NONLINEAR POTENTIAL THEORY AND RICCI-PINCHED 3-MANIFOLDS

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Abstract. Let (M, g) be a complete, connected, noncompact Riemannian 3-manifold. In this short note, we give an alternative proof, based on the nonlinear potential theory, of the fact that if (M, g) satisfies the *Ricci-pinching condition* and superquadratic volume growth, then it is flat. This result is one of the building blocks of the proof of *Hamilton's pinching conjecture*.

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1. Introduction

A Riemannian manifold (M, g) is said to be *Ricci-pinched* if $\mathrm{Ric} \geq 0$ and there exists a constant $\varepsilon > 0$ such that $\mathrm{Ric} \geq \varepsilon \mathrm{R} g$, where Ric and R are the Ricci and scalar curvature of (M, g), respectively. Ricci-pinched manifolds are the focus of *Hamilton's pinching conjecture*, which is now established as a theorem:

Theorem 1.1. *Let (M, g) be a complete, connected Riemannian 3-manifold. Suppose that M is Ricci-pinched. Then, M is flat or compact.*

The theorem can be interpreted as an extension of Bonnet-Myers' theorem in three dimensions: if (M, g) is a complete and connected n-dimensional Riemannian manifold such that $\mathrm{Ric} \geq (n-1)kg$, for some constant $k > 0$, then M is compact and $\mathrm{diam}(M, g) \leq \pi/k^2$.

Chen and Zhu [CZ00] took the initial step in proving Theorem 1.1. They showed that a complete noncompact Riemannian 3-manifold, which has bounded and nonnegative sectional curvature and is Ricci-pinched, is flat. Then, Lott [Lot24] enhanced their result, requiring less stringent conditions on the sectional curvature. Deruelle-Schulze-Simon [DSS22] proved that the conjecture holds if the curvature is merely bounded. Finally, Lee and Topping [LT22] removed the assumption on the curvature. All these results rely on the Ricci flow. A proof based on different geometric flows is conceivable; however, current attempts in the literature require an extra assumption on the volume growth at infinity [HK24; Ben+24].

We say that (M, g) complete, connected, noncompact Riemannian 3-manifold with nonnegative Ricci has *superlinear* volume growth if there exist $q \in M$, $\mathrm{C}_{\mathrm{vol}} > 0$, $\tilde{r} > 0$ and $\alpha > 0$ such that for every radius $r > \tilde{r}$ it holds

$$\mathrm{C}_{\mathrm{vol}}^{-1} r^{1+\alpha} \leq |B_r(q)| \leq \mathrm{C}_{\mathrm{vol}} r^{1+\alpha}. \tag{1.1}$$

By the Bishop–Gromov theorem the parameter α is in $(0, 2]$ and condition (1.1) is independent of the point $q \in M$. We say that M has *superquadratic* volume growth if $\alpha \in (1, 2]$ and *Euclidean volume growth* if $\alpha = 2$. In the latter case (M, g) has strictly positive *asymptotic volume ratio*

$$\mathrm{AVR}(g) \coloneqq \frac{3}{4\pi} \lim_{r \to +\infty} \frac{|B_r(p)|}{r^3}, \qquad \text{with } p \in M.$$

The aim of this short note is to provide a new proof of the following result.

Theorem 1.2. *Let (M, g) be a complete, connected, noncompact, Ricci–pinched Riemannian 3–manifold. Suppose that (M, g) has superquadratic volume growth. Then, (M, g) is flat.*

Theorem 1.2 is contained in Deruelle–Schulze–Simon [DSS22, Theorem 1.3] and proved by Huisken-Körber [HK24, Theorem 12] employing the inverse mean curvature flow. The experts will quickly realize that our proof resemble the one by Huisken-Körber, that is intimately based on the monotonicity of the Willmore functional along the inverse mean curvature flow in Ricci-pinched manifolds. Here, we replace the inverse mean curvature flow with p-harmonic potentials with $p \in (1, 2)$ and the the Willmore functional with a suitable proxy defined in (2.6). Crucial in [HK24] is the use of both Gauss equation and Gauss-Bonnet theorem. The main technical difficulty of our note is the lack of the regularity needed to talk about the Euler characteristic and of the topological properties of the level sets and thus of Gauss-Bonnet theorem. We overcome this issue thanks to a weak version of the theorem recently shown in [BPP24].

