OPTIMAL DOMAINS FOR THE CHEEGER INEQUALITY

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ABSTRACT. In this paper we prove the existence of an optimal domain Ω_{opt} for the shape optimization problem

$$\max\Big\{\lambda_q(\Omega)\ :\ \Omega\subset D,\ \lambda_p(\Omega)=1\Big\},$$

where $q<p$ and D is a prescribed bounded subset of $\mathbb{R}^d$. Here $\lambda_p(\Omega)$ (respectively $\lambda_q(\Omega)$) is the first eigenvalue of the p-Laplacian $-\Delta_p$ (respectively $-\Delta_q$) with Dirichlet boundary condition on $\partial\Omega$. This is related to the existence of optimal sets that minimize the generalized Cheeger ratio

$$\mathcal{F}_{p,q}(\Omega)=\frac{\lambda_p^{1/p}(\Omega)}{\lambda_q^{1/q}(\Omega)}.$$

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1. Introduction

The starting point of this research is the Cheeger [4] inequality

$$\frac{\sqrt{\lambda(\Omega)}}{h(\Omega)}\ge\frac{1}{2}\tag{1.1}$$

valid for every open bounded set $\Omega\subset\mathbb{R}^d$. Here $\lambda(\Omega)$ denotes the first eigenvalue of the Laplace operator $-\Delta$ on the open set Ω, with Dirichlet boundary conditions:

$$\begin{aligned}\lambda(\Omega)&=\inf\left\{\frac{\int|\nabla u|^2\,dx}{\int|u|^2\,dx}\ :\ u\in C_c^1(\Omega)\setminus\{0\}\right\}\\&=\min\left\{\frac{\int|\nabla u|^2\,dx}{\int|u|^2\,dx}\ :\ u\in H_0^1(\Omega)\setminus\{0\}\right\},\end{aligned}$$

where the integrals without the indicated domain are intended over the entire space $\mathbb{R}^d$, and the functions in $H_0^1(\Omega)$ are considered as extended by zero outside the domain Ω. Here $h(\Omega)$ denotes the Cheeger constant

$$h(\Omega)=\inf\left\{\frac{P(E)}{|E|}\ :\ E\Subset\Omega\right\}.$$

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Other equivalent ways to define the Cheeger constant $h(\Omega)$ are:

$$\begin{aligned}h(\Omega)&=\inf\left\{\frac{\int|\nabla u|\,dx}{\int|u|\,dx}\ :\ u\in C^1_c(\Omega)\setminus\{0\}\right\}\\&=\inf\left\{\frac{\int|\nabla u|\,dx}{\int|u|\,dx}\ :\ u\in W^{1,1}_0(\Omega)\setminus\{0\}\right\}.\end{aligned}$$

When Ω is a Lipschitz domain the infimum above coincides with the infimum on $BV(\Omega)$:

$$h(\Omega)=\inf\left\{\frac{\int_\Omega|\nabla u|+\int_{\partial\Omega}|u|\,d\mathcal{H}^{d-1}}{\int_\Omega|u|\,dx}\ :\ u\in BV(\Omega)\setminus\{0\}\right\}$$

and in this case we have

$$h(\Omega)=\inf\left\{\frac{P(E)}{|E|}\ :\ E\subset\Omega\right\}.$$

In this way the Cheeger constant can be seen as the first eigenvalue $\lambda_p(\Omega)$ of the p-Laplacian with Dirichlet boundary conditions:

$$\lambda_p(\Omega)=\inf\left\{\frac{\int|\nabla u|^p\,dx}{\int|u|^p\,dx}\ :\ u\in W^{1,p}_0(\Omega)\setminus\{0\}\right\}\tag{1.2}$$

when $p=1$. The quantity $\lambda_p^{1/p}(\Omega)$ can be defined also for $p=\infty$ since, as it is well-known (see for instance [10]),

$$\lim_{p\to\infty}\lambda_p^{1/p}(\Omega)=\rho(\Omega),$$

where $\rho(\Omega)$ is the so-called *inradius* of Ω, that is the maximal radius of a ball contained in Ω, or equivalently the maximum of the distance function from the boundary $\partial\Omega$.

