STABLE s-MINIMAL CONES IN $\mathbb{R}^2$ ARE FLAT FOR $s \sim 0$

MICHELE CASELLI

Abstract. For $s \in (0,1)$ small, we show that the only cones in $\mathbb{R}^2$ stationary for the s-perimeter and stable in $\mathbb{R}^2 \setminus \{0\}$ are hyperplanes. This is in direct contrast with the case of the classical perimeter or the regime s close to 1, where nontrivial cones as $\{xy > 0\} \subset \mathbb{R}^2$ are stable for inner variations.

Contents

1. Introduction

In this note, we prove that in $\mathbb{R}^2$ and for s close to zero, half-spaces are the only s-minimal cones that are stable in $\mathbb{R}^2 \setminus \{0\}$. Here, by an s-minimal cone, we mean an open cone $E \subset \mathbb{R}^2$ that is an s-minimal surface (that is, stationary for the s-perimeter under inner variations). This result is purely nonlocal since it is in direct contrast with both the classical case (formally $s = 1$) and the regime where s is close to 1, where the cross $\mathbf{X} = \{xy > 0\}$ is a non-flat stationary cone in the plane that is stable for inner variations, in any reasonable sense. Nevertheless, for s close to zero, the cross $\mathbf{X}$ is unstable in $\mathbb{R}^2 \setminus \{0\}$, and has infinite index (see Corollary 1.2). Our proof relies on the behavior of the best constant in Hardy's inequality for the $H^\sigma(\mathbb{R})$ seminorm as $\sigma \downarrow 1/2$.

The classification s-minimal cones in $\mathbb{R}^2$ has been previously studied for $s \in (0,1)$ under stronger hypotheses. Specifically, Savin and Valdinoci [SV13a,SV13b] proved that half-spaces are the unique cones locally minimizing the s-perimeter in $\mathbb{R}^2$ for all $s \in (0,1)$, in accordance with the classical case. Our result achieves the same rigidity for cones that are just stable compactly away from the origin, albeit in the restricted regime of small s.

In higher dimension $3 \le n \le 7$, the classification of stable s-minimal cones in $\mathbb{R}^n$ smooth outside the origin is only expected for s close to 1. Indeed, in [DdPW18], among other things, the authors construct a nontrivial, stable, s-minimal cone in $\mathbb{R}^7$ for s close to zero. On the other hand, this classification for s close to 1 has been proved for $n = 3$ in [CCS20] and recently for $n = 4$ in [CDSV23, Theorem 1.5].

The main result of this note is the following.

Theorem 1.1. *There exists $s_\circ \in (0, 1/2)$ with the following property. Let $s \in (0, s_\circ)$ and $E \subset \mathbb{R}^2$ be an s-minimal cone stable in $\mathbb{R}^2 \setminus \{0\}$. Then E is a half-space.*

Moreover, using the fact that s-minimal cones with finite Morse index outside the origin are stable outside the origin (which is a trivial observation in the classical case of the perimeter, but not entirely trivial for s-minimal cones, see the beginning of Section 5), we deduce that the same holds for finite Morse index cones.

Corollary 1.2. *The classification of Theorem 1.1 holds for s-minimal cones of finite Morse index in $\mathbb{R}^2 \setminus \{0\}$.*

In Appendix 6, we give direct proof that the cross $\mathbf{X}$ is unstable outside the origin, for s close to 0. Since $\mathbf{X}$ is not flat, this fact is already contained in Theorem 1.1. Nevertheless, we provide a very short independent proof since we believe it captures the main idea in the proof of Theorem 1.1.

2. Preliminary tools

Our proof relies on the Hardy inequality for the $H^\sigma(\mathbb{R})$ seminorm and, specifically, on the asymptotic behavior of its optimal constant as $\sigma \downarrow 1/2$. The sharp constant in this inequality has been established in [FS08] (equation (1.6)), and it is also stated in [CCS20, Theorem 3.3]. We will also use the fact that radially symmetric functions in $C^2_c(\mathbb{R} \setminus \{0\})$ nearly saturate Hardy's inequality; this is proved in Section 3.3 of [FS08].

Theorem 2.1 (Hardy's inequality)**.** *Let $n \geq 1$, $\sigma \in (0,1)$ and $u \in H^\sigma_0(\mathbb{R}^n \setminus \{0\})$. Then*

$$\mathcal{H}_{n,\sigma} \int_{\mathbb{R}^n} \frac{u^2}{|x|^{2\sigma}}\, dx \leq c_{n,\sigma} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\sigma}}\, dxdy,$$

where

$$\mathcal{H}_{n,\sigma} = 2^{2\sigma-1}(\sigma - n/2)^2 \frac{\Gamma(n/4+\sigma/2)^2}{\Gamma(n/4-\sigma/2+1)^2},$$

and

$$c_{n,\sigma} = 2^{2\sigma-1}\pi^{-n/2} \frac{\Gamma(n/2+\sigma)}{\Gamma(2-\sigma)} \sigma(1-\sigma).$$

Moreover, the inequality is saturated by radial functions, that is: for every $\varepsilon > 0$ there exists a radial function $\xi(x) = \xi(|x|) \in C^2_c(\mathbb{R}^n \setminus \{0\})$ such that

$$\big(\mathcal{H}_{n,\sigma} + \varepsilon\big) \int_{\mathbb{R}^n} \frac{\xi(x)^2}{|x|^{2\sigma}}\, dx \geq c_{n,\sigma} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{n+2\sigma}}\, dxdy.$$

In particular, for $n = 1$, $s \in (0, 1/2)$ and $\sigma = \frac{1+s}{2}$ the constant reads as

$$\mathcal{H}_{1,\frac{1+s}{2}} = s^2 2^{s-2} \frac{\Gamma(\frac{2+s}{4})^2}{\Gamma(\frac{4-s}{4})^2},$$

and, by elementary properties of the Gamma function, it is easily checked that

$$\frac{\mathcal{H}_{1,\frac{1+s}{2}}}{c_{1,\frac{1+s}{2}}} \leq \frac{Cs^2}{C\frac{1+s}{2}(1-\frac{1+s}{2})} \leq Cs^2,$$

for some absolute $C > 0$ and every $s \in (0, 1/2)$.

