SIAM J. MATH. ANAL.
Vol. 54, No. 1, pp. 723–736
© 2022 Society for Industrial and Applied Mathematics

Downloaded 04/28/22 to 146.83.7.1 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

ON THE STABILITY OF RADIAL SOLUTIONS TO AN ANISOTROPIC GINZBURG–LANDAU EQUATION*

XAVIER LAMY† AND ANDRES ZUNIGA‡

Abstract. We study the linear stability of entire radial solutions $u(re^{i\theta}) = f(r)e^{i\theta}$, with positive increasing profile $f(r)$, to the anisotropic Ginzburg–Landau equation $-\Delta u - \delta(\partial_x + i\partial_y)^2\bar{u} = (1-|u|^2)u$, $-1 < \delta < 1$, which arises in various liquid crystal models. In the isotropic case $\delta = 0$, Mironescu showed that such solution is nondegenerately stable. We prove stability of this radial solution in the range $\delta \in (\delta_1, 0]$ for some $-1 < \delta_1 < 0$ and instability outside this range. In strong contrast with the isotropic case, stability with respect to higher Fourier modes is *not* a direct consequence of stability with respect to lower Fourier modes. In particular, in the case where $\delta \approx -1$, lower modes are stable and yet higher modes are unstable.

Key words. Ginzburg–Landau, liquid crystals, elastic anisotropy

AMS subject classification. 35Q56

DOI. 10.1137/21M1433939

1. Introduction.

Given $\delta \in (-1, 1)$ and $u : \mathbb{R}^2 \to \mathbb{C}$, we consider the anisotropic energy

(1.1)
$$\mathfrak{E}[u] = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 + \frac{\delta}{2}\operatorname{Re}\left\{(\partial_\eta \bar{u})^2\right\} + \frac{1}{4}(1-|u|^2)^2 \,\mathrm{d}x, \qquad \text{where } \partial_\eta = \partial_x + i\partial_y.$$

Minimizers and stable critical points of $\mathfrak{E}$ are relevant in describing two-dimensional point defects (or three-dimensional straight-line defects) in some liquid crystal configurations (e.g., smectic-C^* thin films [4] and nematics close to the Fréedericksz transition [2]). This energy can also be viewed as a toy model to understand intricate phenomena triggered by elastic anisotropy in the more complex Landau–de Gennes energy [12].

Remark 1.1. The anisotropic term $\operatorname{Re}\left\{(\partial_\eta \bar{u})^2\right\}$ can be rewritten as

$$\operatorname{Re}\left\{(\partial_\eta \bar{u})^2\right\} = (\nabla \cdot u)^2 - (\nabla \times u)^2,$$

so that, in view of the identity $|\nabla u|^2 = (\nabla \cdot u)^2 + (\nabla \times u)^2 - 2\det(\nabla u)$, energy (1.1) differs from

$$\widetilde{\mathfrak{E}}[u] = \int \frac{k_s}{2}(\nabla \cdot u)^2 + \frac{k_b}{2}(\nabla \times u)^2 + \frac{1}{4}(1-|u|^2)^2, \qquad k_s = 1+\delta,\ k_b = 1-\delta,$$

*Received by the editors July 15, 2021; accepted for publication October 25, 2021; published electronically January 25, 2022.
https://doi.org/10.1137/21M1433939
Funding: The work of the first author was partially supported by the ANR project ANR-18-CE40-0023 and the COOPINTER project IEA-297303. The work of the second author was supported by ANID Chile under the grant FONDECYT de Iniciación en Investigación 11201259.
†Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France (xlamy@math.univ-toulouse.fr).
‡Instituto de Ciencias de la Ingeniería (ICI), Universidad de O'Higgins (UOH), Rancagua, Chile (andres.zuniga@uoh.cl).

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 04/28/22 to 146.83.7.1 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

only by the integral of the null Lagrangian $\det(\nabla u)$. This is precisely the form that appears in [4] where minimizers of

$$\widetilde{\mathfrak{E}}_\varepsilon[u] = \int_\Omega \frac{k_s}{2}(\nabla \cdot u)^2 + \frac{k_b}{2}(\nabla \times u)^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \tag{1.2}$$

are investigated in the limit as $\varepsilon \to 0^+$ in a bounded planar domain Ω.

Critical points of $\mathfrak{E}$ are solutions of the Euler–Lagrange equation

$$\begin{aligned} &\mathfrak{L}_\delta u = (|u|^2 - 1)u \qquad \text{in } \mathbb{R}^2, \\ &\mathfrak{L}_\delta u := \Delta u + \delta\, \partial_{\eta\eta} \bar{u}. \end{aligned} \tag{1.3}$$

We are interested in symmetric solutions of the form

$$u(re^{i\theta}) = f(r)e^{i\alpha}e^{i\theta} \qquad \text{for some } \alpha \in \mathbb{R}, \tag{1.4}$$

with a radial profile $f(r)$ satisfying

$$f(0) = 0, \quad \lim_{r\to+\infty} f(r) = 1, \quad |f(r)| > 0 \quad \forall r \in (0,\infty). \tag{1.5}$$

Formally, one can always look for solutions of (1.3) in the form (1.4) (as a consequence of the $O(2)$-invariance of $\mathfrak{E}$), and f must solve

$$Tf + \delta e^{-2i\alpha} T\bar{f} = \left(|f|^2 - 1\right) f, \qquad T = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}.$$

At this point we see a fundamental difference with respect to the isotropic case $\delta = 0$. If $\delta = 0$, one can find solutions as above for a real-valued function f, which moreover does not depend on α. In the anisotropic case $\delta \neq 0$, as remarked in [2], the function f can be real-valued only if $\alpha \equiv 0$ modulo $\pi/2$. In that case, the existence and uniqueness of a solution satisfying (1.5) follows from the case $\delta = 0$ (see [1, 7]). Otherwise, the function f must be complex valued.

Remark 1.2. Another difference with respect to the isotropic case is that for $\delta \neq 0$ the ansatz $u(re^{i\theta}) = f(r)e^{im\theta}$ cannot provide a solution when the winding number m is $\neq 1$.

In [2], the core energies of the two symmetric solutions corresponding to $\alpha = 0, \pi/2$ are compared to find that the lowest energy corresponds to $\alpha = 0$ for $\delta < 0$ and $\alpha = \pi/2$ for $\delta > 0$. In view of Remark 1.1 this is consistent with the fact that $\nabla \times e^{i\theta} = 0$, while $\nabla \cdot ie^{i\theta} = 0$; indeed, for $\delta < 0$ the energy $\widetilde{\mathfrak{E}}[u]$ in Remark 1.1 penalizes more strongly the term $(\nabla \times u)^2$ than the term $(\nabla \cdot u)^2$, since in this case $k_b = 1 - \delta > k_s = 1 + \delta$. In [4, Proposition 3.1] the authors use this to show that minimizers of (1.2) behave like $e^{i\alpha}e^{i\theta}$ around point defects, with $\alpha \equiv 0$ (resp., $\pi/2$) modulo π if $\delta < 0$ (resp., $\delta > 0$). These results tell us, for $\delta \neq 0$, which one is the minimizing behavior at infinity.

