Criticality, splitting theorems under spectral Ricci bounds and the topology of stable minimal hypersurfaces

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To the memory of Francesco "Franco" Mercuri and Celso Viana

Abstract

In this paper we prove general criticality criteria for operators $\Delta + V$ on manifolds with more than one end, where V bounds the Ricci curvature, and a related spectral splitting theorem extending Cheeger-Gromoll's one. Our results give new insight on Li-Wang's theory of manifolds with a weighted Poincaré inequality. We apply them to study stable and δ-stable minimal hypersurfaces in manifolds with non-negative bi-Ricci or sectional curvature, in ambient dimension up to 5 and 6, respectively. In the special case where the ambient space is $\mathbb{R}^4$, we prove that a $1/3$-stable minimal hypersurface must either have one end or be a catenoid, and that proper, δ-stable minimal hypersurfaces with $\delta > 1/3$ must be hyperplanes.

1   Introduction

The purpose of the present paper is to investigate the topological properties of complete, two-sided minimal hypersurfaces $M^n \to N^{n+1}$ immersed into a non-negatively curved space. Hereafter, we always implicitly assume that M is connected. We will be interested in hypersurfaces which are stable or, more generally, δ-stable, where δ-stability means that the modified Jacobi operator

$$J_\delta \doteq \Delta + \delta \left(|A|^2 + \overline{\mathrm{Ric}}(\nu, \nu) \right).$$

is non-negative[1]. Here, $\overline{\mathrm{Ric}}$ is the Ricci curvature of N, ν is a global unit normal vector field on M and A is the second fundamental form in direction ν. "Classical" stability corresponds to $\delta = 1$, and one easily checks that δ-stability implies δ'-stability if $\delta' < \delta$. In dimension

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[1]We follow the convention that $\Delta = \mathrm{d}^2/\mathrm{d}x^2$ on $\mathbb{R}$, and that $J_\delta \geq 0$ correponds to the non-negativity of $-(\phi, J_\delta \phi)_{L^2}$ for each test function $\phi \in C_c^\infty(M)$.

$n = 2$, bounds for δ-stable minimal surfaces played a role in the theory developed by Colding & Minicozzi [28, 29]. We here focus on hypersurfaces of dimension $n \geq 3$ in the range

$$\delta \in \left[\frac{n-2}{n}, 1 \right].$$

If $N = \mathbb{R}^{n+1}$, the recent solution to the stable Bernstein problem by Chodosh & Li [24, 25] ($n = 3$), Catino, Mastrolia & Roncoroni [18] ($n = 3$), Chodosh, Li, Minter & Stryker [27] ($n = 4$) and Mazet [52] ($n = 5$), and the results by Bellettini [5] for $n = 6$, provided new powerful tools to investigate the geometry of stable minimal hypersurfaces, with promising applications to treat the problem in more general ambient spaces. Considering the weaker notion of δ-stability is natural in view of the following result:

Theorem 1.1 ([3, 71, 36, 57]). *Let $M^n \to \mathbb{R}^{n+1}$ be a complete, two-sided minimal hypersurface with finite total curvature, i.e. $|A| \in L^n(M)$. Then:*

- *M is δ-stable with $\delta > \frac{n-2}{n}$ if and only if M is a hyperplane;*

- *M is $\frac{n-2}{n}$-stable if and only if M is either a hyperplane or a catenoid.*

The theorem was proved by Anderson [3] and Shen & Zhu [71] for $\delta = 1$. Its extension to the present form is due to Fu & Li [36], based on works in [3, 23]. A different, self-contained proof was found by Pigola & Veronelli [57], which we also recommend for a detailed account of the literature. The $\frac{n-2}{n}$-stability of the higher dimensional catenoid and the finiteness of its total curvature were proved by Tam & Zhou in [72].

It is tempting to ask whether Theorem 1.1 holds in dimension $n \leq 6$ (or in a smaller range) without assuming that M has finite total curvature. In [25], the authors obtained a first relevant step in this direction in dimension $n = 3$ by proving that δ-stability with $\delta > 2/3$ implies flatness. The question was also addressed in the recent [39], a source of various interesting results in the full range $\delta > \frac{n-2}{n}$. As a consequence of [39] and our structure theorems, we obtain:

Theorem 1.2. *Let $M^3 \to \mathbb{R}^4$ be a two-sided, properly immersed minimal hypersurface which is δ-stable for some $\delta > 1/3$. Then, M is a hyperplane.*

A bound on the number of ends of δ-stable hypersurfaces is a main missing information we provide to get Theorem 1.2. Indeed, it is known by [10] that stable minimal hypersurfaces in $\mathbb{R}^{n+1}$ must have only one end, a fact that plays a crucial role in the proofs in [25, 27, 52]. However, although the argument in [10] easily adapts to δ-stability for any

$$\delta > \frac{n-1}{n},$$

justifying the bound $\delta > 2/3$ in [25], it fails for $\delta < \frac{n-1}{n}$. To the best of our knowledge, in full generality no bound on the number of ends of M was established so far for δ close to $\frac{n-2}{n}$, and neither the assertion that M has finitely many ends. We are only aware of the following result due to Cheng & Zhou, [23, Theorem 1.1]:

Theorem 1.3 ([23]). *Let $M^n \to \mathbb{R}^{n+1}$ be a $\frac{n-2}{n}$-stable, two-sided complete minimal hypersurface of dimension $n \geq 3$. If*

$$\lim_{R \to \infty} \left(\sup_{B_R(o)} |A| \right) / \log R = 0 \qquad \text{if } n = 3$$

$$or \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad (1.1)$$

$$\lim_{R \to \infty} \left(\sup_{B_R(o)} |A| \right) / R^{\frac{n-3}{2}} = 0 \qquad \text{if } n \geq 4,$$

where $B_R(o) \subset M$ is the intrinsic ball centered at o with radius R, then either M has only one end or it is a catenoid.

Observe that (1.1) holds, for instance, if $|A|$ is bounded on M. Cheng and Zhou kindly suggested to the first and second author the question whether (1.1) is removable, which indeed was the starting point of the present investigation. We are able to positively anwer in dimension $n = 3$:

Theorem 1.4. *Let $M^3 \to \mathbb{R}^4$ be a complete, two-sided, $1/3$-stable minimal hypersurface. Then, either M has only one end or it is a catenoid.*

Theorem 1.4 is a corollary of a more general result, Theorem 4.1 below, where we study the topology of $1/3$-stable minimal hypersurfaces $M^3 \to N^4$ into a 4-manifold with non-negative sectional curvature. The peculiarity of dimension $n = 3$ will be clear in what follows, see in particular Theorem 3.3.

We focus on ambient spaces $(N^{n+1}, \bar{g})$ with non-negative sectional curvature $\overline{\text{Sec}}$ or, more generally, non-negative bi-Ricci curvature $\overline{\text{BRic}}$. Recall that, given an orthonormal set $\{X, Y\} \subset T_p N$, the bi-Ricci curvature is defined as

$$\overline{\text{BRic}}(X, Y) = \overline{\text{Ric}}(X, X) + \overline{\text{Ric}}(Y, Y) - \overline{\text{Sec}}(X, Y).$$

In particular, if $n + 1 = 3$ then $\overline{\text{BRic}} = \bar{R}/2$, with $\bar{R}$ the scalar curvature of M. Compared to the literature on minimal surfaces in non-negatively curved 3-manifolds, relatively few results on the topology of higher dimensional minimal hypersurfaces appeared so far. The list below collects the ones which are closer to the the present paper. Let $M^n \to N^{n+1}$ be a complete, non-compact, two-sided stable minimal immersion of dimension $n \geq 3$. Assume for simplicity that N is complete as well, although part of the theorems in the list do not need it.

- Li & Wang [45, Thm. 0.1]: if $\overline{\text{Sec}} \geq 0$ and M is properly immersed, then either M has only one end or M is totally geodesic and $M = \mathbb{R} \times P$ with the product metric, for some compact P with non-negative sectional curvature.

- Cheng [21, Thm. 3.2]: if $\overline{\text{Sec}} \geq 0$ and $\overline{\text{Ric}} > 0$, then M is non-parabolic and it has only one end.

- Chodosh, Li & Stryker [26, Thm. 1.3 and Remark 1.4]: if $n = 3$, $0 \leq \overline{\text{Sec}} \leq \kappa$ for some $\kappa \in \mathbb{R}^+$ and the scalar curvature of N satisfies $\bar{R} \geq 1$, then M is totally geodesic and $\overline{\text{Ric}}(\nu, \nu) \equiv 0$.

- Tanno [73, Thm. B]: if $\overline{\text{BRic}} \geq 0$ and M is orientable, then the space of L^2 harmonic 1-forms on M is trivial. In particular, Li-Tam's theory in [42] implies that M has only

one non-parabolic end, and by [31], every cycle of codimension 1 must disconnect M. The result was obtained in [53] by assuming $\overline{\text{Sec}} \geq 0$, see also [56, 74]. Related sharp vanishing theorems on spinnable minimal hypersurfaces in general ambient spaces were recently obtained by Bei & Pipoli [4].

- Shen & Zhu [70, Thm. 1 and Rem. 2]: if $n \in \{3, 4\}$, $\overline{\text{BRic}} \geq 0$ and M is topologically $\mathbb{R} \times P$, where P is compact and admits a metric of non-positive sectional curvature, then M is totally geodesic. Moreover, M is flat if $\overline{\text{Sec}} \geq 0$.

- Cheng [21, Thm. 2.1]: if $n \in \{3, 4\}$ and $\overline{\text{BRic}} > 0$, then M must have only one end[2].

Remark 1.5. For a detailed exposition of Li-Tam's theory of harmonic functions, we suggest [58].

We are ready to state our main results in general target spaces. We first consider ambient manifolds with non-negative bi-Ricci curvature, and obtain the following splitting theorem that improves on [21, 70].

Theorem 1.6. *Let $M^n \to (N^{n+1}, \bar{g})$ be a complete, non-compact, two-sided stable minimal hypersurface of dimension $n \in \{3, 4\}$ in a manifold satisfying*

$$\overline{\text{BRic}} \geq 0.$$

Then, one of the following mutually exclusive cases occurs:

(i) *M has only one end;*

(ii) *M is totally geodesic and $M = \mathbb{R} \times P$ with the product metric $\mathrm{d}t^2 + g_P$, for some compact $(n-1)$-manifold (P, g_P) with non-negative Ricci curvature. Moreover,*

$$\overline{\text{Ric}}(\nu, \nu) \equiv 0, \qquad \overline{\text{BRic}}(\partial_t, \nu) \equiv 0.$$

Furthermore, if $\overline{\text{Ric}} \geq 0$ then in (i) either M is non-parabolic, or M is parabolic, totally geodesic and $\overline{\text{Ric}}(\nu, \nu) \equiv 0$.

Remark 1.7. Observe that $\overline{\text{BRic}} \geq 0$ does not imply $\overline{\text{Ric}} \geq 0$, so the conclusion $\overline{\text{Ric}}(\nu, \nu) \equiv 0$ in (ii) is not obvious. In dimension $n = 2$, $\overline{\text{BRic}} \geq 0$ becomes $\bar{R} \geq 0$, hence Theorem 1.6 can be seen as a higher dimensional analogue of the following well-known result by Fischer-Colbrie & Schoen, [34, Theorem 3]: a stable minimal surface $M^2 \to N^3$ with more than one end in an ambient space with $\bar{R} \geq 0$ must be conformally equivalent to a cylinder, flat and totally geodesic provided it has finite total curvature, i.e.

$$\int_M |K| < \infty.$$

In the same paper, the authors conjectured that the request of finite total curvature should be removable. This is the case, as shown by [53, 64, 6]. As a matter of fact, the argument in Theorem 1.6 works *verbatim* also in dimension $n = 2$ (with $P = \mathbb{S}^1$), thus providing a further different proof of Fischer-Colbrie and Schoen's claim.

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[2]Cheng assumed M and N orientable, but her proof only uses that M is two-sided in N.

We next consider ambient spaces with non-negative sectional curvature. Our goal is to improve on Li-Wang's remarkable splitting theorem [45, Thm. 0.1] quoted above by removing the condition that the immersion $M \to N$ is proper. As a matter of fact, Li-Wang's result is the consequence of a more general statement where properness is not required a priori, see [45, Thm. 4.1]. However, in counting the number of ends the major issue therein is the possible presence of parabolic ends which are extrinsically bounded. In Theorem 1.6 we are able to overcome the problem in dimension $n \le 4$. In the next theorem, we include one more dimension:

Theorem 1.8. *Let $M^n \to (N^{n+1}, \bar{g})$ be a complete, non-compact, two-sided stable minimal hypersurface of dimension $n \in \{3, 4, 5\}$ in a manifold satisfying*

$$\overline{\mathrm{Sec}} \ge 0.$$

Then, one of the following cases occurs:

  (i) *M has only one end. Moreover, either M is non-parabolic, or M is parabolic, totally geodesic and $\overline{\mathrm{Ric}}(\nu, \nu) \equiv 0$;*

  (ii) *M is totally geodesic, $\overline{\mathrm{Ric}}(\nu, \nu) \equiv 0$ and $M = \mathbb{R} \times P$ with the product metric, for some compact $(n-1)$-manifold P with non-negative sectional curvature.*

Moreover, if $n = 3$ then $H^2(M, \mathbb{Z}) = 0$ or $\mathbb{Z}$.

Remark 1.9. Differently from [45], Theorems 1.6 and 1.8 do not need N to be complete.

Remark 1.10. The methods in [10, 45] to bound the number of ends of M ultimately rely on refined constructions of harmonic functions with controlled L^2-energy (more precisely, finite in [10] and slowly growing in [45]), and on the use of a geometric Poincaré formula due to Schoen & Yau, [66], for which stability is required. The formula still allows to conclude if stability is weakened to δ-stability with $\delta > \frac{n-1}{n}$, but to reach lower δ one needs further tools. On the other hand, the method we employ to prove $(i), (ii)$ in Theorem 1.8 works *verbatim* for δ-stable hypersurfaces with

$$\delta > \delta_n \doteq \frac{(n-1)^2}{4n},$$

and $\delta_n \le \frac{n-1}{n}$ if $n \le 5$. Observe that $\delta_3 = 1/3$. The 3-dimensional case of Theorem 1.8 will be strengthened in Theorem 4.1 below.

