FREE BOUNDARY REGULARITY FOR SEMILINEAR VARIATIONAL PROBLEMS WITH A TOPOLOGICAL CONSTRAINT

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ABSTRACT. We study a class of semilinear free boundary problems in which admissible functions u have a topological constraint, or *spanning condition*, on their 1-level set. This constraint forces $\{u = 1\}$, which is the free boundary, to behave like a surface with some special types of singularities attached to a fixed boundary frame, in the spirit of the Plateau problem [HP16b]. Two such free boundary problems are the minimization of capacity among surfaces sharing a common boundary and an Allen-Cahn formulation of the Plateau problem. We establish the existence of minimizers and study the regularity of solutions to the Euler-Lagrange equations, obtaining the optimal Lipschitz regularity for solutions and analytic regularity for the free boundaries away from a codimension two singular set. The singularity models for these problems are given by conical critical points of the minimal capacity problem, which are closely related to spectral optimal partition and segregation problems.

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1. INTRODUCTION

1.1. **Background.** In this paper we study the regularity of solutions to elliptic variational problems with a topological constraint on the level sets of admissible functions. Given a compact set $\mathbf{W} \subset \mathbb{R}^{n+1}$ with $\Omega = \mathbb{R}^{n+1} \setminus \mathbf{W}$ and potentials $F, V : [0,1] \to [0,\infty)$ vanishing at 0, consider the minimization problems

$$\inf\left\{ \int_\Omega |\nabla u|^2 + F(u) \,:\, u \in C^0(\Omega;[0,1]),\, u \text{ vanishes at infinity, } \{u=1\} \text{ “spans” } \mathbf{W} \right\}, \tag{1.1}$$

and

$$\inf\left\{ \int_\Omega |\nabla u|^2 + F(u) \,:\, u \in C^0(\Omega;[0,1]),\, \int_\Omega V(u) = 1,\, \{u=1\} \text{ “spans” } \mathbf{W}\right\}. \tag{1.2}$$

Here the terminology “$\{u = 1\}$ spans $\mathbf{W}$” means that for a homotopically closed family $\mathcal{C}$ of smooth embeddings of $\mathbb{S}^1$ in Ω (which is independent of u), called a spanning class,

$$\{u = 1\} \cap \gamma \neq \varnothing \qquad \text{for every } \gamma \in \mathcal{C}\,. \tag{1.3}$$

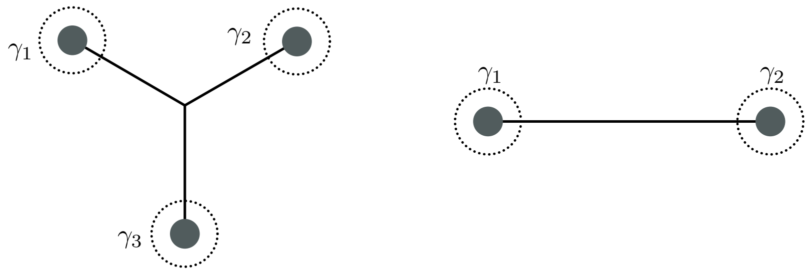

FIGURE 1.1. Shown above are two different configurations of $\mathbf{W} \subset \mathbb{R}^2$, generators for an associated spanning class $\mathcal{C}$, and a spanning set. In both cases, $\mathbf{W}$ is the union of the gray balls, $\mathcal{C}$ is the family of smooth loops homotopic to some γ_i, and the example spanning sets are composed of line segments.

We will refer to any set satisfying (1.3) as $\mathcal{C}$-**spanning**. This type of condition originated in the study of the set-theoretic Plateau problem [HP16b] and at a heuristic level forces the spanning set to behave like a surface bounded by $\mathbf{W}$; see Figure 1.1.

Several examples of these types of problems have appeared in the literature and motivate our work. An early version of the model (1.1)–with a slightly different notion of spanning–and $F = 0$ is the classical problem of finding *surfaces of minimal capacity* spanning a closed curve. In the case when $\mathbf{W}$ is a Jordan curve satisfying some additional restrictions, this problem was solved in $\mathbb{R}^3$ in [Eva40a, Eva40b] using multivalued harmonic functions. A similar method was used in [Caf75] to address the case when $\mathbf{W}$ is a knot, yielding however only local minimizers. Most relevant to our choice of spanning condition is the diffuse interface/Allen-Cahn approximation of the Plateau problem recently introduced in [MNR23a] and which is a prototypical example of (1.2). In that case, $F = W/\varepsilon^2$ is a double-well potential (e.g. $F = u^2(u-1)^2/\varepsilon^2$) and V is a particular volume potential related to F (see (1.6)). It is shown in [MNR23a] that the rescaled problems

$$\inf \left\{ \varepsilon \int_\Omega |\nabla u|^2 + \frac{W(u)}{\varepsilon^2}\, dx \, : \int_\Omega V(u) = v, \; \{u = 1\} \text{ is } \mathcal{C}\text{-spanning} \right\}, \tag{1.4}$$

converge as $\varepsilon, v \to 0$, $\varepsilon \ll v$, to the Plateau problem of Harrison-Pugh [HP16b]. Finally, although it is not immediately obvious from the statements of (1.1)-(1.2), they share some common features with optimal partition/segregation problems from the free boundary literature; see Remark 1.7.

As is to be expected, the spanning constraint significantly affects the behavior of minimizers and the corresponding analysis. For example, with respect to (1.1), in combination with the requirement that u vanishes at infinity it eliminates from contention all constant functions and thus forces non-constant minimizers. Also, for (1.2), it allows for the approximation of minimal surfaces with codimension 1 singularities, for example triple junctions in the plane, by energy minimizing solutions of an Allen-Cahn free boundary problem. This, however, is not possible when considering stationary, stable solutions to the classical Allen-Cahn equation [TW12]. At the level of the Euler-Lagrange equation, the spanning constraint significantly changes even its derivation. This is due to the fact that when $\{u = 1\}$ is $\mathcal{C}$-spanning and the test function φ takes negative values, there is no reason that $\{u + t\varphi = 1\}$ is $\mathcal{C}$-spanning and admissible for (1.1)-(1.2). Setting aside this consideration for the moment, (1.1)-(1.2) formally lead to a free boundary problem with a transmission condition. If u is minimizer and u and the level set $\{u = 1\}$ are sufficiently regular, then there exists a potential Φ such that u solves the free boundary problem

$$\begin{cases} 2\,\Delta u = \Phi'(u)\,, & \text{on } \Omega \cap \{u \neq 1\}\,, \\ |\partial_\nu^+ u| = |\partial_\nu^- u|\,, & \text{on } \Omega \cap \{u = 1\}\,, \\ \text{such that } \{u = 1\} \text{ spans } \mathbf{W}\,, & \end{cases} \tag{1.5}$$

where $\partial_\nu^\pm$ denote the one-sided directional derivative operators with respect to a unit normal ν to $\{u = 1\}$ (cf. [MNR23a, Prop. 1.4]). In the case of (1.1), $\Phi = F$, whereas in (1.2) $\Phi = F - \lambda V$, with $\lambda \in \mathbb{R}$ a Lagrange multiplier associated with the volume constraint $\int V(u) = 1$. The interested reader may refer to [Eva40a, Eq. 1-2] regarding the significance of the transmission condition $|\partial_\nu^+ u| = |\partial_\nu^- u|$ and its integral formulation in the simplest case $F = 0$ with no volume constraint.

1.2. **Main results.** Throughout the paper, we will assume that: F and V are potential functions satisfying the hypotheses

$$F, V \in C^2([0,1];[0,\infty)) \tag{H1}$$

$$0 = F(0) = V(0) = F'(0) = V'(0) = V'(1) = F'(1)\,, \quad \text{and} \tag{H2}$$

$$V \text{ is strictly increasing on } (0,1)\,. \tag{H3}$$

For existence of minimizers in the presence of the volume constraint, we will also assume that

$$\lim_{t\to 0} \frac{V(t)}{F(t)} = 0\,, \tag{H4}$$

which is mild and satisfied for example in the Allen-Cahn setting [MNR23a] where

$$V(t) = \mathcal{F}(t)^{(n+1)/n}\,, \qquad \mathcal{F}(t) = \int_0^t \sqrt{F(s)}\,ds\,. \tag{1.6}$$

Lastly, for the homotopically closed family of smooth loops $\mathcal{C}$ (the spanning class), we assume that

$$\begin{aligned} &\text{no } \gamma \in \mathcal{C} \text{ is homotopic in } \Omega \text{ to a point when } n \geq 2, \text{ and}\\ &\text{no } \gamma \in \mathcal{C} \text{ is homotopic in } \Omega \text{ to a point, or to } \partial B_R \text{ if } \mathbf{W} \subset B_R, \text{ when } n = 1\,. \end{aligned} \tag{1.7}$$

This assumption is sharp in the following sense: if $\mathcal{C}$ contains such a curve γ homotopic (in Ω) to a point x, the problems (1.1)-(1.2) are trivial on the connected component Ω' of Ω containing x, since any u satisfying (1.3) must be 1 on Ω'. Also, in the plane, if there exists $\gamma \in \mathcal{C}$ homotopic to ∂B_R with $\mathbf{W} \subset B_R$, then the admissible class is empty in (1.1) or the infimum is infinite in (1.2). So there is nothing lost in assuming (1.7).

Although our motivation arises from minimizers to (1.1) and (1.2), we are in fact able to develop a regularity theory for general critical points of the Euler-Lagrange equations associated to these minimization problems. More precisely, denoting by $\Phi = F$ in the former case and $\Phi = F - \lambda V$ for suitable $\lambda \in \mathbb{R}$ in the latter, minimizers of (1.1) and (1.2) satisfy the following variational relations (see Theorem 2.1):

$$0 = \int_\Omega \Big(|\nabla u|^2 + \Phi(u)\Big)\operatorname{div} T - 2\langle \nabla u, \nabla u \nabla T\rangle\,dx \qquad \text{for all } T \in C_c^\infty(\Omega, \mathbb{R}^{n+1})\,, \tag{1.8}$$

$$2\int_\Omega |\nabla u|^2\,\varphi\,dx = \int_\Omega (1-u)\Big\{2\nabla u \cdot \nabla\varphi + \Phi'(u)\,\varphi\Big\}\,dx \qquad \text{for all } \varphi \in C_c^\infty(\Omega), \text{ and} \tag{1.9}$$

$$0 \leq \int_\Omega 2\nabla u \cdot \nabla \varphi + \Phi'(u)\,\varphi\,dx \qquad \text{for all } \varphi \in C_c^1(\Omega;[0,\infty)). \tag{1.10}$$

Our main results are sharp regularity for solutions of these equations, together with existence of minimizers for (1.1) and (1.2).

Theorem 1.1 (Regularity)**.** *If $\mathbf{W} = \mathbb{R}^{n+1} \setminus \Omega$ is compact, $\mathcal{C}$ is a spanning class for $\mathbf{W}$ satisfying (1.7), F and V satisfy (H1)-(H3), and $u \in W^{1,2}_{\rm loc}(\Omega;[0,1])$ solves (1.8)-(1.10), then*

(i): *u is locally Lipschitz in Ω, and*

(ii): *on $\{\Omega' \subset \Omega : \Omega'$ is a connected component of Ω not contained in $\{u = 1\}\}$, the free boundary $\{u = 1\}$ decomposes as*

$$\{u = 1\} = \mathcal{R}(u) \sqcup \mathcal{S}(u)\,,$$

where $\mathcal{R}(u)$ is locally an analytic n-dimensional manifold and $\mathcal{S}(u)$ is a closed set with Hausdorff dimension at most $n-1$. If $n=1$, $\{u=1\}$ consists in a locally finite number of analytic curves meeting with equal angles at a discrete number of singular points.

Theorem 1.2 (Existence)**.** *Suppose $\mathbf{W}=\mathbb{R}^{n+1}\setminus\Omega$ is compact, $\mathcal{C}$ is a spanning class for $\mathbf{W}$ satisfying (1.7), and F and V satisfy (H1)-(H4). Then there exists a minimizer for (1.2), and if either*

$$n\geq 2 \qquad \text{or} \qquad \text{there exists } t_j\searrow 0 \text{ such that } F(t_j)>0, \tag{1.11}$$

there exists a minimizer for (1.1).

The proof of Theorem 1.1 comprises the bulk of the paper, and the Lipschitz continuity is used in the proof of the existence of minimizers for (1.2), which is why we have opted to state it first.

Remark 1.3 (Spanning and the Euler-Lagrange equations)**.** The first equation (1.8) is the inner variational equation, the equation (1.9) is the weak formulation of

$$(1-u)\left\{2\Delta u-\Phi'(u)\right\}=0 \qquad \text{in } \Omega\,, \tag{1.12}$$

and (1.10) is the weak form of the differential inequality

$$2\Delta u\leq \Phi'(u) \qquad \text{in } \Omega\,. \tag{1.13}$$

All of the equations in Theorem 2.1 are quite natural in light of the spanning condition (1.3). The inner variational equation utilizes the simple fact that precomposing u with a domain diffeomorphism preserves the spanning constraint; (1.9) is a rigorous version of the intuition that u should satisfy the usual volume-constrained Allen-Cahn equation on $\{u<1\}$, with possible singularities concentrated on $\{u=1\}$; and the differential inequality (1.10) is the manifestation of the fact outer perturbations $u+\varphi$ by *negative* test functions φ may disturb the spanning constraint, so that $u+\varphi$ is not an admissible variation.

Remark 1.4 (Optimality of the assumptions in Theorem 1.2)**.** If $n=1$ and $F(t)=0$ for $t\in[0,t_0]$, then using logarithmic cutoffs it can be shown there does not exist a minimizer for (1.1); see Remark 5.3. An alternative in this case would be to solve the problem on a bounded domain Ω with vanishing Dirichlet conditions; see Remark 1.11.

Remark 1.5 (On the free boundary decomposition)**.** Even with the assumption (1.7) on $\mathcal{C}$, part (ii) of Theorem 1.1 is optimal in terms of its restriction to

$$\{\Omega'\subset\Omega:\Omega' \text{ is a connected component of } \Omega \text{ not contained in } \{u=1\}\}\,.$$

In the case that $\mathbf{W}$ is the union of a solid torus T and some $\overline{B_r\setminus B_s}$ with $T\subset\subset B_s$, the spanning class $\mathcal{C}$ consists of curves contained in B_s whose linking number with T is 1, and the potential F vanishes at 1, the global minimizer for (1.1) is 1 on $B_s\setminus T$ and 0 on $\overline{B_r}^c$; it has zero energy. The same configuration is obviously minimizing in the additional presence of a volume constraint if in addition $\int_{B_s\setminus T}V(1)=1$. However, as long as $\mathbb{R}^{n+1}\setminus\mathbf{W}$ does not have any bounded connected components, minimizers for (1.1) or (1.2) cannot be locally constant.

Remark 1.6 (Expanded formulations of (1.1)-(1.2))**.** As is standard in the calculus of variations, the application of the direct method to prove Theorem 1.2 passes through a "weak" formulation of (1.1)-(1.2), as introduced in [MNR23a] (cf. (5.5)-(5.6)). In the expanded formulation the spanning condition (1.3) for continuous functions is replaced with a more general constraint compatible with weak compactness in $W^{1,2}$ under energy bounds and which agrees with (1.3) for continuous functions (cf. (A.3) and Theorem C.1). We separately prove Lipschitz regularity for critical points of the energies in (5.5)-(5.6) in Theorem 3.1; this shows that the minimizers obtained through this procedure are in fact continuous and therefore minimizers for the original formulations (1.1)-(1.2). As a consequence, (1.1)-(1.2) are equivalent to (5.5)-(5.6), and Theorems 1.1-1.2 could have been stated for (5.5)-(5.6) instead. However, the generalization of (1.3) is longer to state, and while it is certainly important at a conceptual level (without it we would not know that the set of critical points studied in Theorem 1.1 is non-empty!), our regularity arguments do not appeal to it: it is only used in the proof of Theorem 1.2. Therefore, in order to emphasize the original contributions of the paper and to enhance the readability of the introduction, this material is presented in Appendix A. The reader may choose to skim this appendix, treating this machinery as a "black-box" if desired, and safely read the regularity and free boundary analysis in Sections 3-4.

1.3. **Discussion.** We begin with a detailed outline of the proofs of Theorems 1.1-1.2 and comments on the key ideas.

1.3.1. *Commentary on proofs and structure of article.* Since (1.5) is completely formal, in order to rigorously analyze minimizers we do not appeal directly to it but instead base our arguments on several other properties of minimizers of (1.1) and (1.2). First and foremost, in Section 2 we present three criticality conditions, namely (the weak formulations of) the outer variation equation

$$(1-u)\left\{2\Delta u - \Phi'(u)\right\} = 0 \qquad \text{in } \Omega,$$

the inner variation equation for u, and the differential inequality

$$2\Delta u \leq \Phi'(u)$$

(see (1.8)-(1.10) and Remark 1.3). Since this set of conditions comes from considering only first order variations of either (1.1) or (1.2), we adopt them as our notion of critical points or stationary solutions for these models. In Section 3 we prove Theorem 1.1-(i) as a byproduct of our study of general stationary solutions for the problem obtained by the transformation $v = 1-u$. This latter recasting of the model, whose new criticality conditions are given by (3.2)-(3.4), allows us to see solutions as "almost"-subharmonic functions satisfying a weak transmission condition at their zero set. Under this reformulation, our model is quite proximate to those in optimal partition/segregation problems, and the arguments utilize some similar tools. More precisely, the proof of the Lipschitz regularity for stationary solutions, Theorem (3.1), fundamentally relies on the fact that conditions (3.2)-(3.4) are enough to guarantee the almost-monotonicity of the Almgren frequency function which unlocks a series of monotonicity and unique continuation type properties that help us obtain regularity (see, e.g., [CL07, CL08]). The main novel feature in this part of our analysis is a lower bound in the frequency function (Lemma 3.13) which in turn implies that v is Hölder continuous. Via a dimension reduction argument and a blow-up analysis, we improve the lower frequency bound to 1 at all points in $\{v = 0\}$, therefore improving the Hölder regularity to Lipschitz regularity. Note that since the absolute value of any harmonic function satisfies locally (3.2)-(3.4) (with $G = 0$), Lipschitz regularity is the sharp regularity of stationary solutions.

In Section 4, we show Theorem 1.1-(ii) again for any critical point u. The starting point is the fact that points in $\{u = 1\}$ with frequency greater than 1 actually have frequency greater than or equal to $\frac{3}{2}$. This frequency gap is sharp since the frequency $\frac{3}{2}$ is attained at triple junctions as in Figure 1.1, and was obtained in the work [ST15]. From this, we can split $\{u = 1\} = \mathcal{R}(u) \sqcup \mathcal{S}(u)$ where $\mathcal{R}(u)$ consists of the points in $\{u = 1\}$ where $v = 1-u$ has frequency value 1, and $\mathcal{S}(u)$ consists of those points in $\{u = 1\}$ at which v has frequency value greater or equal than $\frac{3}{2}$. After this point, we show that $\mathcal{R}(u)$ is a regular manifold and we derive estimates on the dimension of $\mathcal{S}(u)$. These latter arguments are standard and essentially the same as in [TT12] (see also [CL07, CL08]). Observe that all of the results in Section 3 and Section 4 are only reliant on the validity of the Euler-Lagrange equations for minimizers of (1.1) and (1.2), and do not require the spanning constraint to be satisfied.

Lastly, as a corollary of the Lipschitz continuity, we prove Theorem 1.2 in Section 5. As mentioned previously, this is the only part of the paper that utilizes the generalization of spanning from [MNR23b, MNR23a]. Within this expanded framework, the spanning constraint is preserved under uniform Dirichlet bounds, and so the main difficulty is to rule out volume loss at infinity in (1.2). This can be done by an energy comparison argument that depends on uniform decay at infinity implied by Theorem 3.1, which also implies the continuity of minimizers and thus existence for (1.1)-(1.2).

1.3.2. *Further remarks.* Here we collect some final observations regarding our results and related problems in the literature.

Remark 1.7 (Connection to optimal partition & segregation problems)**.** As pointed out above, models (1.1) and (1.2) share several similarities with those stemming from optimal partition and segregation problems; this is mainly due to the fact that the homotopic spanning condition (1.3) imposes a local separation property at each point $x_0 \in \{u = 1\}$. More precisely, it forces $\{u = 1\}$ to disconnect any small ball centered at x_0. This, coupled with the variational nature of (1.1) and (1.2), suggests that in $\{u = 1\} \cap B_r(x_0)$ (for r small) we should see an optimal partition of the connected components of $\{u < 1\} \cap B_r(x_0)$. However, note that in contrast to general optimal partition problems, the number of these connected components is not prescribed and could be infinitely many a priori. Only in some special cases when we show that nearby certain free boundary points $\{u < 1\}$ has finitely many components (see Lemma 4.1), is our model locally equivalent to the general framework of [TT12]. Indeed, after applying Theorem 1.1 to obtain the Lipschitz regularity of our solutions and localizing to any ball $B \subset \Omega$ for which there are finitely many connected components of $\{u < 1\}$, we observe that for $v = 1 - u$, the functions $v_i = v\,|_{U_i}$ for connected components $U_1, \ldots, U_{N-1}$ of $\{v > 0\}$ and $U_N = B \setminus \bigcup_{i=1}^{N-1} U_i$, satisfy the hypotheses of the main theorem therein.

We also point out that the same methods used in Section 3 show that the class of functions considered in [TT12, Theorem 1.1] are Lipschitz continuous, showing that the latter condition is unnecessary to add as an a priori assumption therein. This observation is immediate if the forcing term $f(x, u)$ considered in [TT12, Theorem 1.1] is x-independent, since in that case it directly falls under our hypotheses. In the case of x dependence, the result holds from straightforward adaptations of our arguments, exploiting fundamentally the assumption $|f(x, u)| \leq Cu$ in [TT12] (compare with (3.1)), combined with the observation that the hypothesis [TT12, (G3)] is sufficient to assume in place of the validity of the general inner variation identity (1.8).

Equipped with the Lipschitz continuity of our solutions u, together with a discrete spectral gap between the two lowest values (1 and $\frac{3}{2}$) of the frequency function, we in turn obtain an analogous structure for the free boundary $\{u = 1\}$ to that for the segregated system considered in [TT12]. Therein, the authors also use the frequency function to distinguish between the regular and singular parts of the free boundary, and characterize the regular part as the points where the solution blows up to a linear function on either side of the free boundary.

Remark 1.8 (Further analysis of singularities)**.** Classifying the types of free boundary singularities, at least in low dimensions, is one example of a natural follow-up question. The singularities correspond to radially homogeneous solutions of (1.8)-(1.10) with $\Phi = 0$. By rewriting in terms of $1 - u$ and restricting to the unit sphere, these solutions may be identified with critical points of the optimal spectral partitioning problem on the sphere considered in [Bog16, HHOT09]. In light of the asymptotic convergence of the rescaled Allen-Cahn problems (1.4) to the Plateau problem as $\varepsilon \to 0$ and $v \to 0$, one may investigate the relationship between the types of singularities in each problem. The limiting Plateau problem singularity models are conical n-dimensional area-minimizing (in the sense of Almgren) sets in $\mathbb{R}^{n+1}$. Since these have been classified as only Y-singularities when $n = 1$ and only Y- and T-singularities when $n = 2$ (see [Tay76]) and these cones coincide with $\{u = 1\}$ for suitable homogeneous solutions of (1.8)-(1.10), this suggests that the other conical singularities that we find for general critical points in these dimensions should not be present for minimizers of (1.4) if ε is small enough with respect to $\mathbf{W}$ and v.

