STRUCTURE OF THE FREE INTERFACES NEAR TRIPLE JUNCTION SINGULARITIES IN HARMONIC MAPS AND OPTIMAL PARTITION PROBLEMS

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Abstract. We consider energy-minimizing harmonic maps into trees and we prove the regularity of the singular part of the free interface near triple junction points. Precisely, by proving a new epiperimetric inequality, we show that around any point of frequency $3/2$, the free interface is composed of three $C^{1,\alpha}$-smooth $(d-1)$-dimensional manifolds (composed of points of frequency 1) with common $C^{1,\alpha}$-regular boundary (made of points of frequency $3/2$) that meet along this boundary at 120 degree angles. Our results also apply to spectral optimal partition problems for the Dirichlet eigenvalues.

1. Introduction

This paper is dedicated to the structure of the singular set of harmonic energy-minimizing maps $u : B_1 \to \Sigma$, defined on $B_1 \subset \mathbb{R}^d$ and with values in singular spaces Σ. We consider the model case in which the target space is of the form

$$\Sigma_N := \Big\{ X \in \mathbb{R}^N : X_i X_j = 0 \text{ for all } j \neq i \Big\} \tag{1.1}$$

endowed with the distance

$$\mathrm{d}_{\Sigma_N}(X,Y) := \begin{cases} |X_i - Y_i|, & \text{if } X_j = Y_j = 0 \text{ for all } j \neq i, \\ |X_i| + |Y_j|, & \text{if } X_i, Y_j > 0 \text{ for some } i \neq j. \end{cases}$$

By the works of Gromov-Schoen [GS92], Caffarelli-Lin [CL07, CL08], Tavares-Terracini [TT12] and Soave-Terracini [ST15] it is known that if

$$\mathcal{F}(u) = \{x \in B_1 \ : \ u(x) = 0\},$$

is the nodal set of a local minimizer $u = (u_1, \dots, u_N)$ of the energy

$$E(u, B_1) := \sum_{i=1}^N \int_{B_1} |\nabla u_i|^2 \,\mathrm{d}x\,,$$

then $\mathcal{F}(u)$ can be decomposed into regular and singular parts according to the values of the Almgren's frequency $\gamma(u,x)$ (see Section 2.2) as follows:

$$\mathrm{Reg}(u) = \{x \in \mathcal{F}(u) \ : \ \gamma(u,x) = 1\} \qquad \text{and} \qquad \mathrm{Sing}(u) = \{x \in \mathcal{F}(u) \ : \ \gamma(u,x) \geq 3/2\}.$$

By [GS92, CL08], in every dimension $d \geq 2$, the regular part $\mathrm{Reg}(u)$ is a smooth $(d-1)$-dimensional manifold, while the structure of the singular set was studied in [GS92] and [Alp20, Dee22], where it was shown that $\mathrm{Sing}(u)$ has Hausdorff dimension $d-2$ ([GS92]) and is $(d-2)$-rectifiable ([Alp20, Dee22]). In dimension $d = 2$, Conti, Terracini and Verzini showed in [CTV03] that the singular set is discrete and gave a complete description of the nodal set $\mathcal{F}(u)$ around any singular point (see also the recent paper [LMNS24] for the case of generic solutions).

In this paper we study the regularity of the lowest stratum of $\mathrm{Sing}(u)$, that is

$$\mathcal{F}_{3/2}(u) := \{x \in \mathcal{F}(u) \ : \ \gamma(u,x) = 3/2\}, \tag{1.2}$$

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and the structure of the entire free interface $\mathcal{F}(u)$ around singular points in $\mathcal{F}_{3/2}(u)$ in any dimension $d \geq 2$. Our main result is the following:

Theorem 1.1. *Let $u : B_1 \to \Sigma_N$ be an energy-minimizing map in $B_1 \subset \mathbb{R}^d$, for some $d \geq 2$ and $N \geq 1$. Then, the singular set of $\mathrm{Sing}(u)$ of the free interface $\mathcal{F}(u)$ can be decomposed as*

$$\mathrm{Sing}(u) = \mathcal{F}_{3/2}(u) \cup \Big\{ x \in \mathcal{F}(u) \ : \ \gamma(u,x) \geq 3/2 + \delta_d \Big\},$$

where $\delta_d > 0$ is dimensional constant. The set $\mathcal{F}_{3/2}(u)$ is an open subset of $\mathrm{Sing}(u)$ and, locally, a $(d-2)$-dimensional $C^{1,\alpha}$-smooth manifold. Moreover, around any $x \in \mathcal{F}_{3/2}(u)$, the nodal set $\mathcal{F}(u)$ consists of three $(d-1)$-manifolds Γ_{12}, Γ_{23}, Γ_{31} with common boundary $\mathcal{F}_{3/2}(u)$, which are all $C^{1,\alpha}$-regular up to $\mathcal{F}_{3/2}(u)$ and form 120 degree angles at $\mathcal{F}_{3/2}(u)$.

Results in the spirit of Theorem 1.1 exist in the context of area-minimizing currents, where the nodal set $\mathcal{F}(u)$ is replaced by the support of an area-minimizing current (mod $p = 3$); in this framework, description of the triple junction singularities is known thanks to the historical works from the 70s of Jean Taylor [Tay73, Tay76], for hypersurfaces in $\mathbb{R}^3$, and from the early 90s of Leon Simon [Sim93], in any dimension $d \geq 3$ (see also the lecture notes [Min24]); we also remark that the analysis for general co-dimension 1 area-minimizing currents mod p was completed only recently in [DHMS20], [DHM$^+$21] and [DHM$^+$22]. To our knowledge Theorem 1.1 is the first instance of a study of triple junction singularities in the contexts of free boundary problems and harmonic maps in dimension $d > 2$.

The main ingredient in the proof of Theorem 1.1 is the following epiperimetric inequality (Theorem 1.2) for the 3/2-Weiss energy

$$W_{\frac{3}{2}}(u) := \sum_{i=1}^{N} \int_{B_1} |\nabla u_i|^2 \,\mathrm{d}x - \frac{3}{2} \sum_{i=1}^{N} \int_{\partial B_1} u_i^2 \,\mathrm{d}S$$

near homogeneous triple junctions $Y(x) = Y(x_{d-1}, x_d)$ with 2-dimensional profile defined as

$$Y_i(r,\theta) = \begin{cases} r^{\frac{3}{2}} \left|\cos\left(\frac{3}{2}\theta\right)\right|, & \text{for } \pi - \frac{2\pi}{3}(i-1) \leq \theta \leq \pi - \frac{2\pi}{3} i, \\ 0, & \text{elsewhere}, \end{cases} \tag{1.3}$$

for $i = 1, 2, 3$.

Theorem 1.2 (Epiperimetric inequality). *There exists $\delta, \varepsilon, \tau \in (0,1)$ depending only on the dimension d such that the following holds. For any $N \in \mathbb{N}$ and any 3/2-homogeneous $c \in H^1(B_1; \Sigma_N) \cap C^{0,1}(B_1; \mathbb{R}^N)$ such that*

$$\sum_{i=1}^{3} \mathrm{d}_{\mathcal{H}}\left(\{c_i > 0\}, \{Y_i > 0\}\right) + \sum_{i=4}^{N} \mathrm{d}_{\mathcal{H}}\left(\{c_i > 0\}, \{x_{d-1} = x_d = 0\}\right) \leq \tau$$

and

$$\|\mathrm{d}_{\Sigma_N}(c, Y)\|_{H^1(B_1)} \leq \delta,$$

there exists $u \in H^1(B_1; \Sigma_N) \cap C^{0,1}(B_1; \mathbb{R}^N)$ such that $u = c$ on ∂B_1 and

$$W_{\frac{3}{2}}(u) \leq (1-\varepsilon) W_{\frac{3}{2}}(c).$$

The proof of Theorem 1.2 is obtained via a contradiction argument in the spirit of Weiss [Wei99] (see also [FS16, GPS16]). The epiperimetric inequality approach to the analysis of the singular points in the context of harmonic maps is, to our knowledge, new in this field.

1.1. **Two consequences of Theorem 1.1.** We now discuss two corollaries of Theorem 1.1 for harmonic maps with values in metric graphs and for spectral optimal partitions.

First of all we notice that an immediate corollary of Theorem 1.1 is that the same result holds for harmonic maps with values in a locally finite tree Σ. Indeed, by [GS92, Dee22], if $u : B_1 \to \Sigma$ is an energy-minimizing map defined in $B_1 \subset \mathbb{R}^d$ with values in a locally finite metric tree, and if p is a vertex of Σ, then the set $\mathcal{F}(u) := u^{-1}(\{p\})$ can be decomposed into a regular and a singular parts, $\mathrm{Reg}(u)$ and $\mathrm{Sing}(u)$, where $\mathrm{Reg}(u)$ is a a smooth $(d-1)$-manifold that consists of the points of order (frequency) $\mathrm{Ord}^u = 1$, while $\mathrm{Sing}(u)$ is $(d-2)$-rectifiable and consists of of points of order (frequency) $\mathrm{Ord}^u > 1$; furthermore, for every $x \in \mathcal{F}(u)$, there is a neighborhood $\Omega \subset B_1$ such that $u : \Omega \to \Sigma$ has values in a subtree Σ_N of the form (1.1). Thus, we have the following.

Theorem 1.3. *Let Σ be a metric tree which is locally finite and let $u : B_1 \to \Sigma$ be an energy-minimizing map in $B_1 \subset \mathbb{R}^d$, for some $d \geq 2$. Then, for every vertex p of Σ the singular set of $\mathcal{F}(u) := u^{-1}(\{p\})$ can be decomposed as*

$$\mathrm{Sing}(u) = \Big\{x \in \mathcal{F}(u) \ : \ \mathrm{Ord}^u(x) = 3/2\Big\} \cup \Big\{x \in \mathcal{F}(u) \ : \ \mathrm{Ord}^u(x) \geq 3/2 + \delta_d\Big\},$$

where $\delta_d > 0$ is a dimensional constant. The set $\mathcal{F}_{3/2}(u) := \{x \in \mathcal{F}(u) \ : \ \mathrm{Ord}^u(x) = 3/2\}$ is an open subset of $\mathrm{Sing}(u)$ and is locally a $(d-2)$-dimensional $C^{1,\alpha}$-smooth manifold. Moreover, around any $x \in \mathcal{F}_{3/2}(u)$, the set $\mathcal{F}(u)$ consists of three $(d-1)$-manifolds with common boundary $\mathcal{F}_{3/2}(u)$, which are all $C^{1,\alpha}$-regular up to $\mathcal{F}_{3/2}(u)$ and form 120 degree angles at $\mathcal{F}_{3/2}(u)$.

Another consequence concerns the regularity of the following optimal partition problem studied in [CL07], [CTV03, CTV05a] and [OV24]. Given a bounded open set $D \subset \mathbb{R}^d$, $d \geq 2$, and a natural number $N \geq 2$ we consider the optimal partition problem

$$\min\Big\{\sum_{j=1}^{N} \lambda_1(\Omega_j) \ : \ \Omega_j \ - \text{ open}, \ \Omega_j \subset D, \ \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j\Big\}, \tag{1.4}$$

where $\lambda_1(\Omega_j)$ is the first eigenvalue on Ω_j with Dirichlet boundary conditions on $\partial\Omega_j$. If the N-uple of disjoint sets $(\Omega_1, \dots, \Omega_N)$ is a minimizer of (1.4) and if $u = (u_1, \dots, u_N)$ is the vector of the normalized first eigenfunctions (each u_j is extended by zero outside Ω_j), then the free interface

$$\mathcal{F}(u) := D \cap \Big(\bigcup_{j=1}^{N} \partial\Omega_j\Big)$$

can be decomposed as a regular part and a singular part as follows (see [CL07, ST15, Alp20]): the regular part $\mathrm{Reg}(u)$ consists of all points $x \in \mathcal{F}(u)$ of frequency $\gamma(u, x) = 1$ and is a smooth manifold of dimension $d - 1$; the singular part $\mathrm{Sing}(u)$ is a closed $(d-2)$-rectifiable set composed of all the points of frequency $\gamma(u, x) \geq 3/2$. We have the following theorem.

Theorem 1.4. *Let $(\Omega_1, \dots, \Omega_N)$ be a minimizer of (1.4) and $u = (u_1, \dots, u_N)$ be the corresponding vector of the normalized first eigenfunctions. Then, the singular set of $\mathrm{Sing}(u)$ of the free interface $\mathcal{F}(u)$ can be decomposed as*

$$\mathrm{Sing}(u) = \Big\{x \in \mathcal{F}(u) \ : \ \gamma(u, x) = 3/2\Big\} \cup \Big\{x \in \mathcal{F}(u) \ : \ \gamma(u, x) \geq 3/2 + \delta_d\Big\},$$

where $\delta_d > 0$ is dimensional constant. The set $\mathcal{F}_{3/2}(u) := \{x \in \mathcal{F}(u) \ : \ \gamma(u, x) = 3/2\}$ is an open subset of $\mathrm{Sing}(u)$ and is locally a $(d-2)$-dimensional $C^{1,\alpha}$-smooth manifold. Moreover, around any $x \in \mathcal{F}_{3/2}(u)$, the set $\mathcal{F}(u)$ consists of three $(d-1)$-manifolds with common boundary $\mathcal{F}_{3/2}(u)$, which are all $C^{1,\alpha}$-regular up to $\mathcal{F}_{3/2}(u)$ and form 120 degree angles at $\mathcal{F}_{3/2}(u)$.