Worth noticing that Theorem 1.2 is a generalization of [Ben+24, Theorem 1.5], that instead requires $\alpha > 4/3$ in (1.1). The extra assumption on α depends on the technique employed. The proof in [Ben+24] uses the harmonic potential, which corresponds to the case $p = 2$. Here we have the freedom to vary the values of p. For each p we obtain a different lower bound on α which approches 1 in the limit as $p \to 1^+$. On the other hand, [Ben+24] has the undeniable value of exploiting the higher regularity of harmonic functions allowing for less technical arguments. This simplification somehow balances the stronger assumption.

We conclude our note by considering the case for manifolds with boundary in Proposition 3.5.

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2. Preliminaries

2.1. **Basic results.** Let (M, g) be a complete noncompact Riemannian 3-manifold with nonnegative Ricci and superquadratic volume growth and take $p \in (1, 2)$. For a given closed and bounded set $\Omega \subset M$ with smooth boundary, we define the function w_p as the solution to

$$\begin{cases} \Delta_p w_p = |\nabla w_p|^p & \text{on } M \smallsetminus \Omega \\ w_p = 0 & \text{on } \partial\Omega \\ w_p \to +\infty & \text{as } d(x,o) \to +\infty \end{cases} \tag{2.1}$$

where $\Delta_p f = \operatorname{div}\left(|\nabla f|^{p-2}\nabla f\right)$ denotes the p-Laplacian operator of (M, g).

Existence, uniqueness and regularity of such a solution come from the classical theory for a slightly different problem:

$$\begin{cases} \Delta_p u_p = 0 & \text{on } M \smallsetminus \Omega \\ u_p = 1 & \text{on } \partial\Omega \\ u_p \to 0 & \text{as } d(x,o) \to +\infty. \end{cases} \tag{2.2}$$

Problem (2.2) has a unique weak solution, which assumes values in $(0, 1]$. The function $u_p \in W^{1,p}_{\mathrm{loc}}(M \smallsetminus \Omega)$ is of class $\mathscr{C}^{1,\beta}$ in any precompact set K, for some $\beta > 0$ depending on K. It is smooth on the open set $\{|\nabla u_p| \neq 0\}$ and attains smoothly the boundary value on $\partial\Omega$ (see [Hol99] and [Ben22, Proposition 2.3.5 and Proposition 2.3.6]). Let $\Omega^{(p)}_t = \{w_p \leq t\} \cup \Omega$, then for almost every $t \in [0, +\infty)$ it holds that the set $\partial\Omega_t \cap \{|\nabla u_p| = 0\}$ is $\mathscr{H}^2$-negligible. Moreover, $|\nabla u_p|^{p-1} \in W^{1,2}_{\mathrm{loc}}$ (see [Lou08]). Crucial in the sequel an *upper bound* for the growth of u_p (see [Hol99, Proposition 5.10] and [Ben22, Proposition 2.3.2]): there exist a positive constant $\mathrm{C} = \mathrm{C}(M, \Omega, p)$ such that, for all $x \in M \smallsetminus \mathrm{Int}(\Omega)$, it holds

$$u(x) \leq \mathrm{C}\, d(x, o)^{-\frac{\alpha+1-p}{p-1}}. \tag{2.3}$$

Then, $w_p = -(p-1)\log u_p$ is a weak solution in $\mathscr{C}^1(M \smallsetminus \Omega)$ of problem (2.1). The solution w_p shares the same properties with u_p. Furthermore, for almost every $\partial\Omega^{(p)}_t$, we have a suitable geometric notion of (weak) mean curvature H and second fundamental form h (see [BPP24]).