Summarizing, taking the ξ relative to $\varepsilon = \mathcal{H}_{1,\frac{1+s}{2}}$ in the saturation statement above, for every $s \in (0, 1/2)$ there exists an even function $\xi \in C^2_c(\mathbb{R} \setminus \{0\})$ such that

$$\int_{\mathbb{R}} \frac{\xi(x)^2}{|x|^{1+s}}\, dx \geq \frac{1}{Cs^2} \iint_{\mathbb{R}\times\mathbb{R}} \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{2+s}}\, dxdy.$$

Lastly, for the same ξ, this directly implies

$$\int_0^\infty \frac{\xi(x)^2}{x^{1+s}}\,dx \geq \frac{1}{Cs^2}\int_0^\infty\int_0^\infty \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{2+s}}\,dxdy. \tag{1}$$

Before stating precisely all the tools that we will need on the first and second variation of the fractional perimeter, we recall the notion of fractional s-perimeter, which was introduced by Caffarelli, Roquejoffre, and Savin in [CRS10].

Definition 2.2. *For $s \in (0,1)$, the fractional perimeter (or s-perimeter) of a measurable set $E \subset \mathbb{R}^n$ is defined as*

$$\mathrm{Per}_s(E) = \frac{1}{2}[\chi_E]^2_{H^{s/2}(\mathbb{R}^n)} = \iint_{E\times E^c} \frac{1}{|x-y|^{n+s}}\,dxdy.$$

The s-perimeter also has a natural localized version in a bounded open set $\Omega \subset \mathbb{R}^n$, in the same spirit of the localized fractional Sobolev spaces $H^s(\Omega)$. This is of use because, for example, one would like to say that a hyperplane in $\mathbb{R}^n$ is an s-minimal surface (see Definition 2.4 below) even though a half-space has infinite s-perimeter for Definition 2.2.

Definition 2.3. *For $s \in (0,1)$, the fractional perimeter (or s-perimeter) of a measurable set $E \subset \mathbb{R}^n$ in a bounded, open set Ω is defined as*

$$\mathrm{Per}_s(E;\Omega) = \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus\Omega^c\times\Omega^c} \frac{|\chi_E(x)-\chi_E(y)|^2}{|x-y|^{n+s}}\,dxdy.$$

Note that for $\Omega = \varnothing$ we recover Definition 2.2.

Definition 2.4 (s-minimal surface)**.** *Let Ω be a bounded open set and $E \subset \mathbb{R}^n$ be a set with locally finite s-perimeter in Ω. Then, E is said to be an s-minimal surface in Ω if, for every vector field X with compact support in Ω there holds*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Per}_s(\phi_t^X(E),\Omega) = 0.$$

By the first variation formula, see for example [FFM+15, FS24], if E is an s-minimal surface and ∂E is C^2 then

$$P.V.\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y)-\chi_E(y)}{|x-y|^{n+s}}\,dy = 0\,, \quad \text{for all } x \in \partial E. \tag{2}$$

The left-hand side is called the s-mean curvature of E at x.

Remark 2.5. *It follows by inspecting the proof of the first variation formula in [FFM+15, FS24] that if ∂E is C^2 only in a neighborhood of some $x \in \partial E$, then (2) holds at x.*

If Ω is a bounded open set and $E \subset \mathbb{R}^n$ is an s-minimal surface in Ω, then E is said to be stable in Ω if, for every vector field X with compact support in Ω, there holds

$$\delta^2\mathrm{Per}_s(E;\Omega)[X] := \left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{Per}_s(\phi_t^X(E),\Omega) \geq 0.$$

If ∂E is smooth, the second variation can be written in a form very reminiscent of the second variation formula for classical minimal surfaces. This second variation formula for smooth s-minimal surfaces was proved in [DdPW18,FFM+15] , and has recently been generalized to ambient Riemannian manifolds in [FS24].

Theorem 2.6. *Let E be an s-minimal surface in $\mathbb{R}^n$, and assume that ∂E is C^2 in some open set Ω. Then, for every $X \in C^2_c(\partial E \cap \Omega; \mathbb{R}^n)$, setting $\varphi := X \cdot \nu_{\partial E}$, we have*

$$\delta^2 \mathrm{Per}_s(E)[X] = \iint_{\partial E \times \partial E} \left(\frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{n+s}} - \frac{|\nu_{\partial E}(x) - \nu_{\partial E}(y)|^2}{|x-y|^{n+s}} \varphi(x)^2 \right) d\sigma(x) d\sigma(y), \tag{3}$$

where $\nu_{\partial E}$ is the outer unit normal to ∂E. In particular, if E is stable in Ω, there holds

$$\iint_{\partial E \times \partial E} \frac{|\nu_{\partial E}(x) - \nu_{\partial E}(y)|^2}{|x-y|^{n+s}} \varphi(x)^2 \, d\sigma(x) d\sigma(y) \leq \iint_{\partial E \times \partial E} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{n+s}} \, d\sigma(x) d\sigma(y),$$

for every $\varphi \in C^2_c(\Omega)$.

3. The BV-estimate for small s

It is known [CSV19, FS24] that stable s-minimal surfaces enjoy a uniform interior BV-estimate, and that the same holds for stable solution of the fractional Allen-Cahn equation [CCS21, CFS23]. The results in these references only control the dependence of the constant from s as $s \to 1$. In this work, we need the same type of BV-estimate with a control of the constant as $s \to 0$.