The rest of the paper is organized as follows. In Section 2 we fix the notation, the functional setting and we recall some known facts about segregated solutions. In Section 3, we classify the $3/2$-homogeneous solutions and we find, as a corollary, the shape of minimizers of a certain optimal partition problem on the sphere (see Corollary 3.6). Section 5 is devoted to the proof of our main tool, the epiperimetric inequality Theorem 1.2 and, finally, in Section 6 we conclude the proof of our main regularity result Theorem 1.1.

2. Preliminaries

In this section we introduce the functional setting and we define the main tools in our framework, such as monotonicity formulas.

For any $N \geq 1$, let Σ_N be as in (1.1). For any open set $D \subseteq \mathbb{R}^d$, we define the Sobolev space

$$H^1(D;\Sigma_N) = \left\{ u \in (H^1(D))^N \colon u_i u_j = 0 \text{ a.e. in } D \text{ for all } i \neq j \right\},$$

with the space $H^1_{\mathrm{loc}}(D;\Sigma_N)$ being analogously defined; we refer to [KS93] for the theory of Sobolev spaces in this setting. Now, to each open $D \subseteq \mathbb{R}^d$ and each function $u \in H^1(D;\Sigma_N)$ we associate the corresponding Dirichlet energy, defined as

$$E(u,D) := \sum_{i=1}^N \int_D |\nabla u_i|^2 \,\mathrm{d}x.$$

In the present paper, we work with certain non-negative stationary points of E, hence we first define the space of admissible functions as

$$\mathcal{A}(D;N) := \left\{ u \in H^1_{\mathrm{loc}}(D;\Sigma_N) \colon u_i \geq 0 \text{ for all } i = 1,\dots,N \right\}.$$

We consider two classes of "critical points". The first one is the class of minimizers: we say that $u \in H^1_{\mathrm{loc}}(D;\Sigma_N)$ is a *local minimizer*, and we write $u \in \mathcal{M}(D;N)$, if

$$E(u,\Omega) \leq E(v,\Omega)$$

for all open $\Omega \Subset D$ and all $v \in H^1(\Omega;\Sigma_N)$ such that $u - v \in H^1_0(\Omega;\Sigma_N)$. We also introduce the class $\mathcal{S}(D;N)$ of functions $u \in \mathcal{A}(D;N)$ satisfying

$$\begin{cases} -\Delta u_i \leq 0, & \text{in } D, \\ -\Delta\Big(u_i - \sum_{j\neq i} u_j\Big) \geq 0, & \text{in } D, \end{cases}$$

in the sense of distributions, for all $i = 1,\dots,N$. In particular, there holds $\mathcal{M}(D;N) \subseteq \mathcal{S}(D;N)$ (see e.g. [CTV05b, Theorem 5.1]).

We now recall the main basic properties of critical points $u \in \mathcal{S}(D;N)$ and energy-minimizers $u \in \mathcal{M}(D;N)$ and of the free boundary $\mathcal{F}(u)$. Most of the properties we recall were proved in a series of papers, namely [GS92, CL07] for the class $\mathcal{M}(D;N)$ and [CTV05b, TT12, ST15] for $\mathcal{S}(D;N)$; for what concerns the regularity of different classes of segregated solutions we refer to [NRS24, ST24b, ST24a].

2.1. **Lipschitz continuity.** If $u \in \mathcal{S}(D;N)$, then it is locally Lipschitz continuous, i.e. $u \in C^{0,1}_{\mathrm{loc}}(D;\mathbb{R}^N)$, which implies that the sets

$$\Omega_i^u := \{x \in D \colon u_i(x) > 0\}$$

are open and well-defined and that u_i are harmonic on Ω_i^u.

2.2. **Almgren and Weiss monotonicity formulas.** For any non-trivial $u \in \mathcal{S}(D;N)$ one can define the Almgren *frequency function* for any $x_0 \in D$ and any $r < \operatorname{dist}(x_0, \partial D)$ as

$$\mathcal{N}(u, x_0, r) := \frac{E(u, x_0, r)}{H(u, x_0, r)},$$

where

$$E(u, x_0, r) := \frac{1}{r^{d-2}} E(u, B_r(x_0)) = \frac{1}{r^{d-2}} \sum_{i=1}^{N} \int_{B_r(x_0)} |\nabla u_i|^2 \,\mathrm{d}x$$

is the scaled energy and

$$H(u, x_0, r) := \frac{1}{r^{d-1}} \sum_{i=1}^{N} \int_{\partial B_r(x_0)} u_i^2 \,\mathrm{d}S$$

is the scaled height function. It is known that for $u \in \mathcal{S}(D;N)$ the function $r \mapsto \mathcal{N}(u, x_0, r)$ is monotone non-decreasing and so the *frequency* of u at x_0

$$\gamma(u, x_0) := \lim_{r \to 0^+} \mathcal{N}(u, x_0, r)$$

is well-defined at any point $x_0 \in \mathcal{F}(u)$. As a consequence, one can derive unique continuation properties implying that $\mathcal{F}(u)$ has empty interior and that

$$\mathcal{F}(u) = \bigcup_{i=1}^{N} \partial\{x \in D \colon u_i(x) > 0\}.$$

Hence, we can also split the points of the free boundary in terms of their frequency; namely, for any $\gamma > 0$ we let

$$\mathcal{F}_\gamma(u) := \{x \in \mathcal{F}(u) \colon \gamma(u, x) = \gamma\}.$$

Finally, we point out that also the Weiss energy

$$W_\gamma(u, x_0, r) := \frac{H(u, x_0, r)}{r^{2\gamma}} \Big(\mathcal{N}(u, x_0, r) - \gamma\Big)$$

is monotone non-decreasing with respect to $r > 0$, for any $\gamma > 0$.

2.3. **Blow-up analysis.** Let $u \in \mathcal{S}(D;N)$, respectively $\mathcal{M}(D;N)$, and let $x_0 \in \mathcal{F}_\gamma(u)$, for some $\gamma \geq 1$. We define the *Almgren rescalings* as

$$u_{x_0,r}(x) := \frac{u(x_0 + rx)}{\sqrt{H(u, x_0, r)}}.$$

As a consequence of the Almgren monotonicity formula, by a standard procedure, we know that for any sequence $r_n \to 0$ there is a subsequence (still denoted by r_n) such that

$$u_{x_0,r_n} \to U \quad \text{uniformly in } B_R \text{ and in } H^1(B_R; \mathbb{R}^N)$$

as $n \to \infty$, for all $R > 0$, for some $U \colon \mathbb{R}^d \to \Sigma_N$ belonging to

$$\mathcal{S}_\gamma(\mathbb{R}^d; N) := \Big\{u \in \mathcal{S}(\mathbb{R}^d; N) \colon u \text{ is } \gamma\text{-homogeneous}\Big\},$$

respectively,

$$\mathcal{M}_\gamma(\mathbb{R}^d; N) := \Big\{u \in \mathcal{M}(\mathbb{R}^d; N) \colon u \text{ is } \gamma\text{-homogeneous}\Big\}.$$

3. Singular blow-ups of minimal frequency

The aim of this section is to classify homogeneous singular solutions with minimal frequency $3/2$, i.e. functions in $\mathcal{S}_{3/2}(\mathbb{R}^d; N)$.

3.1. **A topological lemma.** This section is dedicated to a topological lemma, which we use both in the classification of the $3/2$-homogeneous blow-ups (Proposition 3.5) and in the proof of the no-holes lemma (Lemma 6.6).

Definition 3.1. Let $\mathcal{M}$ be a smooth m-dimensional manifold and let $\mathcal{F} \subset \mathcal{M}$ be a C^1 submanifold of dimension $m-1$ (that is, for every $x_0 \in \mathcal{F}$ there is an open neighborhood $\omega \subset \mathcal{M}$ of x_0 and a C^1-diffeomorphism $\Phi : \omega \to B_1 \subset \mathbb{R}^d$ such that $\Phi(\mathcal{F}) = B_1 \cap \{x_m = 0\}$). For any $x, y \in \mathcal{M} \setminus \mathcal{F}$, we denote by $C(x \to y)$ the family of piecewise C^1 curves transversal to $\mathcal{F}$. Precisely, we say that $\ell \in \mathcal{C}(x \to y)$ if:

(1) $\ell\colon [0,1] \to \mathcal{M}$, $\ell(0) = x$ and $\ell(1) = y$;
(2) $\ell \in C([0,1]; \mathcal{M})$ and there are $k = k(\ell)$ points $0 = T_0 < T_1 < \cdots < T_k = 1$, such that: $\ell \in C^1((T_{j-1}, T_j); \mathcal{M})$ and $\ell' \neq 0$ on (T_{j-1}, T_j) for every $j = 1, \dots, k$, and
$$\ell(T_j) \notin \mathcal{F} \quad \text{for every} \quad j = 0, \dots, k;$$
(3) if $\ell(t) \in \mathcal{F}$ for some $t \in (0,1)$, then ℓ is transversal to $\mathcal{F}$ at $\ell(t)$, that is, $\ell'(t) \notin T_{\ell(t)}\mathcal{F}$, where $T_{\ell(t)}\mathcal{F}$ is the tangent space to $\mathcal{F}$ at $\ell(t)$.

Lemma 3.2. *Let $\mathcal{M}$ be a smooth manifold of dimension m. Suppose that $\mathcal{F}$ is an $(m-1)$-dimensional C^1 submanifold of $\mathcal{M}$ and suppose that $\mathcal{F}$ is a relatively closed subset of $\mathcal{M}$. Let $x, y \in \mathcal{M} \setminus \mathcal{F}$ and let $\ell \in \mathcal{C}(x \to y)$ be a curve in $\mathcal{M}$. Then, the set of intersection points $\mathcal{I}(\ell) := \{t \in (0,1) \ : \ \ell(t) \in \mathcal{F}\}$ is finite.*

Proof. Suppose that the set $\mathcal{I}$ contains an infinite sequence $(t_n)_{n \geq 1}$. Then, we can suppose that t_n converges to some $t_\infty \in (0,1)$. By continuity of γ and the fact that $\mathcal{F}$ is closed, we have $\gamma(t_\infty) \in \mathcal{I}$. But γ is transversal to $\mathcal{F}$ at t_∞. Thus, for $t \neq t_\infty$ in a neighborhood of t_∞, $\gamma(t) \notin \mathcal{F}$, which is a contradiction. □

Lemma 3.3. *Let $\mathcal{M}$ be a simply connected smooth manifold of dimension m. Suppose that $\mathcal{F} \subset \mathcal{M}$ is a relatively closed subset of $\mathcal{M}$ and a C^1-submanifold of dimension $m-1$. Suppose that $x \in \mathcal{M} \setminus \mathcal{F}$ and that $\ell : [0,1] \to \mathcal{M}$ is a closed curve in $\mathcal{C}(x \to x)$. Then the set $\mathcal{I}(\ell)$ has an even number of elements.*

Proof. Since $\mathcal{M}$ is simply connected, we can find a continuous homotopy $\Gamma : [0,1] \times [0,1] \to \mathcal{M}$ that deforms ℓ into the constant curve x. Precisely,
$$\Gamma(0, \cdot) = \ell(\cdot) \qquad \text{and} \qquad \Gamma(1, \cdot) \equiv x.$$
We proceed in several steps.
Step 1. Discrete homotopy. Since $\mathcal{F}$ is a C^1 embedded submanifold, we can find points (s_i, t_j) of the form
$$(s_i, t_j) = \left(\frac{i}{n}, \frac{j}{n}\right) \quad i = 0, \dots, n \ , \ j = 1, \dots, n,$$
such that for every set of "consecutive" 4 points:
$$Q_{i,j} := \Big\{X_{i,j} := \Gamma(s_i, t_j),\ X_{i+1,j} := \Gamma(s_{i+1}, t_j),\ X_{i,j+1} := \Gamma(s_i, t_{j+1}),\ X_{i+1,j+1} := \Gamma(s_{i+1}, t_{j+1})\Big\}$$
we have that either:

(Q1) $Q_{i,j}$ is contained in an open set $\omega_1 \subset \mathcal{M} \setminus \mathcal{F}$ diffeomorphic to a ball;
(Q2) $Q_{i,j}$ is contained in an open set $\omega_2 \subset \mathcal{M}$ for which there is a C^1-diffeomorphism $\Phi : \omega_2 \to B_1 \subset \mathbb{R}^m$ such that $\Phi(\mathcal{F}) = B_1 \cap \{x_m = 0\}$.

