathcal{D}_n$ to some current $T \in \mathcal{D}_n(B^n \times \mathbb{S}^{\mathfrak{p}})$ satisfying the following properties:*

i) *T is i.m. rectifiable in $\mathcal{R}_n(B^n \times \mathbb{S}^{\mathfrak{p}})$;*

ii) *there exist a function $u_T \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ and an i.m. rectifiable current $L_T \in \mathcal{R}_{n-\mathfrak{p}}(B^n)$ such that*

$$T = G_{u_T} + L_T \times [\![\mathbb{S}^{\mathfrak{p}}]\!]; \tag{2.17}$$

iii) *T has finite mass, $\mathbf{M}(T) = \mathbf{M}(G_{v_T}) + \alpha_{\mathfrak{p}} \cdot \mathbf{M}(L_T) < \infty$;*

iv) *T has no interior boundary, i.e.,*

$$\partial T(\eta) := T(\mathrm{d}\eta) = 0 \quad \forall \eta \in \mathcal{D}^{n-1}(B^n \times \mathbb{S}^{\mathfrak{p}}); \tag{2.18}$$

v) *$\{u_k\}$ converges to u_T weakly in $W^{1,\mathfrak{p}}(B^n, \mathbb{R}^{\mathfrak{p}+1})$.*

PROOF: It is based essentially on Federer-Fleming's closure theorem [7]. Compare [13], Sec. 5.2.3 for $\mathfrak{p} = 2$, and Note 6 in Ch. 5 for $\mathfrak{p} \geq 3$. □

Definition 2.7 *We denote by* $\mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$ *the class of n-currents in $B^n \times \mathbb{S}^{\mathfrak{p}}$ satisfying properties* 1.–4. *in Theorem* 2.6.

Therefore, G_u belongs to $\mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$ for every smooth map $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ or, more generally, for every Sobolev map $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ satisfying the null-boundary condition

$$\partial G_u(\eta) := G_u(\mathrm{d}\eta) = 0 \quad \forall \eta \in \mathcal{D}^{n-1}(B^n \times \mathbb{S}^{\mathfrak{p}}), \tag{2.19}$$

a condition automatically satisfied in low dimension $n = \mathfrak{p}$.

More generally (cf. [23, Prop. 6.5] for a proof), in higher dimension we obtain:

Proposition 2.8 *Let $n \geq \mathfrak{p}+1$ and $T \in \mathcal{R}_n(B^n \times \mathbb{S}^{\mathfrak{p}})$ satisfy* (2.17), *where $u_T \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ and $L_T \in \mathcal{R}_{n-\mathfrak{p}}(B^n)$. Then the null-boundary condition* (2.18) *is equivalent to equation*

$$(\partial L_T) \llcorner B^n = -\,\mathbb{P}(u_T), \tag{2.20}$$

where $\mathbb{P}(u_T)$ is given by (2.7).

Example 2.9 In low dimension $n = \mathfrak{p}$, the currents carried by the graphs of the functions $\varphi_{\varepsilon,\delta}$ in Proposition 2.2, where $\delta = \delta_k \searrow 0$, weakly converge to the Cartesian current

$$T = G_P + \delta_{\mathbf{0}} \times [\![\mathbb{S}^{\mathfrak{p}}]\!] \in \operatorname{cart}^{\mathfrak{p},1}(B^{\mathfrak{p}} \times \mathbb{S}^{\mathfrak{p}})$$

where G_P is the graph of the constant map equal to the South Pole P.

In dimension $n = \mathfrak{p} + 1$, a Cartesian current is given e.g. by

$$T = G_{u_V} + L \times [\![\mathbb{S}^{\mathfrak{p}}]\!] \in \operatorname{cart}^{\mathfrak{p},1}(B^{\mathfrak{p}+1} \times \mathbb{S}^{\mathfrak{p}}),$$

where $u_V(x) = x/|x|$ is the vortex map, and L is any current in $\mathcal{R}_1(B^{\mathfrak{p}+1})$ satisfying $(\partial L) \llcorner B^{\mathfrak{p}+1} = \delta_{\mathbf{0}}$.

2.7 A lower semicontinuous functional

In [10] and [11], the authors analysed the *parametric polyconvex l.s.c. extension* of the Euclidean $\mathfrak{p}$-energy integrand

$$e^{\mathfrak{p}}(G) := \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} |G|^{\mathfrak{p}}, \quad G \in M(N,n). \tag{2.21}$$

It is defined for every n-vector $\xi \in \wedge_n \mathbb{R}^{n+N}$ by

$$\begin{aligned} F^{\mathfrak{p}}(\xi) := \sup\{\phi(\xi) \quad | \quad & \phi : \wedge_n \mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+ , \ \phi \text{ linear}, \\ & \phi(M(G)) \le e^{\mathfrak{p}}(G) \quad \forall G \in M(N,n)\}, \end{aligned} \tag{2.22}$$

where $M(G)$ is the n-vector in $\wedge_n \mathbb{R}^{n+N}$ orienting the graph of G, see (2.1).

When dealing with mappings constrained to take values into a smooth manifold $\mathcal{Y}$ isometrically embedded in $\mathbb{R}^N$, the energy density is given by the integrand $\widehat{e}^{\mathfrak{p}} : \mathbb{R}^N \times M(N,n) \to \overline{\mathbb{R}}_+$

$$\widehat{e}^{\mathfrak{p}}(u,G) := \begin{cases} e^{\mathfrak{p}}(G) & \text{if} \quad u \in \mathcal{Y} \quad \text{and} \quad G \in S_u \\ +\infty & \text{otherwise}, \end{cases}$$

where

$$S_u := \{G \in M(N,n) \mid G \in T_u\mathcal{Y}\}, \qquad u \in \mathcal{Y}, \tag{2.23}$$

$T_u\mathcal{Y}$ being the tangent space to $\mathcal{Y}$ at u.

We thus denote by $\widehat{F}^{\mathfrak{p}}(u,\xi) : \mathbb{R}^N \times \wedge_n \mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+$ the parametric polyconvex l.s.c. extension of the integrand $\widehat{e}^{\mathfrak{p}}(u,G)$. Now, the n-vector $M(G)$ corresponding to matrices $G \in S_u$ belongs to the subspace $\wedge_n(\mathbb{R}^n \times T_u\mathcal{Y})$. This implies the following property, compare [13, Sec. 1.2.4] or [17, Sec. 4.8]:

$$\widehat{F}^{\mathfrak{p}}(u,\xi) = \begin{cases} F^{\mathfrak{p}}(\xi) & \text{if} \quad u \in \mathcal{Y}, \ \xi \in \Lambda_n(\mathbb{R}^n \times T_u\mathcal{Y}) \\ +\infty & \text{otherwise}. \end{cases} \tag{2.24}$$

A lower semicontinuous energy can be defined on currents T in $\mathcal{D}_n(B^n \times \mathcal{Y})$ with finite mass, so that the decomposition $T = \overrightarrow{T} \llcorner \|T\|$ yielding to the explicit formula (2.5) holds.

Definition 2.10 *If T is a current in $\mathcal{D}_n(B^n \times \mathcal{Y})$ with finite mass, and $\widehat{F}^{\mathfrak{p}}(u,\xi)$ is given by* (2.24)*, we let*

$$\mathbf{D}^{\mathfrak{p}}(T) := \int_{B^n \times \mathcal{Y}} \widehat{F}^{\mathfrak{p}}(u, \overrightarrow{T}) \, \mathrm{d}\|T\|.$$

By the construction, it turns out that the following *lower semicontinuity* property holds: if $\{T_k\} \subset \mathcal{D}_n(B^n \times \mathcal{Y})$ is a sequence with equibounded masses, $\sup_k \mathbf{M}(T_k) < \infty$, and $T_k \rightharpoonup T$ weakly in $\mathcal{D}_n$ to some $T \in \mathcal{D}_n(B^n \times \mathcal{Y})$, then T has finite mass, and

$$\mathbf{D}^{\mathfrak{p}}(T) \le \liminf_{k\to\infty} \mathbf{D}^{\mathfrak{p}}(T_k).$$

Assume now $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}}$. In that case, the space $T_u\mathbb{S}^{\mathfrak{p}}$ is given by the vectors in $\mathbb{R}^{\mathfrak{p}+1}$ which are orthogonal to u. Moreover, for any $u \in W^{1,p}(B^n, \mathbb{S}^{\mathfrak{p}})$, one clearly has

$$\mathbf{D}^{\mathfrak{p}}(G_u) = \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} \int_{B^n} |Du|^{\mathfrak{p}} \, \mathrm{d}x.$$

With no more information, there is no way to find an explicit formula for the energy $\mathbf{D}^{\mathfrak{p}}(T)$. In case of Cartesian currents, however, it suffices to write more explicitly the action of $F^{\mathfrak{p}}(u,\xi)$ on simple vectors.

More precisely, *if* $\xi = \tau \wedge \eta \in \wedge_{n-\mathfrak{p}}\mathbb{R}^n \otimes \wedge_{\mathfrak{p}}\mathbb{R}^{\mathfrak{p}+1}$ *is simple,* then

$$F^{\mathfrak{p}}(\tau \wedge \eta) = |\tau| \cdot |\eta| \,,$$

compare [13, Sec. 5.4.4], or [17, Sec. 4.8]. As a consequence, if $L \in \mathcal{R}_{n-\mathfrak{p}}(B^n)$ one has:

$$\mathbf{D}^{\mathfrak{p}}(L \times [\![\mathbb{S}^{\mathfrak{p}}]\!]) = \alpha_{\mathfrak{p}} \cdot \mathbf{M}(L) \,, \quad \alpha_{\mathfrak{p}} = \mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}) \,.$$

Definitely, one obtains the following:

Theorem 2.11 *We have:*

(a) the functional $T \mapsto \mathbf{D}^{\mathfrak{p}}(T)$ *is lower semicontinuous in* $\mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$ *with respect to the sequential weak convergence in* $\mathcal{D}_n$*;*

(b) if $T \in \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$ *satisfies* (2.17)*, then* $\mathbf{D}^{\mathfrak{p}}(T) = \mathbf{D}^{\mathfrak{p}}(u_T, B^n) + \alpha_{\mathfrak{p}} \cdot \mathbf{M}(L_T)$*;*

(c) the class $\mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$ *is closed under the weak* $\mathcal{D}_n$*-convergence of sequences* $\{T_k\}$ *of currents with equibounded* $\mathfrak{p}$*-energies,* $\sup_k \mathbf{D}^{\mathfrak{p}}(T_k) < \infty$*;*

(d) if $\{T_k\} \subset \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$ *satisfies* $\sup_k \mathbf{D}^{\mathfrak{p}}(T_k) < \infty$*, possibly passing to a subsequence* $\{T_k\}$ *weakly converges to some current* T *in* $\mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$*;*

(e) for every $T \in \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$*, there exists a sequence of* smooth *maps* $\{u_k\} \subset W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ *such that* $G_{u_k} \rightharpoonup T$ *in* $\mathcal{D}_n$ *and* $\mathbf{D}^{\mathfrak{p}}(u_k, B^n) \to \mathbf{D}^{\mathfrak{p}}(T)$ *as* $k \to \infty$*.*

PROOF: As to properties (a) and (b), see [13, Sec. 1.2.4] and also [17, Sec. 4.9]. Properties (c) and (d) follow by arguing as in [8], where they were proved for the case $\mathfrak{p} = 2$ in any dimension n. The density property (e) is obtained by using the same argument as for the case $\mathfrak{p} = 2$ in [15], see also [17, Ch. 5], on account of Proposition 2.2. It suffices to argue as in the proof of Theorems 4.1 and 4.2 below, where one takes $g_{\alpha\beta}(x) = \delta_{\alpha\beta}$ for all $x \in B^n$. □

2.8 Relaxed energy

According to (1.12), for any $u \in L^{\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ we let:

$$\widetilde{\mathbf{D}}^{\mathfrak{p}}(u, B^n) := \inf\Big\{ \liminf_{k\to\infty} \mathbf{D}^{\mathfrak{p}}(u_k, B^n) \quad | \quad \{u_k\} \subset C^\infty(B^n, \mathbb{S}^{\mathfrak{p}}) \,, \; u_k \to u \text{ strongly in } L^{\mathfrak{p}}(B^n, \mathbb{R}^{\mathfrak{p}+1}) \Big\} \,,$$

so that $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ if $\widetilde{\mathbf{D}}^{\mathfrak{p}}(u, B^n) < \infty$, and in dimension $n = \mathfrak{p}$ one has:

$$\widetilde{\mathbf{D}}^{\mathfrak{p}}(u, B^{\mathfrak{p}}) = \mathbf{D}^{\mathfrak{p}}(u, B^{\mathfrak{p}}) \qquad \forall\, u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}}) \,.$$

In high dimension $n \geq \mathfrak{p} + 1$, for any $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ we denote by $\mathcal{T}_u^{\mathfrak{p},1}$ the class of Cartesian currents with corresponding function u_T equal to u, i.e.,

$$\mathcal{T}_u^{\mathfrak{p},1} := \big\{ T \in \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}}) \mid u_T = u \ \text{ in } (2.17) \big\} \,.$$

By Proposition 2.8, for any $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ we have:

$$\mathcal{T}_u^{\mathfrak{p},1} = \big\{ G_u + L \times [\![\mathbb{S}^{\mathfrak{p}}]\!] \mid L \in \mathcal{R}_{n-\mathfrak{p}}(B^n) \,,\ (\partial L) \llcorner B^n = -\,\mathbb{P}(u) \big\} \,, \tag{2.25}$$

where $\mathbb{P}(u) \in \mathcal{D}_{n-\mathfrak{p}-1}(B^n)$ is given by (2.7). Therefore, Theorem 2.5 yields that the class $\mathcal{T}_u^{\mathfrak{p},1}$ is non-empty whenever $u \in R_{\mathfrak{p}}^\infty(B^n, \mathbb{S}^{\mathfrak{p}})$.