Furthermore, building on the classification of singularities for general critical points when $n = 1$ and corresponding local structure of the free boundary (see Lemma 3.21 and Theorem 1.1.(ii)), it would also be of interest to classify singularity models for critical points when $n = 2$ and analyze the local structure of the free boundary near singularities there. We refer the reader to the recent related work [OV24], where the authors obtain a structural result close to singularities of frequency $3/2$. The arguments in Section 4 imply that after the two lowest frequencies 1 and $3/2$ (corresponding to regular and Y-points), there is a non-explicit gap between $3/2$ and the next lowest frequency, and that if $\{u = 1\} \cap \mathbb{S}^n$ is smooth and $n \geq 2$, then the frequency is at least 2; cf. Proposition 4.3.

Remark 1.9 (Regularity of stationary harmonic functions)**.** We observe that stationary harmonic functions, i.e., functions $v \in W^{1,2}_{\text{loc}}(\Omega)$ satisfying (3.2) with $G = 0$, in 2 dimensions are Lipschitz and thanks to the work of [Rod16], if this is coupled with a condition of the form $\Delta v = \mu$ for a signed Radon measure μ, the support of μ (which coincides with the points where v is not differentiable) is locally contained in the zero set of a harmonic function. In this regard, our work can be seen to an extension of these results to higher dimensions when μ is non-negative and μ is supported in the zero set of v.

Remark 1.10 (Extension of existence from [MNR23a])**.** In [MNR23a, Theorem 1.2.(i)], the existence of minimizers for

$$\inf \left\{ \varepsilon \int_\Omega |\nabla u|^2 + \frac{F(u)}{\varepsilon^2}\, dx \,:\, \int_\Omega V(u) = v,\, \{u = \delta\} \text{ is } \mathcal{C}\text{-spanning in the sense of (A.3)} \right\}, \quad (1.14)$$

is established in a regime for ε, v, and $1/2 < \delta \leq 1$ that ensures the proximity (in the sense of minimizers converging to minimizers) of (1.14) to either the Plateau problem or its "positive volume" extension

$$\inf \left\{ \text{Per}\,(E;\Omega) : E \subset \Omega,\, |E| = v,\, E^{(1)} \cup \partial^* E \text{ is } \mathcal{C}\text{-spanning} \right\}.$$

The proof there relies on the asymptotic connection between (1.14) and various sharp interface problems. Theorem 1.2 strengthens this result significantly in the case $\delta = 1$ by removing any restrictions on the parameters ε and v: by setting $F = W/\varepsilon^2$ and replacing V with V/v, we have the existence of continuous minimizers for any values of ε and v in (1.14) as long as $\delta = 1$.

Remark 1.11 (Extension to bounded domains)**.** Although we do not consider this problem here, one might also formulate (1.1)-(1.2) on e.g. a bounded open set Ω with a corresponding spanning class $\mathcal{C}$ of smooth loops contained in Ω. The arguments in Theorem 1.1 are local in nature and would thus apply equally well in this scenario as well.

1.4. **Notation.** We will use C to denote constants dependent only on the dimension $n+1$ of the ambient Euclidean space and on the fixed wire frame $\mathbf{W}$ in (1.1) or (1.2) throughout. If a constant C has any additional dependencies on quantities $a, b, \ldots$, we will use the notation $C(a, b, \ldots)$. We denote open balls of radius r centered at x in $\mathbb{R}^{n+1}$ by $B_r(x)$. If $x = 0$, we will omit the dependency on the center. $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure (often used on n-dimensional spheres embedded in $\mathbb{R}^{n+1}$), while $\mathcal{L}^{n+1}$ denotes the $(n+1)$-dimensional Lebesgue measure on $\mathbb{R}^{n+1}$.

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2. Preliminaries

Here we derive the criticality conditions (1.8)-(1.10) for minimizers of (1.1) and (1.2). This follows verbatim the proof of the criticality conditions in [MNR23a, Theorem 1.3], so we provide a short summary of the main steps and appropriate references in lieu of complete details.

Theorem 2.1 (Euler-Lagrange equations)**.** *If $\mathbf{W} = \mathbb{R}^{n+1} \setminus \Omega$ is compact, $\mathcal{C}$ is a spanning class for $\mathbf{W}$ satisfying (1.7), F and V satisfy (H1)-(H3), and u is a minimizer for (1.1) or (1.2), then, denoting by $\Phi = F$ in the former case and $\Phi = F - \lambda V$ for suitable $\lambda \in \mathbb{R}$ in the latter, the relations (1.8)-(1.10) hold.*

Outline of Proof of Theorem 2.1. Beginning with (1.8), we first remark that if $\{f_t\}_{-t_0<t<t_0}$ is a smooth one-parameter family of diffeomorphisms with $\{f_t \neq \mathrm{id}\} \subset\subset \Omega$ and $f_0 = \mathrm{id}$, then $f_t^{-1} \circ \gamma \in \mathcal{C}$ whenever $\gamma \in \mathcal{C}$ (by the homotopic closedness of $\mathcal{C}$). Since $\{u = 1\} \cap \gamma \neq \varnothing$ if and only if $\{u \circ f_t = 1\} \cap f_t^{-1} \circ \gamma \neq \varnothing$, we deduce that $u \circ f_t$ satisfies the spanning constraint if and only if u does. Using these variations which preserve the spanning condition, the inner variational equation (1.8) can be deduced from the formulas in Lemma B.1 by a standard computation, following for example the derivation of the constant mean curvature condition for volume-constrained minimizers of perimeter [Mag12, Theorem 17.20].

The idea behind (1.9) and (1.10) is to find a way to make outer variations which preserve the spanning constraint as well as the condition $0 \leq u \leq 1$. The computations may be repeated exactly as in [MNR23a, Proof of Theorem 1.3] by taking $\varepsilon = 1$ there, so we only give the outline. For (1.10), we construct admissible variations via the following observation: if u minimizes (1.1) or (1.2), $\varphi \in C^1_c(\Omega; [0,\infty))$, and $h \in \mathrm{Lip}_c([0,1); [0,\infty))$, then $u + \sigma h(u)\varphi$ satisfies

$$\{u + \sigma h(u)\varphi = 1\} = \{u = 1\} \quad \text{and} \quad 0 \leq u + \sigma h(u)\varphi \leq 1 \text{ for small enough } \sigma > 0.$$

Therefore, after fixing the volume constraint if necessary by using the volume fixing variations in Lemma B.1, we have a one-parameter family $\{u + \sigma h(u)\varphi\}_\sigma$ of admissible outer variations by positive test functions with which to test minimality. The inequality (1.10) is found by testing the minimality of u against $u + \sigma h_k(u)\varphi$, then letting $\sigma \to 0$, and finally sending $h_k \nearrow \mathbf{1}_{[0,1)}$ in the resulting inequality (see [MNR23a, Proof of Theorem 1.3, Steps 1-2]). We remark that this last step of this computation utilizes the fact that

$$\Phi'(1) = F'(1) - \lambda V'(1) = 0\,,$$

which follows from our assumption (H2).

The argument for (1.9) is similar to (1.10) and follows precisely [MNR23a, Proof of Theorem 1.3, Steps 3-7]. The outer variations we wish to use are of the form $u + \sigma h(u)\varphi$ (up to volume constraints, which are handled using Lemma B.1 again), but this time with $\varphi \in C^1_c(\Omega)$, $|\sigma| < \sigma_0$, and $h \in \mathrm{Lip}_c([0,1))$. Now since h is Lipschitz with $\mathrm{spt}\, h \subset\subset [0,1)$, there is $\sigma_0 > 0$ small enough (depending on h and φ) such that

$$\{u + \sigma h(u)\varphi = 1\} = \{u = 1\} \quad \text{and} \quad u + \sigma h(u)\varphi \leq 1 \text{ for } |\sigma| < \sigma_0.$$

However, $\sigma h(u)\varphi$ is no longer non-negative, and an extra argument is necessary to ensure that the variations remain non-negative. This is achieved by showing that

$$u \text{ is lower-semicontinuous and } \Omega' \subset \Omega \text{ open, connected} \implies u \equiv 0 \text{ on } \Omega' \text{ or } u > 0 \text{ on } \Omega'\,. \quad (2.1)$$

(2.1) then allows one to assume without loss of generality that $u > 0$ on $\mathrm{spt}\,\varphi$, so that $u + \sigma h(u)\varphi > 0$ on $\mathrm{spt}\varphi$ for small enough σ. The proof of (2.1) follows from the fact that $e^{-|\sup \Phi''|r} \fint_{B_r(x_0)} u$ is decreasing for small r, which is derived by testing (1.10) with $\{\varphi_k\}$ approximating $[(r^2 - |x - x_0|^2)/2]_+$ and using the property $\Phi'(0) = 0$ (which follows from (H2)). With these variations in hand, one tests the minimality of u against $u + \sigma h(u)\varphi$, sends $\sigma \to 0$, approximates $\mathbf{1}_{[0,t]}$ by $h_k^t \overset{k\to\infty}{\to} \mathbf{1}_{[0,t]}$ for each $t \in (0,1)$, and integrates the resulting equality in t (cf. Lemma 3.7). □

3. Monotonicity properties and regularity of stationary solutions: Proof of Theorem 1.1.(i)

The main result in this section, Theorem 3.1, establishes Lipschitz regularity for general solutions of the criticality conditions (1.8)-(1.10) (which we refer to as stationary solutions). In Section 3.1, we state Theorem 3.1 and show how it implies Theorem 1.1.(i). The remaining subsections constitute the proof of Theorem 3.1.

3.1. **Statement of Theorem 3.1 and application to proof of Theorem 1.1.(i).** Given a function

$$G \in C^2([0,1]) \text{ such that } G(0) = G'(0) = 0, \tag{3.1}$$

we consider functions $v \in W^{1,2}_{\rm loc}(\Omega;[0,1])$ satisfying

$$\int_\Omega \Big(|\nabla v|^2 + G(v)\Big) \operatorname{div} T - 2\langle \nabla v, \nabla v \nabla T\rangle \, dx = 0 \qquad \text{for all } T \in C^\infty_c(\Omega;\mathbb{R}^{n+1}), \tag{3.2}$$

$$2 \int_\Omega |\nabla v|^2 \varphi \, dx = -\int_\Omega v \Big\{ 2\nabla v \cdot \nabla \varphi + G'(v)\,\varphi \Big\} dx \qquad \text{for all } \varphi \in C^\infty_c(\Omega), \text{ and} \tag{3.3}$$

$$\int_\Omega \varphi \, d\mu = \int_\Omega -2\nabla v \cdot \nabla \varphi - G'(v)\,\varphi \, dx \qquad \text{for all } \varphi \in C^\infty_c(\Omega), \tag{3.4}$$

where μ is a non-negative Radon measure on Ω depending only on v. In particular, we have the weak differential inequality

$$2\Delta v \geq G'(v) \quad \text{in } \Omega\,. \tag{3.5}$$

We will verify that these variational identities are satisfied by $v = 1 - u$ when u minimizes (1.1) or (1.2); see Corollary 3.3 below. In particular, they have natural interpretations in the context of (1.1)-(1.2) (see Remark 1.3).

Observe that the assumptions (3.1) together with the mean value theorem imply that

$$|G(t)| \leq kt^2, \qquad |G'(t)| \leq kt\,, \tag{3.6}$$

where $k = \sup_{[0,1]} |G''|$.

The main result of this section establishes the optimal Lipschitz regularity of v when (3.2)-(3.4) hold. We begin by recalling Almgren's frequency function

$$N_{v,x_0}(r) = \frac{r\int_{B_r(x_0)} |\nabla v|^2\,dx}{\int_{\partial B_r(x_0)} v^2\, d\mathcal{H}^n(x)} = \frac{r D_{x_0,r}(r)}{H_{x_0,r}(r)} \qquad x_0 \in \Omega\,,\ r < \operatorname{dist}(x_0, \partial\Omega)\,.$$

Theorem 3.1 (Lipschitz regularity for critical points)**.** *Suppose* $\mathbf{W} = \mathbb{R}^{n+1} \setminus \Omega$ *is closed,* $G \in C^2([0,1])$ *satisfies* (3.1)*, and* $v \in W^{1,2}_{\rm loc}(\Omega;[0,1])$ *satisfies* (3.2)-(3.4)*.*

(i) *There is* $r_{**} = r_{**}(G,n) > 0$ *with the following property: if* $\Omega' \subset\subset \Omega$ *with* $d = \operatorname{dist}(\partial\Omega, \partial\Omega') > 0$*, then there are* $M = M(\Omega', v) > 0$ *such that*

$$N_{v,x}(r) \leq M \qquad \forall x \in \Omega \text{ with } \operatorname{dist}(x, \Omega') \leq d/2, \qquad r < \min\{\operatorname{dist}(\partial\Omega', \partial\Omega), r_{**}\}\,, \tag{3.7}$$

and $C = C(M,n)$ *such that for any* $x_0 \in \Omega'$ *and* $r < \min\{r_{**}, d/3\}$*,*

$$r[v]_{\mathrm{Lip}(B_{r/2}(x_0))} \leq C\Big(\frac{1}{r^{n-1}} \int_{B_{r(x_0)}} |\nabla v|^2\Big)^{\frac{1}{2}}\,. \tag{3.8}$$

Also, $\mathcal{L}^{n+1}(\tilde\Omega \cap \{v = 0\}) = 0$ *for any connected component* $\tilde\Omega \subset \Omega$ *on which* v *is not identically zero.*

(ii) *If in addition* $\mathbf{W}$ *is compact,* $\nabla v \in L^2(\Omega)$*, and* $\mathcal{L}^{n+1}(\{v < t\}) < \infty$ *for all* $t \in (0,1)$*, then given* $d > 0$*, there is* $M(v,d)$ *such that* $N_{v,x}(r) \leq M$ *for all* $x \in \Omega$ *with* $\operatorname{dist}(x, \partial\Omega) \geq d$ *and* $r < \min\{d, r_{**}\}$*.*

Remark 3.2. The hypothesis "$\mathcal{L}^{n+1}(\{v < t\}) < \infty$ for all $t \in (0,1)$" in part **(ii)** of Theorem 3.1 is motivated by the properties of minimizers $u = 1 - v$ of (1.1). When $n \geq 2$, such minimizers satisfy $u \in L^{2(n+1)/(n-1)}(\Omega)$, where $\frac{2(n+1)}{n-1}$ is the Sobolev dual exponent of 2, while for $n = 1$, we no longer have the desired Sobolev embedding so instead we directly verify these measure bounds on the superlevel sets of u (which correspond to sublevel sets of v). See the proof of Theorem 1.2 in Section 5 for more details.

Before starting the proof of Theorem 3.1, which is split into several intermediate results, we give its implications for our original variational problems (1.1)-(1.2). The following corollary is a more precise statement of Theorem 1.1.(i).

Corollary 3.3 (Lipschitz regularity for stationary solutions and minimizers). *Let $\mathbf{W} = \mathbb{R}^{n+1}\setminus\Omega$ be compact and let $\mathcal{C}$ be a spanning class for $\mathbf{W}$ satisfying (1.7). Suppose that F, V satisfy (H1)-(H3), and u satisfies (1.8)-(1.10). Then, setting $\Phi = F$ in the former case and $\Phi = F - \lambda V$ for suitable $\lambda \in \mathbb{R}$ in the latter case, the following properties hold:*

(i): *$G(t) = \Phi(1-t) - \Phi(1)$ and $v = 1-u$ satisfy (3.1) and (3.2)-(3.4), respectively, for an appropriate choice of non-negative Radon measure μ on Ω;*
(ii): *there is $r_{**} = r_{**}(G,n) > 0$ with the following property: if $\Omega' \subset\subset \Omega$ with $d = \mathrm{dist}(\partial\Omega, \partial\Omega') > 0$, then for any $x_0 \in \Omega'$ and $r < \min\{r_{**}, d/3\}$,*

$$r[u]_{\mathrm{Lip}(B_{r/2}(x_0))} \le C\Big(\frac{1}{r^{n-1}}\int_{B_r(x_0)} |\nabla u|^2\Big)^{\frac{1}{2}}\,, \tag{3.9}$$

where $C = C(\Omega', u, n)$ is defined by (3.7), and
(iii): *if u is a minimizer of either (1.1) or (1.2), given $\varepsilon_0 > 0$ there are $C = C(\varepsilon_0, u, n)$ and $r_{**} = r_{**}(n, G)$ such that for any $x_0 \in \Omega_{\varepsilon_0}$ and $r < \min\{r_{**}, \varepsilon_0/3\}$,*

$$r[u]_{\mathrm{Lip}(B_{r/2}(x_0))} \le C\Big(\frac{1}{r^{n-1}}\int_{B_r(x_0)} |\nabla u|^2\Big)^{\frac{1}{2}}\,, \tag{3.10}$$

where $\Omega_{\varepsilon_0} = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \ge \varepsilon_0\}$.

Proof. Let u be a solution of (1.8)-(1.10).

To prove (i) and (ii): We first check that G satisfies (3.1). The fact that G is C^2 follows from (H1), and trivially $G(0) = 0$. Also, due to (H2),

$$G'(0) = -\Phi'(1) = \begin{cases} -F'(1) = 0 & \text{if } u \text{ minimizes (1.1)} \\ -F'(1) + \lambda V'(1) = 0 & \text{if } u \text{ minimizes (1.2)}\,. \end{cases}$$

Next, (3.2)-(3.3) for $v = 1-u$ follow from substituting G and v into the criticality conditions (1.8)-(1.9) from Theorem 2.1, which applies to u since (H1)-(H3) are satisfied. The existence of a measure μ such that (3.4) holds follows directly from the differential inequality (1.10) and the correspondence between monotone linear functionals on $C_c^\infty(\Omega)$ and non-negative Radon measures [EG92, pg 53]. Applying Theorem 3.1.(i) to v, the conclusion **(ii)** follows immediately.

To prove (iii): Now that (3.1) and (3.2)-(3.4) hold for G and v, respectively, we would like to prove (3.10) by applying Theorem 3.1.(ii) to v on compact subsets of Ω'_{ε_0}, which we define to be those $x \in \Omega_{\varepsilon_0}$ such that v does not vanish identically on the connected component of Ω containing x. This boils down to showing that there exists M such that the frequency bound (3.7) holds uniformly on $\Omega'_{\varepsilon_0/2}$ (which is defined analagously to Ω'_{ε_0}), independently of its subcomponents. According to Theorem 3.1.(ii), this uniform upper frequency bound is true if $\nabla v \in L^2$ and $\mathcal{L}^{n+1}(\{v < t\}) < \infty$ for all $t \in (0,1)$. Now in either case that u minimizes (1.1) or (1.2), $\nabla u = \nabla(1-v) \in L^2(\Omega)$. Furthermore, since $u \to 0$ uniformly as $|x| \to \infty$ in (1.1), it is immediate that $\mathcal{L}^{n+1}(\{v < t\}) = \mathcal{L}^{n+1}(\{u > 1-t\}) < \infty$ for all $t \in (0,1)$. Similarly, in (1.2), since $\int V(u) = 1$ and $V > 0$ on $(0,1]$, it must be the case that $\mathcal{L}^{n+1}(\{v < t\}) = \mathcal{L}^{n+1}(\{u > 1-t\}) < \infty$ for all $t \in (0,1)$. ☐

3.2. **Monotonicity formulae.** Our first result is a semilinear version of the classical monotonicity formula for harmonic maps (see, e.g., [LW08, Proposition 3.3.6]).

Lemma 3.4 (Almost-monotonicity of normalized Dirichlet energy)**.** *Let $v \in W^{1,2}_{\rm loc}(\Omega;[0,1])$ satisfy the inner variation equation (3.2). Then, given any $x_0 \in \Omega$ and $r_0 = \operatorname{dist}(x_0, \partial\Omega)$, we have that*

$$\frac{d}{ds}\Big(\frac{1}{s^{n-1}}\int_{B_s(x_0)} |\nabla v|^2\Big) = \frac{2}{s^{n-1}}\int_{\partial B_s(x_0)} |\nabla v\cdot\hat{x}|^2\, d\mathcal{H}^n(x) + M(s) \qquad \text{for a.e. } s\in(0,r_0), \quad (3.11)$$

where $\hat{x} = (x-x_0)/|x-x_0|$ and

$$M(s) = \frac{n+1}{s^n}\int_{B_s(x_0)} G(v) - \frac{1}{s^{n-1}}\int_{\partial B_s(x_0)} G(v)\, d\mathcal{H}^n. \quad (3.12)$$

Proof. Without loss of generality, let us assume $x_0 = 0$, and fix $s < r_0$. Let us consider the vector field $T = \eta(|x|)x$, where $\eta \in C^\infty_c(B_s)$ is a radially symmetric non-negative function. Then,

$$\nabla T = \eta(|x|)\mathrm{Id} + |x|\eta'(|x|)\hat{x}\otimes\hat{x}.$$

Therefore,

$$\langle \nabla v, \nabla v \nabla T\rangle = \eta(|x|)|\nabla v|^2 + \eta'(|x|)|x||\nabla v\cdot\hat{x}|^2,$$

and

$$\operatorname{div} T = (n+1)\eta(|x|) + |x|\eta'(|x|).$$

Plugging in the previous expressions into the inner variation (3.2), we obtain

$$\begin{aligned}(n-1)\int_{B_s} \eta(|x|)|\nabla v|^2\, dx \;&=\; \int_{B_s} \eta'(|x|)|x|(2|\nabla v\cdot\hat{x}|^2 - |\nabla v|^2)\, dx \\ &\quad - \int_{B_s} G(v)\big[(n+1)\eta(|x|) + |x|\eta'(|x|)\big]\, dx\end{aligned} \quad (3.13)$$

Given $s \in (0, r_0)$, after an appropriate regularization procedure, we may consider the following family of Lipschitz test functions:

$$\eta_k(t) = \begin{cases} 1, & t\in[0, s-1/k],\\ k(s-t), & t\in[s-1/k, s],\end{cases}$$

where $k > \frac{1}{s}$. Let us notice that since $\eta_k \to 1_{[0,s]}$ as $k\to\infty$ and $s \mapsto \int_{B_s} |\nabla v|^2$ is absolutely continuous in $(0,r_0)$, we can take $k\to\infty$ in (3.13) for almost every $s\in(0,r_0)$ to deduce

$$(n-1)\int_{B_s} |\nabla v|^2 = s\int_{\partial B_s} \big(|\nabla v|^2 - 2|\nabla v\cdot\hat{x}|^2 + G(v)\big)\, d\mathcal{H}^{n-1}(x) - (n+1)\int_{B_s} G(v) \quad \text{for a.e. } s. \quad (3.14)$$

Dividing by s^n, we can rewrite (3.14) as

$$\frac{d}{ds}\Big(\frac{1}{s^{n-1}}\int_{B_s} |\nabla v|^2\Big) = \frac{2}{s^{n-1}}\int_{\partial B_s} |\nabla v\cdot\hat{x}|^2\, d\mathcal{H}^n(x) + \frac{n+1}{s^n}\int_{B_s} G(v) - \frac{1}{s^{n-1}}\int_{\partial B_s} G(v)\, d\mathcal{H}^n\,.$$

This is precisely the claimed identity (3.11). ☐

The next result shows the almost-subharmonicity of v^2, which is a slight variation of [MNR23b, Theorem 1.3-Step 3].