We recall the definition of the normalized p-capacity of D closed bounded subset of M

$$\mathfrak{c}_p(\partial D) = \inf\left\{ \frac{1}{4\pi}\left(\frac{p-1}{3-p}\right)^{p-1} \int_{M\smallsetminus D} |\nabla\psi|^p \, \mathrm{d}\mathcal{H}^2 \;\middle|\; \psi \in \mathscr{C}^\infty_c(M), \psi \geq \chi_D \right\}.$$

Notice that $\mathfrak{c}_p(\mathbb{S}^2) = 1$.

Lemma 2.1. *A solution to* (2.2) *realizes the p-capacity of the initial set Ω:*

$$\mathfrak{c}_p(\partial\Omega) = \frac{1}{4\pi}\int_{\partial\Omega}\left(\frac{|\nabla w|}{3-p}\right)^{p-1} \mathrm{d}\mathcal{H}^2. \tag{2.4}$$

Moreover, for almost every $t \in [0,\infty)$ it holds

$$\mathfrak{c}_p(\partial\Omega_t) = \mathrm{e}^t\, \mathfrak{c}_p(\partial\Omega). \tag{2.5}$$

Proof. For the proof of these facts we refer to [Tol83, Proposition 3.2.1] and [BFM24, Proposition 2.8, Proposition 2.9]. □

2.2. **Monotone quantities.** Let w_p solution of the problem (2.1), let $\Omega_t^{(p)} = \{w_p \leq t\} \cup \Omega$ and denote by H the (weak) mean curvature with respect to outward pointing unit normal $\nu = \nabla w_p/|\nabla w_p|$. For almost every $t \in [0,+\infty)$ we define

$$\begin{aligned}
\mathscr{F}_p(t) &= \int_{\partial\Omega_t^{(p)}} \frac{\mathrm{H}^2}{4} - \left(\frac{\mathrm{H}}{2} - \frac{|\nabla w_p|}{(3-p)}\right)^2 \mathrm{d}\mathcal{H}^2, \\
\mathscr{G}_p(t) &= \frac{1}{(3-p)^2}\int_{\partial\Omega_t^{(p)}} |\nabla w_p|^2 \, \mathrm{d}\mathcal{H}^2.
\end{aligned} \tag{2.6}$$

Lemma 2.2. *Let (M,g) be a complete, noncompact, Riemannian 3-manifold with nonnegative Ricci curvature and let w_p be a solution to problem* (2.2)*. Then, the function $\mathscr{F}_p$ is a $W^{1,1}_{\mathrm{loc}}$ non-increasing function which admits a continuous representative. The function $\mathscr{G}_p$ is a $W^{2,1}_{\mathrm{loc}}$ non-increasing function which admits a $\mathscr{C}^1$-representative. Moreover, for almost every $t \in [0,+\infty)$, the following holds*

$$\mathscr{F}_p'(t) = -\frac{1}{3-p}\int_{\partial\Omega_t^{(p)}} \mathrm{Ric}(\nu,\nu) + \left|\mathring{\mathrm{h}}\right|^2 + \frac{\left|\nabla^\top|\nabla w_p|\right|^2}{|\nabla w_p|^2} + \frac{3-p}{2(p-1)}\left(\mathrm{H} - \frac{2}{3-p}|\nabla w_p|\right)^2 d\mathcal{H}^2 \leq 0 \tag{2.7}$$

$$\mathscr{G}_p'(t) = \frac{1}{p-1}\int_{\partial\Omega_t^{(p)}} 2|\nabla w_p|^2 - (3-p)\,\mathrm{H}\,|\nabla w_p| \, \mathrm{d}\mathcal{H}^2 \leq 0 \tag{2.8}$$

where h is the second fundamental form of $\partial\Omega_t^{(p)}$, $\mathring{\mathrm{h}}$ is the traceless part of h and $\nabla^\top$ denotes the tangential part of the gradient with respect to $\partial\Omega_t^{(p)}$.