We next notice that by perturbing slightly each of the points $X_{i,j}$ we can construct another family of $n \times n$ points
$$\Big\{\widetilde{X}_{i,j} \in \mathcal{U} \setminus \mathcal{F} \ : \ 0 \leq i \leq n-1,\ 0 \leq j \leq n-1\Big\},$$
such that any set of the form
$$\widetilde{Q}_{i,j} := \Big\{\widetilde{X}_{i,j},\ \widetilde{X}_{i+1,j},\ \widetilde{X}_{i,j+1},\ \widetilde{X}_{i+1,j+1}\Big\},$$

satisfies (Q1) or (Q2) above. Moreover, we can also suppose that

$$\widetilde{X}(0,j) \in \ell([0,1]) \quad \text{for every} \quad j = 0, \dots, n;$$

$$\widetilde{X}(n,j) = x \quad \text{for every} \quad j = 0, \dots, n.$$

Step 2. Construction of a piecewise C^1-regular net with vertices $\widetilde{X}_{ij}$. In what follows we will use the following notation, for every $0 \le i, j \le n$

- ω_{ij} is the open set

$$\omega_{ij} := \begin{cases} \omega_1 & \text{if } (Q1) \text{ is verified for the set } \widetilde{Q}_{ij}\,; \\ \omega_2 & \text{if } (Q2) \text{ is verified for the set } \widetilde{Q}_{ij}\,. \end{cases}$$

- $\alpha_{ij} \in \mathcal{C}(\widetilde{X}_{i,j} \to \widetilde{X}_{i,j+1})$ is a curve lying in ω_{ij} (so α_{ij} does not cross $\mathcal{F}$);
- $\beta_{ij}^+ \in \mathcal{C}(\widetilde{X}_{i,j} \to \widetilde{X}_{i+1,j})$ is a curve lying in ω_{ij} and $\beta_{ij}^-(t) := \beta_{ij}^+(1-t)$.

Finally, for every $i = 1, \dots, n$, we set

$$\ell_i := \alpha_{i1} * \alpha_{i2} * \cdots * \alpha_{i(n-1)},$$

where $*$ denotes the usual concatenation of curves. We notice that, up to a reparametrization, we can choose the curves α_{0j}, $j = 1, \dots, n$, to be pieces of ℓ. Moreover, we can choose α_{nj}, $j = 1, \dots, n$, to be the constant curves $\alpha_{nj}(t) = x$ for all t. Thus, for all t, we have:

$$\ell_0(t) = \ell(t) \qquad \text{and} \qquad \ell_n(t) = x.$$

Step 3. The map M. In what follows, for any curve $\ell : [0,1] \to \mathcal{M}$ with $\ell \in \mathcal{C}(x \to y)$ for some $x, y \in \mathcal{M} \setminus \mathcal{F}$, we will denote by $M(\ell)$ the number of elements of $\mathcal{I}(\ell)$. The map M has the following properties:

(M1) Let $x, y, z \in \mathcal{M} \setminus \mathcal{F}$. If $\ell_1 \in \mathcal{C}(x \to y)$ and $\ell_2 \in \mathcal{C}(y \to z)$, then $\ell_1 * \ell_2 \in \mathcal{C}(x \to z)$ and

$$M(\ell_1 * \ell_2) = M(\ell_1) + M(\ell_2).$$

(M2) Let $x, y \in \mathcal{M} \setminus \mathcal{F}$. If $\ell_+ \in \mathcal{C}(x \to y)$ and ℓ_- is defined as

$$\ell_-(t) = \ell(1-t),$$

then $\ell_- \in \mathcal{C}(y \to x)$ and $M(\ell_+) = M(\ell_-)$.

(M3) Let $x, y, z, w \in \mathcal{M} \setminus \mathcal{F}$ be 4 points in a set ω_2 as in (Q2) and let the curves $\alpha, \beta, \gamma, \delta$ be as follows:

$$\alpha \in \mathcal{C}(x \to y)\ , \quad \beta \in \mathcal{C}(y \to z)\ , \quad \gamma \in \mathcal{C}(z \to w)\ , \quad \delta \in \mathcal{C}(x \to w)\ .$$

Then, we have that

$$M(\delta) = M(\alpha * \beta * \gamma) = M(\alpha) + M(\beta) + M(\gamma).$$

Step 4. Conclusion. Now, using the properties (M1), (M2) and (M3), we obtain that, for every pair of consecutive indices i and $i+1$ with $0 \le i < i+1 \le n$, the following holds:

$$\begin{aligned} M(\ell_{i+1}) = \sum_{j=0}^{n-1} M(\alpha_{(i+1)j}) &= \sum_{j=0}^{n-1} M(\beta_{ij}^- * \alpha_{ij} * \beta_{ij}^+) \\ &= \sum_{j=0}^{n-1} \Big(M(\beta_{ij}^-) + M(\alpha_{ij}) + M(\beta_{ij}^+) \Big) \\ &= \sum_{j=0}^{n-1} \Big(M(\alpha_{ij}) + 2M(\beta_{ij}^+) \Big) \\ &= M(\ell_i) + 2\sum_{j=0}^{n-1} M(\beta_{ij}^+), \end{aligned}$$

so that

$$(-1)^{M(\ell_{i+1})} = (-1)^{M(\ell_i)}.$$

Since i is arbitrary, this implies

$$(-1)^{M(\ell_n)} = (-1)^{M(\ell_0)} = (-1)^{M(\ell)}.$$

Finally, since ℓ_n is the constant curve $\ell_n(t) = x$ for every t, we get that $M(\ell_n) = 0$. This concludes the proof since $M(\ell)$ by definition is the number of elements of $\mathcal{I}(\ell)$. □

3.2. **Classification of the blow-ups of frequency** $3/2$. In this section we prove that any $3/2$-homogeneous function $u \in \mathcal{S}_{3/2}(\mathbb{R}^d; N)$ is of the form Y (see Proposition 3.5). We stress that even if this result is not explicitly stated in [ST15], the key elements (the dimension reduction argument and Lemma 3.4) are already contained in the proof of [ST15, Lemma 4.2]; here below we give the details since we need Proposition 3.5 in Theorem 1.1.

Lemma 3.4. *Let $d \geq 3$ and $N \geq 2$. Suppose that $u = (u_1, \dots, u_N) \in \mathcal{S}_\gamma(\mathbb{R}^d; N)$ is a non-trivial γ-homogeneous function. Suppose that all the free boundary points on the unit sphere are regular, that is*

$$\mathcal{F}(u) \equiv \mathcal{F}_1(u) \quad \text{on} \quad \partial B_1,$$

and that all the sets $\Omega_i^u \cap \partial B_1$, $i = 1, \dots, N$ are connected. Then, there is a function

$$\sigma : \{1, \dots, N\} \to \{-1, 1\},$$

such that the function $\sigma \cdot u$, given by

$$(\sigma \cdot u)(x) := \sum_{i=1}^{N} \sigma(i) u_i(x)$$

is harmonic in $\mathbb{R}^d$. In particular, γ is an integer.

Proof. Without loss of generality we can suppose that

$$\Omega_i^u \cap \partial B_1 \neq \emptyset \quad \text{for every} \quad i = 1, \dots, N.$$

We define the function σ as follows. We set $\sigma(1) = 1$ and we fix a point $x_0 \in \Omega_1^u \cap B_r$. For every point $x_1 \in \partial B_1 \setminus \mathcal{F}(u)$ we take a curve $\ell \in \mathcal{C}(x_0 \to x_1)$ (where $\mathcal{C}(x_0 \to x_1)$ is as in Definition 3.1 with $\mathcal{M} = \partial B_1$ and $\mathcal{F} = \mathcal{F}(u)$) and we define

$$\sigma(x_1) := (-1)^{M(\ell)},$$

$M(\ell)$ being the number of times ℓ crosses $\mathcal{F}(u)$:

$$M(\ell) := \sharp \Big\{ t \in [0, 1] \ : \ \ell(t) \in \mathcal{F}(u) \Big\}. \tag{3.1}$$

By Lemma 3.3 (applied to the sphere $\mathcal{M} = \partial B_1$ and $\mathcal{F} = \mathcal{F}(u)$) the value of $\sigma(x_1)$ does not depend on the choice of the curve ℓ. In order to prove the harmonicity of $\sigma \cdot u$ we notice that if $B_r(x_0)$ is a ball such that $\mathcal{F}(u) \cap B_r(x_0) = \partial\Omega_i \cap \partial\Omega_j \cap B_r(x_0)$ is a smooth graph in $B_r(x_0)$, then for every pair of points $x_i \in \Omega_i \cap B_r(x_0)$ and $x_j \in \Omega_j \cap B_r(x_0)$, we can find a curve in $\mathcal{C}(x_i \to x_j)$ that crosses $\mathcal{F}(u)$ only once. Thus, by construction $\sigma(x_i) = -\sigma(x_j)$ and so the function $\sigma \cdot u$ is harmonic across $\mathcal{F}(u)$. This concludes the proof of Lemma 3.4. □

Proposition 3.5. *Let $u \in \mathcal{S}_{3/2}(\mathbb{R}^d; N)$. Then, up to relabeling the components of u, we have that $u_j \equiv 0$ for $j \geq 4$, and, up to a rotation of the coordinate system, (u_1, u_2, u_3) is of the form $(u_1, u_2, u_3) = cY$ for some constant $c \in \mathbb{R}$ and $Y = (Y_1, Y_2, Y_3)$ as in (1.3).*

Proof. Let $u \in \mathcal{S}_{3/2}(\mathbb{R}^d; N)$. By Lemma 3.4, there is a point $x_0 \in (\mathcal{F}(u) \setminus \mathcal{F}_1(u)) \cap \partial B_1$. But then, the upper semicontinuity of $\gamma(u, \cdot)$ and the homogeneity of u give

$$\frac{3}{2} = \gamma(u, 0) \geq \lim_{t \to +\infty} \gamma(u, t x_0) = \gamma(u, x_0).$$

Since, by [ST15], $3/2$ is the minimal frequency, we get that

$$\frac{3}{2} = \gamma(u, 0) = \gamma(u, x_0),$$

which implies that u is invariant in the direction of x_0. Up to a rotation of the coordinate system, we can suppose that $x_0 = \vec{e}_1$. Then $\widetilde{u}(x_2, \dots, x_d) := u(0, x_2, \dots, x_d)$ is a $3/2$-homogeneous global solution in $\mathbb{R}^{d-1}$, i.e. it belongs to $\mathcal{S}_{3/2}(\mathbb{R}^{d-1}; N)$. By repeating this argument in every dimension $d-k$, $k \geq 1$, we finally get that, up to a rotation of the coordinate axes, we have

$$u(x_1, \dots, x_{d-1}, x_d) = w(x_{d-1}, x_d),$$

where w is a $3/2$-homogeneous solution in dimension 2. This gives the specific form of w and the fact that w has exactly three non-zero components. □

We notice that as a consequence of Proposition 3.5, we obtain the uniqueness of the minimizer (in any dimension) to the optimal 3-partition of the sphere for the min-max problem studied in [CTV05a, HHT09, HHT10, ST15].

Corollary 3.6. *The Y-configuration is the unique minimizer of*

$$\mathcal{L}_3 := \min\left\{ \max_{i\in\{1,2,3\}} \lambda_1(\omega_i) \colon \omega_i \subseteq \partial B_1 \text{ open, connected and s.t. } \omega_i \cap \omega_j = \emptyset \text{ for } i \neq j \right\},$$

where $\lambda_1(\omega_i)$ is the first eigenvalue of the spherical Dirichlet Laplacian on ω_i.

Proof. Let $(\omega_1, \omega_2, \omega_3)$ be a triple achieving the minimum (see for instance [HHT09]). We know that $\omega_i \subset \partial B_1$ is measurable and $\mathcal{H}^{d-1}(\omega_i) > 0$, for every i. Moreover, if ϕ_i is a first normalized eigenfunction on ω_i, then by [HHT09, Theorem 3.4], there are constants $a_i \geq 0$, $i = 1, 2, 3$, such that $\psi := (a_1\phi_1, a_2\phi_2, a_3\phi_3)$ satisfies

$$\begin{cases} -\Delta\psi_i \leq \mathcal{L}_3\psi_i, & \text{in } \partial B_1, \\ -\Delta\Big(\psi_i - \sum_{j\neq i}\psi_j\Big) \geq \mathcal{L}_3\Big(\psi_i - \sum_{j\neq i}\psi_j\Big), & \text{in } \partial B_1. \end{cases}$$

By [HHT09, Theorem 3.4] that $(a_1, a_2, a_3) \neq (0, 0, 0)$. We now claim that $a_i > 0$ for all i. Suppose by contradiction that $a_3 = 0$, so that also $\psi_3 = 0$. By, the equation above, we get that $-\Delta(\psi_1 - \psi_2) = \mathcal{L}_3\,(\psi_1 - \psi_2)$ on ∂B_1. By the unique continuation principle for eigenfunctions, we get that $\mathcal{H}^{d-1}(\{\psi_1 - \psi_2 = 0\}) = 0$, which leads to a contradiction since $\mathcal{H}^{d-1}(\omega_3) > 0$. Now, in view of [HHT09, Remark 3.14 (d)] and [ST15, Theorem 1.10], we know that $\mathcal{L}_3 = \lambda_1(\omega_i) = \frac{3}{2}\left(\frac{3}{2} + d - 2\right)$ for all i. Thus, the $3/2$ homogeneous extension u of ψ belongs to the class $\mathcal{S}_{3/2}(\mathbb{R}^d; 3)$, so the conclusion follows from Proposition 3.5. □

4. Linearization around triple junctions

We now introduce the linearized problem at points of frequency $3/2$, which plays a crucial role in the proof of the epiperimetric inequality. For $i = 1, 2, 3$, we set

$$\Omega_i := \left\{ (r\cos\theta, r\sin\theta) \colon r > 0 \text{ and } \theta \in \left(\pi - (i-1)\frac{2\pi}{3}, \pi - i\frac{2\pi}{3}\right) \right\},$$

so that $\mathbb{R}^2$ is the disjoint union

$$\mathbb{R}^2 = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup Y.$$

Moreover, we may identify Ω_i with its cylindrical extension $\mathbb{R}^{d-2} \times \Omega_i$. Now, the linearized problem around the triple junction solution $Y = (Y_1, Y_2, Y_3) \in \mathcal{S}_{3/2}(\mathbb{R}^2; 3)$, which we recall to be defined as

$$Y_i(r, \theta) = \begin{cases} r^{\frac{3}{2}} \left|\cos\left(\frac{3}{2}\theta\right)\right|, & \text{for } \pi - \frac{2\pi}{3}(i-1) \leq \theta \leq \pi - \frac{2\pi}{3}i, \\ 0, & \text{elsewhere,} \end{cases}$$

is the following:

$$\begin{cases} -\Delta w_k = 0, & \text{in } \Omega_k \text{ for every } k \in \{1,2,3\}, \\ w_i = -w_j, & \text{on } \partial\Omega_i \cap \partial\Omega_j, \text{ for every } i \neq j \in \{1,2,3\}, \\ \nabla w_i = -\nabla w_j & \text{on } \partial\Omega_i \cap \partial\Omega_j, \text{ for every } i \neq j \in \{1,2,3\}. \end{cases} \tag{4.1}$$

More precisely, we say that $w = (w_1, w_2, w_3)$ is a solution of (4.1) if

$$\begin{cases} w_i - w_j \in H^1_{\rm loc}(\Omega_{ij}), \\ -\Delta(w_i - w_h) = 0 \quad \text{in } \Omega_{ij}, \end{cases}$$

where

$$\Omega_{ij} := \Omega_i \cup \Omega_j \cup (\partial\Omega_i \cap \partial\Omega_j) \setminus \{0\}$$

for all $i, j \in \{1, 2, 3\}$, $i \neq j$.