Theorem 3.1. *Let $R > 0$, $s \in (0, 1/2)$, and $E \subset \mathbb{R}^n$ be an s-minimal surface which stable in $B_R(x)$ and such that ∂E is C^2 in $B_R(x)$. Then*

$$\mathrm{Per}(E, B_{R/2}(x)) \leq \frac{C}{s},$$

for some dimensional constant $C > 0$.

We first recall a useful interpolation inequality for the s-perimeter that accounts for the dependence on s both for $s \to 1$ and $s \to 0$. The inequality in the form we use here is not explicitly stated anywhere; it is written throughout the lines of the proof of Theorem 3.1 in [Jac24], or it follows from [FS24, Lemma 3.13] and Young's inequality.

Lemma 3.2. *Let $s \in (0, 1)$, $R > 0$ and E be a set with locally finite perimeter. Then*

$$\mathrm{Per}_s(E; B_R(x)) \leq C \left(\frac{R^{1-s}}{1-s} \mathrm{Per}(E; B_{5R}(x)) + \frac{R^{n-s}}{s} \right),$$

for some dimensional constant $C > 0$.

With this inequality, we can deduce the BV-estimate for small s.

Proof of Theorem 3.1. Similarly to [CCS20], the theorem follows by inspection of the proof of Theorem 1.7 in [CSV19], taking care of the explicit dependence of the constants as $s \to 0$. For the sake of clarity, we rewrite here the crucial estimates in the proof of Theorem 1.7 in [CSV19], with the precise dependence of all constants on s, as $s \to 0$. In the proof that follows $C > 0$ is a dimensional constant that can change from line to line.

Since the statement is scaling and translation invariant, we can assume $R = 1$ and $x = 0$. Since E is stable in B_1, by Theorem 1.9 in [CSV19] applied to the kernel $K(z) = 1/|z|^{n+s}$, we get

$$\mathrm{Per}(E; B_1) \leq C \left(1 + \sqrt{\mathrm{Per}_s(E; B_4)} \right). \tag{4}$$

Moreover, by Lemma 3.2 applied with $R = 4$ we have

$$\mathrm{Per}_s(E; B_4) \leq C \left(\frac{1}{1-s} \mathrm{Per}(E; B_{20}) + \frac{1}{s} \right) \leq C \left(\mathrm{Per}(E; B_{20}) + \frac{1}{s} \right).$$

Thus, by (4) and Young's inequality we get

$$\mathrm{Per}(E, B_1) \le C\left(1 + \sqrt{\mathrm{Per}(E; B_{20}) + \frac{1}{s}}\right) \le C\left(1 + \delta \mathrm{Per}(E; B_{20}) + \frac{\delta}{s} + \frac{1}{\delta}\right)$$
$$= C\left(1 + \frac{\delta}{s} + \frac{1}{\delta}\right) + \delta \mathrm{Per}(E; B_{20}),$$

for every $\delta > 0$. From here, arguing exactly as the end of the proof of [CSV19, Theorem 1.7] or [CFS23, Proposition 3.14], choosing δ smaller than a dimensional constant $\delta_\circ = \delta_\circ(n) > 0$ and a covering argument one concludes the uniform bound

$$\mathrm{Per}(E; B_{1/2}) \le C\left(1 + \frac{\delta_\circ}{s} + \frac{1}{\delta_\circ}\right) \le \frac{C}{s}.$$

□

4. Classification of stable cones for small s

Let us fix the notation that we will use for cones in the plane. For a cone $E \subset \mathbb{R}^2$, we write $\Sigma = \partial E$ and observe that Σ is a union of half-lines from the origin. Write

$$E = \bigcup_{i=1}^{N} E_i\,, \quad \Sigma = \bigcup_{i=1}^{2N} \Sigma_i\,, \quad \partial E_i = \{\Sigma_i, \Sigma_{i+1}\},$$

with the convention that $\Sigma_{2N+1} = \Sigma_1$. Here E_i are disjoint conical sectors from the origin, that is $\lambda E_i = E_i$ for every $\lambda > 0$, Σ_i are rays from the origin with the induced orientation from E_i, and the number N could be $+\infty$ in general, but will be finite in our proof.

We also denote by θ_i^j the counterclockwise angle from Σ_i and Σ_j.

Lemma 4.1. *In the notation above, there is $c > 0$ such that for every $x \in \Sigma_j$ there holds*

$$\int_{\Sigma_i} \frac{1}{|x-y|^{2+s}}\, d\sigma(y) \ge \frac{c}{|x|^{1+s}(1-\cos(\theta_i^j))^{1+s}}.$$

Proof. We have

$$\int_{\Sigma_i} \frac{1}{|x-y|^{2+s}}\, d\sigma(y) = \int_{\Sigma_i} \frac{d\sigma(y)}{(|x|^2 + |y|^2 - 2\cos(\theta_i^j)|x||y|)^{\frac{2+s}{2}}}$$
$$= \int_0^\infty \frac{dz}{(|x|^2 + z^2 - 2\cos(\theta_i^j)|x|z)^{\frac{2+s}{2}}} = \frac{1}{|x|^{1+s}} \int_0^\infty \frac{dt}{(1 + t^2 - 2t\cos(\theta_i^j))^{\frac{2+s}{2}}},$$