Theorem 2.12 shows that the energy gap is finite for any $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$, and it is given (up to the factor $\alpha_{\mathfrak{p}}$) by the integral mass of the current of the singularities. For future use, its proof is reported below.

Theorem 2.12 *Let $n \geq \mathfrak{p}+1 \geq 3$ be integer. For every $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^\mathfrak{p})$ the relaxed energy $\widetilde{\mathbf{D}}^\mathfrak{p}(u, B^n)$ is finite. Moreover, the class $\mathcal{T}_u^{\mathfrak{p},1}$ is non-empty, and we have:*

$$\begin{aligned} \widetilde{\mathbf{D}}^\mathfrak{p}(u, B^n) &= \inf\{\mathbf{D}^\mathfrak{p}(T) \mid T \in \mathcal{T}_u^{\mathfrak{p},1}\} \\ &= \mathbf{D}^\mathfrak{p}(v, B^n) + \alpha_\mathfrak{p} \cdot m_{i,B^n}(\mathbb{P}(u)) < \infty \,. \end{aligned} \tag{2.26}$$

Proof: Let $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^\mathfrak{p})$. By Theorem 2.3, there exists a sequence $\{u_h\} \subset R_\mathfrak{p}^\infty(B^n, \mathbb{S}^\mathfrak{p})$ strongly converging to u in $W^{1,\mathfrak{p}}(B^n, \mathbb{R}^{\mathfrak{p}+1})$. Moreover, by Theorem 2.5 and formula (2.25), for each h there exists a current $T_h \in \mathcal{T}_{u_h}^{\mathfrak{p},1}$ such that

$$\mathbf{D}^\mathfrak{p}(T_h) = \mathbf{D}^\mathfrak{p}(u_h, B^n) + \alpha_\mathfrak{p} \cdot m_{i,B^n}(\mathbb{P}(u_h)) \leq 2\,\mathbf{D}^\mathfrak{p}(u_h, B^n)\,.$$

Therefore, since $\sup_h \mathbf{D}^\mathfrak{p}(T_h) < \infty$, by property (d) in Theorem 2.11, possibly passing to a (not relabeled) subsequence, we find a current $\bar{T} \in \mathrm{cart}^{1,\mathfrak{p}}(B^n \times \mathbb{S}^\mathfrak{p})$ such that $T_h \rightharpoonup \bar{T}$ weakly in $\mathcal{D}_n$. Moreover, by the strong convergence $u_h \to u$, we infer that $u_{\bar{T}} = u$ in (2.17), whence $\bar{T} \in \mathcal{T}_u^{\mathfrak{p},1}$. As a consequence, by applying the strong density property (e) in Theorem 2.11 to the current $\bar{T}$, we infer that $\widetilde{\mathbf{D}}^\mathfrak{p}(u, B^n) < \infty$.

Now, for any sequence $\{u_k\} \subset C^\infty(B^n, \mathbb{S}^\mathfrak{p})$ converging to u in $L^\mathfrak{p}$ and satisfying $\sup_k \mathbf{D}^\mathfrak{p}(u_k, B^n) < \infty$, recalling that $\mathbf{D}^\mathfrak{p}(G_{u_k}) = \mathbf{D}^\mathfrak{p}(u_k, B^n)$, we can again extract a (not relabeled) subsequence such that the graph currents G_{u_k} weakly converge in $\mathcal{D}_n$ to a current $T \in \mathcal{T}_u^{\mathfrak{p},1}$. Therefore, by the lower semicontinuity of the $\mathfrak{p}$-energy functional on currents, we infer that

$$\mathbf{D}^\mathfrak{p}(T) \leq \liminf_{k\to\infty} \mathbf{D}^\mathfrak{p}(u_k, B^n)\,,$$

whence the lower bound "$\geq$" holds true in the first line of formula (2.26). The upper bound "$\leq$" follows by applying the strong density property (e) in Theorem 2.11 to any current $T \in \mathcal{T}_u^{\mathfrak{p},1}$. Finally, the second equality follows from Eq. (2.25) and Def. (2.14). □

In dimension $n = \mathfrak{p}+1$, we recover the expression of the energy gap due to Brezis-Coron-Lieb [5], who defined the *flat norm* of a function $u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^\mathfrak{p})$ through the formula:

$$\mathbf{L}(u, B^{\mathfrak{p}+1}) := \frac{1}{\alpha_\mathfrak{p}} \sup\Big\{\int_{B^{\mathfrak{p}+1}} D(u)\cdot D\phi \,\mathrm{d}x \mid \phi \in C_c^\infty(B^{\mathfrak{p}+1})\,,\ \|D\phi\|_{\infty,B^{\mathfrak{p}+1}} \leq 1\Big\}\,, \tag{2.27}$$

where $D(u) \in L^1(B^{\mathfrak{p}+1}, \mathbb{R}^{\mathfrak{p}+1})$ is the D-field of u, see (2.10). In fact, by a duality argument, see [13, Sec. 4.2.6], the flat norm of u agrees with the *real mass* of $\mathbb{P}(u)$, that is defined by

$$m_{r,B^{\mathfrak{p}+1}}(\mathbb{P}(u)) := \inf\{\mathbf{M}(D) \mid D \in \mathcal{D}_1(B^n)\,, \quad (\partial D) \llcorner B^{\mathfrak{p}+1} = \mathbb{P}(u)\}\,.$$

Since moreover the integral mass of $\mathbb{P}(u)$ is finite, a theorem by Federer [6] implies that it agrees with the real mass of $\mathbb{P}(u)$. Therefore, we obtain

$$\alpha_\mathfrak{p} \cdot m_{i,B^{\mathfrak{p}+1}}(\mathbb{P}(u)) = \alpha_\mathfrak{p} \cdot \mathbf{L}(u, B^{\mathfrak{p}+1}) < \infty\,, \quad \forall\, u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^\mathfrak{p})\,. \tag{2.28}$$

Corollary 2.13 *For every $u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^\mathfrak{p})$, one has*

$$\widetilde{\mathbf{D}}^\mathfrak{p}(u, B^{\mathfrak{p}+1}) \leq 2\,\mathbf{D}^\mathfrak{p}(u, B^{\mathfrak{p}+1})\,.$$

Proof: As a consequence of the cited Federer's theorem [6], arguing as e.g. in [13, Sec. 4.2.6, Prop. 4], by Theorem 2.3 we find a sequence $\{u_k\} \subset R_\mathfrak{p}^\infty(B^{\mathfrak{p}+1}, \mathbb{S}^\mathfrak{p})$ strongly converging to u in $W^{1,\mathfrak{p}}$, and such that

$$\mathbb{P}(u) - \mathbb{P}(u_k) = (\partial L_k) \llcorner B^{\mathfrak{p}+1} \quad \forall\, k\,, \tag{2.29}$$

where $\{L_k\} \subset \mathcal{R}_1(B^{\mathfrak{p}+1})$ satisfies $\mathbf{M}(L_k) \to 0$ as $k \to \infty$. By applying Theorem 2.5 to each u_k, and letting $\mathbf{D}^\mathfrak{p}(v) := \mathbf{D}^\mathfrak{p}(v, B^{\mathfrak{p}+1})$, we get

$$m_{i,\mathbb{P}(u)}(B^{\mathfrak{p}+1}) \leq m_{i,\mathbb{P}(u_k)}(B^{\mathfrak{p}+1}) + \mathbf{M}(L_k) \leq \frac{1}{\alpha_\mathfrak{p}}\,\mathbf{D}^\mathfrak{p}(u_k) + \varepsilon_k \leq \frac{1}{\alpha_\mathfrak{p}}\,\mathbf{D}^\mathfrak{p}(u) + 2\,\varepsilon_k$$

where $\varepsilon_k \searrow 0$ as $k \to \infty$. Therefore, the assertion follows from Theorem 2.12. □

Remark 2.14 Due to the lack of validity of Federer's theorem [6], we do not know whether Corollary 2.13 extends to high dimensions $n \geq \mathfrak{p}+2$.

3 The $\mathfrak{p}$-energy on Cartesian currents

In this section, we analyze the parametric polyconvex l.s.c. envelop of the integrand $e_g^{\mathfrak{p}}(x,G)$. We then write an explicit formula for the corresponding energy functional $T \mapsto \mathbf{D}_g^{\mathfrak{p}}(T)$ on currents in $\text{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$. The results here obtained extend the case $\mathfrak{p}=2$ analyzed in [9].

3.1 The parametric polyconvex l.s.c. envelop

Define $F_g^{\mathfrak{p}} : B^n \times \wedge_n \mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+$ by

$$\mathcal{F}_g^{\mathfrak{p}}(x,\xi) := \sup\big\{\phi(\xi) \quad | \quad \begin{array}{l}\phi : \wedge_n\mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+ \,,\ \phi \text{ linear,} \\ \phi(M(G)) \le e_g^{\mathfrak{p}}(x,G) \quad \forall\, G \in M(N,n)\big\}\,,\end{array} \tag{3.1}$$

where $e_g^{\mathfrak{p}} : B^n \times M(N,n) \to \mathbb{R}$ is the $\mathfrak{p}$-energy density (1.6).

In the Euclidean case $g_{\alpha\beta}(x) \equiv \delta_{\alpha\beta}$, we recover the notation (2.22) for $F^{\mathfrak{p}}$. More generally, if $g(x)$ is constant, then $F_g^{\mathfrak{p}}$ does not depend on x. However, since $e_g^{\mathfrak{p}}$ is continuous, then $F_g^{\mathfrak{p}}(x,\xi)$ *is l.s.c. in all variables and convex in* ξ *for any* x.

The following explicit formula extends the case $\mathfrak{p}=2$ proved in [9]:

Proposition 3.1 *Let* $\mathfrak{p} \ge 2$ *be integer. For every* $x \in B^n$*, we have*

$$F_g^{\mathfrak{p}}(x,\xi) = F^{\mathfrak{p}}(\mathcal{L}_L(\xi)) \qquad \forall\, \xi \in \wedge_n \mathbb{R}^{n+N}\,,$$

where $L = L(x)$ *is the unique symmetric positive definite square matrix in* $M(n,n)$ *satisfying*

$$L(x)L(x)^\top = \det g(x)^{1/\mathfrak{p}} g(x)^{-1}\,, \tag{3.2}$$

and $\mathcal{L}_L$ *is given by Def.* (2.3).

PROOF: If $A = A(x) \in M(n,n)$ is the positive definite symmetric matrix given by (1.9), we actually have $LL^\top = A$, i.e., $L := \sqrt{A}$ in (3.2). Therefore, by (1.8) and (1.10) we infer that

$$e_g^{\mathfrak{p}}(x,G) = \Big(\frac{1}{\mathfrak{p}}\operatorname{tr}((GL)(GL)^\top)\Big)^{\mathfrak{p}/2} = \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}}\,|GL|^{\mathfrak{p}} \qquad \forall\, G \in M(N,n)\,. \tag{3.3}$$

Because of (3.1), this yields that for every $x \in B^n$ and $\xi \in \wedge_n \mathbb{R}^{n+N}$

$$F_g^{\mathfrak{p}}(x,\xi) = \sup\Big\{\phi(\xi) \mid \phi \text{ linear, } \phi(M(G)) \le \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}}\,|GL|^{\mathfrak{p}} \ \ \forall\, G \in M(N,n)\Big\}\,.$$

Since the matrix $L = L(x)$ in (3.2) is invertible, recalling the notation (2.21), by (2.4) we get

$$\begin{aligned} F_g^{\mathfrak{p}}(x,\xi) &= \sup\big\{\phi(\xi) \mid \phi \text{ linear, } \phi(M(GL^{-1})) \le e^{\mathfrak{p}}(G) \ \forall\, G \in M(N,n)\big\} \\ &= \sup\big\{\phi(\xi) \mid \phi \text{ linear, } \phi \circ \mathcal{L}_{L^{-1}}(M(G)) \le e^{\mathfrak{p}}(G) \ \forall\, G \in M(N,n)\big\} \\ &= \sup\big\{\phi \circ \mathcal{L}_{L^{-1}}(\mathcal{L}_L\xi) \mid \phi \text{ linear, } \phi \circ \mathcal{L}_{L^{-1}}(M(G)) \le e^{\mathfrak{p}}(G) \ \forall\, G \in M(N,n)\big\} \\ &= \sup\big\{\widetilde{\phi}(\mathcal{L}_L\xi) \mid \widetilde{\phi} \text{ linear, } \widetilde{\phi}(M(G)) \le e^{\mathfrak{p}}(G) \ \forall\, G \in M(N,n)\big\} =: F^{\mathfrak{p}}(\mathcal{L}_L(\xi))\,, \end{aligned}$$

on account of (2.22), as required. □

Similarly as before, dealing with manifold constrained mappings, we introduce the integrand

$$\widehat{e}_g^{\mathfrak{p}} : B^n \times \mathbb{R}^N \times M(N,n) \to \overline{\mathbb{R}}_+$$

defined by

$$\widehat{e}_g^{\mathfrak{p}}(x,u,G) := \begin{cases} e_g^{\mathfrak{p}}(x,G) & \text{if } \ u \in \mathcal{Y} \ \text{ and } \ G \in S_u \\ +\infty & \text{otherwise}\,, \end{cases}$$

where $e_g^{\mathfrak{p}}(x,G)$ is given by (1.6), and S_u by (2.23). Therefore, denoting by

$$\widehat{F}_g^{\mathfrak{p}}(x,u,\xi) : B^n \times \mathbb{R}^N \times \wedge_n \mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+$$

the parametric polyconvex l.s.c. envelop of the integrand $\widehat{e}_g^{\mathfrak{p}}(x,u,G)$, we readily obtain:

Proposition 3.2 *For every $x \in B^n$ we have:*

$$\widehat{F}^{\mathfrak{p}}_g(x,u,\xi) = \begin{cases} F_g(x,\xi) & \text{if} \quad u \in \mathcal{Y},\ \xi \in \Lambda_n(\mathbb{R}^n \times T_u\mathcal{Y}) \\ +\infty & \text{otherwise}, \end{cases} \tag{3.4}$$

where $F^{\mathfrak{p}}_g(x,\xi)$ is given by (3.1)*, and $T_u\mathcal{Y}$ is the tangent space to $\mathcal{Y}$ at u.*

As in the Euclidean case, when $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}}$, we finally give the following

Definition 3.3 *The $\mathfrak{p}$-energy integral* (1.1) *is extended to currents T in $\mathcal{D}_n(B^n \times \mathbb{S}^{\mathfrak{p}})$ with finite mass, by letting*

$$\mathbf{D}^{\mathfrak{p}}_g(T) := \int_{B^n \times \mathbb{S}^{\mathfrak{p}}} \widehat{F}^{\mathfrak{p}}_g(x,u,\overrightarrow{T})\, d\|T\|\,,$$

where $\widehat{F}^{\mathfrak{p}}_g(x,u,\xi)$ is given by (3.4)*. For any Borel set $B \subset B^n$, we also define*

$$\mathbf{D}^{\mathfrak{p}}_g(T, B \times \mathbb{S}^{\mathfrak{p}}) := \mathbf{D}^{\mathfrak{p}}_g(T \llcorner (B \times \mathbb{S}^{\mathfrak{p}}))\,.$$

3.2 The p-energy on Cartesian currents

In order to obtain a nice formula for the energy $\mathbf{D}^{\mathfrak{p}}_g(T)$ on Cartesian currents, we have to write more explicitly the polyconvex extension of the energy density $e^{\mathfrak{p}}_g$ on simple n-vectors ξ in $\Lambda_{n-\mathfrak{p}}\mathbb{R}^n \otimes \Lambda_{\mathfrak{p}}\mathbb{R}^{\mathfrak{p}+1}$.

In Theorem 3.5, where we take $N = \mathfrak{p}+1$, we show that for every $x \in B^n$, it agrees with the length of ξ in the metric induced by the product of the metric $g(x)$ on B^n and of the Euclidean metric on $\mathbb{R}^{\mathfrak{p}+1}$.

To this purpose, we recall that a metric on $\mathbb{R}^n \simeq T_xB^n$ induces a metric on the whole exterior algebra. In particular, we have

$$\langle \tau, \eta \rangle_{g(x)} = \langle \Lambda_k(g(x))\tau, \eta \rangle \qquad \forall\, \tau,\, \eta \in \Lambda_k\mathbb{R}^n\,,$$

for every $x \in B^n$, so that

$$|\tau|_{g(x)} = |\Lambda_k(g(x)^{1/2})(\tau)| \qquad \forall\, \tau \in \Lambda_k\mathbb{R}^n\,, \tag{3.5}$$

where $g(x)^{1/2} := \sqrt{g(x)}$ is the unique symmetric positive definite square matrix $\widetilde{g}$ such that $\widetilde{g}^2 = g(x)$.

Proposition 3.4 *Let $n, N \geq \mathfrak{p} \geq 2$ be integer. If $\xi = \tau \wedge \eta \in \Lambda_{n-\mathfrak{p}}\mathbb{R}^n \otimes \Lambda_{\mathfrak{p}}\mathbb{R}^N$ and $L = L(x) \in M(n,n)$ is the non-singular matrix given by* (3.2) *for some $x \in B^n$, then*

$$\mathcal{L}_L(\tau \wedge \nu) = (\det L)(\Lambda_{n-\mathfrak{p}}L^{-1}(\tau) \wedge \eta) = \Lambda_{n-\mathfrak{p}}(g^{1/2})(\tau) \wedge \eta\,,$$

where $g = g(x)$ and $\mathcal{L}_L$ is given by Def. (2.3)*.*

Proof: For any $\alpha \in I(n-\mathfrak{p}, n)$ and $\beta \in I(\mathfrak{p}, N)$, by Def. (2.3) we have

$$\mathcal{L}_L(e_\alpha \wedge \epsilon_\beta) = \sum_{\gamma \in I(n-\mathfrak{p},n)} \sigma(\gamma,\overline{\gamma})\, \sigma(\alpha,\overline{\alpha})\, M^{\overline{\alpha}}_{\overline{\gamma}}(L)\, e_\gamma \wedge \varepsilon_\beta\,,$$

whereas

$$\wedge_{n-\mathfrak{p}}L^{-1}(e_\alpha) = \sum_{\gamma \in I(n-\mathfrak{p},n)} M^{\gamma}_{\alpha}(L^{-1})\, e_\gamma\,.$$

By Lemma 2.1 we thus obtain

$$\mathcal{L}_L(e_\alpha \wedge \varepsilon_\beta) = (\det L) \sum_{\gamma \in I(n-\mathfrak{p},n)} M^{\gamma}_{\alpha}(L^{-1})\, e_\gamma \wedge \varepsilon_\beta = (\det L)\, (\Lambda_{n-\mathfrak{p}}L^{-1}(e_\alpha) \wedge \varepsilon_\beta)\,.$$

The first equality follows by using an argument by linearity on the two factors $\Lambda_{n-\mathfrak{p}}\mathbb{R}^n$ and $\Lambda_{\mathfrak{p}}\mathbb{R}^N$. Moreover, by (3.2) we have

$$\det L = (\det g)^{(n-\mathfrak{p})/2\mathfrak{p}}\,, \quad L^{-1} = (\det g)^{-1/2\mathfrak{p}}\, g^{1/2}.$$

This yields

$$(\det L)\, \Lambda_{n-\mathfrak{p}}L^{-1} = \Lambda_{n-\mathfrak{p}}(g^{1/2})$$

and hence the second equality. □

As a consequence of Propositions 3.1 and 3.4, on account of (3.5) we immediately obtain:

Theorem 3.5 *Let $\xi = \tau \wedge \eta \in \wedge_{n-\mathfrak{p}}\mathbb{R}^n \otimes \wedge_{\mathfrak{p}}\mathbb{R}^N$ be a simple n-vector, and let $F_g^{\mathfrak{p}}$ be given by (3.1). For every $x \in B^n$ we have*

$$F_g^{\mathfrak{p}}(x, \tau \wedge \eta) = F^{\mathfrak{p}}(\Lambda_{n-\mathfrak{p}}(g(x)^{1/2})(\tau) \wedge \eta) = |\Lambda_{n-\mathfrak{p}}(g(x)^{1/2})(\tau)| \cdot |\eta| = |\tau|_{g(x)} \cdot |\eta| \,.$$

We now recall that the *g-comass* $\|\omega\|_g$ of a k-form $\omega \in \mathcal{D}^k(B^n)$ is defined by

$$\|\omega(x)\|_{g(x)} := \sup\{\langle \omega(x), \xi\rangle \mid \xi \in \Lambda_k(B^n) \text{ simple}, \ |\xi|_{g(x)} \le 1\}\,, \qquad x \in B^n\,, \tag{3.6}$$

and the g-mass of a current $\Gamma \in \mathcal{D}_k(B^n)$ by

$$\mathbf{M}_g(\Gamma) := \sup\{\Gamma(\omega) \mid \omega \in \mathcal{D}^k(B^n)\,,\ \|\omega(x)\|_{g(x)} \le 1 \ \forall\, x \in B^n\}\,. \tag{3.7}$$

If $g(x) \equiv \delta_{\alpha\beta}$, they agree with the standard comass and mass, respectively. Moreover, if Γ is an i.m. rectifiable current in $\mathcal{R}_k(B^n)$, writing as above $\Gamma = [\![\, \mathcal{G}, \theta, \xi \,]\!]$, where $|\xi| \equiv 1$ in the Euclidean metric, we have

$$\begin{aligned}\mathbf{M}_g(\Gamma) &= \sup\Big\{\int_{\mathcal{G}} \theta(x)\, \langle \omega(x), \xi(x)\rangle \, \mathrm{d}\mathcal{H}^k(x) \mid \omega \in \mathcal{D}^k(B^n)\,,\ \|\omega(x)\|_{g(x)} \le 1 \ \forall\, x \in B^n\Big\} \\ &= \int_{\mathcal{G}} \theta(x)\, |\xi(x)|_{g(x)} \, \mathrm{d}\mathcal{H}^k(x)\,.\end{aligned}$$

Assume now that $T \in \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$, so that the decomposition (2.17) holds. Write $L_T = [\![\, \mathcal{L}, \theta, \tau \,]\!]$, where $\mathcal{L}$ is $(n-\mathfrak{p})$-rectifiable in B^n, $\theta(x)$ is an integer-valued multiplicity function on $\mathcal{L}$, and $\tau(x)$ is a simple $(n-\mathfrak{p})$-vector in $\wedge_{n-\mathfrak{p}}\mathbb{R}^n$ orienting $\mathcal{L}$ at x, with $|\tau(x)| = 1$. In this case, for every Borel set $B \subset B^n$ we have

$$\mathbf{M}_g(L_T \llcorner B) = \int_{\mathcal{L}\cap B} \theta(x)\, |\tau(x)|_{g(x)} \, d\mathcal{H}^{n-\mathfrak{p}}(x) < \infty\,. \tag{3.8}$$

Arguing as for the Euclidean $\mathfrak{p}$-energy integral $\mathbf{D}^{\mathfrak{p}}(T)$, we then compute explicitly:

Proposition 3.6 *For every Borel set $B \subset B^n$ we have*

$$\mathbf{D}_g^{\mathfrak{p}}(T, B \times \mathbb{S}^{\mathfrak{p}}) = \mathbf{D}_g(u_T, B) + \alpha_{\mathfrak{p}} \cdot \mathbf{M}_g(L_T \llcorner B)\,. \tag{3.9}$$

Proof: Recalling that $L_T = [\![\, \mathcal{L}, \theta, \tau \,]\!]$, if $\eta \in \wedge_{\mathfrak{p}}\mathbb{R}^{\mathfrak{p}+1}$ yields an orientation to the tangent space to $\mathbb{S}^{\mathfrak{p}}$ at $u \in \mathbb{S}^{\mathfrak{p}}$, and $|\eta| = 1$, the simple n-vector $\tau \wedge \eta$ yields an orientation to $L_T \times \mathbb{S}^{\mathfrak{p}}$ at (x, u), for $\mathcal{H}^{n-\mathfrak{p}}$-almost every $x \in \mathcal{L}$. By Proposition 3.2 and Theorem 3.5, where $N = \mathfrak{p}+1$, we have

$$\widehat{F}_g^{\mathfrak{p}}(x, u, \tau \wedge \eta) = |\tau|_{g(x)} \cdot |\eta| = |\tau|_{g(x)}\,.$$

Due to Def. 3.3, using the same argument as for the Euclidean $\mathfrak{p}$-integral, compare [13, Sec. 5.4.4] or [17, Sec. 4.9], we obtain

$$\mathbf{D}_g^{\mathfrak{p}}(T, B \times \mathbb{S}^{\mathfrak{p}}) = \int_B e_g^{\mathfrak{p}}(x, Du_T)\, \mathrm{d}x + \alpha_{\mathfrak{p}} \cdot \int_{\mathcal{L}\cap B} \theta(x)\, |\tau(x)|_{g(x)} \, \mathrm{d}\mathcal{H}^{n-\mathfrak{p}}(x)\,.$$

The assertion follows from (3.8). $\square$

By the bound (1.2), for each integer $k \in \{0, \ldots, n\}$ there exists a constant $c = c(k) > 0$ such that

$$c\, \mathbf{M}_g(\Gamma) \le \mathbf{M}(\Gamma) \le \frac{1}{c}\, \mathbf{M}_g(\Gamma)\,, \quad \forall\, \Gamma \in \mathcal{R}_k(B^n)\,. \tag{3.10}$$

In particular, we infer the existence of a real constant $C > 0$, only depending on n and $\mathfrak{p}$, such that

$$C\, \mathbf{D}^{\mathfrak{p}}(T) \le \mathbf{D}_g^{\mathfrak{p}}(T) \le \frac{1}{C}\, \mathbf{D}^{\mathfrak{p}}(T)\,, \qquad \forall\, T \in \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})\,.$$

As a consequence, on account of Theorem 2.11, we readily check the validity of the following properties:

i) $\mathbf{D}_g^{\mathfrak{p}}(T) < \infty$ for every $T \in \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$;

ii) the $\mathfrak{p}$-energy functional $T \mapsto \mathbf{D}_g^{\mathfrak{p}}(T)$ is lower semicontinuous in $\mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$ with respect to the sequential weak $\mathcal{D}_n$-convergence;

iii) the class $\mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$ is closed in the weak $\mathcal{D}_n$-convergence along sequences with equibounded $\mathbf{D}_g^{\mathfrak{p}}$-energies;

iv) sequences in $\mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$ with equibounded $\mathbf{D}_g^{\mathfrak{p}}$-energies are relatively compact in the $\mathcal{D}_n$-topology.