Idea of the proof. For the well-posedness and regularity of $\mathscr{F}_p$ and $\mathscr{G}_p$ we refer, for instance, to [BPP24, Section 3.3] and to [BFM24, Theorem 3.1 and Proposition 3.5]. We sketch the (formal) computations that lead to (2.7) and (2.8). Call

$$\begin{aligned}
X &= |\nabla w_p|\nabla w_p, \\
Y &= \frac{\Delta w_p \nabla w_p}{|\nabla w_p|} - \frac{\nabla|\nabla w_p|^2}{2|\nabla w_p|} - \frac{X}{3-p}.
\end{aligned}$$

Then, by direct computation,

$$\begin{aligned}
\mathrm{div}(X) &= \frac{|\nabla w_p|}{p-1}\left(2|\nabla w_p|^2 - (3-p)\,\mathrm{H}\right), \\
\mathrm{div}(Y) &= -\frac{\mathrm{Ric}(\nabla w_p,\nabla w_p)}{|\nabla w_p|} - \left|\mathring{\mathrm{h}}\right|^2|\nabla w_p| - \frac{\left|\nabla^\top|\nabla w_p|\right|^2}{|\nabla w_p|} - \frac{3-p}{2(p-1)}|\nabla w_p|\left(\mathrm{H} - \frac{2}{3-p}|\nabla w_p|\right)^2.
\end{aligned}$$

Then, we can write

$$\mathscr{G}_p(t) = \frac{1}{(3-p)^2} \int_{\partial\Omega_t^{(p)}} \left\langle X \,\middle|\, \frac{\nabla w_p}{|\nabla w_p|} \right\rangle \mathrm{d}\mathcal{H}^2, \qquad \mathscr{F}_p(t) = \int_{\partial\Omega_t^{(p)}} \left\langle Y \,\middle|\, \frac{\nabla w_p}{|\nabla w_p|} \right\rangle \mathrm{d}\mathcal{H}^2.$$

Hence for $s < t$ in $[0, +\infty)$ by the divergence theorem $\mathscr{F}_p(t) - \mathscr{F}_p(s) = \int_{\{s<w<t\}} \operatorname{div}(X)\,\mathrm{d}\mathcal{H}^2$ and $\mathscr{G}_p(t) - \mathscr{G}_p(s) = \int_{s<w<t} \operatorname{div}(Y)\,\mathrm{d}\mathcal{H}^2$. Thus, at least formally, we obtain (2.7) and (2.8). To give a precise meaning to this formal computation we need quite of extra work. To this aim, we refer to [BPP24, Theorem 3.1]. □

3. Proof of the main theorem

From now on, (M, g) is a complete, connected, noncompact, Ricci-pinched Riemannian 3-manifold with superquadratic volume growth.

Suppose by contradiction that (M, g) is not flat. Then there must exist a point $o \in M$ with $\mathrm{R}(o) > 0$ and a radius $r \ll 1$ such that $\partial B_r(o)$ is a smooth surface with

$$\int_{\partial B_r(o)} \mathrm{H}^2 \,\mathrm{d}\mathcal{H}^2 < 16\pi.$$

This can be obtained by the expansion under normal coordinate in perturbed spheres, see, for instance, [Mon10, Proposition 3.1] and [FST09].

In problem (2.2) we set $\Omega = \overline{B}_r(o)$. By Hopf lemma, zero is a regular value for w_p. Then, $\mathscr{F}_p(0) < 4\pi$ follows from the very definition of $\mathscr{F}_p$ since $\partial\Omega_0 = \partial B_R(o)$.