We remark that the linearized problem can also be rephrased with the cut space as domain, drawing a parallel with the thin obstacle problem. Namely, if we let

$$\mathcal{P} := \{(x'', x_{d-1}, x_d) \in \mathbb{R}^d \colon x_{d-1} \leq 0,\ x_d = 0\}$$

and $\widehat{w} := -w_1 + w_2 - w_3$, then we have the following:

- $\widehat{w} \in H^1_{\rm loc}(\mathbb{R}^d \setminus \mathcal{P})$ solves

$$\begin{cases} -\Delta w = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{P}, \\ \lim\limits_{x_d \to 0^+} w + \lim\limits_{x_d \to 0^-} w = 0, & \text{on } \mathcal{P}, \\ \lim\limits_{x_d \to 0^+} \partial_{x_d} w + \lim\limits_{x_d \to 0^-} \partial_{x_d} w = 0, & \text{on } \mathcal{P}; \end{cases}$$

- we can write $\widehat{w} = w_{\rm e} + w_{\rm o}$, where $w_{\rm e}$ and $w_{\rm o}$ are, respectively, even and odd with respect to x_d:

$$w_{\rm e}(x', x_d) := \frac{w(x', x_d) + w(x', -x_d)}{2}, \qquad w_{\rm o}(x', x_d) := \frac{w(x', x_d) - w(x', -x_d)}{2};$$

- the even part $w_{\rm e}$ solves

$$\begin{cases} -\Delta w_{\rm e} = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{P}, \\ w_{\rm e} = 0, & \text{on } \mathcal{P}; \end{cases}$$

- the odd part $w_{\rm o}$ solves

$$\begin{cases} -\Delta w_{\rm o} = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{P}, \\ \partial_{x_d} w_{\rm o} = 0, & \text{on } \mathcal{P}. \end{cases}$$

A key observation is that, if we consider the even extension of the restriction of $w_{\rm o}$ to $\mathbb{R}^d_+$ and exchange x_{d-1} with $-x_{d-1}$, that is we let

$$\widetilde{w}(x'', x_{d-1}, x_d) := \begin{cases} w_{\rm o}(x'', -x_{d-1}, x_d), & \text{for } x_d > 0, \\ w_{\rm o}(x'', -x_{d-1}, -x_d), & \text{for } x_d \leq 0, \end{cases}$$

then one can see that $\widetilde{w}$ solves the same equation as the even part $w_{\rm e}$, namely

$$\begin{cases} -\Delta \widetilde{w} = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{P}, \\ \widetilde{w} = 0, & \text{on } \mathcal{P}. \end{cases}$$

On the other hand, this coincides with the linearized problem of the thin obstacle problem and so, for what concerns the classification of $3/2$-homogeneous solutions, we can appeal to the known results in this setting (see e.g. [GPS16, FS16]). Therefore, we proved the following.

Proposition 4.1. *Let $\widehat{w} \in H^1_{\rm loc}(\mathbb{R}^d \setminus \mathcal{P})$ be a $3/2$-homogeneous solution of*

$$\begin{cases} -\Delta w = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{P}, \\ \lim\limits_{x_d \to 0^+} w + \lim\limits_{x_d \to 0^-} w = 0, & \text{on } \mathcal{P}, \\ \lim\limits_{x_d \to 0^+} \partial_{x_d} w + \lim\limits_{x_d \to 0^-} \partial_{x_d} w = 0, & \text{on } \mathcal{P}. \end{cases}$$

Then $\widehat{w} = w_{\rm e} + w_{\rm o}$, where $w_{\rm e}$ is even with respect to x_d and takes the form

$$w_{\rm e} = a\widehat{Y}(x_{d-1}, x_d) + U_0(x_{d-1}, x_d) \sum_{i=1}^{d-2} a_i x_i$$

and $w_{\rm o}$ *is odd with respect to* x_d *and takes the form*

$$w_{\rm o} = b\widehat{Y}(-x_{d-1}, x_d) + U_0(-x_{d-1}, x_d)\sum_{i=1}^{d-2} b_i x_i \quad \text{for } x_d > 0,$$

where

$$\widehat{Y}(r,\theta) = r^{\frac{3}{2}}\cos\left(\frac{3}{2}\theta\right) = -Y_1 + Y_2 - Y_3 \quad \text{and} \quad U_0(r,\theta) = r^{\frac{1}{2}}\cos\left(\frac{\theta}{2}\right).$$

In particular, $w_{\rm e}$ *and* $w_{\rm o}$ *are* $3/2$*-homogeneous solutions of*

$$\begin{cases} -\Delta w_{\rm e} = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{P}, \\ w_{\rm e} = 0, & \text{on } \mathcal{P} \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta w_{\rm o} = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{P}, \\ \partial_{x_d} w_{\rm o} = 0, & \text{on } \mathcal{P}. \end{cases}$$

5. Epiperimetric inequality at points of frequency $3/2$

Lemma 5.1 (Distance on Σ_N)**.** *For any* $u, v \in H^1(B_1;\Sigma_N)\cap C^{0,1}(B_1;\mathbb{R}^N)$ *such that* $u_i, v_i \geq 0$ *in* B_1 *for all* $i = 1,\dots,N$*, we have*

$$\mathrm{d}_{\Sigma_N}(u,v) = \Big|v_i - u_i + \sum_{j\neq i} u_j\Big| = |v_i - \sigma_i \cdot u| \quad \text{in } \overline{\{v_i > 0\}},$$

Moreover, we also have

$$\mathrm{d}_{\Sigma_N}(u,v) = \sum_{j=1}^N |u_j - v_j| \quad \text{in } B_1.$$

Proof. Let $x \in \overline{\{v_i > 0\}}$. If $x \in \{u_i > 0\}$, then

$$\mathrm{d}_{\Sigma_N}(u,v)(x) = |v_i(x) - u_i(x)| = \Big|v_i(x) - u_i(x) + \sum_{j\neq i} u_j\Big| = \sum_{j=1}^N |v_j(x) - u_j(x)|,$$

while, if $x \in \overline{\{u_k > 0\}}$, with $k \neq i$, then

$$\mathrm{d}_{\Sigma_N}(u,v)(x) = v_i + u_k = \Big|v_i(x) - u_i(x) + \sum_{j\neq i} u_j(x)\Big| = \sum_{j=1}^N |v_j(x) - u_j(x)|.$$

If $u_k(x) = 0$ for all $k = 1,\dots,N$ the statement is trivial. □

Lemma 5.2. *Suppose that* $v \in \mathcal{S}_\gamma(\mathbb{R}^d;N)$*. Then, for every* $u \in C^{0,1}(B_1)$ *and every* $i = 1,\dots,N$*, we have that*

$$\int_{B_1\cap\{v_i>0\}} \nabla v_i \cdot \nabla u \,\mathrm{d}x = \gamma\int_{\partial B_1\cap\{v_i>0\}} v_i u \,\mathrm{d}S + \int_{B_1\cap\partial\{v_i>0\}} u\partial_\nu v_i \,\mathrm{d}S,$$

where ν *is the normal to* $\mathrm{Reg}(v)\cap\partial\{v_i>0\}$ *pointing outwards* $\{v_i > 0\}$*.*

Proof. Before starting, we first recall that we know some basic structure of the free boundary of v; more precisely, from [TT12] we know that $v \in C^{0,1}(B_1;\mathbb{R}^N)$ and that we can decompose

$$\mathcal{F}(v) = \bigcup_{i=1}^N \partial\{v_i > 0\} = \mathrm{Reg}(v) \cup \mathrm{Sing}(v),$$

where $\mathrm{Reg}(v)$ is the union of $(d-1)$-dimensional manifolds of class $C^{1,\alpha}$ and $\mathrm{Sing}(v)$ has zero $\mathcal{H}^{d-1}$ measure. Moreover,

$$|\nabla v_i| = |\nabla v_j| \quad \text{on } \mathrm{Reg}(v) \qquad \text{and} \qquad |\nabla v_i| = 0 \quad \text{on } \mathrm{Sing}(v). \tag{5.1}$$

Hence, for every $\varepsilon > 0$ we can find a finite family of balls $B_{r_k}(x_k)$ such that:

$$\mathrm{Sing}(v)\cap\overline{B}_1 \subset \bigcup_k B_{r_k}(x_k) \qquad \text{and} \qquad \sum_k r_k^{d-1} \leq \varepsilon.$$

For every ball we consider a smooth function $\phi_k \in C^\infty_c(\mathbb{R}^d)$ such that:

$$\phi_k \equiv 1 \ \text{ on } \ B_{r_k}(x_k), \qquad \phi_k \equiv 0 \ \text{ on } \ \mathbb{R}^d \setminus B_{2r_k}(x_k), \qquad |\nabla\phi_k| \leq 2r_k^{-1}.$$

and we set $\phi = \phi_\varepsilon := \sup_k \phi_k$. Then, since $u(1-\phi)$ is identically zero in a neighborhood of $\mathrm{Sing}(v)$, we have that

$$\int_{B_1\cap\{v_i>0\}} \nabla\big(u(1-\phi)\big)\cdot\nabla v_i = \gamma\int_{\partial B_1\cap\{v_i>0\}}(1-\phi)v_i u\,\mathrm{d}S + \int_{B_1\cap\partial\{v_i>0\}}(1-\phi)u\partial_{\boldsymbol{\nu}} v_i\,\mathrm{d}S.$$

Thus, in order to conclude the proof, it is sufficient to estimate the error term

$$e := \int_{B_1\cap\{v_i>0\}}(\phi\nabla u + u\nabla\phi)\cdot\nabla v_i + \gamma\int_{\partial B_1\cap\{v_i>0\}}\phi v_i u\,\mathrm{d}S + \int_{B_1\cap\partial\{v_i>0\}}\phi u\partial_{\boldsymbol{\nu}} v_i\,\mathrm{d}S.$$

By the definition of ϕ we have that

$$\begin{aligned}|e| &\le \sum_k\Big(\int_{B_{2r_k}(x_k)\cap B_1\cap\{v_i>0\}}|\phi_k||\nabla u||\nabla v_i| + \int_{B_{2r_k}(x_k)\cap B_1\cap\{v_i>0\}}|\nabla\phi_k||u||\nabla v_i| \\ &\qquad + \gamma\int_{B_{2r_k}(x_k)\cap\partial B_1\cap\{v_i>0\}}|\phi_k||v_i||u|\,\mathrm{d}S + \int_{B_{2r_k}(x_k)\cap B_1\cap\partial\{v_i>0\}}|\phi_k||u||\nabla v_i|\,\mathrm{d}S\Big) \\ &\le C\sum_k\left(r_k^d + r_k^d r_k^{-1} + \gamma r_k^{d-1} + r_k^{d-1}\right) \le C\sum_k r_k^{d-1} \le C\varepsilon,\end{aligned}$$

where the constant C depends on d, $\|v\|_{L^\infty}$, $\|\nabla v\|_{L^\infty}$, $\|u\|_{L^\infty}$, and $\|\nabla u\|_{L^\infty}$. ∎

Lemma 5.3. *For any $u\in H^1(B_1;\Sigma_N)$ and any $\gamma\ge 0$, we let*

$$W_\gamma(u) := \sum_{i=1}^N\int_{B_1}|\nabla u_i|^2\,\mathrm{d}x - \gamma\sum_{i=1}^N\int_{\partial B_1}u_i^2\,\mathrm{d}S.$$