3.3 The case of constant metrics

For future use, we now discuss the case when the metric g is constant, $g_{\alpha\beta}(x) \equiv g_{\alpha\beta}$ for any $x \in B^n$, so that $e_g^{\mathfrak{p}}(x,G) \equiv e_g^{\mathfrak{p}}(G)$. Equivalently, assume that $A(x) \equiv A$ is a constant positive definite symmetric matrix in $M(n,n)$. If $g \equiv \delta_{\alpha\beta}$, i.e., if A is the identity matrix, then $e_g^{\mathfrak{p}}(G)$ is given by (2.21), and in general, the following link with the Euclidean $\mathfrak{p}$-energy holds true.

Let L be the unique symmetric positive definite matrix in $M(n,n)$ satisfying $LL^\top = (\det g)^{1/\mathfrak{p}} g^{-1}$, see (3.2). For T in $\text{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$, we denote by $T_L := (L^{-1} \bowtie \text{Id}_{\mathbb{R}^{\mathfrak{p}+1}})_\# T$ the current given by the push forward of T through the linear map $(L^{-1} \bowtie \text{Id}_{\mathbb{R}^{\mathfrak{p}+1}})(x,y) := (L^{-1}x, y)$, so that

$$T_L(\widetilde{\omega}) := T((L^{-1} \bowtie \text{Id}_{\mathbb{R}^{\mathfrak{p}+1}})^\# \widetilde{\omega}), \qquad \widetilde{\omega} \in \mathcal{D}^n(L^{-1}(B^n) \times \mathbb{S}^{\mathfrak{p}}), \tag{3.11}$$

and $T_L \in \text{cart}^{\mathfrak{p},1}(L^{-1}(B^n) \times \mathbb{S}^{\mathfrak{p}})$. Note that if $T = G_{u_T}$ for some Sobolev map $u_T \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$, then

$$(L^{-1} \bowtie \text{Id}_{\mathbb{R}^{\mathfrak{p}+1}})_\# G_{u_T} = G_{v_T},$$

where $v_T : L^{-1}(B^n) \to \mathbb{S}^{\mathfrak{p}}$ is given by $v_T(\widetilde{x}) := u_T(L\widetilde{x})$. This yields that the function v_T corresponding to T_L agrees with $u_T \circ L$. Arguing as in the proof of [9, Prop. 3.11], we obtain:

Proposition 3.7 *Assume that the metric g is constant on B^n. Let T be a current in $\text{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$, so that (2.17) holds. For every Borel set $B \subset B^n$, we have*

$$\mathbf{D}_g^{\mathfrak{p}}(T, B \times \mathbb{S}^{\mathfrak{p}}) = (\det L) \cdot \mathbf{D}^{\mathfrak{p}}(T_L, L^{-1}(B) \times \mathbb{S}^{\mathfrak{p}}),$$

where L is given by (3.2), with $g(x) \equiv g$. In particular, if $T = G_u$ for some $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$, then

$$\int_{B^n} e_g^{\mathfrak{p}}(Du(x))\, dx = (\det L) \cdot \int_{L^{-1}(B^n)} e^{\mathfrak{p}}(Dv(\widetilde{x}))\, d\widetilde{x}, \qquad v(\widetilde{x}) := u(L\widetilde{x}).$$

4 Density results, relaxed energy, coarea formula

In this section, we discuss the *relaxed* $\mathfrak{p}$*-energy* $\widetilde{\mathbf{D}}_g^{\mathfrak{p}}(u, B^n)$ of maps $u \in L^{\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$, that is defined by (1.12). We also prove a coarea formula, Theorem 4.4, and its consequences.

4.1 A strong density result

The explicit formula of the energy gap (1.13) relies on the following:

Theorem 4.1 *Let $n \geq \mathfrak{p} \geq 2$ be integer, and let $g(x)$ be a metric on B^n satisfying the bound (1.2). Then, for any $T \in \text{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$, there exists a sequence of smooth maps $\{u_k\} \subset C^\infty(B^n, \mathbb{S}^{\mathfrak{p}})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^{\mathfrak{p}})$ and $\mathbf{D}_g^{\mathfrak{p}}(u_k, B^n) \to \mathbf{D}_g^{\mathfrak{p}}(T)$, as $k \to \infty$.*

If the metric g is constant on B^n, we immediately deduce Theorem 4.1. In fact, setting T_L by (3.11), Theorem 2.11 yields the existence of a sequence $\{v_k\} \subset C^1(L^{-1}(B^n), \mathbb{S}^{\mathfrak{p}})$ such that $G_{v_k} \rightharpoonup T_L$ weakly in $\mathcal{D}_n$ and $\mathbf{D}^{\mathfrak{p}}(v_k, L^{-1}(B^n)) \to \mathbf{D}^{\mathfrak{p}}(T_L, L^{-1}(B^n) \times \mathbb{S}^{\mathfrak{p}})$ as $k \to \infty$. It then suffices to apply Proposition 3.7, by taking $u_k := v_k \circ L^{-1}$.

In general, we first observe that since the metric tensor function $x \mapsto g(x)$ is continuous in B^n, whereas

$$G \mapsto \frac{e_g^{\mathfrak{p}}(x,G) - e_g^{\mathfrak{p}}(x_0,G)}{|G|^{\mathfrak{p}}}$$

is positively homogeneous of degree zero, it turns out that there exists a continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\omega(t) \to 0$ if $t \to 0^+$, such that for every $x, x_0 \in B^n$ and every $G \in M(\mathfrak{p}+1, n)$,

$$|e_g^{\mathfrak{p}}(x,G) - e_g^{\mathfrak{p}}(x_0,G)| \leq \omega(|x - x_0|) \cdot |G|^{\mathfrak{p}}. \tag{4.1}$$

In low dimension $n = \mathfrak{p}$, the proof of Theorem 4.1 is an easy adaptation of the one from [14], by using the continuity estimate (4.1) and Proposition 3.7, so we omit to write it.

In case $n \geq \mathfrak{p}+1$, for every current $T \in \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$, on account of (2.17), we denote by μ_T^g the finite Radon measure on B^n given for every Borel set $B \subset B^n$ by

$$\mu_T^g(B) := \alpha_{\mathfrak{p}} \cdot \mathbf{M}_g(L_T \llcorner B)\,, \quad \alpha_{\mathfrak{p}} := \mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})\,, \tag{4.2}$$

see (3.8), so that we have

$$\mathbf{D}_g^{\mathfrak{p}}(T, B \times \mathbb{S}^{\mathfrak{p}}) = \mathbf{D}_g^{\mathfrak{p}}(u_T, B) + \mu_T^g(B)\,.$$

We also denote by $\mathbf{F}(T)$ the *flat norm*

$$\mathbf{F}(T) := \sup\bigl\{T(\omega) \mid \omega \in \mathcal{D}^n(B^n \times \mathbb{S}^{\mathfrak{p}})\,,\ \mathbf{F}(\omega) \leq 1\bigr\}\,,$$

where for every $\omega \in \mathcal{D}^n(B^n \times \mathbb{S}^{\mathfrak{p}})$

$$\mathbf{F}(\omega) := \max\Bigl\{\sup_{z \in B^n \times \mathbb{S}^{\mathfrak{p}}} \|\omega(z)\|\,,\ \sup_{z \in B^n \times \mathbb{S}^{\mathfrak{p}}} \|\mathrm{d}\omega(z)\|\Bigr\}\,.$$

As $|T(\omega)| \leq \mathbf{F}(T)\,\mathbf{F}(\omega)$, we infer that $T_k \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^{\mathfrak{p}})$ provided that $\mathbf{F}(T_k - T) \to 0$.

Theorem 4.1 is a consequence of the following *approximation theorem*, the proof of which is postponed to Sec. 5.

Theorem 4.2 *Let* $T \in \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$, $\varepsilon \in (0, 1/2)$, *and* $k \in \mathbb{N}$. *We can find a current* $\widetilde{T} \in \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$ *such that*

$$\mathbf{D}_g^{\mathfrak{p}}(\widetilde{T}) \leq \mathbf{D}_g^{\mathfrak{p}}(T) + \varepsilon^k\,,$$
$$\mathbf{F}(\widetilde{T} - T) \leq \varepsilon^k \qquad \textit{and} \qquad \mu_{\widetilde{T}}^g(B^n) \leq \frac{1}{2} \cdot \mu_T^g(B^n)\,.$$

Proof: [Proof of Theorem 4.1] By Theorem 4.2, using a diagonal argument, we find a sequence $\{T_k\} \subset \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$ that weakly converges to T in $\mathcal{D}_n$, with $\mathbf{D}_g^{\mathfrak{p}}(T_k) \to \mathbf{D}_g^{\mathfrak{p}}(T)$ as $k \to \infty$, and such that $\mu_{T_k}^g(B^n) = 0$ for each k. Therefore, T_k agrees with the graph current G_{u_k}, for some $u_k \in W^{\mathfrak{p},1}(B^n, \mathbb{S}^{\mathfrak{p}})$, and hence $\mathbf{D}_g^{\mathfrak{p}}(T_k) = \mathbf{D}_g^{\mathfrak{p}}(u_k)$. Moreover, by means of Bethuel's density theorem [2], for every k we find a smooth sequence $\{u_h^{(k)}\}_h \subset C^1(B^n, \mathbb{S}^{\mathfrak{p}})$ that strongly converges to u_k in the $W^{1,\mathfrak{p}}$-sense, as $h \to \infty$. In fact, the null-boundary condition (2.19), where $u = u_k$, and the bound (1.2) for the energy, allow us to remove the $(n-\mathfrak{p})$-dimensional singularities, compare e.g. [17, Sec. 5.3]. Lower dimensional singularities are removed as in [2]. By the dominated convergence theorem, we infer that the strong convergence yields $G_{u_h^{(k)}} \rightharpoonup G_{u_k}$ with $\mathbf{D}_g^{\mathfrak{p}}(u_h^{(k)}) \to \mathbf{D}_g^{\mathfrak{p}}(u_k)$. Theorem 4.1 then follows by means of a diagonal argument. □

4.2 The relaxed energy

In dimension $n = \mathfrak{p}$, there is no energy gap in $W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$, see (1.14), so that we now assume $n \geq \mathfrak{p}+1$.

Due to the bound (1.2) on the metric g, a map $u \in L^{\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ has finite relaxed $\mathfrak{p}$-energy if and only if $\widetilde{\mathbf{D}}^{\mathfrak{p}}(u, B^n) < \infty$. Therefore, Theorem 2.12 yields that (1.15) holds true. We now come to the explicit formula (1.18) for the energy gap (1.13).

On account of Def. (3.7) of g-mass, and similarly to (2.14), we denote by

$$\begin{array}{rcl} m_{r,B^n}^g(\mathbb{P}) &:=& \inf\bigl\{\mathbf{M}_g(D) \mid D \in \mathcal{D}_{n-\mathfrak{p}}(B^n)\,,\quad (\partial D) \llcorner B^n = \mathbb{P}\bigr\} \\ m_{i,B^n}^g(\mathbb{P}) &:=& \inf\bigl\{\mathbf{M}_g(L) \mid L \in \mathcal{R}_{n-\mathfrak{p}}(B^n)\,,\quad (\partial L) \llcorner B^n = \mathbb{P}\bigr\} \end{array} \tag{4.3}$$

the *real* and *integral g-mass* of a current $\mathbb{P}$ in $\mathcal{D}_{n-\mathfrak{p}-1}(B^n)$ *relative to* B^n, respectively.

By the bound (1.2), it turns out that $\mathbb{P}$ is an integral flat chain if and only if $m_{i,B^n}^g(\mathbb{P}) < \infty$. In this case, again by Federer-Fleming's closure theorem [7] the minimum is always attained, and an i.m. rectifiable current $L \in \mathcal{R}_{n-\mathfrak{p}}(B^n)$ is called an *integral minimal connection for the g-mass of* $\mathbb{P}$ *allowing connections to the boundary of* B^n if $(\partial L) \llcorner B^n = \mathbb{P}$ and $\mathbf{M}_g(L) = m_{i,B^n}^g(\mathbb{P})$.

Proof: [Proof of formula (1.18)] Let $u \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$, where $n \geq \mathfrak{p}+1$. By Theorem 2.5, we already know that the class $\mathcal{T}_u^{\mathfrak{p},1}$ is non-empty, and it is given by formula (2.25), where $\mathbb{P}(u) \in \mathcal{D}_{n-\mathfrak{p}-1}(B^n)$ is defined by (2.7). Since moreover $\widetilde{\mathbf{D}}^{\mathfrak{p}}(u,B^n)<\infty$, arguing as in the proof of Theorem 2.12, by the sequential lower semicontinuity of the $\mathfrak{p}$-energy functional on currents we readily obtain the lower bound:

$$\widetilde{\mathbf{D}}^{\mathfrak{p}}_g(u,B^n)\geq \inf\bigl\{\mathbf{D}^{\mathfrak{p}}_g(T)\mid T\in\mathcal{T}^{\mathfrak{p},1}_u\bigr\}\,,\quad \forall\, u\in W^{1,\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})\,.$$

Moreover, as in the Euclidean case, by Theorem 4.1 we infer the validity of the opposite inequality in the latter centered formula, so that definitely:

$$\widetilde{\mathbf{D}}^{\mathfrak{p}}_g(u,B^n)= \inf\bigl\{\mathbf{D}^{\mathfrak{p}}_g(T)\mid T\in\mathcal{T}^{\mathfrak{p},1}_u\bigr\}\,,\quad \forall\, u\in W^{1,\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})\,. \tag{4.4}$$

In conclusion, recalling by Proposition 3.6 the expression of the $\mathfrak{p}$-energy of a current in $\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathbb{S}^{\mathfrak{p}})$ satisfying (2.17), Eq. (4.4) gives the explicit formula (1.18), as required. □

Remark 4.3 In dimension $n=\mathfrak{p}+1$, as in (2.27), the flat g-norm of a function $u\in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1},\mathbb{S}^{\mathfrak{p}})$ is defined by (1.20). Again by a duality argument, it turns out that the flat g-norm of u agrees with the real g-mass of $\mathbb{P}(u)$, that in turn (again by Federer's theorem [6]) agrees with the integral g-mass of $\mathbb{P}(u)$. Therefore, we readily obtain the equivalent formula (1.19) for the $\mathfrak{p}$-energy gap.