Remark 1.11. Theorem 0.1 in [45] has a counterpart for properly immersed minimal hypersurfaces $M \to N$ with finite index, which are shown to have finitely many ends, see [45, Thm. 4.1]. We think it would be very interesting to prove an analogous result in dimension $n \le 5$ by removing the properness assumption.

Our applications to minimal hypersurfaces are consequences of some structure theorems for complete manifolds (M^n, g) whose operator

$$\beta \Delta - \lambda_{\mathrm{Ric}}$$

is non-negative in the spectral sense, where $\lambda_{\mathrm{Ric}} : M \to \mathbb{R}$ is the lowest eigenvalue of the Ricci tensor and $\beta > 0$ is a suitable constant. In other words, we assume

$$\beta \int_M |\nabla \varphi|^2 + \int_M \lambda_{\mathrm{Ric}} \varphi^2 \ge 0 \qquad \forall \, \varphi \in \mathrm{Lip}_c(M).$$

Note that the above can be rephrased as the existence of $V \in C(M)$ so that

$$\text{Ric} \geq -\beta V g, \qquad L \doteq \Delta + V \geq 0. \qquad (1.2)$$

It is known that λ_{Ric} is a locally Lipschitz function (cf. [38]), so assuming $V \in C^{0,\alpha}_{\text{loc}}(M)$ is not restrictive. Manifolds with property (1.2) were studied in depth by Li & Wang in [46], a continuation of their analysis of the topology of manifolds with positive spectral radius in [43, 44]. More recently, Bour & Carron [9], Carron [11], Carron & Rose [14] and Antonelli & Xu [2] also obtained relevant results. The investigation highlighted three ranges for β:

(i) $\beta \leq \frac{n-1}{n-2}$: in this range, under some further assumptions on M one is able to control the number of non-parabolic ends, see [46, Theorem A and Corollary 4.2].

(ii) $\beta \leq \frac{4}{n-1}$: this range also allows to control the number of parabolic ends of M, see [46, Theorems B,C] and [21].

(iii) $\beta < \frac{1}{n-2}$: in this range, M looks close to a manifold with $\text{Ric} \geq 0$, as it will become apparent in Theorem 3.11 below. Analogies become tighter if one further assumes the *gaugeability* of L, i.e. the existence of a positive supersolution bounded below and above by positive constants, see [11, 14, 40].

Remark 1.12. Note that $\frac{n-1}{n-2} \geq \frac{4}{n-1}$, with equality if and only if $n = 3$. This accounts for most of the extra information we obtain in dimension 3.

In this paper, we consider each of the ranges (i),(ii),(iii). For β as in (ii), we obtain the following spectral splitting theorem extending Cheeger-Gromoll's one [20]:

Theorem 1.13. *Let (M, g) be a complete Riemannian manifold of dimension $n \geq 3$, and assume that there exists $V \in C^{0,\alpha}_{\text{loc}}(M)$ such that*

$$\text{Ric} \geq -\beta V g \qquad \text{on } M, \qquad L \doteq \Delta + V \geq 0, \qquad (1.3)$$

for some $0 < \beta < \frac{4}{n-1}$. Then, either

(i) *M has only one end, or*

(ii) *$V \equiv 0$ and $M = \mathbb{R} \times P$ with the product metric, for some compact P with $\text{Ric}_P \geq 0$.*

Remark 1.14. The result was also obtained, independently and at the same time, by Antonelli, Pozzetta & Xu [1] with a different technique, see below.

Differently from [46] and from [9, 11, 14, 2], our structure theorems rely on a new point of view on the "conformal method" pioneered by Schoen & Yau [65], Fischer-Colbrie [33], Shen & Ye [68] and Shen & Zhu [70]. Loosely speaking, the method exploits a positive supersolution to $Lu \leq 0$, whose existence is granted by the second in (1.2), to conformally change the metric via $\bar{g} = u^{2\beta} g$. This allows to "push-up" the curvature contribution in the second variation formulas for $\bar{g}$-geodesics. However, so far the idea was mostly used to prove compactness results in the spirit of Bonnet-Myers' one, see [64, 68, 69, 22, 32, 50, 19]. Exceptions are [33, 21, 18] and [17], where it was employed to control the geometry of complete minimal hypersurfaces and to prove the rigidity of critical metrics for a quadratic curvature functional, respectively. Still, the splitting problem was not addressed there, and requires to rework the method from the onset.

A major issue appearing, for instance, in the proof of Theorem 1.13 is to extend the information obtained on a single $\bar{g}$-geodesic to the whole manifold. To this aim, an original

contribution of the present paper is to link the conformal method to the *criticality theory* for operators of the type $\Delta + V$ developed by Murata [54] and Pinchover & Tintarev [59, 60, 61, 62] (see also Zhao [77]). More precisely, we elaborate on [71, 21] to get a criticality criterion for L. The conclusions of our main Theorems 3.3 and 3.9 (which contains Theorem 1.13) can be summarized as follows: either M has only one end, or L is critical and the first inequality in (1.2) is saturated at every point by at least one vector. More precisely:

- in Theorem 3.3, we obtain the above conclusion in the range (i). However, we need a further constraint in dimension $n \geq 4$ (see Example 3.8), in accordance to statement (i) above and Remark 1.12: condition $\beta \leq \frac{n-1}{n-2}$ hardly detects parabolic ends if $n \geq 4$. The constraint is not necessary if V_+ is compactly supported, i.e. if $\mathrm{Ric} \geq 0$ outside of a compact set, see Theorem 3.5;

- in Theorem 3.9, the conclusion in Theorem 1.13 is proved both if

$$\beta < \frac{4}{n-1} \qquad \text{and if}$$

$$\frac{4}{n-1} \leq \beta < \frac{n-1}{n-2} \qquad \text{with } V_+ \text{ compactly supported.}$$

In the range

$$\beta < \frac{1}{n-2},$$

we prove in Theorem 3.11 that for *any* choice of a positive (smooth) solution to $Lu \leq 0$, the metric

$$\bar{g} = u^{2\beta} g$$

is a *complete* metric with non-negative modified Bakry-Émery Ricci tensor

$$\overline{\mathrm{Ric}}_f^N \doteq \overline{\mathrm{Ric}} + \bar{\nabla}^2 f - \frac{1}{N-n} \mathrm{d}f \otimes \mathrm{d}f,$$

for $f = (n-2)\beta \log u$ and a suitable $N = N(n, \beta) > n$. The result, which we obtain by adapting [18], establishes a higher-dimensional analogue of Fischer-Colbrie's [33, Theorem 1] and enables to exploit the extensive literature on manifolds with $\overline{\mathrm{Ric}}_f^N \geq 0$ and algebraic dimension $N > n$. These manifolds exhibit strong analogies to those with non-negative Ricci curvature: for instance, Corollaries 3.13 and 3.14 guarantee bounds on the Betti numbers of M and further topological consequences.

Remark 1.15. The inequality $\overline{\mathrm{Ric}}_f^N \geq 0$ was also pointed out in [15, Corollary 2.3].

Remark 1.16. The criteria in Theorems 3.3, 3.5 and 3.11 also relate to an elegant rigidity result by Castillon [16] for surfaces satisfying

$$\Delta - \mu K \geq 0$$

for some $\mu \geq 0$, where K is the Gaussian curvature. The result was later extended by various authors, see in particular [6].

The paper is organized as follows: in Section 2, we recall the main results in criticality theory needed throughout the paper. In Section 3, we prove the abstract criteria in Theorems 3.3, 3.5, 3.9 and 3.11 and their corollaries. In Section 4, we deduce our geometric consequences.

Note 1.17. *When we were completing this paper, we learned that the authors of [1] had independently obtained results that partially overlap with ours, with different techniques. We agreed with them to post the results on arXiv independently at the same time.*

Inspecting the proof of the spectral splitting theorem in [1], there seems to be an intriguing duality between the combined conformal method+criticality approach in the present paper and the μ-bubble technique by Gromov [37]. We think it could be interesting to investigate if this connection can be given a more rigorous formulation.

2 Criticality theory for $\Delta + V$

Let M be a Riemannian manifold. Hereafter, an exaustion $\{\Omega_j\}$ of M is a collection of relatively compact open sets with smooth boundary satisfying

$$\Omega_j \Subset \Omega_{j+1} \Subset M, \qquad \bigcup_{j=1}^{\infty} \Omega_j = M.$$

Let $V \in L^{\infty}_{\mathrm{loc}}(M)$ and consider the operator

$$L_V \doteq \Delta + V.$$

Assume that L_V is non-negative in the spectral sense (we write $L_V \geq 0$), namely, that the associated quadratic form

$$Q_V(\phi) \doteq \int_M \left[|\nabla \phi|^2 - V\phi^2 \right]$$

is non-negative for each $\phi \in \mathrm{Lip}_c(M)$, the set of compactly supported Lipschitz functions.

Notation 2.1. Hereafter, given $V_1, V_2 \in L^{\infty}_{\mathrm{loc}}(M)$ we will say that $L_{V_1} \geq L_{V_2}$ whenever $Q_{V_1}(\phi) \geq Q_{V_2}(\phi)$ for each $\phi \in \mathrm{Lip}_c(M)$. This is equivalent to say that $V_1 \leq V_2$ a.e. on M.

It is well-known, see for instance [58, Lemma 3.10], that $L_V \geq 0$ is equivalent to the existence of a positive weak solution $0 < u \in H^1_{\mathrm{loc}}(M)$ to $L_V u \leq 0$, and also equivalent to the existence of a weak solution $0 < u \in C^1(M)$ to $L_V u = 0$. The criticality theory discussed below, due to Murata [54] and Pinchover & Tintarev [59, 60, 61], describes the geometry of the cone of positive solutions to $L_V u \leq 0$, see also the work of Zhao [77].

Definition 2.2. For $V \in L^{\infty}_{\mathrm{loc}}(M)$, define $L_V = \Delta + V$ and let $\Omega \subseteq M$ be an open set.

- L_V is **subcritical** in Ω if there exists $w \geq 0$, $w \not\equiv 0$ in Ω (called a **Hardy weight**) such that

$$\int_{\Omega} w|\phi|^2 \leq Q_V(\phi) \qquad \forall\, \phi \in \mathrm{Lip}_c(\Omega). \tag{2.1}$$

Otherwise, the operator L_V is said to be **critical**.

- L_V has a **weighted spectral gap** in Ω if there exists $W \in C(\Omega)$, $W > 0$ on Ω such that

$$\int_{\Omega} W|\phi|^2 \leq Q_V(\phi) \qquad \forall\, \phi \in \mathrm{Lip}_c(\Omega). \tag{2.2}$$

- A sequence $\{\phi_j\} \in L_c^\infty(\Omega) \cap H^1(\Omega)$ is said to be a **null sequence** if $\phi_j \geq 0$ a.e. for each j, $Q_V(\phi_j) \to 0$ as $j \to \infty$ and there exists a relatively compact open set $B \Subset M$ and $C > 1$ such that $C^{-1} \leq \|\phi_j\|_{L^2(B)} \leq C$ for each j.

- A function $0 \leq \eta \in H^1_{\mathrm{loc}}(\Omega)$, $\eta \geq 0$, $\eta \not\equiv 0$ is a **ground state** for L_V on Ω if it is the $L^2_{\mathrm{loc}}(\Omega)$ limit of a null sequence.

We have the following fundamental result, known as the ground state alternative, [54, 61]. The version stated below also includes some further implications, which can be found in [7, Theorem 4.1].

Theorem 2.3. *Let M be connected and non-compact, and consider an operator $L_V \geq 0$ with $V \in L^\infty_{\mathrm{loc}}(M)$. Then, either L_V has a weighted spectral gap or a ground state on M, and the two possibilities mutually exclude. Moreover, the following properties are equivalent:*

  (i) *L_V is subcritical on M;*

  (ii) *L_V has a weighted spectral gap on M;*

  (iii) *There exist two positive solutions $u_1, u_2 \in C(M) \cap H^1_{\mathrm{loc}}(M)$ of $L_V u \leq 0$ which are not proportional;*

  (iv) *for some (equivalently, each) $o \in M$, there exists a minimal positive distributional solution G to $L_V G = -\delta_o$;*

  (v) *For some (equivalently, any) $K \Subset M$ compact with non-empty interior, and for some (any) $0 < \xi \in C(M) \cap H^1_{\mathrm{loc}}(M)$ solving $L_V \xi \leq 0$, it holds*

$$\mathrm{cap}_V(K,\xi) \doteq \inf_{\phi \in \mathscr{D}(K,\xi)} Q_V(\phi) > 0,$$

where

$$\mathscr{D}(K,\xi) = \Big\{ \phi \in \mathrm{Lip}_c(M) \ : \ \phi \geq \xi \ in \ K \Big\}.$$

If L_V has a ground state η, then:

- *$\eta > 0$ on M and solves $L_V \eta = 0$ (in particular, $\eta \in C^{1,\mu}_{\mathrm{loc}}(M)$). Moreover, every other positive solution ξ to $L_V \xi \leq 0$ is a multiple of η, hence a ground state;*

- *there exists a positive function $W > 0$ such that for each $\psi \in C_c^\infty(M)$ satisfying*

$$\int \psi\eta \neq 0,$$

there exists a constant $C = C(M, W, \psi, V)$ such that

$$C^{-1} \int_M W\phi^2 \leq Q_V(\phi) + C \left| \int_M \psi\phi \right|^2 \qquad \forall \phi \in \mathrm{Lip}_c(M). \qquad (2.3)$$

Proof (recap). All the statements, apart from those regarding property (iv), are contained in [7, Theorem 4.1]. The equivalence $(i) \Leftrightarrow (iv)$ is the content of [54, Theorem 2.4]. Inequality (2.3) was proved in [61, 62], see also [76] for a related result. $\qquad\square$

Remark 2.4. Theorem 2.3 extends, to operators with a potential, the dichotomy between parabolic and non-parabolic manifolds, which are exactly the manifolds for which Δ is critical and subcritical, respectively. If Δ is subcritical, by [46] a Hardy weight can be constructed from any non-constant positive supersolution u of $\Delta u \leq 0$ as

$$w = \frac{|\nabla u|^2}{4u^2}.$$

In particular, if u is the Green kernel of Δ on $\mathbb{R}^n$, $n \geq 3$ and r is the distance to a fixed point, we recover the well-known Hardy weight

$$\frac{(n-2)^2}{4r^2}, \tag{2.4}$$

As shown in [12], the same Hardy weight also occurs for minimal submanifolds $M^n \to N^p$ in a Cartan-Hadamard manifold, where now r is the restriction to M of the extrinsic distance from a fixed origin of N. More general Hardy weights, including sharpened ones for minimal submanifolds in hyperbolic space, can be found in [7, Section 5] and [46, Example 1.8], see also [8] for geometric applications.