More generally, defining the shape functional

$$\mathcal{F}_{p,q}(\Omega)=\frac{\lambda_p^{1/p}(\Omega)}{\lambda_q^{1/q}(\Omega)},\tag{1.3}$$

the inequality (1.1) can be seen as a particular case of the more general inequality, valid for every $1\le q\le p\le+\infty$:

$$\mathcal{F}_{p,q}(\Omega)\ge\frac{q}{p}\qquad\text{for every }1\le q\le p\le+\infty.\tag{1.4}$$

This can be also rephrased as a monotonicity property:

the map $p\mapsto p\lambda_p^{1/p}(\Omega)$ is monotonically nondecreasing.

The proof of the inequalities above is rather simple and relies on a suitable use of the Hölder inequality (see [1]). The constant q/p in (1.4) is not sharp, although it becomes asymptotically sharp as the dimension d approaches infinity (see [1]).

Due to the scaling properties of the eigenvalue $\lambda_p(\Omega)$:

$$\lambda_p(t\Omega)=t^{-p}\lambda_p(\Omega)\qquad\text{for every }t>0,$$

the functional $\mathcal{F}_{p,q}$ is scaling invariant, that is

$$\mathcal{F}_{p,q}(t\Omega)=\mathcal{F}_{p,q}(\Omega)\qquad\text{for every }t>0.$$

By the scaling invariance above, in the minimization problem

$$\min\Big\{\mathcal{F}_{p,q}(\Omega)\ :\ \Omega \text{ bounded subset of } \mathbb{R}^d\Big\} \tag{1.5}$$

it is not restrictive to add the constraint $\lambda_p(\Omega) = 1$. Therefore the minimization problem (1.5) can be rewritten as

$$\max\Big\{\lambda_q(\Omega)\ :\ \Omega \text{ bounded subset of } \mathbb{R}^d,\ \lambda_p(\Omega) = 1\Big\}. \tag{1.6}$$

The existence of an optimal domain Ω_{opt} for problem (1.6) is not known; in this paper we prove it assuming that all the competing domains are contained in a given bounded set D of $\mathbb{R}^d$. Although it is not the focus of this paper, we mention that the shape optimization problem (1.5) remains interesting even if we restrict the class of competing domains Ω by adding suitable additional geometric constraints. In [14] the existence of optimal domains is shown, when $q = 1$, in the case where the competing domains Ω are assumed to be convex.

The present paper is organized as follows. In Section 2 we recall the tools that are crucial in the proof: the notions of p-capacitary measures and the γ_p convergence. The key property, proved in Theorem 2.5 is that if a sequence (Ω_n) γ_p converges to a p-capacitary measure μ and simultaneously γ_q converges to a q-capacitary measure ν, with $q < p$, then ν vanishes on the set where μ is finite. In Section 3 we show how this result implies the existence of an optimal domain Ω_{opt} for problem (1.6). Finally, in Section 4 we add some concluding remarks and list some open questions that, in our opinion, deserve further investigation.

2. Preliminary tools

In the rest of the paper D will be a bounded open set in $\mathbb{R}^d$ and all the domains Ω we consider are supposed to be contained in D. In the following we consider quasi open sets whose definition is below.

Definition 2.1. *A set $\Omega \subset \mathbb{R}^d$ is said p-quasi open if there exists a function $u \in W^{1,p}(\mathbb{R}^d)$ such that $\Omega = \{u > 0\}$.*

By the Sobolev embedding theorem, if $p > d$, the p-quasi open sets are nothing but open sets. If Ω is p-quasi open, we may define the Sobolev space $W^{1,p}_0(\Omega)$ as the space of all functions $u \in W^{1,p}(\mathbb{R}^d)$ such that $u = 0$ outside Ω; therefore, the first eigenvalue $\lambda_p(\Omega)$ can be defined as in (1.2) for every p-quasi open set Ω. The class of admissible domains we consider is then

$$\mathcal{A}(D) = \big\{\Omega \subset D\ :\ \Omega\ p\text{-quasi open}\big\}.$$

An essential tool in the proof of existence of optimal domains for the functional $\mathcal{F}_{p,q}$ is the notion of p-capacitary measure and of γ_p convergence. Below by cap_p we indicate the p-capacity:

$$\mathrm{cap}(E) = \inf\Big\{\int\big(|\nabla u|^p + |u|^p\big)\,dx\ :\ u \in W^{1,p}_0(\mathbb{R}^d),\ u = 1 \text{ in a neighborhood of } E\Big\}.$$