4.3 A coarea formula

In this section, we extend the coarea formula from Theorem 2.5 to the $\mathfrak{p}$-energy of "smooth" maps with values into $\mathbb{S}^{\mathfrak{p}}$. As a consequence, in dimension $n=\mathfrak{p}+1$ we obtain an upper bound for the relaxed $\mathfrak{p}$-energy, Corollary 4.5.

Theorem 4.4 *Let $n\geq\mathfrak{p}+1\geq 3$, and let $g_{\alpha\beta}(x)$ be a smooth metric tensor on B^n. Then, for every $u\in R^\infty_{\mathfrak{p}}(B^n,\mathbb{S}^{\mathfrak{p}})$, we have:*

$$\alpha_{\mathfrak{p}}\cdot m^g_{i,B^n}(\mathbb{P}(u))\leq \mathbf{D}^{\mathfrak{p}}_g(u,B^n)\,,\quad \alpha_{\mathfrak{p}}=\mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})\,,$$

where $m^g_{i,B^n}(\mathbb{P}(u))$ is the integral g-mass of $\mathbb{P}(u)$ relative to B^n, see (4.3).

PROOF: For any $x\in B^n$, let $L=L(x)$ be the unique symmetric positive definite square matrix in $M(n,n)$ satisfying (3.2), so that $LL^\top=A$, with $A=A(x)\in M(n,n)$ given by (1.9). Furthermore, by (1.9) we can write $L=\lambda\,\widehat{L}$, where $\widehat{L}\widehat{L}^\top=g^{-1}(x)$ and $\lambda:=(\det g(x))^{1/2\mathfrak{p}}$, where, we recall, $g^{-1}(x)=(g^{\alpha\beta}(x))$. Let $G\in M(\mathfrak{p}+1,n)$. On account of (3.3), we infer that

$$e^{\mathfrak{p}}_g(x,G)=\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}}\,|GL|^{\mathfrak{p}}=\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}}\,\lambda^{\mathfrak{p}}\,|G\hat{L}|^{\mathfrak{p}}\,,$$

where by the parallelogram inequality

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}}\,|G\hat{L}|^{\mathfrak{p}}\geq|\det((G\hat{L})(G\hat{L})^\top)|^{1/2},$$

and hence:

$$e^{\mathfrak{p}}_g(x,G)\geq(\det g(x))^{1/2}\,|\det\bigl(G\,g^{-1}(x)\,G^\top|^{1/2}.$$

We now wish to apply the *coarea formula* for maps $u:\mathcal{X}\to\mathbb{S}^{\mathfrak{p}}$, where $\mathcal{X}=(B^n,g)$, see e.g. [24]. To this purpose, we observe that the Jacobian $J^g_u(x)$ of the map u with respect to the metric induced by g on B^n (and by the Euclidean metric of $\mathbb{R}^{\mathfrak{p}+1}$ on $\mathbb{S}^{\mathfrak{p}}$) satisfies

$$J^g_u(x)=|\det\bigl(\nabla u(x)g^{-1}(x)\nabla u(x)^\top|^{1/2}$$

for a.e. $x\in B^n$. Furthermore, denoting by $\mathcal{H}^k_g$ the k-dimensional Hausdorff measure on B^n with respect to the metric induced by g, we have $\mathrm{d}\mathcal{H}^n_g=\sqrt{\det g}\,\mathrm{d}x$. Therefore, we have obtained the inequality

$$\begin{aligned}\mathbf{D}^{\mathfrak{p}}_g(u,B^n)=&\int_{B^n}e^{\mathfrak{p}}_g(x,Du(x))\,\mathrm{d}x\\ \geq&\int_{B^n}|\det\bigl(Du(x)\,g^{-1}(x)\,Du(x)^\top|^{1/2}\sqrt{\det g(x)}\,\mathrm{d}x=\int_{B^n}J^g_u(x)\,\mathrm{d}\mathcal{H}^n_g(x)\,,\end{aligned}$$

whereas (by the smoothness of u) the coarea formula [24] gives

$$\int_{B^n} J_u^g(x)\, \mathrm{d}\mathcal{H}_g^n(x) = \int_{\mathbb{S}^\mathfrak{p}} \mathcal{H}_g^{n-\mathfrak{p}}(u^{-1}(y))\, \mathrm{d}\mathcal{H}^\mathfrak{p}(y)\,.$$

Furthermore, by the definition of g-mass, and by the properties of the measure $\mathcal{H}_g^{n-\mathfrak{p}} \llcorner u^{-1}(y)$, for $\mathcal{H}^\mathfrak{p}$-almost every $y \in \mathbb{S}^\mathfrak{p}$, the current L_y defined in the proof of Theorem 2.5 satisfies

$$\mathbf{M}_g(L_y) = \mathcal{H}_g^{n-\mathfrak{p}}(u^{-1}(y)) < \infty\,.$$

We thus can choose $y_0 \in \mathbb{S}^\mathfrak{p}$ in such a way that

$$\alpha_\mathfrak{p} \cdot \mathbf{M}_g(L_{y_0}) \le \int_{\mathbb{S}^\mathfrak{p}} \mathbf{M}_g(L_y)\, \mathrm{d}\mathcal{H}^\mathfrak{p}(y) = \int_{\mathbb{S}^\mathfrak{p}} \mathcal{H}_g^{n-\mathfrak{p}}(u^{-1}(y))\, \mathrm{d}\mathcal{H}^\mathfrak{p}(y)\,,$$

and definitely

$$\alpha_\mathfrak{p} \cdot \mathbf{M}_g(L_{y_0}) \le \mathbf{D}_g^\mathfrak{p}(u, B^n)\,.$$

Since by (2.9) and (2.11) we have $(\partial L_{y_0}) \llcorner B^n = \mathbb{P}(u)$, the assertion follows from Def. (4.3). □

Corollary 4.5 *For every $u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}+1}, \mathbb{S}^\mathfrak{p})$, one has*

$$\widetilde{\mathbf{D}}_g^\mathfrak{p}(u, B^{\mathfrak{p}+1}) \le 2\, \mathbf{D}_g^\mathfrak{p}(u, B^{\mathfrak{p}+1})\,.$$

PROOF: We follow the line of the proof of Corollary 2.13. In fact, by dominated convergence and by the bound (1.2), Theorem 2.3 gives a sequence $\{u_k\} \subset R_\mathfrak{p}^\infty(B^{\mathfrak{p}+1}, \mathbb{S}^\mathfrak{p})$ such that in addition $\mathbf{D}_g^\mathfrak{p}(u_k, B^n) \le \mathbf{D}_g^\mathfrak{p}(u, B^n) + \varepsilon_k$, whereas by (3.10), the currents in (2.29) satisfy $\mathbf{M}_g(L_k) \to$ as $k \to \infty$. Therefore, the assertion follows from the explicit formula (1.18) and Theorem 4.4. □

4.4 A case with a non-integer exponent

If $p > 2$ is a non integer exponent, the p-energy of maps $u : \mathcal{X} \to \mathcal{Y} \subset \mathbb{R}^N$, where $\mathcal{X} = (B^n, g)$, is given by

$$\mathbf{D}_g^p(u, B^n) := \int_{B^n} \Big(\frac{1}{p} \sum_{\alpha,\beta=1}^n \sum_{i,j=1}^N g^{\alpha\beta}(x) \delta_{ij} \frac{\partial u^i}{\partial x_\alpha} \frac{\partial u^j}{\partial x_\beta} \cdot (\det g(x))^{1/p} \Big)^{p/2} \mathrm{d}x\,.$$

By the previous results, we can analyse the expression of the relaxed p-energy in the particular case when the target manifold is the unit $\mathfrak{p}$-sphere $\mathbb{S}^\mathfrak{p}$, where $\mathfrak{p} \ge 2$ is the integer part of p. For any $u \in L^p(B^n, \mathbb{S}^\mathfrak{p})$, we thus denote

$$\widetilde{\mathbf{D}}_g^p(u, B^n) := \inf \Big\{ \liminf_{k\to\infty} \mathbf{D}_g^p(u_k, B^n) \quad | \quad \{u_k\} \subset C^\infty(B^n, \mathbb{S}^\mathfrak{p})\,, \; u_k \to u \text{ strongly in } L^p(B^n, \mathbb{R}^{\mathfrak{p}+1}) \Big\}\,.$$

By the bound (1.2), we again infer that $u \in W^{1,p}(B^n, \mathbb{S}^\mathfrak{p})$ if $\widetilde{\mathbf{D}}_g^p(u, B^n) < \infty$. Also, by a density argument, in low dimension $n = \mathfrak{p}$ we have $\widetilde{\mathbf{D}}_g^p(u, B^\mathfrak{p}) = \mathbf{D}_g^p(u, B^\mathfrak{p})$ for any $u \in W^{1,p}(B^n, \mathbb{S}^\mathfrak{p})$.

For $n \ge \mathfrak{p}+1$, Hölder's inequality and the bound (1.2) imply the inclusion $W^{1,p}(B^n, \mathbb{S}^\mathfrak{p}) \subset W^{1,\mathfrak{p}}(B^n, \mathbb{S}^\mathfrak{p})$, and hence the current of the singularities $\mathbb{P}(u)$ is well-defined. However, this time we have:

Theorem 4.6 *Let $n \ge \mathfrak{p}+1 \ge 3$ be integer, and let $\mathfrak{p} < p < \mathfrak{p}+1$. Then, for every $u \in W^{1,p}(B^n, \mathbb{S}^\mathfrak{p})$*

$$\widetilde{\mathbf{D}}_g^p(u, B^n) = \begin{cases} \mathbf{D}_g^p(u, B^n) & \text{if } \mathbb{P}(u) = 0 \\ +\infty & \text{if } \mathbb{P}(u) \ne 0\,. \end{cases} \tag{4.5}$$

Proof: If $\mathbb{P}(u) = 0$, the graph current $T = G_u$ satisfies the null-boundary condition (2.18). Therefore, arguing as in the proof of Theorem 4.1, we can find a sequence of smooth maps $\{u_k\} \subset W^{1,p}(B^n, \mathbb{S}^{\mathfrak{p}})$ strongly converging to u in $W^{1,p}(B^n, \mathbb{R}^{\mathfrak{p}+1})$, whence $\widetilde{\mathbf{D}}^p_g(u, B^n) = \mathbf{D}^p_g(u, B^n)$.

Conversely, we now show that if $\widetilde{\mathbf{D}}^p_g(u, B^n) < \infty$, then $\mathbb{P}(u) = 0$.

Let $\{u_k\} \subset C^\infty(B^n, \mathbb{S}^{\mathfrak{p}})$ be such that $\sup_h \mathbf{D}^p_g(u_k, B^n) < \infty$ and $u_k \to u$ in $L^p(B^n, \mathbb{R}^{\mathfrak{p}+1})$. We then find a (not relabeled) subsequence such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_n$ to some Cartesian current $T \in \mathcal{T}^{\mathfrak{p},1}_u$.

Assume by contradiction that $\mathbb{P}(u) \neq 0$. Then, $T = G_u + L_T \times [\![\mathbb{S}^{\mathfrak{p}}]\!]$ for some $L_T \in \mathcal{R}_{n-\mathfrak{p}}(B^n)$ with positive mass, $\mathbf{M}(L_T) > 0$. Therefore, if $\mathcal{L}_T$ is the set of points of positive density for L_T, we have $\mathcal{H}^{n-\mathfrak{p}}(\mathcal{L}_T) > 0$. For $\mathcal{H}^{n-\mathfrak{p}}$-almost every $x \in \mathcal{L}_T$, we denote by $D^{\mathfrak{p}}(x)$ the $\mathfrak{p}$-dimensional "disk" given by the intersection of B^n with the affine $\mathfrak{p}$-space of $\mathbb{R}^n$ containing x and orthogonal to the approximate tangent $(n-\mathfrak{p})$-space to $\mathcal{L}_T$ at x. We also let $v_k := u_{k|D^{\mathfrak{p}}(x)} : D^{\mathfrak{p}}(x) \to \mathbb{S}^{\mathfrak{p}}$. Then, we have

$$\sup_k \int_{D^{\mathfrak{p}}(x,r)} |Dv_k|^p \, d\mathcal{H}^{\mathfrak{p}} \le C < \infty \,, \tag{4.6}$$

and hence, possibly passing to a not relabeled subsequence, the graph $\mathfrak{p}$-currents G_{v_k} in $D^{\mathfrak{p}}(x) \times \mathbb{S}^{\mathfrak{p}}$ have equibounded $\mathfrak{p}$-energies.