3.1. Weak Gauss-Bonnet theorem.

If Σ is a smooth closed surface in M with normal ν, mean curvature H and traceless second fundamental form $\mathring{\mathrm{h}}$, it holds

$$2\int_\Sigma \mathrm{Ric}(\nu,\nu)\,\mathrm{d}\mathcal{H}^2 \geq \varepsilon\left(16\pi - \int_\Sigma \mathrm{H}^2\,\mathrm{d}\mathcal{H}^2\right) \qquad \text{if } \mathrm{genus}(\Sigma) = 0,$$

$$2\int_\Sigma \mathrm{Ric}(\nu,\nu) + \left|\mathring{\mathrm{h}}\right|^2 \mathrm{d}\mathcal{H}^2 \geq \int_\Sigma \mathrm{H}^2\,\mathrm{d}\mathcal{H}^2 \qquad \text{if } \mathrm{genus}(\Sigma) \geq 1.$$

These two inequalities follow combining Gauss equation

$$\mathrm{R}^\top = \mathrm{R} - 2\,\mathrm{Ric}(\nu,\nu) + \mathrm{H}^2 - |\mathrm{h}|^2 \tag{3.1}$$

together with Gauss-Bonnet Theorem, and the pinching condition [HK24, Lemma 8]. These inequalities are crucial both in [HK24] and in [Ben+24]. Notice that in both papers the considered flow is regular enough to perform the desired estimates. Indeed, in the former case, the level sets of the weak inverse mean curvature flow are $C^{1,\beta}$, and they can be approximated in $W^{2,2}$ by smooth surfaces. In the latter case, harmonic functions are smooth and Sard's theorem applies. Unfortunately in the current situation we cannot infer the regularity on the level sets needed to apply Gauss-Bonnet theorem. We will make use of the following weak replacement (see [BPP24, Theorem 1.3]).

Theorem 3.1. *Let (M, g) be a complete, noncompact 3-dimensional Riemannian manifold. Let $p \in (1, 2)$, w_p be the proper solution to (2.1). Then, for almost every $t \in [0, +\infty)$ it holds*

$$\int_{\partial\Omega_t^{(p)}} \mathrm{R}^\top \,\mathrm{d}\mathcal{H}^{n-1} \in 8\pi\mathbb{Z}.$$

Lemma 3.2. *Under the same assumptions of the previous theorem, whenever (M, g) is also a connected, Ricci-pinched manifold, we get*

$$2\int_{\partial\Omega_t^{(p)}} \mathrm{Ric}(\nu,\nu)\,\mathrm{d}\mathcal{H}^2 \geq \varepsilon\left(16\pi - \int_{\partial\Omega_t^{(p)}} \mathrm{H}^2\,\mathrm{d}\mathcal{H}^2\right) \qquad \textit{if } \int_{\partial\Omega_t^{(p)}} \mathrm{R}^\top\,\mathrm{d}\mathcal{H}^2 \geq 8\pi, \tag{3.2}$$

$$2\int_{\partial\Omega_t^{(p)}} \mathrm{Ric}(\nu,\nu) + \left|\mathring{\mathrm{h}}\right|^2 \mathrm{d}\mathcal{H}^2 \geq \int_{\partial\Omega_t^{(p)}} \mathrm{H}^2\,\mathrm{d}\mathcal{H}^2 \qquad \textit{if } \int_{\partial\Omega_t^{(p)}} \mathrm{R}^\top\,\mathrm{d}\mathcal{H}^2 \leq 0. \tag{3.3}$$

Proof. We can write the Gauss equation (3.1) in a integral from, that is

$$\int_{\partial\Omega_t^{(p)}} \mathrm{H}^2 \, \mathrm{d}\mathcal{H}^2 = \int_{\partial\Omega_t^{(p)}} 2\mathrm{R}^\top - 2\mathrm{R} + 4\,\mathrm{Ric}(\nu,\nu) + 2\big|\mathring{h}\big|^2 \, \mathrm{d}\mathcal{H}^2.$$