Then, for any $u\in H^1(B_1;\Sigma_N)\cap C^{0,1}(B_1;\mathbb{R}^N)$ and $v\in\mathcal{S}_\gamma(\mathbb{R}^d;N)$ such that $u_i, v_i\ge 0$ for all $i=1,\dots,N$, there holds

$$W_\gamma(u) = W_\gamma(\mathrm{d}_{\Sigma_N}(u,v)) + \beta(u,v),$$

where

$$\beta(u,v) := 4\sum_{1\le i<j\le N}\int_{B_1\cap\partial\{v_i>0\}\cap\partial\{v_j>0\}}|\nabla v_i|\Big(\sum_{k\ne i,j}u_k\Big)\,\mathrm{d}S. \tag{5.2}$$

Proof. We prove the result by direct computations. In view of Lemma 5.1 we have that

$$\begin{aligned}W_\gamma(\mathrm{d}_{\Sigma_N}(u,v)) = \sum_{i=1}^N\Bigg[&\int_{B_1\cap\{v_i>0\}}\Big|\nabla\Big(v_i - u_i + \sum_{k\ne i}u_k\Big)\Big|^2\,\mathrm{d}x \\ &- \gamma\int_{\partial B_1\cap\{v_i>0\}}\Big|v_i - u_i + \sum_{k\ne i}u_k\Big|^2\,\mathrm{d}S\Bigg]\end{aligned}$$

and so, by explicit computations, we derive that

$$\begin{aligned}W_\gamma(\mathrm{d}_{\Sigma_N}(u,v)) &= W_\gamma(u) + W_\gamma(v) \\ &\quad - 2\sum_{i=1}^N\int_{B_1\cap\{v_i>0\}}\nabla v_i\cdot\nabla\Big(u_i - \sum_{k\ne i}u_k\Big)\,\mathrm{d}x + 2\gamma\int_{\partial B_1\cap\{v_i>0\}}v_i\Big(u_i - \sum_{k\ne i}u_k\Big)\,\mathrm{d}S.\end{aligned}$$

Now, in view of Lemma 5.2, (5.1) and by homogeneity, we have that

$$\begin{aligned}\sum_{i=1}^N \int_{B_1\cap\{v_i>0\}} \nabla v_i\cdot\nabla\Big(u_i-\sum_{k\neq i}u_k\Big)\,\mathrm{d}x &= \sum_{i=1}^N \int_{B_1\cap\partial\{v_i>0\}} \partial_{\boldsymbol\nu} v_i\Big(u_i-\sum_{k\neq i}u_k\Big)\,\mathrm{d}x\\ &\quad+\sum_{i=1}^N \int_{\partial B_1} \partial_{\boldsymbol\nu} v_i\Big(u_i-\sum_{k\neq i}u_k\Big)\,\mathrm{d}S\\ &= -\sum_{i=1}^N \int_{B_1\cap\partial\{v_i>0\}} |\nabla v_i|\Big(u_i-\sum_{k\neq i}u_k\Big)\,\mathrm{d}x\\ &\quad+\gamma\sum_{i=1}^N \int_{\partial B_1} v_i\Big(u_i-\sum_{k\neq i}u_k\Big)\,\mathrm{d}S\end{aligned}$$

Therefore, since also $W_\gamma(v)=0$ (again from Lemma 5.2), we have

$$\begin{aligned}W_\gamma(\mathrm{d}_{\Sigma_N}(u,v)) &= W_\gamma(u)+2\sum_{i=1}^N \int_{B_1\cap\partial\{v_i>0\}} |\nabla v_i|\Big(u_i-\sum_{k\neq i}u_k\Big)\,\mathrm{d}x\\ &= W_\gamma(u)+2\sum_{i=1}^N\sum_{\substack{j=1\\ j\neq i}}^N \int_{B_1\cap\partial\{v_i>0\}\cap\partial\{v_j>0\}} |\nabla v_i|\Big(u_i-\sum_{k\neq i}u_k\Big)\,\mathrm{d}x\\ &= W_\gamma(u)-4\sum_{1\le i<j\le N} \int_{B_1\cap\partial\{v_i>0\}\cap\partial\{v_j>0\}} |\nabla v_i|\Big(\sum_{k\neq i,j}u_k\Big)\,\mathrm{d}x,\end{aligned}$$

thus concluding the proof. □

Theorem 5.4. *There exists $\delta,\varepsilon,\tau\in(0,1)$ depending only on d such that the following holds. For any $N\in\mathbb{N}$ and any $3/2$-homogeneous $c\in H^1(B_1;\Sigma_N)\cap C^{0,1}(B_1;\mathbb{R}^N)$ such that $c_{n,i}\ge 0$ for all $i=1,\dots,N$,*

$$\sum_{i=1}^3 \mathrm{d}_{\mathcal H}\left(\{c_i>0\},\{Y_i>0\}\right)+\sum_{i=4}^N \mathrm{d}_{\mathcal H}\left(\{c_i>0\},\{x_{d-1}=x_d=0\}\right)\le\tau$$

and

$$\|\mathrm{d}_{\Sigma_N}(c,Y)\|_{H^1(B_1)}\le\delta,$$

there exists $u\in H^1(B_1;\Sigma_N)\cap C^{0,1}(B_1;\mathbb{R}^N)$ such that $u_i\ge 0$ for all i, $u=c$ on ∂B_1, and

$$W_{\frac32}(u)\le(1-\varepsilon)W_{\frac32}(c).$$

Proof. In the whole proof, with an abuse of notation, for the sake of simplicity, we write Ω_i in place of $\Omega_i\cap B_1$, for $i=1,2,3$.

We reason by contradiction, and assume that there exist sequences $\{\delta_n\}_n\subseteq\mathbb{R}_+$, $\{\varepsilon_n\}_n\subseteq\mathbb{R}_+$, $\{\tau_n\}_n\subseteq\mathbb{R}_+$, $\{N_n\}_n\subseteq\mathbb{N}$ and $\{c_n\}_n\subseteq H^1(B_1;\Sigma_{N_n})\cap C^{0,1}(B_1;\mathbb{R}^{N_n})$ such that

$$\begin{aligned}&c_{n,i}\ge 0\quad\text{for all } n\in\mathbb{N}\text{ and all } i=1,\dots,N_n,\\ &\delta_n,\varepsilon_n,\tau_n\to 0\quad\text{as } n\to\infty,\\ &\sum_{i=1}^3 \mathrm{d}_{\mathcal H}\left(\{c_{n,i}>0\},\{Y_i>0\}\right)+\sum_{i=4}^{N_n} \mathrm{d}_{\mathcal H}\left(\{c_i>0\},\{x_{d-1}=x_d=0\}\right)=\tau_n, \qquad (5.3)\\ &\left\|\mathrm{d}_{\Sigma_{N_n}}(c_n,Y)\right\|_{H^1(B_1)}=\delta_n\end{aligned}$$

and

$$(1-\varepsilon_n)W_{\frac32}(c_n)<W_{\frac32}(u) \qquad (5.4)$$

for all $u \in H^1(B_1;\Sigma_{N_n}) \cap C^{0,1}(B_1;\mathbb{R}^{N_n})$ such that $u = c_n$ on ∂B_1. Moreover, without loss of generality, we can assume that

$$\delta_n = \left\| \mathrm{d}_{\Sigma_{N_n}}(c_n, Y) \right\|_{H^1(B_1)} = \inf\left\{ \left\| \mathrm{d}_{\Sigma_{N_n}}(c_n, h)) \right\|_{H^1(B_1)} : h \in \mathcal{M}_{\frac{3}{2}}(\mathbb{R}^d; 3) \right\}.$$

Now, first of all, we apply Lemma 5.3 with $v = Y$ and divide both sides by δ_n^2, thus obtaining that

$$(1-\varepsilon_n)\left(W_{\frac{3}{2}}(\xi_n) + \frac{1}{\delta_n^2}\beta(c_n, Y) \right) < W_{\frac{3}{2}}\left(\frac{\mathrm{d}_{\Sigma_{N_n}}(u, Y)}{\delta_n} \right) + \frac{1}{\delta_n^2}\beta(u, Y),$$

for all $u \in H^1(B_1;\Sigma_{N_n}) \cap C^{0,1}(B_1;\mathbb{R}^{N_n})$ such that $u = c_n$ on ∂B_1, where $\xi_n := \mathrm{d}_{\Sigma_{N_n}}(c_n, Y)/\delta_n$. In other words,

$$(1-\varepsilon_n)\left(\int_{B_1} |\nabla \xi_n|^2 \,\mathrm{d}x + \frac{1}{\delta_n^2}\beta(c_n, Y) \right) < \int_{B_1} \left| \nabla \Big(\frac{\mathrm{d}_{\Sigma_{N_n}}(u, Y)}{\delta_n} \Big) \right|^2 \mathrm{d}x + \frac{1}{\delta_n^2}\beta(u, Y) + \frac{3\varepsilon_n}{2} \int_{\partial B_1} \xi_n^2 \,\mathrm{d}S.$$

We also let

$$w_{n,i} := \begin{cases} \dfrac{Y_i - c_{n,i}}{\delta_n} + \displaystyle\sum_{j \neq i} \frac{c_{n,j}}{\delta_n}, & \text{in } \Omega_i, \\ 0, & \text{in } B_1 \setminus \Omega_i, \end{cases}$$

for $i = 1, 2, 3$. Since

$$|w_{n,i}| = \frac{\mathrm{d}_{\Sigma_{N_n}}(c_n, Y)}{\delta_n} = \xi_n, \quad \text{in } \Omega_i,$$

see Lemma 5.1, in particular,

$$\sum_{i=1}^{3} \|w_{n,i}\|_{H^1(\Omega_i)}^2 = \|\xi_n\|_{H^1(B_1)}^2 = 1.$$

Therefore, there exist functions $w_i \in H^1(\Omega_i)$ such that, up to a subsequence,

$$w_{n,i} \rightharpoonup w_i \quad \text{weakly in } H^1(\Omega_i), \text{ as } n \to \infty,$$

for $i = 1, 2, 3$. Hence, we can rephrase the contradiction assumption as

$$(1-\varepsilon_n)\left(\sum_{i=1}^{3} \int_{\Omega_i} |\nabla w_{n,i}|^2 \,\mathrm{d}x + \frac{1}{\delta_n^2}\beta(c_n, Y) \right) < \int_{B_1} \left| \nabla \Big(\frac{\mathrm{d}_{\Sigma_{N_n}}(u, Y)}{\delta_n} \Big) \right|^2 \mathrm{d}x + \frac{1}{\delta_n^2}\beta(u, Y) + o(1), \tag{5.5}$$

as $n \to \infty$, for all $u \in H^1(B_1;\Sigma_{N_n}) \cap C^{0,1}(B_1;\mathbb{R}^{N_n})$ such that $u = c_n$ on ∂B_1, where the reminder term $o(1)$ is independent from u. We now proceed in the following steps, which, at the end, lead to a contradiction:

- **Step 1:** $\delta_n^{-2}\beta(c_n, Y)$ is uniformly bounded with respect to n;
- **Step 2:** $w = (w_1, w_2, w_3)$ is a solution of the linearized problem, see Section 4;
- **Step 3:** $w_i = 0$ for all $i = 1, 2, 3$;
- **Step 4:** $w_{n,i} \to 0$ strongly in $H^1(\Omega_i)$, as $n \to \infty$, for all $i = 1, 2, 3$.

Proof of **Step 1**. For any $n \in \mathbb{N}$, we define a suitable "competitor" $u_n \in H^1(B_1;\Sigma_{N_n})$ such that $u_n = c_n$ on ∂B_1 as follows. First, in an inner ball, we let $u_n = Y$, namely

$$u_n = Y \quad \text{in } B_{\frac{1}{2}}.$$

In the outer annulus $B_1 \setminus B_{\frac{1}{2}}$, we consider the interpolation between Y and c_n; namely, we first set

$$t = t_r := 2\left(r - \frac{1}{2} \right), \quad \text{with } r = |x|.$$

Now, we let

$$u_n(r,\theta) = r^{\frac{3}{2}} \left(t\, c_{n,i}(1,\theta) + (1-t) Y_i(1,\theta) \right) \vec{\boldsymbol{e}}_i$$

if $1/2 < r < 1$ and $Y_j(\theta) = c_{n,j}(1,\theta) = 0$ for all $j \neq i$ and, on the other hand, we let

$$u_n(r,\theta) = \begin{cases} r^{\frac{3}{2}}(Y_i(1,\theta) - t(Y_i(1,\theta) + c_{n,j}(1,\theta)))\,\vec{\boldsymbol{e}}_i, & \text{for } 0 < t < \frac{Y_i(1,\theta)}{Y_i(1,\theta)+c_{n,j}(1,\theta)}, \\ r^{\frac{3}{2}}(-Y_i(1,\theta) + t(Y_i(1,\theta) + c_{n,j}(1,\theta)))\,\vec{\boldsymbol{e}}_j, & \text{for } \frac{Y_i(1,\theta)}{Y_i(1,\theta)+c_{n,j}(1,\theta)} \leq t < 1, \end{cases}$$

if $1/2 < r < 1$ and $Y_i(\theta) > 0$ and $c_{n,j}(1,\theta) > 0$ for some $j \neq i$. If we now let

$$\widetilde{u}_{n,i} := \frac{Y_i - u_{n,i} + \sum_{j\neq i} u_{n,j}}{\delta_n} \quad \text{in } \Omega_i,$$

then there holds

$$|\widetilde{u}_{n,i}| = \frac{\mathrm{d}_{\Sigma_{N_n}}(u_n, Y)}{\delta_n} \quad \text{in } \Omega_i,$$

see Lemma 5.1, and one can easily check that

$$\widetilde{u}_{n,i} = t w_{n,i} \quad \text{in } \Omega_i \cap \left(B_1 \setminus B_{\frac{1}{2}}\right) \quad \text{and} \quad \widetilde{u}_{n,i} = 0 \quad \text{in } B_{\frac{1}{2}}.$$