Therefore, we can find a neighborhood $J^{\mathfrak{p}}_x$ of x in $D^{\mathfrak{p}}(x)$ such that the $\mathfrak{p}$-currents $G_{v_k} \llcorner (J^{\mathfrak{p}}_x \times \mathbb{S}^{\mathfrak{p}})$ have to converge near the point x to the sliced current $G_{u_{|D^{\mathfrak{p}}(x)}} \llcorner (J^{\mathfrak{p}}_x \times \mathbb{S}^{\mathfrak{p}}) + d \cdot \delta_x \times [\![\mathbb{S}^{\mathfrak{p}}]\!]$, where $d \in \mathbb{Z} \setminus \{0\}$ agrees (up to the sign) with the density of the current L_T at x. Setting now

$$D^{\mathfrak{p}}(x,r) := \{z \in D^{\mathfrak{p}}(x) \, : \, |z-x| < r\} \,, \quad r_0 := \sup\{r > 0 \mid D^{\mathfrak{p}}(x,r) \subset B^n\} > 0 \,,$$

by lower semicontinuity, and again by the bound (1.2), we have:

$$\liminf_{k\to\infty} \int_{D^{\mathfrak{p}}(x,r)} |Dv_k|^{\mathfrak{p}} \, d\mathcal{H}^{\mathfrak{p}} \ge \mathfrak{p}^{1/\mathfrak{p}} \, \mathbf{D}^{\mathfrak{p}}(d \cdot \delta_x \times [\![\mathbb{S}^{\mathfrak{p}}]\!]) = \mathfrak{p}^{1/\mathfrak{p}} \, d \, \alpha_{\mathfrak{p}} > 0 \tag{4.7}$$

for each $r \in (0, r_0)$. On the other hand, Hölder's inequality and the bound (4.6) give for $r \in (0, r_0)$

$$\sup_k \int_{D^{\mathfrak{p}}(x,r)} |Dv_k|^{\mathfrak{p}} \, d\mathcal{H}^{\mathfrak{p}} \le \sup_k c_p \, r^{p-\mathfrak{p}} \Big(\int_{D^{\mathfrak{p}}(x,r)} |Dv_k|^p \, d\mathcal{H}^{\mathfrak{p}} \Big)^{\mathfrak{p}/p} \le c_p \, C^{\mathfrak{p}/p} \, r^{p-\mathfrak{p}} \,, \tag{4.8}$$

where $C_p \, C^{\mathfrak{p}/p} \, r^{p-\mathfrak{p}} \to 0$ as $r \to 0^+$. Since (4.7) is in contradiction with (4.8), we must have $\mathbf{M}(L_T) = 0$, which yields $\mathbb{P}(u) = 0$, by (2.20), as required. Further details are omitted. □

5 The approximation theorem

We give the proof of Theorem 4.2. Firstly, Proposition 5.2, we show how to "deform" a current satisfying suitable energy estimates on the boundary of a ball, into a current satisfying a bound on the oscillation. Secondly, we use a local approximation argument, Proposition 5.3, and describe the dipole construction, Theorem 5.4. In the sequel we denote by $c > 0$ an absolute real constant, possibly varying from line to line. Moreover, recall that we assume $n \ge \mathfrak{p} + 1$.

5.1 Notation

For every $d \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$, we denote by $\mathcal{C}_d$ the integral cycle in $\mathcal{R}_{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})$ given by $\mathcal{C}_d(\eta) := d \int_{\mathbb{S}^{\mathfrak{p}}} \eta$ for all $\eta \in \mathcal{D}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})$, so that $\partial \mathcal{C}_d = 0$, $\mathbf{M}(\mathcal{C}_d) = d \, \alpha_{\mathfrak{p}}$ for each d, and $\mathcal{C}_1 = [\![\mathbb{S}^{\mathfrak{p}}]\!]$.

Let $T \in \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$, so that (2.17) holds. Writing as before $L_T = [\![\mathcal{L}_T, \theta, \tau]\!]$, we let

$$\mathbb{L}_d = [\![\mathcal{L}_d, 1, \tau]\!] \,, \quad \mathcal{L}_d := \{x \in \mathcal{L}_T \mid \theta(x) = d\} \,, \quad \forall \, d \in \mathbb{N}^+ \,.$$

With $u = u_T \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$, we thus can write

$$T = G_u + S_T \,, \quad S_T := \sum_{d \in \mathbb{N}^+} \mathbb{L}_d \times \mathcal{C}_d \,, \tag{5.1}$$

where every $\mathbb{L}_d$ is an i.m. rectifiable current in $\mathcal{R}_{n-\mathfrak{p}}(B^n)$ with *multiplicity one*, the $(n-\mathfrak{p})$-rectifiable sets $\mathcal{L}_d$ are pairwise disjoint, and, we recall, $|\tau(x)|=1$ for all $x\in\mathcal{L}_d$ and for each d. Therefore, on account of (3.9), the $\mathfrak{p}$-energy of T can be equivalently written as

$$\mathbf{D}^{\mathfrak{p}}_g(T,B):=\mathbf{D}^{\mathfrak{p}}_g(T,B\times\mathbb{S}^{\mathfrak{p}})=\mathbf{D}^{\mathfrak{p}}_g(u,B)+\alpha_{\mathfrak{p}}\sum_{d\in\mathbb{N}^+}d\cdot\mathbf{M}_g(\mathbb{L}_d\llcorner B)\,, \tag{5.2}$$

for every Borel set $B\subset B^n$, where, according to (3.8),

$$\mathbf{M}_g(\mathbb{L}_d\llcorner B)=\int_{\mathcal{L}_d\cap B}|\tau(x)|_{g(x)}\,d\mathcal{H}^{n-\mathfrak{p}}(x)\,.$$

As a consequence, the rectifiable measure μ^g_T can be written as

$$\mu^g_T=\theta_T\,\mathcal{H}^{n-\mathfrak{p}}\llcorner\mathcal{L}_T\,,\quad \mathcal{L}_T:=\bigcup\nolimits_{d\in\mathbb{N}^+}\mathcal{L}_d\,,$$

where the $(n-\mathfrak{p})$-rectifiable set $\mathcal{L}_T$ satisfies $\mathcal{H}^{n-\mathfrak{p}}(\mathcal{L}_T)<\infty$, and the density $\theta_T:\mathcal{L}_T\to]0,+\infty)$ is the non-negative $\mathcal{H}^{n-\mathfrak{p}}\llcorner\mathcal{L}_T$-measurable function on $\mathcal{L}_T$ given by

$$\theta_T(x):=\alpha_{\mathfrak{p}}\,d\,|\tau(x)|_{g(x)}\qquad\text{if}\quad x\in\mathcal{L}_d\,,\quad d\in\mathbb{N}^+\,.$$

Moreover, since (4.2) and (5.2) give $\mu_T(B^n)=\alpha_{\mathfrak{p}}\sum_{d\in\mathbb{N}^+}d\cdot\mathbf{M}_g(\mathbb{L}_d)<\infty$, there exists $\bar d\in\mathbb{N}^+$ such that

$$\alpha_{\mathfrak{p}}\sum_{d=\bar d+1}^{\infty}d\cdot\mathbf{M}_g(\mathbb{L}_d)\le\frac{1}{4}\,\mu_T(B^n)\,. \tag{5.3}$$

Therefore, on account of the bound (1.2), there exist two real constants $C_1,C_2>0$ such that

$$0<C_1\le\theta_T(x)\le C_2<\infty\qquad\forall\,x\in\bigcup\nolimits_{d\le\bar d}\mathcal{L}_d\,. \tag{5.4}$$

5.2 Slicing and projection formulas

Similarly to the case of normal currents, for every point $x_0\in B^n$ and for a.e. radius $r\in(0,r_0)$, where $r_0=r_0(x)>0$ is sufficiently small, in dependence of x, the *sliced* $(n-1)$*-current*

$$\langle T,\mathbf{d}_{x_0},r\rangle=\langle G_u,\mathbf{d}_{x_0},r\rangle+\langle S_T,\mathbf{d}_{x_0},r\rangle\,,$$

where $\mathbf{d}_{x_0}(x,y)=\mathbf{d}_{x_0}(x):=|x-x_0|$, is a well-defined Cartesian current in $\operatorname{cart}^{\mathfrak{p},1}(\partial B_r(x_0)\times\mathbb{S}^{\mathfrak{p}})$, where $B_r(x_0)$ denotes the ball of radius r centered at x_0, and $\partial B_r(x_0)$ its boundary. More precisely, we have

$$\langle G_{u_T},\mathbf{d}_{x_0},r\rangle(\omega)=\int_{\partial B_r(x_0)}(\mathrm{Id}\bowtie u_{|\partial B_r(x_0)})^{\#}\omega\,,\qquad\omega\in\mathcal{D}^{n-1}(\partial B_r(x_0)\times\mathbb{S}^{\mathfrak{p}})\,,$$

where $u_{|\partial B_r(x_0)}$ is the restriction of u to $\partial B_r(x_0)$, which is a Sobolev function in $W^{1,\mathfrak{p}}(\partial B_r(x_0),\mathbb{S}^{\mathfrak{p}})$. Also,

$$\langle S_T,\mathbf{d}_{x_0},r\rangle=\sum_{d\in\mathbb{N}^+}\langle\mathbb{L}_d,\mathbf{d}_{x_0},r\rangle\times\mathcal{C}_d\qquad\text{on}\quad\mathcal{D}^{n-1}(\partial B_r(x_0)\times\mathbb{S}^{\mathfrak{p}})\,.$$

As a consequence, we infer that for every Borel set $B\subset B^n$ the $\mathfrak{p}$-energy of $\langle T,\mathbf{d}_{x_0},r\rangle$ on $B\times\mathbb{S}^{\mathfrak{p}}$ is given by

$$\mathbf{D}^{\mathfrak{p}}_g(\langle T,\mathbf{d}_{x_0},r\rangle,B\times\mathbb{S}^{\mathfrak{p}})=\mathbf{D}^{\mathfrak{p}}_g(u_{|\partial B_r(x_0)},B)+\alpha_{\mathfrak{p}}\sum_{d\in\mathbb{N}^+}d\cdot\mathbf{M}_g(\langle\mathbb{L}_d,\mathbf{d}_{x_0},r\rangle\llcorner B)\,, \tag{5.5}$$

where $\mathbf{D}^{\mathfrak{p}}_g(u_{|\partial B_r(x_0)},B)$ can be written in a way similar to (1.6), by using the distributional derivative D_τ w.r.t. an orthonormal frame τ tangential to $\partial B_r(x_0)$. For example, in the case $g_{\alpha\beta}(x)\equiv\delta_{\alpha\beta}$, we have

$$\mathbf{D}^{\mathfrak{p}}(u_{|\partial B_r(x_0)},B)=\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}}\int_{\partial B_r(x_0)\cap B}|D_\tau u_{(r,x_0)}|^{\mathfrak{p}}\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

We also let

$$\mathbf{D}^{\mathfrak{p}}_g(\langle T,\mathbf{d}_{x_0},r\rangle):=\mathbf{D}^{\mathfrak{p}}_g(\langle T,\mathbf{d}_{x_0},r\rangle,\partial B_r(x_0)\times\mathbb{S}^{\mathfrak{p}})\,.$$

Remark 5.1 For future use, we denote by

$$\mathbb{S}^{\mathfrak{p}}{}_{\varepsilon} := \{y \in \mathbb{R}^{\mathfrak{p}+1} \mid \operatorname{dist}(y, \mathbb{S}^{\mathfrak{p}}) \le \varepsilon\}$$

the ε-neighborhood of $\mathbb{S}^{\mathfrak{p}}$. For $0 < \varepsilon \le 1/2$, the *nearest point projection* Π_ε of $\mathbb{S}^{\mathfrak{p}}{}_e$ onto $\mathbb{S}^{\mathfrak{p}}$ is a well defined Lipschitz map with Lipschitz constant $L_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$. For $y \in \mathbb{S}^{\mathfrak{p}}$ and $0 < \varepsilon \le 1/2$, we denote by

$$B_{\mathbb{S}^{\mathfrak{p}}}(y, \varepsilon) := \bar{B}^{\mathfrak{p}+1}(y, \varepsilon) \cap \mathbb{S}^{\mathfrak{p}}$$

the intersection of $\mathbb{S}^{\mathfrak{p}}$ with the closed $(\mathfrak{p}+1)$-ball of radius ε centered at y, so that $\Pi_\varepsilon(\bar{B}^{\mathfrak{p}+1}(y,\varepsilon)) = B_{\mathbb{S}^{\mathfrak{p}}}(y,\varepsilon)$. Finally, we let $\Psi_{(y,\varepsilon)} : \mathbb{R}^{\mathfrak{p}+1} \to B_{\mathbb{S}^{\mathfrak{p}}}(y,\varepsilon)$ be the retraction map given by $\Psi_{(y,\varepsilon)}(z) := \Pi_\varepsilon \circ \xi_{(y,\varepsilon)}$, where

$$\xi_{(y,\varepsilon)}(z) := \begin{cases} z & \text{if} \quad z \in \bar{B}^{\mathfrak{p}+1}(y,\varepsilon) \\ \varepsilon \dfrac{z-y}{|z-y|} & \text{if} \quad z \in \mathbb{R}^{\mathfrak{p}+1} \setminus \bar{B}^{\mathfrak{p}+1}(y,\varepsilon) \end{cases} \tag{5.6}$$

so that $\Psi_{(y,\varepsilon)}$ is a Lipschitz continuous function with $\operatorname{Lip}\Psi_{(y,\varepsilon)} = \operatorname{Lip}\Pi_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$.