Lemma 3.5 (Almost-subharmonicity of v^2)**.** *Let $v \in W^{1,2}_{\rm loc}(\Omega;[0,1])$ satisfy the outer variation equation (3.3) with G satisfying (3.1). If $x_0\in\Omega$ and $r_0 = \operatorname{dist}(x_0,\partial\Omega)$, then, for $k = \sup_{[0,1]}|G''|$, the function*

$$g(r) = e^{\frac{kr^2}{4}} \fint_{B_r(x_0)} v^2\, dx \quad (3.15)$$

is increasing on $(0, r_0)$*. Furthermore, if* v^* *is the precise representative of* v *(c.f. (A.2)),*

$$\int_{B_r(x_0)} v^2\, dx \le \frac{2r}{n+1} \int_{\partial B_r(x_0)} (v^*)^2 d\mathcal{H}^n \tag{3.16}$$

for every $r \in (0, \min\{r_*, r_0\})$ *with* $r_* = \sqrt{(n+1)/k}$*.*

Remark 3.6. Note that an immediate consequence of (3.16) is the following statement: if $v \in W^{1,2}(\Omega; [0,1])$ satisfies the outer variation equation (3.3) with G satisfying (3.1) and $\int_{\partial B_r(x)} v^2\, d\mathcal{H}^n = 0$ for some $x \in \Omega$ and $r < \min\{\operatorname{dist}(x, \partial\Omega), r_*\}$, then $v \equiv 0$ on $B_r(x)$.

Proof. Assume without loss of generality $x_0 = 0$ and let $r \in (0, r_0)$. Testing (3.3) with $\{\varphi_k\}_k \subset C_c^1(\Omega; [0,\infty))$ such that $\varphi_k \to \varphi(x) := [(r^2 - |x|^2)/2]_+$ uniformly and $\nabla\varphi_k \to \nabla\varphi = -x\chi_{[0,r]}$ in L^2, we obtain

$$\int_{B_r} 2\, v\, (\nabla v \cdot x)\, dx \ge \int_{B_r} G'(v) v \varphi\, dx\,. \tag{3.17}$$

On the other hand, recalling from (3.6) that $|G'(v)v| \le kv^2$ and using that $0 \le \varphi \le r^2/2$, the estimate (3.17) in turn yields

$$\int_{B_r} 2\, v\, (\nabla v \cdot x)\, dx \ge -\frac{kr^2}{2} \int_{B_r} v^2\, dx. \tag{3.18}$$

So, introducing the notation

$$\phi(r) := \fint_{B_r} v^2\, dx,$$

and using (3.18) leads us to the lower bound

$$\phi'(r) = \frac{1}{r} \fint_{B_r} 2\, v\, (\nabla v \cdot x)\, dx \ge -\frac{kr}{2} \fint_{B_r} v^2\, dx = -\frac{kr}{2} \phi(r)\,,$$

for a.e. $r \in (0, r_0)$. By combining this estimate with the absolute continuity of g, we deduce its monotonicity. Lastly, towards (3.16), we differentiate g to get

$$0 \le \frac{1}{r^{n+1}} \int_{\partial B_r} v^2\, d\mathcal{H}^n - \frac{n+1}{r^{n+2}} \int_{B_r} v^2\, dx + \frac{kr}{2r^{n+1}} \int_{B_r} v^2\, dx \quad \text{for a.e. } r < r_0,$$

which in turns implies that

$$\int_{B_r} v^2\, dx \le \frac{2r}{n+1} \int_{\partial B_r(x_0)} v^2 d\mathcal{H}^n \quad \text{for almost every } r < \min\{r_0, \sqrt{(n+1)/k}\}\,. \tag{3.19}$$

To prove (3.16) for *all* small r, we choose a more careful Lebesgue representative of v, since the measure zero set for which (3.19) fails depends on the "choice" of v. We thus consider the precise representative

$$v^*(x) = \lim_{s\to 0} \fint_{B_s(x)} v(y)\, dy\,,$$

with the limit existing for every $x \in \Omega$ due to the monotonicity (3.15). Furthermore, $v^*(ty)$ is absolutely continuous as a function of $t \in (0, r_0)$ for $\mathcal{H}^n$-a.e. $y \in \mathbb{S}^n$ [EG92, Section 4.9.2]. So, by fixing small r, letting $t_j \to r$ be a sequence of radii for which (3.19) holds for t_j and v^*, and applying the dominated convergence theorem with $v(t_j y) \to v(ry)$ for $\mathcal{H}^n$-.a.e $y \in \partial B_1$, we find that

$$\int_{B_r} v^2\, dx = \lim_{j\to\infty} \int_{B_{t_j}} v^2\, dx \le \limsup_{j\to\infty} \frac{2t_j}{n+1} \int_{\partial B_{t_j}} v^2\, d\mathcal{H}^n = \frac{2r}{n+1} \int_{\partial B_r} (v^*)^2\, dx\,.$$

Thus (3.16) holds as claimed for all $r < \min\{r_0, r_*\}$. ☐

We also have the almost-subharmonicity of v by [MNR23a, Proof of Theorem 1.3, Step 3], which is written for minimizers u of (1.2) but in fact only relies on the Euler-Lagrange equations (1.8)-(1.10), and may be rewritten in terms of $v = 1 - u$. The proof follows analogous reasoning to that of Lemma 3.5 but since it is short, only exploits the outer variation equation (3.3) and allows us to define the precise representative of v as the limit of integral averages at all points in Ω, we include it here for the convenience of the reader.

Lemma 3.7 (Almost-subharmonicity of v)**.** *Let $v \in W^{1,2}_{\mathrm{loc}}(\Omega; [0,1])$ satisfy the outer variation equation (3.3) with G satisfying (3.1). If $x_0 \in \Omega$ and $r_0 = \mathrm{dist}(x_0, \partial\Omega)$, then for $k = \sup_{[0,1]} |G''|$ the function*

$$r \mapsto e^{\frac{kr^2}{8}} \fint_{B_r(x_0)} v\, dx \tag{3.20}$$

is increasing on $(0, r_0)$. As a consequence, with v^* denoting the precise representative (see (A.2)), we have

$$v^*(x_0) = \lim_{r\to 0} \fint_{B_r(x_0)} v\, dx \qquad \textit{for every } x_0 \in \Omega\,. \tag{3.21}$$

Proof. We assume $x_0 = 0$ again, and once again recall the estimate (3.6) for G. By the same regularization procedure as in the proof of Lemma 3.5, we now test (3.4) with $\varphi := [(r^2 - |x|^2)/2]_+$ and estimate

$$\int_{B_r} 2\,\nabla v \cdot x\, dx \geq \int_{B_r} G'(v)\, \varphi\, dx \geq -\frac{kr^2}{2} \int_{B_r} v\,. \tag{3.22}$$

Since the function

$$\psi(r) := \fint_{B_r} v\, dx$$

satisfies

$$\psi'(r) = \frac{1}{r} \fint_{B_r} \nabla v \cdot x\, dx$$

for a.e. $r \in (0, \mathrm{dist}(0, \partial\Omega))$, the estimate (3.22) implies that for a.e. $r \in (0, \mathrm{dist}(0, \partial\Omega)\})$, we have

$$\psi'(r) \geq -\frac{kr}{4} \fint_{B_r} v\, dx = -\frac{kr}{4} \psi(r)\,.$$

From this inequality we easily conclude (3.20), and (3.21) in turn follows immediately, since we have in particular just demonstrated the limit therein exists for all points x_0 in Ω. The upper-semicontinuity of v^* is a standard consequence of the monotonicity (3.20). □

Remark 3.8 (Identification of v with its precise representative)**.** Given a function $v \in W^{1,2}_{\mathrm{loc}}(\Omega; [0,1])$ satisfying (3.3), Lemma 3.5 allows us to make a canonical choice for $v(x)$ via (3.21), by identifying v with its precise representative v^*. **For the rest of the paper, we identify v with v^*.**

We address now the key monotonicity property satisfied by v. Given $x_0 \in \Omega$, and $r \in (0, \mathrm{dist}(x_0, \partial\Omega))$ we recall Almgren's frequency function

$$N_{v,x_0}(r) = \frac{\int_{B_r(x_0)} |\nabla v|^2}{\int_{\partial B_r(x_0)} v^2 d\mathcal{H}^n} = \frac{D_{v,x_0}(r)}{H_{v,x_0}(r)}\,, \tag{3.23}$$

where

$$\begin{aligned} D_{v,x_0}(r) &= \frac{1}{r^{n-1}} \int_{B_r(x_0)} |\nabla v|^2\,, \\ H_{v,x_0}(r) &= \frac{1}{r^n} \int_{\partial B_r(x_0)} v^2 d\mathcal{H}^n\,, \end{aligned} \tag{3.24}$$

so that $N_{v,x_0}(r)$ is well defined when $H_{v,x_0}(r) > 0$. When it is clear from context, we will omit the dependency on v and/or x_0 for the frequency function. The next lemma shows the almost-monotonicity of (3.23) for v.

Lemma 3.9 (Almost-monotonicity of the frequency)**.** *Let $v \in W^{1,2}_{\rm loc}(\Omega; [0,1])$ satisfy both (3.2) and (3.3) with G satisfying (3.1). Then, there exists $\kappa = \kappa(\sup_{[0,1]} |G''|, n) \geq 0$ such that for any $x_0 \in \Omega$ with $H_{v,x_0}(r) > 0$ for some $r < \min\{r_0, r_*\}$, the function*

$$r \to e^{\frac{\kappa r^2}{2}}(N_{v,x_0}(r) + 1) \tag{3.25}$$

is well-defined (absolutely continuous) and non-decreasing on $(\inf\{r : H_{v,x_0}(r) > 0\}, \min\{r_0, r_*, 1\})$ with r_0, r_* as in Lemma 3.5. Moreover, if $G = 0$ then N_{v,x_0} is increasing on $(\inf\{r : H_{v,x_0}(r) > 0\}, r_0)$ and is constant if and only if v is homogeneous of degree $N_{v,x_0}(0^+)$.

Remark 3.10. The conclusion of Lemma 3.9, together with Remark 3.6 in particular allows one to make sense of the limit $N_{v,x_0}(0^+) := \lim_{r\to 0^+} N_{v,x_0}(r)$, provided that $H_{v,x_0}(r) > 0$ for all $r > 0$ sufficiently small.

Proof. We assume, without loss of generality that $x_0 = 0$ and omit dependency of N, D and H on x_0. If $H(r) > 0$ for some $r < \min\{r_0, r_*\}$, then by Remark 3.6, $H(s) > 0$ for all $r < s < \min\{r_0, r_*\}$. Thus $\{r < \min\{r_0, r_*\} : H(r) = 0\}$ coincides with the interval $(0, \inf\{r : H(r) > 0\}]$, so we may as well restrict ourselves to the interval $(\inf\{r : H(r) > 0\}, \min\{r_0, r_*\})$ where $H > 0$. Clearly on this interval N is absolutely continuous, since both H and D are. Since

$$N'(r) = \frac{D'(r)H(r) - H'(r)D(r)}{H(r)^2}$$

the monotonicity of (3.25) is equivalent to the bound $\partial_r[\log(N(r) + 1)] \geq -\kappa r$, which may in turn be rewritten as

$$D'(r)H(r) - H'(r)D(r) \geq -\kappa r\Big(H(r)^2 + D(r)H(r)\Big). \tag{3.26}$$

Having (3.26) in mind as our target, we compute each one of the terms, starting with $D'(r)$ which, thanks to (3.11), has the form

$$D'(r) = \frac{2}{r^{n-1}} \int_{\partial B_r} |\nabla v \cdot \hat{x}|^2 d\mathcal{H}^n + M(r)\,. \tag{3.27}$$

On the other hand, testing (3.3) with $\varphi \to \chi_{B_r}$ we deduce

$$\int_{B_r} \left(2|\nabla v|^2 + G'(v)v\right) dx = 2\int_{\partial B_r} (\nabla v \cdot \hat{x})v \, d\mathcal{H}^n. \tag{3.28}$$

Thus, differentiating H and using (3.28) yields

$$\begin{aligned} H'(r) &= \frac{2}{r^n}\int_{\partial B_r} (\nabla v \cdot \hat{x})v \, d\mathcal{H}^n \\ &= \frac{1}{r^n}\int_{B_r} (2|\nabla v|^2 + G'(v)v)\, dx\,, \end{aligned} \tag{3.29}$$

or equivalently,

$$D(r) = \frac{1}{r^{n-1}} \int_{\partial B_r} (\nabla v \cdot \hat{x})v \, d\mathcal{H}^n - I(r), \tag{3.30}$$

where

$$I(r) = \frac{1}{2r^{n-1}} \int_{B_r} G'(v)v \, dx. \tag{3.31}$$

Altogether, (3.27), (3.29), and (3.30) yield the estimate

$$D'(r)H(r) - H'(r)D(r) = \frac{2}{r^{n-1}} H(r) \int_{\partial B_r} |\nabla v \cdot \hat{x}|^2 d\mathcal{H}^n - \frac{2}{r^n} D(r) \int_{\partial B_r} (\nabla v \cdot \hat{x})v \, d\mathcal{H}^n$$

$$
\begin{aligned}
&+M(r)H(r) && (3.32)\\
&= \frac{2}{r^{2n-1}}\Bigg(\int_{\partial B_r} v^2 d\mathcal{H}^{n-1}\int_{\partial B_r}|\nabla v\cdot\hat{x}|^2 d\mathcal{H}^n - \Big(\int_{\partial B_r}(\nabla v\cdot\hat{x})v\,d\mathcal{H}^n\Big)^2\Bigg)\\
&\quad +H(r)M(r)+\frac{2I(r)}{r^n}\int_{\partial B_r}(\nabla v\cdot\hat{x})v\,d\mathcal{H}^n\\
&\geq H(r)M(r)+\frac{2I(r)}{r^n}\int_{\partial B_r}(\nabla v\cdot\hat{x})v\,d\mathcal{H}^n && (3.33)
\end{aligned}
$$

where in the last line we have used Cauchy-Schwarz.

Our goal now is to bound the last line in (3.33) for $r<\min\{r_*,r_0,1\}$. With this idea in mind, let us notice that the bound $|G(t)|\leq kt^2$ (from (3.6)) and (3.16) (which applies since $r<\min\{r_*,r_0\}$) together imply that

$$
\begin{aligned}
|M(r)| &\leq \frac{n+1}{r^n}\int_{B_r} G(v)\,dx+\frac{1}{r^{n-1}}\int_{\partial B_r} G(v)d\mathcal{H}^n && (3.34)\\
&\leq \frac{k(n+1)}{r^n}\int_{B_r} v^2\,dx+\frac{k}{r^{n-1}}\int_{\partial B_r} v^2 d\mathcal{H}^n\\
&\leq \frac{k(n+1)}{r^n}\int_{\partial B_r}\frac{2r}{n+1}v^2\,d\mathcal{H}^n+\frac{k}{r^{n-1}}\int_{\partial B_r}v^2 d\mathcal{H}^n = 3krH(r),
\end{aligned}
$$

where $k=\sup_{[0,1]}|G''|$, and similarly

$$
|I(r)|\leq \frac{1}{2r^{n-1}}\int_{B_r} kv^2\,dx \leq \frac{1}{2r^{n-1}}\cdot\frac{2r}{n+1}\int_{\partial B_r} kv^2\,d\mathcal{H}^n = \frac{kr^2}{n+1}H(r). \qquad (3.35)
$$

Additionally, from (3.30) and (3.35), we deduce that

$$
\frac{1}{r^n}\int_{\partial B_r}|(\nabla v\cdot\hat{x})v|\,d\mathcal{H}^n \leq \frac{kr}{n+1}H(r)+\frac{D(r)}{r}. \qquad (3.36)
$$

Thus, by combining (3.33), (3.34), (3.35), and (3.36) we deduce

$$
D'(r)H(r)-H'(r)D(r)\geq -3krH(r)^2-\frac{kr^2}{n+1}H(r)\Big[\frac{2kr}{n+1}H(r)+\frac{D(r)}{r}\Big]
$$

which, since $r<1$, yields (3.26) for suitable κ depending on k and n.

Let us finish by observing that in the absence of potential G, the classical frequency monotonicity formula holds, which amounts to the inequality

$$
D'(r)H(r)-H'(r)D(r)\geq 0\,.
$$

Furthermore, one has the usual characterization of the case when $r\mapsto N(r)$ is constant by analyzing the case when this is an equality. ☐

Later, we will need information on functions satisfying a lower frequency bound. Towards this end, we give the following almost-monotonicity result for the normalized L^2 spherical averages of functions satisfying both the inner and outer variation equations (see, for instance, [FR22, Lemma 4.2] or [DLS11, Corollary 3.18]).

Corollary 3.11. *Let $v\in W^{1,2}_{\mathrm{loc}}(\Omega;[0,1])$ satisfy both (3.2) and (3.3) with G satisfying (3.1). Then given $\alpha>0$, there exists a constant $\kappa_1>0$ depending on n, $\sup_{[0,1]}|G''|$, and α such that the following holds: if $x_0\in\Omega$ and $N_{v,x_0}(0^+)\geq\alpha$, then the function*

$$
\phi(r)=\frac{e^{\kappa_1 r}}{r^{2\alpha+n}}\int_{\partial B_r(x_0)} v^2\,d\mathcal{H}^n
$$

is non-decreasing for $r\in(0,\min\{\mathrm{dist}(x_0,\partial\Omega),r_*,1\})$.

Proof. As usual, we omit dependency on v and x_0 for H and N. Also, recalling Remark 3.6 and the triviality of the claim if $H \equiv 0$, we may as well assume we are working on an interval of scales r where $H(r) > 0$. We will choose κ_1 at the end. Noticing that $\phi(r) = \frac{e^{\kappa_1 r} H(r)}{r^{2\alpha}}$, we can compute directly its derivative using (3.29), yielding

$$
\begin{aligned}
\phi'(r) &= \frac{\phi(r)}{r}\Big(\frac{rH'(r)}{H(r)} - 2\alpha + \kappa_1 r\Big), \\
&= \frac{\phi(r)}{r}\Big(2N(r) + \frac{1}{r^{n-1}H(r)}\int_{B_r} G'(v)v\,dx - 2\alpha + \kappa_1 r\Big). \qquad (3.37)
\end{aligned}
$$

On the other hand, thanks to Lemma 3.5 and (3.6), we deduce that

$$
\frac{1}{r^{n-1}H(r)}\int_{B_r} |G'(v)v|\,dx \le \frac{2k}{n+1}r^2, \qquad (3.38)
$$

for $k = \sup_{[0,1]}|G''|$. Additionally, from Lemma 3.9 and our assumption that $\lim_{r\to 0^+} N_{v,x_0}(r) \ge \alpha$, we have $N(r) \ge e^{-\frac{\kappa r^2}{2}}(\alpha+1) - 1$. Combining this with (3.37) and (3.38) yields

$$
\begin{aligned}
\phi'(r) &\ge \frac{\phi(r)}{r}\Big(2(e^{-\frac{\kappa r^2}{2}} - 1)(\alpha+1) - \frac{2k}{n+1}r^2 + \kappa_1 r\Big) \\
&\ge \frac{\phi(r)}{r}\Big(-C(\alpha+1)r^2 - \frac{2k}{n+1}r^2 + \kappa_1 r\Big) \\
&= \phi(r)\Big(-C(\alpha+1)r - \frac{2k}{n+1}r + \kappa_1\Big), \qquad (3.39)
\end{aligned}
$$

for C depending on $\kappa = \kappa(k,n)$ from Lemma 3.9. Finally, since $r \le 1$, we can take κ_1 large enough (with the claimed dependencies) in (3.39) to make the right-hand side positive and thus conclude the proof. □

3.3. **A criterion for Hölder regularity.** Here we show that locally uniform lower and upper bounds on the frequency function yield a locally uniform Hölder bound.

Lemma 3.12 (Local Hölder regularity from frequency bounds)**.** *Suppose that G satisfies (3.1), $v \in W^{1,2}_{\mathrm{loc}}(\Omega;[0,1])$ satisfies (3.2)-(3.3), $\{v > 0\}$ is relatively open in Ω, and $2\Delta v = G'(v)$ in the classical sense in $\{v > 0\}$. Then there exists $r_{**}(\alpha, n, \sup_{[0,1]}|G''|) \in (0, r_*]$ with the following property:*

if $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ and there are $\alpha \in (0,1]$, and $M > 0$, such that

$$
\alpha \le N_{v,x_0}(0^+) \quad \text{for each } x_0 \in \Omega_2 \cap \partial\{v = 0\}, \quad \text{and} \qquad (3.40)
$$

$$
N_{v,x_0}(r) \le M \quad \text{for each } x_0 \in \Omega_2 \cap \overline{\{v > 0\}} \text{ and } 0 < r < \min\{r_{**}, \operatorname{dist}(\partial\Omega_2, \partial\Omega)/2\} \qquad (3.41)
$$

then there is $C = C(\alpha, M, n)$ such that given $x_0 \in \overline{\Omega_1}$ and $0 < 2r \le r_0 \le \min\{\operatorname{dist}(\partial\Omega_1, \partial\Omega_2)/3, r_{**}\}$,

$$
r^\alpha [v]_{C^\alpha(B_r(x_0))} \le C\Big(\frac{1}{r_0^{n-1}}\int_{B_{r_0}} |\nabla v|^2\Big)^{\frac{1}{2}} \quad \text{if } \alpha < 1 \quad \text{and} \qquad (3.42)
$$

$$
r\|v\|_{\mathrm{Lip}(B_r(x_0))} \le C\Big(\frac{1}{r_0^{n-1}}\int_{B_{r_0}} |\nabla v|^2\Big)^{\frac{1}{2}} \quad \text{if } \alpha = 1. \qquad (3.43)
$$

Proof. The estimates (3.42)-(3.43) would follow from obtaining $C(\alpha, M, n) > 0$ such that

$$
\int_{B_r(x_0)} |\nabla v|^2 \le C\Big(\frac{r}{r_0}\Big)^{2\alpha+n-1}\int_{B_{r_0}(x_0)} |\nabla v|^2, \qquad (3.44)
$$

for all $x_0 \in \overline{\Omega_1}$ and $0 < r \le r_0 \le \min\{\operatorname{dist}(\partial\Omega_1, \partial\Omega_2)/3, r_{**}\}$. Indeed, from this, a direct application of Campanato's criterion (see e.g. [Mag12, Theorem 6.1]) yields (3.42). For reasons which will be evident from the arguments, we choose $r_{**} < \min\{1, r_*\}$ (cf. Lemma 3.5) small enough so that, again setting $k = \sup_{[0,1]} |G''|$ and recalling κ from Lemma 3.9 and κ_1 from Lemma 3.11, we have

$$\alpha/2 \le e^{-\kappa r^2/2}(\alpha+1) - 1\,, \quad e^{kr^2/8} \le 2\,, \quad \text{and} \quad e^{\kappa_1 r} \le 2 \qquad \text{for all } r \le r_{**}\,. \tag{3.45}$$

Since r_* and κ depend on n and k and κ_1 depends on n, k, and α, we observe that r_{**} depends on n, k, and α.

