

On the equivalence of static and dynamic weak optimal transport

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Abstract

We show that there is a PDE formulation in terms of Fokker-Planck equations for weak optimal transport problems. The main novelty is that we introduce a minimization problem involving Fokker-Planck equations in the extended sense of measure-valued solutions and prove that it is equal to the associated weak transport problem.

Résumé

Nous montrons qu'il existe une formulation EDP en termes d'équations de Fokker-Planck pour des problèmes de transport optimal faible. La principale nouveauté est que nous introduisons un problème de minimisation dont la contrainte est constituée des équations de Fokker-Planck pour les mesures générales et prouvons qu'il est égal au problème de transport optimal faible correspondant.

Keywords: Fokker-Planck equation; weak transport problem; duality; Laplace operator; Green function; subharmonic function; viscosity solution; Hamilton-Jacobi-Bellman equation.

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1. Introduction

Given probability measures μ, ν on \mathbb{R}^d , we denote by $\Pi(\mu, \nu)$ the set of all probability measures (called transport plans) on $\mathbb{R}^d \times \mathbb{R}^d$ whose first and second marginals are μ and ν , respectively. Disintegrating a transport plan $\gamma \in \Pi(\mu, \nu)$ with respect to its first marginal μ , we obtain a μ -almost everywhere uniquely determined family of probability measures $\{\gamma^x\}_{x \in \mathbb{R}^d} \subset \mathcal{P}(\mathbb{R}^d)$ such that $\gamma = \gamma^x \otimes \mu$. Let $G : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \rightarrow [0, +\infty]$ be lower semicontinuous in an appropriate sense such that for each $x \in \mathbb{R}^d$, $G(x, \cdot)$ is convex on $\mathcal{P}(\mathbb{R}^d)$. The weak transport problem is then defined by

$$H(\mu, \nu) = \inf \left\{ \int_{\mathbb{R}^d} G(x, \gamma^x) d\mu(x) \mid \gamma \in \Pi(\mu, \nu) \right\}.$$

This problem was first introduced by Gozlan, Roberto, Samson and Tetali [16], and shortly thereafter by Alibert, Bouchitté and Champion [2]. The weak transport problem has been extensively studied in the literature: the results of existence and duality are established in [2, 5, 16]; the concept of C -monotonicity, which is analogous to cyclical monotonicity, was developed in [4, 5, 15] in order to provide a characterization of optimizers; in [1, 2, 4, 7] the weak transport viewpoint is used to investigate martingale optimal transport problems; for applications of weak transport theory, the reader may consult [6].

In [18], Huesmann and Trevisan introduced the following minimization problem

$$f_{\text{FPE}}(\mu, \nu) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^d} f(a_t(x)) d\varrho_t(x) dt \mid \partial_t \varrho = \text{tr} \left(\frac{1}{2} \nabla^2 a \varrho \right), \varrho_0 = \mu, \varrho_1 = \nu \right\},$$

where μ, ν are probabilities on \mathbb{R}^d with finite second moment, the Fokker-Planck equation $\partial_t \varrho = \text{tr}(\frac{1}{2} \nabla^2 a \varrho)$ in $(0, 1) \times \mathbb{R}^d$ holds in the weak sense and f is p -admissible for some $p > 1$ (namely, $f(a)$ behaves like

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$f(a) \sim |a|^p$ for a nonnegative symmetric real $d \times d$ matrix a ; see Sections 2, 3 in [18]). They proved that

$$f_{\text{FPE}}(\mu, \nu) = \inf \left\{ \int_0^1 \mathbb{E} [f(\langle \dot{X} \rangle_t)] dt \right\}, \quad (1.1)$$

where the infimum is taken over all martingales connecting μ and ν whose quadratic variation process $\langle X \rangle$ is absolutely continuous with respect to the Lebesgue measure (see [18, Theorem 3.3]). It is worth noting the following. Let $G(x, p) = f_{\text{FPE}}(\delta_x, p)$ for each $x \in \mathbb{R}^d$ and for each probability measure p on \mathbb{R}^d with finite second moment. Then, in view of [18, Theorem 4.3], the functional G is lower semicontinuous in an appropriate sense and $G(x, \cdot)$ is convex for each $x \in \mathbb{R}^d$. Taking into account (1.1) and using a standard measurable selection argument (see, for instance, Section 7.7 in [8]), we have

$$f_{\text{FPE}}(\mu, \nu) = \inf \left\{ \int_{\mathbb{R}^d} f_{\text{FPE}}(\delta_x, \gamma^x) d\mu(x) \mid \gamma \in \Pi(\mu, \nu) \right\} = H(\mu, \nu).$$

This provides an equivalent PDE formulation in terms of Fokker-Planck equations for the weak transport problem $H(\mu, \nu)$, where $G(x, p) = f_{\text{FPE}}(\delta_x, p)$ and f is p -admissible for some $p > 1$ (the reader may also consult the proof of [6, Theorem 8.2], where an equivalent SDE formulation for the weak transport problem is established using a measurable selection argument). We emphasize that the p -admissibility property of the function f was crucial in the proof of (1.1) in [18]. Namely, the equality (1.1) was proved in [18] only for strictly convex functions f behaving like $f(a) \sim |a|^p$ for some $p > 1$ and for each nonnegative symmetric real $d \times d$ matrix a . The purpose of this paper is to provide an equivalent PDE formulation for $H(\mu, \nu)$ in terms of Fokker-Planck equations when the cost function f is not strictly convex and behaves like $f(a) \sim |a|$ (i.e., for example, for the trace or spectral radius of a nonnegative symmetric real $d \times d$ matrix), thereby narrowing the “gap” between the classical static formulation for $H(\mu, \nu)$ and the dynamic formulation involving a PDE.

In particular, we introduce the minimization problem

$$F(\mu, \nu) := \inf \left\{ \int_0^1 \int_{\mathbb{R}^d} f \left(\frac{d\lambda_t}{d|\lambda_t|} \right) d|\lambda_t| dt \mid \partial_t \varrho = \text{tr} \left(\frac{1}{2} \nabla^2 \lambda \right), \varrho_0 = \mu, \varrho_1 = \nu \right\},$$

where f is a nonnegative sublinear function and the Fokker-Planck equation $\partial_t \varrho = \text{tr}(\frac{1}{2} \nabla^2 \lambda)$ in $(0, 1) \times \mathbb{R}^d$ holds in the weak extended sense of measure-valued solutions (see Section 2.2), namely, λ does not have to be absolutely continuous with respect to ϱ , in contrast to the Fokker-Planck equations $\partial_t \varrho = \text{tr}(\frac{1}{2} \nabla^2 a \varrho)$ in $(0, 1) \times \mathbb{R}^d$ considered by Huesmann and Trevisan in [18], where the analysis is carried out for p -admissible costs. By defining $G(x, p) = F(\delta_x, p)$, our goal is to prove the equality $H(\mu, \nu) = F(\mu, \nu)$. Let us highlight that, unlike the case of p -admissible costs [18], we do not have an equivalent Benamou-Brenier type formulation (1.1) for $F(\mu, \nu)$ (see Remark 3.5), and hence we cannot use a standard measurable selection argument to prove the equality between $F(\mu, \nu)$ and $H(\mu, \nu)$. Therefore, we implement a new analytical approach. We derive dual formulations for $F(\mu, \nu)$ and $H(\mu, \nu)$ in terms of G -invariant functions. Since the dual competitors for $H(\mu, \nu)$ are less regular than the dual competitors for $F(\mu, \nu)$, the equality $F(\mu, \nu) = H(\mu, \nu)$ is proved when G -invariant functions can be approximated by smooth G -invariant functions in a suitable sense (see Theorems 4.1, 4.2). Under the coercivity assumption on f , we characterize bounded continuous G -invariant functions as viscosity solutions of the Hamilton-Jacobi-Bellman equation. This, under the coercivity and growth assumptions on f , yields the dual formulation for $H(\mu, \nu)$, where the competitors are viscosity solutions of the Hamilton-Jacobi-Bellman equation (see Corollary 4.13). Then we perform a smoothing procedure and obtain the equality $F(\mu, \nu) = H(\mu, \nu)$ (see Theorem 4.14). We consider in detail four examples (i)-(iv) for which we compute $F(\mu, \nu)$ in terms of μ and ν .

Let us outline the contents of the paper. In Section 2, we introduce our main notation and recall some definitions. In Section 3, we announce our basic assumptions on the cost function f and introduce the minimization problem (3.1) involving Fokker-Planck equations in the extended sense of measure-valued solutions. The key assumption is the positive 1-homogeneity of f , using which, we eliminate the time variable in the statement of the problem (3.1) via the equation (3.3) and obtain its equivalent formulation (3.5) without time (see Proposition 3.2). We obtain the dual formulation (3.12) for the problem (3.1)

and prove the existence of a (primal) minimizer when $F(\mu, \nu) < +\infty$ (see Theorem 3.6). It is worth noting that our dual formulation (3.12) cannot be derived from the results of [24], established using stochastic control theory. Firstly, in view of Remark 3.5, the problem (3.1) and the semimartingale transportation problems studied in [24] are different. Secondly, the existence and duality results in [24] are established under the crucial coercivity assumption (see Assumption 3.3 in [24]), which guarantees tightness of any minimizing sequences of the (primal) problems in [24] and does not hold for the cost functions behaving like $c(a) \sim |a|$ for $a \in S_d^+$, in particular, for our cost f (it is also worth noting that the coercivity Assumption 8.1 (3) in [6] does not hold for our cost f). Thirdly, the coercivity of f is not required in Theorem 3.6. We deduce that the functional F is convex, subadditive and lower semicontinuous in an appropriate sense (see Proposition 3.9). Next, we introduce the weak transport problem (3.19) (static formulation) associated with the problem (3.1) (PDE formulation). We prove that the functional H is greater than or equal to F (see Proposition 3.10), and, using the existing theory of weak optimal transport, we deduce that H is convex, lower semicontinuous in an appropriate sense and obtain a dual formulation for $H(\mu, \nu)$ (see Proposition 3.12). Under the coercivity assumption on f , we, on the one hand, relax and, on the other hand, strengthen the dual constraint in Proposition 3.12 (ii) using bounded lower semicontinuous functions (see Proposition 3.16). Next, we develop the theory of subadditive cost functionals that appeared in [2, Section 6], where the role of \mathbb{R}^d is replaced by the closure of a bounded open convex subset of \mathbb{R}^d . Unlike [2], this paper considers the set of probability measures on \mathbb{R}^d that is not compact, which leads to additional difficulties. We obtain new results either under the coercivity assumption on f (see Propositions 3.17-3.19) or under the growth assumption on f (see Propositions 3.24, 3.25). Dual formulations for $H(\mu, \nu)$ in terms of G -invariant functions are obtained: under the coercivity assumption on f (see Theorem 3.20), under the growth assumption on f (see Theorem 3.26), under both coercivity and growth assumptions on f (see Theorem 3.27). The dual formulation for $F(\mu, \nu)$ in terms of G -invariant functions is obtained in Theorem 3.29. In Section 4, the equality between $F(\mu, \nu)$ and $H(\mu, \nu)$ is proved: under the approximation assumptions on G -invariant functions (see Theorems 4.1, 4.2); under the coercivity and growth assumptions on f (see Theorem 4.14).

2. Preliminaries

2.1. Conventions and Notation

Conventions: in this paper, we say that a value is positive if it is strictly greater than zero, and a value is nonnegative if it is greater than or equal to zero. Euclidean spaces are endowed with the Euclidean inner product $a \cdot b = a^T b$ and the induced norm $|a| = \sqrt{a^T a}$. By d we denote a positive integer.

Notation: for $r > 0$, $B_r(x)$, $\bar{B}_r(x)$, and $\partial B_r(x)$ denote, respectively, the open ball, the closed ball, and the $(d-1)$ -sphere with center x and radius r . Let $C_b(\mathbb{R}^d)$, $C_b^{1,2}((0,1) \times \mathbb{R}^d)$, $C_b^2(\mathbb{R}^d)$ be, respectively, the spaces of bounded continuous functions on \mathbb{R}^d , functions continuously differentiable once in t and twice in x in $(0,1) \times \mathbb{R}^d$ with uniformly bounded $\partial_t \varphi$, $\nabla_x^2 \varphi$, and functions twice continuously differentiable on \mathbb{R}^d whose Hessian is uniformly bounded. We denote by $\Phi_2(\mathbb{R}^d)$ the space of all functions $\varphi \in C(\mathbb{R}^d)$ such that there exists a constant $C > 0$ such that $|\varphi(x)| \leq C(1 + |x|^2)$ for each $x \in \mathbb{R}^d$, and by $\Phi_{b,2}(\mathbb{R}^d)$ the subset of functions in $\Phi_2(\mathbb{R}^d)$ which are bounded from below. Let S_d be the space of real $d \times d$ symmetric matrices endowed with the Hilbert–Schmidt (or Frobenius) inner product $A : B = \text{tr}(A^T B)$ and the induced norm $\|A\| = \sqrt{A : A}$. We write I_d for the $d \times d$ identity matrix and S_d^+ (S_d^{++}) for the set of symmetric nonnegative (positive) definite real $d \times d$ matrices. We write $\mathcal{P}_2((0,1) \times \mathbb{R}^d)$ ($\mathcal{P}_2(\mathbb{R}^d)$) for the set of probability measures μ on $(0,1) \times \mathbb{R}^d$ (\mathbb{R}^d) such that $\int_{(0,1) \times \mathbb{R}^d} |x|^2 d\mu(t, x)$ ($\int_{\mathbb{R}^d} |x|^2 d\mu(x)$) is finite. We endow $\mathcal{P}_2(\mathbb{R}^d)$ with the topology generated by the 2-Wasserstein distance (i.e., in duality with $\Phi_2(\mathbb{R}^d)$). We write $\mathcal{M}((0,1) \times \mathbb{R}^d, S_d^+)$ ($\mathcal{M}(\mathbb{R}^d, S_d^+)$) for the set of measures λ on $(0,1) \times \mathbb{R}^d$ (\mathbb{R}^d) with values in S_d such that for each Borel set $E \subset (0,1) \times \mathbb{R}^d$ ($E \subset \mathbb{R}^d$), $\lambda(E) \in S_d^+$. For a measure λ , we denote by $|\lambda|$ its total variation. By $C_b((0,1) \times \mathbb{R}^d, S_d)$ ($C_b(\mathbb{R}^d, S_d)$) we shall denote the space of all bounded continuous functions on $(0,1) \times \mathbb{R}^d$ (\mathbb{R}^d) with values into S_d . By $\mathcal{S}_b(\mathbb{R}^d)$ ($\mathcal{U}_b(\mathbb{R}^d)$) we shall

denote the set of all bounded lower (upper) semicontinuous functions on \mathbb{R}^d . By $\mathcal{S}_{b,2}(\mathbb{R}^d)$ we denote the set of all lower semicontinuous functions $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ such that φ is bounded from below and for some constant $C > 0$, $|\varphi(x)| \leq C(1 + |x|^2)$ for each $x \in \mathbb{R}^d$. If $U \subset \mathbb{R}^d$ is Lebesgue measurable, then $L^1(U)$ will denote the space consisting of all real measurable functions on U that are integrable on U . By $L^1_{loc}(U)$ we denote the space of functions u such that $u \in L^1(V)$ for all $V \Subset U$. We use the standard notation for Sobolev spaces. For an open set $U \subset \mathbb{R}^d$, denote by $W_0^{1,p}(U)$ the closure of $C_c^\infty(U)$ in the Sobolev space $W^{1,p}(U)$, where $C_c^\infty(U)$ is the space of functions in $C^\infty(U)$ with compact support in U . By \mathcal{L}^r and \mathcal{H}^r we denote the r -dimensional Lebesgue and Hausdorff measure, respectively. For each $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $[\mu]$ and $\text{var}(\mu)$ will denote the barycenter and the variance of μ , respectively. Namely, $[\mu] = \int_{\mathbb{R}^d} x \, d\mu(x)$ and $\text{var}(\mu) = \int_{\mathbb{R}^d} |x|^2 \, d\mu(x) - |[\mu]|^2$.

2.2. Fokker-Planck equation for general measures

Given a pair of measures $(\varrho, \lambda) \in \mathcal{P}_2((0,1) \times \mathbb{R}^d) \times \mathcal{M}((0,1) \times \mathbb{R}^d, S_d^+)$ such that one can disintegrate $\varrho = \varrho_t \otimes \mathcal{L}^1 \llcorner (0,1)$, $\lambda = \lambda_t \otimes \mathcal{L}^1 \llcorner (0,1)$, we say that the Fokker-Planck equation in the weak extended sense of measure-valued solutions

$$\partial_t \varrho = \text{tr} \left(\frac{1}{2} \nabla^2 \lambda \right) \quad \text{in } (0,1) \times \mathbb{R}^d \quad (\text{GFPE})$$

holds if

$$\int_0^1 |\lambda_t|(\mathbb{R}^d) \, dt < +\infty$$

and for each $\varphi \in C_b^{1,2}((0,1) \times \mathbb{R}^d)$ with (closed) support in $(0,1) \times \mathbb{R}^d$,

$$\int_0^1 \int_{\mathbb{R}^d} \partial_t \varphi(t, x) \, d\varrho_t(x) \, dt = - \int_0^1 \int_{\mathbb{R}^d} \frac{1}{2} \nabla_x^2 \varphi(t, x) : d\lambda_t(x) \, dt. \quad (2.1)$$

If $(\varrho, \lambda) \in \mathcal{P}_2((0,1) \times \mathbb{R}^d) \times \mathcal{M}((0,1) \times \mathbb{R}^d, S_d^+)$ is a solution to (GFPE), then there exists a narrowly continuous curve $(\tilde{\varrho}_t)_{t \in [0,1]} \subset \mathcal{P}_2(\mathbb{R}^d)$ such that $\tilde{\varrho}_t = \varrho_t$ for \mathcal{L}^1 -a.e. $t \in (0,1)$. Moreover, if $\psi \in C_b^2(\mathbb{R}^d)$ and $0 \leq t_1 \leq t_2 \leq 1$, then

$$\int_{\mathbb{R}^d} \psi(x) \, d\tilde{\varrho}_{t_2}(x) - \int_{\mathbb{R}^d} \psi(x) \, d\tilde{\varrho}_{t_1}(x) = \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{1}{2} \nabla^2 \psi(x) : d\lambda_t(x) \, dt \quad (2.2)$$

(the reader may consult [3, Lemma 8.1.2] and [25, Remark 2.3]). Thus, for a solution (ϱ, λ) to (GFPE), without loss of generality, we shall assume that $(\varrho_t)_{t \in (0,1)}$ is narrowly continuous.

Proposition 2.1. *Let (ϱ, λ) solve (GFPE), $\mu = w * -\lim_{t \searrow 0} \varrho_t$ and $\nu = w * -\lim_{t \nearrow 1} \varrho_t$. If $\psi \in C_b^2(\mathbb{R}^d)$ is convex, the function $t \in [0,1] \mapsto \int_{\mathbb{R}^d} \psi \, d\varrho_t$ is nondecreasing. In particular, $\int_{\mathbb{R}^d} \psi \, d\mu \leq \int_{\mathbb{R}^d} \psi \, d\nu$.*

Proof. For every $0 \leq t_1 \leq t_2 \leq 1$, using (2.2) and the fact that ψ is convex and $\lambda_t \in \mathcal{M}(\mathbb{R}^d, S_d^+)$, we have

$$\int_{\mathbb{R}^d} \psi(x) \, d\varrho_{t_2}(x) = \int_{\mathbb{R}^d} \psi(x) \, d\varrho_{t_1}(x) + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{1}{2} \nabla^2 \psi(x) : d\lambda_t(x) \, dt \geq \int_{\mathbb{R}^d} \psi(x) \, d\varrho_{t_1}(x),$$

which completes our proof of Proposition 2.1. \square

For convenience, we recall the next definition (see [23]).