Remark 2.5. As a direct application of the ground state alternative, if M satisfies a weighted Sobolev inequality

$$\left(\int_M \eta |\phi|^{\frac{2\nu}{\nu-2}} \right)^{\frac{\nu-2}{\nu}} \leq \mathcal{S} \int |\nabla \phi|^2 \qquad \forall \, \phi \in \mathrm{Lip}_c(M)$$

for some $0 < \eta \in C(M)$, $\mathcal{S} > 0$ and $\nu > 2$, then Δ is subcritical.

Remark 2.6. We stress that Theorem 2.3 also holds for homogeneous, p-Laplace operators

$$\mathrm{div}\left(|\nabla u|^{p-2} \nabla u \right) + V |u|^{p-2} u,$$

see [62, 7]. Note that for $p \neq 2$ one cannot avail of the Doob transform to reduce the result to the case $V \equiv 0$ up to adding suitable weights to the Laplacian. See [11, Section 2.3] for more information on the Doob transform.

The next strict convexity property, which will be crucial for us in what follows, was shown in [60, Theorem 3.1] in the Euclidean setting. An alternative proof can be found in [62, Proposition 4.3]. Both arguments hold verbatim on manifolds, and we reproduce it here for the sake of completeness.

Theorem 2.7 ([60, 62]). *Let $V_0, V_1 \in L^\infty_{\mathrm{loc}}(M)$ and assume that $L_{V_0} \geq 0$, $L_{V_1} \geq 0$. Then, setting $V_t = (1-t)V_0 + tV_1$ for $t \in [0,1]$, it holds $L_{V_t} \geq 0$. Moreover, if V_0 does not coincide with V_1 a.e., for each $t \in (0,1)$ the operator L_{V_t} is subcritical on M. In other words,*

$$\mathcal{K} \doteq \{ V \in L^\infty_{\mathrm{loc}}(M) : L_V \geq 0 \}$$

is a convex set whose extremal points are those V for which L_V is critical.

Proof. Since

$$Q_{V_t}(\phi) = (1-t)Q_{V_0}(\phi) + tQ_{V_1}(\phi),$$

the assertion $L_{V_t} \geq 0$ is immediate. Moreover, if any of L_{V_0}, L_{V_1} is subcritical, say L_{V_1}, and W_1 is a Hardy weight, L_{V_t} is subcritical for $t \in (0,1)$ with Hardy weight tW. We are left to

consider the case $V_0 \not\equiv V_1$ and L_{V_0}, L_{V_1} critical. For $j \in \{0, 1\}$ pick ground states η_j for L_{V_j}. Assume by contradiction that L_{V_t} is critical for some $t \in (0, 1)$, consider a null sequence $\{\phi_k\}$ and its L^2_{loc}-limit η, a ground state. Since $V_0 \not\equiv V_1$, η is neither proportional to η_0 nor to η_1, thus there exist $\psi_i \in C^\infty_c(M)$, $i \in \{0, 1\}$ satisfying

$$\int_M \psi_i \eta_i \neq 0, \qquad \int_M \psi_i \eta = 0 \qquad \text{if } i \neq j. \tag{2.5}$$

By Theorem 2.3, there exist $0 < W \in C(M)$ and a constant $C = C(\psi_0, \psi_1, V_0, V_1, M, W)$ such that

$$C^{-1} \int_M W \phi^2 \leq Q_{V_i}(\phi) + C \left| \int_M \psi_i \phi \right|^2 \qquad \forall \phi \in C^\infty_c(M).$$

By taking convex combination, we deduce

$$C^{-1} \int_M W \phi^2 \leq Q_{V_t}(\phi) + C(1-t) \left| \int_M \psi_1 \phi \right|^2 + Ct \left| \int_M \psi_2 \phi \right|^2 \qquad \forall \phi \in C^\infty_c(M).$$

Hence, using $\phi = \phi_k$, from $Q_{V_t}(\phi_k) \to 0$ and (2.5) we get

$$C^{-1} \int_M W \eta^2 \leq C(1-t) \left| \int_M \psi_1 \eta \right|^2 + Ct \left| \int_M \psi_2 \eta \right|^2 = 0, \tag{2.6}$$

contradiction. □

The notion of criticality can be localized on each end of M as follows: assume that $L \doteq \Delta + V$ satisfies $L \geq 0$ on M. Hereafter, a pair (K, ξ) is the data of a solution $0 < \xi \in C^{1,\alpha}_{\text{loc}}(M)$ to $L\xi = 0$ (equality here is important) and a compact set K with non-empty interior and smooth boundary. Let E be an end of M with respect to K, that is, a connected component of $M \backslash K$ with non-compact closure. Fix an exhaustion $\{\Omega_j\}$ of M with $K \subset \Omega_1$, set $E_j = E \cap \Omega_j$ and consider the family of solutions u_j to

$$\begin{cases} Lu_j = 0 & \text{on } E_j, \\ u_j = \xi & \text{on } \partial E, \qquad u_j = 0 \ \text{ on } E \cap \partial \Omega_j. \end{cases} \tag{2.7}$$

The maximum and comparison principles imply that $0 < u_j \leq u_{j+1} \leq \xi$, thus by elliptic estimates $u_j \uparrow u_\xi$ in $C^1_{\text{loc}}(\overline{E})$ for some $u_\xi > 0$ solving

$$\begin{cases} Lu_\xi = 0 & \text{on } E, \\ u_\xi = \xi & \text{on } \partial E, \qquad 0 < u_\xi \leq \xi \ \text{ on } E. \end{cases}$$

By comparison, u_ξ does not depend on the chosen exhaustion. We call u_ξ the minimal solution on E with respect to ξ. The strong maximum principle implies that either $u_\xi \equiv \xi$ or $u_\xi < \xi$ on E.

Definition 2.8. We say that L is ξ-critical on E if $u_\xi \equiv \xi$ on E, and ξ-subcritical otherwise.

We have the following

Theorem 2.9. *Let (M, g) be a Riemannian manifold, $V \in L^\infty_{\text{loc}}(M)$ and assume that $L \doteq \Delta + V \geq 0$. Then, the following are equivalent:*

(i) L is critical on M;

(ii) for every pair (K, ξ), L is ξ-critical on each end E with respect to K;

(iii) for some pair (K, ξ), L is ξ-critical on each end E with respect to K;

Proof. We introduce the following construction. Fix a pair (K, ξ), and let $E^{(1)}, \dots, E^{(k)}$ be the ends with respect to K. Up to attaching to K the bounded connected components of $M \backslash K$, we can assume that $M \backslash K = E^{(1)} \cup \dots \cup E^{(k)}$. For each i, let $u^{(i)}$ be the minimal solution on $E^{(i)}$ with respect to ξ. Let $\{\Omega_j\}$ be a smooth exhaustion of M, and for each i let $\{u_j^{(i)}\}$ be the sequence constructed in (2.7) on $E^{(i)}$. Define

$$\xi_j \doteq \begin{cases} \xi & \text{on } K, \\ u_j^{(i)} & \text{on } E^{(i)} \cap \Omega_j. \end{cases}$$

Then, $0 < \xi_j \le \xi$ on Ω_j. Moreover, integrating by parts $0 = \xi L \xi$ on K and $0 = u_j^{(i)} L u_j^{(i)}$ on $E^{(i)} \cap \Omega_j$ and subtracting the resulting identities we deduce

$$Q_V(\xi_j) = \sum_{i=1}^{k} \int_{\partial E^{(i)}} \xi \left[\partial_\eta \xi - \partial_\eta u_j^{(i)} \right].$$

Moreover,

$$\xi_j \uparrow \hat{\xi} \doteq \begin{cases} \xi & \text{on } K, \\ u^{(i)} & \text{on } E^{(i)}, \end{cases}$$

and from $u^{(i)} \le \xi$ one easily deduces that $L\hat{\xi} \le 0$ in the weak sense on M. We are ready to prove our equivalence.

$(i) \Rightarrow (ii)$. By contradiction, assume that for some pair (K, ξ) the operator L is subcritical on some end. Then, by construction $\hat{\xi} \not\equiv \xi$. Since $\hat{\xi} = \xi$ on K, ξ and $\hat{\xi}$ are distinct supersolutions for L which are not proportional. By (iii) in Theorem 2.3 we deduce that L is subcritical on M, contradiction.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$. Fix the pair (K, ξ) in (iii), and assume by contradiction that L is subcritical on M. Then, the energy $Q_V(\xi_j)$ cannot vanish as $j \to \infty$ (otherwise, $\{\xi_j\}$ would be a null sequence). Since

$$Q_V(\xi_j) \to \sum_{i=1}^{k} \int_{\partial E^{(i)}} \xi \left[\partial_\eta \xi - \partial_\eta u^{(i)} \right],$$

we deduce that $u^{(i)} \ne \xi$ for some i, so L is ξ-subcritical on $E^{(i)}$, contradiction. $\square$

3   The conformal method, criticality and splitting criteria

Let us consider a conformal deformation

$$\bar{g} = e^{2\varphi} g, \qquad \varphi \in C^\infty(M),$$

and denote with a bar superscript quantities in the metric $\bar{g}$. The Ricci curvatures of g and $\bar{g}$ relate as follows:

$$\overline{\text{Ric}} = \text{Ric} - (n-2)(\nabla^2 \varphi - d\varphi \otimes d\varphi) - \left[\Delta\varphi + (n-2)|d\varphi|^2 \right] g. \tag{3.1}$$

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Let γ be a $\bar{g}$-geodesic, namely, a geodesic in the metric $\bar{g}$. Denoting by s and $\bar{s}$, respectively, the g-arclength and the $\bar{g}$-arclength of γ, we have $d\bar{s} = e^{\varphi}ds$ and $\partial_{\bar{s}} = e^{-\varphi}\partial_s$, where φ is evaluated at $\gamma(s)$. Since the g and $\bar{g}$-unit tangent vectors to γ satisfy $\gamma_{\bar{s}} = e^{-\varphi}\gamma_s$, we have

$$0 = \bar{\nabla}_{\gamma_{\bar{s}}}\gamma_{\bar{s}} = e^{-2\varphi}\left(\nabla_{\gamma_s}\gamma_s + \gamma_s(\varphi)\gamma_s - \nabla\varphi\right)$$

Therefore,

$$
\begin{aligned}
(\varphi\circ\gamma)_{ss} &= \nabla^2\varphi(\gamma_s,\gamma_s) + d\varphi(\nabla_{\gamma_s}\gamma_s) \\
&= \nabla^2\varphi(\gamma_s,\gamma_s) + d\varphi\left(\nabla\varphi - \gamma_s(\varphi)\gamma_s\right) = \nabla^2\varphi(\gamma_s,\gamma_s) + |d\varphi|^2 - d\varphi(\gamma_s)^2
\end{aligned}
\tag{3.2}
$$

Therefore, when restricted to γ, (3.1) implies the remarkable identity

$$\overline{\mathrm{Ric}}(\gamma_s,\gamma_s) = \mathrm{Ric}(\gamma_s,\gamma_s) - (n-2)(\varphi\circ\gamma)_{ss} - \Delta\varphi, \tag{3.3}$$

appearing implicitly in [68] and proved in the Appendix of [32]. Hereafter, the composition with γ will tacitly be assumed, thus we will write φ_s, φ_{ss} rather than $(\varphi\circ\gamma)_s, (\varphi\circ\gamma)_{ss}$ and so forth.

We examine more closely the metric properties of the metric $\bar{g} = e^{2\varphi}g$. If $f, \psi : M \to \mathbb{R}$ and smooth functions, note that

$$
\begin{aligned}
\bar{\nabla}^2 f &= \nabla^2 f - (df\otimes d\varphi + d\varphi\otimes df) + g(df,d\varphi)g, \\
\bar{\Delta}\psi &= e^{-2\varphi}\left(\Delta\psi + (n-2)g(d\varphi,d\psi)\right)
\end{aligned}
$$

Hence, from (3.1), choosing

$$f = (n-2)\varphi$$

the ∞-Bakry-Emery Ricci curvature

$$\overline{\mathrm{Ric}}_f \doteq \overline{\mathrm{Ric}} + \bar{\nabla}^2 f$$

satisfies:

$$\overline{\mathrm{Ric}}_f = \mathrm{Ric} - (n-2)d\varphi\otimes d\varphi - (\Delta\varphi)g. \tag{3.4}$$

Moreover, the associated weighted Laplacian

$$\bar{\Delta}_f\psi \doteq \bar{\Delta}\psi - \bar{g}(df,d\psi)$$

of a function ψ satisfies

$$\bar{\Delta}_f\psi = e^{-2\varphi}\Delta\psi. \tag{3.5}$$

Observe that, in dimension $n = 2$, formulas (3.4) and (3.5) become the well-known identities relating the Gaussian curvature and the Laplacian of conformally deformed metrics. The main computation, repeatedly used throughout the paper and essentially due to [68], is the following. Hereafter, we write

$$C_0^k([a,b]) = \left\{\eta \in C^k([a,b]) : \eta(a) = \eta(b) = 0\right\}.$$

Proposition 3.1. *Assume that (M^n, g) is a Riemannian manifold, and let $V \in C(M)$ satisfy*

$$L \doteq \Delta + V \geq 0.$$

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Let $0 < u \in C^2(M)$ solve $Lu \leq 0$, and for $\beta > 0$ consider the metric

$$\bar{g} = u^{2\beta} g.$$

Assume that $\gamma : [a, b] \to M$ is a curve parametrized by g-arclength s, and assume that γ is $\bar{g}$-length minimizing. Then, for each $\eta \in C_0^1([a, b])$ it holds

$$\beta \int_a^b \frac{-Lu}{u} \frac{\eta^2}{u^\beta} ds + \int_a^b \left(\mathrm{Ric}(\gamma_s, \gamma_s) + \beta V \right) \frac{\eta^2}{u^\beta} ds + \beta \int_a^b |\nabla^\perp \log u|^2 \frac{\eta^2}{u^\beta} ds$$

$$\leq (n-1) \int_a^b \frac{(\eta_s)^2}{\eta^2} \frac{\eta^2}{u^\beta} ds - \beta \int_a^b \frac{(u_s)^2}{u^2} \frac{\eta^2}{u^\beta} ds - \beta(n-2) \int_a^b \frac{u_s}{u} \left(\frac{\eta^2}{u^\beta} \right)_s ds,$$

(3.6)

where $\nabla^\perp$ is the component of the g-gradient ∇ orthogonal to γ_s.