Moreover, we have that

$$u_n = t c_n \quad \text{on } \{(Y_1, Y_2, Y_3) = (0,0,0)\}.$$

Therefore, from (5.5) we derive that

$$\frac{\beta((1-\varepsilon_n - t)c_n, Y)}{\delta_n^2} < (\varepsilon_n - 1)\sum_{i=1}^{3}\int_{\Omega_i} |\nabla w_{n,i}|^2 \,\mathrm{d}x + \sum_{i=1}^{3}\int_{\Omega_i} |\nabla(t w_{n,i})|^2 \,\mathrm{d}x + o(1), \tag{5.6}$$

as $n \to \infty$. On one hand, being $w_{n,i}$ bounded in $H^1(\Omega_i)$, the right hand side is uniformly bounded with respect to $n \in \mathbb{N}$, while, on the other hand, by direct computations we have

$$\beta((1-\varepsilon_n - t)c_n, Y) = (d+1)\int_0^1 r^d(1-\varepsilon_n - t)\,\mathrm{d}r\, \beta(c_n, Y). \tag{5.7}$$

Therefore since $t < 1$ in $B_1 \setminus B_{1/2}$, from (5.6) we conclude the proof of the step.
Proof of **Step 2**. We are going to prove that

$$-\Delta(w_i - w_j) = 0 \quad \text{in } \Omega_{ij}, \tag{5.8}$$

for all $i, j = 1, 2, 3$, $i \neq j$, where

$$\Omega_{ij} := (\Omega_i \cup \Omega_j \cup (\partial\Omega_i \cap \partial\Omega_j)) \cap B_1.$$

For the sake of simplicity, we prove it for $i = 1$ and $j = 2$. First of all, we fix any $x_0 \in \Omega_{12}$ and $r > 0$ in such a way that $\overline{B_{2r}(x_0)} \subseteq \Omega_{12}$. Moreover, for fixed $\tau > 0$, we let

$$C_\tau := \left\{(x'', x_{d-1}, x_d) \in \mathbb{R}^d \colon \sqrt{x_{d-1}^2 + x_d^2} < \tau\right\}$$

and we assume that $B_{2r}(x_0) \cap C_\tau = \emptyset$ and that $c_{n,j} = 0$ in $B_{2r}(x_0)$, for $j = 3, \ldots, N_n$. This is possible by choosing τ sufficiently small and n sufficiently large, in view of the Hausdorff convergence assumption (5.3). We now use (5.5) with $u \in H^1(B_1; \Sigma_{N_n})$ being such that $u = c_n$ in $B_1 \setminus B_{2r}(x_0)$ and $u_j = 0$ in $B_{2r}(x_0)$ for $j = 3, \ldots, N_n$. Since $c_{n,j} = u_{n,j} \equiv 0$ in $B_{2r}(x_0)$, we have that

$$\beta(u, Y) = \beta(c_n, Y),$$

which, in view of Step 1, gives

$$\frac{1}{\delta_n^2}\beta(u, Y) - \frac{1-\varepsilon_n}{\delta_n^2}\beta(c_n, Y) = \frac{\varepsilon_n}{\delta_n^2}\beta(c_n, Y) = o(1), \tag{5.9}$$

as $n \to +\infty$. Moreover, we observe that

$$\frac{\mathrm{d}_{\Sigma_{N_n}}(u_n, Y)}{\delta_n} = \frac{1}{\delta_n}\left|Y_i - c_{n,i} + \sum_{j\neq i} c_{n,j}\right| = w_{n,i} \quad \text{in } \Omega_i \setminus B_{2r}(x_0),$$

for $i = 1, 2, 3$. Now, from (5.5) and (5.9) we deduce that

$$\int_{\Omega_1\cap B_{2r}(x_0)} |\nabla w_{n,1}|^2\,\mathrm{d}x + \int_{\Omega_2\cap B_{2r}(x_0)} |\nabla w_{n,2}|^2\,\mathrm{d}x + o(1)$$
$$< \int_{\Omega_1\cap B_{2r}(x_0)} \left|\nabla\left(\frac{Y_1 - u_1 + u_2}{\delta_n}\right)\right|^2 \mathrm{d}x + \int_{\Omega_2\cap B_{2r}(x_0)} \left|\nabla\left(\frac{Y_2 - u_2 + u_1}{\delta_n}\right)\right|^2 \mathrm{d}x,$$

for all $u \in H^1(B_1; \Sigma_2)$ such that $u = c_n$ on $\partial B_{2r}(x_0)$, where the reminder term does not depend on u. In particular, if we let

$$W_n := w_{n,1} - w_{n,2} \quad \text{and} \quad U := u_1 - u_2$$

this implies that

$$\int_{B_{2r}(x_0)} |\nabla W_n|^2\,\mathrm{d}x < \int_{B_{2r}(x_0)} \left|\nabla\left(\frac{Y_1 - Y_2 - U}{\delta_n}\right)\right|^2 \mathrm{d}x + o(1) \tag{5.10}$$

for all $U \in H^1(B_{2r}(x_0))$ such that $U = c_{n,1} - c_{n,2}$ on $\partial B_{2r}(x_0)$. One can also see that, by definition,

$$w_{n,1} = -w_{n,2} \quad \text{and} \quad \nabla w_{n,1} = -\nabla w_{n,2} \quad \text{on } (\partial\Omega_1 \cap \partial\Omega_2) \cap B_{2r}(x_0),$$

so that $W_n \in H^1(B_{2r}(x_0)) \cap C^{0,1}(B_{2r}(x_0))$; moreover, its H^1 norm is bounded and so, up to a subsequence,

$$W_n \rightharpoonup w_1 - w_2 \quad \text{weakly in } H^1(B_{2r}(x_0)), \text{ as } n \to \infty.$$

At this point, we define

$$U = \eta(Y_1 - Y_2 - \delta_n\phi) + (1-\eta)(c_{n,1} - c_{n,2}),$$

where $\eta \in C_c^\infty(B_{2r}(x_0))$ is such that $\eta = 1$ in $B_r(x_0)$ and $\phi \in H^1(B_{2r}(x_0))$. Plugging this choice into (5.10), we get

$$\int_{B_{2r}(x_0)} |\nabla W_n|^2\,\mathrm{d}x < \int_{B_{2r}(x_0)} |\nabla((1-\eta)W_n + \eta\phi)|^2\,\mathrm{d}x + o(1).$$

By direct computations, one can see that

$$\int_{B_{2r}(x_0)} (1 - (1-\eta)^2)|\nabla W_n|^2\,\mathrm{d}x + o(1)$$
$$< \int_{B_{2r}(x_0)} \Big(W_n^2|\nabla\eta|^2 - 2(1-\eta)W_n\nabla W_n\cdot\nabla\eta + |\nabla(\eta\phi)|^2 + 2\nabla((1-\eta)W_n)\cdot\nabla(\eta\phi)\Big)\,\mathrm{d}x$$

and, since we can pass now to the limit as $n \to \infty$ in the inequality above, after rearranging the terms back we obtain that

$$\int_{B_{2r}(x_0)} |\nabla(w_1 - w_2)|^2\,\mathrm{d}x \le \int_{B_{2r}(x_0)} |\nabla((1-\eta)(w_1 - w_2) + \eta\phi)|^2\,\mathrm{d}x.$$

Finally, we take $\phi = w_1 - w_2$ in $B_{2r}(x_0) \setminus B_r(x_0)$, so that

$$\int_{B_r(x_0)} |\nabla(w_1 - w_2)|^2\,\mathrm{d}x \le \int_{B_r(x_0)} |\nabla\phi|^2\,\mathrm{d}x.$$

Hence, we proved that $\Delta(w_1 - w_2) = 0$ in $B_r(x_0)$. Up to moving the ball, we get (5.8).

Proof of **Step 3**. We use the classification of the $3/2$-homogeneous solutions of the linearized problem, namely Proposition 4.1. More precisely, if we denote

$$\widehat{w} := -w_1 + w_2 - w_3,$$

which, we recall, belongs to $H^1(B_1 \setminus \mathcal{P})$, then in view of Proposition 4.1 we have that

$$\widehat{w} = w_{\mathrm{e}} + w_{\mathrm{o}},$$

where $w_{\rm e}$ is even with respect to x_d and is of the form

$$w_{\rm e}(x) = a_0\widehat{Y}(x_{d-1},x_d) + U_0(x_{d-1},x_d)\sum_{j=1}^{d-2} a_j x_j, \quad \text{for } x_d>0,$$

for some $a_j\in\mathbb{R}$, $j=0,\dots,d-2$, and $w_{\rm o}$ is odd with respect to x_d and is of the form

$$w_{\rm o}(x) = b_0\widehat{Y}(-x_{d-1},x_d) + U_0(-x_{d-1},x_d)\sum_{j=1}^{d-2} b_j x_j \quad \text{for } x_d>0,$$

for some $b_j\in\mathbb{R}$, $j=0,\dots,d-2$, where

$$\widehat{Y}(r,\theta) = r^{\frac{3}{2}}\cos\left(\frac{3}{2}\theta\right) = -Y_1+Y_2-Y_3 \quad\text{and}\quad U_0(r,\theta)=r^{\frac12}\cos\left(\frac{\theta}{2}\right).$$

Moreover, if we denote by Z and V_0, respectively, the odd extensions of $\widehat{Y}(-x_{d-1},x_d)$ and $U_0(-x_{d-1},x_d)$ and if we pass to polar coordinates in the last two variables, i.e.

$$\begin{cases} x_{d-1} = r\cos\theta \\ x_d = r\sin\theta, \end{cases} \qquad \text{with } r\in(0,1) \text{ and } \theta\in(-\pi,\pi)\,,$$

then we have

$$Z(r,\theta) = -r^{\frac32}\sin\left(\frac{3}{2}\theta\right) \quad\text{and}\quad V_0(r,\theta) = r^{\frac12}\sin\left(\frac{\theta}{2}\right). \tag{5.11}$$

We also observe that, by explicit computations

$$\partial_x\widehat{Y}(r,\theta) = \frac{3}{2}r^{\frac12}\cos\left(\frac{\theta}{2}\right) \quad\text{and}\quad \partial_y\widehat{Y}(r,\theta) = -\frac{3}{2}r^{\frac12}\sin\left(\frac{\theta}{2}\right). \tag{5.12}$$

We now prove that all the coefficients a_j, b_j, with $j=0,\dots,d-2$ vanish, by exploiting the fact that Y is the projection of c_n onto the space of blow-up limits $\mathcal{M}_{\frac32}(\mathbb{R}^d;3)$. We know that

$$\left\|{\rm d}_{\Sigma_{N_n}}(c_n,Y)\right\|_{H^1(B_1)} \le \left\|{\rm d}_{\Sigma_{N_n}}(c_n,h)\right\|_{H^1(B_1)} \quad \text{for all } h\in\mathcal{M}_{\frac32}(\mathbb{R}^d;3)$$

and, since

$${\rm d}_{\Sigma_{N_n}}(c_n,h) = \Big|h_i - c_{n,i} + \sum_{j\neq i} c_{n,j}\Big| \quad \text{on } \Omega_i,$$

we can rephrase it as follows

$$\sum_{i=1}^{3}\left(w_{n,i}, \frac{Y_i-h_i}{\delta_n}\right)_{H^1(\Omega_i)} \le \frac{\delta_n}{2}\sum_{i=1}^{3}\left\|\frac{h_i-Y_i}{\delta_n}\right\|^2_{H^1(\Omega_i)} \quad \text{for all } h\in\mathcal{M}_{\frac32}(\mathbb{R}^d;3). \tag{5.13}$$

At this point, we consider as h suitable perturbations of the blow-up Y (depending on the parameter δ_n) and pass to the limit as $n\to\infty$. This will provide a set of orthogonality conditions which the limit w must satisfy; in view of the classification, this leads to conditions on the coefficients a_j and b_j. More precisely, we are going to choose a multiple of the blow-up limit Y and the infinitesimal rotations inside the coordinate planes x_ix_j, with $i=1,\dots,d-1$ and $j=d-1,d$ $(i\neq j)$. This way, we get a total of $2(d-2)$ equations which coincide with the degrees of freedom of the problem. We proceed in the following steps.

1. $a_0=0$: we choose $h(x)=(1\pm\delta_n)Y(x_{d-1},x_d)$.
2. $a_j=0$ for $j=1,\dots,d-2$: we choose $h(x)=Y(x_{d-1}\cos(\delta_n)\pm x_i\sin(\delta_n),x_d)$.
3. $b_0=0$: we choose $h(x)=Y(x_{d-1}\cos(\delta_n)\mp x_d\sin(\delta_n), x_d\cos(\delta_n)\pm x_{d-1}\sin(\delta_n))$.
4. $b_j=0$ for $j=1,\dots,d-2$: we choose $h(x)=Y(x_{d-1},x_d\cos(\delta_n)\pm x_j\sin(\delta_n))$.

We observe that perturbations (1) and (2) leave the last coordinate x_d untouched, and so the generate conditions only on the even part of w. Let us now perform the various computations.