Definition 2.2. We say that measures $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ are in convex order, and we write $\mu \leq_c \nu$, if for each convex function $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$, it holds

$$\int_{\mathbb{R}^d} \psi(x) \, d\mu(x) \leq \int_{\mathbb{R}^d} \psi(x) \, d\nu(x).$$

Remark 2.3. Notice that if $\mu \leq_c \nu$, then $\int_{\mathbb{R}^d} \psi \, d\mu = \int_{\mathbb{R}^d} \psi \, d\nu$ for each affine map $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$. Since every convex function is nonnegative up to the addition of some affine map, $\mu \leq_c \nu$ if and only if for each

convex function $\varphi : \mathbb{R}^d \rightarrow [0, +\infty)$, $\langle \nu - \mu, \varphi \rangle \geq 0$. For each convex function $\varphi : \mathbb{R}^d \rightarrow [0, +\infty)$, there exists a sequence $(\varphi_k)_{k \in \mathbb{N}}$ of convex functions $\varphi_k : \mathbb{R}^d \rightarrow [0, +\infty)$ such that $\varphi_k(\cdot) = \inf\{\varphi(y) + k|\cdot - y| \mid y \in \mathbb{R}^d\}$ is k -Lipschitz and $\varphi_k \nearrow \varphi$ as $k \rightarrow +\infty$. For each $k \in \mathbb{N}$, using convolution with $\eta_\varepsilon(\cdot) = \varepsilon^{-d}\eta(\cdot/\varepsilon)$, where $\varepsilon > 0$ and η is a standard mollifier (namely, $\eta \in C_c^\infty(\mathbb{R}^d)$, $\text{supp}(\eta) = \overline{B}_1(0)$, $\eta \geq 0$, $\eta(x) = \eta(-x)$ and $\int_{\mathbb{R}^d} \eta \, dx = 1$), φ_k can be uniformly approximated on \mathbb{R}^d by a sequence of nonnegative convex smooth k -Lipschitz functions. Again, using the above mollification procedure, each convex smooth Lipschitz function on \mathbb{R}^d can be uniformly approximated by a sequence of convex functions lying in $C_b^2(\mathbb{R}^d)$. After all, taking into account the above monotone and uniform convergences, for each convex function $\varphi : \mathbb{R}^d \rightarrow [0, +\infty)$ we can find a sequence $(\varphi_k)_{k \in \mathbb{N}} \subset C_b^2(\mathbb{R}^d)$ of nonnegative convex functions such that $\int_{\mathbb{R}^d} \varphi_k \, d\mu \rightarrow \int_{\mathbb{R}^d} \varphi \, d\mu$ and $\int_{\mathbb{R}^d} \varphi_k \, d\nu \rightarrow \int_{\mathbb{R}^d} \varphi \, d\nu$ as $k \rightarrow +\infty$. Therefore, $\mu \leq_c \nu$ if and only if $\langle \nu - \mu, \varphi \rangle \geq 0$ for each convex function $\varphi \in C_b^2(\mathbb{R}^d)$.

2.3. Subharmonic functions

Definition 2.4. A function $u : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty\}$ is said to be subharmonic if it satisfies the following conditions.

- (i) u is not identically equal to $-\infty$.
- (ii) u is upper semicontinuous.
- (iii) For each $x \in \mathbb{R}^d$ and $r > 0$,

$$u(x) \leq \int_{\partial B_r(x)} u(y) \, d\mathcal{H}^{d-1}(y),$$

$$\text{where } \int_{\partial B_r(x)} u(y) \, d\mathcal{H}^{d-1}(y) = \frac{1}{\mathcal{H}^{d-1}(\partial B_r(x))} \int_{\partial B_r(x)} u(y) \, d\mathcal{H}^{d-1}(y).$$

A function $v : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be superharmonic if $u = -v$ is subharmonic. A function is harmonic if and only if it is both subharmonic and superharmonic. If $u \in C^2(\mathbb{R}^d)$, then u is subharmonic if and only if $\Delta u \geq 0$ in \mathbb{R}^d (see, for instance, [21, Section 3.2]). In one dimension, a function is subharmonic if and only if it is convex; however, in dimension $d \geq 2$ the notions of subharmonicity and convexity are not equivalent (for more details, the reader may consult, for instance, [21]). We shall denote by $\mathcal{SH}(\mathbb{R}^d)$ the cone of all subharmonic functions on \mathbb{R}^d .

It is also worth recalling the notion of the so-called *subharmonic order*, which is stronger and more natural than the convex order in dimension $d \geq 2$ and is equivalent to the convex order in one dimension.

Definition 2.5. We say that measures $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ are in subharmonic order, and we write $\mu \leq_{sh} \nu$, if for each $u \in C_b^2(\mathbb{R}^d) \cap \mathcal{SH}(\mathbb{R}^d)$ the following holds

$$\int_{\mathbb{R}^d} u \, d\mu \leq \int_{\mathbb{R}^d} u \, d\nu.$$

If $\mu \leq_{sh} \nu$, using a standard mollification procedure, we observe that $\int_{\mathbb{R}^d} u \, d\mu \leq \int_{\mathbb{R}^d} u \, d\nu$ for each bounded or Lipschitz $u \in \mathcal{SH}(\mathbb{R}^d)$.

3. A PDE constrained optimization problem

For each $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, we consider the following minimization problem

$$F(\mu, \nu) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^d} f \left(\frac{d\lambda_t}{d|\lambda_t|} \right) \, d|\lambda_t| \, dt \mid \partial_t \varrho = \text{tr} \left(\frac{1}{2} \nabla^2 \lambda \right), \varrho_0 = \mu, \varrho_1 = \nu \right\}, \quad (3.1)$$

where the function $f : S_d \rightarrow [0, +\infty]$ with $\text{dom}(f) := \{M \in S_d \mid f(M) < +\infty\} \subset S_d^+$ such that $\text{dom}(f) \cap S_d^{++}$ is dense in $\text{dom}(f)$ satisfies the following assumptions.

$$f \text{ is sublinear.} \tag{A1}$$

$$f \text{ is lower semicontinuous.} \tag{A2}$$

Recall that $f : S_d \rightarrow [0, +\infty]$ is said to be sublinear (see [14] or Section 4 in [23]) if it is subadditive and positively 1-homogeneous, which means that

$$f(A + B) \leq f(A) + f(B) \text{ and } f(tA) = tf(A) \text{ for all } A, B \in S_d \text{ and } t \geq 0.$$

Among the interesting examples of such functions, we emphasize the following.

- (i) $f(A) = t$ if $A = tI_d$ for some $t \geq 0$ and $f(A) = +\infty$ otherwise.
- (ii) For some $\xi \in \mathbb{R}^d$, $f(A) = A : \xi \otimes \xi$ if $A \in S_d^+$ and $f(A) = +\infty$ otherwise.
- (iii) $f(A) = \text{tr}(A)$ if $A \in S_d^+$ and $f(A) = +\infty$ otherwise.
- (iv) $f(A)$ is the largest eigenvalue of A if $A \in S_d^+$ and $f(A) = +\infty$ otherwise.

Notice that in the example (i) $\text{dom}(f)$ is a proper convex subset of S_d^+ , and in the example (ii) the function f is not coercive.

Using the 1-homogeneity of f , we compute its Fenchel conjugate. For each $A \in S_d$,

$$f^*(A) = \sup \{A : M - f(M) \mid M \in S_d\} = \begin{cases} 0 & \text{if } A : M \leq f(M) \ \forall M \in \text{dom}(f) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.2}$$

Remark 3.1. If $F(\mu, \nu) < +\infty$, then $\mu \leq_c \nu$ (see Proposition 2.1 and Remark 2.3), which implies that $[\mu] = [\nu]$.

Using the 1-homogeneity of f , we eliminate the time variable in (3.1) via the equation

$$\text{tr} \left(\frac{1}{2} \nabla^2 \lambda \right) = \nu - \mu \text{ in } \mathbb{R}^d, \tag{3.3}$$

which means that $\lambda \in \mathcal{M}(\mathbb{R}^d, S_d^+)$, $|\lambda|(\mathbb{R}^d) < +\infty$ and

$$\int_{\mathbb{R}^d} \frac{1}{2} \nabla^2 \varphi(x) : d\lambda(x) = \int_{\mathbb{R}^d} \varphi(x) d\nu(x) - \int_{\mathbb{R}^d} \varphi(x) d\mu(x) \quad \forall \varphi \in C_b^2(\mathbb{R}^d). \tag{3.4}$$

Proposition 3.2. For each $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ the following holds

$$F(\mu, \nu) = \inf \left\{ \int_{\mathbb{R}^d} f \left(\frac{d\lambda}{d|\lambda|} \right) d|\lambda| \mid \text{tr} \left(\frac{1}{2} \nabla^2 \lambda \right) = \nu - \mu \right\}. \tag{3.5}$$

Remark 3.3. Since f is positively 1-homogeneous, for each positive finite measure m on \mathbb{R}^d such that $|\lambda| \ll m$, it holds

$$\int_{\mathbb{R}^d} f \left(\frac{d\lambda}{dm} \right) dm = \int_{\mathbb{R}^d} f \left(\frac{d\lambda}{d|\lambda|} \right) d|\lambda|.$$

Proof of Proposition 3.2. Let $\lambda \in \mathcal{M}(\mathbb{R}^d, S_d^+)$ be a solution to (3.3). For each $t \in [0, 1]$, define

$$\varrho_t = (1 - t)\mu + t\nu \in \mathcal{P}_2(\mathbb{R}^d). \tag{3.6}$$

Fix an arbitrary $\varphi \in C_b^{1,2}((0, 1) \times \mathbb{R}^d)$ with (closed) support in $(0, 1) \times \mathbb{R}^d$. Then, using Fubini's theorem and integrating by parts, we have

$$\int_0^1 \int_{\mathbb{R}^d} \partial_t \varphi(t, x) d\varrho_t(x) dt = \int_{\mathbb{R}^d} \int_0^1 \varphi(t, x) dt d\mu(x) - \int_{\mathbb{R}^d} \int_0^1 \varphi(t, x) dt d\nu(x). \tag{3.7}$$

Since $\text{tr}(\frac{1}{2}\nabla^2\lambda) = \nu - \mu$ and $\int_0^1 \varphi(t, \cdot) dt \in C_b^2(\mathbb{R}^d)$, using (3.4) and Fubini's theorem, we deduce that

$$\int_0^1 \int_{\mathbb{R}^d} \frac{1}{2} \nabla_x^2 \varphi(t, x) : d\lambda(x) dt = \int_{\mathbb{R}^d} \int_0^1 \varphi(t, x) dt d\nu(x) - \int_{\mathbb{R}^d} \int_0^1 \varphi(t, x) dt d\mu(x). \quad (3.8)$$

Combining (3.7) and (3.8), we get

$$\int_0^1 \int_{\mathbb{R}^d} \partial_t \varphi(t, x) d\varrho_t(x) dt = - \int_0^1 \int_{\mathbb{R}^d} \frac{1}{2} \nabla_x^2 \varphi(t, x) : d\lambda(x) dt,$$

and hence the pair $(\varrho_t \otimes \mathcal{L}^1 \llcorner (0, 1), \lambda \otimes \mathcal{L}^1 \llcorner (0, 1))$ is a solution to (GFPE). Thus,

$$F(\mu, \nu) \leq \inf \left\{ \int_{\mathbb{R}^d} f \left(\frac{d\lambda}{d|\lambda|} \right) d|\lambda| \mid \text{tr} \left(\frac{1}{2} \nabla^2 \lambda \right) = \nu - \mu \right\}. \quad (3.9)$$

Now assume that $F(\mu, \nu) < +\infty$ and let $(\varrho_t \otimes \mathcal{L}^1 \llcorner (0, 1), \lambda_t \otimes \mathcal{L}^1 \llcorner (0, 1))$ be an admissible pair for (3.1). Using (2.2) with $t_1 = 0$ and $t_2 = 1$, for each $\varphi \in C_b^2(\mathbb{R}^d)$, we obtain

$$\int_0^1 \int_{\mathbb{R}^d} \frac{1}{2} \nabla^2 \varphi(x) : d\lambda_t(x) dt = \int_{\mathbb{R}^d} \varphi(x) d\nu(x) - \int_{\mathbb{R}^d} \varphi(x) d\mu(x). \quad (3.10)$$

Define

$$\lambda = \pi_{\#}^x (\lambda_t \otimes \mathcal{L}^1 \llcorner (0, 1)) \in \mathcal{M}(\mathbb{R}^d, S_d^+),$$

where $\pi^x(t, x) = x$ for each $(t, x) \in [0, 1] \times \mathbb{R}^d$. Then $|\lambda|(\mathbb{R}^d) < +\infty$ and for each $\xi \in C_b(\mathbb{R}^d, S_d)$,

$$\int_{\mathbb{R}^d} \xi(x) : d\lambda(x) = \int_0^1 \int_{\mathbb{R}^d} \xi(\pi^x(t, x)) : d\lambda_t(x) dt. \quad (3.11)$$

This, together with (3.10), implies that

$$\int_{\mathbb{R}^d} \frac{1}{2} \nabla^2 \varphi(x) : d\lambda(x) = \int_{\mathbb{R}^d} \varphi(x) d\nu(x) - \int_{\mathbb{R}^d} \varphi(x) d\mu(x) \quad \forall \varphi \in C_b^2(\mathbb{R}^d).$$

Thus, λ is a solution to (3.3). Next, we define the convex functional $\Psi : (C_b(\mathbb{R}^d, S_d))' \rightarrow [0, +\infty]$ by

$$\Psi(\sigma) = \begin{cases} \int_{\mathbb{R}^d} f \left(\frac{d\sigma}{d|\sigma|} \right) d|\sigma| & \text{if } \sigma \in \mathcal{M}(\mathbb{R}^d, S_d^+), |\sigma|(\mathbb{R}^d) < +\infty \\ +\infty & \text{otherwise.} \end{cases}$$

For each $\xi \in C_b(\mathbb{R}^d, S_d)$,

$$\Psi^*(\xi) = \begin{cases} 0 & \text{if } \xi(\mathbb{R}^d) \subset \text{dom}(f^*) \\ +\infty & \text{otherwise.} \end{cases}$$

Indeed, if $\xi(x_0) \notin \text{dom}(f^*)$, then there exists $A \in \text{dom}(f)$ such that $\xi(x_0) : A > f(A)$. Since ξ is continuous, there exists $r > 0$ such that $\xi(x) : A > f(A)$ for each $x \in B_r(x_0)$. Defining $\sigma_n = nA\mathcal{L}^d \llcorner B_r(x_0)$, we have

$$\Psi^*(\xi) \geq \int_{\mathbb{R}^d} \xi(x) : d\sigma_n(x) - \int_{\mathbb{R}^d} f \left(\frac{d\sigma_n}{d|\sigma_n|} \right) d|\sigma_n| = n \int_{B_r(x_0)} (\xi(x) : A - f(A)) dx > 0.$$

Letting n tend to $+\infty$, we deduce that $\Psi^*(\xi) = +\infty$.

Observe that for each $\sigma \in \mathcal{M}(\mathbb{R}^d, S_d^+)$ such that $|\sigma|(\mathbb{R}^d) < +\infty$, Ψ is (weakly) lower semicontinuous at σ and hence $\Psi^{**}(\sigma) = \Psi(\sigma)$ (see [9, Theorem 2.1 (i)]). Thus, the following holds

$$\begin{aligned} \int_{\mathbb{R}^d} f \left(\frac{d\lambda}{d|\lambda|} \right) d|\lambda| &= \sup \left\{ \int_{\mathbb{R}^d} \xi(x) : d\lambda(x) \mid \xi \in C_b(\mathbb{R}^d, S_d), \xi(\mathbb{R}^d) \subset \text{dom}(f^*) \right\} \\ &= \sup \left\{ \int_0^1 \int_{\mathbb{R}^d} \xi(\pi^x(t, x)) : d\lambda_t(x) dt \mid \xi \in C_b(\mathbb{R}^d, S_d), \xi(\mathbb{R}^d) \subset \text{dom}(f^*) \right\} \\ &\leq \sup \left\{ \int_0^1 \int_{\mathbb{R}^d} \xi(t, x) : d\lambda_t(x) dt \mid \xi \in C_b((0, 1) \times \mathbb{R}^d, S_d), \xi((0, 1) \times \mathbb{R}^d) \subset \text{dom}(f^*) \right\} \\ &\leq \int_0^1 \int_{\mathbb{R}^d} f \left(\frac{d\lambda_t}{d|\lambda_t|} \right) d|\lambda_t| dt. \end{aligned}$$

Using this together with the fact that λ is a solution to (3.3) and taking into account (3.9), we deduce (3.5), which completes our proof of Proposition 3.2. \square

As a byproduct of the proof of Proposition 3.2, we obtain the next

Corollary 3.4. If the infimum in (3.1) is achieved on $(\varrho_t \otimes \mathcal{L}^1 \llcorner (0, 1), \lambda_t \otimes \mathcal{L}^1 \llcorner (0, 1))$, then the infimum in (3.5) is achieved on $\lambda = \pi_{\#}^x(\lambda_t \otimes \mathcal{L}^1 \llcorner (0, 1))$. Conversely, if the infimum in (3.5) is achieved on λ , then the infimum in (3.1) is achieved on $(\varrho_t \otimes \mathcal{L}^1 \llcorner (0, 1), \lambda_t \otimes \mathcal{L}^1 \llcorner (0, 1))$, where $\varrho_t = (1-t)\mu + t\nu$ and $\lambda_t = \lambda$.

Remark 3.5. In general, one cannot find a martingale $(X_t)_{t \in [0,1]}$ (for the definition, see, for instance, Section 2 in [18]) with continuous paths whose marginals are $\varrho_t = (1-t)\mu + t\nu$ for $t \in [0,1]$. Indeed, if $\mu = \delta_{\frac{x+y}{2}}$ and $\nu = \frac{1}{2}(\delta_x + \delta_y)$, the optimal curve $\varrho_t = (1-t)\mu + t\nu$ is absolutely continuous in the 1-Wasserstein distance, but it is not absolutely continuous in the p -Wasserstein distance with $p > 1$, which from the particle point of view means that the particles must jump from $\frac{x+y}{2}$ to x or y at a certain rate. This phenomenon occurs because, in contrast to [18], in our case the cost f is not p -admissible, which correlates with the fact that for a solution (ϱ, λ) of the problem (3.1), λ does not have to be absolutely continuous with respect to ϱ (the reader may also consult [10, Remark 3.3]).

3.1. Existence and duality

We introduce a dual formulation to (3.1) and prove the existence of a minimizer when $F(\mu, \nu) < +\infty$.