Here, in contrast, we fix the far-field behavior and investigate the local stability of radial solutions with respect to compactly supported perturbations. For the isotropic case $\delta = 0$, this study has been performed in [13] (see also [5]), and the radial solution is stable. Recently, these linear stability estimates have been improved to nonlinear stability in [6]. In the anisotropic situation $\delta \neq 0$ we find that the corresponding symmetric solution stays stable for negative δ close to zero and it loses stability for δ either positive or close to minus one (see Theorem 1.3 for precise statements).

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 04/28/22 to 146.83.7.1 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

It can be readily seen that the case $\alpha = \pi/2$ corresponds to $\alpha = 0$ after changing the sign of δ. Accordingly, we only treat the case where $\alpha = 0$. That is, we investigate the linear stability of solutions u of the form

$$u^\delta_{\rm rad}(r,\theta) = f(r)e^{i\theta}, \quad f\colon (0,+\infty) \to (0,+\infty) \quad \text{with} \quad f(0)=0, \quad \lim_{r\to+\infty} f(r) = 1. \tag{1.6}$$

Let us note that the equation satisfied by $u^\delta_{\rm rad}$, (1.3), reduces to the following ODE for f:

$$(1+\delta)Tf = (f^2-1)f, \qquad T = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}. \tag{1.7}$$

As pointed out in [2], the rescaling of the variable by $(1+\delta)^{\frac{1}{2}}$ simplifies (1.7) to the standard ODE corresponding to the isotropic case $\delta = 0$. Whence, existence and uniqueness of f follow from [1, 7]. Moreover, it is known that f takes values in $(0,1)$ and is strictly increasing.

The second variation of the energy $\mathfrak{E}$ around $u^\delta_{\rm rad}$ is the quadratic form

$$\begin{aligned}\mathfrak{Q}^\delta_{\rm rad}[v] &= \int_{\mathbb{R}^2} |\nabla v|^2 + \delta\,\mathrm{Re}\left\{(\partial_\eta \bar v)^2\right\} - (1-|u^\delta_{\rm rad}|^2)|v|^2 + 2\,(u^\delta_{\rm rad}\cdot v)^2\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2} |\nabla v|^2 + \delta\,\mathrm{Re}\left\{(\partial_\eta \bar v)^2\right\} - (1-f^2)|v|^2 + 2f^2\,(e^{i\theta}\cdot v)^2\,\mathrm{d}x \end{aligned} \tag{1.8}$$

associated to the linear operator obtained by linearizing (1.3) around $u^\delta_{\rm rad}$:

$$\mathcal{L}(u^\delta_{\rm rad})[v] = -\mathfrak{L}_\delta v - (1-|u^\delta_{\rm rad}|^2)v + 2\,(u^\delta_{\rm rad}\cdot v)\,u^\delta_{\rm rad},$$

where $u\cdot v := \mathrm{Re}\,\{u\bar v\}$ denotes the standard inner product of complex-valued functions.

Taking into account the asymptotic expansion $f(r) = 1 + O(r^{-2})$ as $r \to \infty$ (see [1, 7]), it follows that the energy space of $\mathfrak{Q}^\delta_{\rm rad}$ naturally corresponds to

$$\mathcal{H} := \left\{ v \in H^1_{loc}(\mathbb{R}^2)\colon \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{1}{r^2}|v|^2 + (e^{i\theta}\cdot v)^2\,\mathrm{d}x < +\infty \right\}.$$

Also, the translational invariance of $\mathfrak{E}$ readily provides two elements of $\mathcal{H}$ at which $\mathfrak{Q}^\delta_{\rm rad}$ vanishes, namely,

$$\partial_x u^\delta_{\rm rad} = e^{i\theta}\left(f'\cos\theta - i\frac{f}{r}\sin\theta\right), \qquad \partial_y u^\delta_{\rm rad} = e^{i\theta}\left(f'\sin\theta + i\frac{f}{r}\cos\theta\right),$$

and the linear space they generate is denoted by

$$K_0 = \mathrm{span}\{\partial_x u^\delta_{\rm rad}, \partial_y u^\delta_{\rm rad}\}.$$

Our main result shows that the symmetric solution $u^\delta_{\rm rad}$ is stable when $\delta \le 0$ is small and unstable otherwise.

THEOREM 1.3. *Let $u^\delta_{\rm rad}$ denote the radial solution* (1.6) *of the anisotropic Ginzburg–Landau equation* (1.3), *and let $\mathfrak{Q}^\delta_{\rm rad}$ denote the quadratic form* (1.8) *associated to the energy $\mathfrak{E}$ around $u^\delta_{\rm rad}$. Then, there exists a unique number $\delta_1 \in (-1,0)$ such that*

- *for every $\delta \in (\delta_1, 0]$, $u^\delta_{\rm rad}$ is nondegenerately stable: namely,*

$$\mathfrak{Q}^\delta_{\rm rad}[v] > 0 \qquad \forall\ v \in H \setminus K_0,$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

- *for every* $\delta \in (-1, \delta_1) \cup (0, 1)$, u^δ_{rad} *is linearly unstable: namely,*

$$\mathfrak{Q}^\delta_{\mathrm{rad}}[v] < 0 \qquad \textit{for some } v \in H.$$

Remark 1.4. The most relevant range from the standpoint of physics is $\delta \in (-1, 0]$ since for $\delta > 0$ the far-field behavior corresponding to $\alpha = 0$ is nonminimizing, and this translates here into instability of the radial solution.

Remark 1.5. In the stability range $\delta \in (\delta_1, 0]$, a contradiction argument as in [5, Lemma 3.1] provides a coercivity estimate of the form

$$\mathfrak{Q}^\delta_{\mathrm{rad}}[v] \geq C(\delta) \int_{\mathbb{R}^2} |\nabla v|^2 \,\mathrm{d}x \qquad \forall v \in K_0^\perp \colon \int_{\mathbb{S}^1} (ie^{i\theta}) \cdot v(re^{i\theta})\, d\theta = 0 \;\; \forall r > 0,$$

where $\perp$ denotes orthogonality in $\mathcal{H}$. Using this coercivity for $\delta = 0$, one can deduce stability for small negative δ via a relatively simple perturbation argument, combined with properties of the lower modes in section 3. Instead, we will give a more quantitative proof, which provides an explicit range for stability: we deduce that $\delta_1 \leq -1/\sqrt{5}$.

Remark 1.6. Very recently the linear stability estimates of [13] in the isotropic case $\delta = 0$ have been improved in [6] to establish nonlinear stability. It would be interesting to extend these nonlinear stability estimates to the anisotropic case in the stability range $\delta \in (\delta_1, 0]$.