Before proving (3.44), we introduce the notations

$$\varepsilon := \operatorname{dist}(\partial\Omega_1, \partial\Omega_2) \quad \text{and} \quad \Omega_2^t := \{x \in \Omega_2 : \operatorname{dist}(x, \partial\Omega_2) \ge t\}$$

and make a preliminary observation. We claim that if $x \in \{v > 0\}$, then

$$\int_{B_t(x)} |\nabla v|^2 \le 2\Big(\frac{t}{s}\Big)^{n+1} \int_{B_s(x)} |\nabla v|^2 \qquad \forall\, 0 < s < t \le \min\{\operatorname{dist}(x, \{v=0\}), r_{**}\}\,. \tag{3.46}$$

To prove (3.46), we can use the equation $2\Delta v = G'(v)$ combined with Bochner's formula to deduce that in $\{v > 0\}$,

$$\frac{1}{2}\Delta|\nabla v|^2 = |D^2 v|^2 + \frac{1}{2}G''(v)|\nabla v|^2 \ge -\frac{k}{2}|\nabla v|^2\,.$$

Thus $|\nabla v|^2$ is almost subharmonic, and so thanks to Lemma 3.7 (note that the latter merely relies on an estimate of the above form), we see that

$$t \to e^{\frac{kt^2}{8}} \fint_{B_t(x)} |\nabla v|^2$$

is non-decreasing on $(0, \operatorname{dist}(x, \{v=0\}))$. Consequently, for $t < s < r_{**}$, (3.45) implies (3.46).

The proof of (3.44) at $x_0 \in \overline{\Omega}_1$ is split into five cases depending on $d = \operatorname{dist}(x_0, \partial\{v=0\})$, r, and r_0: i) $d = 0$, ii) $r \ge r_0/10$, iii) $0 < d \le r < r_0/10$, iv) $0 < r < d < r_0/10$ and v) $0 < r < r_0/10 \le d$.

To prove (3.44) *if* $d = 0$: Since it will be useful later, we prove (3.44) for any $x_0 \in \Omega_2^{2\varepsilon/3}$ such that $d = 0$. Note that $v(x_0) = 0$ since $\{v > 0\}$ is relatively open. Then by (3.45), Lemma 3.9, the lower frequency bound (3.40), and the upper frequency bound (3.41) we deduce that

$$\alpha/2 \le e^{-\kappa r^2/2}(\alpha+1) - 1 \le N_{v,x_0}(r) \le M \quad \forall\, 0 < r \le r_{**}\,. \tag{3.47}$$

By combining (3.47) with Corollary 3.11 and Lemma 3.5, we may estimate

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla v|^2\,dx &\le \frac{M}{r}\int_{\partial B_r(x_0)} v^2\,d\mathcal{H}^n \\
&\le \frac{M}{r}\Big(\frac{r}{r_0}\Big)^{2\alpha+n} e^{\kappa_1 r_0} \int_{\partial B_{r_0}(x_0)} v^2\,d\mathcal{H}^n \\
&\le M\Big(\frac{r}{r_0}\Big)^{2\alpha+n-1} \frac{2e^{\kappa_1 r_0}}{\alpha} \int_{B_{r_0}(x_0)} |\nabla v|^2.
\end{aligned} \tag{3.48}$$

Since $e^{\kappa_1 r_0} \le e^{\kappa_1 r_{**}}$ and κ_1 from Corollary 3.11 depends on n, k, and α, after decreasing r_{**} depending on κ_1 if needed, we have proved (3.44) with $C(\alpha, M) = \frac{2M}{\alpha}$.

To prove (3.44) *if* $r \ge r_0/10$: Since $10r/r_0 \ge 1$, we have

$$\int_{B_r(x_0)} |\nabla v|^2 \le \int_{B_{r_0}(x_0)} |\nabla v|^2 \le 10^{2\alpha+n-1}\Big(\frac{r}{r_0}\Big)^{2\alpha+n-1} \int_{B_{r_0}(x_0)} |\nabla v|^2\,,$$

which is (3.44) with constant $C(\alpha, n) = 10^{2\alpha+n-1}$.

To prove (3.44) *if* $0 < d \le r < r_0/10$. First, since $0 < d \le r < \varepsilon/3$ and $x_0 \in \overline{\Omega_1}$ (so $\text{dist}(x, \partial\Omega_2) \ge \varepsilon$), we may choose $y \in \{v = 0\} \cap \Omega_2^{2\varepsilon/3}$ such that $d = \text{dist}(x_0, \{v = 0\}) = |x_0 - y| \le r$. Since also $r < r_0/10$, we thus have the inclusions

$$B_r(x_0) \subset B_{2r}(y) \subset B_{r_0}(y)\,.$$

By these inclusions and (3.48) applied at $y \in \{v = 0\} \cap \Omega_2^{2\varepsilon/3}$,

$$\int_{B_r(x_0)} |\nabla v|^2 \le \int_{B_{2r}(y)} |\nabla v|^2 \le \frac{2M}{\alpha}\Big(\frac{2r}{r_0}\Big)^{2\alpha+n-1} \int_{B_{r_0}(y)} |\nabla v|^2\,,$$

which is (3.44) with constant $C(\alpha, M, n) = \frac{2^{2\alpha+n}M}{\alpha}$.

To prove (3.44) *if* $0 < r < d < r_0/10$: First, note that since $0 < r < d$, then either $B_r(x_0) \subset \{v = 0\}$ or $B_r(x) \subset \{v > 0\}$. If $B_r(x) \subset \{v = 0\}$, the the estimate (3.44) is trivial, so we may as well assume that $B_r(x_0) \subset \{v > 0\}$. Similar to the previous case, since $0 < d \le r_0 < \varepsilon/3$ and $x_0 \in \overline{\Omega_1}$, we may choose $y \in \{v = 0\} \cap \Omega_2^{2\varepsilon/3}$ such that $|x_0 - y| = d$. Now, observe that $\text{dist}(y, \partial B_{r_0}(x_0)) = r_0 - d > \frac{9r_0}{10}$, so $B_{9r_0/10}(y) \subset B_{r_0}(x_0)$. Combining this elementary observation with the estimate (3.48) already established for $y \in \{v = 0\} \cap \Omega_2^{2\varepsilon/3}$, we find that

$$\begin{aligned}
\int_{B_d(x_0)} |\nabla v|^2 &\le \int_{B_{2d}(y)} |\nabla v|^2 \\
&\le \frac{2M}{\alpha}\Big(\frac{18d}{10r_0}\Big)^{2\alpha+n-1} \int_{B_{\frac{9r_0}{10}}(y)} |\nabla v|^2 \\
&\le C(\alpha, M, n)\Big(\frac{2d}{r_0}\Big)^{2\alpha+n-1} \int_{B_{r_0}(x_0)} |\nabla v|^2.
\end{aligned} \tag{3.49}$$

Hence, by combining (3.46) at scales $r < d$ and (3.49), we deduce that for $r < d$,

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla v|^2 &\le 2\Big(\frac{r}{d}\Big)^{n+1} C(\alpha, M, n)\Big(\frac{2d}{r_0}\Big)^{2\alpha+n-1} \int_{B_{r_0}(x_0)} |\nabla v|^2 \\
&= 2^{2\alpha+n} C(\alpha, M, n)\Big(\frac{d}{r}\Big)^{2\alpha-2}\Big(\frac{r}{r_0}\Big)^{2\alpha+n-1} \int_{B_{r_0}(x_0)} |\nabla v|^2\,.
\end{aligned}$$

Since $\alpha \in (0, 1]$ and $r < d$ together imply that $(d/r)^{2\alpha-2} \le 1$, this yields (3.44) when $0 < r < d < r_0/10$ with constant $2^{2\alpha+n}C(\alpha, M, n)$.

To prove (3.44) *if* $0 < r < r_0/10 \le d$: Again, we have either $B_r(x_0) \subset \{v = 0\}$ or $B_r(x_0) \subset \{v > 0\}$, and (3.44) is trivial in the former case. So we take $B_r(x_0) \subset \{v > 0\}$. By (3.46) applied at scales r and $r_0/10$ (which applies since $r_0/10 \le \min\{d, r_{**}\}$), we have

$$\int_{B_r(x_0)} |\nabla v|^2 \le 2\Big(\frac{10r}{r_0}\Big)^{n+1} \int_{B_{r_0/10}(x_0)} |\nabla v|^2 \le 2\Big(\frac{10r}{r_0}\Big)^{2\alpha+n-1} \int_{B_{r_0}(x_0)} |\nabla v|^2\,,$$

where we have used $10r/r_0 < 1$ and $\alpha \in (0, 1]$. This is precisely (3.44) with constant $2 \cdot 10^{2\alpha+n-1}$. □

3.4. **Frequency bounds and Hölder regularity.** In this subsection we establish local lower and upper bounds for the frequency function for solutions of (3.2)-(3.3), then use them together with Lemma 3.12 to establish local Hölder regularity.

Lemma 3.13 (Locally uniform frequency lower bound)**.** *Let* $v \in W^{1,2}_{\text{loc}}(\Omega; [0, 1])$ *satisfy* (3.3) *with* G *satisfying* (3.1)*. Then*

(i): *every $x_0 \in \Omega$ is a Lebesgue point of v and*

$$\lim_{r\to 0^+} \frac{1}{r^{n-1}} \int_{B_r(x_0)} |\nabla v|^2 = 0\,, \qquad \textit{and} \tag{3.50}$$

(ii): *given $\Omega' \subset\subset \Omega$, there exists a constant $\alpha = \alpha(n, \Omega', v) \in (0,1]$ such that if $v(x_0) = 0$ and $x \in \Omega' \cap \overline{\{v > 0\}}$, then*

$$\alpha \int_{\partial B_r(x_0)} v^2 d\mathcal{H}^n \le r \int_{B_r(x_0)} |\nabla v|^2 \qquad \forall\, 0 < r < \operatorname{dist}(x, \partial\Omega)\,. \tag{3.51}$$

Proof. First of all, recall that we are identifying v with its precise representative v^* (which requires only the equation (3.3), cf. Remark 3.8) so that (3.21) holds. Before proving items (i) and (ii), we make a preliminary computation testing (3.3) with the mollified fundamental solution. Assuming for notational convenience that $0 \in \Omega$, let us consider the functions $\Gamma_t = \eta_t \star \Gamma$ and $\Gamma_t^\sigma = \eta_t \star \Gamma^\sigma$, where $\{\eta_t\}_t \subset C_c^\infty(B_1)$ are an approximation to the identity, $t, \sigma > 0$ and

$$\Gamma(x) = \begin{cases} |x|^{1-n} & \text{when } n \ge 2 \\ -\ln(|x|) & \text{when } n = 1 \end{cases}, \qquad\qquad \Gamma^\sigma(x) = \Gamma(x/\sigma)\,.$$

Fix $\psi \in C_c^\infty(B_1;[0,1])$ with $\psi = 1$ on $B_{1/2}$, so that for each $r > 0$, $\psi_r := \psi(\cdot/r) \in C_c^\infty(B_r;[0,1])$, $\psi_r = 1$ on $B_{r/2}$, and $r|\nabla\psi_r| + r^2|\Delta\psi_r| \le C$. By testing (3.3) with $\Gamma_t^\sigma\psi_r \in C_c^\infty(B_r)$ and then integrating by parts, we obtain

$$\begin{aligned} 2\int_{B_r} \Gamma_t^\sigma \psi_r |\nabla v|^2 &= -\int_{B_r} \left(\nabla v^2 \cdot \nabla(\Gamma_t^\sigma\psi_r) + G'(v) v \psi_r \Gamma_t^\sigma\right) \\ &= \int_{B_r} v^2\, \Delta(\Gamma_t^\sigma\psi_r) - G'(v)v\psi_r\Gamma_t^\sigma\,. \end{aligned} \tag{3.52}$$

Since $\nabla\psi_r = 0$ in $B_{r/2}$, we have that

$$\int_{B_r} v^2\,\Delta(\Gamma_t^\sigma\psi_r) = \int_{B_r} v^2(\Delta\Gamma_t^\sigma)\psi_r + 2\int_{B_r\setminus B_{r/2}} v^2 \nabla\Gamma_t^\sigma\cdot\nabla\psi_r + \int_{B_r\setminus B_{r/2}} v^2\Gamma_t^\sigma\Delta\psi_r. \tag{3.53}$$

In summary, we have found

$$\begin{aligned} 2\int_{B_r} \Gamma_t^\sigma\psi_r|\nabla v|^2 &= \int_{B_r} v^2(\Delta\Gamma_t^\sigma)\psi_r + 2\int_{B_r\setminus B_{r/2}} v^2\nabla\Gamma_t^\sigma\cdot\nabla\psi_r + \int_{B_r\setminus B_{r/2}} v^2\Gamma_t^\sigma\Delta\psi_r \\ &\quad - \int_{B_r} G'(v)v\psi_r\Gamma_t^\sigma\,; \end{aligned} \tag{3.54}$$

observe that the same equality for the translation $v(\cdot + x)$ replacing v holds whenever $B_r(x) \subset \Omega$.

To prove (3.50): Let us assume without loss of generality that $x_0 = 0$. Since (3.50) is trivial if $n = 1$ (by the continuity of the Lebesgue integral), for this step we assume $n \ge 2$. We notice now that $\Delta\Gamma_t = \bar{c}\eta_t$ for some dimensional constant $\bar{c} > 0$ (which only depends on n, not $\mathbf{W}$), since $\Delta\Gamma = \bar{c}\delta_0$, where δ_0 is the Dirac mass at 0. Altogether, bearing in mind that η_t is an approximation to the identity and that $0 \le v, \psi_r \le 1$, we have that

$$\limsup_{t\to 0}\left|\int_{B_r} v^2(\Delta\Gamma_t)\psi_r\right| \le \bar{c}\limsup_{t\to 0}\int_{B_r}\eta_t \le \bar{c}. \tag{3.55}$$

On top of this, since $n \ge 2$, the estimates for $\nabla\psi_r$ and $\Delta\psi_r$ yield

$$\lim_{t\to 0}\int_{B_r\setminus B_{r/2}} \Big(|\Delta\psi_r||\Gamma_t(x)| + |\nabla\psi_r||\nabla\Gamma_t(x)|\Big)v^2 \le \frac{C}{r^{n+1}}\int_{B_r\setminus B_{r/2}} v^2\,, \tag{3.56}$$

after updating the constant C. Thus, by combining (3.54)-(3.56), Fatou's lemma, and the estimate (3.6) for $|G'|$, we deduce

$$\int_{B_{r/2}} \Gamma|\nabla v|^2 \leq C\left(1 + \frac{1}{r^{n+1}}\int_{B_r\setminus B_{r/2}} v^2 + \int_{B_r} \Gamma v^2\right). \tag{3.57}$$

In particular, again exploiting the fact that $|v| \leq 1$, we have shown that $\Gamma|\nabla v|^2$ is locally integrable.

From this integrability we may conclude (3.50). Indeed, (3.57) and the dominated convergence theorem applied to the 1-parameter family of functions $f_s = \chi_{B_s}\Gamma|\nabla v|^2$ give

$$0 = \lim_{s\to 0^+}\int_{B_1} f_s \geq \limsup_{s\to 0^+}\frac{1}{s^{n-1}}\int_{B_s}|\nabla v|^2 = 0.$$

To prove that x_0 is a Lebesgue point: Again working at the origin for convenience, we set $v_r = \fint_{B_r} v$. By Poincaré's inequality, we have that

$$\begin{aligned}\frac{1}{r^{n+1}}\int_{B_r}|v(x)-v(0)|^2 &\leq \frac{C}{r^{n+1}}\int_{B_r}|v(x)-v_r|^2 + |v_r - v(0)|^2\,dx \\ &\leq \frac{C}{r^{n-1}}\int_{B_r}|\nabla v|^2\,dx + C|v_r - v(0)|^2 \to 0,\end{aligned} \tag{3.58}$$

as $r \to 0^+$ in virtue of (3.50) and (3.21); recall that here $v(0)$ is defined via (3.21).

To prove (3.51): Fix $\Omega' \subset\subset \Omega$ and assume, for a contradiction, that (3.51) fails to be true. Then at least one of following two statements holds: either there exists $x \in \Omega' \cap \{v = 0\} \cap \overline{\{v > 0\}}$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$ such that

$$\int_{B_r(x)}|\nabla v|^2 = 0\,, \tag{3.59}$$

or there exists a sequence of points $x_k \in \Omega' \cap \{v = 0\} \cap \overline{\{v > 0\}}$ with (by the compactness of Ω') $x_k \to x \in \Omega'$ and radii $r_k > 0$ satisfying $B_{r_k}(x_k) \subset \Omega$ and $r_k \to r \geq 0$ and such that

$$\frac{r_k\int_{B_{r_k}(x_k)}|\nabla v|^2}{\int_{\partial B_{r_k}(x_k)} v^2\,d\mathcal{H}^n} \to 0. \tag{3.60}$$

Since (3.59) is impossible if $x \in \{v = 0\}\cap\overline{\{v > 0\}}$ and *every* point is a Lebesgue point of v, it must be the case that (3.60) holds. Furthermore, it must be the case that

$$r_k \to r = 0\,. \tag{3.61}$$

Indeed, if $r_k \to r > 0$, then we would have $\int_{B_r(x)}|\nabla v|^2 = 0$, which again is in contradiction with every point being a Lebesgue point and $x_k \in \{v = 0\} \cap \overline{\{v > 0\}} \cap B_r(x)$.

Let $w_k(x) = v(x + x_k)$. Since each x_k is a Lebesgue point for v by item (i), we observe that 0 is a Lebesgue point for $w_k^2\psi$, and since $\nabla\psi_{r_k} = \Delta\psi_{r_k} = 0$ in $B_{r_k/2}$, we have

$$\lim_{t\to 0^+}\int_{B_{r_k}} w_k^2(\Delta\Gamma_t^{r_k})\psi_{r_k}\,dx = \lim_{t\to 0^+}\bar{c}r_k^{-2}\int_{B_{r_k}} w_k^2\eta_t\psi_{r_k}\,dx = 0. \tag{3.62}$$

Then, sending $t \to 0^+$ in (3.54), using Fatou's lemma on the left hand side, and using (3.62) and the Dominated Convergence Theorem on the right, we obtain

$$\begin{aligned}2\int_{B_{r_k}}\Gamma^{r_k}\psi_{r_k}|\nabla w_k|^2 \leq\; & 2\int_{B_{r_k}\setminus B_{r_k/2}} w_k^2\nabla\Gamma^{r_k}\cdot\nabla\psi_{r_k} + \int_{B_{r_k}\setminus B_{r_k/2}} w_k^2\Gamma^{r_k}\Delta\psi_{r_k} \\ & - \int_{B_{r_k}} G'(w_k)w_k\psi_{r_k}\Gamma^{r_k}.\end{aligned} \tag{3.63}$$

Note that the left hand side is positive (even when $n = 1$) since $\Gamma^{r_k} \geq 0$ and $\psi_k \geq 0$ on B_{r_k}.

We introduce the re-scalings $v_k = \frac{v(x_k + r_k x)}{c(k)}$, with $c(k) = \left(\fint_{\partial B_{r_k}(x_k)} v^2 \, d\mathcal{H}^n \right)^{\frac{1}{2}}$. By using the fact that $|G'(t)| \leq kt$ and re-scaling accordingly, we can rewrite (3.63) with v_k as

$$0 \leq 2 \int_{B_1} v_k^2 \nabla\Gamma \cdot \nabla\psi + \int_{B_1} v_k^2 \Gamma \Delta\psi + k(r_k)^2 \int_{B_1} v_k^2 \psi \Gamma. \tag{3.64}$$

On the other hand, from (3.60) we observe that $|\nabla v_k| \to 0$ in $L^2(B_1)$ which, by the Rellich Compactness Theorem, together with Lemma 3.5 and the normalization of v_k, implies that up to subsequences, v_k converges strongly in $L^2(B_1)$ to some function $\bar{v}$. The continuity of the trace operator on $W^{1,2}(B_1)$ further implies that $\bar{v} \equiv 1$. Therefore, taking limits in (3.64) yields

$$0 \leq 2 \int_{B_1} \nabla\Gamma \cdot \nabla\psi + \int_{B_1} \Gamma \Delta\psi. \tag{3.65}$$

Finally, integrating by parts in (3.65), recalling that $\psi(0) = 1$, and using that $\Delta\Gamma = \bar{c}\delta_0$ for $\bar{c} > 0$ as before, we derive the contradiction

$$0 \leq -2\bar{c}\psi(0) + \bar{c}\psi(0) = -\bar{c}.$$

$\square$

Lemma 3.14 (Locally uniform frequency upper bound and consequences)**.** *Let $v \in W^{1,2}_{\mathrm{loc}}(\Omega; [0,1])$ satisfy both (3.2) and (3.3), with G satisfying (3.1). Then given any $\Omega' \subset\subset \Omega$, there exist constants $M > 0$, $K > 0$ and $\zeta \in (0,1)$, all depending on v, Ω', $\sup_{[0,1]} |G''|$, and n, such that for any $x_0 \in \Omega' \cap \overline{\{v > 0\}}$ and $r \leq \min\{\mathrm{dist}(\partial\Omega', \partial\Omega)/2, r_*, 1\}$, we have*

$$N_{v,x_0}(r) \leq M \tag{3.66}$$

$$\int_{\partial B_r(x_0)} v^2 \, d\mathcal{H}^n \leq K \int_{\partial B_{r/2}(x_0)} v^2 \, d\mathcal{H}^n \,, \tag{3.67}$$

$$\mathcal{L}^{n+1}(B_{r/2}(x_0) \cap \{v = 0\}) \leq \zeta \omega_{n+1} (r/2)^{n+1} \quad \textit{and} \tag{3.68}$$

$$\|v\|_{L^\infty(B_{r/2}(x_0))} \leq K H_{v,x_0}(r/2)^{1/2} \,. \tag{3.69}$$

Proof. We set $\tilde{r} = \min\{\mathrm{dist}(\partial\Omega', \partial\Omega)/2, r_*, 1\}$.

To prove (3.66): The function $x \mapsto H_{v,x}(\tilde{r})$ is continuous in x, so it achieves a minimum on the compact set $\Omega \cap \overline{\{v > 0\}}$ at some y_0. By Remark 3.6, if it were the case that $H_{v,y_0}(\tilde{r}) = 0$, we would have $v \equiv 0$ on $B_{\tilde{r}}(y_0)$, contradicting the fact that $y_0 \in \overline{\{v > 0\}}$ (recall that we are taking $v = v^*$). It then follows by the frequency almost-monotonicity in Lemma 3.9 that

$$\begin{aligned} \sup_{B_r(x) \,:\, x \in \overline{\{v>0\}} \cap \Omega', r \leq \tilde{r}} N_{v,x}(r) &\leq \sup_{x \in \overline{\{v>0\}} \cap \Omega'} e^{\kappa \tilde{r}^2/2} N_{v,x}(\tilde{r}) + e^{\kappa \tilde{r}^2/2} - 1 \\ &\leq e^{\kappa \tilde{r}^2/2} + \frac{e^{\kappa \tilde{r}^2/2}}{\tilde{r}^{n-1} H_{v,y_0}(\tilde{r})} \int_{\{\mathrm{dist}(x,\Omega') < \tilde{r}\}} |\nabla v|^2 \, dx =: M \,. \end{aligned} \tag{3.70}$$

To prove (3.67): Again using Remark 3.6, we have $H_{v,x_0}(r) > 0$ for all $0 < r < \tilde{r}$ and $x_0 \in \Omega' \cap \overline{\{v > 0\}}$. Therefore, given $r \in (0, \tilde{r}]$, in virtue of (3.29), Lemma 3.5 and (3.6), we compute

$$\frac{d}{dr} \ln(H_{v,x_0}(r)) = \frac{2}{r} N_{v,x_0}(r) + \frac{1}{r^n H_{v,x_0}(r)} \int_{B_r(x_0)} G'(v) v \leq \frac{2}{r} N_{v,x_0}(r) + Cr \,, \tag{3.71}$$

where C depends on $\sup_{[0,1]}|G''|$ and n. So, using Lemma 3.9, and integrating (3.71) between $r/2$ and r, with $r \in (0,\tilde{r}]$, on both sides, we deduce

$$\ln\Big(\frac{H_{v,x_0}(r)}{H_{v,x_0}(r/2)}\Big) \le 2\ln(2)e^{2\kappa\tilde{r}^2}(N_{v,x_0}(\tilde{r})+1) + \frac{C\tilde{r}^2}{2},$$

which in turns implies

$$\int_{\partial B_r(x_0)} v^2\,d\mathcal{H}^n \le e^{C\tilde{r}^2/2}2^{2\psi(x_0,\tilde{r})}\int_{\partial B_{r/2}(x_0)} v^2\,d\mathcal{H}^n, \tag{3.72}$$

where $\psi(x_0,\tilde{r}) := e^{2\kappa\tilde{r}^2}(N_{v,x_0}(\tilde{r})+1)$. After exploiting the definition of M given by (3.70), this is precisely the claimed doubling estimate (3.67) on spherical shells, with $K = e^{C\tilde{r}^2/2}2^{2M}$.