Theorem 3.6. For each $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ the following equality holds

$$F(\mu, \nu) = \sup \left\{ \langle \nu - \mu, \psi \rangle \mid \psi \in C_b^2(\mathbb{R}^d), -f^* \left(\frac{1}{2} \nabla^2 \psi \right) = 0 \text{ in } \mathbb{R}^d \right\}. \quad (3.12)$$

Moreover, if $F(\mu, \nu) < +\infty$, then the infimum in (3.1) (and in (3.5)) is actually a minimum.

Remark 3.7. The function f is not strictly convex and, generally speaking, is not coercive (see, for instance, the example (ii)). Notice that in [18, Theorem 4.3] the strict convexity and p -coercivity (for some $p > 1$; see Section 2 in [18] for the definition) of the cost function are used to prove the existence of a minimizer of the primal problem. In particular, the fact that the Fenchel conjugate of the cost function is finite on $\{tI_d \mid t \geq 0\}$ (since the cost function in [18] is p -coercive) is used to prove that an optimum of the primal problem in [18, Theorem 4.3] solves the Fokker-Planck equation, where the diffusion term is weighted accordingly with the mass. In our setting, if $\text{dom}(f^*)$ contained $\{tI_d \mid t \geq 0\}$, we would have $f = +\infty$ identically on S_d^{++} . It is also worth noting that our dual formulation (3.12) cannot be derived from the results of [24], established using stochastic control theory. Indeed, in view of Remark 3.5, the problem (3.1) and the semimartingale transportation problems studied in [24] are different. Furthermore, the existence and duality results in [24] are established under the crucial coercivity assumption (see Assumption 3.3 in [24]), which guarantees tightness of any minimizing sequences of the (primal) problems in [24] and does not hold for the cost functions behaving like $c(a) \sim |a|$ for $a \in S_d^+$, in particular, for our cost f (notice also that the Assumption 8.1 (3) in [6] does not hold for f). Thirdly, the coercivity of f is not required in Theorem 3.6.

Proof of Theorem 3.6. Define the functional $\Psi : (C_b(\mathbb{R}^d, S_d))' \rightarrow [0, +\infty]$ by

$$\Psi(\sigma) = \begin{cases} \int_{\mathbb{R}^d} f \left(\frac{d\sigma}{d|\sigma|} \right) d|\sigma| & \text{if } \sigma \in \mathcal{M}(\mathbb{R}^d, S_d^+), |\sigma|(\mathbb{R}^d) < +\infty \\ +\infty & \text{otherwise.} \end{cases}$$

Observe that Ψ is convex and for each $\sigma \in \mathcal{M}(\mathbb{R}^d, S_d^+)$ such that $|\sigma|(\mathbb{R}^d) < +\infty$, Ψ is lower semicontinuous at σ and hence $\Psi^{**}(\sigma) = \Psi(\sigma)$ (see [9, Theorem 2.1 (i)]). Proceeding in the same way as in the proof of Proposition 3.2, for each $\xi \in C_b(\mathbb{R}^d, S_d)$, we deduce that

$$\Psi^*(\xi) = \begin{cases} 0 & \text{if } \xi(\mathbb{R}^d) \subset \text{dom}(f^*) \\ +\infty & \text{otherwise.} \end{cases}$$

We write the dual pairing as $\sigma(\xi)$ for $\sigma \in (C_b(\mathbb{R}^d, S_d))'$ and $\xi \in C_b(\mathbb{R}^d, S_d)$. Assume by contradiction that $\Psi^{**}(\sigma) < +\infty$ and $\sigma(\xi) > 0$ for some $\xi \in C_b(\mathbb{R}^d, S_d)$ such that $-\xi \in C_b(\mathbb{R}^d, S_d^+)$. Since $\Psi^*(t\xi) = 0$ for each $t \geq 0$, $\Psi^{**}(\sigma) \geq t\sigma(\xi) > 0$. Letting t tend to $+\infty$, we obtain $\Psi^{**}(\sigma) = +\infty$, which leads to a contradiction. This proves the following fact: if $\Psi^{**}(\sigma) < +\infty$, then $\sigma(\xi) \geq 0$ for each $\xi \in C_b(\mathbb{R}^d, S_d^+)$.

Next, following [18], we say that $\xi \in C_b(\mathbb{R}^d, S_d)$ is represented by $\varphi \in C_b^2(\mathbb{R}^d)$ if $\xi = -\frac{1}{2}\nabla^2\varphi$. We define $\Theta : C_b(\mathbb{R}^d, S_d) \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\Theta(\xi) = \begin{cases} \langle \mu - \nu, \varphi \rangle & \text{if } \xi \text{ is represented by } \varphi \\ +\infty & \text{otherwise.} \end{cases}$$

If φ_1 and φ_2 represent ξ , then $\frac{1}{2}\nabla^2(\varphi_1 - \varphi_2) = 0$, and hence $\varphi_1(x) = a + y \cdot x + \varphi_2(x)$ for some fixed $a \in \mathbb{R}$ and $y \in \mathbb{R}^d$. Without loss of generality, we can assume that $\int_{\mathbb{R}^d}(a + y \cdot x) d\mu(x) = \int_{\mathbb{R}^d}(a + y \cdot x) d\nu(x)$, because otherwise $F(\mu, \nu)$ and the supremum in (3.12) are equal to $+\infty$: $F(\mu, \nu) = +\infty$ because μ and ν would not be in convex order; the supremum in (3.12) is equal to $+\infty$ by letting $\psi(x) = ty \cdot x$, which satisfies $-f^*(\frac{1}{2}\nabla^2\psi) = 0$ in \mathbb{R}^d , and letting t tend to $\pm\infty$ depending on the sign of the difference. This implies that $\langle \mu - \nu, \varphi_1 - \varphi_2 \rangle = 0$. Thus, Θ does not depend on the choice of φ . It is also worth noting that the set of represented mappings $\xi \in C_b(\mathbb{R}^d, S_d)$ is a linear subspace on which Θ is linear. In particular, Θ is positively 1-homogeneous with the Fenchel conjugate

$$\Theta^*(\sigma) = \sup \{ \sigma(\xi) + \langle \nu - \mu, \varphi \rangle \mid \xi \text{ is represented} \}$$

taking values in $\{0, +\infty\}$ and $\Theta^*(\sigma) = 0$ if and only if

$$\int_{\mathbb{R}^d} \frac{1}{2}\nabla^2\varphi : d\sigma = \int_{\mathbb{R}^d} \varphi d\nu - \int_{\mathbb{R}^d} \varphi d\mu \quad \forall \varphi \in C_b^2(\mathbb{R}^d), \quad (3.13)$$

where we interpret the integral on the left-hand side of the above equality as a duality pairing.

We have proved that if $\Psi^{**}(\sigma) < +\infty$, then σ is a nonnegative bounded linear functional. We claim that σ is tight and hence induced by a measure. Let $g : \mathbb{R} \rightarrow [0, 1]$ be a smooth nondecreasing function such that $g(t) = 0$ for $t \in (-\infty, 1/2]$, $g(t) = 1$ for $t \in [1, +\infty)$ and $|g'(t)|, |g''(t)| \leq 4$ for $t \in \mathbb{R}$. Define the function $h : \mathbb{R} \rightarrow [0, +\infty)$ by $h(t) = \int_{-\infty}^t g(s) ds$. Then h is a smooth function, $h'(t) = g(t)$ and $h''(t) = g'(t) \geq 0$ for $t \in \mathbb{R}$. For $L > 0$ and $x \in \mathbb{R}^d$, we set $\varphi^L(x) = h(|x|^2 - L^2)$. Notice that $\xi^L(x) = -(g(|x|^2 - L^2)I_d + 2g'(|x|^2 - L^2)x \otimes x)$ is represented by φ^L and $-\xi^L \geq I_d$ on $\mathbb{R}^d \setminus B_{\sqrt{L^2+1}}(0)$. Since $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $h(|x|^2 - L^2) \leq |x|^2 - L^2 \leq |x|^2$ on $\mathbb{R}^d \setminus B_L(0)$, for each fairly small $\varepsilon > 0$, there exists $L > 0$ large enough such that

$$\int_{\mathbb{R}^d} h(|x|^2 - L^2) d\nu(x) < \varepsilon. \quad (3.14)$$

For each $\xi \in C_b(\mathbb{R}^d, S_d)$ such that $|\xi| \leq 1$ and $\text{supp}(\xi) \subset \mathbb{R}^d \setminus B_{\sqrt{L^2+1}}(0)$, it holds $\xi \leq -\xi^L$ and $-\xi \leq -\xi^L$. Using this, the facts that σ is a nonnegative linear functional and μ is a nonnegative measure, (3.13) and (3.14), we deduce the following

$$|\sigma(\xi)| \leq \sigma(-\xi^L) = \int_{\mathbb{R}^d} h(|x|^2 - L^2) d(\nu(x) - \mu(x)) < \varepsilon,$$

which proves that σ is tight and hence induced by a measure.

The mapping I_d is represented by $\varphi(x) = -|x|^2$, Ψ^* is continuous at $-I_d$ and $\Theta(I_d) < +\infty$. After all, applying the formula for the conjugate of the sum $\Psi^*(-\cdot) + \Theta(\cdot)$ at $0 \in (C_b(\mathbb{R}^d, S_d))'$ (see, for instance, [9, Proposition 2.3 (i)] or [11, Theorem 1.12]), we obtain

$$\inf \{ \Psi^{**}(\sigma) + \Theta^*(\sigma) \mid \sigma \in (C_b(\mathbb{R}^d, S_d))' \} = \sup \{ -\Psi^*(-\xi) - \Theta(\xi) \mid \xi \in C_b(\mathbb{R}^d, S_d) \}, \quad (3.15)$$

where the infimum is actually a minimum if the supremum, coinciding with the supremum in (3.12), is finite. The latter holds if and only if $F(\mu, \nu) < +\infty$. Indeed, if the supremum in (3.15) is finite, then according to [9, Proposition 2.3 (ii)] (or [11, Theorem 1.12]), the infimum in (3.15) is actually a minimum

and if σ is a minimizer, then we have proved that $\sigma \in \mathcal{M}(\mathbb{R}^d, S_d^+)$, $|\sigma|(\mathbb{R}^d) < +\infty$, σ solves (3.3) for μ and ν (see (3.13)) and $\Psi(\sigma) = \Psi^{**}(\sigma)$. This, together with Proposition 3.2, implies that the left-hand side in (3.15) coincides with $F(\mu, \nu)$, and the infimum in (3.1) (and in (3.5)) is actually a minimum. On the other hand, if $F(\mu, \nu) < +\infty$, then there exists $\sigma \in \mathcal{M}(\mathbb{R}^d, S_d^+)$ solving (3.3) for μ and ν . For each $\psi \in C_b^2(\mathbb{R}^d)$ such that $-f^*(\frac{1}{2}\nabla^2\psi) = 0$ in \mathbb{R}^d , we have $\frac{1}{2}\nabla^2\psi : \frac{d\sigma}{d|\sigma|} \leq f(\frac{d\sigma}{d|\sigma|})$ $|\sigma|$ -a.e. on \mathbb{R}^d . Hence

$$\langle \nu - \mu, \psi \rangle = \int_{\mathbb{R}^d} \frac{1}{2} \nabla^2 \psi : d\sigma \leq \int_{\mathbb{R}^d} f\left(\frac{d\sigma}{d|\sigma|}\right) d|\sigma|,$$

which implies that the supremum in (3.15) is less than or equal to $\int_{\mathbb{R}^d} f(\frac{d\sigma}{d|\sigma|}) d|\sigma| < +\infty$. This completes our proof of Theorem 3.6. \square

Corollary 3.8. For each $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ the following estimate holds

$$F(\mu, \nu) \geq f\left(\int_{\mathbb{R}^d} x \otimes x (d\nu(x) - d\mu(x))\right). \quad (3.16)$$

Proof of Corollary 3.8. Let $A \in \text{dom}(f^*)$ be arbitrary and $\psi(x) = Ax \cdot x$ for each $x \in \mathbb{R}^d$. Then we have $-f^*(\frac{1}{2}\nabla^2\psi) = -f^*(A) = 0$ in \mathbb{R}^d , which, in view of Theorem 3.6, yields

$$F(\mu, \nu) \geq \int_{\mathbb{R}^d} Ax \cdot x d\nu(x) - \int_{\mathbb{R}^d} Ax \cdot x d\mu(x) = A : \left(\int_{\mathbb{R}^d} x \otimes x d\nu(x) - \int_{\mathbb{R}^d} x \otimes x d\mu(x)\right). \quad (3.17)$$

Since $A \in \text{dom}(f^*)$ was arbitrarily chosen, $f^* = 0$ on $\text{dom}(f^*)$ and $f = f^{**}$ (this comes from (A1), (A2) and [9, Theorem 2.1 (i)]), (3.17) implies (3.16), which completes our proof of Corollary 3.8. \square

3.2. Lower semicontinuity, convexity and subadditivity

Proposition 3.9. *The following assertions hold.*

(i) *F is convex on $\mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d)$ and lower semicontinuous with respect to the weak topology on $\mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d)$ in duality with $\Phi_2(\mathbb{R}^d) \times \Phi_2(\mathbb{R}^d)$.*

(ii) *For each choice of $\mu_1, \mu_2, \mu_3 \in \mathcal{P}_2(\mathbb{R}^d)$,*

$$F(\mu_1, \mu_3) \leq F(\mu_1, \mu_2) + F(\mu_2, \mu_3).$$

Proof. It is a direct consequence of Theorem 3.6 that F is convex and lower semicontinuous with respect to the specified product topology, since it is represented as the supremum of the family consisting of linear functionals that are continuous with respect to this topology. This proves (i).

Let us prove (ii). For each $\psi \in C_b^2(\mathbb{R}^d)$ such that $-f^*(\frac{1}{2}\nabla^2\psi) = 0$ in \mathbb{R}^d , using Theorem 3.6, we have

$$\langle \mu_3 - \mu_1, \psi \rangle = \langle \mu_2 - \mu_1, \psi \rangle + \langle \mu_3 - \mu_2, \psi \rangle \leq F(\mu_1, \mu_2) + F(\mu_2, \mu_3),$$

which implies (ii) and completes our proof of Proposition 3.9. \square

3.3. Associated weak transport problem

We define the cost function $G : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty]$ by

$$G(x, p) = F(\delta_x, p). \quad (3.18)$$

By Proposition 3.9, G is lower semicontinuous in (x, p) and convex in p . For each $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, we consider the following weak transport problem

$$\inf \left\{ \int_{\mathbb{R}^d} G(x, \gamma^x) d\mu(x) \mid \gamma \in \Pi(\mu, \nu) \right\} \quad (3.19)$$

and define the functional $H : \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty]$ by

$$H(\mu, \nu) = \inf \left\{ \int_{\mathbb{R}^d} G(x, \gamma^x) d\mu(x) \mid \gamma \in \Pi(\mu, \nu) \right\}. \quad (3.20)$$

Since $G(x, \delta_x) = 0$ for each $x \in \mathbb{R}^d$ and $\gamma^x \otimes \mu \in \Pi(\mu, \mu)$ when $\gamma^x = \delta_x$ for μ -a.e. $x \in \mathbb{R}^d$,

$$H(\mu, \mu) = F(\mu, \mu) = 0 \quad \forall \mu \in \mathcal{P}_2(\mathbb{R}^d), \quad (3.21)$$

which implies that the functional H is proper (i.e., $\text{dom}(H) \neq \emptyset$).

We shall prove the equality $F = H$. First, we show that $H \geq F$, which is a consequence of Theorem 3.6. To establish the converse inequality, which is a delicate matter, we develop the dual result of [5] and the theory for subadditive costs that appeared in [2, Section 6], where the role of \mathbb{R}^d is replaced by the closure of a bounded open convex subset of \mathbb{R}^d . The main difficulty is that, unlike [2], we work with the set of probability measures on \mathbb{R}^d , which is not compact with respect to the weak topology.

Proposition 3.10. *For each $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, $H(\mu, \nu) \geq F(\mu, \nu)$.*

Proof. Fix an arbitrary $\gamma = \gamma^x \otimes \mu \in \Pi(\mu, \nu)$. According to the definition of G and Theorem 3.6, for μ -a.e. $x \in \mathbb{R}^d$ and for each $\psi \in C_b^2(\mathbb{R}^d)$ such that $-f^*(\frac{1}{2}\nabla^2\psi) = 0$ in \mathbb{R}^d , we have

$$G(x, \gamma^x) \geq \int_{\mathbb{R}^d} \psi(y) d\gamma^x(y) - \psi(x).$$

Integrating both sides of the above inequality over \mathbb{R}^d with respect to μ , we obtain

$$\int_{\mathbb{R}^d} G(x, \gamma^x) d\mu(x) \geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(y) d\gamma^x(y) d\mu(x) - \int_{\mathbb{R}^d} \psi(x) d\mu(x) = \langle \nu - \mu, \psi \rangle.$$

This, since $\gamma \in \Pi(\mu, \nu)$ and $\psi \in C_b^2(\mathbb{R}^d)$ satisfying $-f^*(\frac{1}{2}\nabla^2\psi) = 0$ in \mathbb{R}^d , according to Theorem 3.6, completes our proof of Proposition 3.10. \square

For each $x \in \mathbb{R}^d$ and for each universally measurable function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying the estimate $|\varphi(\cdot)| \leq C(1 + |\cdot|^2)$ for some constant $C > 0$, we define

$$\varphi^G(x) := \inf \left\{ \int_{\mathbb{R}^d} \varphi dp + G(x, p) \mid p \in \mathcal{P}_2(\mathbb{R}^d) \right\}. \quad (3.22)$$

Remark 3.11. If $\varphi \in \Phi_{b,2}(\mathbb{R}^d)$, then φ^G is lower semianalytic (and hence universally measurable, see [8, Proposition 7.47]), bounded from below, $|\varphi^G(\cdot)| \leq C(1 + |\cdot|^2)$ for some constant $C > 0$ and the integral $\int_{\mathbb{R}^d} \varphi^G dp$ is well defined for all $p \in \mathcal{P}_2(\mathbb{R}^d)$. In particular, for each $x \in \mathbb{R}^d$, we can define $\varphi^{GG}(x)$.