where we have substituted $z = t|x|$ in the last line. Moreover

$$\int_0^\infty \frac{dt}{(1 + t^2 - 2t\cos(\theta_i^j))^{\frac{2+s}{2}}} = \int_0^\infty \frac{dt}{((t-1)^2 + 2t(1-\cos(\theta_i^j)))^{\frac{2+s}{2}}}$$
$$\ge \int_{1/2}^{3/2} \frac{dt}{((t-1)^2 + 2t(1-\cos(\theta_i^j)))^{\frac{2+s}{2}}} \ge \int_{1/2}^{3/2} \frac{dt}{((t-1)^2 + 3(1-\cos(\theta_i^j)))^{\frac{2+s}{2}}}$$
$$= \int_{-1/2}^{1/2} \frac{dt}{((t-1)^2 + 3(1-\cos(\theta_i^j)))^{\frac{2+s}{2}}} \ge \int_{1/2}^{3/2} \frac{dt}{(t^2 + 3(1-\cos(\theta_i^j)))^{\frac{2+s}{2}}}$$
$$\ge \frac{1}{(1-\cos(\theta_i^j))^{1+s}} \int_{-1/10}^{1/10} \frac{dt}{(t^2+3)^{\frac{2+s}{2}}} \ge \frac{c}{(1-\cos(\theta_i^j))^{1+s}}.$$

This concludes the proof. □

Now, we have all the ingredients to prove our main result Theorem 1.1. In the proof, we plug in the stability inequality a radial test function that nearly saturates Hardy's inequality on $(0, \infty)$, in the sense that (1) holds.

Proof of Theorem 1.1. By Theorem 3.1, that is the BV-estimate for stable s-minimal surfaces for $s \in (0, 1/2)$, and a standard covering argument we get that

$$2N = \mathrm{Per}(E; B_2 \setminus B_1) \leq \frac{C}{s}.$$

Hence $\Sigma = \partial E$ is a finite number of rays from the origin, whose number is bounded by

$$2N \leq \frac{C}{s}. \tag{5}$$

Recall the stability inequality (3), and let ν_i be the outer unit normal to Σ_i from E_i. For the left hand side, for every $\varphi \in C^2_c(\mathbb{R}^2 \setminus \{0\})$, we have

$$\begin{aligned}\iint_{\Sigma\times\Sigma} \frac{|\nu_\Sigma(x) - \nu_\Sigma(y)|^2}{|x-y|^{2+s}} \varphi(x)^2 \, d\sigma(x) d\sigma(y) &= \sum_{i,j} \iint_{\Sigma_i\times\Sigma_j} \frac{|\nu_i(x) - \nu_j(y)|^2}{|x-y|^{2+s}} \varphi(x)^2 \, d\sigma(x) d\sigma(y) \\ &= 2\sum_{i\neq j} (1 - (-1)^{i+j} \cos(\theta_i^j)) \iint_{\Sigma_i\times\Sigma_j} \frac{\varphi(x)^2}{|x-y|^{2+s}} \, d\sigma(x) d\sigma(y).\end{aligned}$$

By Lemma 4.1 we can estimate

$$\iint_{\Sigma_i\times\Sigma_j} \frac{\varphi(x)^2}{|x-y|^{2+s}} \, d\sigma(x) d\sigma(y) \geq \frac{c}{(1-\cos(\theta_i^j))^{1+s}} \int_{\Sigma_j} \frac{\varphi(x)^2}{|x|^{1+s}} \, dx.$$

Now, taking $\varphi(x) = \xi(|x|)$ with ξ saturating Hardy's inequality on $(0, \infty)$ as in (1), gives

$$\int_{\Sigma_j} \frac{\xi(x)^2}{|x|^{1+s}} \, d\sigma(x) \geq \frac{1}{Cs^2} \iint_{\Sigma_j\times\Sigma_j} \frac{|\xi(x) - \xi(y)|^2}{|x-y|^{2+s}} \, d\sigma(x) d\sigma(y),$$

thus

$$\begin{aligned}&\iint_{\Sigma\times\Sigma} \frac{|\nu_\Sigma(x) - \nu_\Sigma(y)|^2}{|x-y|^{1+s}} \xi(|x|)^2 \, d\sigma(x) d\sigma(y) \\ &\qquad\geq \frac{c}{s^2} \sum_{j=1}^{2N} \left(\iint_{\Sigma_j\times\Sigma_j} \frac{|\xi(x) - \xi(y)|^2}{|x-y|^{2+s}} \, d\sigma(x) d\sigma(y) \right) \sum_{1\leq i\leq 2N,\, i\neq j} \frac{1 - (-1)^{i+j}\cos(\theta_i^j)}{(1-\cos(\theta_i^j))^{1+s}}. \\ &\qquad= \frac{c}{s^2} \left(\int_0^\infty \int_0^\infty \frac{|\xi(x) - \xi(y)|^2}{|x-y|^{2+s}} \, dx dy \right) \sum_{j=1}^{2N} \sum_{\{1\leq i\leq 2N,\, i\neq j\}} \frac{1 - (-1)^{i+j}\cos(\theta_i^j)}{(1-\cos(\theta_i^j))^{1+s}}. \end{aligned} \tag{6}$$

Claim. There exists $s_\circ < 1/2$ sufficiently small with the following property. For every $s \in (0, s_\circ)$, there exists $j \in \{1, 2, \ldots, 2N\}$ such that

$$\sum_{\{1\leq i\leq 2N,\, i\neq j\}} \frac{1 - (-1)^{i+j}\cos(\theta_i^j)}{(1-\cos(\theta_i^j))^{1+s}} \leq \frac{1}{100}. \tag{7}$$

Indeed, suppose this is not the case. Then, for s arbitrary small, (6) implies

$$\iint_{\Sigma\times\Sigma} \frac{|\nu_\Sigma(x) - \nu_\Sigma(y)|^2}{|x-y|^{1+s}} \xi(|x|)^2 \, d\sigma(x) d\sigma(y) \geq \frac{cN}{100s^2} \int_0^\infty \int_0^\infty \frac{|\xi(x) - \xi(y)|^2}{|x-y|^{2+s}} \, dx dy.$$