5.3 Projecting the image of a current

For $n \ge \mathfrak{p}+1$, we set

$$B^n_\rho := B^n(\mathbf{0}, \rho), \qquad x = (\widetilde{x}, \widehat{x}) \in \mathbb{R}^{n-\mathfrak{p}} \times \mathbb{R}^{\mathfrak{p}}, \qquad D_\rho := B^{n-\mathfrak{p}}(0_{\mathbb{R}^{n-\mathfrak{p}}}, \rho).$$

Proposition 5.2 *Let $0 < R < R_0 < 1$ and $T \in \operatorname{cart}^{\mathfrak{p},1}(B^n_{R_0} \times \mathbb{S}^{\mathfrak{p}})$ be such that*

$$\begin{aligned} &\mathbf{D}^{\mathfrak{p}}_g(\langle T, \mathbf{d_0}, R\rangle, \partial B^n_R \setminus (\overline{D}_R \times \{0_{\mathbb{R}^{\mathfrak{p}}}\})) \le c\,\sigma\,\theta_T(\mathbf{0}) R^{n-\mathfrak{p}-1}, \\ &\mathbf{D}^{\mathfrak{p}}_g(\langle T, \mathbf{d_0}, R\rangle) \le c\,\theta_T(\mathbf{0})\, R^{n-\mathfrak{p}-1}, \\ &\int_{\partial B^n_R} |u_T(x) - y|^{\mathfrak{p}}\, d\mathcal{H}^{n-1} \le c\,\sigma\, R^{n-1}, \end{aligned} \tag{5.7}$$

for some $y \in \mathbb{S}^{\mathfrak{p}}$ and for $\sigma > 0$ small enough. Then there exists an absolute constant $c > 0$ such that, if $q \in \mathbb{N}^+$ is the integer part of $c\,\sigma^{\alpha(n,\mathfrak{p})}$, where

$$\alpha(n,\mathfrak{p}) := -\frac{1}{6(n-\mathfrak{p})(\mathfrak{p}-1)} < 0, \tag{5.8}$$

we can find a Cartesian current $\widetilde{T} \in \operatorname{cart}^{\mathfrak{p},1}((B^n_R \setminus \bar{B}^n_r) \times \mathbb{S}^{\mathfrak{p}})$, where $r = R(1-1/q)$, such that:

(a) $\langle \widetilde{T}, \mathbf{d_0}, R\rangle = \langle T, \mathbf{d_0}, R\rangle$ *and* $\langle \widetilde{T}, \mathbf{d_0}, r\rangle = (\psi_{R,r} \bowtie \Psi_{(y,\varepsilon_\sigma)})_\# \langle T, \mathbf{d_0}, R\rangle$, *where* $\varepsilon_\sigma := c \cdot \sigma^{2/3}$, $\psi_{R,r}(x) := rx/R$, *and* $\Psi_{(y,\varepsilon_\sigma)}(z) := \Pi_{\varepsilon_\sigma} \circ \xi_{(y,\varepsilon_\sigma)}$, *see* (5.6), *so that* $\operatorname{spt}\langle \widetilde{T}, \mathbf{d_0}, r\rangle \subset \partial B^n_r \times B_{\mathbb{S}^{\mathfrak{p}}}(y, \varepsilon_\sigma)$;

(b) $\widetilde{T}$ *has small $\mathfrak{p}$-energy on $B^n_R \setminus B^n_r$, i.e.,*

$$\mathbf{D}^{\mathfrak{p}}_g(\widetilde{T}, B^n_R \setminus B^n_r) \le c\,\frac{R}{q}\,\mathbf{D}^{\mathfrak{p}}_g(\langle T, \mathbf{d_0}, R\rangle); \tag{5.9}$$

(c) we finally have

$$\mathbf{F}((\widetilde{T} - G_y) \llcorner (B^n_R \setminus \bar{B}^n_r) \times \mathbb{S}^{\mathfrak{p}}) \le c\,\frac{\sigma}{q}\, R^n \le c\,\sigma\, R^{n-1}. \tag{5.10}$$

Proof: It is a readaptation of the one of [9, Prop. 4.6], where $\mathfrak{p} = 2$, in case $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}}$. We thus only sketch the main modifications to the construction, where we work with the k-skeleton of triangulations, for $k = \mathfrak{p}, \ldots, n-1$. This time, using the first inequality in (5.7), we find the estimate

$$\int_{\Sigma^{\mathfrak{p}-1}_R} |Dw_{|\Sigma^{\mathfrak{p}-1}_R}|^{\mathfrak{p}}\, d\mathcal{H}^{\mathfrak{p}-1} \le c\,\sigma^{2/3}\,\frac{1}{R},$$

so that since $\mathcal{H}^{\mathfrak{p}}(\Sigma^1_R) \le c\,R^{\mathfrak{p}-1}\,q^{n-\mathfrak{p}}$, by Hölder's inequality we get

$$\int_{\Sigma_R^{\mathfrak{p}-1}} |Dw_{|\Sigma^1_R}|\,d\mathcal{H}^{\mathfrak{p}-1} \le c\,q^{(n-\mathfrak{p})(\mathfrak{p}-1)/\mathfrak{p}}\,\sigma^{2/3\mathfrak{p}} \le c\,\sigma^{1/2\mathfrak{p}}\,,$$

provided that $q \in \mathbb{N}^+$ is chosen as in the thesis. By the third inequality in (5.7), we may and do assume that the oscillation of $w_{|\Sigma_R^{\mathfrak{p}-1}}$ is smaller than $c\,\sigma^{1/2\mathfrak{p}}$ and that the image $w(\Sigma_R^{\mathfrak{p}-1})$ is contained in the geodesic ball $B_{\mathbb{S}^{\mathfrak{p}}}(y,\varepsilon_\sigma)$. Therefore, as in Step 3 of [18], we may and do define the current $\widetilde{T}$ satisfying the above properties. In fact, when extending $\widetilde{T}$ from the $\mathfrak{p}$-skeleton to the $(\mathfrak{p}+1)$-skeleton of a partition of $B^n_R \setminus B^n_r$ in "cubes", in principle $\widetilde{T}$ has a non-zero boundary of the type $m\,\delta_{x_l} \times [\![\, \mathbb{S}^{\mathfrak{p}} \,]\!]$ for each $(\mathfrak{p}+1)$-face F_l of such a cubeulation, where x_l is the barycenter of F_l and $m \in \mathbb{Z}$. However, since by the construction the mass of such a current is small with σ, then necessarily $m = 0$. In dimension $n \ge \mathfrak{p}+2$, and for $k = \mathfrak{p}+2,\ldots,n$, no extra-boundary is produced when extending $\widetilde{T}$ from the $(k-1)$-skeleton to the k-skeleton of the cubes of the partition of $B^n_R \setminus B^n_r$. Further details are omitted □

5.4 Approximation on a ball

Let $y(\widetilde{x}) := (r - |\widetilde{x}|)$ denote the distance of $\widetilde{x}$ from the boundary of the $(n-\mathfrak{p})$-disk D_r, and

$$\phi_\delta(x) := (\widetilde{x}, \varphi_\delta(y(\widetilde{x}))\,\widehat{x})\,, \quad x \in D_r \times \bar{B}^{\mathfrak{p}}\,, \quad \varphi_\delta(y) := \min\{y,\delta\}\,, \tag{5.11}$$

so that $\Omega_\delta := \phi_\delta(D_r \times \bar{B}^{\mathfrak{p}})$ is a small neighborhood of the interior of the disk $D_r \times \{0_{\mathbb{R}^{\mathfrak{p}}}\}$ in B^n_R. Also, let

$$\widetilde{\Omega}_\delta := \phi_\delta(D_r \times \bar{B}^{\mathfrak{p}}_{1/2}) = \{(\widetilde{x},\widehat{x}) \mid \widetilde{x} \in D_r\,,\ \rho \le \varphi_\delta(y(\widetilde{x}))/2\}\,, \tag{5.12}$$

where in the sequel $\rho := |\widehat{x}| = \sqrt{{x_{n-\mathfrak{p}+1}}^2 + \cdots + {x_n}^2}$, and

$$\Omega_{(r,\delta)} := \Omega_\delta \setminus (D_r \times \{0_{\mathbb{R}^{\mathfrak{p}}}\})\,.$$

Proposition 5.3 *Let $T \in \mathrm{cart}^{\mathfrak{p},1}(B^n_r \times \mathbb{S}^{\mathfrak{p}})$, so that the decomposition (5.1) holds. Assume that $\mathrm{spt}\,T \subset \bar{B}^n_r \times B_{\mathbb{S}^{\mathfrak{p}}}(y,\varepsilon_\sigma)$, where $y \in \mathbb{S}^{\mathfrak{p}}$ and $\varepsilon_\sigma = c\cdot\sigma^{2/3}$, with $\sigma > 0$ small, and that $D_r \times \{0_{\mathbb{R}^{\mathfrak{p}}}\} \subset \mathcal{L}_{d_0}$ for some $d_0 \in \mathbb{N}^+$. For $\delta > 0$ small enough, we can find a current $\widetilde{T} \in \mathrm{cart}^{\mathfrak{p},1}((B^n_r \setminus \widetilde{\Omega}_\delta) \times \mathbb{S}^{\mathfrak{p}})$ satisfying:*

i) $\partial(\widetilde{T} \llcorner (B^n_r \setminus \widetilde{\Omega}_\delta) \times \mathbb{S}^{\mathfrak{p}}) = \partial(T \llcorner B^n_r \times \mathbb{S}^{\mathfrak{p}}) - [\![\, \widetilde{\Omega}_\delta \,]\!] \times \delta_y - [\![\, \partial D_r \times \{0_{\mathbb{R}^{\mathfrak{p}}}\} \,]\!] \times \mathcal{C}_{d_0}$;

ii) $\mathbf{D}^{\mathfrak{p}}_g(\widetilde{T}, (B^n_r \setminus \widetilde{\Omega}_\delta) \times \mathbb{S}^{\mathfrak{p}}) \le \mathbf{D}^{\mathfrak{p}}_g(u, (B^n_r \setminus \Omega_\delta)) + c\,\sigma\,r^{n-\mathfrak{p}} + c\,\mu^g_T(\Omega_{(r,\delta)})$;

iii) $\mathbf{F}((\widetilde{T} - T) \llcorner (B^n_r \setminus \widetilde{\Omega}_\delta) \times \mathbb{S}^{\mathfrak{p}}) \le c\,\sigma\,r^{n-\mathfrak{p}}$.

PROOF: It suffices to argue in a way very similar to the proof of [9, Prop. 4.7] in case $\mathfrak{p} = 2$, with $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}}$, on account of the bound (1.2). Therefore, it is omitted. □

5.5 The dipole construction

Theorem 5.4 *Let $d \in \mathbb{N}^+$ and $y \in \mathbb{S}^{\mathfrak{p}}$. For every $\sigma > 0$, there exists a function $v_\sigma \in W^{1,\mathfrak{p}}(\widetilde{\Omega}_\delta, \mathbb{S}^{\mathfrak{p}})$, with $\delta > 0$ sufficiently small, such that $G_{v_\sigma} \in \mathrm{cart}^{\mathfrak{p},1}(\mathrm{int}(\widetilde{\Omega}_\delta) \times \mathbb{S}^{\mathfrak{p}})$ and*

$$\int_{\widetilde{\Omega}_\delta} e^{\mathfrak{p}}_g(\mathbf{0}, Dv_\sigma)\,dx \le \sigma\,r^{n-\mathfrak{p}} + |\tau|_{g(\mathbf{0})} \cdot \mathcal{H}^{n-\mathfrak{p}}(D_r) \cdot \alpha_{\mathfrak{p}}\,d\,, \tag{5.13}$$

where $\tau := e_1 \wedge \cdots \wedge e_{n-\mathfrak{p}} \in \wedge_{n-\mathfrak{p}}\mathbb{R}^n$. Moreover, $v_{\sigma\#}[\![\, \widetilde{\Omega}_\delta \,]\!] = \mathcal{C}_d$ and

$$\partial G_{v_\sigma} = \partial[\![\, \widetilde{\Omega}_\delta \,]\!] \times \delta_y + [\![\, \partial D_r \times \{0_{\mathbb{R}^{\mathfrak{p}}}\} \,]\!] \times \mathcal{C}_d\,. \tag{5.14}$$

PROOF: Set $\Omega := D_r \times B^{\mathfrak{p}}_{1/2}$, and assume that $u \in W^{1,\mathfrak{p}}(\Omega, \mathbb{S}^{\mathfrak{p}})$ only depends on the last $\mathfrak{p}$ variables, i.e., $u = u(\widehat{x})$, where $x = (\widetilde{x}, \widehat{x}) \in \mathbb{R}^{n-\mathfrak{p}} \times \mathbb{R}^{\mathfrak{p}}$. By Fubini's theorem, for every $0 < \rho < r$ we have

$$\int_{D_\rho \times B^{\mathfrak{p}}_{1/2}} e^{\mathfrak{p}}_g(\mathbf{0}, Du(x))\,\mathrm{d}x = \mathcal{H}^{n-\mathfrak{p}}(D_\rho) \cdot \int_{B^{\mathfrak{p}}_{1/2}} e^{\mathfrak{p}}_g(\mathbf{0}, Du(\widehat{x}))\,\mathrm{d}\widehat{x}\,.$$

Now, writing $u := \widetilde{u} \circ L^{-1}$, where $L = L(\mathbf{0})$ is given by (3.2), by (3.3) we obtain:

$$e^{\mathfrak{p}}_g(\mathbf{0}, Du(\widehat{x})) = \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} |D\widetilde{u}(z)|^{\mathfrak{p}}\,, \qquad z := L^{-1}x\,.$$