Our proof of Theorem 1.3 follows the general strategy of [13]: we decompose v into Fourier modes

$$v = e^{i\theta} \sum_{n \in \mathbb{Z}} w_n(r) e^{in\theta},$$

and we are led to studying the sign of $\mathfrak{Q}^\delta_{\mathrm{rad}}$, separately, for each mode

$$e^{i\theta}\left(w_n(r)e^{in\theta} + w_{-n}(r)e^{-in\theta}\right).$$

As in [13], the lower modes $n = 0$ and $n = 1$ play a special role. They can be studied via an appropriate decomposition already used in [13] (see also [5]). For any $\delta \in (-1, 0]$ we find that these lower modes are stable, while for $\delta > 0$ the mode corresponding to $n = 0$ is unstable.

A major difference of the present work compared to [13] (or similar results in [9, 10, 11]) pertains to the higher modes $n \geq 2$. In contrast with the cited works, stability for the higher modes is not an obvious consequence of stability for the lower modes. More precisely in the isotropic case we have

$$\mathfrak{Q}^0_{\mathrm{rad}}\left[e^{i\theta}\left(w_+(r)e^{in\theta} + w_-(r)e^{-in\theta}\right)\right] \geq \mathfrak{Q}^0_{\mathrm{rad}}\left[e^{i\theta}\left(w_+(r)e^{i\theta} + w_-(r)e^{-i\theta}\right)\right] \quad \forall n \geq 1,$$

but for $\delta \neq 0$ this is not valid anymore; see (4.1). This feature is new and specific to the anisotropic case $\delta \neq 0$. Our strategy to study the sign of these higher modes is based on the same decomposition used for $n = 1$ and a careful balance of the contributions of additional terms, which end up causing instability for $\delta \approx -1$. We are, however, not able to estimate precisely the Morse index of the solution in the instability range.

The article is organized as follows. In section 2 we recall the splitting property of the quadratic form $\mathfrak{Q}^\delta_{\mathrm{rad}}$ with respect to Fourier expansion. In section 3 we study the stability of lower modes and in section 4 the instability of higher modes. In section 5 we give the proof of Theorem 1.3. In addition, we included Appendix A to recall the details of the decomposition used to study the lower modes, adapted to our notations.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 04/28/22 to 146.83.7.1 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Downloaded 04/28/22 to 146.83.7.1 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

2. Fourier splitting. Recall that $f(r) = f_0((1+\delta)^{-\frac{1}{2}}r)$ where f_0 is the classical Ginzburg–Landau vortex profile corresponding to the case $\delta = 0$. That is, the unique solution of

$$f_0'' + \frac{1}{r}f_0' - \frac{1}{r^2}f_0 = -(1-f_0^2)f_0, \quad f_0 > 0 \text{ on } (0,+\infty), \quad f_0(0) = 0, \quad \lim_{r\to+\infty} f_0(r) = 1. \tag{2.1}$$

We rescale variables and consider $\mathcal{Q}^\delta[v] = \mathfrak{Q}^\delta_{\mathrm{rad}}[\tilde{v}]$, where $\tilde{v}(\tilde{x}) = v((1+\delta)^{-\frac{1}{2}}\tilde{x})$, so that

$$\mathcal{Q}^\delta[v] = \int_{\mathbb{R}^2} |\nabla v|^2 + \delta \,\mathrm{Re}\left\{(\partial_\eta \bar{v})^2\right\} + (1+\delta)\left\{2f_0^2\,(e^{i\theta}\cdot v)^2 - (1-f_0^2)|v|^2\right\}\,\mathrm{d}x, \tag{2.2}$$

which corresponds to the second variation of the appropriately rescaled energy around u^0_{rad}. Following [13] we decompose v using Fourier series, as

$$v = e^{i\theta} w = e^{i\theta}\sum_{n\in\mathbb{Z}} w_n(r)e^{in\theta}, \tag{2.3}$$

where we have conveniently shifted the index $n-1 \mapsto n$.

This decomposition provides a "diagonalization" of the linearized operator.

LEMMA 2.1. *The quadratic form* (2.2) *splits as*

$$\mathcal{Q}^\delta[v] = \mathcal{Q}^\delta\left[w_0(r)e^{i\theta}\right] + \sum_{n\geq 1}\mathcal{Q}^\delta\left[e^{i\theta}\left(w_n(r)e^{in\theta} + w_{-n}(r)e^{-in\theta}\right)\right].$$

Proof of Lemma 2.1. Lemma 2.1 essentially asserts that the family of functions

$$w_0(r)e^{i\theta}, \quad \{e^{i\theta}\left(w_n(r)e^{in\theta} + w_{-n}(r)e^{-in\theta}\right) : \ n\geq 1\}, \tag{2.4}$$

is orthogonal for the quadratic form $\mathcal{Q}$. This quadratic form (2.2) is composed of three terms. For the first term,

$$\int_{\mathbb{R}^2} |\nabla v|^2\,\mathrm{d}x,$$

the orthogonality of (2.4) is a standard fact (recall, e.g., in [13]). For the third term,

$$\int_{\mathbb{R}^2} \left\{f_0^2\,(e^{i\theta}\cdot v)^2 - (1-f_0^2)|v|^2\right\}\,\mathrm{d}x,$$

the orthogonality of (2.4) is proved in [13]. The novelty here, with respect to [13], concerns the anisotropic term

$$\int_{\mathbb{R}^2} \mathrm{Re}\left\{(\partial_\eta\bar{v})^2\right\}\,\mathrm{d}x.$$

The orthogonality of (2.4) for this anisotropic term, as a matter of fact, follows from the calculations in [3, section 3.2]. As our notations are different, we sketch a proof here for the reader's convenience.

We compute

$$\partial_\eta \bar{v} = e^{i\theta}\partial_r\bar{v} + \frac{ie^{i\theta}}{r}\partial_\theta\bar{v} = \sum_{n\in\mathbb{Z}}\left(\bar{w}_n' + \frac{1+n}{r}\bar{w}_n\right)e^{-in\theta}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 04/28/22 to 146.83.7.1 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

and deduce, using the orthogonality of $\{e^{in\theta}\}$ in $L^2(\mathbb{S}^1)$,

$$\begin{aligned}
&\fint_{\mathbb{S}^1} \mathrm{Re}\left\{(\partial_\eta \bar{v})^2\right\} d\theta \\
&= \mathrm{Re}\left\{\sum_{n,m\in\mathbb{Z}} \left(\bar{w}_n' + \frac{1+n}{r}\bar{w}_n\right)\left(\bar{w}_m' + \frac{1+m}{r}\bar{w}_m\right) \fint_{\mathbb{S}^1} e^{-i(n+m)\theta} d\theta\right\} \\
&= \mathrm{Re}\left\{\sum_{n\in\mathbb{Z}} \left(\bar{w}_n' + \frac{1+n}{r}\bar{w}_n\right)\left(\bar{w}_{-n}' + \frac{1-n}{r}\bar{w}_{-n}\right)\right\} \\
&= \sum_{n\in\mathbb{Z}} \mathrm{Re}\left\{\left(\bar{w}_n' + \frac{1+n}{r}\bar{w}_n\right)\left(\bar{w}_{-n}' + \frac{1-n}{r}\bar{w}_{-n}\right)\right\}.
\end{aligned}$$