To prove (3.68): Fix $x_0 \in \overline{\{v>0\}} \cap \Omega'$. We first integrate (3.67) with respect to the radius to deduce the doubling property

$$\int_{B_r(x_0)} v^2\,dx \le 2K\int_{B_{r/2}(x_0)} v^2\,dx \quad \forall\, 0<r\le\tilde{r}, \tag{3.73}$$

on balls. On the other hand, let us notice that by the almost-subharmonicity in Lemma 3.5,

$$\|v\|^2_{L^\infty(B_{r/2}(x_0))} \le \frac{C(n,\sup|G''|)}{r^{n+1}}\int_{B_r(x_0)} v^2\,dx \qquad \forall\, 0<r\le\tilde{r}. \tag{3.74}$$

The estimates (3.73) and (3.74) together imply

$$\|v\|^2_{L^\infty(B_{r/2}(x_0))} \le \frac{2CK}{r^{n+1}}\int_{B_{r/2}(x_0)} v^2\,dx \qquad \forall\, 0<r\le\tilde{r}. \tag{3.75}$$

Applying Hölder's inequality on both sides of (3.75) we deduce the reverse Hölder type inequality

$$\Big(\frac{1}{r^{n+1}}\int_{B_{\frac{r}{2}}(x_0)} v^p\,dx\Big)^{\frac{1}{p}} \le 2CK\Big(\frac{1}{r^{n+1}}\int_{B_{\frac{r}{2}}(x_0)} v^q\,dx\Big)^{\frac{1}{q}}, \tag{3.76}$$

for any $1\le p,q\le\infty$, where the constants are independent of $x_0 \in \overline{\{v>0\}}\cap\Omega'$. To deduce the Lebesgue density upper bound from (3.76), we first apply Hölder's inequality to estimate

$$⨍_{B_{r/2}(x_0)} v\,dx \le \Big(⨍_{B_{r/2}(x_0)} v^2\,dx\Big)^{1/2}\Big(⨍_{B_{r/2}(x_0)} \mathbf{1}_{\{v>0\}}\,dx\Big)^{1/2} \qquad \forall\, 0<r\le\tilde{r}\,.$$

After rearranging this inequality and applying (3.76) with $p=2$ and $q=1$, we arrive at

$$\Big(⨍_{B_{r/2}(x_0)} \mathbf{1}_{\{v>0\}}\,dx\Big)^{-1/2} \le \Big(⨍_{B_{r/2}(x_0)} v\,dx\Big)^{-1}\Big(⨍_{B_{r/2}(x_0)} v^2\,dx\Big)^{1/2} \le 2CK\,,$$

which implies (3.68) with $\zeta = 1-(2CK)^{-2}$.

To prove (3.69): We use (3.75) followed by (3.16) to estimate

$$\|v\|^2_{L^\infty(B_{r/2}(x_0))} \le \frac{2CK}{r^{n+1}}\int_{B_{r/2}(x_0)} v^2\,dx \le \frac{2CK}{r^{n+1}}\cdot\frac{r}{2}C\big(\sup_{[0,1]}|G''|,n\big)\int_{\partial B_{r/2}(x_0)} (v^*)^2\,d\mathcal{H}^n\,.$$

By taking square root on both sides and renaming the constant, we obtain (3.69). □

Remark 3.15. When combined with the results in [Lee23] and [DLS11], for any Dir-stationary Q-valued map $f:\Omega\to\mathcal{A}_Q(\mathbb{R}^n)$, the argument in the proof of Lemma 3.13 in fact provides a multi-valued Campanato-type estimate locally around Q-points x with $f(x) = Q[\![p]\!]$ for some $p\in\mathbb{R}^n$, of the form

$$\inf_{P\in\mathcal{A}_Q(\mathbb{R}^n)}\int_{B_r(x)} \mathcal{G}(f,P)^2 \le Cr^{n+1+2\alpha} \qquad \text{in } B_r(x)\subset\Omega.$$

However, note that this does not easily provide Hölder continuity of such maps, since the inner variation is not preserved under splitting of the sheets; see e.g. [HS22] for further discussion on this matter.

As an intermediate step towards Theorem 3.1, we prove interior Hölder regularity of solutions.

Lemma 3.16 (Interior Hölder regularity)**.** *Let* $\mathbf{W} = \mathbb{R}^{n+1} \setminus \Omega$ *be closed,* $G \in C^2([0,1])$ *satisfy* (3.1)*, and* $v \in W^{1,2}_{\mathrm{loc}}(\Omega;[0,1])$ *satisfy* (3.2)*-*(3.4)*. If* $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$*, and* $\alpha \in (0,1]$ *and* M *are the lower and upper frequency bounds on* Ω_2 *from* (3.66) *and* (3.51) *respectively, then there exists* $C(\alpha, M, n) > 0$ *and* $r_{**} = r_{**}(\alpha, n, \sup_{[0,1]}|G''|)$*, such that for every* $x_0 \in \Omega_1$ *and* $r < \min\{r_{**}, \mathrm{dist}(\partial\Omega_1, \partial\Omega)/3\}$*, we have*

$$r^\alpha [v]_{C^\alpha(B_{r/2}(x_0))} \leq C\Big(\frac{1}{r^{n-1}} \int_{B_r} |\nabla v|^2\Big)^{\frac{1}{2}}. \tag{3.77}$$

Proof. Firstly, note that Lemma 3.13 guarantees that every point of v is a Lebesgue point, so v is defined pointwise as a limit of its integral averages. Our goal is to show that v satisfies the hypotheses of Lemma 3.12. Applying Lemma 3.12 will then yield the estimate (3.77). Since the frequency bounds (3.40)-(3.41) hold due to (3.51) and (3.66) as noted in the statement of the lemma, we must therefore demonstrate that $\{v > 0\}$ is relatively open in Ω, and that v solves $2\Delta v = G'(v)$ in the classical sense in $\{v > 0\}$.

To verify that $\{v > 0\}$ *is relatively open*: It suffices to show that $\{v = 0\}$ is relatively closed in Ω. Indeed, first of all, thanks to Lemma 3.9, the mapping $x \mapsto N_x(0^+) = \lim_{r\to 0^+} N_x(r)$ is upper-semicontinuous. Thus, by Lemma 3.13.(ii), any accumulation point x' of $\{v = 0\}$ in the interior of Ω satisfies $N_{x'}(0^+) > 0$, namely (3.51) holds with center x' and constant depending on x'. On the other hand, (3.50), (3.16), and (3.21) together imply that $N_x(0^+) = 0$ for $x \notin \{v = 0\}$, implying that we must have $v(x') = 0$.

To verify $2\Delta v = G'(v)$ *in the classical sense in* $\{v > 0\}$: Let μ be the non-negative measure from (3.4), so that in particular

$$\int_\Omega \varphi \, d\mu = \int_\Omega -2\nabla\varphi \cdot \nabla v - \varphi G'(v)\, dx \qquad \forall \varphi \in C^\infty_c(\{v > 0\}). \tag{3.78}$$

If $\mu = 0$ on $\{v > 0\}$, then v would solve the equation $2\Delta v = G'(v)$ in the usual weak sense in the open set $\{v > 0\}$, at which point the standard elliptic regularity theory shows that v is a classical solution there. To prove that $\mu = 0$ on $\{v > 0\}$, we claim that it suffices to show that

$$\int_{\{v>0\}} \varphi\, v \, d\mu = 0 \qquad \forall \varphi \in C^\infty_c(\{v > 0\}), \tag{3.79}$$

Indeed, (3.79) implies that the non-negative Radon measure $v\mu \llcorner \{v > 0\}$ is the zero measure, but since $v(x) > 0$ for *every* $x \in \{v > 0\}$, this forces $\mu = 0$ there. So our task is reduced to proving (3.79).

Given an arbitrary test function $\varphi \in C^\infty_c(\{v > 0\})$, let us consider the mollifications $(\varphi v)_\varepsilon := (\varphi v) * \eta_\varepsilon$ for a family $\{\eta_\varepsilon\}$ of smooth mollifiers. By the property $0 \leq v \leq 1$ and the fact that every point of v is a Lebesgue point (Lemma 3.13.(i), we have

$$0 \leq (\varphi v)_\varepsilon \leq 1 \quad \text{and} \quad \big((\varphi v) * \eta_\varepsilon\big)(x) \to (\varphi v)(x) \text{ for all } x \in \{v > 0\}. \tag{3.80}$$

Then, since $(\varphi v)_\varepsilon \in C^\infty_c(\{v > 0\}$, we may test (3.78) with $(\varphi v)_\varepsilon$ and apply the Dominated Convergence Theorem to compute

$$\int_{\{v>0\}} \varphi v \, d\mu = \lim_{\varepsilon\to 0} \int_{\{v>0\}} (\varphi v)_\varepsilon \, d\mu = \lim_{\varepsilon \to 0} \int_{\{v>0\}} -2\nabla(\varphi v)_\varepsilon \cdot \nabla v - (\varphi v)_\varepsilon G'(v)\, dx$$

$$= \int_{\{v>0\}} -2\nabla(\varphi v)\cdot\nabla v - (\varphi v)G'(v)\,dx\,, \tag{3.81}$$

where in the last equality we have used the strong $W^{1,2}$-convergence of $(\varphi v)_\varepsilon$ to φv. Now by the product rule for products of C_c^∞ and $W^{1,2}$ functions and then (3.3), the right hand side expands as

$$\int_{\{v>0\}} -2\nabla(\varphi v)\cdot\nabla v - (\varphi v)G'(v)\,dx = \int_{\{v>0\}} -2\varphi|\nabla v|^2 - 2v\nabla\varphi\cdot\nabla v - (\varphi v)G'(v)\,dx = 0\,. \tag{3.82}$$

Putting (3.81)-(3.82) together yields (3.79), as desired. □

3.5. **Compactness, tangent functions, and unique continuation.** In this subsection we first show that solutions of (3.2)-(3.4) enjoy strong compactness in $W^{1,2}$. We then use this compactness to study blow-ups and tangent functions. Lastly, we prove a unique continuation-type result.

Lemma 3.17 (Compactness for solutions of (3.2)-(3.4))**.** *Let $B_{3r_0}(x_0)\subset\mathbb{R}^{n+1}$, $v_k\in(W^{1,2}_{\rm loc}\cap C^0)(B_{3r_0};[0,\infty))$ satisfy (3.2)-(3.4) for some $G_k\in C^1(\overline{v_k(B_{3r_0})})$ and non-negative Radon measures μ_k. If, furthermore, there exists a function $v\in(W^{1,2})\cap C^0)(\overline{B_{2r_0}(x_0)})$ such that*

$$v_k \rightharpoonup v \quad \text{weakly in } W^{1,2}(B_{2r_0}(x_0))\,, \tag{3.83}$$

$$\|v-v_k\|_{L^\infty(B_{2r_0}(x_0))}\to 0 \qquad \text{and} \tag{3.84}$$

$$\|G_k'(v_k)-G_0'(v)\|_{L^2(B_{2r_0}(x_0))}\to 0 \tag{3.85}$$

for some $G_0\in C^1(v(B_{2r_0}(x_0)))$, then there exists a non-negative Radon measure $\bar\mu$ in B_{2r_0} such that, up to extracting a subsequence,

$$\mu_k \overset{*}{\rightharpoonup} \bar\mu \quad \text{as measures in } B_{2r_0}(x_0), \tag{3.86}$$

$$v_k\to\bar v \quad \text{strongly in } W^{1,2}(B_{r_0}(x_0)), \quad \text{and} \tag{3.87}$$

$$\bar v \text{ satisfies (3.2)-(3.4) with } \mu=\bar\mu,\ G=G_0 \text{ and } \Omega=B_{r_0}(x_0)\,. \tag{3.88}$$

Remark 3.18. The uniform convergence assumption is not restrictive, as it is satisfied in the case of blow-ups, as we shall verify shortly, or in any case where $v_k\in W^{1,2}_{\rm loc}(B_{3r_0};[0,1])$ enjoy uniform upper and lower frequency bounds according to Lemma 3.12.

Proof. Thanks to a translation and rescaling argument we can take, without loss of generality, $x_0=0$ and $r_0=1$. Now test (3.4) with $\varphi\in C_c^\infty(B_2)$ such that $\varphi\geq\chi_{B_{\frac32}}$ and combine with (3.83) and the uniform bounds on $\|\nabla v_k\|_{L^2(B_2)}$ and $\|G_k'\|_{L^1(B_2}$ (consequences of (3.83) and (3.85)) to deduce

$$\mu_k(B_{\frac32})\leq C+\int_{B_2}|\nabla\varphi\cdot\nabla v_k|\leq Cr^2+C\Big(\int_{B_2}|\nabla v_k|^2\Big)^{\frac12}\leq C\,. \tag{3.89}$$

Thus, we may conclude (3.86) from (3.89). Moreover, taking the limit as $k\to\infty$ in (3.4) for v_k using that $\mu_k\overset{*}{\rightharpoonup}\bar\mu$, (3.83), and (3.85), we find that

$$2\Delta\bar v=\bar\mu+G_0'(\bar v) \qquad \text{distributionally in } B_{\frac32}, \tag{3.90}$$

which verifies the validity of (3.4) for $\bar v$ on B_1 with $G=G_0$ and $\mu=\bar\mu$.

To complete the proof, it suffices to verify the strong $W^{1,2}$-convergence. Indeed, this would readily imply the validity of both (3.2) and (3.3) for $\bar v$ with $G=G_0$ and $\Omega=B_1$. To prove the strong convergence, fix $\varphi\in C_c^\infty(B_{\frac32};[0,1])$ with $\varphi\equiv1$ on B_1, and choose, for each k, $w_k\in C_c^\infty(B_2)$ approximating $v_k-\bar v$ well enough in $(L^\infty\cap W^{1,2})(B_{3/2})$ (which is possible since we are assuming that v_k and $\bar v$ are continuous) so that

$$\left|\int_{B_2}\varphi\nabla(v_k-\bar v)\cdot\nabla w_k\,dx-\int_{B_2}\varphi|\nabla(v_k-\bar v)|^2\,dx\right|\leq\frac1k\,, \tag{3.91}$$

and

$$\|w_k\|_{L^2(B_{3/2})} + \|w_k\|_{L^\infty(B_{3/2})} \le \frac{1}{k} + \|\bar{v} - v_k\|_{L^\infty(B_{3/2})}\,. \tag{3.92}$$

Next, if we use (3.91), then subtract (3.4) for v_k from (3.90) and test it with φw_k, we deduce that

$$\begin{aligned}\int_{B_1} |\nabla(v_k - \bar{v})|^2\,dx &\le \int_{B_2} |\nabla(v_k - \bar{v})|^2\varphi\,dx \\ &\le \int_{B_2} \varphi\nabla(v_k - \bar{v})\cdot\nabla w_k\,dx + \frac{1}{k} \\ &= -\int_{B_2} w_k\nabla(v_k - \bar{v})\cdot\nabla\varphi\,dx - \int_{B_2}\varphi w_k\,d(\mu_k - \bar{\mu}) \\ &\quad - \int_{B_2} (G_k'(v_k) - G_0'(\bar{v}))(v_k - \bar{v})\varphi + \frac{1}{k}\,.\end{aligned}$$

As $k\to\infty$, the first integral vanishes due to the fact that $w_k \to 0$ in L^∞ from (3.92) and $\nabla v_k - \nabla\bar{v} \rightharpoonup 0$ in L^2, the second integral vanishes because of the vanshing L^∞-norms of w_k again, combined with $\mu_k \overset{*}{\rightharpoonup} \bar{\mu}$, and the last vanishes by the Hölder convergence of $v_k \to v$ and the L^2 convergence (3.85). □

To apply the preceding compactness arguments, we introduce the rescalings

$$v_{x_0,r} := \frac{v(x_0 + r\cdot)}{H_{v,x_0}(r)^{1/2}} \quad \text{for } x_0 \in \Omega\cap\overline{\{v>0\}} \text{ and } r\in(0,\operatorname{dist}(x_0,\partial\Omega)), \tag{3.93}$$

where $H_{v,x_0}(r)$ is the L^2 height function of v centered at x_0 as introduced in (3.24). Note that by Remark 3.6, $x_0 \in \Omega\cap\overline{\{v>0\}}$ implies that

$$H_{v,x_0}(r) > 0 \qquad \text{for } 0 < r < \min\{\operatorname{dist}(x,\partial\Omega), r_*, 1\},$$

so that $v_{x_0,r}$ is well-defined for all small enough r.

Lemma 3.19 (Compactness for $v_{x_0,r}$)**.** *Suppose that* $\mathbf{W} = \mathbb{R}^{n+1}\setminus\Omega$ *is closed,* $G \in C^2([0,1])$ *satisfies* (3.1)*, and* $v \in W^{1,2}_{\mathrm{loc}}(\Omega;[0,1])$ *satisfies* (3.2)-(3.4)*.*

(1) *Given* $x_0 \in \Omega\cap\overline{\{v>0\}}$ *and* $d = \operatorname{dist}(x_0,\partial\Omega)$*, if* $r < \min\{d, r_*, 1\}$ *the rescaling* $v_{x_0,r}$ *satisfies*

$$\Delta v_{x_0,r} = \frac{r^2}{2H_{x_0}(r)^{1/2}} G'(v_{x_0,r}H_{x_0}(r)) + \mu_{x_0,r} \qquad \textit{distributionally in } B_{\frac{d}{r}}, \tag{3.94}$$

with

$$\mu_{x_0,r} = \frac{(\Psi_{x_0,r})_\#\,\mu}{2H_{x_0}(r)^{1/2} r^{n-1}}$$

and where $(\Psi_{x_0,r})_\#\mu$ *represents the push-forward of the measure* μ *with respect to the function* $\Psi_{x_0,r}(y) = \frac{y-x_0}{r}$.

(2) *Let* $\{x_k\} \subset \Omega\cap\partial\{v>0\}$ *such that* $x_k \to \bar{x}\in\Omega$ *and* $r_k \to 0$*. Then, up to subsequences, there exists a non-negative Radon measure* $\bar{\mu}$ *in* B_1 *and a function* $\bar{v} \in (C^\alpha\cap W^{1,2})(\overline{B}_1)$ *for some* $\alpha\in(0,1]$ *such that*

$$\mu_{x_k,r_k} \overset{*}{\rightharpoonup} \bar{\mu}, \tag{3.95}$$

as measures and

$$v_{x_k,r_k} \to \bar{v} \tag{3.96}$$

strongly in $W^{1,2}(B_1)$ *and locally uniformly as* $k\to\infty$.

(3) $\bar{v}$ *satisfies the criticality conditions* (3.2)*,* (3.3)*, and* (3.4) *with* $\Omega = B_1$*,* $G = 0$ *and* $\mu = \bar{\mu}$.

Proof. We start with (1). Let $x_0 \in \Omega \cap \overline{\{v>0\}}$. Given $\varphi \in C_c^\infty(B_{\frac{d}{r}})$, testing (3.4) with $\varphi \circ \Psi_{x_0,r} \in C_c^\infty(B_d(x_0))$ (extended by zero to Ω), we obtain

$$\int_{B_d(x_0)} \varphi \circ \Psi_{x_0,r}\, d\mu = \int_{B_d(x_0)} -2\nabla v \cdot \nabla(\varphi \circ \Psi_{x_0,r}) - G'(v)\, \varphi \circ \Psi_{x_0,r}\, dx,$$

which, using the definition of push-forward measure on the left hand side and applying the change of variables $z = \Psi_{x_0,r}(x)$ on the right, can be rewritten as follows:

$$\int_{B_{\frac{d}{r}}} \varphi\, d(\Psi_{x_0,r})_\# \mu = r^n \int_{B_{\frac{d}{r}}} -2\Big(\nabla v \circ \Psi_{x_0,r}^{-1}\Big) \cdot \nabla \varphi - rG'(v_{x_0,r} H_{x_0}(r)^{1/2})\, \varphi\, dz\,.$$

Dividing both sides by $2H_{x_0}(r)^{1/2} r^{n-1}$, we obtain (3.94) in distributional form.

We now prove (2) and (3). Then by combining (3.66), (3.69), and (3.77) on a suitable open set containing $\bar{x}$ and compactly contained in Ω, we obtain $\alpha \in (0,1]$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$\|v_{x_k,r_k}\|_{C^\alpha(B_2)} \leq C, \tag{3.97}$$

$$\|v_{x_0,r}\|_{W^{1,2}(B_2)} \leq C, \tag{3.98}$$

for some C depending on the open set, v, G, etc. but not on x_k. In particular, up to a subsequence, the weak $W^{1,2}$ and uniform convergence of v_k to some $\bar{v} \in (C^0 \cap W^{1,2})(\overline{B}_1)$ is immediate. Also, as a consequence of (3.97) and the Lipschitz bound (3.6) for G', we have the estimate

$$\Big|\frac{r^2}{2H_{x_0}(r)^{1/2}} G'(v_{x_0,r} H_{x_0}(r)^{1/2})\Big| \leq Ckr^2 \tag{3.99}$$

on B_2. From here, we notice that, up to a subsequence, $\{v_{x_k,r_k}\}_k$ satisfies the hypotheses of Lemma 3.17 with $G_k \to 0$ in $C^1([0,1])$, which finishes the proof. $\square$

The main consequence of Lemma 3.19 is the subsequential $W^{1,2}$-compactness of the rescalings $\{v_{x_0,r}\}_r$ as defined by (3.93) as $r \downarrow 0$. This will allow us to deduce some fundamental properties of the subsequential limits, which we will refer to from now on as *tangent functions* (of v).

In the next lemma, we exploit the compactness properties derived in Lemma 3.19 to prove in our setting some well-known properties of tangent functions and the behavior of Almgren's frequency function for them.