Let $\{v_1, \ldots, v_n\} \subset \mathbb{R}^n$ be a $g(\mathbf{0})$-orthogonal basis given by eigenvectors of the matrix $g(\mathbf{0})$, and let $S \in M(n,n)$ be given by $S^i_j := v^i_j$, where $v_j := (v^1_j, \ldots, v^n_j)$. Since τ orients the $(n-\mathfrak{p})$-disk D_r, it turns out that $\widetilde{u} \in W^{1,\mathfrak{p}}(L^{-1}(\Omega), \mathbb{S}^{\mathfrak{p}})$ only depends on the orthogonal directions to $S^\top \tau$. Setting $\widetilde{e}_i := S^\top e_i$, this means that

$$\widetilde{u}(z) = F(z^{n-\mathfrak{p}+1}, \ldots, z^n)\,, \qquad z = \sum_{i=1}^n z^i\,\widetilde{e}_i \tag{5.15}$$

for some function $F \in W^{1,\mathfrak{p}}(\widetilde{D}, \mathbb{S}^{\mathfrak{p}})$, where $\widetilde{D} := L^{-1}(\{0_{\mathbb{R}^{n-\mathfrak{p}}}\} \times B^{\mathfrak{p}}_{1/2})$. On the other hand, since $\widehat{x} = \widehat{L}z$, where $\widehat{L} \in M(\mathfrak{p}, n)$ is the matrix of the last $\mathfrak{p}$ rows of L, by a change of variable we find that

$$\int_{B^{\mathfrak{p}}_{1/2}} e^{\mathfrak{p}}_g(\mathbf{0}, Du(\widehat{x}))\,\mathrm{d}\widehat{x} = |M_{(\mathfrak{p})}\widehat{L}| \cdot \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} \int_{\widetilde{D}} |DF|^{\mathfrak{p}}\, d\mathcal{H}^{\mathfrak{p}}\,, \tag{5.16}$$

where $|M_{(\mathfrak{p})}\widehat{L}|$ is the $\mathfrak{p}$-*dimensional Jacobian* of $\widehat{L}$. More precisely, setting $\alpha_0 := (1, \ldots, n-\mathfrak{p}) \in I(n-\mathfrak{p}, n)$,

$$|M_{(\mathfrak{p})}\widehat{L}|^2 = \sum_{\gamma \in I(n-\mathfrak{p},n)} M^{\overline{\alpha}_0}_{\overline{\gamma}}(L)^2\,. \tag{5.17}$$

Lemma 5.5 *We have $|M_{(\mathfrak{p})}\widehat{L}| = |\tau|_g$, where $g = g(\mathbf{0})$.*

PROOF: By (3.5) and Proposition 3.4, we infer that

$$|\tau|_g = (\det L)\,|\Lambda_{n-\mathfrak{p}} L^{-1}(\tau)|\,, \qquad L = L(\mathbf{0})\,, \quad g = g(\mathbf{0})\,.$$

Since $\Lambda_{n-\mathfrak{p}} L^{-1}(\tau) = L^{-1}e_1 \wedge \cdots \wedge L^{-1}e_{n-\mathfrak{p}}$, we compute

$$\Lambda_{n-\mathfrak{p}} L^{-1}(\tau) = \sum_{\gamma \in I(n-\mathfrak{p},n)} M^{\gamma}_{\alpha_0}(L^{-1})\, e_\gamma\,.$$

Moreover, Lemma 2.1 yields

$$(\det L)\, M^{\gamma}_{\alpha_0}(L^{-1}) = \sigma(\gamma, \overline{\gamma})\, \sigma(\alpha_0, \overline{\alpha}_0)\, M^{\overline{\alpha}_0}_{\overline{\gamma}}(L)\,,$$

so that we obtain

$$|\tau|^2_g = \sum_{\gamma \in I(n-\mathfrak{p},n)} (\det L)^2\, M^{\gamma}_{\alpha_0}(L^{-1})^2 = \sum_{\gamma \in I(n-\mathfrak{p},n)} M^{\overline{\alpha}_0}_{\overline{\gamma}}(L)^2$$

and hence the assertion follows from (5.17). □

Now, on account of Proposition 2.2, arguing as e.g. in [17, Sec. 5.1], we readily obtain:

Proposition 5.6 *Let $d \in \mathbb{N}^+$ and $y \in \mathbb{S}^{\mathfrak{p}}$ be a given point. There exists a family of Lipschitz functions $F^y_\varepsilon : \widetilde{D} \to \mathbb{S}^{\mathfrak{p}}$ such that $F^y_{\varepsilon\,|\partial\widetilde{D}} \equiv y$ and*

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} \int_{\widetilde{D}} |DF^y_\varepsilon|^{\mathfrak{p}}\,\mathrm{d}\mathcal{H}^{\mathfrak{p}} \le \alpha_{\mathfrak{p}}\, d + c\,\varepsilon\,,$$

where $\widetilde{D} := L^{-1}(\{0_{\mathbb{R}^{n-\mathfrak{p}}}\} \times B^{\mathfrak{p}}_{1/2})$. Moreover, $F^y_{\varepsilon\#}[\![\,\widetilde{D}\,]\!] = \mathcal{C}_d$.

As a consequence, taking $F = F^y_\varepsilon$ in (5.15), by (5.16) and Lemma 5.5 we obtain $u_\varepsilon \in W^{1,\mathfrak{p}}(\Omega, \mathbb{S}^\mathfrak{p})$ such that for every $\rho \in (0, r]$

$$\int_{D_\rho \times B^\mathfrak{p}_{1/2}} e^\mathfrak{p}_g(\mathbf{0}, Du_\varepsilon)\,\mathrm{d}x \le \mathcal{H}^{n-\mathfrak{p}}(D_\rho) \cdot |\tau|_{g(\mathbf{0})} \cdot (\alpha_\mathfrak{p}\, d + c\,\varepsilon)\,. \tag{5.18}$$

Moreover, arguing as in a way similar e.g. to [17, Sec. 5.5], by using the bound (1.2) we obtain:

Lemma 5.7 *Let $0 < \delta < 1$ and $u^\varepsilon_\delta := u_\varepsilon \circ \phi_\delta^{-1} : \widetilde{\Omega}_\delta \to \mathbb{S}^\mathfrak{p}$, where ϕ_δ is given by (5.11). Then we have*

$$\int_{\widetilde{\Omega}_\delta} e^\mathfrak{p}_g(\mathbf{0}, Du^\varepsilon_\delta)\,\mathrm{d}x \le \int_{D_r \times B^\mathfrak{p}_{1/2}} e^\mathfrak{p}_g(\mathbf{0}, Du_\varepsilon)\,\mathrm{d}x + c\int_{(D_r \setminus D_{r-\delta}) \times B^\mathfrak{p}_{1/2}} e^\mathfrak{p}_g(\mathbf{0}, Du_\varepsilon)\,\mathrm{d}x\,.$$

Definitely, on account of (5.18), we obtain the energy estimate

$$\int_{\widetilde{\Omega}_\delta} e^\mathfrak{p}_g(\mathbf{0}, Du^\varepsilon_\delta)\,\mathrm{d}x \le (\mathcal{H}^{n-\mathfrak{p}}(D_r) + c\,\mathcal{H}^{n-\mathfrak{p}}(D_r \setminus D_{r-\delta})) \cdot |\tau|_{g(\mathbf{0})} \cdot (\alpha_\mathfrak{p}\, d + \varepsilon)\,,$$

and hence, setting $v_\sigma := u^\varepsilon_\delta$ for $\varepsilon > 0$ sufficiently small, and for δ sufficiently small in dependence of ε and of the Lipschitz constant of F^y_ε, we get (5.13), whereas (5.14) follows from the construction. ☐

5.6 Proof of the approximation theorem

PROOF: [Proof of Theorem 4.2] It is very similar to the proof of [9, Thm. 4.4], taking account the preliminary results already obtained in this section. For that reason, it is only sketched, and we simply outline the main differences. Incidentally, we also point out a flaw in the cited theorem, that is adjusted here. Precisely, the upper bound in [9, Eq. (4.5)] fails to hold, and in order to correct the argument, one has to proceed in a way similar to the argument yielding to (5.4).

Referring to [9, pp. 28–32], we essentially replace the exponent 2 with $\mathfrak{p}$, e.g., $\mathrm{cart}^{2,1}$ with $\mathrm{cart}^{\mathfrak{p},1}$, and choose $\mathcal{X} = B^n$, $\mathcal{Y} = \mathbb{S}^\mathfrak{p}$, so that $N = \mathfrak{p} + 1$, and $H^{sph}_2(\mathcal{Y})$ becomes $H_\mathfrak{p}(\mathbb{S}^\mathfrak{p}) \simeq \mathbb{Z}$, whence $R_q \in H^{sph}_2(\mathcal{Y})$ is replaced with the $\mathfrak{p}$-cycle $\mathcal{C}_d$, for some $d \in \mathbb{N}^+$. Therefore, $set(\mathbb{L}_{q_j})$ becomes $\mathcal{L}_{d_j}$. Moreover, the terms $\mathbf{D}_g$, e_g, and μ_T, become $\mathbf{D}^\mathfrak{p}_g$, $e^\mathfrak{p}_g$, and μ^g_T, respectively.

More precisely, on pp. 28–29, where we take $(n-\mathfrak{p})$-submanifolds $\mathcal{M}_j$, we modify properties i)–xiii) as follows. The exponent $n-2$ becomes $n-\mathfrak{p}$ in Eqs. (4.14), (4.17), and (4.18). Moreover, v) becomes "$\mathcal{M}_j \subset \mathcal{L}_d$ for some $d = d_j \in \mathbb{N}^+$", whereas in viii), taking r_j small so that $|Du(p_j)|^\mathfrak{p}\, {r_j}^\mathfrak{p} \le \sigma\,\theta_T(p_j)$, we obtain Eq. (4.19), with r^{n-3}_j replaced by $r^{n-\mathfrak{p}-1}_j$, and similarly for Eq. (4.20). In x), by the continuity property (4.1), we get Eq. (4.21) with $|G|^\mathfrak{p}$ instead of $|G|^2$. Finally, in xi), Eq. (4.22) becomes:

$$|\mu^g_T(B_j) - \alpha_\mathfrak{p}\, d_j \cdot \omega_{n-\mathfrak{p}}\, {r_j}^{n-\mathfrak{p}}| \le \sigma\,\omega_{n-\mathfrak{p}}\, {r_j}^{n-\mathfrak{p}}\,, \quad \omega_{n-\mathfrak{p}} := \mathcal{H}^{n-\mathfrak{p}}(B^{n-\mathfrak{p}})\,.$$

In addition, on account of (5.3) and (5.4), in the sequel we let $\mathcal{I}(\bar d)$ denote the set of indexes j such that property v) holds true with $d = d_j \le \bar d$, and we work with the restriction of T to the balls B_j, where $j \in \mathcal{I}(\bar d)$. Therefore, we now fix $j \in \mathcal{I}(\bar d)$ in the definition of T^σ_j on p. 29, and we follow the lines up to Eq. (4.30), with the following modifications.

We replace d with R_0, and apply Proposition 5.2 instead of [9, Prop. 4.6]. Furthermore, since the negative constant $\alpha(n) = \alpha(n,2)$ is replaced by $\alpha(n,\mathfrak{p})$ in (5.8), instead of $\beta(n) = \beta(n,2)$ we let:

$$\beta(n,\mathfrak{p}) := \frac{1}{12(n-\mathfrak{p})(\mathfrak{p}-1)} > 0\,,$$

so that Eq. (4.23) holds with $n-\mathfrak{p}$ instead of $n-2$. Following the lines of the proof, this time the current $\breve{T}^\sigma_j$ satisfies the hypotheses of Proposition 5.3, instead of [9, Prop. 4.7]. We then apply Theorem 5.4 instead of [9, Thm. 4.8]. Therefore, e.g. the last centered formula on p. 30 becomes:

$$\int_{\widetilde{\Omega}_\delta} e^\mathfrak{p}_g(x, Dv^\sigma_j)\,\mathrm{d}x \le c\,\sigma\, {r_j}^{n-\mathfrak{p}} + (1 + c\,\sigma)\,\mu^g_T(B_j)\,.$$

Now, after [9, Eq. (4.30)], we conclude in a different way, and define $T^\sigma \in \mathrm{cart}^{\mathfrak{p},1}(B^n \times \mathbb{S}^{\mathfrak{p}})$ by

$$T^\sigma := \sum_{j \in \mathcal{I}(\overline{d})} T_j^{(\sigma)} + T \llcorner \big(B^n \setminus \bigcup_{j \in \mathcal{I}(\overline{d})} \mathrm{int}(B_j)\big) \times \mathbb{S}^{\mathfrak{p}} .$$

This way, the first centered formula on p. 32 is replaced by:

$$\begin{aligned} \mu^g_{T^\sigma}(B^n) &\le c \sum_{j \in \mathcal{I}(\overline{d})} \mu^g_T(B(p_j, r_j) \setminus (B(p_j, tr_j) \cap \mathcal{M}_j)) \\ &\quad + \sum_{j \notin \mathcal{I}(\overline{d})} \mu^g_T(B(p_j, r_j)) + \mu^g_T(B^n \setminus \mathcal{L}_T) \\ &\le c\,\sigma\,\mu^g_T(B^n) + \frac{1}{4}\,\mu^g_T(B^n) < \frac{1}{2}\,\mu^g_T(B^n) , \end{aligned}$$

and the second one by

$$\mathbf{F}(T^\sigma - T) \le \sum_{j=1}^\infty \mathbf{F}((T_j^{(\sigma)} - T) \llcorner B_j \times \mathbb{S}^{\mathfrak{p}}) \le c\,\sigma \sum_{j=1}^\infty {r_j}^{n-\mathfrak{p}} < \varepsilon^k ,$$

if $\sigma = \sigma(\varepsilon, k, \mathcal{L}_T, \mu^g_T) > 0$ is small. Taking $\widetilde{T} = T^\sigma$, the proof is complete. Further details are omitted. $\square$

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