From this inequality, using that E is stable gives

$$\begin{aligned}
&\frac{cN}{100s^2}\int_0^\infty\!\!\int_0^\infty \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{2+s}}\,dxdy \\
&\qquad\le \iint_{\Sigma\times\Sigma} \frac{|\nu_\Sigma(x)-\nu_\Sigma(y)|^2}{|x-y|^{1+s}}\xi(|x|)^2\,d\sigma(x)d\sigma(y) \\
&\qquad\overset{\text{(Stability)}}{\le} \iint_{\Sigma\times\Sigma} \frac{|\xi(|x|)-\xi(|y|)|^2}{|x-y|^{2+s}}\,d\sigma(x)d\sigma(y) \\
&\qquad= N\int_0^\infty\!\!\int_0^\infty \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{2+s}}\,dxdy + \sum_{i\neq j}\iint_{\Sigma_i\times\Sigma_j} \frac{|\xi(|x|)-\xi(|y|)|^2}{|x-y|^{2+s}}\,d\sigma(x)d\sigma(y).
\end{aligned}$$

Moreover, since $|x-y|\ge ||x|-|y||$, we have

$$\begin{aligned}
\iint_{\Sigma_i\times\Sigma_j} \frac{|\xi(|x|)-\xi(|y|)|^2}{|x-y|^{2+s}}\,d\sigma(x)d\sigma(y) &\le \iint_{\Sigma_i\times\Sigma_j} \frac{|\xi(|x|)-\xi(|y|)|^2}{||x|-|y||^{2+s}}\,d\sigma(x)d\sigma(y) \\
&= \int_0^\infty\!\!\int_0^\infty \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{2+s}}\,dxdy,
\end{aligned}$$

for every $i\neq j$. Thus

$$\frac{cN}{100s^2}\int_0^\infty\!\!\int_0^\infty \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{2+s}}\,dxdy \le N^2\int_0^\infty\!\!\int_0^\infty \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{2+s}}\,dxdy,$$

which implies, together with (5), that

$$s^2 \ge \frac{c}{100N} \ge \frac{c}{50C}s,$$

which gives a contradiction if s is small. Hence, the claim is proved.

Now, we conclude the proof of our theorem by contradiction, with $s_\circ$ the one given by the claim above. Assume by contradiction that $N\ge 2$. Then, for the index j such that (7) holds we get that (here the indices are modulo $2N$)

$$\frac{1-(-1)^{j+(j+2)}\cos(\theta_j^{j+2})}{(1-\cos(\theta_j^{j+2}))^{1+s}} = \frac{1}{(1-\cos(\theta_j^{j+2}))^s} \le \frac{1}{100},$$

holds for every $s\le s_\circ$. Clearly, this is not possible for any value of $\theta_j^{j+2}\in[0,2\pi)$, and hence we conclude that $N=1$ and E is made only of one conical sector of angle θ.

With no loss of generality, up to rotation and complementation, assume that $\theta\in(0,\pi]$ and

$$E = \{(\rho,\varphi)\,:\,\rho>0, \varphi\in(0,\theta)\}.$$

Consider the set $\widetilde{E} := E + e_1$. Since $\widetilde{E}\subset E$ and $e_1\in\partial E\cap\partial\widetilde{E}$ we have, for every $\varepsilon>0$, that

$$\begin{aligned}
\int_{B_\varepsilon(e_1)} \frac{\chi_{E^c}(y)-\chi_E(y)}{|y-e_1|^{2+s}}\,dy &\ge \int_{B_\varepsilon(e_1)} \frac{\chi_{\widetilde{E}^c}(y)-\chi_{\widetilde{E}}(y)}{|y-e_1|^{2+s}}\,dy \\
&= \int_{B_\varepsilon(0)} \frac{\chi_{E^c}(z)-\chi_E(z)}{|z|^{2+s}}\,dz \\
&= 2\pi\int_\varepsilon^\infty \frac{(2\pi-\theta)-\theta}{r^{1+s}}\,dr = \frac{4\pi(\pi-\theta)}{s\varepsilon^s}.
\end{aligned}$$

Since E is an s-minimal surface and is smooth in a neighborhood of $e_1 \in \partial E$, by the first variation formula (2) at $x = e_1$ (see Remark 2.5) we get

$$0 = \lim_{\varepsilon \to 0} \int_{B_\varepsilon(e_1)} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y - e_1|^{2+s}} \, dy \geq \lim_{\varepsilon \to 0} \frac{4\pi(\pi - \theta)}{s\varepsilon^s}.$$

Since $\theta \in (0, \pi]$, this implies $\theta = \pi$ and thus E is flat.

□

5. Extension to cones of finite index

First, let us recall the definition of finite Morse index for smooth s-minimal surfaces as introduced in [CFS23] or [FS24].

Definition 5.1. *Let E be an s-minimal surface in $\mathbb{R}^n$, and assume that ∂E is C^2 in some open set Ω. Then, E is said to have Morse index at most m in Ω if for every $(m+1)$ vector fields $X_1, \dots, X_{m+1}$ with compact support in Ω, there exists coefficients $a_1, \dots, a_{m+1}$ such that $a_1^2 + \dots + a_{m+1}^2 = 1$ and*

$$\delta^2 \mathrm{Per}_s(E; \Omega)[a_1 X_1 + \dots + a_{m+1} X_{m+1}] \geq 0.$$

In this section, we show that our classification for s-minimal cones stable in $\mathbb{R}^2 \setminus \{0\}$ implies the classification of cones with finite Morse index in $\mathbb{R}^2 \setminus \{0\}$. However, to establish this implication, we must verify that any regular cone $E \subset \mathbb{R}^n$ stationary for the s-perimeter and with finite Morse index in $\mathbb{R}^n \setminus \{0\}$ is stable in $\mathbb{R}^n \setminus \{0\}$.