Definition 2.2. *Let $p \leq d$. We say that a nonnegative Borel measure μ (possibly taking the $+\infty$ value) is of a p-capacitary type if*

$$\mu(E) = 0 \qquad \text{for every Borel set } E \text{ with } p\text{-capacity zero.}$$

When $p > d$ every nonempty set has a positive p-capacity, hence p-capacitary measures simply reduce to Borel measures. Measures of p-capacitary type generalize p-quasi open sets; indeed, if Ω is a p-quasi open set, the Borel measure

$$\infty_{\mathbb{R}^d\setminus\Omega}(E) = \begin{cases} 0 & \text{if } \operatorname{cap}_p(E \setminus \Omega) = 0 \\ +\infty & \text{otherwise} \end{cases}$$

is of p-capacitary type. More generally, the eigenvalues λ_p can be defined for a p-capacitary measure μ as

$$\lambda_p(\mu) = \inf\left\{\frac{\int |\nabla u|^p\,dx + \int |u|^p d\mu}{\int |u|^p\,dx}\ :\ u \in W_0^{1,p}(D) \setminus \{0\}\right\},$$

and we have

$$\lambda_p(\infty_{\mathbb{R}^d\setminus\Omega}) = \lambda_p(\Omega) \quad \text{for every } p\text{-quasi open set } \Omega.$$

From the definition above we see immediately that

$$\lambda_p(\mu_1) \leq \lambda_p(\mu_2) \qquad \text{whenever } \mu_1 \leq \mu_2. \tag{2.1}$$

Definition 2.3. *We say that a sequence (μ_n) of p-capacitary measures γ_p converges to a p-capacitary measure μ if the sequence of functionals*

$$F_n(u) = \int_D |\nabla u|^p dx + \int |u|^p d\mu_n \qquad u \in W_0^{1,p}(D)$$

Γ-converges in $L^p(D)$ to the functional

$$F(u) = \int_D |\nabla u|^p dx + \int |u|^p d\mu.$$

For all the details concerning Γ-convergence we refer to the book [6], and for shape optimization problems, p-quasi open sets, and capacitary measures, we refer to the book [3] and references therein. What is important here is to recall the following facts.

- When $p > d$ the γ_p convergence of a sequence (Ω_n) of open sets simply reduces to the Hausdorff convergence of the closed sets $\overline{D} \setminus \Omega_n$.
- The γ_p convergence is compact, that is every sequence (μ_n) of p-capacitary measures admits a subsequence that γ_p converges to a p-capacitary measure μ.
- The class of capacitary measures, endowed with the γ_p convergence, is a metrizable space and the γ_p convergence of a sequence (μ_n) is equivalent to the L^p convergence of the solutions $w(\mu_n)$ of the PDEs

$$\begin{cases} -\Delta_p w + \mu_n w^{p-1} = 1 & \text{in } D \\ w \in W_0^{1,p}(D), \quad w \geq 0. \end{cases} \tag{2.2}$$

This allows to define the γ_p-distance

$$d_{\gamma_p}(\mu,\nu)=\|w(\mu)-w(\nu)\|_{L^p}.$$

- The eigenvalue $\lambda_p(\mu)$ is continuous with respect to the γ_p convergence.
- When $p\le d$ the class of all the γ_p limits of sequences (Ω_n) is exactly the class of p-capacitary measures. When $p>d$ on the contrary, the class of open sets Ω with the Hausdorff convergence of $\overline{D}\setminus\Omega$ is already compact. The first example of a sequence (Ω_n) for which the γ_2 limit is the Lebesgue measure was obtained in [5], while the full characterization above in terms of capacitary measures, when $p=2$, was obtained in [8]. Finally the full proof for any $p\le d$ was obtained in [7].

Definition 2.4. *Given a p-capacitary measure μ we define the set Ω_μ where μ is finite as*

$$\Omega_\mu=\{w(\mu)>0\},$$

being $w(\mu)$ the solution of (2.2).

The key result we need in order to prove the existence of an optimal set Ω_{opt} for the shape functional $\mathcal{F}_{p,q}$ in (1.3) is the following.