Proposition 3.12. *The following assertions hold.*

(i) *H is convex on $\mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d)$ and lower semicontinuous with respect to the weak topology on $\mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d)$ in duality with $\Phi_2(\mathbb{R}^d) \times \Phi_2(\mathbb{R}^d)$. If $H(\mu, \nu) < +\infty$, then the weak transport problem (3.19) admits a solution.*

(ii) *For each $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$,*

$$H(\mu, \nu) = \sup \left\{ \int_{\mathbb{R}^d} \varphi^G d\mu - \int_{\mathbb{R}^d} \varphi d\nu \mid \varphi \in \Phi_{b,2}(\mathbb{R}^d) \right\}. \quad (3.23)$$

Proof. According to Proposition 3.9 (i), $(x, p) \mapsto G(x, p)$ is convex in p and lower semicontinuous in (x, p) with respect to the product topology on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, where the topology on \mathbb{R}^d is generated by the Euclidean distance and the topology on $\mathcal{P}_2(\mathbb{R}^d)$ is generated by the 2-Wasserstein distance (see Definition 6.8 and Theorem 6.9 in [26]). Then, using [5, Theorem 2.9], we deduce that H is lower semicontinuous with respect to the weak topology on $(\mathcal{P}_2(\mathbb{R}^d))^2$ in duality with $(\Phi_2(\mathbb{R}^d))^2$ and prove that (3.19) admits a solution whenever $H(\mu, \nu) < +\infty$. Applying [5, Theorem 3.1], we obtain the dual formulation (3.23), which implies the convexity of H . This completes our proof of Proposition 3.12. \square

To develop the theory of subadditive cost functionals, which appeared earlier in [2, Section 6], where the role of \mathbb{R}^d is replaced by the closure of a bounded open convex subset of \mathbb{R}^d , we need to introduce some additional assumptions on f , namely either the coercivity (see the example (i)), or the growth assumption (see the example (ii)), or both (see the examples (iii), (iv)). In particular, we introduce the following assumptions.

$$f \text{ is coercive.} \tag{A3}$$

$$\text{dom}(f) = S_d^+ \text{ and there exists } \kappa_1 > 0 \text{ such that } f(A) \leq \kappa_1 \text{tr}(A) \text{ for all } A \in S_d^+. \tag{A4}$$

Remark 3.13. Since f satisfies (A1) and (A2), the assumption (A3) holds if and only if there exists a constant $\kappa_0 > 0$ such that $f(A) \geq \kappa_0 \text{tr}(A)$ for each $A \in S_d^+$. Indeed, assume that (A3) holds. If $A \in S_d^+$ and $\text{tr}(A) > 0$, then $|A| > 0$ and, in view of (A1), $f(A) = |A|f(\frac{A}{|A|})$. Using this, (A2) and the fact that $\text{dom}(f) \cap S_d^{++}$ is dense in $\text{dom}(f) \neq \emptyset$, we deduce that there exists $\tilde{E} \in S_d^+$ such that $|\tilde{E}| = 1$ and $f(\tilde{E}) = \min\{f(E) \mid E \in S_d^+, |E| = 1\} < +\infty$. Then, defining $\kappa_0 = f(\tilde{E})/\sqrt{d}$, we have $f(A) \geq f(\tilde{E})|A| \geq \kappa_0 \text{tr}(A)$ for each $A \in S_d^+$. Clearly, the last inequality implies the coercivity of f .

Next, we prove that G is *narrowly* lower semicontinuous if (A3) holds. Recall that $(p_n)_{n \in \mathbb{N}} \subset \mathcal{P}(\mathbb{R}^d)$ narrowly converges to $p \in \mathcal{P}(\mathbb{R}^d)$ if $\int_{\mathbb{R}^d} \varphi dp_n \rightarrow \int_{\mathbb{R}^d} \varphi dp$ as $n \rightarrow +\infty$ for each $\varphi \in C_b(\mathbb{R}^d)$.

Proposition 3.14. *Let (A3) hold, $x_n \rightarrow x \in \mathbb{R}^d$, $(p_n)_{n \in \mathbb{N}} \subset \mathcal{P}_2(\mathbb{R}^d)$ and p_n narrowly converges to $p \in \mathcal{P}_2(\mathbb{R}^d)$. Then*

$$G(x, p) \leq \liminf_{n \rightarrow +\infty} G(x_n, p_n).$$

Proof. Without loss of generality, there exists a constant $C > 0$ (independent of n) such that for each $n \in \mathbb{N}$, $G(x_n, p_n) \leq C$. Then, by Theorem 3.6, for each $n \in \mathbb{N}$, there exists $\lambda_n \in \mathcal{M}(\mathbb{R}^d, S_d^+)$ such that $|\lambda_n|(\mathbb{R}^d) < +\infty$, $\text{tr}(\frac{1}{2}\nabla^2 \lambda_n) = p_n - x_n$ and $G(x_n, p_n) = \int_{\mathbb{R}^d} f(\frac{d\lambda_n}{d|\lambda_n|}) d|\lambda_n|$. Using this and (A3), we obtain

$$|\lambda_n|(\mathbb{R}^d) \leq \eta_0 \int_{\mathbb{R}^d} f\left(\frac{d\lambda_n}{d|\lambda_n|}\right) d|\lambda_n| \leq \eta_0 C$$

for some constant $\eta_0 > 0$ independent of n (see Remark 3.13). Then, according to the Banach-Alaoglu theorem, there exists $\lambda \in \mathcal{M}(\mathbb{R}^d, S_d^+)$ such that $|\lambda|(\mathbb{R}^d) < +\infty$ and λ_n converges weakly to λ . Since, for each $\varphi \in C_c^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \varphi dp_n - \varphi(x_n) = \int_{\mathbb{R}^d} \frac{1}{2} \nabla^2 \varphi : d\lambda_n,$$

letting n tend to $+\infty$ and using the weak convergences, we deduce that

$$\int_{\mathbb{R}^d} \varphi dp - \varphi(x) = \int_{\mathbb{R}^d} \frac{1}{2} \nabla^2 \varphi : d\lambda. \tag{3.24}$$

By direct adaptation of the density argument in [25, Remark 2.3], (3.24) implies that $\text{tr}(\frac{1}{2}\nabla^2 \lambda) = p - \delta_x$. Thus, λ is a competitor for $G(x, p) = F(\delta_x, p)$ (see (3.5)), which, in view of the lower semicontinuity of the function $\sigma \mapsto \int_{\mathbb{R}^d} f(\frac{d\sigma}{d|\sigma|}) d|\sigma|$ on the subset of finite measures in $\mathcal{M}(\mathbb{R}^d, S_d^+)$, yields the estimate

$$G(x, p) \leq \int_{\mathbb{R}^d} f\left(\frac{d\lambda}{d|\lambda|}\right) d|\lambda| \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^d} f\left(\frac{d\lambda_n}{d|\lambda_n|}\right) d|\lambda_n| = \liminf_{n \rightarrow +\infty} G(x_n, p_n)$$

and completes our proof of Proposition 3.14. \square

Proposition 3.15. *Let (A3) hold and $\varphi \in \mathcal{S}_{b,2}(\mathbb{R}^d)$. Then the infimum for φ in (3.22) is actually a minimum and $\varphi^G \in \mathcal{S}_{b,2}(\mathbb{R}^d)$.*

Proof. Let $x \in \mathbb{R}^d$ and $(p_n)_{n \in \mathbb{N}} \subset \mathcal{P}_2(\mathbb{R}^d)$ be a minimizing sequence for $\varphi^G(x) \in \mathbb{R}$. Then there exists a constant $C > 0$ (independent of n) such that for each $n \in \mathbb{N}$ large enough, $G(x, p_n) \leq C$. Using this, together with (A3) (see Remark 3.13) and Corollary 3.8, we deduce that

$$A := \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^d} |y|^2 dp_n(y) < +\infty,$$

which, in view of [3, Remark 5.1.5], implies that $(p_n)_{n \in \mathbb{N}}$ is tight. Then, by Prokhorov's theorem (see [3, Theorem 5.1.3]), there exists a probability measure p on \mathbb{R}^d such that, up to a subsequence (not relabeled), p_n converges narrowly to p . Thus,

$$\int_{\mathbb{R}^d} |y|^2 dp(y) \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^d} |y|^2 dp_n(y) \leq A$$

and $p \in \mathcal{P}_2(\mathbb{R}^d)$. By the narrow convergence (recall that $\varphi \in \mathcal{S}_{b,2}(\mathbb{R}^d)$) and Proposition 3.14,

$$\int_{\mathbb{R}^d} \varphi dp + G(x, p) \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^d} \varphi dp_n + G(x, p_n),$$

which says that p is a minimizer in the definition of $\varphi^G(x)$ (see (3.22)).

Next, we prove the lower semicontinuity of φ^G . Let $x_n \rightarrow x$, $p_n \in \mathcal{P}_2(\mathbb{R}^d)$ be a minimizer in the definition of $\varphi^G(x_n)$ and $\liminf_{n \rightarrow +\infty} \varphi^G(x_n) < +\infty$. Proceeding as before, we can assume that there exists $p \in \mathcal{P}_2(\mathbb{R}^d)$ such that, up to a subsequence (not relabeled), p_n converges narrowly to p . Using the narrow convergence, the fact that $\varphi \in \mathcal{S}_{b,2}(\mathbb{R}^d)$ and Proposition 3.14, we obtain the following

$$\varphi^G(x) \leq \int_{\mathbb{R}^d} \varphi dp + G(x, p) \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^d} \varphi dp_n + G(x_n, p_n) = \liminf_{n \rightarrow +\infty} \varphi^G(x_n),$$

which proves the lower semicontinuity of φ^G and completes our proof of Proposition 3.15. \square

Under the assumption (A3), we can, on the one hand relax and, on the other hand, strengthen the dual constraint in (3.23) using bounded lower semicontinuous functions.

Proposition 3.16. *Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and (A3) hold. Then*

$$H(\mu, \nu) = \sup \left\{ \int_{\mathbb{R}^d} \varphi^G d\mu - \int_{\mathbb{R}^d} \varphi d\nu \mid \varphi \in \mathcal{S}_b(\mathbb{R}^d) \right\}. \quad (3.25)$$

Proof. Let $\varphi \in \Phi_{b,2}(\mathbb{R}^d)$ and $\varphi_n = \min\{n, \varphi\}$ for each $n \in \mathbb{N}$. Then $\varphi_n \in C_b(\mathbb{R}^d)$ and $\varphi_n \nearrow \varphi$ as $n \rightarrow +\infty$. According to Proposition 3.15, for each $n \in \mathbb{N}$ and for each $x \in \mathbb{R}^d$, there exists $p_n \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\varphi_n^G(x) = \int_{\mathbb{R}^d} \varphi_n dp_n + G(x, p_n)$. Since $\varphi_n^G(x) \leq \varphi(x) < +\infty$, arguing by the same way as in the proof of Proposition 3.15, we deduce that there exists $p \in \mathcal{P}_2(\mathbb{R}^d)$ such that, up to a subsequence (not relabeled), p_n converges narrowly to p . For each $k \in \mathbb{N}$, using the weak convergence and Proposition 3.14, we obtain

$$\int_{\mathbb{R}^d} \varphi_k dp + G(x, p) \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^d} \varphi_k dp_n + G(x, p_n) \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^d} \varphi_n dp_n + G(x, p_n) = \liminf_{n \rightarrow +\infty} \varphi_n^G(x).$$

Letting k tend to $+\infty$, by the monotone convergence theorem, we have

$$\varphi^G(x) \leq \int_{\mathbb{R}^d} \varphi dp + G(x, p) = \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^d} \varphi_k dp + G(x, p) \leq \liminf_{n \rightarrow +\infty} \varphi_n^G(x).$$

On the other hand, $\varphi_n^G(x) \leq \varphi^G(x)$ for each $n \in \mathbb{N}$ and hence $\varphi_n^G(x) \nearrow \varphi^G(x)$ as $n \rightarrow +\infty$. Thus, by the monotone convergence theorem, $\int_{\mathbb{R}^d} \varphi_n^G d\mu \rightarrow \int_{\mathbb{R}^d} \varphi^G d\mu$ and $\int_{\mathbb{R}^d} \varphi_n d\nu \rightarrow \int_{\mathbb{R}^d} \varphi d\nu$ as $n \rightarrow +\infty$. This, together with (3.23), implies that

$$H(\mu, \nu) = \sup \left\{ \int_{\mathbb{R}^d} \varphi^G d\mu - \int_{\mathbb{R}^d} \varphi d\nu \mid \varphi \in C_b(\mathbb{R}^d) \right\}.$$

Since for each $\varphi \in \mathcal{S}_b(\mathbb{R}^d)$, there exists a sequence $(\varphi_n)_{n \in \mathbb{N}} \subset C_b(\mathbb{R}^d)$ such that $\varphi_n \nearrow \varphi$ as $n \rightarrow +\infty$, repeating the above procedure, we complete our proof of Proposition 3.16. \square

If the G -transform $\varphi \mapsto \varphi^G$ is idempotent on $\mathcal{S}_b(\mathbb{R}^d)$, the following dual formulation for $H(\mu, \nu)$ holds.

Proposition 3.17. *Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, (A3) hold and $\varphi^{GG} = \varphi^G$ for each $\varphi \in \mathcal{S}_b(\mathbb{R}^d)$. Then*

$$H(\mu, \nu) = \sup \{ \langle \nu - \mu, \psi \rangle \mid \psi \in \mathcal{U}_b(\mathbb{R}^d), -\psi = (-\psi)^G \}. \quad (3.26)$$

Proof. In view of Proposition 3.16, $H(\mu, \nu)$ is greater than or equal to the supremum in (3.26). On the other hand, using the estimate $\varphi^G \leq \varphi$ in (3.25), we deduce the following

$$H(\mu, \nu) \leq \sup\{\langle \mu - \nu, \varphi^G \rangle \mid \varphi \in \mathcal{S}_b(\mathbb{R}^d)\} \leq \sup\{\langle \mu - \nu, \varphi \rangle \mid \varphi \in \mathcal{S}_b(\mathbb{R}^d), \varphi = \varphi^G\},$$

where the latter estimate comes from the assumption that $\varphi^{GG} = \varphi^G$ for each $\varphi \in \mathcal{S}_b(\mathbb{R}^d)$, since in this case, taking into account Proposition 3.15, we have $\{\varphi^G \mid \varphi \in \mathcal{S}_b(\mathbb{R}^d)\} \subset \{\varphi \mid \varphi \in \mathcal{S}_b(\mathbb{R}^d), \varphi = \varphi^G\}$. Thus, the supremum in (3.26) is greater than or equal to $H(\mu, \nu)$ and the dual formulation (3.26) holds, which completes our proof of Proposition 3.17. \square

The following proposition describes some situations in which the G -transform is idempotent on $\mathcal{S}_b(\mathbb{R}^d)$, which in particular happens when H is subadditive.

Proposition 3.18. *Let (A3) hold. Then the following assertions are equivalent.*

(i) For each choice of $\mu, \nu, p \in \mathcal{P}_2(\mathbb{R}^d)$, $H(\mu, \nu) \leq H(\mu, p) + H(p, \nu)$.

(ii) For each $\nu, p \in \mathcal{P}_2(\mathbb{R}^d)$ and analytically measurable probability kernel $y \in \mathbb{R}^d \mapsto \gamma^y \in \mathcal{P}_2(\mathbb{R}^d)$,

$$G(x, \nu) \leq G(x, p) + \int_{\mathbb{R}^d} G(y, \gamma^y) dp(y) \quad \text{whenever } \nu = \int_{\mathbb{R}^d} \gamma^y dp(y). \quad (3.27)$$

(iii) For each $\varphi \in \mathcal{S}_b(\mathbb{R}^d)$, $\varphi^{GG} = \varphi^G$.

Proof. The proof of the implication (i) \Rightarrow (ii) follows by choosing $\mu = \delta_x$, $\nu = \int_{\mathbb{R}^d} \gamma^y dp(y)$ and using the definition of $H(p, \nu)$ as the infimum (see (3.20)).

Now we prove the implication (ii) \Rightarrow (iii). In view of Proposition 3.15 and since $\varphi^G \leq \varphi$, for each $\varphi \in \mathcal{S}_b(\mathbb{R}^d)$, we have $\varphi^G, \varphi^{GG} \in \mathcal{S}_b(\mathbb{R}^d)$ and $\varphi^{GG} \leq \varphi^G$. Thus, it is enough to prove the estimate

$$\varphi^G(x) \leq \int_{\mathbb{R}^d} \varphi^G(y) dp(y) + G(x, p) \quad (3.28)$$

for each $x \in \mathbb{R}^d$ and $p \in \mathcal{P}_2(\mathbb{R}^d)$. Using the lower semicontinuity of G and the fact that $\varphi \in \mathcal{S}_b(\mathbb{R}^d)$, we deduce that the function $(x, p) \mapsto \int_{\mathbb{R}^d} \varphi dp + G(x, p)$ is lower semicontinuous on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, where the topology on \mathbb{R}^d is generated by the Euclidean distance and the topology on $\mathcal{P}_2(\mathbb{R}^d)$ is generated by the 2-Wasserstein distance. Then, according to [8, Proposition 7.50], for each $\varepsilon > 0$ there exists an analytically measurable probability kernel $y \in \mathbb{R}^d \mapsto \gamma^y \in \mathcal{P}_2(\mathbb{R}^d)$ such that

$$\varphi^G(y) + \varepsilon \geq \int_{\mathbb{R}^d} \varphi(z) d\gamma^y(z) + G(y, \gamma^y).$$

Then defining $\nu(dz) = \int_{\mathbb{R}^d} \gamma^y(dz) dp(y)$, integrating both sides of the above inequality with respect to $p \in \mathcal{P}_2(\mathbb{R}^d)$ and using (ii), we obtain

$$\begin{aligned} G(x, p) + \int_{\mathbb{R}^d} \varphi^G(y) dp(y) + \varepsilon &\geq G(x, p) + \int_{\mathbb{R}^d} G(y, \gamma^y) dp(y) + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(z) d\gamma^y(z) dp(y) \\ &\geq G(x, \nu) + \int_{\mathbb{R}^d} \varphi(z) d\nu(z) \\ &\geq \varphi^G(x), \end{aligned}$$

which yields (3.28) and completes our proof of the implication (ii) \Rightarrow (iii).

The implication (iii) \Rightarrow (i) is a direct consequence of Proposition 3.17, since for each $\psi \in \mathcal{U}_b(\mathbb{R}^d)$ such that $-\psi = (-\psi)^G$,

$$\langle \nu - \mu, \psi \rangle \leq \langle p - \mu, \psi \rangle + \langle \nu - p, \psi \rangle \leq H(\mu, p) + H(p, \nu).$$

This completes our proof of Proposition 3.18. \square

Under the assumption (A3) the functional H is subadditive, and hence the G -transform is idempotent on $\mathcal{S}_b(\mathbb{R}^d)$.

Proposition 3.19. *Let (A3) hold. Then the assertions (i)-(iii) of Proposition 3.18 hold.*

Proof. In view of Proposition 3.18, it is enough to prove that for each $p \in \mathcal{P}_2(\mathbb{R}^d)$, $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ and analytically measurable probability kernel $y \in \mathbb{R}^d \mapsto \gamma^y \in \mathcal{P}_2(\mathbb{R}^d)$,

$$G(x, \nu) \leq G(x, p) + \int_{\mathbb{R}^d} G(y, \gamma^y) dp(y) \quad \text{whenever } \nu = \int_{\mathbb{R}^d} \gamma^y dp(y). \quad (3.29)$$

Using the definition of H (see (3.20)) and Proposition 3.10, we have

$$\int_{\mathbb{R}^d} G(y, \gamma^y) dp(y) \geq H(p, \nu) \geq F(p, \nu). \quad (3.30)$$

Combining Proposition 3.9 (ii) and (3.30), yields (3.29), namely

$$G(x, \nu) \leq G(x, p) + F(p, \nu) \leq G(x, p) + \int_{\mathbb{R}^d} G(y, \gamma^y) dp(y)$$

(see (3.18)). This completes our proof of Proposition 3.19. \square

Theorem 3.20. *Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and (A3) hold. Then*

$$H(\mu, \nu) = \sup\{\langle \nu - \mu, \psi \rangle \mid \psi \in \mathcal{U}_b(\mathbb{R}^d), -\psi = (-\psi)^G\}.$$

Proof. The proof follows from Propositions 3.17 and 3.19. \square

Remark 3.21. Let $E \subset \mathbb{R}^d$. Then in view of (3.18) and (3.22),

$$\begin{aligned} & (-\psi)^G(x) = -\psi(x) \quad \forall x \in E \\ \Leftrightarrow & \int_{\mathbb{R}^d} -\psi dp + G(x, p) \geq -\psi(x) \quad \forall (x, p) \in E \times \mathcal{P}_2(\mathbb{R}^d) \\ \Leftrightarrow & \int_{\mathbb{R}^d} -\psi dp + \int_{\mathbb{R}^d} f\left(\frac{d\lambda}{d|\lambda|}\right) d|\lambda| \geq -\psi(x) \\ & \forall (x, p, \lambda) \in E \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d, S_d^+), \quad \text{tr}\left(\frac{1}{2}\nabla^2\lambda\right) = p - \delta_x. \end{aligned}$$

Assuming (A4) instead of (A3), we also obtain the dual formulation for H , where the dual competitors are invariant under the G -transform but belong to $\Phi_{b,2}(\mathbb{R}^d)$ (see Theorem 3.26). We first prove the following key result.