This implies the announced orthogonality and completes the proof of Lemma 2.1. □

According to the decomposition of Lemma 2.1, we define the quadratic forms

$$\begin{aligned}
Q_0^\delta[\varphi] &= \frac{1}{2\pi}\mathcal{Q}^\delta\left[\varphi(r)e^{i\theta}\right] && \text{for } \varphi \in \mathcal{H}_0, \\
Q_n^\delta[\varphi,\psi] &= \frac{1}{2\pi}\mathcal{Q}^\delta\left[e^{i\theta}\left(\varphi(r)e^{in\theta} + \psi(r)e^{-in\theta}\right)\right] && \text{for } (\varphi,\psi) \in \mathcal{H}_1,
\end{aligned}$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are the natural spaces corresponding to the conditions $\varphi(r)e^{i\theta} \in \mathcal{H}$ and $e^{i\theta}(\varphi(r)e^{in\theta} + \psi(r)e^{-in\theta}) \in \mathcal{H}$ for $n \geq 1$, respectively,

$$\mathcal{H}_0 = \left\{\varphi \in H^1_{loc}(0,\infty)\colon \int_0^{+\infty}\left(|\varphi'|^2 + \frac{|\varphi|^2}{r^2} + \mathrm{Re}\{\varphi\}^2\right) r\,dr < +\infty\right\},$$

$$\mathcal{H}_1 = \left\{(\varphi,\psi) \in (H^1_{loc}(0,\infty))^2\colon \int_0^{+\infty}\left(|\varphi'|^2 + |\psi'|^2 + \frac{|\varphi|^2+|\psi|^2}{r^2} + |\varphi+\bar{\psi}|^2\right) r\,dr < +\infty\right\}.$$

Remark 2.2. Using the density of smooth functions in H^1_{loc} and cut-off functions χ_ε such that $\mathbf{1}_{2\varepsilon<r<\varepsilon^{-1}} \leq \chi_\varepsilon(r) \leq \mathbf{1}_{\varepsilon<r<2\varepsilon^{-1}}$ and $|\chi_\varepsilon'(r)| \leq C/r$, we see that smooth test functions with compact support in $(0,\infty)$ are dense in $\mathcal{H}_0$ and $\mathcal{H}_1$. Hence, in what follows, we will always be able to perform calculations assuming, without loss of generality, that φ and ψ are such test functions.

The quadratic forms Q_0^δ and Q_n^δ are explicitly given by

$$\begin{aligned}
Q_0^\delta[\varphi] = \int_0^\infty \Bigg[&|\varphi'|^2 + \frac{1}{r^2}|\varphi|^2 + \delta\,\mathrm{Re}\left\{\left(\bar{\varphi}' + \frac{1}{r}\bar{\varphi}\right)^2\right\} \\
&+ (1+\delta)\left\{2f_0^2(\mathrm{Re}\{\varphi\})^2 - (1-f_0^2)|\varphi|^2\right\}\Bigg] rdr,
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
Q_n^\delta[\varphi,\psi] = \int_0^\infty \Bigg[&|\varphi'|^2 + |\psi'|^2 + \frac{(1+n)^2}{r^2}|\varphi|^2 + \frac{(1-n)^2}{r^2}|\psi|^2 \\
&+ 2\delta\,\mathrm{Re}\left\{\left(\bar{\varphi}' + \frac{1+n}{r}\bar{\varphi}\right)\left(\bar{\psi}' + \frac{1-n}{r}\bar{\psi}\right)\right\} \\
&+ (1+\delta)\left\{f_0^2|\varphi+\bar{\psi}|^2 - (1-f_0^2)\left(|\varphi|^2+|\psi|^2\right)\right\}\Bigg] rdr.
\end{aligned} \tag{2.6}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 04/28/22 to 146.83.7.1 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Remark 2.3. For every $n \geq 1$ there is a further splitting, namely,

$$Q_n^\delta[\varphi, \psi] = Q_n^\delta\left[\operatorname{Re}\{\varphi\}, \operatorname{Re}\{\psi\}\right] + Q_n^\delta\left[\operatorname{Im}\{\varphi\}, -\operatorname{Im}\{\psi\}\right].$$

Consequently, it will be sufficient to consider real-valued test functions φ, ψ.

3. Study of the lower modes Q_0^δ and Q_1^δ.

We show that Q_0^δ is positive for $\delta \leq 0$, but it can become negative for $\delta > 0$. In addition, we prove that Q_1^δ is nonnegative $\forall \delta \in (-1, 0]$.

3.1. Positivity of Q_0^δ for $\delta \in (-1, 0]$.

Let us recall from (2.5) that Q_0^δ is given by

$$Q_0^\delta[\varphi] = \int_0^\infty \left[|\varphi'|^2 + \frac{1}{r^2}|\varphi|^2 + \delta \operatorname{Re}\left\{ \left(\bar{\varphi}' + \frac{1}{r}\bar{\varphi} \right)^2 \right\} + (1+\delta)\left\{ 2f_0^2 (\operatorname{Re}\{\varphi\})^2 - (1-f_0^2)|\varphi|^2 \right\} \right] r dr.$$

We now introduce the quadratic form

$$A_0[\varphi] := Q_0^0[\varphi] = \int_0^\infty \left[|\varphi'|^2 + \frac{1}{r^2}|\varphi|^2 + 2f_0^2(\operatorname{Re}\{\varphi\})^2 - (1-f_0^2)|\varphi|^2 \right] r dr.$$

It is known that $A_0[\varphi] > 0$, unless $\varphi = 0$ (see Appendix A for more details). Moreover, we have the identity

$$\begin{aligned} Q_0^\delta[\varphi] &= (1+\delta)A_0[\operatorname{Re}\{\varphi\}] + (1-\delta)A_0[i \operatorname{Im}\{\varphi\}] - 2\delta \int (1-f_0^2)(\operatorname{Im}\{\varphi\})^2\, r\, dr \\ &\quad + \delta \int_0^\infty \frac{d}{dr}\left[(\operatorname{Re}\{\varphi\})^2 - (\operatorname{Im}\{\varphi\})^2 \right] dr \\ &= (1+\delta)A_0[\operatorname{Re}\{\varphi\}] + (1-\delta)A_0[i \operatorname{Im}\{\varphi\}] - 2\delta \int (1-f_0^2)(\operatorname{Im}\{\varphi\})^2\, r\, dr, \end{aligned}$$

which is valid for any $\varphi \in C_c^\infty(0, \infty)$, hence for $\varphi \in \mathcal{H}_0$ thanks to Remark 2.2. Since $1 - f_0^2 \geq 0$, we deduce the positivity of Q_0^δ for every $\delta \in (-1, 0]$.