Proof. Setting $\varphi = \beta \log u$, we deduce from (3.3) the identity

$$\overline{\mathrm{Ric}}(\gamma_s, \gamma_s) = \mathrm{Ric}(\gamma_s, \gamma_s) - (n-2)\beta(\log u)_{ss} - \beta \frac{Lu}{u} + \beta V + \beta|d \log u|^2.$$

Let $\bar{s}$ denotes the $\bar{g}$-arclength. Since γ is $\bar{g}$-minimizing, the second variation formula in the metric $\bar{g}$ yields

$$
\begin{aligned}
0 &\leq \int_{\bar{s}(a)}^{\bar{s}(b)} \left[(n-1)(\eta_{\bar{s}})^2 - \overline{\mathrm{Ric}}(\gamma_{\bar{s}}, \gamma_{\bar{s}})\eta^2 \right] d\bar{s} \\
&= \int_a^b \left[(n-1)(\eta_s)^2 - \overline{\mathrm{Ric}}(\gamma_s, \gamma_s)\eta^2 \right] u^{-\beta} ds \\
&= (n-1) \int_a^b (\eta_s)^2 u^{-\beta} ds - \int_a^b \left(\mathrm{Ric}(\gamma_s, \gamma_s) + \beta V \right) \eta^2 u^{-\beta} ds \\
&\quad - \beta \int_a^b \left(|d \log u|^2 - (n-2)(\log u)_{ss} \right) \eta^2 u^{-\beta} ds + \beta \int_a^b \frac{Lu}{u} \eta^2 u^{-\beta} ds.
\end{aligned}
$$

(3.7)

Using $|d \log u|^2 = |\nabla^\perp \log u|^2 + (u_s)^2/u^2$ and integrating by parts the term with $(\log u)_{ss}$ we get (3.6). $\qquad \square$

The following proposition enables to construct a "shortest line" γ in a Riemannian manifold (M, h) with more than one end, so that the completeness of h is detected by the fact that γ is defined on the whole of $\mathbb{R}$ once parametrized by h-arclength. The result improves on [21, Lemma 1.1].

Proposition 3.2. *Let (M, h) be a Riemannian manifold with more than one end. Then, there exist $T, T^* \in (0, \infty]$ and a constant speed curve $\gamma : (-T, T^*) \to M$ which satisfies:*

(i) *γ is length minimizing for every pair of its points,*

(ii) *γ is a divergent curve, i.e. $\gamma^{-1}(K)$ is compact for each $K \subset M$ compact. Moreover, for ε small enough $\gamma_{[T^*-\varepsilon, T^*)}$ and $\gamma_{(-T, -T+\varepsilon]}$ belong to different ends of M.*

(iii) *(M, h) is complete if and only if both the curves $\gamma_{[0, T^*)}$ and $\gamma_{(-T, 0]}$ have infinite length, that is, $T = T^* = \infty$.*

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Proof. To construct γ, fix a compact set K with smooth boundary such that $M \setminus K$ has at least two ends. Up to including in K the components of $M \setminus K$ with compact closure, we can assume that $M \setminus K = \coprod_{i=1}^{\ell} E_i$, with E_i an end. Pick an exhaustion $\{\Omega_n\}$ of M with $K \Subset \Omega_1$. For each i, n, define $E_i(n) = E_i \setminus \Omega_n$. Define

$$\tau_n = \min \left\{ d\big(\partial E_i(n), \partial E_j(n)\big) : i, j \in \{1, \ldots, \ell\}, \ i \neq j. \right\}.$$

Since $(\overline{\Omega}_n, h)$ is a compact manifold with boundary, there exists a segment (i.e. a unit speed, minimizing geodesic) γ_n in $\overline{\Omega}_n$ realizing τ_n, joining $\partial E_{i_n}(n)$ to $\partial E_{j_n}(n)$ for some indices $i_n \neq j_n$. By construction, γ_n intersects K. Thus, up to translation we can assume that $\gamma : [-T_n, T_n^*] \to \overline{\Omega}_n$ for some positive T_n, T_n^*, and that

$$\gamma_n(0) \in K, \qquad \gamma_n(-T_n) \in \partial E_{i_n}(n), \qquad \gamma_n(T_n^*) \in \partial E_{j_n}(n).$$

By minimality $\gamma_n((-T_n, T_n^*)) \subset \Omega_n$, hence γ_n is a geodesic. Furthermore, by construction there exist indices i, j such that γ_n joins $\partial E_i(n)$ and $\partial E_j(n)$ for infinitely many n. Up to passing to a subsequence in n (still labelled the same), the family $\{\gamma_n\}$ locally converges to a minimizing geodesic $\gamma : (-T, T^*) \to M$ which crosses every $\partial E_i(n)$ and $\partial E_j(n)$, say at instants t_n and t_n^*, respectively, and satisfies $\gamma(0) = o \in K$. In particular, γ satisfies (*i*). To prove (*ii*), it is enough to show that $\gamma_{[0, T^*)}$ is divergent and eventually valued in E_j, the case of $\gamma_{(-T, 0]}$ being analogous. By contradiction, assume that there exists a compact set K (which we can assume contains o in its interior) and a sequence $\tau_n \to T^*$ so that $\gamma(\tau_n) \in K$. Pick n_0 such that $K \subset \Omega_{n_0}$. Then, for each $n \geq n_0$ we can estimate

$$|t_n^* - \tau_n| = d\big(\gamma(t_n^*), \gamma(\tau_n)\big) \geq d(\partial E_j(n), K) \geq d(\partial E_j(n_0), K).$$

From $t_n, \tau_n \to T^*$ we infer that necessarily $T^* = \infty$. However, from

$$\tau_n = d(o, \gamma(\tau_n)) \leq \operatorname{diam}(K)$$

and letting $n \to \infty$ we get a contradiction. Thus, $\gamma_{[0, T^*)}$ is divergent, and since $\gamma(t_n) \in \partial E_j(n)$ we conclude that γ is eventually contained in E_j. This proves (*ii*).

To show (*iii*), assume that (M, h) is complete. Then, there exist $R_n \to \infty$ such that $B_{R_n}(o) \subset \Omega_n$. Since γ is minimizing, by the definition of t_n, t_n^* it holds $t_n \leq -R_n$, $t_n^* \geq R_n$, hence $T = T^* = \infty$.

Viceversa, assume that $T = T^* = \infty$. In particular, $T_n^* \to \infty$. Let $\sigma : [0, R) \to M$ be a maximally extended unit speed geodesic issuing from o. We claim that $R = \infty$, from which the completeness follows. Assume by contradiction that $R < \infty$. Then, by ODE theory σ is a divergent curve, and we can fix $k \in \{1, \ldots, \ell\}$ so that $\sigma([R_0, R)) \subset E_k$ for some $R_0 > 0$. Choose an index between i, j which is different from k, say i. For each n, let s_n be the first instant such that $\sigma(s_n) \in \partial E_k(n)$. We claim that

$$d(\gamma_n(0), \partial E_j(n)) \leq d(\gamma_n(0), \partial E_k(n)). \tag{3.8}$$

Otherwise, concatenating $\gamma_n : (-T_n, 0] \to M$ to a segment η from $\gamma_n(0)$ to $\partial E_k(n)$ we would get a curve joining $\partial E_i(n)$ to $\partial E_k(n)$ shorter than γ_n, contradiction. From (3.8) we deduce

$$\begin{aligned} T_n^* &= d(\gamma_n(0), \partial E_j(n)) \leq d(\gamma_n(0), \partial E_k(n)) \\ &\leq d(\gamma_n(0), \gamma(0)) + d(\gamma(0), \partial E_k(n)) \\ &\leq d(\gamma_n(0), \gamma(0)) + s_n < d(\gamma_n(0), \gamma(0)) + R. \end{aligned}$$

Letting $n \to \infty$ we get a contradiction. $\qquad \square$

Here is our first main result:

Theorem 3.3. *Let (M, g) be a complete Riemannian manifold, and assume that there exists $V \in C^{0,\alpha}_{\mathrm{loc}}(M)$ such that*

$$\mathrm{Ric} \geq -\beta V g \qquad \text{on } M, \qquad L \doteq \Delta + V \geq 0, \tag{3.9}$$

where

$$0 < \beta \leq \frac{n-1}{n-2}.$$

If either

   (i) $n = 3$, or

  (ii) $n \geq 4$, and there exists a compact set Z and a weak solution $0 < w \in C(\overline{M \backslash Z})$ to $Lw \leq 0$ satisfying

$$w(x) = o\left(r(x)\right)^{\frac{n-1}{\beta(n-3)}} \qquad \text{as } r(x) \to \infty, \tag{3.10}$$

where $r(x)$ is the distance from a fixed origin, then the following holds: either

- *M has only one end, or*

- *L is critical and, for each $x \in M$,*

$$\min_{X \in T^1_x M} \left(\mathrm{Ric}(X, X) + \beta V(x) \right) = 0. \tag{3.11}$$

Proof. Assume that M has at least two ends, and, by contradiction, that L is subcritical. Pick an exhaustion $\{\Omega_j\}$ of M with $Z \subset \Omega_1$ and so that $M \backslash \Omega_1$ has at least two ends E_1 and E_2. By $(i) \iff (ii)$ in Theorem 2.3, there exists $0 \leq W \in C^\infty_c(\Omega_3)$ which is positive on $\overline{\Omega}_2$ and satisfies

$$\int_\Omega 2W |\eta|^2 \mathrm{d}x \leq Q_V(\eta) \qquad \forall \eta \in \mathrm{Lip}_c(M).$$

Thus, $L + W \geq 0$ and is subcritical. Picking a point $o \in \Omega_1$, by (iv) in Theorem 2.3 there exists a minimal positive solution $G \in C^{2,\alpha}_{\mathrm{loc}}(M \backslash \{o\})$ to $LG + WG = -\delta_o$. Fix also a global solution $0 < v \in C^{2,\alpha}_{\mathrm{loc}}(M)$ to $Lv = 0$ on M. Since G is constructed as the limit of a sequence of kernels G_j on Ω_j with Dirichlet boundary conditions, comparing G_j to w and v on $\Omega_j \backslash \Omega_3$ (recall, $LG_j = 0$ there) and letting $j \to \infty$, there exists a constant $C > 0$ such that $G \leq C \min \{w, v\}$ on $M \backslash \Omega_3$. Thus, up to increasing C,

$$G \leq C \min \{w, v\} \qquad \text{on } M \backslash \Omega_1. \tag{3.12}$$

The function $z \doteq G/v$ therefore solves

$$\begin{cases} \mathrm{div}\left(v^2 \nabla z\right) \leq 0 & \text{on } M \backslash \{o\} \\ z(x) \to \infty & \text{as } x \to o \\ z \leq C & \text{on } M \backslash \Omega_1. \end{cases}$$

Consider a concave function $\eta \in C^2([0, \infty))$ satisfying $\eta(t) = t$ for $t \in [0, 2C]$, $\eta(t) = 3C$ for $t \geq 4C$. Then, $\eta(z) \in C^2(M)$ solves

$$\mathrm{div}\left(v^2 \nabla \eta(z)\right) \leq 0 \qquad \text{on } M,$$

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and thus

$$u \doteq v\eta(z) \in C^2(M) \qquad \text{solves} \qquad \begin{cases} Lu \leq 0 & \text{on } M \\ u = G & \text{on } M \backslash \Omega_1. \end{cases}$$

Consider the conformal deformation

$$\bar{g} = u^{2\beta} g.$$

We construct a $\bar{g}$-line $\gamma : (-T, T^*) \to M$ as in Proposition 3.2. By property (ii) of γ and since g is complete, it follows that once reparametrized by its g-arclength s the curve γ is defined on the whole of $\mathbb{R}$. Also, because $M \backslash \Omega_1$ is disconnected and again by property (ii) we deduce that $\gamma \cap \overline{\Omega}_1 \neq \emptyset$ and $\gamma \cap (\Omega_2 \backslash \overline{\Omega}_1) \neq \emptyset$. By Proposition 3.1, for each $\eta \in C_c^1(\mathbb{R})$ it holds

$$\beta \int_{-\infty}^{\infty} \frac{-Lu}{u} \frac{\eta^2}{u^\beta} + \int_{-\infty}^{\infty} \left(\text{Ric}(\gamma_s, \gamma_s) + \beta V \right) \frac{\eta^2}{u^\beta}$$

$$\leq (n-1) \int_{-\infty}^{\infty} \frac{(\eta_s)^2}{\eta^2} \frac{\eta^2}{u^\beta} - \beta \int_{-\infty}^{\infty} \frac{(u_s)^2}{u^2} \frac{\eta^2}{u^\beta} - \beta(n-2) \int_{-\infty}^{\infty} \frac{u_s}{u} \left(\frac{\eta^2}{u^\beta} \right)_s,$$

where we omitted to write ds for the ease of notation. From (3.9),

$$\beta \int_{-\infty}^{\infty} \frac{-Lu}{u} \frac{\eta^2}{u^\beta} \leq (n-1) \int_{-\infty}^{\infty} \frac{(\eta_s)^2}{\eta^2} \frac{\eta^2}{u^\beta} - \beta \int_{-\infty}^{\infty} \frac{(u_s)^2}{u^2} \frac{\eta^2}{u^\beta} - \beta(n-2) \int_{-\infty}^{\infty} \frac{u_s}{u} \left(\frac{\eta^2}{u^\beta} \right)_s.$$