Proposition 5.2. *Let $s \in (0,1)$ and $E \subset \mathbb{R}^n$ be cone with C^2 boundary in $\mathbb{R}^n \setminus \{0\}$, stationary for the s-perimeter, and with finite Morse index in $\mathbb{R}^n \setminus \{0\}$. Then E is stable in $\mathbb{R}^n \setminus \{0\}$.*

In the classical case of the perimeter (formally $s = 1$), this property follows easily by a scaling argument; if E were unstable in $\mathbb{R}^n \setminus \{0\}$, one could construct infinitely many disjoint scaled copies of an unstable variation, contradicting the finite index assumption. The fractional setting presents additional difficulty since the nonlocal interactions between different scales prevent such a direct argument, as functions with disjoint support are not orthogonal for the H^s scalar product.

This kind of result has been previously established in [CFS23] for blow-ups of s-minimal surfaces arising as limits of the fractional Allen-Cahn equation. Our proof follows a similar strategy and is essentially already contained in the union of [CFS23] and [FS24]. Nevertheless, since the result as needed in this work is never stated explicitly nor proved, we provide a proof in this section.

Lemma 5.3. *Let $E \subset \mathbb{R}^n$ be a set stationary for the s-perimeter, and with finite Morse index m in Ω. Assume also that $\Sigma = \partial E$ is C^2 in Ω. Let $X_1, X_2, ..., X_{m+1}$ be smooth vector fields on Ω with disjoint compact supports $A_1, ..., A_{m+1} \subset \Omega$, and denote $D_{k\ell} := \mathrm{dist}(A_k, A_\ell)$. For $1 \leq i < \ell \leq m+1$, fix positive weights $\lambda_{i\ell} > 0$. Then, for at least one of the i (depending on E) we have that*

$$\delta^2 \mathrm{Per}_s(E)[X_i] \geq -C \|X_i\|_{L^\infty}^2 \mathrm{diam}(A_i)^{2(n-1)} \left(\sum_{\ell < i} \frac{1}{\lambda_{\ell i}} D_{i\ell}^{-(n+s)} + \sum_{\ell > i} \lambda_{i\ell} D_{i\ell}^{-(n+s)} \right),$$

for some constant $C = C(n, s, m) > 0$.

Proof. The statement is a more precise version of Lemma 5.8 in [FS24], and the proof proceeds similarly. Using the second variation formula, we compute the second variation of E for linear combinations of $m+1$ vector fields X_i , supported each in the corresponding A_i. We denote by $\xi_i := X_i \cdot \nu_\Sigma$ the scalar normal component of these vector fields.

By the second variation formula of Theorem 2.6 we have

$$\begin{aligned}
&\delta^2 \mathrm{Per}_s(E)[a_1 X_1 + a_2 X_2 + \ldots + a_{m+1} X_{m+1}] \\
&\qquad = a_1^2 \delta^2 \mathrm{Per}_s(E)[X_1] + \ldots + a_{m+1}^2 \delta^2 \mathrm{Per}_s(E)[X_{m+1}] \\
&\qquad + 2a_1 a_2 \iint_{(\Sigma\cap A_1)\times(\Sigma\cap A_2)} \frac{(\xi_1(x)-\xi_1(x))(\xi_2(y)-\xi_2(y))}{|x-y|^{n+s}} d\sigma(x) d\sigma(y) \\
&\qquad + \ldots \\
&\qquad + 2a_m a_{m+1} \iint_{(\Sigma\cap A_m)\times(\Sigma\cap A_{m+1})} \frac{(\xi_m(x)-\xi_m(x))(\xi_{m+1}(y)-\xi_{m+1}(y))}{|x-y|^{n+s}} d\sigma(x) d\sigma(y).
\end{aligned}$$

Recall that the supports of ξ_i and ξ_j are the disjoint subsets of A_i and A_j, respectively. Then, the term containing the double integral over $A_i \times A_j$ with $i < j$ can be bounded as

$$\begin{aligned}
&2a_i a_j \iint_{(\Sigma\cap A_i)\times(\Sigma\cap A_j)} \frac{(\xi_i(x)-\xi_i(x))(\xi_j(y)-\xi_j(y))}{|x-y|^{n+s}} d\sigma(x) d\sigma(y) \\
&\qquad\qquad = -2a_i a_j \iint_{(\Sigma\cap A_i)\times(\Sigma\cap A_j)} \frac{\xi_i(x)\xi_j(y)}{|x-y|^{n+s}} d\sigma(x) d\sigma(y) \\
&\qquad\qquad \le 2|a_i a_j| D_{ij}^{-(n+s)} \|\xi_i\|_{L^1(\Sigma\cap A_i)} \|\xi_j\|_{L^1(\Sigma\cap A_j)} \\
&\qquad\qquad \le \lambda_{ij} a_i^2 D_{ij}^{-(n+s)} \|\xi_i\|^2_{L^1(\Sigma\cap A_i)} + \frac{1}{\lambda_{ij}} a_j^2 D_{ij}^{-(n+s)} \|\xi_j\|^2_{L^1(\Sigma\cap A_j)},
\end{aligned}$$

where we have used Young's inequality in the last line. Substituting this into the second variation expression above gives

$$\begin{aligned}
&\delta^2 \mathrm{Per}_s(E)[a_1 X_1 + a_2 X_2 + \ldots + a_{m+1} X_{m+1}] \\
&\qquad \le \sum_{i=1}^{m+1} a_i^2 \left[\delta^2 \mathrm{Per}_s(E)[X_i] + \|\xi_i\|^2_{L^1(\Sigma\cap A_i)} \left(\sum_{j<i} \frac{1}{\lambda_{ji}} D_{ij}^{-(n+s)} + \sum_{j>i} \lambda_{ij} D_{ij}^{-(n+s)} \right) \right].
\end{aligned}$$