3.2. Instability for $\delta > 0$.

Using the formula (A.1) obtained for A_0 in Appendix A, we see that for any compactly supported real-valued test function χ we have

$$Q_0^\delta[if_0\chi] = (1-\delta)\int f_0^2 (\chi')^2\, r\, dr - 2\delta \int (1-f_0^2) f_0^2 \chi^2\, r\, dr.$$

Applying this to $\chi_n(r) = \chi_1(r/n)$, for a fixed test function χ_1, and using the asymptotic expansion [1, 7]

$$f_0(r) = 1 - \frac{1}{2r^2} + O(r^{-4}) \qquad \text{as } r \to \infty,$$

we see that

$$\lim_{n\to\infty} Q_0^\delta[if_0\chi_n] = (1-\delta)\int (\chi_1')^2\, r\, dr - 2\delta \int \frac{\chi_1^2}{r^2}\, r\, dr.$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

When $\delta > 0$, this expression must be negative for some χ_1, since Hardy's inequality is known to fail in two dimensions. Explicitly, by choosing

$$\chi_1(r) = \sin(\sqrt{\lambda}\ln r)\mathbf{1}_{(1,e^{\pi/\sqrt{\lambda}})}(r) \qquad \text{for } \lambda = \frac{\delta}{1-\delta} > 0,$$

we have that $\chi_1 \in H^1(0,\infty)$ is compactly supported, and

$$\lim_{n\to\infty} Q_0^\delta[if_0\chi_n] = -\delta \int \frac{\chi_1^2}{r^2}\, r\, dr < 0.$$

Whence, for $\delta > 0$, the mode of order 0 already brings instability. This comes as no surprise as this mode corresponds to infinitesimal rotations (see Appendix A), and we know that the far-field behavior $e^{i\theta}$ is unstable: rotating this far-field behavior decreases the energy.

3.3. Positivity of Q_1^δ for $\delta \leq 0$. Recall, according to (2.6), that Q_1^δ is given by

$$\begin{aligned} Q_1^\delta[\varphi,\psi] = \int_0^\infty \Bigg[& |\varphi'|^2 + |\psi'|^2 + \frac{4}{r^2}|\varphi|^2 \\ & + 2\delta\, \mathrm{Re}\left\{ \left(\bar\varphi' + \frac{2}{r}\bar\varphi\right)\bar\psi' \right\} \\ & + (1+\delta)\left\{ f_0^2|\varphi+\bar\psi|^2 - (1-f_0^2)\left(|\varphi|^2+|\psi|^2\right)\right\} \Bigg] r dr. \end{aligned}$$

We introduce the quadratic form $A_1 := Q_1^0$, namely,

$$\begin{aligned} A_1[\varphi,\psi] = \int_0^\infty \Bigg[& |\varphi'|^2 + |\psi'|^2 + \frac{4}{r^2}|\varphi|^2 \\ & + f_0^2|\varphi+\bar\psi|^2 - (1-f_0^2)\left(|\varphi|^2+|\psi|^2\right) \Bigg] r dr. \end{aligned}$$

It is a known fact that A_1 is nonnegative on $\mathcal{H}_1$ and vanishes exactly at pairs (φ,ψ) corresponding to maps v which are linear combinations of $\partial_x u^0_{\mathrm{rad}}$ and $\partial_y u^0_{\mathrm{rad}}$ (see Appendix A for more details). Moreover, we have

(3.1)

$$\begin{aligned} Q_1^\delta[\varphi,\psi] - (1+\delta)A_1[\varphi,\psi] &= -\delta \int_0^\infty \left[|\varphi'|^2 + |\psi'|^2 + \frac{4}{r^2}|\varphi|^2 \right] r dr \\ &\quad + 2\delta \int_0^\infty \mathrm{Re}\left\{ \left(\bar\varphi' + \frac{2}{r}\bar\varphi\right)\bar\psi' \right\} r dr \\ &= -\delta \int_0^\infty \left| \varphi' + \frac{2}{r}\varphi - \bar\psi' \right|^2 r dr - 2\delta \int_0^\infty \frac{d}{dr}\left[|\varphi|^2\right] dr \\ &= -\delta \int_0^\infty \left| \varphi' + \frac{2}{r}\varphi - \bar\psi' \right|^2 r dr \end{aligned}$$

for $(\varphi,\psi) \in (C_c^\infty(0,\infty))^2$, hence $\forall(\varphi,\psi) \in \mathcal{H}_1$. From this identity we infer that $Q_1^\delta \geq 0$ for every $\delta \in (-1,0]$, and equality can only occur when v is a linear combination of $\partial_x u^0_{\mathrm{rad}}$ and $\partial_y u^0_{\mathrm{rad}}$.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 04/28/22 to 146.83.7.1 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

4. Study of the higher modes Q_n^δ for $n \geq 2$.

4.1. Positivity of Q_n^δ for $n \geq 2$ and $\delta \in [-1/\sqrt{5}, 0]$. Let us recall: in the isotropic case, the positivity of Q_n^δ (any $n \geq 2$) is a consequence of the fact that $Q_n^0 \geq Q_1^0$. Here, from the definition (2.6) of Q_n^δ, we have

$$Q_n^\delta[\varphi,\psi] - Q_1^\delta[\varphi,\psi] = (n-1)\int_0^\infty \left[\frac{n+3}{r^2}|\varphi|^2 + \frac{n-1}{r^2}|\psi|^2 - 2\delta\frac{n+1}{r^2}\operatorname{Re}\{\bar\varphi\bar\psi\} + 2\frac{\delta}{r}\operatorname{Re}\{\bar\varphi\bar\psi' - \bar\varphi'\bar\psi\}\right] rdr. \tag{4.1}$$

Unlike what happens in the isotropic case, this does not obviously have a sign (because of the last term which contains derivatives).

It seems reasonable to use a decomposition for φ, ψ adapted to Q_1^δ, as in Appendix A. Accordingly, we define for any real-valued test functions ζ, η, the adapted quadratic form

$$B_n^\delta[\zeta,\eta] = \frac{1}{2}Q_n^\delta\left[f_0'\zeta - r^{-1}f_0\eta, f_0'\zeta + r^{-1}f_0\eta\right].$$

Decomposing

$$Q_n^\delta = (1+\delta)A_1 + Q_1^\delta - (1+\delta)A_1 + Q_n^\delta - Q_1^\delta$$

and using the above expressions of $Q_n^\delta - Q_1^\delta$ ((4.1)) and $Q_1^\delta - (1+\delta)A_1$ ((3.1)), we have, for real-valued $(\varphi,\psi) \in \mathcal{H}_1$

$$\begin{aligned} Q_n^\delta[\varphi,\psi] &= (1+\delta)A_1[\varphi,\psi] \\ &\quad - \delta\int_0^\infty \left(\varphi' + \frac{2}{r}\varphi - \psi'\right)^2 rdr \\ &\quad + (n-1)\int_0^\infty \left[\frac{n+3}{r^2}\varphi^2 + \frac{n-1}{r^2}\psi^2 - 2\delta\frac{n+1}{r^2}\varphi\psi\right] \\ &\quad + 2\delta(n-1)\int_0^\infty \frac{1}{r}\left(\varphi\psi' - \varphi'\psi\right) rdr. \end{aligned}$$