Theorem 2.5. *Let $q<p$ and let (Ω_n) be a sequence of open sets such that:*

$$\Omega_n\overset{\gamma_p}{\to}\mu\qquad\textit{and}\qquad\Omega_n\overset{\gamma_q}{\to}\nu.$$

Then $\nu=0$ on the p-quasi open set Ω_μ where μ is finite.

Proof. Let w be the solution of (2.2) related to μ and let (w_n) be an optimal sequence for w in the Γ-convergence, that is such that $w_n\in W_0^{1,p}(\Omega_n)$ and

$$\lim_n\int|\nabla w_n|^p dx=\int|\nabla w|^p dx+\int w^p d\mu.$$

Let $\alpha>0$ be fixed and let

$$\varphi(x)=\big(w(x)-\alpha\big)^+.$$

We also denote by $H(s)$ the function

$$H(s)=(s/\alpha)\wedge 1.$$

Taking $u_n=\varphi\,H(w_n)$ we have that $u_n\in W_0^{1,p}(\Omega_n)$ and $u_n\to u=\varphi\,H(w)$ in L^q, so that, by the Γ-liminf inequality, we have

$$\begin{aligned}\liminf_n\int|\nabla u_n|^q dx&\ge\int|\nabla u|^q dx+\int u^q d\nu\\&=\int\big|H(w)\nabla\varphi+\varphi H'(w)\nabla w\big|^q dx+\int\big(\varphi H(w)\big)^q d\nu.\end{aligned}$$

By the definition of φ and H we have $H(w)\nabla\varphi=\nabla\varphi$, $\varphi H'(w)=0$, and $\varphi H(w)=(w-\alpha)^+$, so that

$$\begin{aligned}\int\big((w-\alpha)^+\big)^q d\nu&\le-\int|\nabla\varphi|^q dx+\liminf_n\int|\nabla u_n|^q dx\\&\le\limsup_n\int\Big[\big|H(w_n)\nabla\varphi+\varphi H'(w_n)\nabla w_n\big|^q-|\nabla\varphi|^q\Big]dx.\end{aligned}$$

By using the inequality

$$|b|^q - |a|^q \le C|b-a|\big(|a|^{q-1} + |b|^{q-1}\big)$$

we obtain

$$\int \big((w-\alpha)^+\big)^q d\nu \le C \limsup_n \int \Big[\big|H(w_n)-1\big||\nabla\varphi| + \big|\varphi H'(w_n)\nabla w_n\big|\Big] \cdot \Big[|\nabla\varphi|^{q-1} + \big|\varphi H'(w_n)\nabla w_n\big|^{q-1}\Big]\,dx.$$

Hölder inequality gives

$$\begin{aligned}\int &\Big[\big|H(w_n)-1\big||\nabla\varphi| + \big|\varphi H'(w_n)\nabla w_n\big|\Big] \\ &\cdot \Big[|\nabla\varphi|^{q-1} + \big|\varphi H'(w_n)\nabla w_n\big|^{q-1}\Big]\,dx. \\ \le& C\Big[\int |\nabla\varphi|^q\big|H(w_n)-1\big|^q + \big|\varphi H'(w_n)\nabla w_n\big|^q dx\Big]^{1/q} \\ &\cdot \Big[\int |\nabla\varphi|^q + \big|\varphi H'(w_n)\nabla w_n\big|^q dx\Big]^{(q-1)/q}.\end{aligned}$$

Now, Hölder inequality again provides

$$\int \big|\varphi H'(w_n)\nabla w_n\big|^q dx \le \Big[\int |\nabla w_n|^p\Big]^{q/p}\Big[\int \big|\varphi H'(w_n)\big|^{p/(p-q)}\Big]^{(p-q)/p}.$$