Proposition 3.22. *Let (A4) hold. Then for each $p \in \mathcal{P}_2(\mathbb{R}^d)$,*

$$G([p], p) \leq \kappa_1 \text{var}(p). \quad (3.31)$$

Furthermore, if $f = \text{tr}$ on S_d^+ , then for each $p \in \mathcal{P}_2(\mathbb{R}^d)$,

$$G([p], p) = \text{var}(p). \quad (3.32)$$

Proof. In view of Proposition 3.2 and (A4),

$$G([p], p) \leq \kappa_1 \inf \left\{ \int_{\mathbb{R}^d} \text{tr} \left(\frac{d\lambda}{d|\lambda|} \right) d|\lambda| \mid \text{tr} \left(\frac{1}{2} \nabla^2 \lambda \right) = p - \delta_{[p]} \right\}.$$

Thus, it suffices to prove (3.32). Assume that $f = \text{tr}$ on S_d^+ . By Jensen's inequality, for each concave function $u : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\langle p - \delta_{[p]}, u \rangle \leq 0. \quad (3.33)$$

Since for each $A \in S_d$, $-\text{tr}^*(A) = 0 \Leftrightarrow A - I_d \leq 0$ (see (4.22)), according to Theorem 3.6,

$$\begin{aligned} G([p], p) &= \sup \left\{ \langle p - \delta_{[p]}, \psi \rangle \mid \psi \in C_b^2(\mathbb{R}^d), \left(\frac{1}{2} \nabla^2 \psi - I_d \right) \leq 0 \text{ on } \mathbb{R}^d \right\} \\ &= \langle p - \delta_{[p]}, |\cdot|^2 \rangle + \sup \left\{ \langle p - \delta_{[p]}, u \rangle \mid u \in C_b^2(\mathbb{R}^d), u \text{ is concave on } \mathbb{R}^d \right\} \\ &= \text{var}(p), \end{aligned}$$

where the last equality comes from (3.33). This completes our proof of Proposition 3.22. \square

Corollary 3.23. Let (A4) hold. Then for each lower semianalytic function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ bounded from below such that $\varphi = \varphi^G$, the function $\varphi(\cdot) + \kappa_1 |\cdot|^2$ is convex and locally Lipschitz on \mathbb{R}^d .

Proof of Corollary 3.23. Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be lower semianalytic, bounded from below and $\varphi = \varphi^G$. According to Remark 3.21, for each $p \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\varphi([p]) \leq \int_{\mathbb{R}^d} \varphi dp + G([p], p) \leq \int_{\mathbb{R}^d} \varphi dp + \kappa_1 \left(\int_{\mathbb{R}^d} |y|^2 dp(y) - |[p]|^2 \right),$$

where the latter estimate comes from the estimate (3.31) of Proposition 3.22. Thus,

$$\varphi([p]) + \kappa_1 |[p]|^2 \leq \int_{\mathbb{R}^d} (\varphi(y) + \kappa_1 |y|^2) dp(y). \quad (3.34)$$

Since

$$\frac{x+y}{2} = \left[\frac{\delta_x}{2} + \frac{\delta_y}{2} \right] \quad \forall x, y \in \mathbb{R}^d,$$

(3.34) yields

$$\varphi\left(\frac{x+y}{2}\right) + \kappa_1 \left| \frac{x+y}{2} \right|^2 \leq \frac{\varphi(x) + \varphi(y)}{2} + \frac{\kappa_1(|x|^2 + |y|^2)}{2},$$

which, together with the local boundedness of φ , implies the convexity of $\varphi(\cdot) + \kappa_1 |\cdot|^2$ on \mathbb{R}^d . For the fact that a convex function is locally Lipschitz on the interior of its proper domain, the reader may consult [22, Theorem 10.4]. This completes our proof of Corollary 3.23. \square

Proposition 3.24. Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, (A4) hold and $\varphi^{GG} = \varphi^G$ for each $\varphi \in \Phi_{b,2}(\mathbb{R}^d)$. Then

$$H(\mu, \nu) = \sup\{\langle \nu - \mu, \psi \rangle \mid -\psi \in \Phi_{b,2}(\mathbb{R}^d), -\psi = (-\psi)^G\}. \quad (3.35)$$

Proof. By Proposition 3.12 (ii), $H(\mu, \nu)$ is greater than or equal to the supremum in (3.35). In view of Remark 3.11 and Corollary 3.23, for each $\varphi \in \Phi_{b,2}(\mathbb{R}^d)$, the functions φ^G and φ^{GG} are well defined and $\varphi^G \in \Phi_{b,2}(\mathbb{R}^d)$ whenever $\varphi^{GG} = \varphi^G$. Then $\{\varphi^G \mid \varphi \in \Phi_{b,2}(\mathbb{R}^d)\} \subset \{\varphi \mid \varphi \in \Phi_{b,2}(\mathbb{R}^d), \varphi = \varphi^G\}$, since $\varphi^{GG} = \varphi^G$ for each $\varphi \in \Phi_{b,2}(\mathbb{R}^d)$ by our assumption. Using this and the estimate

$$\int_{\mathbb{R}^d} \varphi^G d\mu - \int_{\mathbb{R}^d} \varphi d\nu \leq \int_{\mathbb{R}^d} \varphi^G d\mu - \int_{\mathbb{R}^d} \varphi^G d\nu$$

for each $\varphi \in \Phi_{b,2}(\mathbb{R}^d)$ in (3.23), we deduce that $H(\mu, \nu)$ is less than or equal to the supremum in (3.35). This completes our proof of Proposition 3.24. \square

The next proposition is a counterpart of Proposition 3.18, where we replace the assumption (A3) by (A4) and describe some situations in which the G -transform is idempotent on $\Phi_{b,2}(\mathbb{R}^d)$.

Proposition 3.25. Let (A4) hold. Then the following assertions are equivalent.

(i) For each choice of $\mu, \nu, p \in \mathcal{P}_2(\mathbb{R}^d)$, $H(\mu, \nu) \leq H(\mu, p) + H(p, \nu)$.

(ii) For each $\nu, p \in \mathcal{P}_2(\mathbb{R}^d)$ and analytically measurable probability kernel $y \in \mathbb{R}^d \mapsto \gamma^y \in \mathcal{P}_2(\mathbb{R}^d)$,

$$G(x, \nu) \leq G(x, p) + \int_{\mathbb{R}^d} G(y, \gamma^y) dp(y) \quad \text{whenever } \nu = \int_{\mathbb{R}^d} \gamma^y dp(y). \quad (3.36)$$

(iii) For each $\varphi \in \Phi_{b,2}(\mathbb{R}^d)$, $\varphi^{GG} = \varphi^G$.

Proof. The proof follows by reproducing the arguments of the proof of Proposition 3.18 with minor modifications, in particular, using Proposition 3.24 in the proof of the implication (iii) \Rightarrow (i). \square

Theorem 3.26. Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and (A4) hold. Then

$$H(\mu, \nu) = \sup\{\langle \nu - \mu, \psi \rangle \mid -\psi \in \Phi_{b,2}(\mathbb{R}^d), -\psi = (-\psi)^G\}.$$

Proof. Reproducing the proof of Proposition 3.19, we deduce that the assertions (i)-(iii) of Proposition 3.25 hold. Then, applying Proposition 3.24, we complete our proof of Theorem 3.26. \square

If (A3) and (A4) hold simultaneously, we have the following dual formulation.

Theorem 3.27. *Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and (A3), (A4) hold. Then*

$$H(\mu, \nu) = \sup\{\langle \nu - \mu, \psi \rangle \mid \psi \in C_b(\mathbb{R}^d), -\psi = (-\psi)^G\}. \quad (3.37)$$

Proof. The proof follows from Theorem 3.20 and Corollary 3.23. \square

3.4. Duality for F in terms of invariant functions under G -transform

Proposition 3.28. *Let $\psi \in C_b^2(\mathbb{R}^d)$. Then $-f^*(\frac{1}{2}\nabla^2\psi) = 0$ in \mathbb{R}^d if and only if $-\psi = (-\psi)^G$.*

Proof. According to Theorem 3.6, for each $x \in \mathbb{R}^d$, $p \in \mathcal{P}_2(\mathbb{R}^d)$ and $\psi \in C_b^2(\mathbb{R}^d)$ such that $-f^*(\frac{1}{2}\nabla^2\psi) = 0$ in \mathbb{R}^d ,

$$G(x, p) \geq \int_{\mathbb{R}^d} \psi dp - \psi(x)$$

and hence

$$(-\psi)^G(x) = \inf\left\{\int_{\mathbb{R}^d} -\psi dp + G(x, p) \mid p \in \mathcal{P}_2(\mathbb{R}^d)\right\} \geq -\psi(x).$$

On the other hand, $(-\psi)^G(x) \leq -\psi(x)$ for each $x \in \mathbb{R}^d$, since $G(x, \delta_x) = 0$. Thus, $-\psi = (-\psi)^G$.

Let us now assume that $\psi \in C_b^2(\mathbb{R}^d)$ and $-\psi = (-\psi)^G$. Fix arbitrary $A = (a_{ij})_{i,j=1}^d \in S_d^{++}$ and $x_0 \in \mathbb{R}^d$. Let $g_A(x, y)$ be the Green function of the elliptic operator $L = -\sum_{i,j=1}^d a_{ij}\partial_{ij} = -\operatorname{div}(A\nabla)$ on $B_r(x_0)$, namely, for each $y \in B_r(x_0)$, $g_A(\cdot, y) \in W_0^{1,p}(B_r(x_0))$ whenever $p < d/(d-1)$ and

$$-A : \nabla^2 g_A = \delta_y \quad \text{in } \mathcal{D}'(B_r(x_0)),$$

which means that

$$-\int_{B_r(x_0)} A : \nabla^2 \varphi(x) g_A(x, y) dx = \varphi(y) \quad \forall \varphi \in C_c^2(B_r(x_0))$$

(see, for instance, [19]). Since $A \in S_d^{++}$, $A = P \operatorname{diag}(\lambda_1, \dots, \lambda_d) P^T$ for some orthogonal real $d \times d$ matrix P and positive numbers $\lambda_i > 0$. Also $A^{-1} \in S_d^{++}$ and there exists the unique matrix $B \in S_d^{++}$ such that $A = B^{-2}$, namely $B = P \operatorname{diag}(1/\sqrt{\lambda_1}, \dots, 1/\sqrt{\lambda_d}) P^T$. Fix an arbitrary $v \in C_b^2(\mathbb{R}^d)$ and define $u(\cdot) = v(B^{-1}\cdot)$ so that $u \in C_b^2(\mathbb{R}^d)$. Then $A : \nabla^2 v(x) = \Delta u(Bx)$ for each $x \in \mathbb{R}^d$. If g is the Green function of the Laplace operator on $BB_r(x_0) = \{Bx \mid x \in B_r(x_0)\}$, then $g_A(x, y) = \det(B)g(Bx, By)$. To lighten the notation, define $U = BB_r(x_0)$. Next, using the Green representation formula, changing the variables and using that $A : \nabla^2 v(x) = \Delta u(Bx)$ and $g_A(x, x_0) = \det(B)g(Bx, Bx_0)$, we have

$$\begin{aligned} v(x_0) &= u(Bx_0) = -\int_U \Delta u(x) g(x, Bx_0) dx - \int_{\partial U} u(x) \nabla g(x, Bx_0) \cdot \nu_{\partial U}(x) d\mathcal{H}^{d-1}(x) \\ &= -\det(B) \int_{B_r(x_0)} \Delta u(Bx) g(Bx, Bx_0) dx \\ &\quad - \det(B) \int_{\partial B_r(x_0)} u(Bx) \nabla g(Bx, Bx_0) \cdot \nu_{\partial U}(Bx) |B^{-T}\nu(x)| d\mathcal{H}^{d-1}(x) \\ &= -\int_{B_r(x_0)} A : \nabla^2 v(x) g_A(x, x_0) dx - \int_{\partial B_r(x_0)} v(x) A \nabla g_A(x, x_0) \cdot \nu(x) d\mathcal{H}^{d-1}(x), \end{aligned} \quad (3.38)$$

where $\nu_{\partial U}$ and ν denote the outward pointing unit normal vector fields along ∂U and $\partial B_r(x_0)$, respectively.

By Hopf's lemma (see [17, 20]), $-A \nabla g_A(x, x_0) \cdot \nu(x) > 0$ for each $x \in \partial B_r(x_0)$. Then, using (3.38) with $v = 1$, we obtain $p = -A \nabla g_A(\cdot, x_0) \cdot \nu(\cdot) \mathcal{H}^{d-1} \llcorner \partial B_r(x_0) \in \mathcal{P}_2(\mathbb{R}^d)$.

Altogether, due to (3.38) and the fact that $v \in C_b^2(\mathbb{R}^d)$ was arbitrarily chosen, we have

$$\operatorname{tr}\left(\frac{1}{2}\nabla^2\lambda\right) = p - \delta_{x_0},$$

where $\lambda = 2Ag_A(\cdot, x_0)\mathcal{L}^d \llcorner B_r(x_0) \in \mathcal{M}(\mathbb{R}^d, S_d^+)$ and $|\lambda|(\mathbb{R}^d) < +\infty$. Since $-\psi(x_0) = (-\psi)^G(x_0)$, according to Remark 3.21,

$$\int_{\mathbb{R}^d} f\left(\frac{d\lambda}{d|\lambda|}\right) d|\lambda| \geq \int_{\mathbb{R}^d} \psi dp - \psi(x_0).$$

This, together with the positive 1-homogeneity of f (see (A1)) and the fact that $\psi \in C_b^2(\mathbb{R}^d)$, implies that

$$\int_{B_r(x_0)} f(A)g_A(x, x_0) dx \geq \int_{B_r(x_0)} A : \frac{1}{2}\nabla^2\psi(x)g_A(x, x_0) dx. \quad (3.39)$$

Assume by contradiction that $A : \frac{1}{2}\nabla^2\psi(x_0) - f(A) > 0$. Then there exist $\varepsilon, r > 0$ such that for each $x \in B_r(x_0)$, $A : \frac{1}{2}\nabla^2\psi(x) - f(A) \geq \varepsilon$. But this contradicts (3.39), since $\int_{B_r(x_0)} g_A(x, x_0) dx > 0$. Thus, $A : \frac{1}{2}\nabla^2\psi(x_0) - f(A) \leq 0$, which implies that $\frac{1}{2}\nabla^2\psi(x_0) \in \text{dom}(f^*)$, because $A \in S_d^{++}$ was arbitrarily chosen and $S_d^{++} \cap \text{dom}(f)$ is dense in $\text{dom}(f)$. Since $x_0 \in \mathbb{R}^d$ was arbitrary, $\frac{1}{2}\nabla^2\psi(\mathbb{R}^d) \subset \text{dom}(f^*)$, which holds if and only if $-f^*(\frac{1}{2}\nabla^2\psi) = 0$ in \mathbb{R}^d . This completes our proof of Proposition 3.28. \square

Theorem 3.29. *For each $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, we have*

$$F(\mu, \nu) = \sup\{\langle \nu - \mu, \psi \rangle \mid \psi \in C_b^2(\mathbb{R}^d), -\psi = (-\psi)^G\}. \quad (3.40)$$

Proof. The proof is a direct consequence of Theorem 3.6 and Proposition 3.28. \square

4. F versus H

4.1. $F = H$: the approximation assumptions

Our first type of approximation assumption is related to the assumption (A3).

Theorem 4.1. *Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and (A3) hold. Assume that for each $\psi \in \mathcal{U}_b(\mathbb{R}^d)$ such that $-\psi = (-\psi)^G$ there exist $(\psi_n)_{n \in \mathbb{N}} \subset C_b^2(\mathbb{R}^d)$ such that $-\psi_n = (-\psi_n)^G$ for each $n \in \mathbb{N}$ and $\langle \nu - \mu, \psi_n \rangle \rightarrow \langle \nu - \mu, \psi \rangle$ as $n \rightarrow +\infty$. Then $F(\mu, \nu) = H(\mu, \nu)$.*

Proof. By Proposition 3.10, $F(\mu, \nu) \leq H(\mu, \nu)$. Next, using the assumption of Theorem 4.1, together with Theorem 3.20 and Theorem 3.29, we have $H(\mu, \nu) \leq F(\mu, \nu)$, which completes our proof of Theorem 4.1. \square

Example (i). For each $A \in S_d$,

$$f(A) = \begin{cases} t & \text{if } A = tI_d \\ +\infty & \text{otherwise.} \end{cases} \quad (4.1)$$

Clearly, f satisfies (A1)-(A3). Then for each $A \in S_d$,

$$f^*(A) = \sup\{A : M - f(M) \mid M \in \text{dom}(f)\} = \begin{cases} 0 & \text{if } \text{tr}(A) \leq 1 \\ +\infty & \text{otherwise.} \end{cases} \quad (4.2)$$

Given $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, using Theorem 3.6 and (4.2), we have

$$\begin{aligned} F(\mu, \nu) &= \sup\left\{\langle \nu - \mu, \psi \rangle \mid \psi \in C_b^2(\mathbb{R}^d), \text{tr}\left(\frac{1}{2}\nabla^2\psi(x)\right) \leq 1 \ \forall x \in \mathbb{R}^d\right\} \\ &= \sup\left\{\langle \nu - \mu, \psi \rangle \mid \psi \in C_b^2(\mathbb{R}^d), \Delta\left[\frac{1}{2}\psi(x) - \frac{|x|^2}{2d}\right] \leq 0 \ \forall x \in \mathbb{R}^d\right\} \\ &= \left\langle \nu - \mu, \frac{1}{d}|\cdot|^2 \right\rangle + \sup\{\langle \nu - \mu, \varphi \rangle \mid \varphi \in C_b^2(\mathbb{R}^d), \Delta\varphi(x) \leq 0 \ \forall x \in \mathbb{R}^d\} \\ &= \begin{cases} \frac{1}{d}\text{var}(\nu) - \frac{1}{d}\text{var}(\mu) & \text{if } \mu \leq_{sh} \nu \\ +\infty & \text{otherwise} \end{cases} \end{aligned} \quad (4.3)$$

$$= \begin{cases} f\left(\int_{\mathbb{R}^d} x \otimes x d(\nu(x) - \mu(x))\right) & \text{if } \mu \leq_{sh} \nu \\ +\infty & \text{otherwise,} \end{cases} \quad (4.4)$$

where to obtain (4.3) we have used the fact that if $\mu \leq_{sh} \nu$ (see Definition 2.5), then $[\mu] = [\nu]$ and hence $\langle \nu - \mu, \frac{1}{d} |\cdot|^2 \rangle = \frac{1}{d} \text{var}(\nu) - \frac{1}{d} \text{var}(\mu)$. To obtain (4.4) we have used the following. Notice that a competitor λ for (3.5) such that $\int_{\mathbb{R}^d} f \left(\frac{d\lambda}{d|\lambda|} \right) d|\lambda| < +\infty$ with f defined by (4.1) has the form $\lambda = u I_d m$, where m is a nonnegative measure on \mathbb{R}^d , $u \geq 0$ m -a.e. on \mathbb{R}^d and $u \in L^1(\mathbb{R}^d, dm)$. Furthermore, such a measure λ solves $\text{tr} \left(\frac{1}{2} \nabla^2 \lambda \right) = \nu - \mu$ if and only if $\Delta \left(\frac{1}{2} u m \right) = \nu - \mu$ in the weak sense (see (3.4)). Thus, for each $i, j \in \{1, \dots, d\}$ such that $i \neq j$, we have

$$\int_{\mathbb{R}^d} (x_i^2 - x_j^2) d\nu(x) - \int_{\mathbb{R}^d} (x_i^2 - x_j^2) d\mu(x) = \int_{\mathbb{R}^d} \frac{1}{2} \Delta [x_i^2 - x_j^2] u(x) dm(x) = 0 \quad (4.5)$$

and

$$\int_{\mathbb{R}^d} x_i x_j d\nu(x) - \int_{\mathbb{R}^d} x_i x_j d\mu(x) = \int_{\mathbb{R}^d} \frac{1}{2} \Delta [x_i x_j] u(x) dm(x) = 0. \quad (4.6)$$

If $\mu \leq_{sh} \nu$, by (4.3), $F(\mu, \nu) < +\infty$. Then, (4.5) and (4.6) imply that $\int_{\mathbb{R}^d} x \otimes x d(\nu(x) - \mu(x)) \in S_d^+$ is a diagonal matrix equal to $\int_{\mathbb{R}^d} \frac{1}{d} |x|^2 d(\nu(x) - \mu(x)) I_d$ and hence

$$f \left(\int_{\mathbb{R}^d} x \otimes x d(\nu(x) - \mu(x)) \right) = \int_{\mathbb{R}^d} \frac{1}{d} |x|^2 d(\nu(x) - \mu(x)) = \frac{1}{d} \text{var}(\nu) - \frac{1}{d} \text{var}(\mu).$$

Thus, for each $a \in \mathbb{R}^d$, for each $\xi \in \mathbb{R}^d$ and for each $H \in S_d$ such that $\text{tr}(H) = 0$, the function

$$\psi(x) = a + x \cdot \xi + \left(H + \frac{I_d}{d} \right) : x \otimes x$$

is the dual optimizer for $F(\mu, \nu)$ when f is defined by (4.1).