When plugging in $\varphi = f_0'\zeta - r^{-1}f_0\eta$, $\psi = f_0'\zeta + r^{-1}f_0\eta$, the first term significantly simplifies thanks to the formula (A.2) for A_1 in Appendix A. For the other terms we directly expand

$$\begin{aligned} &\varphi' + \frac{2}{r}\varphi - \psi' = 2f_0'\frac{\zeta-\eta}{r} - 2\frac{f_0}{r}\eta', \\ &\frac{n+3}{r^2}\varphi^2 + \frac{n-1}{r^2}\psi^2 - 2\delta\frac{n+1}{r^2}\varphi\psi \\ &= 2(1-\delta)\frac{n+1}{r^2}(f_0'\zeta)^2 + 2(1+\delta)\frac{n+1}{r^2}\left(\frac{f_0}{r}\eta\right)^2 - \frac{8}{r^2}f_0'\zeta\frac{f_0}{r}\eta, \\ &\varphi\psi' - \varphi'\psi = 2\left(\frac{f_0}{r}\eta\right)' f_0'\zeta - 2(f_0'\zeta)'\frac{f_0}{r}\eta, \end{aligned}$$

Downloaded 04/28/22 to 146.83.7.1 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

from which it follows that $B_n^\delta[\zeta,\eta] = (1/2)Q_n^\delta[f_0'\zeta - r^{-1}f_0\eta, f_0'\zeta + r^{-1}f_0\eta]$ can be rewritten as

$$B_n^\delta[\zeta,\eta] = (1+\delta)\int_0^\infty \left[\frac{f_0^2}{r^2}(\eta')^2 + (f_0')^2(\zeta')^2 + \frac{2}{r^3}f_0f_0'(\eta-\zeta)^2\right] rdr \tag{4.2}$$
$$- 2\delta\int_0^\infty \left[\frac{f_0'}{r}(\eta-\zeta) + \frac{f_0}{r}\eta'\right]^2 rdr$$
$$+ (n-1)\int_0^\infty \Bigg[(1-\delta)\frac{n+1}{r^2}(f_0'\zeta)^2 + (1+\delta)\frac{n+1}{r^2}\left(\frac{f_0}{r}\eta\right)^2 - \frac{4}{r^2}(f_0'\zeta)\left(\frac{f_0}{r}\eta\right)\Bigg] rdr$$
$$+ 2\delta(n-1)\int_0^\infty \frac{1}{r}\left[\left(\frac{f_0}{r}\eta\right)' f_0'\zeta - (f_0'\zeta)'\,\frac{f_0}{r}\eta\right] rdr.$$

Integrating by parts, the last integral becomes

$$\int_0^\infty \frac{1}{r}\left[\left(\frac{f_0}{r}\eta\right)' f_0'\zeta - (f_0'\zeta)'\,\frac{f_0}{r}\eta\right] rdr = 2\int_0^\infty \left(\frac{f_0}{r}\eta\right)' f_0'\frac{\zeta}{r}\, rdr$$
$$= 2\int_0^\infty \left[\left(f_0' - \frac{f_0}{r}\right) f_0'\frac{\eta}{r}\frac{\zeta}{r} + \frac{f_0}{r}\eta' f_0'\frac{\zeta}{r}\right] rdr.$$

We use the first positive term in (4.2) in order to absorb this latter term: thanks to the identity

$$(1+\delta)\frac{f_0^2}{r^2}(\eta')^2 + 4\delta(n-1)\frac{f_0}{r}\eta' f_0'\frac{\zeta}{r} = (1+\delta)\left(\frac{f_0}{r}\eta' + \frac{2\delta}{1+\delta}(n-1)f_0'\frac{\zeta}{r}\right)^2 - 4\frac{\delta^2}{1+\delta}(n-1)^2\,(f_0')^2\left(\frac{\zeta}{r}\right)^2,$$

we rewrite (4.2) as

$$B_n^\delta[\zeta,\eta] = B_n^{\delta,1}[\zeta,\eta] + (n-1)B_n^{\delta,2}[\zeta,\eta],$$
$$B_n^{\delta,1}[\zeta,\eta] = (1+\delta)\int_0^\infty \left[\left(\frac{f_0}{r}\eta' + \frac{2\delta}{1+\delta}(n-1)f_0'\frac{\zeta}{r}\right)^2 + (f_0')^2(\zeta')^2\right] rdr$$
$$+ 2\int_0^\infty \left\{(1+\delta)f_0'\frac{f_0}{r}\frac{(\eta-\zeta)^2}{r^2} - \delta\left[\frac{f_0'}{r}(\eta-\zeta) + \frac{f_0}{r}\eta'\right]^2\right\} rdr,$$
$$B_n^{\delta,2}[\zeta,\eta] = \int_0^\infty q_n^\delta(r)\left[f_0'\frac{\zeta}{r}, \frac{f_0}{r}\frac{\eta}{r}\right] rdr,$$

and $q_n^\delta(r)$ is the quadratic form on $\mathbb{R}^2$ given by

$$q_n^\delta(r)[X,Y] = a_nX^2 + b_nY^2 + 2c(r)XY,$$
$$a_n = (1-\delta)(n+1) - 4\frac{\delta^2}{1+\delta}(n-1),$$
$$b_n = (1+\delta)(n+1),$$
$$c(r) = -2 - 2\delta\left(1 - r\frac{f_0'}{f_0}\right).$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 04/28/22 to 146.83.7.1 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Downloaded 04/28/22 to 146.83.7.1 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

We readily see that $B_n^{\delta,1}$ is nonnegative for $\delta \le 0$. Moreover, since $1 > rf_0'/f_0 > 0$ [8, Proposition 2.2], for $\delta \le 0$, it follows that

$$|c(r)| \le 2.$$

As $b_n > 0$, a sufficient condition for $q_n^\delta(r)$ to be positive definite $\forall r > 0$ is

$$4 < a_n b_n = (1-\delta^2)(n+1)^2 - 4\delta^2(n^2-1).$$

This amounts to the condition

$$0 < \alpha(\delta)n^2 + \beta(\delta)n + \gamma(\delta),$$

where

$$\alpha(\delta) = 1 - 5\delta^2,$$
$$\beta(\delta) = 2(1-\delta^2),$$
$$\gamma(\delta) = -3(1-\delta^2).$$

For $\delta \in [-1/\sqrt{5}, 0]$ we have $\alpha(\delta), \beta(\delta) \ge 0$ so that the above polynomial in n is nondecreasing on $[0, +\infty)$. Hence, it is positive for all values of $n \ge 2$ if and only if it is positive for $n = 2$. That is,

$$0 < 4\alpha(\delta) + 2\beta(\delta) + \gamma(\delta) = 5 - 21\,\delta^2.$$

We deduce that q_n^δ is a positive definite quadratic form $\forall n \ge 2$ whenever $\delta \in [-1/\sqrt{5}, 0]$. In particular, $B_n^{\delta,2} \ge 0$, and therefore $Q_n^\delta \ge 0$ for $\delta \in [-1/\sqrt{5}, 0]$, with equality only at $(0, 0)$.