Lemma 3.20 (Tangent functions)**.** *Suppose that $\mathbf{W} = \mathbb{R}^{n+1} \setminus \Omega$ is closed, $G \in C^2([0,1])$ satisfies* (3.1)*, and $v \in W^{1,2}_{\mathrm{loc}}(\Omega;[0,1])$ satisfies* (3.2)*-*(3.4)*. If $x_0 \in \Omega \cap \overline{\{v>0\}}$, r_j is a sequence of scales with $r_j \downarrow 0$, then, up to extracting a subsequence, there exists*

$$\bar{v}(x) = \lim_{j\to\infty} \frac{v(x_0 + r_j x)}{H_{x_0}(r_j)^{1/2}} \tag{3.100}$$

which is a non-zero tangent function of v at x_0, with the limit taken in $W^{1,2}(B_1)$ and locally uniformly (see Lemma 3.19). Also, $\bar{v}$ satisfies the criticality conditions (3.2), (3.3), *and* (3.4) *with $\Omega = B_1$, $G = 0$ and a non-negative Radon measure $\bar{\mu}$, and*

(1) *$N_{v,x_0}(0^+) = N_{\bar{v},0}(0^+)$,*
(2) *$\bar{v}$ is radially homogeneous of degree $N_{\bar{v},0}(0^+)$, and*
(3) *$N_{\bar{v},e}(0^+) \leq N_{\bar{v},0}(0^+)$ for any $e \in \mathbb{S}^n$ with equality if and only if $\bar{v}(x+te) = \bar{v}(x)$ for any $t \in \mathbb{R}$.*

Proof. Let us assume without loss of generality that $x_0 = 0$. In virtue of Lemma 3.19, we have that $\bar{v} \in (W^{1,2} \cap C^\alpha)(B_1)$ and we can assume that the limit $\bar{v}$ as defined in (3.100) indeed exists. From Lemma 3.19 we also have that $\bar{v}$ satisfies the criticality conditions (3.2), (3.3), and (3.4) with

$\Omega = B_1$, $G = 0$, and the measure $\bar{\mu}$ as given by (3.95) with $x_k \equiv 0$. Furthermore, $\|\bar{v}\|_{L^2(\partial B_1)} = 1$, by our choice of normalization.

Now, for any $\rho \in (0,1)$, in virtue of the strong convergence of

$$v_{r_j} := \frac{v(r_j\cdot)}{H(r_j)^{1/2}}$$

to $\bar{v}$ in $W^{1,2}(B_1)$, we have that $\int_{B_\rho} |\nabla v_{r_j}|^2 \to \int_{B_\rho} |\nabla \bar{v}|^2$ and $\int_{\partial B_\rho} v_{r_j}^2 \, d\mathcal{H}^n \to \int_{\partial B_\rho} \bar{v}^2 \, d\mathcal{H}^n$. Additionally, since $x \in \Omega \cap \partial\{v > 0\}$, Remark 3.6 implies that $\int_{\partial B_\rho} \bar{v}^2 \, d\mathcal{H}^n \neq 0$ for any such ρ. Thus,

$$N_{\bar{v},0}(\rho) = \frac{\rho \int_{B_\rho} |\nabla \bar{v}|^2}{\int_{\partial B_\rho} |\bar{v}|^2} = \lim_{j\to\infty} \frac{\rho \int_{B_\rho} |\nabla v_{r_j}|^2}{\int_{\partial B_\rho} v_{r_j}^2} = \lim_{j\to\infty} \frac{\int_{B_1} |\nabla v_{\rho r_j}|^2}{\int_{\partial B_1} v_{\rho r_j}^2 \, d\mathcal{H}^n} = \lim_{j\to\infty} N_{v,0}(\rho r_j) = N_{v,0}(0^+).$$

Thus, in virtue of the constancy case in Lemma 3.9, we deduce that $\bar{v}$ must be radially α-homogeneous with $\alpha = N_{\bar{v},0}(0^+)$. From here we also deduce that $N_{\bar{v},0}(0^+) = N_{v,0}(0^+)$. Meanwhile, the conclusion (3) is a simple consequence of the upper-semicontinuity of Almgren's frequency function and a blowdown argument; see, for instance, [Sim96, Section 3.3]. ☐

Lemma 3.21 (Classification of planar tangent functions). *If $\mathbf{W} = \mathbb{R}^{n+1}\setminus\Omega$ is closed, $G \in C^2([0,1])$ satisfies (3.1), and $v \in W^{1,2}_{\mathrm{loc}}(\Omega;[0,1])$ satisfies (3.2)-(3.4), then up to rotation any tangent function $\bar{v}$ at $x_0 \in \Omega \cap \overline{\{v > 0\}}$ is given by*

$$\bar{v}(r,\theta) = \frac{1}{\sqrt{\pi}} r^{N/2} \left| \sin\left(\frac{N\theta}{2}\right) \right|, \qquad N \in \mathbb{N}_{\geq 2}\,, \tag{3.101}$$

where (r,θ) denote polar coordinates in $\mathbb{R}^2$.

Proof. If $\bar{v}$ is any tangent function to v at x_0, thanks to Lemma 3.20 we have that $\bar{v}\Delta\bar{v} = 0$ weakly in B_1 (recall that this is (3.3) with $G = 0$). Furthermore, the homogeneity of $\bar{v}$ and the fact that $v \in W^{1,2}(\Omega)$ together imply that $\bar{v}|_{\partial B_r}$ belongs to $W^{1,2}(\partial B_r)$ for every $0 < r < 1$ instead of just almost every r. Thus by the Morrey-Sobolev embedding in one dimension and the homogeneity, $\bar{v}$ is continuous on B_1 and thus $\{\bar{v} = 0\}$ is closed. As a consequence, $\Delta\bar{v} = 0$ on the open set $\{\bar{v} > 0\}$ in the classical sense, and writing the equation in polar coordinates (r,θ) yields

$$\Delta\bar{v} = \partial_{rr}\bar{v} + r^{-1}\partial_r\bar{v} + r^{-2}\partial_{\theta\theta}\bar{v} = 0 \qquad \text{on } B_1 \setminus \{\bar{v} = 0\}.$$

In virtue of Lemma 3.20, we can exploit the radial homogeneity of $\bar{v}$ to conclude that for some $\gamma > 0$ we have

$$r^{-2}[\gamma^2\bar{v} + \partial_{\theta\theta}\bar{v}] = 0\,,$$

in any open, convex cone $\mathcal{C}$ formed from a single connected component of $\mathbb{R}^2 \setminus \{\bar{v} = 0\}$. Solving this ODE in θ, we obtain

$$\bar{v}(r,\theta) = r^\gamma[a\sin(\gamma\theta) + b\cos(\gamma\theta)] \quad \text{in } \mathcal{C}\,, \tag{3.102}$$

for some $a, b \in \mathbb{R}$. Up to rotation, we may without loss of generality assume that $\bar{v} = 0$ when $\theta = 0$. Thus, $b = 0$. Furthermore, observe that the exponent γ is the radial homogeneity of $\bar{v}$, so is *the same* for any such convex cone that is a connected component of $\mathbb{R}^2 \setminus \{\bar{v} = 0\}$. Additionally, $a = \frac{1}{\sqrt{\pi}}$ since $\bar{v} \geq 0$ and $\|\bar{v}\|_{L^2(\partial B_1)} = 1$.

We claim that $\{\bar{v} = 0\}$ consists of finitely many half-lines emanating from the origin. Indeed, observe that we have already demonstrated the fact that $\bar{v}$ has radial homogeneity of fixed degree γ in each open, convex, connected conical component of $\mathbb{R}^2 \setminus \{\bar{v} = 0\}$. This in particular implies that the angle of any such conical component must be an integer multiple of $\frac{\pi}{\gamma}$, in order to ensure that $\bar{v} = 0$ on the boundary of the cone. This in turn implies that there are only finitely many such connected components. Their complement will thus consist of finitely many closed, convex cones $K_1, \dots, K_N$, on each of which $\bar{v} = 0$. By a standard argument based on the inner variational equation (3.2) (see e.g. [ACF84, Theorem 2.4] or [MNR23a, Proposition 1.4]), on $\partial K_i \setminus \{0\}$ for each i we have the transmission condition

$$|\partial_\nu^- u| = |\partial_\nu^+ u|\,,$$

for the one-sided normal derivatives of u. If K_i had non-empty interior for some i, this gives a contradiction, since, coming from the side where $\bar{v} > 0$, the one-sided derivative normal derivative does not vanish (as is seen by direct computation using (3.102)). So each K_i must have empty interior and be a half-line. This observation combined with the periodicity of $\sin(\gamma\theta)$ and the fact that $\bar{v}$ is given by (3.102) on connected components of $\{\bar{v} > 0\}$ implies that $\gamma = \frac{N}{2}$, with $N \geq 1$.

We complete the proof by observing that $\bar{v}(r,\theta) = \frac{1}{\sqrt{\pi}} r^{\frac{1}{2}} \sin(\frac{\theta}{2})$ cannot arise as a tangent function. This is the case because $N_{\bar{v},0}(0^+) = \frac{1}{2}$ whereas $N_{\bar{v},te_1}(0^+) = 1$ for any $t \in (0,1)$, simply because $\bar{v}$ is Lipschitz at any of those points (this can be explicitly verified). This clearly contradicts the upper semicontinuity of $x \mapsto N_{\bar{v},x}(0^+)$. □

Finally, we prove a sort of unique continuation result for solutions of (3.2)-(3.4) that, roughly speaking, says the free boundary $\{v = 0\}$ is Lebesgue negligible if v is non-constant.

Lemma 3.22 (Unique continuation)**.** *Let $v \in W^{1,2}_{\mathrm{loc}}(\Omega; [0,1])$ satisfy (3.2)-(3.4) with G satisfying (3.1). Then for any connected component Ω' of Ω, either $\mathcal{L}^{n+1}(\{v = 0\} \cap \Omega') = 0$ or $v = 0$ $\mathcal{L}^{n+1}$-a.e. in Ω'. As a consequence, if $v|_{\Omega'}$ is not the zero function, then $\Omega' \cap \overline{\{v > 0\}} = \Omega'$ and the upper frequency bound (3.66), doubling estimate (3.67), and the L^∞-bound (3.69) hold on U with constants independent of $x \in U$.*

Proof of Lemma 3.22. The validity of (3.66), (3.67), and (3.69) on $U \subset\subset \Omega'$ if $v|_{\Omega'}$ is not the zero function follow immediately from Lemma 3.14 since $\Omega' \cap \overline{\{v > 0\}} = \Omega'$. So we prove that either $\mathcal{L}^{n+1}(\{v = 0\} \cap \Omega') = 0$ or $v = 0$ $\mathcal{L}^{n+1}$-a.e. in Ω'. Suppose, for contradiction, that for some connected component Ω' of Ω,

$$0 < \mathcal{L}^{n+1}(\Omega' \cap \{v = 0\}) < \mathcal{L}^{n+1}(\Omega')\,. \tag{3.103}$$

Then the perimeter $P(\{v = 0\}; \Omega')$ of $\{v = 0\}$ in Ω' is either infinity or strictly positive; it cannot be zero. Letting

$$\Omega' \cap \partial^e\{v = 0\} = \{x \in \Omega' : x \notin \{v = 0\}^{(1)} \cup \{v = 0\}^{(0)}\}$$

denote the essential boundary of $\{v = 0\}$ relative to Ω, where $\{v = 0\}^{(i)}$ denote the points in $\{v = 0\}$ of Lebesgue density i, we claim that it is non-empty. Indeed, if it were empty, then by Federer's criterion for sets of finite perimeter, we must have $P(\{v = 0\}; \Omega') = 0$, which is impossible. So there exists $x \in \Omega' \cap \partial^e\{v = 0\}$. By the containment $\Omega' \cap \partial^e\{v = 0\} \subset \Omega' \cap \partial\{v > 0\}$ and (3.68), we have

$$\limsup_{r \to 0} \frac{\mathcal{L}^{n+1}(\{v = 0\} \cap B_r(x))}{\omega_{n+1} r^{n+1}} < 1\,.$$

Since the limsup vanishing would imply that $x \in \{v = 0\}^{(0)}$ (against $x \in \partial^e\{v = 0\}$) there must exist $r_j \to 0$ and $\beta \in (0,1)$ such that

$$\limsup_{j \to \infty} \frac{\mathcal{L}^{n+1}(\{v = 0\} \cap B_{r_j}(x))}{\omega_{n+1} r_j^{n+1}} = \beta \in (0,1)\,;$$

by restricting to a further subsequence using Lemma 3.20, we may obtain a tangent function $\bar{v}$ such that

$$0 < \mathcal{L}^{n+1}(B_1 \cap \{\bar{v} = 0\}) < \mathcal{L}^{n+1}(B_1)\,. \tag{3.104}$$

We use (3.104) to obtain a contradiction, first in two dimensions and then in higher dimensions.

Contradiction in 2D: The equation (3.104) directly contradicts the classification of tangent functions in Lemma 3.21, since it implies that $\bar{v}$ has Lebesgue non-trivial zero set.

Contradiction in higher dimensions: By the same perimeter argument as above, (3.104) implies the existence of $y \in B_1 \cap \partial^e\{v > 0\}$, and again (3.68) implies the existence of $s_j \to 0$ such that

$$\limsup_{j\to\infty} \frac{\mathcal{L}^{n+1}(\{\bar{v} = 0\} \cap B_{s_j}(y))}{\omega_{n+1} s_j^{n+1}} \in (0,1)\,.$$

Up to a further subsequence, we therefore have a non-zero tangent function w to $\bar{v}$ at y with $\mathcal{L}^{n+1}$-nontrivial zero set. Furthermore, since $N_{\bar{v},ty}(0^+)$ is constant for $t \in (0,\infty)$, parts one and three of Lemma 3.20 show that w is independent of y. Thus the restriction $w : y^\perp \to \mathbb{R}$ is a homogeneous solution of (3.2)-(3.4) with $G = 0$ and $\mu = 0$ in $\mathbb{R}^n$, and $\mathcal{L}^n$-nontrivial zero set. By induction, since there is no such solution in $\mathbb{R}^2$, it is impossible in $\mathbb{R}^{n+1}$ and we have a contradiction. □

3.6. **Sharp frequency lower bound and the proof of Theorem 3.1.** Our final step is to improve the initial Hölder regularity to Lipschitzianity via a blow-up analysis. In this order of ideas, given any $x_0 \in \{v = 0\}$, we recall the blowups

$$v_{x_0,r} := \frac{v(x_0 + r\cdot)}{H_{x_0}(r)^{1/2}}.$$

from (3.93) for $r \in (0, \mathrm{dist}(x_0, \partial\Omega)$, where $H_{x_0}(r)$ is the L^2 height function of v centered at x as introduced in (3.24). The next result is a classification of tangent functions which, in particular, completes our regularity analysis.

Proposition 3.23. *Let $\mathbf{W} = \mathbb{R}^{n+1}\setminus\Omega$ be closed, $G \in C^2([0,1])$ satisfy (3.1), and $v \in W^{1,2}_{\mathrm{loc}}(\Omega;[0,1])$ satisfy (3.2)-(3.4). Then, $N_{v,x_0}(0^+) \geq 1$ for any $x_0 \in \{v = 0\}$ such that $v \not\equiv 0$ on the connected component of Ω containing x_0. Moreover, for any sequence $\{v = 0\} \ni x_k \to x_0$ with $N_{v,x_0}(0^+) = 1$, any subsequential limit*

$$\bar{v} := \lim_{k\to\infty} \frac{v(x_k + r_k\cdot)}{H_{x_k}(r_k)^{1/2}} \tag{3.105}$$

satisfies

$$\bar{v}(x) = \frac{1}{\sqrt{\omega_{n+1}}}|x \cdot e|\,, \tag{3.106}$$

for some $e \in \mathbb{S}^n$ and where ω_{n+1} is the Euclidean volume of the unit ball in $\mathbb{R}^{n+1}$. In particular, (3.106) holds true for any tangent function $\bar{v}$ to v at x_0.

Proof of Proposition 3.23. The proof is divided into steps.

Step 1. In this step we carry out a dimension reduction argument to show that $N_{\bar{v},0}(0^+) \geq 1$ in all dimensions for any tangent function $\tilde{v}$.

Let us start by noticing that the class of non-zero homogeneous solutions of (3.2)-(3.4) with $G = 0$ and $\Omega = \mathbb{R}^{n+1}$ is closed under taking tangent functions at any point, in virtue of Lemma 3.20 (note that by Lemma 3.22, any such solution $\bar{v}$ satisfies $\overline{\{\bar{v} > 0\}} = \mathbb{R}^{n+1}$, so Lemma 3.20 holds at every point). This class contains, in particular, all tangent functions to v. Let us denote this class as τ_{n+1} and let us define

$$m_{n+1} = \inf_{\bar{v}\in\tau_{n+1}} \inf_{x\in B_1} N_{\bar{v},x}(0^+)\,,$$

which, thanks to the closure of τ_{n+1} with respect to blow-ups and property (1) in Lemma 3.20, can be written as

$$m_{n+1} = \inf_{\bar{v}\in\tau_{n+1}} N_{\bar{v},0}(0^+)\,. \tag{3.107}$$

Our goal is to show, by induction on the dimension $n+1$, that $m_{n+1} \geq 1$. Let us notice that by Lemma 3.21 the base case $n=1$ is already covered. Let us assume now that $n \geq 2$. Suppose that for every dimension $\bar{n} \leq n$,

$$m_{\bar{n}} \geq 1.$$

First of all, we claim that (3.107) is attained for some $\bar{v} \in \tau_{n+1}$. Indeed, if $\bar{v}_k$ is an infimizing sequence for (3.107), by homogeneity of $\bar{v}_k$ we have

$$N_{\bar{v}_k,0}(1) = N_{\bar{v}_k,0}(0^+) \to m_{n+1} \tag{3.108}$$

as $k \to \infty$. In particular, the functions $\tilde{v}_k = \frac{\bar{v}_k}{H_{\bar{v}_k}(1)}$ satisfies the hypotheses of Lemma 3.17, implying that $\tilde{v}_k$ converges locally strongly in $W^{1,2}$, up to subsequences, to a non-zero function $\bar{v}_0 \in \tau_{n+1}$ such that $N_{\bar{v}_0,0}(0^+) = N_{\bar{v}_0,0}(1) = m_{n+1}$.

Now take $\bar{v}_0$ attaining (3.107); we claim that $\bar{v}_0$ is translation-invariant along some line through the origin. In other words, up to rotation, $\bar{v}_0(x_1,\ldots,x_{n+1}) = w(x_1,\ldots,x_n)$ for an m_{n+1}-homogeneous solution w of (3.2)-(3.4) on $\mathbb{R}^{n+1}$. Observe that after proving this, the inductive hypothesis would imply that $m_{\bar{n}} \geq 1$. Turning into the proof of the claim, let us notice that 0 cannot be an isolated zero for $\bar{v}_0$, otherwise $\bar{v}_0$ would be a continuous function in B_1, harmonic in $B_1 \setminus \{0\}$, which implies that $\bar{v}_0$ is harmonic in B_1 yielding a contradiction to the minimum principle- since $\bar{v}_0(0) = 0$. Hence, we have deduced the existence of a ray of zeros with frequency greater or equal than $m_{\bar{n}}$ which combined with Lemma 3.20 proves the claim.

Step 2. We complete the proof by characterizing limiting functions $\bar{v}$ given by (3.105) whenever $N_{v,x_0}(0^+) = 1$. Given such a function $\bar{v}$, we begin by demonstrating that $\bar{v}$ is still radially 1-homogeneous in this case, despite the varying centers. In light of Lemma 3.19 and Lemma 3.9, it suffices to demonstrate that $r \mapsto N_{\bar{v},0}(r)$ is identically equal to 1. Fix $\varepsilon > 0$ arbitrarily. Since $N_{v,x_0}(0^+) = 1$, the absolute continuity of N guarantees that there exists $\bar{\rho} \in (0, \mathrm{dist}(x_0, \partial\Omega))$ such that

$$N_{v,x_0}(\rho) \leq 1 + \frac{\varepsilon}{4} \qquad \forall \rho \in (0, \bar{\rho}]\,.$$

In particular, when combined with the Lemma 3.19, we have

$$N_{v,x_k}(\bar{\rho}) \leq 1 + \frac{\varepsilon}{2}\,, \tag{3.109}$$

for every k sufficiently large. Now for any given $r > 0$, up to taking k even larger if necessary so that $r_k \leq \frac{\bar{\rho}}{r}$, we further have

$$N_{v,x_k}(r_k r) + 1 \leq e^{\frac{\kappa(\bar{\rho}^2 - r_k r)}{2}} (N_{v,x_k}(\bar{\rho}) + 1)\,.$$

By further decreasing $\bar{\rho}$ if necessary and combining with (3.109), we can therefore ensure that

$$N_{v,x_k}(r_k r) \leq 1 + \varepsilon\,,$$

for all k sufficiently large. Letting $v_k := \frac{v(x_k + r_k \cdot)}{H_{x_k}(r_k)^{1/2}}$, we have $N_{v,x_k}(r_k r) = N_{v_k,0}(r)$, and so Lemma 3.19 guarantees that $N_{\bar{v},0}(r) \leq 1 + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we deduce that $N_{\bar{v},0}(r) \leq 1$.

Now, in Step 1 we verified that $N_{\bar{v},0}(0^+) \geq 1$, which, combined with the monotonicity in the $G = 0$ case of Lemma 3.9 yields $N_{\bar{v},0}(r) \geq 1$. The desired conclusion that $N_{\bar{v},0} \equiv 1$ follows.

Thus, any such limit $\bar{v}$ lies in the class τ_{n+1} and attains m_{n+1} as in (3.107). The argument in Step 1, iterated inductively, in fact implies that up to rotation, $\bar{v} = \bar{w}(x_1, x_2)$ for a 1-homogeneous function $\bar{w}$ satisfying (3.2)-(3.4) with $G = 0$ in $B_1 \subset \mathbb{R}^2$. Hence, by the classification in $\mathbb{R}^2$, we find that $\bar{v}$ must be a rotation of $L|x_1|$. Finally, since $\|\bar{v}\|_{L^2(\partial B_1)} = 1$, we have that $L = \frac{1}{\sqrt{\omega_{n+1}}}$. □

We conclude with the proof of the main theorem of this section.

Proof of Theorem 3.1. First of all, by Lemma 3.22, $\{v = 0\} \cap \Omega'$ is Lebesgue null whenever $\Omega' \subset \Omega$ is a connected component on which $v \not\equiv 0$. So we may as well assume that $\overline{\{v > 0\}} \cap \Omega = \Omega$, since the conclusions are trivial when $v \equiv 0$ on a given connected component of Ω. To prove item (i), thanks to Proposition 3.23, we note that $N_{v,x_0}(0^+) \geq 1$ for any $x_0 \in \{v = 0\}$, thus a direct application of Lemma 3.12 implies local Lipschitz continuity for v together with the estimate (3.8). To prove (ii), by the monotonicity of the frequency and the local frequency bound (Lemma 3.14), it suffices to show that (up to renaming r_{**})

$$\limsup_{|x|\to\infty} N_{v,x}(r_{**}) < \infty \,. \tag{3.110}$$

Now since $\nabla v \in L^2$, $r_{**}D_{v,x}(r_{**})$ decays uniformly as $|x| \to \infty$. Furthemore, since $\mathcal{L}^{n+1}(\{v < t\}) < \infty$ for all $t \in (0,1)$ (in particular for $t = 1/2$), Chebyshev's inequality yields

$$\int_{B_{r_{**}}(x)} v^2\, dy \geq \mathcal{L}^{n+1}(\{v > 1/2\} \cap B_{r_{**}(x)})/4 \to \omega_{n+1} r_{**}^{n+1}/4 \qquad \text{uniformly as } |x| \to \infty$$

also. Thus by Fubini's theorem, there exists $c > 0$ such that for all large enough $|x|$, $H_{v,x}(r) > c$ for some $r \in (r_{**}/2, r_{**})$. After replacing r_{**} with $r_{**}/2$, these two observations and the monotonicity of N imply (3.110). □

4. Regularity and structure of the free boundary: Proof of Theorem 1.1.(ii)

In this section, we begin our description of the structure of the free boundary $\{u = 1\}$ for solutions $u \in W^{1,2}_{\mathrm{loc}}(\Omega; [0,1])$ of (1.8)-(1.10). In order to carry out our analysis of the free boundary, we crucially rely on the following proposition, which establishes a local separation property for the set $\{v > 0\}$ of $v = 1 - u$, into two components near points of frequency 1.