Write $\eta = u^\theta \psi$ for some $\theta \geq 0$ to be determined. Then,

$$\frac{\beta}{n-1} \int_{-\infty}^{\infty} \frac{-Lu}{u} u^{2\theta - \beta} \psi^2 \leq \int_{-\infty}^{\infty} \left[\theta^2 - \frac{2\theta\beta(n-2)}{n-1} + \frac{\beta^2(n-2)}{n-1} - \frac{\beta}{n-1} \right] \frac{(u_s)^2}{u^2} u^{2\theta - \beta} \psi^2$$

$$+ \int_{-\infty}^{\infty} (\psi_s)^2 u^{2\theta - \beta} + 2 \left[\theta - \frac{\beta(n-2)}{n-1} \right] \int_{-\infty}^{\infty} \frac{u_s \psi \psi_s}{u} u^{2\theta - \beta}.$$

Choosing $\theta = \frac{\beta(n-2)}{n-1}$, we conclude

$$\frac{\beta}{n-1} \int_{-\infty}^{\infty} \frac{-Lu}{u} u^{\frac{\beta(n-3)}{n-1}} \psi^2 \leq \int_{-\infty}^{\infty} (\psi_s)^2 u^{\frac{\beta(n-3)}{n-1}} \tag{3.13}$$

Let $0 \leq \psi_R \leq 1$ be an even function satisfying $\psi_R \equiv 1$ on $[-R, R]$, $\psi_R \in C_c^\infty((-2R, 2R))$ and $|(\psi_R)_s| \leq 4/R$. Then,

$$\frac{\beta}{n-1} \int_{-R}^{R} \frac{-Lu}{u} u^{\frac{\beta(n-3)}{n-1}} \leq \frac{16}{R^2} \left[\int_{R}^{2R} u^{\frac{\beta(n-3)}{n-1}} + \int_{-2R}^{-R} u^{\frac{\beta(n-3)}{n-1}} \right]. \tag{3.14}$$

If $n = 3$, letting $R \to \infty$ we deduce $Lu \equiv 0$ along γ. Similarly, if $n \geq 4$, by (3.12) there exists C such that

$$u(\gamma(s)) \leq C(1 + w(\gamma(s))) \qquad \forall s \in \mathbb{R}. \tag{3.15}$$

Assumption (3.10) (without loss of generality, we can assume that r is the distance from $\gamma(0)$) and $|\gamma_s| = 1$ imply

$$r(\gamma(s)) \leq |s|,$$

whence

$$u(\gamma(s)) = o\left(|s|^{\frac{n-1}{\beta(n-3)}}\right) \qquad \text{as } |s| \to \infty.$$

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Plugging into (3.14) and letting $R \to \infty$ we again deduce $Lu \equiv 0$ along γ.
To conclude, we observe that

$$Lu = LG = -WG < 0 \qquad \text{on } \gamma \cap (\Omega_2 \backslash \overline{\Omega}_1) \neq \emptyset,$$

leading to the desired contradiction. To prove (3.11), assume by contradiction that

$$\mathrm{Ric} + \beta V g > 0 \qquad \text{at some } x_0 \in M,$$

and fix $0 \leq w \in C^\infty(M)$ not identically zero satisfying

$$\mathrm{Ric} + \beta(V - w)g \geq 0 \qquad \text{on } M.$$

Since $L - w \geq L \geq 0$, and $w \not\equiv 0$, $L - w$ is subcritical. However, we can apply the first part of the proof with $V - w$ replacing V to deduce that $L - w$ is critical, contradiction. $\qquad \square$

Example 3.4. We denote by $C^n \to \mathbb{R}^{n+1}$, $n \geq 3$, the higher-dimensional catenoid. Following [72] (see also [30]), C^n depends on a real parameter $\phi_0 > 0$ and is defined by the map

$$F : (-S, S) \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1},$$
$$(s, \omega) \mapsto (\phi(s)\omega, s),$$

where

$$S = \int_{\phi_0}^\infty \frac{\mathrm{d}t}{\left[(t/\phi_0)^{2(n-1)} - 1\right]^{1/2}} < \infty$$

and ϕ solves

$$\begin{cases} \phi'' \left[1 + \left(\phi'\right)^2\right]^{-\frac{3}{2}} - \frac{n-1}{\phi}\left[1 + \left(\phi'\right)^2\right]^{-\frac{1}{2}} = 0 \\ \phi(0) = \phi_0 > 0 \\ \phi'(0) = 0. \end{cases}$$

By [72], the second fundamental $|A|$ is positive on C^n, globally bounded (in fact, C^n has finite total curvature) and solves

$$\Delta |A|^{\frac{n-2}{n}} + \frac{n-2}{n}|A|^2 |A|^{\frac{n-2}{n}} = 0.$$

Therefore, C^n is $\frac{n-2}{n}$-stable:

$$L \doteq \Delta + V \geq 0, \qquad \text{where } V \doteq \frac{n-2}{n}|A|^2.$$

By Gauss equations and the refined Kato inequality,

$$\mathrm{Ric} \geq -\frac{n-1}{n}|A|^2 = \frac{n-1}{n-2}V,$$

Hence, we can apply Theorem 3.3 with $w = |A|^{\frac{n-2}{n}}$ to deduce that L is critical on C^n.

We next examine the case where V_+ has compact support, namely, $\mathrm{Ric} \geq 0$ outside of a compact set. In this case, we can drop the dimensional restriction and get:

Theorem 3.5. *Let (M, g) be a complete Riemannian manifold, and assume that there exists $V \in C^{0,\alpha}_{\mathrm{loc}}(M)$ with V_+ compactly supported such that*

$$\mathrm{Ric} \geq -\frac{n-1}{n-2} V g \quad \text{on } M, \qquad L \doteq \Delta + V \geq 0. \tag{3.16}$$

then, either

- *M has only one end, or*

- *L is critical and, for each $x \in M$,*

$$\min_{X \in T^1_x M} \left(\mathrm{Ric}(X, X) + \frac{n-1}{n-2} V(x) \right) = 0.$$

Proof. Assuming, by contradiction, that M has at least two ends and that L is subcritical, we follow the proof of Theorem 3.3 and construct u, γ. For $\tau \geq 1$, let $\xi : [0, \infty) \to \mathbb{R}$ be a concave, non-decreasing function satisfying

$$\xi(x) = x \quad \text{for } x \in [0, 1], \qquad \xi(x) = 2 \quad \text{for } x \geq 4, \qquad \xi'(x) \leq 1 \ \forall x \in [0, \infty)$$

and define

$$\xi_\tau(x) = \tau \xi(x/\tau), \qquad u_\tau = \xi_\tau(u)$$

Then, $0 \leq \xi'_\tau \leq 1$, $\xi_\tau - x\xi'_\tau \geq 0$, $u_\tau \in L^\infty(M)$ and $\xi'_\tau \uparrow 1$, $u_\tau \uparrow u$ as $\tau \to \infty$. Moreover,

$$L u_\tau \leq \xi'_\tau(u) L u + V(u_\tau - u\xi'_\tau(u)) \leq \xi'_\tau(u) L u + V_+(u_\tau - u\xi'_\tau(u))$$

By using $\beta = \frac{n-1}{n-2}$ and u_τ in place of u, and proceeding as in Theorem 3.3, we obtain the following analogue of (3.13):

$$\frac{1}{n-2} \int_{-\infty}^{\infty} \frac{-\xi'_\tau(u) L u}{u_\tau} u_\tau^{\frac{n-3}{n-2}} \psi^2 - \frac{1}{n-2} \int_{-\infty}^{\infty} \frac{V_+(u_\tau - u\xi'_\tau(u))}{u_\tau} u_\tau^{\frac{n-3}{n-2}} \psi^2 \leq \int_{-\infty}^{\infty} (\psi_s)^2 u_\tau^{\frac{n-3}{n-2}} \tag{3.17}$$

Consider the family of cutoffs ψ_R as above, so $\psi_R \uparrow 1$ pointwise as $R \to \infty$. Taking into account that the first integral in the left-hand side of (3.17) is non-negative and the second is restricted to $\{s \in \mathbb{R} : V_+(\gamma(s)) \neq 0\}$, a compact set since γ diverges, letting $R \to \infty$ and applying monotone and Lebesgue convergence theorems we get

$$\int_{-\infty}^{\infty} \frac{-\xi'_\tau(u) L u}{u_\tau} u_\tau^{\frac{n-3}{n-2}} - \int_{-\infty}^{\infty} \frac{V_+(u_\tau - u\xi'_\tau(u))}{u_\tau} u_\tau^{\frac{n-3}{n-2}} \leq 0.$$

In particular, the first integral is finite. Letting $\tau \to \infty$ and applying Fatou's Lemma (to the first integral) and Lebesgue's Theorem (to the second), we conclude

$$\int_{-\infty}^{\infty} \frac{-L u}{u} u^{\frac{n-3}{n-2}} \leq 0.$$

thus $L u \equiv 0$ along γ. The rest of the argument follows verbatim the one in Theorem 3.3. $\quad\square$

Remark 3.6. Theorem 3.5 should be compared to [46, Corollary 4.2]. There, the authors assume (3.16) with $V \geq 0$ and the polynomial volume growth

$$|B_r| \leq C r^{2(n-1)} \quad \text{for } r \geq 1,$$

a condition satisfied when V_+ is compactly supported because of Bishop-Gromov volume comparison. Applying [46] we deduce that, in the setting of Theorem 3.5 with $V \geq 0$, either M has only one *non-parabolic* end, or M splits as Example 3.7 below.

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To further comment on the assumptions in Theorem 3.3 and introduce our next result, denote by $\lambda_1(M)$ the bottom of the Laplace spectrum on M.

Example 3.7. This example is taken from [46, Proposition 6.1]. Given $n \geq 3$, a manifold (P^{n-1}, h) and a function $0 < \eta \in C^\infty(\mathbb{R})$, consider the warped product

$$M = \mathbb{R} \times P, \qquad g = \mathrm{d}t^2 + \eta(t)^2 h$$

Under the assumptions

$$\eta'' > 0, \qquad (n-2)(\log \eta)'' + \eta^{-2}\mathrm{Ric}_P \geq 0,$$

the manifold M satisfies

$$\mathrm{Ric} \geq -\frac{n-1}{n-2}Vg \qquad \text{with} \qquad V = (n-2)\frac{\eta''}{\eta}.$$

The function

$$v = \int_0^t \frac{\mathrm{d}s}{\eta(s)^{n-1}}$$

is harmonic and $w = |\nabla v|^{\frac{n-2}{n-1}} = c \cdot \eta(t)^{2-n}$ is a positive solution to

$$Lw \doteq \Delta w + Vw = 0, \qquad \text{so } L \geq 0.$$

Therefore, if η is bounded from below by a positive constant on $\mathbb{R}$ and P is compact, by Theorem 3.3 the operator L is critical. An example is given by the choices

$$P \text{ compact with } \mathrm{Ric}_P \geq -(n-2)h, \qquad \eta(t) = \mathrm{ch}\, t$$

considered in [43]. In this case, the resulting warped product metric $\mathrm{d}t^2 + (\mathrm{ch}\, t)^2 h$ satisfies

$$\mathrm{Ric} \geq -(n-1)g, \qquad \Delta + (n-2) \geq 0 \quad \text{and is critical on } M.$$

In particular, $\lambda_1(M) = n - 2$.

Example 3.8. The following example shows that a growth condition on w in Theorem 3.3 is necessary to prove criticality in dimension $n \geq 4$. Consider in Example 3.7 the choices

$$P \text{ compact with } \mathrm{Ric}_P \geq 0, \qquad \eta(t) = e^t,$$

see [43, Example 2.2]. The resulting warped product metric $\mathrm{d}t^2 + e^{2t}h$ satisfies

$$\mathrm{Ric} \geq -(n-1)g, \qquad L \doteq \Delta + (n-2) \geq 0.$$

In particular, $\lambda_1(M) \geq n-2$. However, the positive solution $w = e^{(2-n)t}$ to $Lw = 0$ singled out in Example 3.7 does not satisfy (3.10), thus in dimension $n \geq 4$ Theorem 3.3 is not applicable to guarantee that L is critical. Indeed, we show that L is subcritical on M. To see this, for functions u of t only, equation $\Delta u + (n-2)u = 0$ becomes

$$u''(t) + (n-1)u'(t) + (n-2)u(t) = 0,$$

whose general solution is spanned by $\{e^{-(n-2)t}, e^{-t}\}$ in dimension $n \geq 4$ and by $\{e^{-t}, te^{-t}\}$ in dimension 3. Thus, if $n \geq 4$ the operator L admits the positive solutions $e^{-(n-2)t}$ and e^{-t} which are not proportional, hence it is subcritical. On the other hand, if $n = 3$ the only positive

solution just depending on t is, up to scaling, the function e^{-t}. In fact, Theorem 3.3 guarantees that L is critical.