Keeping in mind that $\mathrm{Ric} \geq 0$, we have $\mathrm{Ric}(\nu,\nu) - \mathrm{R} \leq 0$. Then, if $\int_{\partial\Omega_t^{(p)}} \mathrm{R}^\top \leq 0$ we trivially get

$$\int_{\partial\Omega_t^{(p)}} \mathrm{H}^2 \, \mathrm{d}\mathcal{H}^2 = \int_{\partial\Omega_t^{(p)}} 2\mathrm{R}^\top - 2\mathrm{R} + 2\,\mathrm{Ric}(\nu,\nu) + 2\,\mathrm{Ric}(\nu,\nu) + 2\big|\mathring{h}\big|^2 \, \mathrm{d}\mathcal{H}^2 \leq \int_{\partial\Omega_t^{(p)}} 2\,\mathrm{Ric}(\nu,\nu) + 2\big|\mathring{h}\big|^2 \, \mathrm{d}\mathcal{H}^2.$$

Differently, if $\int_{\partial\Omega_t^{(p)}} \mathrm{R}^\top \geq 8\pi$, using the pinching condition we obtain

$$\int_{\partial\Omega_t^{(p)}} \mathrm{H}^2 \, \mathrm{d}\mathcal{H}^2 = \int_{\partial\Omega_t^{(p)}} 2\mathrm{R}^\top - 2\mathrm{R} + 4\,\mathrm{Ric}(\nu,\nu) + 2\big|\mathring{h}\big|^2 \, \mathrm{d}\mathcal{H}^2 \geq 16\pi - \frac{1}{\varepsilon}\int_{\partial\Omega_t^{(p)}} 2\,\mathrm{Ric}(\nu,\nu) \, \mathrm{d}\mathcal{H}^2,$$

as desired. □

3.2. Asymptotic behavior of $\mathscr{F}_p$ and $\mathscr{G}_p$.

Lemma 3.3. *There exists $T_0 \in [0,+\infty)$ and there exists a positive constant $\mathrm{C} = \mathrm{C}(T_0)$ such that for almost every $t \geq T_0$, there holds $\mathscr{F}_p(t) \leq \mathrm{C}\,\mathrm{e}^{-\frac{2}{3-p}t}$.*

Proof. For almost every $t \in [0,+\infty)$, we know that $\int_{\partial\Omega_t^{(p)}} \mathrm{R}^\top \, \mathrm{d}\mathcal{H}^2 \in 8\pi\mathbb{Z}$. If $\int_{\partial\Omega_t^{(p)}} \mathrm{R}^\top \, \mathrm{d}\mathcal{H}^2 \leq 0$, combining the expression (2.7) with the estimate (3.3), we obtain

$$-2(3-p)\mathscr{F}_p^{'}(t) \geq 2\int_{\partial\Omega_t^{(p)}} \mathrm{Ric}(\nu,\nu) + \big|\mathring{\mathrm{h}}\big|^2 \, \mathrm{d}\mathcal{H}^2 \geq \int_{\partial\Omega_t^{(p)}} \mathrm{H}^2 \, \mathrm{d}\mathcal{H}^2 \geq 4\mathscr{F}_p(t).$$

If $\int_{\partial\Omega_t^{(p)}} \mathrm{R}^\top \, \mathrm{d}\mathcal{H}^2 \geq 8\pi$, by (2.7) and the estimate (3.2), we have

$$\begin{aligned}
-2(3-p)\mathscr{F}_p^{'}(t) &\geq \int_{\partial\Omega_t^{(p)}} 2\,\mathrm{Ric}(\nu,\nu) + \frac{3-p}{p-1}\left(\mathrm{H} - \frac{2}{3-p}|\nabla w_p|\right)^2 \mathrm{d}\mathcal{H}^2 \\
&\geq \int_{\partial\Omega_t^{(p)}} 2\,\mathrm{Ric}(\nu,\nu) + \varepsilon\left(\mathrm{H} - \frac{2}{3-p}|\nabla w_p|\right)^2 \mathrm{d}\mathcal{H}^2 \\
&\geq \varepsilon\left(16\pi - \frac{4}{3-p}\int_{\Omega_t^{(p)}} \mathrm{H}\,|\nabla w_p| - \frac{|\nabla w_p|^2}{3-p}\,\mathrm{d}\mathcal{H}^2\right) \\
&\geq \varepsilon\,(16\pi - 4\mathscr{F}_p(t))
\end{aligned}$$