Next, we check the assumptions of Theorem 4.1. For each $r > 0$ and $y \in \mathbb{R}^d$, we define the measure $\lambda_{y,r} = 2I_d g(\cdot, y) \mathcal{L}^d \llcorner B_r(y) \in \mathcal{M}(\mathbb{R}^d, S_d^+)$, where g is the Green function of the Laplacian on $B_r(y)$. Let $\psi \in \mathcal{U}_b(\mathbb{R}^d)$ and $-\psi = (-\psi)^G$. Using the facts that $\text{tr} \left(\frac{1}{2} \nabla^2 \lambda_{y,r} \right) = \mathcal{H}^{d-1}(\partial B_r(y))^{-1} \mathcal{H}^{d-1} \llcorner \partial B_r(y) - \delta_y$ and

$$\int_{\mathbb{R}^d} f \left(\frac{d\lambda_{y,r}}{d|\lambda_{y,r}|} \right) d|\lambda_{y,r}| = \int_{B_r(y)} 2g(x, y) dx = \frac{r^2}{d},$$

according to Remark 3.21, we have

$$\int_{\partial B_r(y)} -\psi(x) d\mathcal{H}^{d-1}(x) + \frac{r^2}{d} \geq -\psi(y).$$

This implies that

$$\int_{\partial B_r(y)} \left(-\psi(x) + \frac{|x|^2}{d} \right) d\mathcal{H}^{d-1}(x) \geq -\psi(y) + \frac{|y|^2}{d},$$

since

$$\int_{\partial B_r(y)} \frac{|x|^2}{d} d\mathcal{H}^{d-1}(x) = \frac{r^2}{d} + \frac{|y|^2}{d}.$$

Then, by Definition 2.4, $\Psi(\cdot) := -\psi(\cdot) + \frac{1}{d} |\cdot|^2 \in \mathcal{SH}(\mathbb{R}^d)$. Fix an arbitrary $\varepsilon > 0$. Defining for each $x \in \mathbb{R}^d$,

$$\Psi_\varepsilon(x) = \int_{\mathbb{R}^d} \Psi(y) \eta_\varepsilon(x - y) dy,$$

where $\eta_\varepsilon(\cdot) = \varepsilon^{-d} \eta(\cdot/\varepsilon) \in C_c^\infty(\mathbb{R}^d)$ and η is a standard mollifier (namely, $\eta \in C_c^\infty(\mathbb{R}^d)$, $\text{supp}(\eta) = \bar{B}_1(0)$, $\eta \geq 0$, $\eta(x) = \eta(-x)$ and $\int_{\mathbb{R}^d} \eta dx = 1$), we observe that $\Psi_\varepsilon \in C_b^2(\mathbb{R}^d)$ is subharmonic on \mathbb{R}^d (the reader may consult the proof of Lemma 4.9). For each $p \in \mathcal{P}_2(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$ such that $\Delta \left(\frac{1}{2} u m \right) = p - \delta_y$ for some nonnegative measure m on \mathbb{R}^d and $u \in L^1(\mathbb{R}^d, dm)$ such that $u \geq 0$ m -a.e. on \mathbb{R}^d , it holds $\delta_y \leq_{sh} p$. Thus,

$$\int_{\mathbb{R}^d} \Psi_\varepsilon(x) dp(x) \geq \Psi_\varepsilon(y), \quad (4.7)$$

since $\Psi_\varepsilon \in C_b^2(\mathbb{R}^d) \cap \mathcal{SH}(\mathbb{R}^d)$ (see Definition 2.5). Next, observing that $\frac{1}{d} |\cdot|^2 \in C_b^2(\mathbb{R}^d) \cap \mathcal{SH}(\mathbb{R}^d)$, where

$$\frac{|x|_\varepsilon^2}{d} = \int_{\mathbb{R}^d} \frac{|y|^2}{d} \eta_\varepsilon(x - y) dy \quad \text{and} \quad \Delta \frac{|x|_\varepsilon^2}{d} = 2 \quad \forall x \in \mathbb{R}^d,$$

if $\Delta(\frac{1}{2}um) = p - \delta_y$, we have

$$\int_{\mathbb{R}^d} u \, dm = \int_{\mathbb{R}^d} \frac{1}{2} \Delta \left(\frac{|x|_\varepsilon^2}{d} \right) u(x) \, dm(x) = \int_{\mathbb{R}^d} \frac{|x|_\varepsilon^2}{d} \, dp(x) - \frac{|y|_\varepsilon^2}{d}.$$

Using (4.7), the above formula and the fact that

$$\int_{\mathbb{R}^d} u \, dm = \int_{\mathbb{R}^d} f \left(\frac{d\lambda}{d|\lambda|} \right) d|\lambda|,$$

where $d\lambda = uI_d \, dm$, we get

$$\int_{\mathbb{R}^d} \left(\Psi_\varepsilon(x) - \frac{|x|_\varepsilon^2}{d} \right) dp(x) + \int_{\mathbb{R}^d} f \left(\frac{d\lambda}{d|\lambda|} \right) d|\lambda| \geq \Psi_\varepsilon(y) - \frac{|y|_\varepsilon^2}{d}.$$

This, in view of Remark 3.21, implies that $\Psi_\varepsilon - \frac{1}{d}|\cdot|_\varepsilon^2 = (\Psi_\varepsilon - \frac{1}{d}|\cdot|_\varepsilon^2)^G$ on \mathbb{R}^d . Fix now a sequence of sufficiently small positive numbers $(\varepsilon_n)_{n \in \mathbb{N}}$ such that $\varepsilon_n \rightarrow 0+$ as $n \rightarrow +\infty$. Observe that

$$\Psi_{\varepsilon_n}(x) - \frac{|x|_{\varepsilon_n}^2}{d} \rightarrow -\psi(x) \quad \forall x \in \mathbb{R}^d$$

(here we use that the convolution of a subharmonic function converges to this function everywhere, in view of the monotonicity condition of subharmonic functions; see, for instance, [21, Section 2.9]). Altogether, we have defined the approximation sequence $(\psi_n)_{n \in \mathbb{N}} = (-\Psi_{\varepsilon_n} + \frac{1}{d}|\cdot|_{\varepsilon_n}^2)_{n \in \mathbb{N}}$ for ψ in the sense of Theorem 4.1. Therefore, according to Theorem 4.1, $F = H$ when f is defined by (4.1).

Our second type of approximation assumption is related to the assumption (A4).

Theorem 4.2. *Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and (A4) hold. Assume that for each $\varphi \in \Phi_{b,2}(\mathbb{R}^d)$ such that $\varphi = \varphi^G$ there exists a sequence $(\varphi_n)_{n \in \mathbb{N}} \subset C_b^2(\mathbb{R}^d)$ such that $\varphi_n = \varphi_n^G$ for each $n \in \mathbb{N}$ and $\langle \nu - \mu, \varphi_n \rangle \rightarrow \langle \nu - \mu, \varphi \rangle$ as $n \rightarrow +\infty$. Then $F(\mu, \nu) = H(\mu, \nu)$.*

Proof. By Proposition 3.10, $F(\mu, \nu) \leq H(\mu, \nu)$. Next, using the assumption of Theorem 4.2, together with Theorem 3.26 and Theorem 3.29, we have $H(\mu, \nu) \leq F(\mu, \nu)$, which completes our proof of Theorem 4.2. \square

Example (ii). For some $\xi \in \mathbb{R}^d$ and for each $A \in S_d$,

$$f(A) = \begin{cases} A : \xi \otimes \xi & \text{if } A \in S_d^+ \\ +\infty & \text{otherwise.} \end{cases} \quad (4.8)$$

Then f satisfies (A1), (A2), (A4) and for each $A \in S_d$,

$$f^*(A) = \sup\{(A - \xi \otimes \xi) : M \mid M \in S_d^+\} = \begin{cases} 0 & \text{if } A - \xi \otimes \xi \leq 0 \\ +\infty & \text{otherwise.} \end{cases} \quad (4.9)$$

Given $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, using Theorem 3.6 and (4.9), we obtain

$$\begin{aligned} F(\mu, \nu) &= \sup \left\{ \langle \nu - \mu, \psi \rangle \mid \psi \in C_b^2(\mathbb{R}^d), \frac{1}{2} \nabla^2 \psi(x) - \xi \otimes \xi \leq 0 \quad \forall x \in \mathbb{R}^d \right\} \\ &= \int_{\mathbb{R}^d} |x \cdot \xi|^2 \, d\nu(x) - \int_{\mathbb{R}^d} |x \cdot \xi|^2 \, d\mu(x) + \sup \left\{ \langle \nu - \mu, \varphi \rangle \mid \varphi \in C_b^2(\mathbb{R}^d), \varphi \text{ is concave on } \mathbb{R}^d \right\} \\ &= \begin{cases} \int_{\mathbb{R}^d} |x \cdot \xi|^2 \, d\nu(x) - \int_{\mathbb{R}^d} |x \cdot \xi|^2 \, d\mu(x) & \text{if } \mu \leq_c \nu \\ +\infty & \text{otherwise,} \end{cases} \\ &= \begin{cases} f \left(\int_{\mathbb{R}^d} x \otimes x \, d(\nu(x) - \mu(x)) \right) & \text{if } \mu \leq_c \nu \\ +\infty & \text{otherwise,} \end{cases} \end{aligned} \quad (4.10)$$

where to obtain (4.10) we have used Remark 2.3. Notice that for each $a \in \mathbb{R}$ and for each $\zeta \in \mathbb{R}^d$, the function

$$\psi(x) = a + x \cdot \zeta + \xi \otimes \xi : x \otimes x$$

is the dual optimizer for $F(\mu, \nu)$ when f is defined by (4.8).

Next, we check the assumptions of Theorem 4.2. Let $\varphi \in \Phi_2(\mathbb{R}^d)$. We claim that $\varphi = \varphi^G$ if and only if the function $\varphi_\xi \in \Phi_2(\mathbb{R}^d)$ defined by $\varphi_\xi(x) = \varphi(x) + |x \cdot \xi|^2$ is convex on \mathbb{R}^d . Indeed, if φ_ξ is convex, using Jensen's inequality and (4.10), for each $p \in \mathcal{P}_2(\mathbb{R}^d)$, we have

$$\varphi([p]) \leq \int_{\mathbb{R}^d} (\varphi(y) + |y \cdot \xi|^2) dp(y) - |[p] \cdot \xi|^2 = \int_{\mathbb{R}^d} \varphi dp + G([p], p), \quad (4.11)$$

which, in view of Remark 3.21 and (4.10), yields $\varphi^G = \varphi$ on \mathbb{R}^d . On the other hand, if $\varphi = \varphi^G$ on \mathbb{R}^d , then, according to Remark 3.21 and (4.10), for each $p \in \mathcal{P}_2(\mathbb{R}^d)$, (4.11) holds and hence

$$\varphi_\xi([p]) \leq \int_{\mathbb{R}^d} \varphi_\xi dp. \quad (4.12)$$

For each $x, y \in \mathbb{R}^d$, choosing $p = \frac{1}{2}(\delta_x + \delta_y)$ in (4.12), we obtain

$$\varphi_\xi\left(\frac{x+y}{2}\right) \leq \frac{\varphi_\xi(x)}{2} + \frac{\varphi_\xi(y)}{2},$$

which, since φ_ξ is continuous on \mathbb{R}^d , implies that φ_ξ is convex on \mathbb{R}^d . This completes the proof of our claim.

Let $\varphi \in \Phi_{b,2}(\mathbb{R}^d)$ satisfy $\varphi = \varphi^G$. Then φ_ξ is convex on \mathbb{R}^d . Furthermore, there exists a sequence $(\psi_k)_{k \in \mathbb{N}} \subset C_b^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \psi_k d\mu \rightarrow \int_{\mathbb{R}^d} \varphi_\xi d\mu$ and $\int_{\mathbb{R}^d} \psi_k d\nu \rightarrow \int_{\mathbb{R}^d} \varphi_\xi d\nu$ as $k \rightarrow +\infty$ (we refer to Remark 2.3). For each $k \in \mathbb{N}$, define $\varphi_k(x) = \psi_k(x) - |x \cdot \xi|^2$ for each $x \in \mathbb{R}^d$. Since $\psi_k \in C_b^2(\mathbb{R}^d)$ is convex, $\varphi_k = \varphi_k^G \in C_b^2(\mathbb{R}^d)$. This defines the approximation sequence for φ in the sense of Theorem 4.2, since $\int_{\mathbb{R}^d} \varphi_k d\mu \rightarrow \int_{\mathbb{R}^d} (\varphi_\xi(x) - |x \cdot \xi|^2) d\mu(x) = \int_{\mathbb{R}^d} \varphi d\mu$ and $\int_{\mathbb{R}^d} \varphi_k d\nu \rightarrow \int_{\mathbb{R}^d} (\varphi_\xi(x) - |x \cdot \xi|^2) d\nu(x) = \int_{\mathbb{R}^d} \varphi d\nu$ as $k \rightarrow +\infty$. Therefore, according to Theorem 4.2, $F = H$ when f is defined by (4.8).

4.2. Viscosity solutions

In this subsection, under the assumption (A3), we characterize the functions $\psi \in C_b(\mathbb{R}^d)$ such that $-\psi = (-\psi)^G$ as viscosity solutions of the Hamilton-Jacobi-Bellman equation $-f^*(\frac{1}{2}\nabla^2 u) = 0$ in \mathbb{R}^d (notice that f^* is discontinuous and takes its values in $\{0, +\infty\}$, see (3.2)).

Remark 4.3. Let (A3) hold. Since $\text{dom}(f) \cap S_d^{++}$ is dense in $\text{dom}(f) \neq \emptyset$, for each $A \in S_d$, we have

$$f^*(A) = \sup_{t>0} t(\mathcal{T}(A) - 1) = \begin{cases} 0 & \text{if } \mathcal{T}(A) \leq 1 \\ +\infty & \text{otherwise,} \end{cases} \quad (4.13)$$

where

$$\mathcal{T}(A) = \sup\{A : E \mid E \in S_d^{++}, f(E) = 1\}. \quad (4.14)$$

In view of Proposition 3.28 and (4.13), if $\psi \in C_b^2(\mathbb{R}^d)$, then $-\psi = (-\psi)^G$ on \mathbb{R}^d if and only if $1 - \mathcal{T}(\frac{1}{2}\nabla^2 \psi) \geq 0$ in \mathbb{R}^d . However, a function $\psi \in C_b(\mathbb{R}^d)$ such that $-\psi = (-\psi)^G$ may not be regular enough to define $\nabla^2 \psi$ in the classical sense. Using the theory of viscosity solutions, we can define $\nabla^2 \psi$ in the viscosity sense. Taking into account Theorems 3.27 and 3.29, to derive the equality between F and H , we first show that each $\psi \in C_b(\mathbb{R}^d)$ such that $-\psi = (-\psi)^G$ is a viscosity supersolution of the equation $1 - \mathcal{T}(\frac{1}{2}\nabla^2 u) = 0$ in \mathbb{R}^d (see Definition 4.5).

Following [12], we shall say that a function $\mathcal{F} : S_d \rightarrow \mathbb{R}$ is *degenerate elliptic* and, actually, *proper* if

$$\mathcal{F}(A_2) \leq \mathcal{F}(A_1) \text{ whenever } A_1 \leq A_2. \quad (4.15)$$

Remark 4.4. For each $c \in \mathbb{R}$ and $A \in S_d^{++}$, the function $\mathcal{F}(B) = c - A : B$, $B \in S_d$ is proper. Furthermore, for each $B_1, B_2 \in S_d$ satisfying $B_1 \leq B_2$, we have

$$\sup \left\{ \frac{1}{2} B_2 : E \mid E \in S_d^{++}, f(E) = 1 \right\} \geq \sup \left\{ \frac{1}{2} B_1 : E \mid E \in S_d^{++}, f(E) = 1 \right\}$$

and hence $c - \mathcal{T}(\frac{1}{2}B_2) \leq c - \mathcal{T}(\frac{1}{2}B_1)$, where $\mathcal{T} : S_d \rightarrow \mathbb{R}$ is defined in (4.14). Thus, $\mathcal{F}(\cdot) = c - \mathcal{T}(\frac{1}{2}\cdot)$ is proper for each $c \in \mathbb{R}$.