The condition that the Morse index is at most m implies that the expression cannot be < 0 for all $(a_1, \ldots, a_{m+1}) \neq 0$. Hence, we find that there must exist some i such that

$$\delta^2 \mathrm{Per}_s(E)[X_i] \ge -\|\xi_i\|^2_{L^1(\Sigma\cap A_i)} \left(\sum_{j<i} \frac{1}{\lambda_{ji}} D_{ij}^{-(n+s)} + \sum_{j>i} \lambda_{ij} D_{ij}^{-(n+s)} \right)$$

holds for all $\xi_i \in C_c^1(\Sigma \cap A_i)$. Moreover, for $z \in A_i$, we have

$$\begin{aligned}
\|\xi_i\|^2_{L^1(\Sigma\cap A_i)} &= \left(\int_{A_i\cap\Sigma} X_i \cdot \nu \, d\sigma \right)^2 \le \|X_i\|^2_{L^\infty} \mathcal{H}^{n-1}(A_i \cap \Sigma)^2 \\
&\le \|X_i\|^2_{L^\infty} \mathrm{Per}(E; B_{\mathrm{diam}(A_i)}(z))^2 \le C \|X_i\|^2_{L^\infty} \mathrm{diam}(A_i)^{2(n-1)}.
\end{aligned}$$

Putting everything together concludes the proof. □

Recall also the following result, which is [CFS23, Lemma 4.15].

Lemma 5.4. *Let $s \in (0,1)$ and $E \subset \mathbb{R}^n$ be a cone with $\mathrm{Per}_s(E; B_1(0)) < +\infty$. Assume that E is stationary for the s-perimeter and that it satisfies the property in Lemma 5.3. Then E is stable in $\mathbb{R}^n \setminus \{0\}$.*

With this, we can easily deduce Proposition 5.2.

Proof of Proposition 5.2. Since E has finite Morse index, it satisfies the conclusion of Lemma 5.3. Moreover, by [FS24, Theorem 5.4] and a simple covering argument we have that

$$\mathrm{Per}_s(E; B_1(0)) < +\infty.$$

From here, the result follows by Lemma 5.4. □

6. Appendix

Here we give a direct proof that

$$\mathbf{X} = \{xy > 0\} \subset \mathbb{R}^2$$

is unstable for variations compactly supported in $\mathbb{R}^2 \setminus \{0\}$. First, note that by symmetry, it is clear that $\mathbf{X}$ satisfies (2) at every $x \in \partial\mathbf{X}$ and for every $s \in (0,1)$. Hence, since $\mathbf{X}$ is smooth away from the origin, $\mathbf{X}$ is an s-minimal surface for every $s \in (0,1)$.

Proposition 6.1. *There exists $s_\circ \in (0, 1/2)$ such that, for $s \in (0, s_\circ)$, $\boldsymbol{X}$ is unstable in $\mathbb{R}^2 \setminus \{0\}$.*

Proof. Let $L_x := \{(x,0) \in \mathbb{R}^2 : x > 0\}$ and $L_y := \{(0,y) \in \mathbb{R}^2 : y > 0\}$. Choose a radial test function $\varphi = \varphi(|x|) \in C^2_c(\mathbb{R}^2 \setminus \{0\})$. With a little abuse of notation, we still denote by φ its trace on the lines in $\partial\mathbf{X}$.

On the one hand

$$\begin{aligned}\iint_{\partial\mathbf{X}\times\partial\mathbf{X}} &\frac{|\nu(x)-\nu(y)|^2}{|x-y|^{2+s}}\varphi(|x|)^2\, d\sigma(x)d\sigma(y)\\ &= 4\int_0^\infty\!\!\int_{-\infty}^0 \frac{2^2}{|x-y|^{2+s}}\varphi(x)^2 dxdy + 8\int_{L_x}\int_{L_y}\frac{(\sqrt{2})^2}{|x-y|^{2+s}}\varphi(|x|)^2 d\sigma(x)d\sigma(y)\\ &= 16\int_0^\infty\!\!\int_{-\infty}^0 \frac{\varphi(x)^2}{|x-y|^{2+s}}\, dxdy + 16\int_{L_x}\int_{L_y}\frac{\varphi(|x|)^2}{|x-y|^{2+s}}\, d\sigma(x)d\sigma(y).\end{aligned}$$

Note that

$$\begin{aligned}\int_{L_x}\int_{L_y}\frac{\varphi(|x|)^2}{|x-y|^{2+s}}\, d\sigma(x)d\sigma(y) &= \int_{L_x}\varphi(|x|)^2\, d\sigma(x)\int_{L_y}\frac{d\sigma(y)}{(x^2+y^2)^{\frac{2+s}{2}}}\\ &= \int_0^\infty \varphi(x)^2\left(\frac{1}{x^{1+s}}\int_0^\infty\frac{dt}{(1+t^2)^{\frac{2+s}{2}}}\right)dx\\ &= \frac{\sqrt{\pi}\,\Gamma\left(\frac{1+s}{2}\right)}{2\Gamma\left(\frac{2+s}{2}\right)}\int_0^\infty\frac{\varphi(x)^2}{x^{1+s}}\, dx,\end{aligned}$$

and similarly

$$\int_0^\infty\!\!\int_{-\infty}^0 \frac{\varphi(x)^2}{|x-y|^{2+s}}\, dxdy = \frac{1}{1+s}\int_0^\infty\frac{\varphi(x)^2}{x^{1+s}}\, dx.$$