where we have used that the pinching constant is $\varepsilon \leq 1/3$, by tracing the pinching condition, and hence $\varepsilon < \frac{3-p}{p-1}$. Then, we conclude that for almost every $t \in [0,+\infty)$,

$$\mathscr{F}_p^{'} \leq \max\left\{-\frac{2}{3-p}\mathscr{F}_p(t), -\frac{\varepsilon}{3-p}\,(8\pi - 2\mathscr{F}_p(t))\right\}$$

holds. Since $\mathscr{F}_p$ is a $W^{1,1}_{\mathrm{loc}}$ function and it is non-increasing, then either $\mathscr{F}_p(t) \geq 8\pi\varepsilon/(2+2\varepsilon)$ for every $t \geq 0$, or there exists $\tau \geq 0$ such that $\mathscr{F}_p(t) \leq 8\pi\varepsilon/(2+2\varepsilon)$ for every $t \geq \tau$. However, in the former case, we know that $\mathscr{F}_p^{'} \leq \varepsilon\,(8\pi - 2\mathscr{F}_p(t))$, for almost every $t \geq 0$ and $\mathscr{F}_p(t) \leq \mathscr{F}_p(0) < 4\pi$. Hence, these has to exists some $t \geq 0$ such that $\mathscr{F}_p(t) < 8\pi\varepsilon/(2+2\varepsilon)$, leading to a contradiction.

In the latter case, $\mathscr{F}_p^{'}(t) \leq -\frac{2}{3-p}\mathscr{F}_p(t)$ for almost every $t \geq \tau$, implying $\mathscr{F}_p(t) \leq C(T_0)\,\mathrm{e}^{-\frac{2}{3-p}t}$. □

Lemma 3.4. *For almost every $t \in [0,+\infty)$, there holds $0 \leq \mathscr{G}_p(t) \leq \mathscr{F}_p(t)$. In particular,*

$$\lim_{t\to+\infty} \mathscr{F}_p(t) = \lim_{t\to+\infty} \mathscr{G}_p(t) = 0.$$

Proof. As stated previously and as a consequence of [BFM24, Theorem 3.1], the function $\mathscr{G}_p$ admits a non-increasing $\mathscr{C}^1$-extension on all $[0,+\infty)$. It can be seen from (2.8) that at almost every $t \in [0,+\infty)$ of w_p, the following is satisfied

$$0 \geq \mathscr{G}_p^{'}(t) = \frac{(3-p)^2}{p-1}\left(\mathscr{G}_p(t) - \mathscr{F}_p(t)\right),$$

thus giving the thesis. □

3.3. **Proof of theorem 1.2.** By equations (2.5) and (2.4) and Hölder inequality, for almost every $t \in [0,+\infty)$, we obtain

$$\begin{aligned}\mathrm{e}^t\,\mathfrak{c}_p(\partial\Omega^{(p)}) &\overset{(2.5)}{=} \mathfrak{c}_p(\partial\Omega_t^{(p)}) \overset{(2.4)}{=} \frac{1}{4\pi}\int_{\partial\Omega_t^{(p)}}\left(\frac{|\nabla w_p|}{3-p}\right)^{p-1}\mathrm{d}\mathcal{H}^2 \\ &\leq \frac{1}{4\pi}\frac{1}{(3-p)^{p-1}}\left(\int_{\partial\Omega_t^{(p)}}|\nabla w_p|^2\,\mathrm{d}\mathcal{H}^2\right)^{\frac{p}{3}}\left(\int_{\partial\Omega_t^{(p)}}|\nabla w_p|^{-1}\,\mathrm{d}\mathcal{H}^2\right)^{\frac{3-p}{3}}.\end{aligned}$$