(1) Plugging h into (5.13), we get that

$$\mp\sum_{i=1}^{3}(w_{n,i},Y_i)_{H^1(\Omega_i)}\leq\frac{\delta_n}{2}\sum_{i=1}^{3}\|Y_i\|^2_{H^1(\Omega_i)}\,,$$

and, passing to the limit as $n\to\infty$, this leads to

$$\sum_{i=1}^{3}(\widehat{w},\widehat{Y})_{H^1(\Omega_i)}=(\widehat{w},\widehat{Y})_{H^1(B_1\setminus\mathcal{P})}=0,$$

being $\widehat{w},\widehat{Y}\in H^1(B_1\setminus\mathcal{P})$. In view of the classification of the solutions of the linearized problem, since $\widehat{Y}$ is even with respect to x_d and independent from x_j, for $j=1,\dots,d-2$, the terms in $\widehat{w}$ which linearly depend on x_j do not contribute; thus, we have

$$a_0\|\widehat{Y}\|^2_{H^1(B_1\setminus\mathcal{P})}=(\widehat{w},\widehat{Y})_{H^1(B_1\setminus\mathcal{P})}=0,$$

which implies that $a_0=0$.

(2) First of all, we observe that

$$\frac{\widehat{Y}(x_{d-1}\cos(\delta_n)\pm x_j\sin(\delta_n),x_d)-\widehat{Y}(x_{d-1},x_d)}{\delta_n}\to\pm\frac{3}{2}x_jU_0(x_{d-1},x_d)$$

strongly in $H^1(\Omega_i)$, as $n\to\infty$, for all $i=1,2,3$ and $j=1,\dots,d-2$. Therefore, plugging h into (5.13) and passing to the limit as $n\to\infty$ yields

$$\sum_{i=1}^{3}(\widehat{w},x_jU_0)_{H^1(\Omega_i)}=(\widehat{w},x_jU_0)_{H^1(B_1\setminus\mathcal{P})}=0.$$

On the other hand, since $\widehat{Y}$ is even with respect to x_d and since x_jU_0 is independent from the variable x_k for $k=1,\dots,d-2$, $k\neq j$, thanks to the explicit form of $\widehat{w}$ we infer

$$(\widehat{w},x_jU_0)_{H^1(B_1\setminus\mathcal{P})}=a_j\,\|x_jU_0\|^2_{H^1(B_1\setminus\mathcal{P})}=0,$$

thus proving that $a_j=0$.

(3) By (5.11) and (5.12) we deduce that

$$\frac{\widehat{Y}(x_{d-1}\cos(\delta_n)\mp x_d\sin(\delta_n),x_d\cos(\delta_n)\pm x_{d-1}\sin(\delta_n))-\widehat{Y}(x_{d-1},x_d)}{\delta_n}$$

converges to $\pm Z(x_{d-1},x_d)$ weakly in $H^1(\Omega_i)$, for $i=1,2,3$, as $n\to\infty$. Hence, reasoning analogously to the previous step, from (5.13) we get

$$(\widehat{w},Z)_{H^1(B_1\setminus\mathcal{P})}=0.$$

Since the second term of the scalar product is odd with respect to x_d and since the right term does not depend on x_j for any $j=1,\dots,d-2$, in view also of the explicit form of $\widehat{w}$, this implies that

$$b_0\,\|Z\|^2_{H^1(B_1\setminus\mathcal{P})}=0.$$

thus implying that $b_0=0$.

(4) Again, in view of (5.11) and (5.12) we have that

$$\frac{\widehat{Y}(x_{d-1},x_d\cos(\delta_n)\pm x_j\sin(\delta_n))-\widehat{Y}(x_{d-1},x_d)}{\delta_n}\to\mp\frac{3}{2}x_jV_0(x_{d-1},x_d)$$

weakly in $H^1(\Omega_i)$, for $i=1,2,3$, as $n\to\infty$. Passing to the limit in (5.13), reasoning as in step (2) and (3), we get

$$b_j\,\|x_jV_0\|^2_{H^1(B_1\setminus\mathcal{P})}=0.$$

which forces $b_j=0$, thus completing the proof of the claim.

Proof of **Step 4**. For any $n \in \mathbb{N}$, we define $u_n \in H^1(B_1; \Sigma_{N_n})$ and $\widetilde{u}_{n,i}$, $i = 1, 2, 3$, as in Step 1. Now, we compute the energy of $w_{n,i}$ and $\widetilde{u}_{n,i}$ on each Ω_i (setting $\Omega_i' := \partial\Omega_i \cap \partial B_1$, $\gamma := 3/2$ and extending $t = 0$ for $r < 1/2$), obtaining that

$$\int_{\Omega_i} |\nabla w_{n,i}|^2 \,\mathrm{d}x = \int_0^1 r^{d-1} |\partial_r(r^\gamma)|^2 \,\mathrm{d}r \int_{\Omega_i'} w_{n,i}^2 \,\mathrm{d}S + \int_0^1 r^{d+2\gamma-3} \,\mathrm{d}r \int_{\Omega_i'} |\nabla_{\partial B_1} w_{n,i}|^2 \,\mathrm{d}S$$

and that

$$\begin{aligned}\int_{\Omega_i} |\nabla \widetilde{u}_{n,i}|^2 \,\mathrm{d}x &= \int_{\Omega_i \setminus B_{\frac{1}{2}}} |\nabla (t w_{n,i})|^2 \,\mathrm{d}x \\ &= \int_0^1 r^{d-1} |\partial_r(t r^\gamma)|^2 \,\mathrm{d}r \int_{\Omega_i'} w_{n,i}^2 \,\mathrm{d}S + \int_0^1 r^{d+2\gamma-3} t^2 \,\mathrm{d}r \int_{\Omega_i'} |\nabla_{\partial B_1} w_{n,i}|^2 \,\mathrm{d}S.\end{aligned}$$

Plugging these computations into (5.5) and recalling (5.7), we get that

$$\begin{aligned}&\int_0^1 r^{d+2\gamma-3}(1-t^2) \,\mathrm{d}r \sum_{i=1}^3 \int_{\Omega_i'} |\nabla_{\partial B_1} w_{n,i}|^2 \,\mathrm{d}S \\ &\quad < \int_0^1 r^{d-1} \left(|\partial_r(t r^\gamma)|^2 - |\partial_r(r^\gamma)|^2 \right) \mathrm{d}r \sum_{i=1}^3 \int_{\Omega_i'} w_{n,i}^2 \,\mathrm{d}S \\ &\qquad - (d+1) \int_0^1 r^d (1 - \varepsilon_n - t) \,\mathrm{d}r \, \frac{\beta(c_n, Y)}{\delta_n^2} + o(1),\end{aligned}$$

as $n \to \infty$. Since

$$\int_0^1 r^{d+2\gamma-3}(1-t^2) \,\mathrm{d}r > 0, \quad (d+1) \int_0^1 r^d (1 - \varepsilon_n - t) \,\mathrm{d}r \, \frac{\beta(c_n, Y)}{\delta_n^2} \geq 0$$

for n sufficiently large and since, by compact trace embedding,

$$w_{n,i} \to 0 \quad \text{strongly in } L^2(\Omega_i'), \text{ as } n \to \infty,$$

we deduce that $w_{n,i} \to 0$ strongly in $H^1(\Omega_i)$, for all $i = 1, 2, 3$, thus reaching a contradiction and concluding the proof. □

6. Regularity of the free boundary around points of frequency $3/2$

In this section we complete the proof of Theorem 1.1 by proving the regularity of the free interface $\mathcal{F}(u)$ around points of frequency $3/2$. The main ingredients are the classification of the $3/2$-homogeneous blow-ups (see Proposition 3.5) and the epiperimetric inequality Theorem 1.2.

6.1. Rate of convergence of the blow-up sequence.

We will show that we can apply the epiperimetric inequality to the rescalings of u of the form $u^{r,x_0}(x) = r^{-3/2} u(x_0 + rx)$. In order to do so, we need to show that the conditions of Theorem 1.2 are fulfilled uniformly, at all points x_0 and at all small scales r, by multiples of u^{r,x_0}. We start with the following.

Lemma 6.1 (Uniform Hausdorff distance estimate)**.** *Let $d \geq 2$ and $N \geq 1$ be fixed. For any $\varepsilon > 0$ there exists $\delta > 0$ (depending on d, N and ε) such that, if $u \in H^1(B_1; \Sigma_N) \cap C^{0,1}(B_1; \mathbb{R}^N)$ is a minimizer, $u \in \mathcal{M}(B_1; N)$, that satisfies*

$$\|\mathrm{d}_{\Sigma_N}(u, Y)\|_{L^\infty(B_1)} \leq \delta,$$

then

$$\sum_{i=1}^3 \mathrm{d}_{\mathcal{H}}\Big(\{u_i > 0\} \cap B_{1/2}, \{Y_i > 0\} \cap B_{1/2}\Big) \leq \varepsilon, \tag{6.1}$$

and

$$\sum_{i=4}^N \mathrm{d}_{\mathcal{H}}\Big(\{u_i > 0\} \cap B_{1/2}, \{x_{d-1} = x_d = 0\} \cap B_{1/2}\Big) \leq \varepsilon. \tag{6.2}$$

Moreover, in $B_1 \cap \left\{x \in \mathbb{R}^d\colon \sqrt{x_{d-1}^2 + x_d^2} > \varepsilon\right\}$ *the free boundary* $\mathcal{F}(u)$ *is the disjoint union of the three surfaces* $\Gamma_{ij} = \partial\{u_i > 0\} \cap \partial\{u_j > 0\}$*, for* $i, j \in \{1, 2, 3\}$*,* $i \neq j$*, which are* $C^{1,\alpha}$ *graphs over* $\partial\{Y_i > 0\} \cap \partial\{Y_j > 0\}$*.*

Proof. Suppose by contradiction that there are $\varepsilon > 0$ and a sequence $u_n \in H^1(B_1; \Sigma_N) \cap C^{0,1}(B_1; \mathbb{R}^N) \cap \mathcal{M}(B_1; N)$ such that

$$\|\mathrm{d}_{\Sigma_N}(u_n, Y)\|_{L^\infty(B_1)} = \delta_n \to 0,$$

and for which (6.1) or (6.2) fail. For any $\tau > 0$, we denote

$$\mathcal{O}_\tau := \{x \in \mathbb{R}^d\colon \operatorname{dist}(x, \{Y = 0\}) < \tau\},$$
$$C_\tau := \left\{x \in \mathbb{R}^d\colon \sqrt{x_{d-1}^2 + x_d^2} < \tau\right\}.$$

Notice there exists $t = t_\varepsilon > 0$ such that $\mathrm{d}_{\Sigma_N}(Y, 0) > t$ on the set $B_1 \setminus \mathcal{O}_{\varepsilon/6}$. On the other hand, by the uniform convergence of u_n to Y we get that for n large enough

$$\{Y_i > 0\} \cap B_1 \setminus \mathcal{O}_{\varepsilon/6} \subset \{Y_i > t\} \cap B_1 \subset \{u_{n,i} > 0\} \cap B_1 \quad \text{for} \quad i = 1, 2, 3.$$

In particular, this excludes the possibility that (6.1) fails for u_n as $n \to +\infty$. In order to show that (6.2) cannot fail for large n, we notice that for every ball $B_{\varepsilon/3}(x)$ with center $x \in B_{1/2} \cap \partial\{Y_i > 0\} \cap \partial\{Y_j > 0\} \setminus C_{\varepsilon/2}$ (for $i, j \in \{1, 2, 3\}$ with $i \neq j$), we have that $Y = Y_i\vec{\boldsymbol{e}}_i + Y_j\vec{\boldsymbol{e}}_j$ and

$$\|\mathrm{d}_{\Sigma_N}(u_n, Y)\|_{L^\infty(B_{\varepsilon/3}(x))} \to 0 \quad \text{as } n \to \infty.$$

Thus, by the epsilon-regularity lemma [OV24, Lemma 9.8], we get that $u_k \equiv 0$ in $B_{\varepsilon/6}(x)$ for all $k \neq i, j$ and $\mathcal{F}(u) \cap B_{\varepsilon/6}(x) = \Gamma_{ij} \cap B_{\varepsilon/6}(x)$ is a $C^{1,\alpha}$ graph (with small C^1 norm) over the hyperplane $\partial\{Y_i > 0\} \cap \partial\{Y_j > 0\}$. In particular, this implies that (2) cannot happen and it proves that second part of the claim. □

Lemma 6.2 (Uniform convergence of rescalings with variable center)**.** *Let* $u \in \mathcal{M}(B_1; N)$ *be such that* $\gamma(u, 0) = 3/2$*. Then, for all* $\varepsilon > 0$ *there exists* $\rho, r_0 > 0$ *(depending on* d*,* N *and* ε*) such that for all* $z \in B_\rho \cap \mathcal{F}_{3/2}(u)$ *and for all* $r < r_0$ *there exists* $Y \in \mathcal{M}_{3/2}(\mathbb{R}^d; 3)$ *such that*

$$\|\mathrm{d}_{\Sigma_N}(u_{z,r}, Y)\|_{L^\infty(B_{1/2})} \leq \varepsilon,$$

where $u_{z,r}(x) := H(u, z, r)^{-1/2} u(rx + z)$*.*

Proof. Let us assume by contradiction that there exists sequences $r_n \to 0$ and $z_n \in \mathcal{F}_{3/2}(u)$, with $|z_n| \to 0$, such that

$$\|\mathrm{d}_{\Sigma_N}(u^{z_n,r_n}, Y)\|_{L^\infty(B_{1/2})} > \varepsilon \quad \text{for all } n \in \mathbb{N} \tag{6.3}$$

for all $Y \in \mathcal{M}_{3/2}(\mathbb{R}^d; 3)$, for some $\varepsilon > 0$, and that $\gamma(u^{z_n,r_n}, 0) = 3/2$ for all $n \in \mathbb{N}$. First of all, we observe that

$$\sum_{i=1}^N \int_{B_1} |\nabla(u_{z_n,r_n})_i|^2 \, \mathrm{d}x = \mathcal{N}(u_{z_n,r_n}, 0, 1). \tag{6.4}$$