4.2. Instability for $\boldsymbol{\delta \approx -1}$. In this section we show that Q_n^δ can take negative values for $\delta \approx -1$ and $n \ge 1$ large enough. To this end, we choose $\eta = \zeta$ in (4.2) to obtain

$$\begin{aligned}
\hat{B}_n^\delta[\zeta] &= B_n^\delta[\zeta, \zeta] \\
&= (1-\delta)\int_0^\infty \frac{f_0^2}{r^2}(\zeta')^2\, r dr + (1+\delta)\int_0^\infty (f_0')^2(\zeta')^2\, r dr \\
&\quad + (n-1)\int_0^\infty \frac{\zeta^2}{r^2}\alpha_n^\delta(r)\, r dr \\
\alpha_n^\delta(r) &= (1-\delta)(n+1)(f_0')^2 + (1+\delta)(n+1)\left(\frac{f_0}{r}\right)^2 \\
&\quad - 2(2+\delta)f_0'\frac{f_0}{r} + 2\delta(f_0')^2 - 2\delta f_0 f_0''.
\end{aligned}$$

Using the asymptotics of f_0 ([1, 7])

$$f_0(r) = 1 - \frac{1}{2}r^{-2} + O(r^{-4}), \quad f_0'(r) = r^{-3} + O(r^{-5}), \quad f_0''(r) = -3r^{-4} + O(r^{-6}),$$

we find, for $r \to +\infty$,

$$\alpha_n^\delta(r) = \frac{(1+\delta)(n+1)}{r^2}\left(1 - \frac{1}{r^2}\right) - 4\frac{1-\delta}{r^4} + O(r^{-6}).$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 04/28/22 to 146.83.7.1 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

For $\delta = -1$ the leading order is negative. Hence, there exists $\varepsilon > 0$ and a compact interval $[r_0, r_0 + 1]$ on which $\alpha_n^{-1} \le -2\varepsilon$. Thus, we deduce that $\forall n \ge 2$ there exists $\delta_n > -1$ such that $\forall \delta \in (-1, \delta_n]$,

$$-\varepsilon \ge \alpha_n^\delta(r) \qquad \forall r \in [r_0, r_0 + 1].$$

Choosing a nonzero test function ζ_0 with support in $[r_0, r_0 + 1]$, we obtain

$$\hat{B}_n^\delta[\zeta_0] \le C_1(\zeta_0) - (n-1)\varepsilon C_2(\zeta_0) \qquad \forall \delta \in (-1, \delta_n]$$

for some $C_1(\zeta_0), C_2(\zeta_0) > 0$. If n is large enough this becomes negative. Compared to the isotropic case this is a really new situation: lower modes are positive but higher modes can bring instability.

5. Proof of Theorem 1.3. In what precedes we have shown that u_{rad}^δ is nondegenerately stable for small $\delta \le 0$, and unstable for $\delta > 0$ and δ close to -1. In particular, setting

$$\delta_1 = \sup\{\delta \in (-1, 0) \colon u_{\mathrm{rad}}^\delta \text{ is unstable}\},$$

we know that $-1 < \delta_1 < 0$. It remains to show that u_{rad}^δ is unstable $\forall \delta \in (-1, \delta_1)$ and nondegenerately stable for $\delta \in (\delta_1, 0]$.

Let $\delta' \in (-1, \delta_1)$ be such that u_{rad}^δ is unstable; that is, $\mathcal{Q}^{\delta'}[v] < 0$ for some choice of $v \in H$. Given that $\delta \mapsto \mathcal{Q}^\delta[v]$ is an affine function which is nonnegative for $\delta = 0$ and negative for $\delta = \delta'$, we deduce that $\mathcal{Q}^\delta[v] < 0 \ \forall \delta \le \delta'$. Therefore, u_{rad}^δ is unstable $\forall \delta \in (-1, \delta')$. By arbitrariness of δ' we deduce that u_{rad}^δ is unstable $\forall \delta \in (-1, \delta_1)$.

Let us now fix $\delta \in (\delta_1, 0]$. By definition of δ_1, u_{rad}^δ is not unstable $\forall \delta \in (\delta_1, 0]$. In other words, $\mathcal{Q}^\delta[v]$ is nonnegative $\forall v \in \mathcal{H}$. It remains to show that, in fact, $\mathcal{Q}^\delta[v] > 0 \forall\ v \in \mathcal{H} \setminus \mathrm{span}(\partial_x u_{\mathrm{rad}}^0, \partial_y u_{\mathrm{rad}}^0)$. We observe that the function $\delta \mapsto \mathcal{Q}^\delta[v]$ is affine for any given $v \in \mathcal{H} \setminus \mathrm{span}(\partial_x u_{\mathrm{rad}}^0, \partial_y u_{\mathrm{rad}}^0)$; it is positive for $\delta = 0$ because u_{rad}^0 is nondegenerately stable, and it is nonnegative for $\delta \in (\delta_1, 0)$. Thus, it must be strictly positive for $\delta \in (\delta_1, 0)$. This proves the desired nondegenerate stability in the announced range.

Appendix A. Positivity of A_0, A_1.

We sketch here the approach in [13], adapted to our notation (see also [5]), based on Hardy-type decompositions to show positivity of the two following quadratic forms:

$$A_0[\varphi] = \int_0^\infty \left[|\varphi'|^2 + \frac{1}{r^2}|\varphi|^2 + 2f_0^2(\mathrm{Re}\,\{\varphi\})^2 - (1 - f_0^2)|\varphi|^2 \right] r dr,$$

$$A_1[\varphi, \psi] = \int_0^\infty \Bigg[|\varphi'|^2 + |\psi'|^2 + \frac{4}{r^2}|\varphi|^2 + f_0^2|\varphi + \bar{\psi}|^2 - (1 - f_0^2)\left(|\varphi|^2 + |\psi|^2\right) \Bigg] r dr.$$

Testing (2.1), solved by f_0, against $f_0|\tilde{\varphi}|^2$ for any smooth compactly supported $\tilde{\varphi} \in C_c^\infty(\mathbb{R}; \mathbb{C})$, one obtains

$$\int_0^\infty \left[(f_0')^2|\tilde{\varphi}|^2 + 2f_0 f_0' \tilde{\varphi} \cdot \tilde{\varphi}' + \frac{f_0^2}{r^2}|\tilde{\varphi}|^2 - (1 - f_0^2)f_0^2|\tilde{\varphi}|^2 \right] r dr = 0,$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 04/28/22 to 146.83.7.1 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

so that

$$A_0[f_0\tilde{\varphi}] = \int_0^\infty \left[f_0^2|\tilde{\varphi}'|^2 + 2f_0^4(\mathrm{Re}\{\tilde{\varphi}\})^2\right] rdr. \tag{A.1}$$

By density of test functions, and since $f_0 > 0$, we deduce that $A_0[\varphi] > 0$ for any nonzero $\varphi \in \mathcal{H}_0$. Moreover $A_0[\varphi] \approx 0$ exactly when $\varphi \approx if_0$. This corresponds to the fact that in the isotropic case $\delta = 0$,

$$\partial_\alpha[e^{i\alpha}u_{\mathrm{rad}}^\delta]_{\lfloor\alpha=0} = if_0e^{i\theta}$$

solves the linearized equation due to rotational invariance.