Let us examine this example further. The function

$$\hat{w} = e^{-\frac{n-1}{2}t} \quad \text{solves} \quad \Delta\hat{w} + \frac{(n-1)^2}{4}\hat{w} = 0;$$

hence,

$$\lambda_1(M) \geq \frac{(n-1)^2}{4} \tag{3.18}$$

(in fact, equality holds in (3.18) by Cheng's eigenvalue estimate). Equivalently, since $\mathrm{Ric} \geq -(n-1)$ this can be rewritten as follows:

$$\mathrm{Ric} \geq -\frac{4}{n-1}\hat{V}, \qquad \hat{L} \doteq \Delta + \hat{V} \geq 0 \quad \text{with } \hat{V} = \frac{(n-1)^2}{4}.$$

Note that $\frac{(n-1)^2}{4} \geq (n-2)$, with equality iff $n = 3$. We claim that $\hat{L}$ is critical, and to see this we check each end separately. Let $E^{(1)} = (-\infty, 0) \times P$ and $E^{(2)} = (1, \infty) \times P$ be the ends with respect to $K = [0, 1] \times P$. Consider $\hat{w}$ as a supersolution, and let $u^{(i)}$ be the minimal solution on $E^{(i)}$ with respect to $\hat{w}$. Then, by construction $u^{(i)} = \lim_{j \to \infty} u_j^{(i)}$ where

$$
\begin{cases}
\hat{L}u_j^{(1)} = 0 & \text{on } [-j, 0] \times P, \\
u_j^{(1)} = 1 & \text{on } \{0\} \times P, \\
u_j^{(1)} = 0 & \text{on } \{-j\} \times P.
\end{cases}
\qquad
\begin{cases}
\hat{L}u_j^{(2)} = 0 & \text{on } [1, j] \times P, \\
u_j^{(2)} = e^{-\frac{n-1}{2}} & \text{on } \{1\} \times P, \\
u_j^{(2)} = 0 & \text{on } \{j\} \times P.
\end{cases}
$$

By uniqueness of solutions, $u_j^{(i)}$ only depends on t (since its average over P is still a solution). The general solution to the ODE

$$\hat{L}u = u''(t) + (n-1)u'(t) + \frac{(n-1)^2}{4}u(t) = 0$$

is $e^{-\frac{n-1}{2}t}(a + bt)$ with $a, b \in \mathbb{R}$, whence

$$u_j^{(1)}(t) = \frac{j+t}{j}e^{-\frac{n-1}{2}t}, \qquad u_j^{(2)}(t) = \frac{j-t}{j-1}e^{-\frac{n-1}{2}t}.$$

Letting $j \to \infty$ we get $u^{(1)} = \hat{w}$ and $u^{(2)} = \hat{w}$, therefore $\hat{L}$ is $\hat{w}$-critical on each end. By Proposition 2.9, $\hat{L}$ is critical on M.

Example 3.8 may suggest that if

$$\mathrm{Ric} \geq -\frac{4}{n-1}Vg \quad \text{on } M, \qquad L \doteq \Delta + V \geq 0,$$

then the dichotomy in the conclusion of Theorem 3.3 holds in any dimension $n \geq 3$ without the growth condition (3.10). It is interesting, in this respect, to compare to [46, Theorem C] and to the detailed ODE analysis in Sections 6 and 8 therein. In the next result, which improves on [21, Theorem 1.1], we are able to obtain the following stronger statement when $\beta < \frac{4}{n-1}$:

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Theorem 3.9. *Let (M,g) be a complete Riemannian manifold of dimension $n \geq 3$, and assume that there exists $V \in C^{0,\alpha}_{loc}(M)$ such that*

$$\mathrm{Ric} \geq -\beta V g \qquad on \ M, \qquad L \doteq \Delta + V \geq 0, \tag{3.19}$$

where either

$$0 < \beta < \frac{4}{n-1}, \qquad or$$

$$\frac{4}{n-1} \leq \beta < \frac{n-1}{n-2} \qquad and \ V_+ \ is \ compactly \ supported.$$

Then, either

(i) M has only one end, or

(ii) $V \equiv 0$ and $M = \mathbb{R} \times P$ with the product metric, for some compact P with $\mathrm{Ric}_P \geq 0$.

Proof. We first examine the case $\beta \in \left(0, \frac{4}{n-1}\right)$. Let us choose $\sigma \in \left(\beta, \frac{4}{n-1}\right)$ and set

$$\hat{V} \doteq \frac{\beta}{\sigma} V.$$

Then, by Theorem 2.7

$$\mathrm{Ric} \geq -\sigma \hat{V} g, \qquad \hat{L} \doteq \Delta + \hat{V} \geq 0. \tag{3.20}$$

Moreover, since $\Delta \geq 0$ and $\beta/\sigma \in (0,1)$, $\hat{L}$ is subcritical unless $\hat{V} \equiv 0$, that is, unless $V \equiv 0$. Let $0 < u \in C^{2,\alpha}_{loc}(M)$ solve $\hat{L}u \leq 0$, and consider the conformal deformation $\bar{g} = u^{2\sigma} g$. If we assume that M has at least two ends, using Proposition 3.2 we guarantee the existence of a $\bar{g}$-line $\gamma : (T, T^*) \to M$ in $(M, \bar{g})$. By (ii) in Proposition 3.2 and since g is complete, it follows that once reparametrized by g-arclength s the curve γ is defined on the whole of $\mathbb{R}$. Proposition 3.1 implies

$$\sigma \int_{-\infty}^{\infty} \frac{-\hat{L}u}{u} \frac{\eta^2}{u^\sigma} + \int_{-\infty}^{\infty} \left(\mathrm{Ric}(\gamma_s, \gamma_s) + \sigma \hat{V}\right) \frac{\eta^2}{u^\sigma}$$

$$\leq (n-1) \int_{-\infty}^{\infty} \frac{(\eta_s)^2}{\eta^2} \frac{\eta^2}{u^\sigma} - \sigma \int_{-\infty}^{\infty} \frac{(u_s)^2}{u^2} \frac{\eta^2}{u^\sigma} - \sigma(n-2) \int_{-\infty}^{\infty} \frac{u_s}{u} \left(\frac{\eta^2}{u^\sigma}\right)_s,$$

holds for every $\eta \in C^1_c(\mathbb{R})$. Writing $\eta = u^{\sigma/2}\psi$ and rearranging we get

$$\sigma \int_{-\infty}^{\infty} \frac{-\hat{L}u}{u} \psi^2 + \int_{-\infty}^{\infty} \left(\mathrm{Ric}(\gamma_s, \gamma_s) + \sigma \hat{V}\right) \psi^2$$

$$\leq \sigma \left[\frac{(n-1)\sigma}{4} - 1\right] \int_{-\infty}^{\infty} \frac{(u_s)^2}{u^2} \psi^2 - \sigma(n-3) \int_{-\infty}^{\infty} \frac{u_s \psi \psi_s}{u} + (n-1) \int_{-\infty}^{\infty} (\psi_s)^2.$$

By (3.20) and since $\sigma < \frac{4}{n-1}$, applying Young's inequality to the second term of the right-hand side we deduce the inequality

$$\int_{-\infty}^{\infty} \frac{-\hat{L}u}{u} \psi^2 \leq C \int_{-\infty}^{\infty} (\psi_s)^2$$

for some constant $C > 0$. By choosing an even cut-off $0 \leq \psi_R \leq 1$ such that $\psi_R \equiv 1$ on $[-R, R]$, $\psi_R \in C^\infty_c((-2R, 2R))$ and $|(\psi_R)_s| \leq 4/R$, letting $R \to \infty$ we deduce that $\hat{L}u = 0$

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along γ. As u can be any supersolution, we actually proved that any $C^{2,\alpha}_{\text{loc}}$ supersolution u satisfy $\hat{L}u = 0$ on some curve. If $\hat{L}$ were subcritical, then by Theorem 2.3 there would exist $0 < W \in C(M)$ such that $\hat{L} + W \geq 0$. By approximating W from below, we can assume that $W \in C^\infty(M)$. It would therefore exist a positive solution $u \in C^{2,\alpha}_{\text{loc}}(M)$ to $\hat{L}u = -Wu < 0$ on the entire M, contradiction. We conclude that $\hat{L}$ is critical, thus $V \equiv 0$. The conclusion in (ii) thus follows from Cheeger-Gromoll's splitting theorem [20].

If $\beta \in \left[\frac{4}{n-1}, \frac{n-1}{n-2}\right)$ and V_+ is compactly supported, fix $\sigma \in \left(\beta, \frac{n-1}{n-2}\right)$ and $\hat{V}, \hat{L}$ as above. We follow the proof of Theorem 3.5 with σ replacing $\frac{n-1}{n-2}$ up to (3.17), which in view of (3.14) now becomes

$$\frac{\sigma}{n-1}\int_{-\infty}^{\infty} \frac{-\xi'_\tau(u)\hat{L}u}{u_\tau} u_\tau^{\frac{\sigma(n-3)}{n-1}} \psi^2 - \frac{\sigma}{n-1}\int_{-\infty}^{\infty}\frac{\hat{V}_+(u_\tau - u\xi'_\tau(u))}{u_\tau}u_\tau^{\frac{\sigma(n-3)}{n-1}}\psi^2 \leq \int_{-\infty}^{\infty}(\psi_s)^2 u_\tau^{\frac{\sigma(n-3)}{n-1}}.$$

Proceeding as in Theorem 3.5 we deduce that $\hat{L}$ is critical. The conclusion follows as in the first part of the proof. $\qquad\square$

In view of Theorem 3.9, some of the applications of Cheeger-Gromoll's theorem to the geometry of manifolds with $\text{Ric} \geq 0$ directly extend to manifolds with spectral lower Ricci bounds. We mention, for instance, the following corollary that generalizes [67, 13]:

Corollary 3.10. *Let (M, g) be a complete, oriented Riemannian manifold of dimension $n \geq 3$ satisfying the assumptions of Theorem 3.9. Then, one of the following cases occurs:*

   (i) $H_{n-1}(M, \mathbb{Z}) = 0$;

   (ii) M is the determinant line bundle of a non orientable compact manifold with $\text{Ric}_N \geq 0$, and in that case $H_{n-1}(M, \mathbb{Z}) = \mathbb{Z}$;

   (iii) $M = N \times \mathbb{R}$ with the product metric, for some compact orientable N with $\text{Ric}_N \geq 0$, and in that case $H_{n-1}(M, \mathbb{Z}) = \mathbb{Z}$.

Proof. The argument follows *verbatim* that in [13, Proposition 5.3], with Theorem 3.9 replacing Cheeger-Gromoll's theorem. In this respect, note that the spectral conditions in Theorem 3.9 lift to any finite covering of M. $\qquad\square$

In general, in case (i) of Theorem 3.9 the metric $\bar{g} = u^{2\beta}g$ may not be complete. To see this, consider Euclidean space, $V \equiv 0$ and u a smoothing of the Green kernel centered at the origin:

$$u(x) = \tau(|x|^{2-n})$$

for a concave function $\tau : [0, \infty) \to \mathbb{R}$ satisfying $\tau(t) = t$ for $t \leq 1$ and $\tau \equiv 2$ on $[4, \infty)$. Then, $\Delta u \leq 0$ on $\mathbb{R}^n$ but $\bar{g}$ is incomplete for each

$$\beta \in \left(\frac{1}{n-2}, \frac{4}{n-1}\right).$$

The next theorem shows that for $\beta < (n-2)^{-1}$ the fact does not occur.

Theorem 3.11. *Let (M, g) be a complete Riemannian manifold of dimension $n \geq 3$, and assume that there exists $V \in C^{0,\alpha}_{\text{loc}}(M)$ such that*

$$\text{Ric} \geq -\beta V g \qquad \text{on } M, \qquad L \doteq \Delta + V \geq 0 \tag{3.21}$$

with

$$0 < \beta < \frac{1}{n-2}.$$

Then, for each $0 < u \in C^2(M)$ solving $Lu \le 0$, the metric $\bar{g} = u^{2\beta}g$ is complete and satisfies

$$\overline{\mathrm{Ric}}_f^N \ge \mathrm{Ric} + \beta V g \ge 0,$$

where

$$f = (n-2)\beta \log u \quad and \quad N = n + \frac{\beta(n-2)^2}{1 - \beta(n-2)} > n.$$

Remark 3.12. In the borderline case $\beta = (n-2)^{-1}$, the same computations show that the ∞-Bakry-Emery Ricci tensor $\overline{\mathrm{Ric}}_f$ is non-negative on M (cf. [40, Proposition 3.3]). However, in this case we are not able to prove that $\bar{g}$ is complete, nor we are aware of counterexamples to this fact.

Proof. By [50, Lemma 2.2], one can construct a "shortest ray" γ in $(M, \bar{g})$ issuing from a fixed origin o, that is, a divergent curve minimizing $\bar{g}$-distance between any pair of its points, with the property that $(M, \bar{g})$ is complete if and only if γ has infinite $\bar{g}$-length. Parametrizing γ by g-arclength s, since g is complete γ is defined for $s \in [0, \infty)$; hence, to get completeness we only have to check that

$$\ell_{\bar{g}}(\gamma_{[0,\infty)}) = \int_0^\infty u^\beta \mathrm{d}s = \infty.$$

Since γ is $\bar{g}$-minimizing, by equation (3.6) in Proposition 3.1 we deduce, using the hypothesis $Lu \le 0$ and $\mathrm{Ric} \ge -\beta V g$, that for every smooth function η with compact support in $[0, \infty)$ and vanishing at 0, and for every $\beta > 0$,

$$0 \le (n-1)\int_0^\infty (\eta_s)^2 u^{-\beta} - \beta \int_0^\infty \frac{(u_s)^2}{u^2}\eta^2 u^{-\beta} - \beta(n-2)\int_0^\infty \frac{u_s}{u}\left(\eta^2 u^{-\beta}\right)_s. \qquad (3.22)$$

Writing $\eta = u^\beta \psi$, we have

$$\eta^2 u^{-\beta} = u^\beta \psi^2,$$
$$\eta_s = \beta\psi u^{\beta-1}u_s + u^\beta\psi_s,$$
$$(\eta_s)^2 u^{-\beta} = \beta^2\psi^2 u^{\beta-2}(u_s)^2 + u^\beta(\psi_s)^2 + 2\beta\psi\psi_s u^{\beta-1}u_s,$$

and substituting in equation (3.22) we get

$$(n-1)\int_0^\infty u^\beta(\psi_s)^2 \ge -2\beta\int_0^\infty \psi\psi_s u^{\beta-1}u_s + \beta(1-\beta)\int_0^\infty \psi^2 u^{\beta-2}(u_s)^2. \qquad (3.23)$$