For the reader's convenience, we recall (the reader may consult [12]) the next

Definition 4.5. Let $\mathcal{F} : S_d \rightarrow \mathbb{R}$ be proper (see (4.15)). A lower semicontinuous function $u : \mathbb{R}^d \rightarrow \mathbb{R}$ is a viscosity supersolution of $\mathcal{F} = 0$ (a viscosity solution of $\mathcal{F} \geq 0$) in \mathbb{R}^d provided that if $\varphi \in C^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ is a local minimum of $u - \varphi$, then

$$\mathcal{F}(\nabla^2 \varphi(x)) \geq 0.$$

Similarly, an upper semicontinuous function $u : \mathbb{R}^d \rightarrow \mathbb{R}$ is a viscosity subsolution of $\mathcal{F} = 0$ (a viscosity solution of $\mathcal{F} \leq 0$) in \mathbb{R}^d provided that if $\varphi \in C^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ is a local maximum of $u - \varphi$, then

$$\mathcal{F}(\nabla^2 \varphi(x)) \leq 0.$$

Finally, $u \in C(\mathbb{R}^d)$ is a viscosity solution of $\mathcal{F} = 0$ in \mathbb{R}^d if it is both a viscosity supersolution and a viscosity subsolution of $\mathcal{F} = 0$ in \mathbb{R}^d .

Notice that the function \mathcal{F} in (4.15) and in Definition 4.5 can be discontinuous. Even more, allowing \mathcal{F} to become infinite (see, for instance, [12, Example 1.11]), we observe that $\mathcal{F} = -f^* : S_d \rightarrow \{0, -\infty\}$ is proper.

Proposition 4.6. Let $\psi \in C_b(\mathbb{R}^d)$ satisfy $-\psi = (-\psi)^G$ on \mathbb{R}^d . Then ψ is a viscosity supersolution of $1 - \mathcal{T}(\frac{1}{2}\nabla^2 u) = 0$ in \mathbb{R}^d .

Proof. Let $x_0 \in \mathbb{R}^d$, $\varphi \in C^2(\mathbb{R}^d)$ and x_0 be a local minimum of $\psi - \varphi$. Then there exists $r > 0$ such that $\psi(x) - \psi(x_0) \geq \varphi(x) - \varphi(x_0)$ for each $x \in \overline{B}_r(x_0)$. Fix an arbitrary $A = (a_{ij})_{i,j=1}^d \in S_d^{++}$ such that $f(A) = 1$. Let g_A be the Green function of the elliptic operator $L = -\sum_{i,j=1}^d a_{ij} \partial_{ij}$ on $B_r(x_0)$. Then, defining $\lambda = 2Ag_A(\cdot, x_0) \mathcal{L}^d \llcorner B_r(x_0) \in \mathcal{M}(\mathbb{R}^d, S_d^+)$ and $p = -A \nabla g_A(\cdot, x_0) \cdot \nu(\cdot) \mathcal{H}^{d-1} \llcorner \partial B_r(x_0) \in \mathcal{P}_2(\mathbb{R}^d)$, where $\nu(x)$ is the outward pointing unit normal to $\partial B_r(x_0)$ at x , we know that

$$\mathrm{tr} \left(\frac{1}{2} \nabla^2 \lambda \right) = p - \delta_{x_0} \quad (4.16)$$

(see the proof of Proposition 3.28). Since $\varphi - \varphi(x_0) \leq \psi - \psi(x_0)$ on $\overline{B}_r(x_0)$,

$$\int_{\mathbb{R}^d} \varphi dp - \varphi(x_0) \leq \int_{\mathbb{R}^d} \psi dp - \psi(x_0) \leq G(x_0, p), \quad (4.17)$$

where the last estimate comes from the fact that $-\psi = (-\psi)^G$ (see Remark 3.21). On the other hand, in view of (4.16), λ is a competitor in the definition of $G(x_0, p) = F(\delta_{x_0}, p)$ (see Proposition 3.2). Using this, (A1) and the fact that $f(A) = 1$, we obtain

$$G(x_0, p) \leq \int_{\mathbb{R}^d} f \left(\frac{d\lambda}{d|\lambda|} \right) d|\lambda| = 2 \int_{B_r(x_0)} f(A) g_A(x, x_0) dx = 2 \int_{B_r(x_0)} g_A(x, x_0) dx.$$

This, together with (4.17) and the fact that we can actually use φ as a test function for (4.16) (since the supports of the measures λ , p and δ_{x_0} are contained in $\overline{B}_r(x_0)$ and we can multiply φ by a cutoff function equal to 1 on $B_{2r}(x_0)$), implies that

$$\int_{B_r(x_0)} A : \nabla^2 \varphi(x) g_A(x, x_0) dx \leq 2 \int_{B_r(x_0)} g_A(x, x_0) dx. \quad (4.18)$$

Then $A : \frac{1}{2}\nabla^2\varphi(x_0) \leq 1$, because otherwise we could choose $r > 0$ small enough such that for some $\varepsilon > 0$ and for each $x \in B_r(x_0)$ we would have $A : \frac{1}{2}\nabla^2\varphi(x) - 1 \geq \varepsilon$, which would lead to a contradiction with (4.18), since $\int_{B_r(x_0)} g_A(x, x_0) dx > 0$. Thus, $1 - A : \frac{1}{2}\nabla^2\varphi(x_0) \geq 0$ for each $A \in S_d^{++}$ satisfying $f(A) = 1$. Therefore,

$$1 - \mathcal{T}\left(\frac{1}{2}\nabla^2\varphi(x_0)\right) = 1 - \sup\left\{A : \frac{1}{2}\nabla^2\varphi(x_0) \mid A \in S_d^{++}, f(A) = 1\right\} \geq 0.$$

This, according to Definition 4.5, completes our proof of Proposition 4.6. \square

Next, we perform a smoothing procedure by convolution with a mollifier for a viscosity solution of $1 - \mathcal{T}\left(\frac{1}{2}\nabla^2u\right) \geq 0$ in \mathbb{R}^d to obtain the classical solution.

Remark 4.7. Let $c \in \mathbb{R}$ and $w : \mathbb{R}^d \rightarrow \mathbb{R}$ be lower semicontinuous. Then w is a viscosity supersolution of $c - \mathcal{T}\left(\frac{1}{2}\nabla^2u\right) = 0$ in \mathbb{R}^d if and only if for each $A \in S_d^{++}$ such that $f(A) = 1$, w is a viscosity supersolution of $c - A : \frac{1}{2}\nabla^2u = 0$ in \mathbb{R}^d . Indeed, if $\varphi \in C^2(\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$ is a local minimum of $w - \varphi$, then

$$\mathcal{T}\left(\frac{1}{2}\nabla^2\varphi(x_0)\right) \leq c \Leftrightarrow A : \frac{1}{2}\nabla^2\varphi(x_0) \leq c \quad \forall A \in S_d^{++}, f(A) = 1,$$

in view of (4.14).

Lemma 4.8. Let $A \in S_d^{++}$, $c \in \mathbb{R}$ and $w : \mathbb{R}^d \rightarrow \mathbb{R}$ be lower semicontinuous. Let $B \in S_d^{++}$ be the unique matrix such that $A = B^{-2}$ and for each $x \in \mathbb{R}^d$, define $v(x) = w(B^{-1}x)$ and $h(x) = v(x) - \frac{c}{d}|x|^2$. Then the following assertions are equivalent.

(i) w is a viscosity supersolution of $c - A : \frac{1}{2}\nabla^2u = 0$ in \mathbb{R}^d .

(ii) v is a viscosity supersolution of $c - \frac{1}{2}\Delta u = 0$ in \mathbb{R}^d .

(iii) h is a viscosity supersolution of $-\frac{1}{2}\Delta u = 0$ in \mathbb{R}^d .

Proof. We first prove that (i) \Leftrightarrow (ii). By definition, $v : \mathbb{R}^d \rightarrow \mathbb{R}$ is lower semicontinuous. Assume that (i) holds. Let $\varphi \in C^2(\mathbb{R}^d)$ and $y_0 \in \mathbb{R}^d$ be a local minimum of $v - \varphi$. Define $\psi(x) = \varphi(Bx)$ for each $x \in \mathbb{R}^d$. Then $\psi \in C^2(\mathbb{R}^d)$ and $x_0 = B^{-1}y_0$ is a local minimum of $w - \psi$, which, by Definition 4.5, yields $c - A : \frac{1}{2}\nabla^2\psi(x_0) \geq 0$. Observing that $A : \frac{1}{2}\nabla^2\psi(x) = \frac{1}{2}\Delta\varphi(Bx)$ for each $x \in \mathbb{R}^d$, we have $c - A : \frac{1}{2}\nabla^2\psi(x_0) = c - \frac{1}{2}\Delta\varphi(Bx_0) = c - \frac{1}{2}\Delta\varphi(y_0) \geq 0$. Then, by Definition 4.5, (ii) holds, which proves the implication (i) \Rightarrow (ii). The proof of the implication (ii) \Rightarrow (i) is similar. Thus, (i) \Leftrightarrow (ii).

Now we prove that (ii) \Leftrightarrow (iii). By definition, $h : \mathbb{R}^d \rightarrow \mathbb{R}$ is lower semicontinuous. Let $\varphi \in C^2(\mathbb{R}^d)$. Then y_0 is a local minimum of $h - \varphi$ if and only if y_0 is a local minimum of $v - \psi$, where $\psi(\cdot) = \frac{c}{d}|\cdot|^2 + \varphi(\cdot)$. This and the fact that $c - \frac{1}{2}\Delta\psi(y_0) \geq 0$ if and only if $-\frac{1}{2}\Delta\varphi(y_0) \geq 0$, according to Definition 4.5, proves the equivalence between (ii) and (iii), which completes our proof of Lemma 4.8. \square

Lemma 4.9. Let $A \in S_d^{++}$, $c \in \mathbb{R}$ and $w \in L_{\text{loc}}^1(\mathbb{R}^d)$ be lower semicontinuous on \mathbb{R}^d . Let $\eta \in C_c^\infty(\mathbb{R}^d)$, $\text{supp}(\eta) = \overline{B}_1(0)$, $\eta \geq 0$, $\eta(x) = \eta(-x)$, $\int_{\mathbb{R}^d} \eta dx = 1$ and $\eta_\varepsilon(\cdot) = \varepsilon^{-d}\eta(\cdot/\varepsilon)$ for each $\varepsilon > 0$. Assume that w is a viscosity supersolution of $c - A : \frac{1}{2}\nabla^2u = 0$ in \mathbb{R}^d . Then for each $\varepsilon > 0$, $c - A : \frac{1}{2}\nabla^2w_\varepsilon \geq 0$ in \mathbb{R}^d , where $w_\varepsilon(\cdot) = \int_{\mathbb{R}^d} w(y)\eta_\varepsilon(\cdot - y) dy$.

Proof. Let $B \in S_d^{++}$ be the unique matrix such that $A = B^{-2}$. For each $x \in \mathbb{R}^d$, define $v(x) = w(B^{-1}x)$. Since w is a viscosity supersolution of $c - A : \frac{1}{2}\nabla^2u = 0$ in \mathbb{R}^d , by Lemma 4.8, v is a viscosity supersolution of $c - \frac{1}{2}\Delta u = 0$ in \mathbb{R}^d . Let us prove that $c - \frac{1}{2}\Delta v_\varepsilon \geq 0$ in \mathbb{R}^d , where $v_\varepsilon(\cdot) = \int_{\mathbb{R}^d} u(y)\tilde{\eta}_\varepsilon(\cdot - y) dy$ and $\tilde{\eta}_\varepsilon(\cdot) = (\det(B))^{-1}\eta_\varepsilon(B^{-1}\cdot)$. By definition, $\tilde{\eta}_\varepsilon \in C_c^\infty(\mathbb{R}^d)$, $\text{supp}(\tilde{\eta}_\varepsilon) \subset \{Bx \mid x \in \overline{B}_\varepsilon(0)\}$, $\tilde{\eta}_\varepsilon \geq 0$, $\tilde{\eta}_\varepsilon(x) = \tilde{\eta}_\varepsilon(-x)$ and $\int_{\mathbb{R}^d} \tilde{\eta}_\varepsilon(x) dx = 1$. By Lemma 4.8, v is a viscosity supersolution of $c - \frac{1}{2}\Delta u = 0$ in \mathbb{R}^d if and only if h is a viscosity supersolution of $-\frac{1}{2}\Delta u = 0$ in \mathbb{R}^d , where $h(x) = v(x) - \frac{c}{d}|x|^2$ for each $x \in \mathbb{R}^d$. The latter holds if and only if $\int_{\partial B_r(x_0)} h(x) d\mathcal{H}^{d-1}(x) \leq h(x)$ for each $x_0 \in \mathbb{R}^d$ and $r > 0$, namely h is superharmonic in \mathbb{R}^d (see Definition 2.4). Let us show that $h_\varepsilon(\cdot) = \int_{\mathbb{R}^d} h(y)\tilde{\eta}_\varepsilon(\cdot - y) dy$ is superharmonic

in \mathbb{R}^d , which is equivalent to the property $c - \frac{1}{2}\Delta v_\varepsilon \geq 0$ in \mathbb{R}^d . Thus, changing the variables and applying Fubini's theorem, we obtain

$$\begin{aligned} \int_{\partial B_r(x)} h_\varepsilon(y) d\mathcal{H}^{d-1}(y) &= \int_{\partial B_r(x)} d\mathcal{H}^{d-1}(y) \int_{\mathbb{R}^d} h(z) \tilde{\eta}_\varepsilon(y-z) dz \\ &= \int_{\mathbb{R}^d} dz \int_{\partial B_r(x)} h(y-z) \tilde{\eta}_\varepsilon(z) d\mathcal{H}^{d-1}(y) \\ &\leq \int_{\mathbb{R}^d} h(x-z) \tilde{\eta}_\varepsilon(z) dz = h_\varepsilon(x), \end{aligned}$$

where we have also used that $\int_{\partial B_r(x-z)} h(y) d\mathcal{H}^{d-1}(y) \leq h(x-z)$, since h is superharmonic in \mathbb{R}^d . Then, by Definition 2.4, h_ε is superharmonic in \mathbb{R}^d . This, since $h_\varepsilon \in C^2(\mathbb{R}^d)$, implies that $-\frac{1}{2}\Delta h_\varepsilon \geq 0$ in \mathbb{R}^d and hence $c - \frac{1}{2}\Delta v_\varepsilon \geq 0$ in \mathbb{R}^d as desired.

Next, observe that $A : \nabla^2 \eta_\varepsilon(x) = \det(B) \Delta \tilde{\eta}_\varepsilon(Bx)$ for each $x \in \mathbb{R}^d$. Using this and changing the variables, for each $x \in \mathbb{R}^d$ we deduce the following

$$\begin{aligned} 0 \leq c - \frac{1}{2}\Delta v_\varepsilon(Bx) &= c - \frac{1}{2} \int_{\mathbb{R}^d} v(y) \Delta \tilde{\eta}_\varepsilon(Bx-y) dy \\ &= c - \frac{(\det(B))^{-1}}{2} \int_{\mathbb{R}^d} v(y) A : \nabla^2 \eta_\varepsilon(x - B^{-1}y) dy \\ &= c - \frac{1}{2} \int_{\mathbb{R}^d} v(By) A : \nabla^2 \eta_\varepsilon(x-y) dy = c - A : \frac{1}{2} \nabla^2 w_\varepsilon(x), \end{aligned}$$

which completes our proof of Lemma 4.9. \square

Corollary 4.10. Let $c \in \mathbb{R}$, $w \in L^1_{\text{loc}}(\mathbb{R}^d)$ be lower semicontinuous on \mathbb{R}^d , $\eta \in C_c^\infty(\mathbb{R}^d)$ be the mollifier of Lemma 4.9 and $\eta_\varepsilon(\cdot) = \varepsilon^{-d} \eta(\cdot/\varepsilon)$ for each $\varepsilon > 0$. Assume that w is a viscosity supersolution of $c - \mathcal{T}(\frac{1}{2}\nabla^2 u) = 0$ in \mathbb{R}^d , where \mathcal{T} is defined in (4.14). Then for each $\varepsilon > 0$, $c - \mathcal{T}(\frac{1}{2}\nabla^2 w_\varepsilon) \geq 0$ in \mathbb{R}^d , where $w_\varepsilon(\cdot) = \int_{\mathbb{R}^d} w(y) \eta_\varepsilon(\cdot - y) dy$.

Proof of Corollary 4.10. According to Remark 4.7 and Lemma 4.9, for each $\varepsilon > 0$ and for each $A \in S_d^{++}$ such that $f(A) = 1$, it holds $c - A : \frac{1}{2} \nabla^2 w_\varepsilon \geq 0$ in \mathbb{R}^d . Therefore,

$$c \geq \sup \left\{ A : \frac{1}{2} \nabla^2 w_\varepsilon(x) \mid A \in S_d^{++}, f(A) = 1 \right\} = \mathcal{T} \left(\frac{1}{2} \nabla^2 w_\varepsilon(x) \right) \quad \forall x \in \mathbb{R}^d,$$

which completes our proof of Corollary 4.10. \square

Now we characterize the dual competitors in (3.37) as viscosity solutions of $-f^*(\frac{1}{2}\nabla^2 u) = 0$ in \mathbb{R}^d .

Proposition 4.11. Let $\psi \in C_b(\mathbb{R}^d)$ and (A3) hold. Then $-\psi = (-\psi)^G$ on \mathbb{R}^d if and only if ψ is a viscosity solution of $-f^*(\frac{1}{2}\nabla^2 u) = 0$ in \mathbb{R}^d .

Remark 4.12. In [24, Theorem 4.2], in the Markovian case, it was proved that the dynamic value function is a viscosity solution of the Hamilton-Jacobi-Bellman equation. It is worth noting that, unlike [24], in our case f^* is not continuous and $\text{dom}(f^*)$ is a closed convex subset of S_d . Furthermore, we provide an analytical proof of Proposition 4.11 that is different from the proof of [24, Theorem 4.2], showing in addition that if ψ is a viscosity solution of the Hamilton-Jacobi-Bellman equation $-f^*(\frac{1}{2}\nabla^2 u) = 0$ in \mathbb{R}^d , then $-\psi$ is invariant under the G -transform.

Proof. By Proposition 4.6, (4.13) and Definition 4.5 (where we allow \mathcal{F} to be discontinuous and, even more, to become infinite), if $\psi \in C_b(\mathbb{R}^d)$ and $-\psi = (-\psi)^G$ on \mathbb{R}^d , then ψ is a viscosity solution of $-f^*(\frac{1}{2}\nabla^2 u) = 0$ in \mathbb{R}^d .