For A_1, it is convenient to start by splitting it as

$$A_1[\varphi,\psi] = A_1[\mathrm{Re}\{\varphi\}, \mathrm{Re}\{\psi\}] + A_1[\mathrm{Im}\{\varphi\}, -\mathrm{Im}\{\psi\}],$$

so we may just treat the case of real-valued test functions φ, ψ. Guided by the fact that

$$\partial_x u_{\mathrm{rad}}^0 = e^{i\theta}\left(f_0'\cos\theta - i\frac{f_0}{r}\sin\theta\right), \qquad \partial_y u_{\mathrm{rad}}^0 = e^{i\theta}\left(f_0'\sin\theta + i\frac{f_0}{r}\cos\theta\right),$$

solve the linearized equation around u_{rad}^0, one uses the ansatz

$$\varphi = f_0'\zeta - \frac{f_0}{r}\eta, \qquad \psi = f_0'\zeta + \frac{f_0}{r}\eta$$

for some real-valued $\eta, \zeta \in C_c^\infty(0,\infty)$. Testing (2.1), solved by f_0, against $f_0r^{-2}\eta^2$ we obtain

$$\int_0^\infty \left[\left(\left(\frac{f_0}{r}\right)'\right)^2\eta^2 + 2\left(\frac{f_0}{r}\right)'\frac{f_0}{r}\eta\eta' + \frac{2}{r^4}f_0^2\eta^2 - \frac{2}{r^3}f_0f_0'\eta^2 - (1-f_0^2)\frac{f_0^2}{r^2}\eta^2\right] rdr = 0,$$

and similarly testing (2.1) against $(f_0'\zeta^2)'$ we find

$$\int_0^\infty \left[(f_0'')^2\zeta^2 + 2f_0'f_0''\zeta\zeta' + \frac{2}{r^2}(f_0')^2\zeta^2 - \frac{2}{r^3}f_0f_0'\zeta^2 + (3f_0^2-1)(f_0')^2\zeta^2\right] rdr = 0.$$

As a consequence of these two identities, we learn

$$\begin{aligned} &A_1\left[f_0'\zeta - r^{-1}f_0\eta, f_0'\zeta + r^{-1}f_0\eta\right] \\ &\quad = 2\int_0^\infty \left[\frac{f_0^2}{r^2}(\eta')^2 + (f_0')^2(\zeta')^2 + \frac{2}{r^3}f_0f_0'(\eta-\zeta)^2\right] rdr. \end{aligned} \tag{A.2}$$

Since $f_0, f_0' > 0$ one may consider the choice

$$\zeta = \frac{1}{2f_0'}(\varphi+\psi), \quad \eta = \frac{r}{2f_0}(\psi-\varphi),$$

and deduce from the above that $A_1[\varphi,\psi] > 0$ for all nonzero $(\varphi,\psi) \in \mathcal{H}_1$. Moreover $A_1[\varphi,\psi] = 0$ exactly when (φ,ψ) is in the real linear span of

$$\left(f_0' - \frac{f_0}{r}, f_0' + \frac{f_0}{r}\right), \quad \left(i\left(f_0' - \frac{f_0}{r}\right), -i\left(f_0' + \frac{f_0}{r}\right)\right),$$

which corresponds to the fact that $\partial_x u_{\mathrm{rad}}^0$ and $\partial_y u_{\mathrm{rad}}^0$ solve the linearized equation.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 04/28/22 to 146.83.7.1 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

REFERENCES

[1] X. Chen, C. M. Elliott, and T. Qi, *Shooting method for vortex solutions of a complex-valued Ginzburg-Landau equation*, Proc. Roy. Soc. Edinburgh Sect. A, 124 (1994), pp. 1075–1088, https://doi.org/10.1017/S0308210500030122.

[2] M. G. Clerc, E. Vidal-Henriquez, J. D. Davila, and M. Kowalczyk, *Symmetry breaking of nematic umbilical defects through an amplitude equation*, Phys. Rev. E, 90 (2014), 012507, https://doi.org/10.1103/PhysRevE.90.012507.

[3] S. Colbert-Kelly, G. B. McFadden, D. Phillips, and J. Shen, *Numerical analysis and simulation for a generalized planar Ginzburg-Landau equation in a circular geometry*, Commun. Math. Sci., 15 (2017), pp. 329–357, https://doi.org/10.4310/CMS.2017.v15.n2.a3.

[4] S. Colbert-Kelly and D. Phillips, *Analysis of a Ginzburg-Landau type energy model for smectic C^* liquid crystals with defects*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 30 (2013), pp. 1009–1026, https://doi.org/10.1016/j.anihpc.2012.12.010.

[5] M. del Pino, P. Felmer, and M. Kowalczyk, *Minimality and nondegeneracy of degree-one Ginzburg-Landau vortex as a Hardy's type inequality*, Int. Math. Res. Not., (2004), pp. 1511–1527, https://doi.org/10.1155/S1073792804133588.

[6] P. Gravejat, E. Pacherie, and D. Smets, *On the Stability of the Ginzburg-Landau Vortex*, preprint, arXiv:2106.02511.

[7] R.-M. Hervé and M. Hervé, *Étude qualitative des solutions réelles d'une équation différentielle liée à l'équation de Ginzburg-Landau*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 11 (1994), pp. 427–440, https://doi.org/10.1016/S0294-1449(16)30182-2.

[8] R. Ignat, L. Nguyen, V. Slastikov, and A. Zarnescu, *Uniqueness results for an ODE related to a generalized Ginzburg-Landau model for liquid crystals*, SIAM J. Math. Anal., 46 (2014), pp. 3390–3425, https://doi.org/10.1137/130948598.

[9] R. Ignat, L. Nguyen, V. Slastikov, and A. Zarnescu, *Stability of the melting hedgehog in the Landau–de Gennes theory of nematic liquid crystals*, Arch. Ration. Mech. Anal., 215 (2015), pp. 633–673, https://doi.org/10.1007/s00205-014-0791-4.

[10] R. Ignat, L. Nguyen, V. Slastikov, and A. Zarnescu, *Instability of point defects in a two-dimensional nematic liquid crystal model*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 33 (2016), pp. 1131–1152, https://doi.org/10.1016/j.anihpc.2015.03.007.

[11] R. Ignat, L. Nguyen, V. Slastikov, and A. Zarnescu, *Stability of point defects of degree $\pm\frac{1}{2}$ in a two-dimensional nematic liquid crystal model*, Calc. Var. Partial Differential Equations, 55 (2016), 119, https://doi.org/10.1007/s00526-016-1051-2.

[12] G. Kitavtsev, J. Robbins, V. Slastikov, and A. Zarnescu, *Liquid crystal defects in the Landau-de-Gennes theory in two dimensions - beyond the one-constant approximation*, Math. Models Methods Appl. Sci., 26 (2016), pp. 2769–2808.

[13] P. Mironescu, *On the stability of radial solutions of the Ginzburg-Landau equation*, J. Funct. Anal., 130 (1995), pp. 334–344, https://doi.org/10.1006/jfan.1995.1073.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.