Proposition 4.1. *Let $v \in W^{1,2}_{\mathrm{loc}}(\Omega; [0,1])$ be a solution of (3.2)-(3.4) and suppose that $x_0 \in \{v = 0\}$. In addition, suppose that $N_{x_0}(0^+) = 1$ and that there exists $R_0 > 0$ such that for each $y \in B_{R_0}(x_0) \cap \{v = 0\}$, $N_y(0^+) = 1$.*

Then there exists $r_0 > 0$ (depending on x_0) such that $\{v > 0\} \cap B_{R_0}(x_0)$ has exactly 2 connected components.

Remark 4.2. The additional requirement that $\{y : N_y(0^+) = 1\}$ is relatively open in $B_{R_0}(x_0) \cap \{v = 0\}$ in Proposition 4.1 will shortly become superfluous; see Corollary 4.7.

Proof. The argument follows the same reasoning as that in the proof of [TT12, Proposition 5.4]. We provide an outline here for the purpose of clarity, and refer the reader to [TT12] for more details.

Step 1. We claim that for any $\delta \in (0,1)$, there exists $r_0 = r_0(x_0, \delta) \in (0, \frac{R_0}{2})$ such that $\{v = 0\} \cap B_{R_0/2}(x_0)$ is (δ, r_0)-Reifenberg flat, namely for each $x \in \{v = 0\} \cap B_{R_0/2}(x_0)$ and $r \in (0, r_0]$, there exists an n-dimensional linear subspace $L_{x,r}$ such that

$$d_H(\{v = 0\} \cap B_r(x), (x + L_{x,r}) \cap B_r(x)) \leq \delta r \,, \tag{4.1}$$

where d_H denotes the Hausdorff distance. To see that this claim holds, we argue by contradiction. Namely, suppose that there exists $\delta > 0$ such that for some sequence $r_k \downarrow 0$ and $\{v = 0\} \cap B_{R_0/2}(x_0) \ni x_k \to \bar{x}$ with $N_{\bar{x}}(0^+) = 1$, the rescalings

$$v_{x_k,r_k}(x) := \frac{v(x_k + r_k x)}{H_{x_k}(r_k)^{1/2}}$$

satisfy

$$d_H(\{v_{x_k,r_k} = 0\} \cap B_1, L \cap B_1) > \delta \,, \tag{4.2}$$

for any n-dimensional linear subspace L. Applying Lemma 3.19 and Lemma 3.23 and recalling that $N_{\bar{x}}(0^+) = 1$ in light of the hypothesis, we conclude that $v_{x_k,r_k} \to \bar{v}$ in $W^{1,2}(B_1)$ and locally uniformly, where $v(x) = \frac{1}{\omega_{n+1}}|x \cdot e|$ for some $e \in \mathbb{S}^n$. In particular, $\{\bar{v} = 0\} = L_0 \cap B_1$ for some n-dimensional linear subspace L_0. This implies that

$$d_H(\{v_{x_k,r_k} = 0\} \cap B_1, L \cap B_1) \to 0 .$$

Indeed, this can be proven directly from the definition; one inclusion is a mere consequence of the uniform convergence, while the other is due to the fact that there must be zeros of v_{x_k,r_k} converging to each zero of $\bar{v}$, in light of the minimum principle for harmonic functions. This directly contradicts (4.2).

Step 2. We may now exploit the local Reifenberg flatness of $\{v = 0\}$ around x_0 to deduce the local separation property as follows. By Step 1, given a fixed absolute $\delta \in (0, \frac{1}{4})$, there exists a linear n-dimensional subspace L_{x_0,r_0} such that (4.1) holds with $x = x_0$ and $r = r_0(x_0, \delta)$. Thus, letting $B_0^\pm$ denote the two connected components of $B_{r_0}(x_0) \setminus B_{\delta r_0}(x_0 + L_{x_0,r_0})$, there exist two connected components $D^\pm$ of $\{v > 0\} \cap B_{r_0}(x_0)$ such that $B_0^+ \subset D^+$ and $B_0^- \subset D^-$. Define a function $\epsilon : B_0^+ \cup B_0^- \to \{+1, -1\}$ by

$$\epsilon = \begin{cases} +1 & \text{in } B_0^+ \\ -1 & \text{in } B_0^- . \end{cases}$$

We may now cover $B_{\delta r_0}(x_0 + L_{x_0,r_0})$ by a finite number of balls $B_{r_0/2}(x_i)$, $i = 1, \dots N$, with $x_i \in \{v = 0\}$, and apply the conclusion of Step 1 to each of these balls. Proceeding as above and exploiting overlaps, this implies that

$$\bigcup_{i=1}^N B_{r_0/2}(x_i) \setminus B_{\delta r_0/2}(x_i + L_{x_i,r_0/2})$$

consists of two mutually disjoint connected components B_1^+ and B_1^-, which are respectively contained in D^+ and D^-. Moreover, we may continuously extend ϵ to $B_0^+ \cup B_0^- \cup B_1^+ \cup B_1^-$. We may now proceed iteratively, using balls of radius $\frac{r_0}{2^k}$ at the k-th stage of the iteration, at each stage extending ϵ continuously to the pair of mutually disjoint connected components $\bigcup_{j=0}^k B_j^+ \cup \bigcup_{j=0}^k B_j^-$ formed at each stage. A final application of the Reifenberg property at the nearest point in $\{v = 0\} \cap B_{r_0}(x_0)$ to an arbitrary given point in $\{v > 0\} \cap B_{r_0}(x_0)$, at a scale comparable to the distance between these two points, guarantees that ϵ extends continuously to the entirety of $\{v > 0\} \cap B_{r_0}(x_0)$; the conclusion follows (see [TT12] for more details). ☐

We now characterize points $x_0 \in \{v = 0\}$ with $N_{v,x_0}(0^+) > 1$.

Proposition 4.3. *Let $v \in W^{1,2}_{\text{loc}}(\Omega)$ be a solution of (3.2)-(3.4). Suppose that $x_0 \in \partial\{v > 0\} \subset \Omega$ and that $N_{v,x_0}(0^+) > 1$. Then, any tangent function $\bar{v}$ at x_0 satisfies the following dichotomy. Either:*

1. *there exists $e_0 \in \{\bar{v} = 0\} \cap \partial B_1$ with $N_{\bar{v},e_0}(0^+) \geq \frac{3}{2}$, or*
2. *$\bar{v} = |h|$ where h is a homogeneous harmonic polynomial of degree at least 2.*

In particular, we have $N_{v,x_0}(0^+) \geq \frac{3}{2}$.

The proof of Proposition 4.3 follows a very similar line of reasoning to that of [ST15, Proof of Lemma 4.2]; however, we provide a proof here for clarity and due to the fact that we learned of the result [ST15, Lemma 4.2] after this article was completed. In order to prove Proposition 4.3, we require the following key characterization of radially homogeneous minimizers of our variational problem.

Lemma 4.4. *Suppose that v, x_0 and $\bar{v}$ are as in Proposition 4.3. Moreover, suppose that $n \geq 2$ and that $N_{\bar{v},e}(0^+) = 1$ for every $e \in \{\bar{v} = 0\} \cap \partial B_1$. Then $\bar{v} = |h|$ for a harmonic polynomial h.*

We remark that tangent functions of the type (2) in Proposition 4.3 do indeed exist. In [Lew77] it is shown that homogeneous harmonic polynomials of even degree must have at least three nodal domains and, moreover, that for every k even there exist a harmonic polynomial of degree k with exactly 3 nodal domains in $\mathbb{S}^2$. Similarly, Lewy showed that for any k odd there exists a polynomial of degree k with exactly 2 nodal domains in $\mathbb{S}^2$ (see [Bad11, Figure 1] for an explicit example of the latter).

Our proof of Lemma 4.4 relies on the following reflection property for v locally around points with frequency 1, which is useful in its own right.

Lemma 4.5. *Suppose that* $\mathbf{W}$*,* G *and* v *are as in Lemma 3.16. In addition, suppose that for some* $B_{\rho_0}(x_0) \subset \Omega$ *centered at a point* $x_0 \in \mathcal{R}(u)$*, there are exactly two connected components* $B^{\pm}$ *of* $\{v > 0\} \cap B_{\rho_0}(x_0)$ *and suppose that* $N_{v,y}(0^+) = 1$ *for every* $y \in \{v = 0\} \cap B_{\rho_0}(x_0)$*. Then the function* $\tilde{v} := v\mathbf{1}_{\overline{B}^+} - v\mathbf{1}_{B^-}$ *is a weak solution of*

$$\Delta \tilde{v} = \frac{1}{2}\tilde{H}(\tilde{v}) \text{ in } B_{\rho_0}(x_0)\,, \tag{4.3}$$

where $\tilde{H}$ *is the odd reflection of* G'*, i.e.,*

$$\tilde{H}(t) = \begin{cases} G'(t), & \text{if } t \in [0,1], \\ -G'(-t), & \text{if } t \in [-1,0)\,. \end{cases} \tag{4.4}$$

In order to prove Lemma 4.5, we require the following basic property of v restricted to its connected components.

Lemma 4.6. *Let* $v \in W^{1,2}_{\mathrm{loc}}(\Omega;[0,1])$ *satisfy* (3.2)-(3.4)*, with* G *satisfying* (3.1)*. Let* $x_0 \in \{v = 0\}$*, let* $r > 0$ *be such that* $B_r(x_0) \subset \Omega$*, and let* $D \subset B_r(x_0)$ *be an open set such that* $v = 0$ *on* $\partial D \cap B_r(x_0)$*. Then, the function*

$$v_1(x) = \begin{cases} v(x), & x \in D, \\ 0, & x \in B_r(x_0) \setminus D \end{cases}$$

is Lipschitz in $B_r(x_0)$ *and satisfies, in the sense of distributions, the equation*

$$2\Delta v_1 = G'(v_1) + \mu_1, \quad \text{in } B_r(x_0) \tag{4.5}$$

for some non-negative Radon measure μ_1*.*

Proof. As usual, we may assume without loss of generality that $x_0 = 0$. In virtue of the Lipschitz regularity of v proved in Theorem 3.1, we immediately have that v_1 is also Lipschitz continuous.

Let us notice that (4.5) amounts to showing that $2\Delta v_1 - G'(v_1) \geq 0$ in the sense of distributions (see, e.g., [CFROS20, Lemma A.1]). On the other hand, since v_1 and v agree on the open set D and we have the validity of (3.4) for v, it suffices to show that $2\Delta v_1 - G'(v_1) \geq 0$ nearby $\partial D \cap B_r$. Let $y_0 \in \partial D \cap B_r$ and let $\varphi \in C^\infty_c(B_r)$ be a non-negative test function suported on a neighborhood of y_0. Since v_1 is Lipschitz, and satisfies $2\Delta v_1 = G'(v_1)$ in D, we have that for almost every $t \in (0,\infty)$, $\{v_1 > t\}$ is a set of finite perimeter and the integration by parts formula

$$\begin{aligned} -2\int_{\{v_1>t\}} \nabla v_1 \cdot \nabla \varphi &= 2\int_{\partial^*\{v_1>t\}} |\nabla v_1|\varphi + 2\int_{\{v_1>t\}} \Delta v_1 \varphi \\ &\geq \int_{\{v_1>t\}} G'(v_1)\varphi \end{aligned} \tag{4.6}$$

holds. Thus, taking a sequence $t_k \downarrow 0$ such that (4.6) holds, we deduce that

$$-2\int_{B_r} \nabla v_1 \cdot \nabla \varphi = -2\int_{\{v_1>0\}} \nabla v_1 \cdot \nabla \varphi \geq \int_{\{v_1>0\}} G'(v_1)\varphi = \int_{B_r} G'(v_1)\varphi,$$

where we have used that $\nabla v_1 = 0$ $\mathcal{L}^{n+1}$-a.e. in $\{v_1 = 0\}$, and $G'(0) = 0$.

□

Proof of Lemma 4.5. First of all, observe that $\tilde{v}$ is Lipschitz in light of Theorem 3.1. Furthermore, thanks to Lemma 4.6, there exists two non-negative Radon measures μ_1 and μ_2 supported in $\{v = 0\} \cap B_{\rho_0}(x_0)$ such that

$$\Delta \tilde{v} = \frac{1}{2}\tilde{H}(\tilde{v}) + \mu_1 - \mu_2 \text{ in } B_{\rho_0}(x_0), \tag{4.7}$$

in the sense of distributions. So, showing (4.3) amounts to proving $\mu_1 = \mu_2$ which, in virtue of the Lebesgue-Besicovitch differentiation theorem (see, e.g., [Mag12, Theorem 5.8]) is equivalent to showing

$$\lim_{r \to 0^+} \frac{\mu_1(B_r(y))}{\mu_2(B_r(y))} = 1 \text{ for } y \in \operatorname{supp}(\mu_2)\,, \tag{4.8}$$

$$\lim_{r \to 0^+} \frac{\mu_2(B_r(y))}{\mu_1(B_r(y))} = 1 \text{ for } y \in \operatorname{supp}(\mu_1)\,. \tag{4.9}$$

We will show (4.8) since the argument for (4.9) is completely analogous. Let $y \in \operatorname{supp}(\mu_2)$ and consider a sequence $\{r_k\}$ with $r_k \to 0^+$ as $k \to \infty$. We will show that, up to taking a further subsequence (which we won't relabel), we have that

$$\lim_{k \to \infty} \frac{\mu_1(B_{r_k}(y))}{\mu_2(B_{r_k}(y))} = 1\,. \tag{4.10}$$

Since the sequence $\{r_k\}$ is arbitrary, the desired conclusion follows immediately. With this goal in mind, recalling the L^2 height function $H_y(r)$ of v centered at y as introduced in (3.24), consider the rescalings $v_{i,r}(x) = \frac{v_i(y+rx)}{H_y(r)^{1/2}}$ and $\mu_{i,r}$ given by $\mu_{i,r}(E) = \frac{\mu_i(rE+y)}{H_y(r)^{1/2} r^{n-1}}$ for any Borel set E and for $i = 1, 2$. Here, we take $r \in (0, \rho_0 - |y - x_0|)$. Clearly we may then rewrite (4.10) as

$$\lim_{k \to \infty} \frac{\mu_{1,r_k}(B_1(y))}{\mu_{2,r_k}(B_1(y))} = 1 \tag{4.11}$$

In addition, recall that by analogous reasoning to that in the proof of Lemma 3.19, the rescalings satisfy

$$\Delta v_{i,r} = \frac{r^2}{2H_y(r)^{1/2}} G'(v_{i,r} H_y(r)^{1/2}) + \mu_{i,r} \tag{4.12}$$

in the sense of distributions for $i = 1, 2$, together with the estimate

$$\left| \frac{r^2}{2H_y(r)^{1/2}} G'(v_{i,r} H_y(r)^{1/2}) \right| \leq Cr^2\,. \tag{4.13}$$

On the other hand, v_1 and v_2 have disjoint supports, we have that for r small enough

$$\int_{B_2} |\nabla v_{i,r}|^2 \leq \int_{B_2} |\nabla v_{y,r}|^2 \leq C\,, \tag{4.14}$$

where $v_{y,r}(x) = \frac{v(y+rx)}{H_y(r)^{1/2}}$ and where we have used (3.67) and the almost monotonicity of the frequency function proved in Lemma 3.9. We now proceed as in the proof of Lemma 3.17 to conclude the weak convergence (up to subsequence) of μ_{i,r_k} and v_{i,r_k}. More precisely, let $\varphi \in C_c^\infty(B_2)$ with $\varphi \geq \chi_{B_{\frac{3}{2}}}$, testing (4.12) and combining it with (4.13) and (4.14), we deduce

$$\mu_{i,r}(B_{\frac{3}{2}}) \leq Cr^2 + \int_{B_2} |\nabla\varphi \cdot \nabla v_{i,r}| \leq Cr^2 + \Big(\int_{B_2} |\nabla v_{i,r}|^2 \Big)^{\frac{1}{2}} \leq C,$$

for $i = 1, 2$ and for r small enough. So, up to extracting a subsequence of $\{r_k\}$, there exist $\tilde{\mu}_i$ and $\tilde{v}_i$ such that $\mu_{i,r_k} \overset{*}{\rightharpoonup} \tilde{\mu}_i$ as Radon measures in $B_{\frac{3}{2}}$ and that $v_{i,r_k} \rightharpoonup \tilde{v}_i$ weakly in $W^{1,2}(B_{\frac{3}{2}})$ as $k \to \infty$

for $i=1,2$. However, since $N_y(0^+)=1$ for every $y\in\{v=0\}\cap B_{\rho_0}(x_0)$, Proposition 3.23 implies that (up to taking a new subsequence) $\tilde{v}_1(x)=L(x\cdot e)_+$ and $\tilde{v}_2(x)=L(x\cdot e)_-$ for some $L>0$ and some $e\in\mathbb{S}^n$. Furthermore, by weak convergence, we have that

$$\Delta\tilde{v}_i=\tilde{\mu}_i \tag{4.15}$$

holds in the sense of distributions for $i=1,2$. From here, since $\tilde{v}_1(x)-\tilde{v}_2(x)=\frac{1}{\sqrt{|\omega_{n+1}|}}(x\cdot e)$, we deduce that $\tilde{\mu}_1=\tilde{\mu}_2$. In addition, by the particular form of $\tilde{v}_i$, we deduce from (4.15) that $\tilde{\mu}_i=\frac{1}{\sqrt{|\omega_{n+1}|}}\mathcal{H}^n\llcorner\{x\in B_{\frac{3}{2}}:x\cdot e=0\}$ and, thus, $\tilde{\mu}_i(\partial B_1)=0$. From here (4.11) follows immediately. □

Proof of Lemma 4.4. We will demonstrate that we may identify the set of connected components of $\{\bar{v}>0\}$ with the set of vertices for a bipartite graph, when $n\geq 2$. Once we show this, we may conclude as follows. Recall that every bipartite graph is two-colorable. Let $\mathcal{F}_1,\mathcal{F}_2$ denote the two mutually disjoint subsets of connected components of $\{\bar{v}>0\}$, each corresponding to the set of vertices of the same color in the graph. Define

$$h=\begin{cases}\bar{v} & \text{on every connected component in } \mathcal{F}_1,\\ -\bar{v} & \text{on every connected component in } \mathcal{F}_2,\\ 0 & \text{on } \{\bar{v}=0\}.\end{cases}$$

Observe that by construction, $\bar{v}=|h|$. Thus, we just need to verify that h is a harmonic polynomial. To see this, first of all notice that the harmonicity of h follows immediately from Lemma 4.5. Indeed, this is due to the fact that the hypotheses of the lemma guarantee that $\{\bar{v}=0\}\cap B_1\setminus\{0\}\subset\{y:N_{\bar{v},y}(0^+)=1\}$, combined with the bipartite graph property of the connected components of $\{\bar{v}>0\}$, and the fact that $\{0\}$ forms a capacity zero subset of B_1. Moreover, note that the function $\tilde{H}$ given by (4.4) associated to the tangent function $\bar{v}$ vanishes identically (see Lemma 3.19). To see that h is a polynomial, we simply exploit the radial homogeneity of $\bar{v}$, together with the well-known classification of radially homogeneous harmonic functions.

It now remains to prove the aforementioned claim that the connected components of $\{\bar{v}>0\}$ identify with the set of vertices of a bipartite graph. Note that this claim crucially requires $n\geq 2$, and is false when $n=1$. First, note our assumption that $N_x(0^+)=1$ for all $x\in\{\bar{v}=0\}\cap\partial B_1$ combined with the radial homogeneity of v implies that $N_x(0^+)=1$ for all $x\in\{\bar{v}=0\}\setminus\{0\}$. Then we can apply Proposition 4.1, Lemma 4.4, and the classification of frequency one blowups to conclude that $\{\bar{v}=0\}\setminus\{0\}$ locally coincides with the zero set of a harmonic function with non-vanishing gradient on its nodal set. As a consequence, the Implicit Function Theorem yields that $\{\bar{v}=0\}\cap\partial B_1$ is a smooth (even analytic), embedded $(n-1)$-manifold. The coloring can be done now by exhaustion as follows. Suppose, without loss of generality, that the two colors are red and blue. Consider a connected component U_0 of $\partial B_1\cap\{\bar{v}>0\}$, and assign this the color red. We assign each connected component of $\{\bar{v}>0\}$ neighboring U_0 the color blue, and call these $\{U_1^i\}_i$. We claim that

$$\text{if } i\neq j,\text{ then } \partial^{\partial B_1}U_1^i\cap\partial^{\partial B_1}U_1^j=\varnothing. \tag{4.16}$$

Assume for contradiction that (4.16) did not hold for some U_1^i and U_1^j. Then by the smoothness of $\{\bar{v}=0\}$, their common boundary is also smooth, and so we can choose a smooth connected component of $\partial^{\partial B_1}U_1^i\cap\partial^{\partial B_1}U_1^j$ and call it M. By the Jordan-Brouwer separation theorem on ∂B_1 (which follows for $n\geq 2$ from e.g. the statement on $\mathbb{R}^n$ [GP74, pg 89] and a stereographic projection, but does not hold on $\mathbb{S}^1$), denoting by A and B the open sets on ∂B_1 with $\partial^{\partial B_1}A\cap\partial^{\partial B_1}B=M$, we have, up to relabelling, $U_1^i\subset A$ and $U_1^j\subset B$. Also, since U_0 is open and connected and does not intersect M, we must have either $U_0\subset A$ or $U_0\subset B$. But either case leads to a contradiction: if $U_0\subset A$, it cannot border U_1^j since $U_1^j\subset B$ and $M\cap\partial U_0=\varnothing$ by the smoothness of $\{\bar{v}=0\}$, with a similar contradiction if $U_0 \subset B$. Next we color in red every open connected component of $\{\bar{v} > 0\}$ bordering some U_1^i; this is well-defined, since by (4.16) no blue sets share a common boundary. We can now proceed inductively in this manner, exhausting all of the connected components of $\{\bar{v} > 0\} \cap \partial B_1$ (of which there are finitely many according to the smooth embeddedness of $\{\bar{v} = 0\} \cap \partial B_1$). ☐

Proof of Proposition 4.3. Fix $x_0 \in \{v = 0\}$ with $N_{v,x_0}(0^+) > 1$ and consider any tangent function $\bar{v}$ at x_0. First of all, recall from Lemma 3.20, $\bar{v}$ is radially homogeneous of degree $\alpha := N_{v,x_0}(0^+) = N_{\bar{v},0}(0^+) > 1$.