Conversely, if $\psi \in C_b(\mathbb{R}^d)$ is a viscosity solution of $-f^*(\frac{1}{2}\nabla^2 u) = 0$ in \mathbb{R}^d , by (4.13) and Definition 4.5, ψ is a viscosity supersolution of $1 - \mathcal{T}(\frac{1}{2}\nabla^2 u) = 0$ in \mathbb{R}^d . Next, applying Corollary 4.10, we deduce that for each $\varepsilon > 0$, $\psi_\varepsilon(\cdot) = \int_{\mathbb{R}^d} \psi(y) \eta_\varepsilon(\cdot - y) dy$, where η is the mollifier of Lemma 4.9, it holds $1 - \mathcal{T}(\frac{1}{2}\nabla^2 \psi_\varepsilon) \geq 0$ in \mathbb{R}^d and hence $-f^*(\frac{1}{2}\nabla^2 \psi_\varepsilon) = 0$ in \mathbb{R}^d (see (4.13)). This, according to Proposition 3.28, yields the

equality $-\psi_\varepsilon = (-\psi_\varepsilon)^G$ on \mathbb{R}^d for each $\varepsilon > 0$. According to Proposition 3.15, for each $x \in \mathbb{R}^d$ and $\varepsilon > 0$, there exist $p, p_\varepsilon \in \mathcal{P}_2(\mathbb{R}^d)$ such that

$$(-\psi_\varepsilon)^G(x) = \int_{\mathbb{R}^d} -\psi_\varepsilon dp_\varepsilon + G(x, p_\varepsilon) \quad (4.19)$$

and

$$(-\psi)^G(x) = \int_{\mathbb{R}^d} -\psi dp + G(x, p). \quad (4.20)$$

Using (4.20), the fact that $\psi \in C_b(\mathbb{R}^d)$, Lebesgue's dominated convergence theorem, (4.19) and the equality $-\psi_\varepsilon = (-\psi_\varepsilon)^G$, for each $x \in \mathbb{R}^d$, we have

$$\begin{aligned} (-\psi)^G(x) &= \int_{\mathbb{R}^d} -\psi dp + G(x, p) \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^d} -\psi_\varepsilon dp + G(x, p) \\ &\geq \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^d} -\psi_\varepsilon dp_\varepsilon + G(x, p_\varepsilon) \\ &= \lim_{\varepsilon \rightarrow 0^+} -\psi_\varepsilon(x) \\ &= -\psi(x). \end{aligned}$$

This, in view of the fact that $-\psi \geq (-\psi)^G$ on \mathbb{R}^d , proves the equality $-\psi = (-\psi)^G$ on \mathbb{R}^d and completes our proof of Proposition 4.11. \square

The next corollary is a direct consequence of Theorem 3.27 and Proposition 4.11.

Corollary 4.13. Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and (A3), (A4) hold. Then

$$H(\mu, \nu) = \sup \left\{ \langle \nu - \mu, \psi \rangle \mid \psi \in C_b(\mathbb{R}^d) \text{ is a viscosity solution of } -f^* \left(\frac{1}{2} \nabla^2 u \right) = 0 \text{ in } \mathbb{R}^d \right\}.$$

4.3. $F = H$ under the assumptions (A3), (A4)

Theorem 4.14. Let (A3) and (A4) hold. Then $F = H$ on $\mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d)$.

Proof. Let $\psi \in C_b(\mathbb{R}^d)$ be a viscosity solution of $-f^*(\frac{1}{2}\nabla^2 u) = 0$ in \mathbb{R}^d . Then, performing the smoothing procedure as in the proof of Proposition 4.11, we obtain a sequence of functions $\psi_n \in C_b^2(\mathbb{R}^d)$ such that $-f^*(\frac{1}{2}\nabla^2 \psi_n) = 0$ in \mathbb{R}^d , $\int_{\mathbb{R}^d} \psi_n d\mu \rightarrow \int_{\mathbb{R}^d} \psi d\mu$ and $\int_{\mathbb{R}^d} \psi_n d\nu \rightarrow \int_{\mathbb{R}^d} \psi d\nu$ as $n \rightarrow +\infty$. Using this, Corollary 4.13 and Theorem 3.6, we obtain $H(\mu, \nu) \leq F(\mu, \nu)$. Therefore, $H(\mu, \nu) = F(\mu, \nu)$, since $H(\mu, \nu) \geq F(\mu, \nu)$ by Proposition 3.10. This completes our proof of Theorem 4.14. \square

Example (iii). For each $A \in S_d$,

$$f(A) = \begin{cases} \text{tr}(A) & \text{if } A \in S_d^+ \\ +\infty & \text{otherwise.} \end{cases} \quad (4.21)$$

Clearly, f satisfies (A1)-(A4). Then for each $A \in S_d$,

$$\mathcal{T}(A) \leq 1 \Leftrightarrow A - I_d \leq 0. \quad (4.22)$$

Indeed, $A : M \leq 1$ for each $M \in S_d^{++}$ such that $\text{tr}(M) = 1$ if and only if $(A - I_d) : M \leq 0$ for each $M \in S_d^+$, which is equivalent to say that $A - I_d \leq 0$. Using this and (4.14), one deduces (4.22).

Given $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, applying Theorem 3.6, (4.13) and (4.22), we compute

$$\begin{aligned} F(\mu, \nu) &= \sup \left\{ \langle \nu - \mu, \psi \rangle \mid \psi \in C_b^2(\mathbb{R}^d), \frac{1}{2} \nabla^2 \psi(x) - I_d \leq 0 \quad \forall x \in \mathbb{R}^d \right\} \\ &= \langle \nu - \mu, |\cdot|^2 \rangle + \sup \{ \langle \nu - \mu, u \rangle \mid u \in C_b^2(\mathbb{R}^d), u \text{ is concave on } \mathbb{R}^d \} \\ &= \begin{cases} \text{var}(\nu) - \text{var}(\mu) & \text{if } \mu \leq_c \nu \\ +\infty & \text{otherwise} \end{cases} \end{aligned} \quad (4.23)$$

$$= \begin{cases} f \left(\int_{\mathbb{R}^d} x \otimes x d(\nu(x) - \mu(x)) \right) & \text{if } \mu \leq_c \nu \\ +\infty & \text{otherwise,} \end{cases} \quad (4.24)$$

where we have used that $\langle \nu - \mu, |\cdot|^2 \rangle = \text{var}(\nu) - \text{var}(\mu)$ if $\mu \leq_c \nu$ (in this case $[\mu] = [\nu]$). For each $a \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$, the function $\psi(x) = a + \xi \cdot x + |x|^2$ is the dual optimizer for $F(\mu, \nu)$ when f is defined by (4.21). According to Theorem 4.14, $F = H$ when f is defined by (4.21).

Example (iv). For each $A \in S_d$,

$$f(A) = \begin{cases} \lambda_{\max}(A) & \text{if } A \in S_d^+ \\ +\infty & \text{otherwise,} \end{cases} \quad (4.25)$$

where $\lambda_{\max}(A) = \max\{Ax \cdot x \mid |x| = 1\}$ is the largest eigenvalue of A . Clearly, f satisfies (A1)-(A4). Every matrix $A \in S_d$ has d real eigenvalues $\lambda_1, \dots, \lambda_d$ such that $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_d|$ with corresponding eigenvectors $v_1, \dots, v_d \in \mathbb{R}^d$ forming an orthonormal basis of \mathbb{R}^d . Denote $J_+ = \{j \in \{1, \dots, d\} \mid \lambda_j \geq 0\}$, $J_- = \{1, \dots, d\} \setminus J_+$, $A^+ = \sum_{j \in J_+} \lambda_j v_j \otimes v_j$, $A^- = \sum_{j \in J_-} \lambda_j v_j \otimes v_j$ so that $A = A^+ + A^-$. We claim that

$$\mathcal{T}(A) \leq 1 \Leftrightarrow \sum_{j \in J_+} \lambda_j \leq 1. \quad (4.26)$$

Indeed, if $A \in S_d$, then $A = PDP^T$, where $D = \text{diag}(\lambda_1, \dots, \lambda_d)$ and $P = (v_1 \dots v_d) \in S_d$ is the orthonormal matrix with columns v_1, \dots, v_d . Let $\varepsilon \in (0, 1)$ be fairly small. Define $M = PEP^T$, where $E = (e_{ij})_{i,j=1}^d \in S_d^+$ is the diagonal matrix satisfying $e_{ii} = \varepsilon$ if $\lambda_i < 0$ and $e_{ii} = 1$ otherwise. Next, we compute

$$A : M = \text{tr}(PDP^T PEP^T) = \text{tr}(DE) = \sum_{j \in J_+} \lambda_j + \varepsilon \sum_{j \in J_-} \lambda_j.$$

Thus, assuming that $\mathcal{T}(A) \leq 1$ and letting $\varepsilon \rightarrow 0+$, we have $\sum_{j \in J_+} \lambda_j \leq 1$. Conversely, suppose that $\sum_{j \in J_+} \lambda_j \leq 1$. For each $M \in S_d^{++}$ such that $\lambda_{\max}(M) = 1$,

$$A : M = (A^+ + A^-) : M \leq \text{tr}(A^+ M) = \sum_{j \in J_+} \lambda_j M v_j \cdot v_j \leq \sum_{j \in J_+} \lambda_j \lambda_{\max}(M) \leq 1.$$

This proves our claim.

Given $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, using Theorem 3.6, (4.13) and (4.26), we deduce that

$$F(\mu, \nu) = \sup \left\{ \langle \nu - \mu, \psi \rangle \mid \psi \in C_b^2(\mathbb{R}^d), \sum_{j \in J_+} \lambda_j \left(\frac{1}{2} \nabla^2 \psi(x) \right) \leq 1 \quad \forall x \in \mathbb{R}^d \right\}. \quad (4.27)$$

Since $\frac{1}{d} \text{tr}(A) \leq \lambda_{\max}(A) \leq \text{tr}(A)$ for each $A \in S_d^+$, we have $\frac{1}{d} \tilde{F}(\mu, \nu) \leq F(\mu, \nu) \leq \tilde{F}(\mu, \nu)$, where

$$\tilde{F}(\mu, \nu) = \inf \left\{ \int_{\mathbb{R}^d} \text{tr} \left(\frac{d\lambda}{d|\lambda|} \right) d|\lambda| \mid \text{tr} \left(\frac{1}{2} \nabla^2 \lambda \right) = \nu - \mu \right\}.$$

In particular, if $d = 1$, then $\tilde{F}(\mu, \nu) = F(\mu, \nu)$ and the expression of $F(\mu, \nu)$ in terms of μ and ν comes from (4.23), (4.24). Then, in view of (4.23),

$$F(\mu, \nu) < +\infty \Leftrightarrow \mu \leq_c \nu.$$

Assume that $\mu \leq_c \nu$. Define

$$\alpha = \sup\{\langle \nu - \mu, \psi \rangle \mid \psi \in C_b^2(\mathbb{R}^d) \text{ is convex and } \Delta\psi = 2 \text{ in } \mathbb{R}^d\}.$$

For each $\xi \in \partial B_1(0)$, $x \in \mathbb{R}^d \mapsto |x \cdot \xi|^2$ belongs to $C_b^2(\mathbb{R}^d)$, is convex and $\Delta|x \cdot \xi|^2 = 2$ in \mathbb{R}^d . Then

$$\begin{aligned} f\left(\int_{\mathbb{R}^d} x \otimes x d(\nu(x) - \mu(x))\right) &= \max_{\xi \in \partial B_1(0)} \xi \otimes \xi : \int_{\mathbb{R}^d} x \otimes x d(\nu(x) - \mu(x)) \\ &= \max_{\xi \in \partial B_1(0)} \int_{\mathbb{R}^d} |x \cdot \xi|^2 d(\nu(x) - \mu(x)) \leq \alpha. \end{aligned} \quad (4.28)$$

Let $\psi \in C_b^2(\mathbb{R}^d)$ be a convex function such that $\Delta\psi = 2$ in \mathbb{R}^d . Define $\varphi = \psi - \frac{1}{d}|\cdot|^2 \in C_b^2(\mathbb{R}^d)$. Then φ is harmonic in \mathbb{R}^d . Since $\varphi \in \Phi_2(\mathbb{R}^d)$, using [13, Theorem 2.10], we deduce that there exists $H \in S_d$ such that $\text{tr}(H) = 0$, $H + \frac{I_d}{d} \geq 0$ and $\nabla^2\varphi(x) = 2H$ for each $x \in \mathbb{R}^d$. Then there exist $a \in \mathbb{R}$ and $\zeta \in \mathbb{R}^d$ such that $\psi(x) = a + \zeta \cdot x + (H + \frac{I_d}{d}) : x \otimes x$ for each $x \in \mathbb{R}^d$. Thus, taking into account that $\mu \leq_c \nu$, we have

$$\alpha = \sup\left\{\int_{\mathbb{R}^d} \left(H + \frac{I_d}{d}\right) : x \otimes x d(\nu(x) - \mu(x)) \mid H + \frac{I_d}{d} \in S_d^+, \text{tr}(H) = 0\right\}. \quad (4.29)$$

For each matrix $H + \frac{I_d}{d} \in S_d^+$, where $\text{tr}(H) = 0$, there exist eigenvalues $\lambda_1 \geq \dots \geq \lambda_d \geq 0$ and corresponding eigenvectors v_1, \dots, v_d forming an orthonormal basis of \mathbb{R}^d such that $H + \frac{I_d}{d} = \sum_{i=1}^d \lambda_i v_i \otimes v_i$ and $\sum_{i=1}^d \lambda_i = 1$. This, together with (4.27), (4.28) and (4.29), yields

$$f\left(\int_{\mathbb{R}^d} x \otimes x d(\nu(x) - \mu(x))\right) = \alpha \leq F(\mu, \nu).$$

Next, let $\psi \in C_b^2(\mathbb{R}^d)$ be such that $\sum_{j \in J_+} \lambda_j (\frac{1}{2} \nabla^2 \psi(x)) \leq 1$ for each $x \in \mathbb{R}^d$. Then there exists a subharmonic function $u \in C_b^2(\mathbb{R}^d)$ such that $\Delta u = 2 - \Delta\psi \geq 0$. Using [13, Theorem 2.10], we obtain a matrix $H \in S_d$ such that $\text{tr}(H) = 0$ and $\nabla^2 u(x) + \nabla^2 \psi(x) = 2H + \frac{2I_d}{d}$ for each $x \in \mathbb{R}^d$. Then there exist $a \in \mathbb{R}$ and $\zeta \in \mathbb{R}^d$ such that $\psi(x) = a + \zeta \cdot x + (H + \frac{I_d}{d}) : x \otimes x - u(x)$ for each $x \in \mathbb{R}^d$. Taking into account the constraint in (4.27) and the assumption $\mu \leq_c \nu$, we obtain

$$\begin{aligned} F(\mu, \nu) = \sup\left\{\int_{\mathbb{R}^d} \left(\left(H + \frac{I_d}{d}\right) : x \otimes x - u(x)\right) d(\nu(x) - \mu(x)) \mid H \in S_d, \text{tr}(H) = 0, u \in C_b^2(\mathbb{R}^d), \right. \\ \left. \Delta u \geq 0 \text{ in } \mathbb{R}^d \text{ and } \sum_{j \in J_+} \lambda_j \left(\left(H + \frac{I_d}{d}\right) - \frac{1}{2} \nabla^2 u(x)\right) \leq 1 \ \forall x \in \mathbb{R}^d\right\}. \end{aligned}$$

Conjecture. If $\mu \leq_c \nu$, then

$$f\left(\int_{\mathbb{R}^d} x \otimes x d(\nu(x) - \mu(x))\right) = \alpha = F(\mu, \nu).$$

As in the example (iii), in our opinion, a dual optimizer for $F(\mu, \nu)$ should be sought among convex functions when f is defined by (4.25) and $\mu \leq_c \nu$. We expect that such an optimizer exists and that its Laplacian is equal to 2 in \mathbb{R}^d . In dimension 1 and in general when $\mu \leq_{sh} \nu$, the above conjecture is true and

$$F(\mu, \nu) = \frac{1}{d} \text{var}(\nu) - \frac{1}{d} \text{var}(\mu),$$

where for each $a \in \mathbb{R}$, for each $\zeta \in \mathbb{R}^d$ and for each $H \in S_d$ such that $H + \frac{I_d}{d} \geq 0$ and $\text{tr}(H) = 0$, the function $\psi(x) = a + \zeta \cdot x + (H + \frac{I_d}{d}) : x \otimes x$ is a dual optimizer for $F(\mu, \nu)$.

By Theorem 4.14, $F = H$ when f is defined by (4.25).

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References

- [1] A. Alfonsi, J. Corbetta, and B. Jourdain, *Sampling of probability measures in the convex order by Wasserstein projection*, Ann. Inst. Henri Poincaré Probab. Stat. **56** (2020), no. 3, 1706–1729.
- [2] J.-J. Alibert, G. Bouchitté, and T. Champion, *A new class of costs for optimal transport planning*, Eur. J. Appl. M. **30** (2019), no. 6, 1229–1263.
- [3] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, 2nd ed., Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008.
- [4] J. Backhoff-Veraguas, M. Beiglböck, M. Huesmann, and S. Källblad, *Martingale Benamou-Brenier: A probabilistic perspective*, Ann. Probab. **48** (2020), no. 5, 2258–2289.
- [5] J. Backhoff-Veraguas, M. Beiglböck, and G. Pammer, *Existence, duality, and cyclical monotonicity for weak transport costs*, Calc. Var. Partial Differential Equations **58** (2019), no. 6, Paper No. 203, 28 pp.
- [6] J. Backhoff-Veraguas and G. Pammer, *Applications of weak transport theory*, Bernoulli **28** (2022), no. 1, 370–394.
- [7] M. Beiglböck and N. Juillet, *Shadow couplings*, Trans. Amer. Math. Soc. **374** (2021), 4973–5002.
- [8] D. P. Bertsekas and S. E. Shreve, *Stochastic optimal control*, Mathematics in Science and Engineering, vol. 139, Academic Press, Inc., New York-London, 1978. The discrete time case.
- [9] G. Bouchitté, *Convex analysis and duality methods*, Encyclopedia of mathematical physics, 2006, pp. 642–652.
- [10] L. Brasco, *A survey on dynamical transport distances*, J. Math. Sci. (N. Y.) **181** (2012), no. 6, 755–781.
- [11] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011.
- [12] M. G. Crandall, H. Ishii, and P.-L. Lions, *User’s guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. **27** (1992), no. 1, 1–67.
- [13] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [14] C. Goffman and J. Serrin, *Sublinear functions of measures and variational integrals*, Duke Math. J. **31** (1964), no. 1, 159–178.
- [15] N. Gozlan and N. Juillet, *On a mixture of brenier and strassen theorems*, Proc. Lond. Math. Soc. **120** (2020), no. 3, 434–463.
- [16] N. Gozlan, C. Roberto, P.-M. Samson, and P. Tetali, *Kantorovich duality for general transport costs and applications*, J. Funct. Anal. **273** (2017), no. 11, 3327–3405.
- [17] E. Hopf, *A remark on linear elliptic differential equations of second order*, Proc. Amer. Math. Soc. **3** (1952), 791–793.
- [18] M. Huesmann and D. Trevisan, *A Benamou-Brenier formulation of martingale optimal transport*, Bernoulli **25** (2019), no. 4A, 2729–2757.
- [19] W. Littman, G. Stampacchia, and H. F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. Pisa **17** (1963), no. 1-2, 43–77.
- [20] O. A. Oleinik, *On properties of solutions of certain boundary problems for equations of elliptic type*, Mat. Sb. (NS) **72** (1952), no. 3, 695–702.
- [21] T. Radó, *Subharmonic functions*, Springer-Verlag, Berlin, 1937.
- [22] R. T. Rockafellar, *Convex analysis*, Princeton Mathematical Series, vol. 28, Princeton University Press, Princeton, NJ, 1970.
- [23] V. Strassen, *The existence of probability measures with given marginals*, Ann. Math. Statist. **36** (1965), 423–439.
- [24] X. Tan and N. Touzi, *Optimal transportation under controlled stochastic dynamics*, Ann. Probab. **41** (2013), 3201–3240.
- [25] D. Trevisan, *Well-posedness of multidimensional diffusion processes with weakly differentiable coefficients*, Electron. J. Probab. **21 Paper No. 22** (2016).
- [26] C. Villani, *Optimal Transport. Old and New*, Grundlehren der Mathematischen Wissenschaften, vol. 338, Springer-Verlag, Berlin, 2009.