Hence

$$\iint_{\partial\mathbf{X}\times\partial\mathbf{X}} \frac{|\nu(x)-\nu(y)|^2}{|x-y|^{2+s}}\varphi^2(x)\, d\sigma(x)d\sigma(y) = 16\left(\frac{1}{1+s}+\frac{\sqrt{\pi}\,\Gamma\left(\frac{1+s}{2}\right)}{2\Gamma\left(\frac{2+s}{2}\right)}\right)\int_0^\infty\frac{\varphi(x)^2}{x^{1+s}}\, dx. \tag{8}$$

On the other hand, for the Sobolev part in the stability inequality

$$\iint_{\partial\mathbf{X}\times\partial\mathbf{X}} \frac{|\varphi(|x|)-\varphi(|y|)|^2}{|x-y|^{2+s}}\, d\sigma(x)d\sigma(y) \le 100\int_0^\infty\!\!\int_0^\infty\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{2+s}}\, dxdy. \tag{9}$$

Now, we use the fact that Hardy's inequality is saturated. Choose $\varphi(x) = \xi(|x|)$ where ξ saturates the Hardy's inequality as in (1). Applying (8) and (9) with $\varphi(x) = \xi(|x|)$ we obtain

$$\begin{aligned}\iint_{\partial\mathbf{X}\times\partial\mathbf{X}} \frac{|\nu(x)-\nu(y)|^2}{|x-y|^{2+s}}\xi^2(x)\,d\sigma(x)d\sigma(y) &= 16\left(\frac{1}{1+s}+\frac{\sqrt{\pi}\,\Gamma\left(\frac{1+s}{2}\right)}{2\Gamma\left(\frac{2+s}{2}\right)}\right)\int_0^\infty \frac{\xi(x)^2}{x^{1+s}}\,dx \\ &\overset{(1)}{\geq} \frac{16}{Cs^2}\left(\frac{1}{1+s}+\frac{\sqrt{\pi}\,\Gamma\left(\frac{1+s}{2}\right)}{2\Gamma\left(\frac{2+s}{2}\right)}\right)\int_0^\infty\int_0^\infty \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{2+s}}\,dxdy \\ &\geq \frac{16}{100Cs^2}\left(\frac{1}{1+s}+\frac{\sqrt{\pi}\,\Gamma\left(\frac{1+s}{2}\right)}{2\Gamma\left(\frac{2+s}{2}\right)}\right)\iint_{\partial\mathbf{X}\times\partial\mathbf{X}} \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{2+s}}\,d\sigma(x)d\sigma(y).\end{aligned}$$

Thus, in order to contradict stability, it is sufficient that

$$\frac{16}{100Cs^2}\left(\frac{1}{1+s}+\frac{\sqrt{\pi}\,\Gamma\left(\frac{1+s}{2}\right)}{2\Gamma\left(\frac{2+s}{2}\right)}\right) \geq 1,$$

which is clearly the case if s is sufficiently small. Hence, $\mathbf{X}$ is unstable in this range. □

Acknowledgements. The author is extremely grateful to Enric Florit-Simon for many valuable comments on a preliminary version of this work.

References

[CCS20] Xavier Cabré, Eleonora Cinti, and Joaquim Serra. Stable s-minimal cones in $\mathbb{R}^3$ are flat for $s \sim 1$. *J. Reine Angew. Math.*, 764:157–180, 2020. 1, 2, 4

[CCS21] Xavier Cabré, Eleonora Cinti, and Joaquim Serra. Stable solutions to the fractional allen-cahn equation in the nonlocal perimeter regime. 2021. 4

[CDSV23] H. Chan, S. Dipierro, J. Serra, and E. Valdinoci. Nonlocal approximation of minimal surfaces: optimal estimates from stability. *arXiv:2308.06328*, 2023. 1

[CFS23] M. Caselli, E. Florit, and J. Serra. Yau's conjecture for nonlocal minimal surfaces. *arXiv:2306.07100*, 2023. 4, 5, 8, 9

[CRS10] L. Caffarelli, J.-M. Roquejoffre, and O. Savin. Nonlocal minimal surfaces. *Comm. Pure Appl. Math.*, 63(9):1111–1144, 2010. 3

[CSV19] Eleonora Cinti, Joaquim Serra, and Enrico Valdinoci. Quantitative flatness results and BV-estimates for stable nonlocal minimal surfaces. *J. Differential Geom.*, 112(3):447–504, 2019. 4, 5

[DdPW18] Juan Dávila, Manuel del Pino, and Juncheng Wei. Nonlocal s-minimal surfaces and Lawson cones. *J. Differential Geom.*, 109(1):111–175, 2018. 1, 3

[FFM+15] A. Figalli, N. Fusco, F. Maggi, V. Millot, and M. Morini. Isoperimetry and stability properties of balls with respect to nonlocal energies. *Comm. Math. Phys.*, 336(1):441–507, 2015. 3

[FS08] Rupert L. Frank and Robert Seiringer. Non-linear ground state representations and sharp Hardy inequalities. *J. Funct. Anal.*, 255(12):3407–3430, 2008. 2

[FS24] E. Florit-Simon. Weyl law and convergence in the classical limit for min-max nonlocal minimal surfaces. *arXiv:2406.12162*, 2024. 3, 4, 8, 10

[Jac24] Thompson Jack. Density estimates and the fractional sobolev inequality for sets of zero s-mean curvature. *arXiv:2406.04618*, 2024. 4

[SV13a] Ovidiu Savin and Enrico Valdinoci. Regularity of nonlocal minimal cones in dimension 2. *Calc. Var. Partial Differential Equations*, 48(1-2):33–39, 2013. 1

[SV13b] Ovidiu Savin and Enrico Valdinoci. Some monotonicity results for minimizers in the calculus of variations. *J. Funct. Anal.*, 264(10):2469–2496, 2013. 1

(M. Caselli) Scuola Normale Superiore
Piazza dei Cavalieri 7, 56126 Pisa, Italy
Email address: michele.caselli@sns.it