Now we follow the same lines of the computations in [18, Lemma 2.3]. Define

$$I \doteq \int_0^\infty \psi\psi_s u^{\beta-1}u_s$$

and observe that

$$I = \frac{1}{\beta}\int_0^\infty \psi\psi_s(u^\beta)_s = -\frac{1}{\beta}\int_0^\infty u^\beta(\psi_s)^2 - \frac{1}{\beta}\int_0^\infty \psi\psi_{ss}u^\beta.$$

Moreover, for every $t > 1$ and using Young's inequality with $\varepsilon > 0$ to be chosen later, we have

$$
\begin{aligned}
2\beta I &= 2\beta t I + 2\beta(1-t)I \\
&= -2t \int_0^\infty u^\beta(\psi_s)^2 - 2t \int_0^\infty \psi\psi_{ss}u^\beta + 2\beta(1-t)\int_0^\infty \psi\psi_s u^{\beta-1}u_s \\
&\leq -2t \int_0^\infty u^\beta(\psi_s)^2 - 2t \int_0^\infty \psi\psi_{ss}u^\beta \\
&\quad + \beta(t-1)\varepsilon \int_0^\infty \psi^2 u^{\beta-2}(u_s)^2 + \frac{\beta(t-1)}{\varepsilon}\int_0^\infty u^\beta(\psi_s)^2.
\end{aligned}
$$

Recalling that $\beta < (n-2)^{-1}$ and choosing

$$
\varepsilon \doteq \frac{1-\beta}{t-1}
$$

we obtain

$$
\begin{aligned}
2\beta I &\leq -2t \int_0^\infty \psi\psi_{ss}u^\beta + \beta(1-\beta)\int_0^\infty \psi^2 u^{\beta-2}(u_s)^2 \\
&\quad + \left[\frac{\beta(t-1)^2}{1-\beta} - 2t\right]\int_0^\infty u^\beta(\psi_s)^2.
\end{aligned}
$$

From (3.23) we get

$$
0 \leq \left[\frac{\beta(t-1)^2}{1-\beta} - 2t + (n-1)\right]\int_0^\infty u^\beta(\psi_s)^2 - 2t\int_0^\infty \psi\psi_{ss}u^\beta \tag{3.24}
$$

for every $t > 1$ and every positive $\beta < (n-2)^{-1}$. Let

$$
P(t,n,\beta) \doteq \frac{\beta(t-1)^2}{1-\beta} - 2t + (n-1) = \frac{1}{1-\beta}\left[\beta t^2 - 2t + (n-1) - \beta(n-2)\right];
$$

it is not difficult to see that there exists $\bar{t} = \bar{t}(\beta, n) > 1$ such that, for $n \geq 3$, $P(\bar{t}, n, \beta) < 0$. Therefore, from (3.24) we deduce

$$
0 \leq -\int_0^\infty u^\beta(\psi_s)^2 - C\int_0^\infty u^\beta \psi\psi_{ss}
$$

for every ψ smooth with compact support in $[0, \infty)$ and vanishing at 0, and for some positive constant C depending on $\bar{t}$, n and β. Now we choose $\psi = s\varphi$ with φ smooth with compact support in $[0, \infty)$: thus

$$
\psi_s = \varphi + s\varphi_s, \quad \psi_{ss} = 2\varphi_s + s\varphi_{ss},
$$

and we get

$$
\int_0^\infty u^\beta \varphi^2 \leq \int_0^\infty u^\beta \left(-2(C+1)s\varphi\varphi_s - Cs^2\varphi\varphi_{ss} - s^2(\varphi_s)^2\right).
$$

Choose φ such that $\varphi \equiv 1$ on $[0, R]$, $\varphi \equiv 0$ on $[2R, \infty)$ and with $|\varphi_s|$ and $|\varphi_{ss}|$ bounded by $\tilde{C}/R$ and $\tilde{C}/R^2$, respectively, for $R \leq s \leq 2R$ ($\tilde{C}$ is a positive constant). Then

$$
\int_0^R u^\beta \leq \int_0^\infty u^\beta \varphi^2 \leq C\int_R^\infty u^\beta
$$

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for some $C > 0$ independent of R. We conclude that necessarily

$$\int_0^\infty u^\beta = \infty,$$

i.e. $\bar{g} = u^{2\beta} g$ is complete. The desired lower bound on the Bakry-Emery Ricci curvature follows by plugging $\varphi = \beta \log u$ into (3.4): in our assumptions, and using the inequality $|\mathrm{d}u|^2 g \geq \mathrm{d}u \otimes \mathrm{d}u$, we get

$$
\begin{aligned}
\overline{\mathrm{Ric}}_f \quad &\geq \quad -\beta V - (n-2)\beta^2 \mathrm{d}\log u \otimes \mathrm{d}\log u + \beta(V + |\mathrm{d}\log u|^2)g \\
&\geq \quad \beta(1 - (n-2)\beta)\mathrm{d}\log u \otimes \mathrm{d}\log u = \frac{1 - (n-2)\beta}{(n-2)^2\beta}\mathrm{d}f \otimes \mathrm{d}f.
\end{aligned}
$$

This concludes the proof. $\qquad\square$

Two consequences of the previous theorem are the following topological properties. We first consider the compact case, exploiting the results in [63, 49, 75].

Corollary 3.13. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, and assume that there exists $V \in C^{0,\alpha}_{\mathrm{loc}}(M)$ such that*

$$\mathrm{Ric} \geq -\beta V g \qquad on\ M, \qquad L \doteq \Delta + V \geq 0, \tag{3.25}$$

where

$$0 < \beta \leq \frac{1}{n-2}.$$

Then:

(i) $\pi_1(M)$ *has a free abelian subgroup of finite index of rank $\leq n$, with equality iff M is a flat torus and $V \equiv 0$.*

(ii) $\pi_1(M)$ *is finite if $\mathrm{Ric} + \beta V g > 0$ at one point.*

Proof. Let $0 < u \in C^2(M)$ solve $Lu = 0$. By Remark 3.12 the metric $\bar{g} = u^{2\beta} g$ satisfies

$$\overline{\mathrm{Ric}}_f \geq \mathrm{Ric} + \beta V g \geq 0,$$

with $f = (n-2)\beta \log u$. Moreover, since M is compact $\bar{g}$ is complete. By [75, Corollary 6.7], (ii) holds and $\pi_1(M)$ has a free abelian subgroup of finite index of rank $\leq n$. Moreover, equality holds if and only if f is constant (i.e. u is constant) and $(M, \bar{g})$ is a flat torus, hence so is (M, g). Moreover, in this case, from $Lu = 0$, we also conclude $V \equiv 0$. $\qquad\square$

It is interesting to compare our theorem with those in [9, 11, 14], where the authors assume similar but different spectral conditions to obtain topological conclusions. In the non-compact case, using [47, 48] we have the following:

Corollary 3.14. *Let (M, g) be a complete, non-compact, Riemannian manifold of dimension $n \geq 3$, and assume that there exists $V \in C^{0,\alpha}_{\mathrm{loc}}(M)$ such that*

$$\mathrm{Ric} \geq -\beta V g \qquad on\ M, \qquad L \doteq \Delta + V \geq 0, \tag{3.26}$$

where

$$0 < \beta < \frac{1}{n-2}.$$

Then:

- $H_{n-1}(M, \mathbb{Z}) = 0$ *or* $\mathbb{Z}$;

- *positive harmonic functions on M are constant.*

Proof. The first property is [47, Theorem 1.6], the second follows from [48, Theorem 1.3] applied to $(M, \bar{g})$ once observed that, by (3.5), a harmonic function u satisfies $\bar{\Delta}_f u = 0$ on $(M, \bar{g})$. $\qquad\square$

Conditions implying $H_{n-1}(M, \mathbb{Z}) = 0$, respectively $H_{n-1}(M, \mathbb{Z}) = \mathbb{Z}$, can be found in [47, Theorem 1.10].

4 Applications to minimal hypersurfaces

Let $M^n \to (N^{n+1}, \bar{g})$ be a complete, two-sided, minimally immersed hypersurface. Denote with a bar tensors associated to $\bar{g}$. Write

$$J_\delta \doteq \Delta + \delta \left(|A|^2 + \overline{\mathrm{Ric}}(\nu, \nu) \right)$$

for the δ-stability operator. Choose a Darboux frame $\{e_i\}$ along M, and set $e_0 = \nu$. As the second fundamental form A in direction ν is traceless, it satisfies the refined Kato inequality

$$\sum_{j=1}^n A_{1j}^2 \le \frac{n-1}{n} |A|^2,$$

see [58, Lemma 1.5]. Moreover, equality holds at a point x where $A \neq 0$ if and only if $A(x)$ has only two eigenvalues λ, μ, of multiplicities 1 and 2 respectively, and $Ae_1 = \lambda e_1$. By Gauss equations,

$$\mathrm{Ric}_{11} \;=\; \sum_{\alpha \ge 2} \bar{R}_{1\alpha 1\alpha} - \sum_{j \ge 1} A_{1j}^2 \ge \sum_{\alpha \ge 2} \bar{R}_{1\alpha 1\alpha} - \frac{n-1}{n}|A|^2. \tag{4.1}$$

- Assume that $\overline{\mathrm{BRic}} \ge 0$. Then,

$$
\begin{aligned}
\mathrm{Ric}_{11} \;\ge\; & \sum_{\alpha \ge 2} \bar{R}_{1\alpha 1\alpha} - \frac{n-1}{n}|A|^2 = \overline{\mathrm{BRic}}_{1\nu} - \overline{\mathrm{Ric}}(\nu, \nu) - \frac{n-1}{n}|A|^2 \\
\;\ge\; & -\left(\overline{\mathrm{Ric}}(\nu, \nu) + |A|^2 \right) + \tfrac{1}{n}|A|^2,
\end{aligned}
\tag{4.2}
$$

- Assume that $\overline{\mathrm{Sec}} \ge 0$. Then,

$$\mathrm{Ric}_{11} \ge \sum_{\alpha \ge 2} \bar{R}_{1\alpha 1\alpha} - \frac{n-1}{n}|A|^2 \ge -\frac{n-1}{n}\left(\overline{\mathrm{Ric}}(\nu, \nu) + |A|^2 \right). \tag{4.3}$$

We are ready to prove our results:

Proof of Theorem 1.6. Assume that M has at least two ends, for otherwise (*i*) holds. Defining $V \doteq \overline{\mathrm{Ric}}(\nu, \nu) + |A|^2$, from (4.2) and the stability assumption we get

$$\mathrm{Ric} \ge -Vg + \frac{1}{n}|A|^2 g, \qquad \Delta + V \ge 0. \tag{4.4}$$

Since $1 < \frac{4}{n-1}$ holds in dimension $n \in \{3,4\}$, applying Theorem 3.9 we obtain that $V \equiv 0$ and that $M = \mathbb{R} \times P$ with the product metric $ds^2 + g_P$, for some compact (P, g_P) with non-negative Ricci curvature. From $0 = \mathrm{Ric}(\partial_s, \partial_s)$ and (4.4) we conclude $|A| \equiv 0$, thus $\overline{\mathrm{Ric}}(\nu, \nu) = V - |A|^2 \equiv 0$. The conclusion $\overline{\mathrm{BRic}}(\partial_s, \nu) \equiv 0$ follows from (4.2). In case (i), if M is parabolic then one cannot deduce $V \equiv 0$ (as the case of surfaces already suggests). However, if $\overline{\mathrm{Ric}} \geq 0$, any positive solution to $Lu = 0$ is superharmonic. Hence, the parabolicity of M forces u to be constant, so $V \equiv 0$. $\qquad\square$

If $\overline{\mathrm{BRic}} \geq 0$ is strengthened to $\overline{\mathrm{Sec}} \geq 0$, we gain one more dimension and further information in the previous result.

Proof of Theorem 1.8. The proof follows the same path as that of Theorem 1.6, we only point out the differences: setting $V \doteq \overline{\mathrm{Ric}}(\nu, \nu) + |A|^2$, Gauss equation (4.3) yields

$$\mathrm{Ric} \geq -\frac{n-1}{n}V, \qquad \Delta + V \geq 0.$$

Inequality $\frac{n-1}{n} < \frac{4}{n-1}$ now holds up to $n = 5$, justifying the dimensional improvement. If M has more than one end, from $V \equiv 0$ one deduces $|A| \equiv 0$ and $\overline{\mathrm{Ric}}(\nu, \nu) \equiv 0$. By Gauss equation, M has non-negative sectional curvature, hence so does (P, g_P).
If $n = 3$, the conclusion $H^2(M, \mathbb{Z}) = 0$ or $\mathbb{Z}$ follows by a direct application of Corollary 3.14, once observed that $\frac{n-1}{n} < \frac{1}{n-2}$ holds in dimension 3. $\qquad\square$

We are left to consider the case of $\frac{n-2}{n}$-stable hypersurfaces. Cheng & Zhou's strategy to obtain Theorem 1.3 is based on a clever refinement of [46, Theorem A], and exploits the Hardy inequality

$$\int_M \frac{(n-2)^2}{4r^2}\phi^2 \leq \int_M |\nabla\phi|^2 \qquad \forall\, \phi \in \mathrm{Lip}_c(M)$$

satisfied by minimal hypersurfaces in Euclidean space, see Remark 2.4. Crucial for the conclusion in [23] are both the completeness of the metric $r^{-2}g$ on M, and a lower bound for the conformal factor r^{-2} in terms of the intrinsic distance on M. When trying to adapt the ideas to more general ambient spaces, this latter control is particularly difficult to achieve. In dimension $n = 3$, we can use Theorem 3.3 to overcome the problem and obtain information on $1/3$-stable hypersurfaces. The next theorem generalizes Theorem 1.4 in the Introduction.

Theorem 4.1. *Let $M^3 \to (N^4, \bar{g})$ be a complete, non-compact, two-sided minimal hypersurface in a manifold satisfying $\overline{\mathrm{Sec}} \geq 0$. Assume that M is δ-stable with $\delta \geq 1/3$. Then, one of the following mutually exclusive cases occurs:*

(i) M has only one end. Moreover, either M is non-parabolic, or M is parabolic, totally geodesic and $\overline{\mathrm{Ric}}(\nu, \nu) \equiv 0$;

(ii) M is totally geodesic, $\overline{\mathrm{Ric}}(\nu, \nu) \equiv 0$ and $M = \mathbb{R} \times P$ with the product metric, for some compact surface P with non-negative Gaussian curvature.