Since (w_n) is an optimal sequence related to w in the Γ-convergence, as $n\to\infty$ the right-hand side above tends to

$$\Big[\int |\nabla w|^p dx + \int w^p d\mu\Big]^{q/p}\Big[\int \big|\varphi H'(w)\big|^{p/(p-q)}\Big]^{(p-q)/p},$$

which vanishes, since $\varphi H'(w) = 0$. The term

$$\int |\nabla\varphi|^q\big|H(w_n)-1\big|^q dx$$

tends, as $n\to\infty$, to

$$\int |\nabla\varphi|^q\big|H(w)-1\big|^q dx$$

which also vanishes, by the definition of φ an H. Then, putting all together, we obtain

$$\int \big((w-\alpha)^+\big)^q d\nu = 0.$$

This concludes the proof, since $\alpha > 0$ was arbitrary. □

3. The existence result

In this section we prove the existence of an optimal p-quasi open domain Ω_{opt} for the problem

$$\max\Big\{\lambda_q(\Omega)\ :\ \Omega\in\mathcal{A}(D),\ \lambda_p(\Omega)=1\Big\}. \tag{3.1}$$

Theorem 3.1. *For every $q<p$ there exists a p-quasi open set Ω_{opt} that that solves the optimization problem* (3.1)*.*

Proof. Let (Ω_n) be a maximizing sequence for the optimization problem (3.1). Since the γ_p and γ_q convergences are compact, possibly passing to subsequences, we may also assume that

$$\Omega_n \xrightarrow{\gamma_p} \mu \qquad \text{and} \qquad \Omega_n \xrightarrow{\gamma_q} \nu,$$

for some p-capacitary measure μ and q-capacitary measure ν. If $p>d$, since the γ_p convergence reduces to the Hausdorff convergence of the complements $\overline{D}\setminus\Omega_n$, the measure μ will be of the form $\mu=\infty_{\overline{D}\setminus\Omega}$ for a suitable open set Ω, and similarly for ν when $q>d$.

By the continuity of λ_p and λ_q with respect to the γ_p and γ_q convergences respectively, we have

$$\lambda_p(\mu)=1,\qquad \lambda_q(\nu)=\lim_n \lambda_q(\Omega_n)=\sup\,(3.1).$$

By Theorem 2.5 we have that $\nu=0$ on the set Ω_μ where μ is finite; therefore, from the monotonicity property (2.1) we deduce that

$$\lambda_p(\Omega_\mu)\le\lambda_p(\mu)=1\qquad\text{and}\qquad \lambda_q(\nu)\le\lambda_q(\Omega_\mu).$$

Take now $t\le 1$ such that $\lambda_p(t\Omega_\mu)=1$; we clain that the p-quasi open set $t\Omega_\mu$ is optimal for the problem (3.1). In fact we have $t\Omega_\mu\subset D$, $\lambda_p(t\Omega_\mu)=1$, and

$$\lambda_q(t\Omega_\mu)\ge\lambda_q(\Omega_\mu)\ge\lambda_q(\nu)=\sup\,(3.1),$$

which achieves the existence proof. □

4. Concluding remarks and open questions

Before entering into comments and open questions, let us summarize the known facts about the optimization problems related to the Cheeger ratio shape functional $\mathcal{F}_{p,q}$. It is convenient to set

$$\begin{aligned} m(p,q)&=\inf\Big\{\mathcal{F}_{p,q}(\Omega)\ :\ \Omega\text{ bounded subset of }\mathbb{R}^d\Big\},\\ M(p,q)&=\sup\Big\{\mathcal{F}_{p,q}(\Omega)\ :\ \Omega\text{ bounded subset of }\mathbb{R}^d\Big\}.\end{aligned}$$

The following facts are known (see [1]).

- When $d=1$ the functional $\mathcal{F}_{p,q}$ is constant, and for every $\Omega\subset\mathbb{R}$ we have

  $$\mathcal{F}_{p,q}(\Omega)=\frac{\pi_p}{\pi_q}$$

  where

  $$\pi_p=\begin{cases}2\pi\dfrac{(p-1)^{1/p}}{p\sin(\pi/p)} & \text{for } 1<p<\infty\\ 2 & \text{for } p=1 \text{ and } p=\infty.\end{cases}$$

- For every $p \geq q$
$$m(p,q) \geq \frac{q}{p}$$
and the inequality above becomes asymptotically sharp as the dimension d tends to infinity. Moreover, the value $m(p,q)$ depends decreasingly on the dimension d.
- For every $q < p$ we have:
$$\begin{cases} M(p,q) = +\infty & \text{for } q \leq d \\ M(p,q) < +\infty & \text{for } q > d. \end{cases}$$