Thus, up to a subsequence

$$u_{z_n,r_n} \to u_0 \quad \text{strongly in } H^1(B_1; \Sigma_N) \text{ and uniformly in } B_{1/2}, \tag{6.5}$$

for some $u_0 \in H^1(B_1; \Sigma_N)$, $u_0 \not\equiv 0$. Moreover, since $u_{z_n,r_n} \in \mathcal{M}(B_1; N)$ for all n, also $u_0 \in \mathcal{M}(B_1; N)$. We now observe that, since $z_n \in \mathcal{F}_{3/2}(u)$ and by monotonicity, we have

$$\frac{3}{2} = \mathcal{N}(u, z_n, 0^+) \leq \mathcal{N}(u, z_n, Rr_n) = \mathcal{N}(u_{z_n,r_n}, 0, R),$$

for any $R > 0$ and n sufficiently large. Passing to the limit as $n \to \infty$, we get

$$\frac{3}{2} \leq \mathcal{N}(u_0, 0, R) \quad \text{for all } R > 0.$$

By the continuity of $x \mapsto \mathcal{N}(u,x,r)$, the monotonicity of $r \mapsto \mathcal{N}(u,x,r)$ and Lemma 3.4, we have that for all $R, \tau > 0$ sufficiently small, there holds

$$\frac{3}{2} \le \mathcal{N}(u_0, 0, R) \le \mathcal{N}(u, 0, \tau R).$$

Sending $\tau \to 0$ implies that u_0 is ${}^3\!/_2$-homogeneous, which, in view of Lemma 3.4 and (6.5) contradicts (6.3). □

We are now in position to prove the uniqueness of the blow-up limit and rate of convergence of the blow-up sequence.

Proposition 6.3. *Let $\alpha := \varepsilon(d+1)$, with $\varepsilon > 0$ being as in Theorem 1.2 and let $u \in \mathcal{M}(B_1; N)$. Then, for all compact $K \subseteq B_1$ there exists $R > 0$ such that for all $x_0 \in K \cap \mathcal{F}_{3/2}(u)$, there exists $C_{\mathrm{rate}} > 0$ depending on d and $\sup_{x \in K} \mathcal{N}(u, x, \mathrm{d}_{\mathcal{H}}(K, \partial B_1))$ such that*

$$\|\mathrm{d}_{\Sigma_N}(u^{x_0,r_2}, u^{x_0,r_1})\|^2_{L^2(\partial B_1)} \le \sum_{i=1}^N \|u_i^{x_0,r_2} - u_i^{x_0,r_1}\|^2_{L^2(\partial B_1)} \le C_{\mathrm{rate}}(r_2^\alpha - r_1^\alpha)$$

for all $0 < r_1 < r_2 < R$ where $u^{x_0,r}(x) := r^{-3/2} u(rx + x_0)$. In particular, there exists $Y^{x_0} \in \mathcal{M}_{3/2}(\mathbb{R}^d; 3)$ such that

$$\|\mathrm{d}_{\Sigma_N}(u^{x_0,r}, Y^{x_0})\|^2_{L^2(\partial B_1)} \le \sum_{i=1}^N \|u_i^{x_0,r} - Y_i^{x_0}\|^2_{L^2(\partial B_1)} \le C_{\mathrm{rate}} r^\alpha,$$

for all $r < R$.

Proof. Thanks to Lemma 6.1 and Lemma 6.2, there is a radius R such that we can apply the epiperimetric inequality to all the rescaling of the form $u^{x_0,r}(x) = r^{-3/2}u(x_0 + rx)$ with $r \le R$ and $x_0 \in K$. The uniqueness of the blow-up and the rate of convergence now follow by a standard argument (see e.g. [OV24, Proposition 8.1]). □

As a corollary, we also obtain the following.

Proposition 6.4 (Oscillation of the ${}^3\!/_2$-blow-up limits)**.** *Let $\bar\alpha := 2\alpha/(\alpha+3)$, where α is as in Proposition 6.3 and let $u \in \mathcal{M}(B_1; N)$. Then, for all compact $K \subseteq B_1$ there exists $r_0 > 0$ (depending on d and K) and $C_{\mathrm{osc}} > 0$ (depending on d and $\sup_{x \in K} \mathcal{N}(u, x, \mathrm{d}_{\mathcal{H}}(K, \partial B_1))$) such that*

$$\sum_{i=1}^3 \|Y_i^{x_0} - Y_i^{z_0}\|^2_{L^2(\partial B_1)} \le C_{\mathrm{osc}} |x_0 - z_0|^{\bar\alpha}$$

for all $x_0, z_0 \in \mathcal{F}_{3/2}(u) \cap K$ such that $|x_0 - z_0| < r_0$, where

$$Y^p = \lim_{r \to 0} u^{p,r} \quad \text{and} \quad u^{p,r}(x) := r^{-\frac{3}{2}} u(rx + p).$$

Proof. We first estimate

$$\sum_{i=1}^3 \|Y^{x_0} - Y^{z_0}\|^2_{L^2(\partial B_1)} \le 2 \sum_{i=1}^N \Big(\|u_i^{x_0,r} - Y_i^{x_0}\|^2_{L^2(\partial B_1)} + \|u_i^{z_0,r} - Y_i^{z_0}\|^2_{L^2(\partial B_1)} + \|u_i^{x_0,r} - u_i^{z_0,r}\|^2_{L^2(\partial B_1)} \Big). \tag{6.6}$$

The first two terms can be bounded in view of Proposition 6.3 as follows

$$\sum_{i=1}^N \Big(\|u_i^{x_0,r} - Y_i^{x_0}\|^2_{L^2(\partial B_1)} + \|u_i^{z_0,r} - Y_i^{z_0}\|^2_{L^2(\partial B_1)} \Big) \le C\, r^\alpha, \tag{6.7}$$

for r sufficiently small depending on d and $\mathrm{d}_{\mathcal{H}}(K, \partial B_1)$ and $C > 0$ depending on d and $\sup_{x \in K} \mathcal{N}(u, x, r_0)$. On the other hand, if $L_K > 0$ denotes the Lipschitz constant of u in K, we have that

$$\sum_{i=1}^{N}\|u_i^{x_0,r}-u_i^{z_0,r}\|^2_{L^2(\partial B_1)}\le \frac{L_K^2}{r^3}|x_0-z_0|^2. \tag{6.8}$$

Therefore, we can choose r in such a way that

$$r^\alpha=\frac{|x_0-z_0|^2}{r^3},\quad \text{that is}\quad r=|x_0-z_0|^{\frac{2}{\alpha+3}}.$$

The claim follows by combining (6.8) and (6.7) with (6.6). □

6.2. **Regularity of $\mathcal{F}_{3/2}$.** In this section we conclude the proof of Theorem 1.1.

Lemma 6.5 (Frequency gap from above)**.** *There exists $\delta_d>0$ such that, if $u\in\mathcal{M}_\gamma(\mathbb{R}^d;N)$ is a γ-homogeneous minimizer with $\gamma>3/2$, then there holds $\gamma\ge 3/2+\delta_d$.*

Proof. Suppose that this is not the case and that there is a sequence of γ_n homogeneous solutions u_n with $\gamma_n\to {}^3/_2$. Then, up to a subsequence, u_n converges to a ${}^3/_2$-homogeneous solution Y. By Lemma 6.1, for n large enough we can apply the epiperimetric inequality from Theorem 1.2 to u_n. But this is a contradiction with the minimality of u_n. □

Lemma 6.6 (No holes lemma)**.** *For any $\varepsilon,\rho\in(0,{}^1/_2)$ there exists $\delta>0$ (depending on d, N and ε) such that, if $u\in H^1(B_1;\Sigma_N)\cap C^{0,1}(B_1;\mathbb{R}^N)\cap\mathcal{M}(B_1;N)$ satisfies*

$$\|\mathrm{d}_{\Sigma_N}(u,Y)\|_{L^\infty(B_1)}\le\delta\quad\textit{and}\quad\sup_{x\in B_{\frac12}}\mathcal{N}\left(u,x,\frac12\right)<\frac32+\delta,$$

then for all $x''\in B''_\rho:=\{y\in\mathbb{R}^{d-2}\colon|y|<\rho\}$ there exists $x=(x'',x_{d-1},x_d)$ such that

$$\sqrt{x_{d-1}^2+x_d^2}<\varepsilon\qquad\textit{and}\qquad\gamma(u,x)={}^3/_2.$$

Proof. We argue by contradiction and assume that there exist $\varepsilon,\rho\in(0,{}^1/_2)$, vanishing sequences $\{\delta_n\}\subseteq\mathbb{R}_+$ and $u_n\in H^1(B_1;\Sigma_N)\cap C^{0,1}(B_1;\mathbb{R}^N)$ such that

$$\|\mathrm{d}_{\Sigma_N}(u_n,Y)\|_{L^\infty(B_1)}\le\delta_n\quad\text{and}\quad\sup_{x\in B_{\frac12}}\mathcal{N}\left(u,x,\frac12\right)<\frac32+\delta_n,$$

for all $n\in\mathbb{N}$ and there exist $x_0''\in B''_\rho$ such that there holds $\gamma(u_n,x)\ne{}^3/_2$ for all points $x\in\mathcal{F}(u_n)$ of the form $x=(x_0'',x_{d-1},x_d)\in B_\rho\cap C_\varepsilon$. In view of Lemma 6.5 and the fact that

$$\gamma(u_n,x)\le\mathcal{N}\left(u_n,x,\frac12\right)<\frac32+\delta_n,$$

for all $x\in B_\rho\cap C_\varepsilon\cap\mathcal{F}(u_n)$, we derive that $\gamma(u_n,x)<{}^3/_2$ for n sufficiently large for all $x\in B_\rho\cap C_\varepsilon\cap\mathcal{F}(u_n)$. On the other hand, by the optimal frequency gap we also know that $\gamma(u_n,x)=1$ for all $x=(x_0'',x_{d-1},x_d)\in B_\rho\cap C_\varepsilon\cap\mathcal{F}(u_n)$. As a consequence, for some n large enough, we can find $t>0$ such that in the set

$$\mathcal{U}:=\Big\{(x'',x_{d-1},x_d)\ :\ |x''-x_0''|<t,\ \sqrt{x_{d-1}^2+x_d^2}<\frac12\Big\},$$

the whole free boundary $\mathcal{F}(u_n)$ consists only of points of frequency 1 and the number of connected components of $\mathcal{U}\setminus\mathcal{F}(u_n)$ is finite. Let now ω be the connected component containing $B_{1/2}\cap\{u_{n,1}>0\}\setminus\mathcal{C}_\varepsilon$ and let $y\in B_{1/2}\cap\{u_{n,1}>0\}\setminus\mathcal{C}_\varepsilon$ be fixed. By Lemma 3.3 (applied to $\mathcal{M}=\mathcal{U}$ and $\mathcal{F}=\mathcal{F}(u_n)$) every closed curve starting from y, contained in $\mathcal{U}$, and intersecting $\mathcal{F}(u_n)$ a finite number of times (and in a transversal way) should cross $\mathcal{F}(u_n)$ an even number of times. This is a contradiction since a closed curve circling around $\mathcal{C}_\varepsilon$ crosses $\mathcal{F}(u_n)$ exactly three times, in view of Lemma 6.1. □

Proof of Theorem 1.1. Let $x_0 \in \mathcal{F}_{3/2}(u)$ and let u_0 be the $3/2$-homogeneous blow-up of u at x_0, which is unique and non-zero by Proposition 6.3. Up to multiplying u with a constant, we can suppose that $u_0 = Y$. By Lemma 6.5 we know that in a neighborhood of x_0 there are only points of frequency 1 or $3/2$. By the no-holes lemma Lemma 6.6, we know that for every $x'' \in \mathbb{R}^{d-2}$ with $|x''|$ small enough, there exist $(x_{d-1}, x_d) \in \mathbb{R}^2$ such that $x = (x'', x_{d-1}, x_d) \in \mathcal{F}_{3/2}(u)$. By Proposition 6.3, we know that the point (x_{d-1}, x_d) is unique, so in a neighborhood of x_0, $\mathcal{F}_{3/2}(u)$ is a graph of a function $\eta : \mathbb{R}^{d-2} \to \mathbb{R}^2$. The $C^{1,\alpha}$-regularity of η follows from Proposition 6.3, Proposition 6.4 and Lemma 6.1 by a standard argument. Finally, the regularity of $\mathcal{F}(u)$ around x_0 is a consequence of the regularity of η, the uniqueness of the blow-up and the epsilon-regularity for $\mathcal{F}_1(u)$ ([OV24, Lemma 9.8]); the argument is standard and for the details we refer for instance to [OV24, Section 9]. □

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