From Lemmas 3.3 and 3.4, we know that there exists a $T_0 \in [0,+\infty)$ and a positive constant C such that for almost all $t \in [T_0,+\infty)$, the following holds

$$\int_{\partial\Omega_t^{(p)}}|\nabla w_p|^2\,\mathrm{d}\mathcal{H}^2 = \mathscr{G}_p(t) \leq \mathrm{C}\,\mathrm{e}^{-\frac{2}{3-p}t}.$$

Thus, by the coarea formula, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}\left(\{w_p \leq t\} \smallsetminus \{|\nabla w_p| = 0\}\right) = \int_{\partial\Omega_t^{(p)}}|\nabla w_p|^{-1}d\mathcal{H}^2 \geq \mathrm{C}\,\mathrm{e}^{\frac{9-p}{(3-p)^2}t}$$

for almost every $t \in [0,+\infty)$.

Consider $R_t = \sup\{\mathrm{d}(x,o) : w_p(x) \leq t\}$ for any $t \in [0,+\infty)$. Let $T_1 \in (T_0,+\infty)$. By integrating the above inequality on $[T_0,T_1]$ and using the superquadratic volume growth condition, we get

$$\mathrm{C}\left(\mathrm{e}^{\frac{9-p}{(3-p)^2}T_1} - \mathrm{e}^{\frac{9-p}{(3-p)^2}T_0}\right) \leq \mathrm{Vol}\left(\{w_p \leq t\} \smallsetminus \{|\nabla w_p| = 0\}\right) \leq \mathrm{Vol}(B_{R_{T_1}}(o)) \leq \mathrm{C}_{\mathrm{vol}}R_{T_1}^{1+\alpha}. \tag{3.4}$$

Since $u_p = \mathrm{e}^{-\frac{w_p}{p-1}}$, by estimate (2.3), we have

$$\mathrm{e}^{-\frac{w_p(x)}{p-1}} \leq \mathrm{C}\mathrm{d}(x,o)^{-\frac{\alpha+1-p}{p-1}}.$$

Taking the supremum among all $x \in \partial\Omega_{T_1}^{(p)}$, we obtain $R_{T_1}^{\alpha+1-p} \leq \mathrm{C}\,\mathrm{e}^{T_1}$, implying $R_{T_1}^{1+\alpha} \leq \mathrm{C}\,\mathrm{e}^{\frac{1+\alpha}{\alpha+1-p}T_1}$, for a positive constant $\mathrm{C} = \mathrm{C}(M, B_r(o))$. Therefore, by inequality (3.4), we conclude that

$$\mathrm{e}^{\frac{9-p}{(3-p)^2}T_1} - \mathrm{e}^{\frac{9-p}{(3-p)^2}T_0} \leq \mathrm{C}R_{T_1}^{1+\alpha} \leq \mathrm{C}\mathrm{e}^{\frac{1+\alpha}{1+\alpha-p}T_1}.$$

This is not possible for T_1 sufficiently large and for $\alpha > 4/(5-p)$. Set $f(p) = 4/(5-p)$. we notice that it holds $f(1) = 1$. Thus, for any $\alpha > 1$, we can find $p \in (1,2)$, sufficiently close to 1, such that $\alpha > f(p)$, therefore we get a contradiction.

Proposition 3.5. *There does not exist a complete, connected, noncompact, Ricci-pinched Riemannian 3-manifold (M,g) with superquadratic volume growth and a compact smooth boundary ∂M satisfying*

$$\int_{\partial M}\mathrm{H}^2\,\mathrm{d}\mathcal{H}^2 < 16\pi.$$

Proof. It follows directly from the above analysis, it is enough to take $\Omega = \partial M$. □

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