4.1. **Existence of optimal domains for $D = \mathbb{R}^d$.** By Theorem 3.1 we obtain the existence of an optimal domain Ω, that minimizes the shape functional $F_{p,q}$, in the case when all competing domains are constrained to stay in a given bounded set D. The question is now to see what happens when $D = \mathbb{R}^d$. Similar problems have been considered in the literature in the framework of *spectral optimization*, where the shape functional depends on the eigenvalues of the Laplace operator $-\Delta$ with Dirichlet conditions on $\partial\Omega$. It is possible that some of the tools developed in [2] (that use concentration compactness arguments) and in [11] (that use suitable surgery techniques) could also be applied to the case of the shape functional $\mathcal{F}_{p,q}$.

Concerning the maximization problem when $q > d$ it is not clear if a domain Ω (possibly unbounded) maximizing $\mathcal{F}_{p,q}$ exists. Good candidates could be the domains of the form $\Omega = \mathbb{R}^d \setminus Z$ with Z a discrete set; in particular with Z periodic. See the comments in Subsection 4.3 below for the case $p = \infty$.

4.2. **Optimization problems for convex domains.** When Ω is convex it is possible to prove that (see [1])
$$\max\left\{\frac{q}{p}, \frac{\pi_p}{d\pi_q}\right\} \leq \mathcal{F}_{p,q}(\Omega) \leq \pi_p \min\left\{\frac{q}{2}, \frac{d}{\pi_q}\right\}.$$
In particular, the supremum
$$M_{conv}(p,q) = \sup\Big\{\mathcal{F}_{p,q}(\Omega) \ : \ \Omega \text{ bounded convex subset of } \mathbb{R}^d\Big\}$$
is always finite. A reasonable conjecture, formulated by Parini in [13] is that $M_{conv}(p,q)$ coincides with π_p/π_q and is asymptotically reached by thin slabs $\Omega_\varepsilon = A\times]0,\varepsilon[$, being A a $d-1$ dimensional open set. Additional remarks on the case Ω convex can be found in [9] and in [12].

In spite of the strong geometrical constraint imposed by the convexity, the existence of optimal convex domains minimizing $\mathcal{F}_{p,q}$ is not yet completely proved. The only available result is for $\mathcal{F}_{p,1}$ in [14]. An interesting conjecture in [12] is that in the case $d = 2$ the optimal convex set minimizing $\mathcal{F}_{2,1}$ is the square.

4.3. **The case $p = \infty$.** Since

$$\mathcal{F}_{\infty,q}(\Omega) = \frac{1}{\rho(\Omega)\lambda_q^{1/q}(\Omega)},$$

we have

$$m(\infty, q) = \mathcal{F}_{\infty,q}(B_1) = \frac{1}{\lambda_q^{1/q}(B_1)},$$

where B_1 is a ball of unitary radius. Concerning $M(\infty, q)$, taking $\Omega_n = B_1 \setminus Z_n$ where Z_n is a set of n points in B_1 "uniformly" distributed, since points have zero q-capacity when $q \le d$, it is easy to see that

$$M(\infty, q) = +\infty \qquad \text{for every } q \le d.$$

On the contrary, $M(\infty, q)$ is finite when $q > d$. It would be interesting to investigate the following questions in the case $q > d$.

- Is there a domain Ω (possibly unbounded) such that

$$M(\infty, q) = \mathcal{F}_{\infty,q}(\Omega)?$$

- Are there optimal domains Ω of the form $\Omega = \mathbb{R}^d \setminus Z$ with Z a discrete set?
- The discrete set Z above can be periodic? In particular, in dimension $d = 2$, is the domain $\Omega = \mathbb{R}^2 \setminus Z$, with Z consisting of the centers of a regular hexagonal tiling of $\mathbb{R}^2$, a domain that maximizes $\mathcal{F}_{\infty,q}$?

4.4. **Regularity issues.** The question that arises now, and which is not addressed in this work, is related to the regularity of the optimal sets Ω_{opt}; here we only prove that they are p-quasi open, but we expect that they are much more regular. As already done for other shape optimization problems, the steps to follow would be: prove that optimal domains Ω_{opt} are open (this is automatic if $p > d$), prove that they have a finite perimeter, and finally obtain higher regularity results.

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