(iii) $\delta = 1/3$, $\overline{\mathrm{Ric}}(\nu, \nu) \equiv 0$, M has at least two ends and is non-parabolic, the set $U \doteq \{|A| > 0\}$ is non-empty and A has only two eigenvalues λ, μ on each connected component of U, of multiplicities 1 and 2 respectively. Moreover, if $\{e_j\}$ is an orthonormal frame of eigenvectors with $Ae_1 = \lambda e_1$ and $Ae_\alpha = \mu e_\alpha$ for $\alpha \geq 2$, and setting $e_0 = \nu$, the components of the curvature tensor of N satisfy

$$\bar{R}_{1010} = \bar{R}_{0\beta0\alpha} = \bar{R}_{1\beta1\alpha} = \bar{R}_{\beta\alpha\beta0} = \bar{R}_{\beta\alpha\beta1} = \bar{R}_{\alpha0\alpha1} = \bar{R}_{101\alpha} = \bar{R}_{010\alpha} = 0. \qquad (4.5)$$

Proof. The restriction to dimension $n = 3$ is only needed at one step of the proof, hence we prefer to keep writing n. The assumed δ-stability and $\overline{\mathrm{Sec}} \geq 0$ therefore imply

$$J_{\frac{n-2}{n}} \doteq \Delta + \frac{n-2}{n}\left(|A|^2 + \overline{\mathrm{Ric}}(\nu, \nu)\right) \geq 0.$$

Assume that M has at least two ends, for otherwise (*i*) holds. Define

$$L = \Delta + \frac{n-2}{n}|A|^2, \qquad V \equiv \frac{n-2}{n}|A|^2.$$

By Gauss equations (4.1),

$$\mathrm{Ric}_{11} = \sum_{\alpha \geq 2} \bar{R}_{1\alpha 1\alpha} - \sum_{j \geq 1} A_{1j}^2 \geq -\frac{n-1}{n}|A|^2 = -\frac{n-1}{n-2}V$$

On the other hand,

$$L \geq J_{\frac{n-2}{n}} \geq 0. \tag{4.6}$$

Therefore, applying Theorem 3.3 we deduce that L is critical and that, for each $x \in M$, there exists an orthonormal basis $\{e_j\}$ whose first element e_1 satisfies

$$\mathrm{Ric}_{11} = \sum_{j \geq 1} A_{1j}^2 = -\frac{n-1}{n}|A|^2. \tag{4.7}$$

Namely, $\bar{R}_{1\alpha 1\alpha} = 0$ for each $\alpha \geq 2$, and A satisfies equality in the refined Kato inequality. From (4.6) and the criticality of L,

$$\overline{\mathrm{Ric}}(\nu, \nu) \equiv 0 \qquad \text{on } M.$$

If $\delta > \frac{n-2}{n}$, we deduce from

$$L \geq J_\delta \geq J_{\frac{n-2}{n}} \geq 0$$

and the criticality of L that $J_\delta = J_{\frac{n-2}{n}}$, which means $|A| \equiv 0$. From Gauss equations we deduce that M has non-negative sectional curvature. Since M has two ends, the splitting theorem implies $M = \mathbb{R} \times P$ for some compact manifold P with non-negative sectional curvature.

It remains to consider the case $\delta = \frac{n-2}{n}$ and $U \doteq \{|A| > 0\} \neq \emptyset$. By (4.7) and the characterization of equality in the refined Kato inequality,

$$A_{1\alpha} = 0, \quad A_{11} = \lambda, \quad A_{\alpha\beta} = \mu\delta_{\alpha\beta} \qquad \text{with } \lambda + (n-1)\mu = 0. \tag{4.8}$$

Since $\lambda \neq \mu$, the two eigenspace distributions have constant multiplicity and are therefore smooth. Thus, $\lambda, \mu \in C^\infty(U)$ and the frame $\{e_1, e_\alpha\}$ satisfying (4.8) can be chosen smoothly around any given point. Furthermore, from $\bar{R}_{1\alpha 1\alpha} = 0$ and $\overline{\mathrm{Ric}}(\nu, \nu) \equiv 0$ we deduce

$$0 = \bar{R}_{1\alpha 1\alpha} = \bar{R}_{1010} = \bar{R}_{\alpha 0\alpha 0} \qquad \forall \alpha \geq 2.$$

For $i \in \{0, 1\}$, set

$$F_i(t) = \bar{R}(e_i, e_\alpha + te_\beta, e_i, e_\alpha + te_\beta).$$

From $F_i \geq 0$ and $F_i(0) = 0$ we deduce $0 = F_i'(0) = 2\bar{R}_{i\beta i\alpha}$. Similarly, setting

$$F_i(t) = \bar{R}(e_\beta, e_i + te_\alpha, e_\beta, e_i + te_\alpha),$$

from $F_i \geq 0$ and $F_i(0) = 0$ we deduce $0 = F_i'(0) = 2\bar{R}_{\beta\alpha\beta i}$. Likewise, differentiating

$$F(t) = \bar{R}(e_\alpha, e_0 + te_1, e_\alpha, e_0 + te_1)$$

we get $0 = \bar{R}_{\alpha 0\alpha 1}$, and differentiating, respectively,

$$F(t) = \bar{R}(e_1, e_0 + te_\alpha, e_1, e_0 + te_\alpha), \qquad F(t) = \bar{R}(e_0, e_1 + te_\alpha, e_0, e_1 + te_\alpha)$$

we deduce, respectively, $0 = \bar{R}_{101\alpha}$ and $0 = \bar{R}_{010\alpha}$. This concludes the proof. $\square$

Remark 4.2. Theorem 4.1 holds in dimension $n \geq 4$, with $(n-2)/n$ replacing $1/3$ and principal eigenspaces of dimensions 1 and $(n-1)$, whenever M supports a positive solution to $J_{\frac{n-2}{n}} w \leq 0$ satisfying

$$w(x) = o\left(r(x)^{\frac{n-2}{n-3}}\right) \qquad \text{as } r(x) \to \infty.$$

However, currently we do not have a manageable geometric condition to guarantee the existence of such w.

In (iii) of Theorem 4.1, it is unclear to us whether the mixed components $\bar{R}_{01\alpha\beta}$ should vanish or not. If this were the case, a computation using the Codazzi-Mainardi equations would imply that the distribution generated by $\{e_\alpha\}$ is integrable, allowing to locally (or even globally) split M as a warped product. This is the case of ambient Euclidean space:

Proof of Theorem 1.4. Denote by $f : M \to \mathbb{R}^4$ the minimal immersion. If M has at least two ends, by Theorem 4.1 and since case (ii) does not occur in Euclidean space we deduce that M is non-parabolic, that $U \doteq \{|A| > 0\}$ is non-empty and that A has only two eigenvalues on any connected component $U' \subset U$. By a result due to Do Carmo and Dajczer [30, Theorem 4.4], $f(U')$ is a piece of a catenoid C, and by continuity so is $f(\overline{U'})$. If by contradiction $U' \not\equiv M$, then A should vanish on $\partial U'$, hence the second fundamental form of C should vanish on $f(\partial U')$. However, the second fundamental form of C is positive everywhere, contradiction. Therefore, $U' \equiv M$ and $f(M) \subset C$. Since M is complete and $f : M \to C$ is a local isometry, by Ambrose theorem f is surjective and a Riemannian covering, hence a diffeomorphism since the 3-dimensional catenoid is simply connected. $\square$

Proof of Theorem 1.2. Up to passing to the orientable double covering of M, which is still properly immersed and δ-stable, we can assume that M^3 is orientable. In our assumptions, setting $V = \delta|A|^2$ we have from (4.1)

$$\text{Ric} \geq -\frac{2}{3}|A|^2 g = -\beta V g, \qquad \Delta + V \geq 0 \qquad \text{with } \beta = \frac{2}{3\delta} < 2.$$

We can thus apply Corollary 3.10 to deduce either $H_2(M, \mathbb{Z}) = 0$ or that M is, in particular, a manifold with non-negative Ricci curvature and linear volume growth. The second case does not occur, as Gauss equations would imply $0 \leq R = -|A|^2$, thus M would be totally geodesic, contradicting the linear volume growth property. We claim that $H_2(M, \mathbb{Z}) = 0$ has the following topological implication:

Claim 1: for any smooth, connected, relatively compact open set $U \Subset M$, each connected component L of $M \backslash U$ has connected boundary.

Proof: assume by contradiction that $\partial_1 L$ and $\partial_2 L$ are distinct components of $\partial L \subset \partial U$. Pick tubular neighbourhoods T_j of $\partial_j L$ and points $x_j, y_j \in T_j$ with $x_j \in U$, $y_j \in L$. Joining y_1 to y_2 with an arc in L, x_1 to x_2 with an arc in U and x_j to y_j with a suitable arc in T_j transverse to

$\partial_j L$, we produce a loop γ with intersection number $\gamma \cdot \partial_j L = \pm 1$. This contradicts the fact that $[\partial_j L] = 0$ in $H_2(M, \mathbb{Z})$. We include a brief argument to see this. Let $\Sigma = \partial_1 L$, pick a chain c so that $\Sigma = \partial c$ and a smooth, connected relatively compact open set V containing Σ and the support of c. By [41, Theorem 18.7], c can be chosen smooth. Let P be the oriented double of $\overline{V}$. Then, $[\Sigma] = 0$ in $H_2(P, \mathbb{Z})$, so the following functional is zero by Stokes theorem for chains in [41, Theorem 18.12]:

$$[\omega] \mapsto \int_\Sigma \omega \quad : \quad H^2(P) \to \mathbb{R}.$$

By Poincaré duality, the Poincaré dual η_Σ is therefore zero in cohomology. Hence, by [55, Proposition 7.3.12], the intersection number satisfies

$$\pm 1 = \gamma \cdot \Sigma = \int_P \eta_\gamma \wedge \eta_\Sigma = 0,$$

contradiction.

Next, by Theorem 1.4, either M has only one end or it is a catenoid. However, the catenoid is not δ-stable for $\delta > 1/3$, as can be deduced by the criticality of $\Delta + 1/3|A|^2$ in Example 3.4. Having shown that M has only one end and that $H_2(M, \mathbb{Z}) = 0$, to prove that M is a hyperplane we follow [25, 27, 39] with some adjustments[3]. Let r be the distance in $\mathbb{R}^4$ from 0 and write $\mathbb{B}(R)$ for the ball of radius R in $\mathbb{R}^4$ centered at 0. Up to translation, we assume $0 \notin M$, $M \cap \mathbb{B}(1) \neq \emptyset$. Fix a connected component M_1^* of $M \cap \mathbb{B}(1)$, and for $R > 1$ let M_R^* be the connected component of $M \cap \mathbb{B}(R)$ containing M_1^*. We shall prove that

$$|M_R^*|_g \leq \Lambda R^3 \qquad \forall R > 1, \tag{4.9}$$

for some $\Lambda \in \mathbb{R}^+$. Write for convenience $c = e^{8\pi\sqrt{2}}$. Let M_{cR} be the union of M_{cR}^* and the relatively compact connected components of its complement. Since M has only one end, $M \setminus M_{cR}$ is connected and thus, by Claim 1, $\partial M_{cR} = \partial(M \setminus M_{cR})$ is connected. Denote by N the manifold M endowed with the complete metric $\tilde{g} = r^{-2}g$, and by N_0 the subset M_{cR}. By [25, Lemma 25], for any $p \in M \setminus M_{cR}$ and $q \in M_R^*$ it holds

$$d_{\tilde{g}}(p, q) \geq \log \frac{r(p)}{r(q)} \geq 8\pi\sqrt{2}. \tag{4.10}$$

We can thus apply [39, Theorem 5.4] to deduce the existence of a smooth, connected, relatively compact open set $\Omega \subset N_0$ with

(i) $\partial N_0 \subset \Omega \subset B_{4\pi\sqrt{2}}^{\tilde{g}}(\partial N_0)$, the tubular neighbourhood of ∂N_0 of $\tilde{g}$-radius $4\pi\sqrt{2}$;

(ii) every connected component of $\partial'\Omega \doteq \partial\Omega \setminus \partial N_0$ is a sphere of $\tilde{g}$-area at most 16π and diameter at most 4π.

By (i), (4.10) and the triangle inequality, M_R^* is disjoint from $(M \setminus M_{cR}) \cup \overline{\Omega}$. Moreover, applying [25, Lemma 25] with $p \in \overline{\Omega}$ and $q \in \partial M_{cR}$ we have $r(p) \leq Re^{12\pi\sqrt{2}}$. In particular,

(iii) $\tilde{g} \geq C_1 R^{-2} g$ on $\partial'\Omega$, for some constant C_1.

---

[3]Indeed, it is not enough to apply [39, Theorem 1.7] because M is assumed to be simply connected there, and properness does not lift to the universal covering unless M has finite fundamental group.

Let L be the union of $(M \setminus M_{cR}) \cup \Omega$ and the connected components of its complement which are disjoint from M_R^*, and let $M' = M \setminus L$. Then, M' is relatively compact and $\partial M' \subset \partial' \Omega$. Moreover, since M_R^* is connected, then $M' \supset M_R^*$ and M' is connected as well. By Claim 1, $\partial M' = \partial L$ is connected, hence by (ii) its $\tilde{g}$-area does not exceed 16π. The isoperimetric inequality for minimal hypersurfaces and (iii) then imply

$$|M_R^*|_g \leq |M'|_g \leq C_2 |\partial M'|_g^{\frac{3}{2}} \leq C_3 R^3 |\partial M'|_{\tilde{g}}^{\frac{3}{2}} \leq C_4 R^3,$$

as required. To conclude, observe that $\{M_R^*\}$ is a family of relatively compact domains exhausting M. Applying the curvature estimate in [39, Corollary 1.5] to each (scaled) M_R^* and letting $R \to \infty$ we deduce that $|A| \equiv 0$. $\qquad\square$

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