We proceed to argue by induction on n, for solutions of (3.2)-(3.4), which in particular includes all tangent functions $\bar{v}$, in light of Lemma 3.19. Let us begin with the base case $n = 1$. In this case, the classification of Lemma 3.21 immediately implies that the alternative (1) holds and $N_{v,x_0}(0^+) \geq \frac{3}{2}$. Note that in this case, the alternative (2) is impossible, since there are exactly two connected components of $\{\bar{v} > 0\} \cap B_1$ if and only if $\bar{v}(x) = \frac{1}{\sqrt{\pi}}|x \cdot e|$ for some $e \in \mathbb{S}^1$, in which case $N_{v,x_0}(0^+) = 1$.

Now fix $n \geq 2$ and suppose that the conclusions of the proposition hold (including the lower frequency bound) in every dimension $m+1 \leq n$, in place of $n+1$. Let x_0, v be as in the statement of the proposition. There are two possibilities. Either

(a) there exists $e_0 \in \{\bar{v} = 0\} \cap \partial B_1$ with $N_{\bar{v},e_0}(0^+) > 1$, or
(b) for every $e \in \{\bar{v} = 0\} \cap \partial B_1$, $N_{\bar{v},e}(0^+) = 1$.

In case (a), by Lemma 3.20, any tangent function $\bar{w}$ of $\bar{v}$ at e_0 is translation-invariant in the direction e_0 and thus identifies with a solution of (3.2)-(3.4) with $G = 0$ that is a function of n real variables. Since we additionally have $N_{\bar{v},0}(0^+) \geq N_{\bar{v},e_0}(0^+) = N_{\bar{w},0}(0^+)$, $\bar{w}$ satisfies the hypotheses of the proposition at the origin (where any tangent function of it will be itself). The inductive hypothesis therefore allows us to conclude in this case. In case (b), we simply apply Lemma 4.4, which implies that $N_{\bar{v},0}(0^+) = N_{h,0}(0^+) \geq 2$.

When $n = 2$ and (2) holds, notice that the alternative (b) from the above dichotomy must hold. Indeed, if $N_{\bar{v},e_0}(0^+) > 1$ for some $e_0 \in \{\bar{v} = 0\} \cap \partial B_1$, then the tangent function $\bar{w}$ at e_0 as above will be a function of 2 real variables. However, we additionally have exactly two connected components of $\{\bar{v} > 0\} \cap B_1(e_0)$, which in turn implies that $\{\bar{w} > 0\}$ has exactly two connected components. This, combined with the fact that $N_{\bar{w},0}(0^+) = N_{\bar{v},e_0}(0^+) > 1$ is in contradiction with the classification given by Proposition 3.23. ☐

4.1. **Proof of Theorem 1.1.** We are now in a position to prove our main regularity result. Proposition 3.23 and Proposition 4.3 allow us to provide the following definitions of the *singular* and *regular* parts of the free boundary $\{u = 1\}$, for solutions u of (1.8)-(1.10) in terms of $v = 1 - u$.

Definition 4.1. Let $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a solution of (3.2)-(3.4). We define the singular set $\mathcal{S}(v)$ of $\{v = 0\}$ as

$$\mathcal{S}(v) := \left\{x : N_{v,x}(0^+) \geq \tfrac{3}{2}\right\},$$

and we define the regular set $\mathcal{R}(u)$ of $\{u = 1\}$ as

$$\mathcal{R}(u) := \{x : N_{v,x}(0) = 1\}.$$

Abusing notation, for $u \in W^{1,2}_{\mathrm{loc}}(\Omega; [0,1])$ a solution of (1.8)-(1.10) and $v = 1 - u$, we in turn define the respective singular and regular sets $\mathcal{S}(u)$, $\mathcal{R}(u)$ of $\{u = 1\}$ as

$$\mathcal{S}(u) := \mathcal{S}(v), \qquad \mathcal{R}(u) := \mathcal{R}(v).$$

We have the following immediate corollary.

Corollary 4.7. *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega; [0,1])$ be a solution of (1.8)-(1.10). Then $\Omega \cap \{u = 1\}$ decomposes as the disjoint union $\mathcal{S}(u) \sqcup \mathcal{R}(u)$, where $\mathcal{S}(u)$ is relatively closed in $\Omega \cap \{u = 1\}$.*

Proof. The decomposition into $\mathcal{R}(u)$ and $\mathcal{S}(u)$ is an immediate consequence of Proposition 3.23, and the fact that $\mathcal{S}(u)$ is relatively closed in Ω follows by the upper semicontinuity of the frequency function. $\square$

We begin our analysis by focusing on $\mathcal{R}(u)$; namely, we will proceed to prove Theorem 1.1. This regularity will essentially follow from noticing that points in $\mathcal{R}(u)$ correspond to regular points in the zero set of a function obtained from a suitable reflection of v. Thus, a priori, from the regularity of this reflected function, one can get initial regularity via the implicit function theorem for $\mathcal{R}(u)$. This argument is rather standard and we present it here for the sake of completeness, we remark that this reflection argument can be traced back, at least in spirit the argument to [Eva40a] where surfaces of minimal capacity were realized as zero sets of multivalued harmonic functions. In our case, we follow the arguments in [TT12]. The only difference lies in the analyticity conclusion which is a direct consequence of [KN15, Theorem 4]. Notice that this latter result is rather surprising, since it guarantees that regular level sets of solutions to semilinear PDEs are analytic regardless of the regularity of the non-linearity.

Let us re-state Theorem 1.1 here for convenience.

Theorem 4.8 (Regularity of $\mathcal{R}(u)$)**.** *If $u \in W^{1,2}_{\mathrm{loc}}(\Omega;[0,1])$ is a solution of (1.8)-(1.10) with $\Phi \in C^2$ and $\Phi'(1)=0$, then $\mathcal{R}(u)$ is locally an n-dimensional analytic submanifold.*

Proof of Theorem 4.8. Let $x_0 \in \mathcal{R}(u)$. It suffices to prove that $\mathcal{R}(u) \cap B_{\rho_0}(x_0)$ has the desired structure for some $0 < \rho_0 < \mathrm{dist}(x_0, \mathcal{S}(u))$, bearing in mind that $\mathrm{dist}(x_0, \mathcal{S}(u)) > 0$, since $\mathcal{S}(u)$ is relatively closed in Ω. We proceed in steps as follows.

Let $v = 1-u$ and let $G(v) := \Phi(1-v)$. First, we observe that, in virtue of Proposition 4.1, there exists $0 < \rho_0 < \mathrm{dist}(x_0, \mathcal{S}(u))$ such that $\{v > 0\} \cap B_{\rho_0}(x_0)$ has exactly two connected components.

We will now proceed to show that the zero set of $\tilde{v} := v\mathbf{1}_{\overline{B^+}} - v\mathbf{1}_{B^-}$ is analytic in $B_{\rho_0}(x_0)$. Firstly, we may apply Lemma 4.5 to conclude that $\tilde{v}$ is a weak solution of

$$\Delta \tilde{v} = \frac{1}{2}\tilde{H}(\tilde{v}) \text{ in } B_{\rho_0}(x_0)\,,$$

for $\tilde{H}$ as in (4.4). Since $G'(0)=0$, and $G \in C^2$, we have that $\tilde{H}$ is C^1. Thus, in virtue of the regularity of G and standard elliptic regularity theory, we deduce from the previous step that $\tilde{v} \in C^2_{\mathrm{loc}}(B_{\rho_0}(x_0))$. In particular, $\nabla\tilde{v}(x)$ exists in the classical sense at any $x \in B_\rho(x_0)$. Let us notice that at any $y \in \{v=0\} \cap B_{\rho_0}(x_0)$, we have $N_{v,y}(0^+) = N_{\tilde{v},y}(0^+) = 1$. Now for any such y, consider a subsequential limit w of the rescalings $\tilde{v}_{y,r}(x) = \frac{\tilde{v}(y+rx)}{H_{\tilde{v},y}(r)^{1/2}}$. Once again exploiting Lemma 4.5 together with Lemma 3.19 and Lemma 3.20 (cf. the proof of Lemma 4.4), we deduce that w is a homogeneous harmonic polynomial of degree $N_{w,0}(0^+) = N_{\tilde{v},y}(0^+)$. Now, if $\nabla\tilde{v}(y) = 0$, the subsequential convergence of $\tilde{v}_{y,r}$ to w guarantees that $N_{w,0}(0^+) > 1$, yielding a contradiction. Thus, $\nabla\tilde{v}$ doesn't vanish anywhere on $\{v=0\} \cap B_{\rho_0}(x_0)$. Finally, we deduce from [KN15, Theorem 4] that $\{\tilde{v}=0\} \cap B_{\rho_0}(x_0)$ is analytic. $\square$

We continue our analysis by providing a dimension bound on $\mathcal{S}(u)$ à la Federer. The argument is standard and appears in the literature in numerous places (for instance [TT12, Theorem 4.6], [DL16]), but we provide a proof here nevertheless, for purpose of clarity, since it is short and elementary. We start by combining Lemma 3.20 and Proposition 3.23 to deduce that when $n=1$, $\mathcal{S}(u)$ consists of isolated points.

Theorem 4.9. *Let $n=1$ and let $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a solution of (3.2)-(3.4). Then $\mathcal{S}(v)$ consists of isolated points.*

Proof. We argue by contradiction. Suppose that there exists a sequence $\{x_k\} \subset \mathcal{S}(v)$ with an accumulation in the interior of Ω. Then up to extracting a subsequence, $x_k \to x_0 \in \mathcal{S}(v)$. Let $r_k := 2|x_k - x_0|$. Applying Lemma 3.20 to the sequence v_{x_0,r_k}, we obtain a limiting radially

$N_{v,x_0}(0^+)$-homogeneous function $\bar{v} \in W^{1,2} \cap \mathrm{Lip}(\bar{B}_1)$, which, up to rotation, has the structure (3.101) for some integer $N \geq 2$. However, observe that the points $y_k = \frac{x_k - x_0}{r_k}$ satisfy $|y_k| = \frac{1}{2}$ and again by upper semicontinuity of the frequency, $y_k \to y_0$ with $N_{\bar{v},y_0}(0) > 1$. However, this contradicts the classification in Lemma 3.21 established for $\bar{v}$; indeed, it is easy to explicitly check that $N_{\bar{v},y}(0^+) = 1$ for any $y \neq 0$. ☐

Corollary 4.10. *Let $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a solution of (3.2)-(3.4). Then*

$$\dim_{\mathcal{H}}(\mathcal{S}(v)) \leq n - 1.$$

Proof. We will argue by induction on n, following Federer's dimension reduction argument in this setting. Observe that the $n = 1$ case is automatically covered by Theorem 4.9, which already provides a sharper statement. Now suppose that $n \geq 2$ and that we have established the the dimension estimate in $\mathbb{R}^n$, but suppose for a contradiction that it is false in $\mathbb{R}^{n+1}$. Then there exists v satisfying (3.2)-(3.4), an exponent $\alpha > 0$ and a compact subset $K \subset \mathcal{S}(v)$ such that

$$\mathcal{H}^{n-1+\alpha}(K) > 0.$$

Recall the notion of $(n-1+\alpha)$-dimensional Hausdorff content $\mathcal{H}^{n-1+\alpha}_\infty$ (see e.g. [Sim83]), which has the same negligible sets as $\mathcal{H}^{n-1+\alpha}$, but unlike the Hausdorff measure itself, is upper semicontinuous with respect to Hausdorff convergence of compact sets.

In particular, there exists $x_0 \in K$ and $\eta > 0$ such that

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^{n-1+\alpha}_\infty(B_r(x_0) \cap K)}{r^{n-1+\alpha}} \geq \lim_{r \downarrow 0} \mathcal{H}^{n-1+\alpha}_\infty(B_1 \cap K_{x_0,r}) \geq \eta.$$

where $K_{x_0,r} \subset \mathcal{S}(v_{x_0,r})$ denotes the rescaling $(K - x_0)r^{-1}$, with $v_{x_0,r}$ as defined in (3.93). Therefore, there exists a subsequence $r_k \downarrow 0$ and a compact set K_∞ such that $K_{x_0,r_k} \to K_\infty$ in Hausdorff distance, and

$$\mathcal{H}^{n-1+\alpha}_\infty(B_1 \cap K_\infty) \geq \eta. \tag{4.17}$$

In particular, we argue as above to deduce that there must exist a point $y_0 \in K_\infty \cap B_1 \setminus \{0\}$ with

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^{n-1+\alpha}_\infty(B_r(y_0) \cap K_\infty)}{r^{n-1+\alpha}} > 0$$

Furthermore, letting $\bar{v}$ denote a tangent function of v at x_0 along the sequence $\{r_k\}$; the conclusions of Lemma 3.20 imply that $K_\infty \cap B_1 \subset \mathcal{S}(\bar{v})$.

Repeating the above steps, we may now apply Lemma 3.20 to take a tangent function $\bar{v}_\infty$ to $\bar{v}$ at y_0, along some sequence $\rho_k \downarrow 0$, so that we additionally have

$$\mathcal{H}^{n-1+\alpha}_\infty(B_1 \cap \mathcal{S}(\bar{v}_\infty)) > 0.$$

Since $y_0 \neq 0$ and $\bar{v}$ is radially homogeneous, this implies that $\bar{v}_\infty$ is translation-invariant along some line through the origin. In other words, up to rotation, $\bar{v}_\infty(x_1, \ldots, x_{n+1}) = \bar{w}_\infty(x_1, \ldots, x_n)$, with

$$\mathcal{H}^{n-2+\alpha}_\infty(B_1 \cap \mathcal{S}(\bar{w}_\infty)) > 0$$

However, by our inductive hypothesis, we must have $\dim_{\mathcal{H}}(\mathcal{S}(\bar{w}_\infty)) \leq n-2$, which yields the desired contradiction. ☐

Lemma 4.11. *Let $n = 1$ and let u be a solution of (1.8)-(1.10). Suppose that $x_0 \in \{v = 0\}$. Then there exists $r_0 > 0$ (depending on x_0) such that $\{v > 0\} \cap B_{r_0}(x_0)$ has finitely many connected components.*

Proof. We may without loss of generality assume that $x_0 = 0$. We divide the proof into steps as follows.

Step 1. We first demonstrate that $\{v = 0\}$ has finite length inside any annulus centered at the origin contained in B_{r_0}, for any $r_0 > 0$ small enough. In light of Theorem 4.8 and Theorem 4.9, there exists $r_0 > 0$ such that $\mathcal{S}(u) \cap B_{r_0} = \{0\}$, and $\mathcal{R}(u) \cap B_{r_0}$ consists of analytic curves (possibly infinitely many) terminating at the origin. Let $0 < r < s \leq \frac{r_0}{2}$ and let $\varphi \in C_c^\infty(B_{r_0}; [0,\infty))$ be such that $\chi_{B_s \setminus \overline{B}_r} \leq \varphi \leq \chi_{B_{2s} \setminus \overline{B}_{r/2}}$. Let $\{D_i\}_{i\in\mathbb{N}}$ denote the connected components of B_{r_0} and let $v_i = v|_{D_i}$, extended by zero to B_{r_0}. Then each v_i is Lipschitz, Lemma 4.5 and an analogous computation to (4.6) together guarantee that, since $2\Delta v_i = G'(v_i)$ in D_i, we have

$$2\sum_i \int_{\partial\{v_i=0\}} |\nabla v_i| \varphi d\mathcal{H}^n = -2\sum_i \int_{\{v_i>0\}} \nabla v_i \cdot \nabla \varphi - \sum_i \int_{\{v_i>0\}} G'(v_i)\varphi\,.$$

Since $N_{v,x}(0^+) = 1$ for each $x \in B_{2s} \setminus \overline{B}_{r/2}$, the same argument as in Step 2 of Theorem 4.8 guarantees that $|\nabla v|$ does not vanish anywhere on $\{v=0\} \cap (B_s \setminus \overline{B}_r)$, and thus

$$\mathcal{H}^n(\{v=0\} \cap (B_s \setminus \overline{B}_r)) \leq C(r,s)\,.$$

Step 2. Let us now conclude that there exists $r_1 \leq \frac{r_0}{2}$ such that for any $0 < r < r_1$, under the additional assumption that $\{v=0\}$ has transverse intersection with ∂B_r, then $\{v=0\}$ consists of finitely many curves in B_r. From this, the conclusion will follow, in light of the transversality of smooth parametric families of maps to a given smooth submanifold (which follows from Sard's Theorem). Indeed, the latter together with the regularity of $\{v=0\}$, tells us that for almost-every $\rho \in (0,r_1)$, $\{v=0\}$ is transverse to ∂B_ρ.

Fix r_1 arbitrarily, to be determined later. First of all, observe that the conclusion of Step 1 guarantees that $\{v=0\} \cap (\overline{B}_{r_1} \setminus B_r)$ consists of countably many disjoint curves $\gamma_i : [0,1] \to \overline{B}_{r_1} \setminus B_r$, $i \in \mathbb{N}$, and at most finitely many of them have $\gamma_i(0) \in \partial B_{r_1}$ and $\gamma_i(1) \in \partial B_r$ (or vice versa).

In addition, we claim that only finitely many of them can have both $\gamma_i(0)$ and $\gamma_i(1)$ lying on ∂B_r. Indeed, if there are infinitely many, then the transversality assumption combined with an additional application of the conclusion of Step 1 implies that there must exist a closed embedded curve $\mathcal{C} \subset \{v=0\}$ contained in the interior of B_{r_1}. This in turn produces a connected component U of $\{v > 0\}$ contained strictly in the interior of B_{r_1}. We claim that for r_1 sufficiently small (depending implicitly on x_0 which we have taken to be the origin), this is not possible. This follows the reasoning of [CTV05, Proposition 6.2], which we repeat here for convenience. First of all, consider the rescaling $v_{r_1} \equiv v_{0,r_1}$ as in (3.93). In light of Lemma 3.19, we have the identity

$$\Delta v_{r_1} = \frac{r_1^2}{2H(r_1)^{1/2}} G'\big(v_{r_1} H(r_1)^{1/2}\big)\,,$$

inside the rescaled component $\tilde{U} := r_1^{-1}U$. Testing this against v_{0,r_1} (which can be done since v has zero boundary data in U) and integrating by parts, we obtain the Poincaré inequality

$$\int_{\tilde{U}} |\nabla v_{r_1}|^2 \leq \frac{kr_1^2}{2} \int_{\tilde{U}} v_{r_1}^2\,,$$

where $k = \sup_{[0,1]} |G''|$. Choosing r_1 sufficiently small such that $\frac{kr_1^2}{2} < \lambda_1(B_1)$, where $\lambda_1(B_1)$ denotes the lowest Dirichlet eigenvalue of the unit ball (which is an explicitly computable constant), we arrive at a contradiction.

Observe that this argument further tells us that we cannot have any connected components of $\{v > 0\}$ in B_r, and thus, again combining with the transverse intersection assumption, we deduce that the only possibility is that $\{v=0\} \cap B_r$ consists of a finite number of curves with either both endpoints on ∂B_r, or with one endpoint on ∂B_r and one endpoint at the origin. ☐

We finish this section with the proof of our main theorem. Our proof of the uniqueness of blow-ups at singular points in the planar case is a well know argument (see, e.g, [TT12]) which exploits the expansion of solutions to elliptic equations around critical points in the plane [HW53, Theorem 1].

Proof of Theorem 1.1. Part **(i)** is consequence of Corollary 3.3. The conclusions of Part **(ii)** when $n \geq 2$, together with the regularity of $\mathcal{R}(u)$ when $n = 1$, follow immediately from Corollary 4.7, Theorem 4.8 and Corollary 4.10. It merely remains to characterize the behavior of $\{u = 1\}$ at points in $\mathcal{S}(u)$ when $n = 1$. Letting $v = 1 - u$, from Theorem 4.9 we know that $\mathcal{S}(u)$ is discrete. Thus, for $x_0 \in \mathcal{S}(u)$, in virtue of Lemma 4.11, there exists $r_0 > 0$ such that $\{v > 0\} \cap B_{r_0}(x_0)$ has a finite number ℓ of connected components. Assuming without loss of generality that $x_0 = 0$, let us consider the function $w(\rho, \theta) = v(\rho^2, 2\theta)$ written in polar coordinates (ρ, θ). Notice that $\{w > 0\} \cap B_{r_0}$ has 2ℓ connected components $\{C_i\}_{i=1}^{2\ell}$ labelled so that $\partial C_i \cap \partial C_{i+1} \cap B_{r_0} \neq \varnothing$ for $i = 1, \cdots, 2\ell - 1$ and $\partial C_{2\ell} \cap \partial C_1 \cap B_{r_0} \neq \varnothing$. Consider now the function $z = \sum_{i=1}^{2\ell} (-1)^i w|_{C_i}$. We claim that

$$\Delta z(x) = 2|x|^2 \tilde{H}(z(x)) \quad x \in B_{r_0}, \tag{4.18}$$

with $\tilde{H}$ given by (4.4), and that (4.18) implies the desired conclusion.

Assuming for a moment the validity of the claim (4.18), since $f(x) = 2|x|^2 \frac{\tilde{H}(z(x))}{z(x)}$ is continuous, (4.18) falls under the hypotheses of [HW53, Theorem 1] with this choice of f, and $d = e = 0$ (see also [HW55] for a "modern" formulation of the result), implying that z admits a unique asymptotic expansion in polar coordinates of the form

$$z(\rho, \theta) = c_1 \rho^L \sin(L\theta) + c_2 \rho^L \cos(L\theta) + o\Big(\rho^L\Big), \tag{4.19}$$

as $\rho \to 0^+$ for some $c_1, c_2 \in \mathbb{R}$ and $L \in \mathbb{N}$. Notice that this combined with Lemma 3.21 and Lemma 3.20 implies that the tangent function $\bar{v}$ of v at 0 is unique and that $c_1 = \frac{1}{\sqrt{\pi}}$ and $L = 2\ell = 2N_{v,0}(0^+)$ in the expansion (4.19), as desired.

We finish the argument by proving (4.18). Let us observe first that when $z > 0$,

$$\begin{aligned}
\Delta z(\rho, \theta) &= \partial_{\rho\rho} z + \frac{\partial_\rho z}{\rho} + \frac{1}{\rho^2} \partial_{\theta\theta} z \\
&= 4\rho^2 \Big(\partial_{\rho\rho} v(\rho^2, 2\theta) + \frac{1}{\rho^2} \partial_\rho v(\rho^2, 2\theta) + \frac{1}{\rho^4} \partial_{\theta\theta} v(\rho^2, 2\theta) \Big) \\
&= 2\rho^2 G'(v)(\rho^2, 2\theta),
\end{aligned}$$

similarly we have that if $z < 0$, $\Delta z = -2\rho^2 G'(-v)(\rho^2, 2\theta)$. Lastly, since for each connected component C_i of $w > 0$, $\partial C_i \cap (B_{r_0} \setminus \{0\})$ is a union of regular curves in virtue of Theorem 4.8 and the normal derivatives of z on each side of ∂C_i match for $i = 1, \cdots, 2\ell$, we have that $\Delta z(x) = 2|x|^2 \tilde{H}(z(x))$ holds in $B_{r_0} \setminus \{0\}$ but since z is continuous up to the origin, we conclude that actually (4.18) holds. □

5. Existence of Minimizers: Proof of Theorem 1.2

For Theorem 1.2, we will need some basic information regarding the auxiliary variational problem

$$\Psi(v_0) = \inf \Big\{ \int_{\mathbb{R}^{n+1}} |\nabla u|^2 + F(u)\, dx : u \in W^{1,2}(\mathbb{R}^{n+1}; [0,1]),\ \int_{\mathbb{R}^{n+1}} V(u)\, dx = v_0 \Big\}. \tag{5.1}$$

This problem was introduced in [MR24] with the volume potential V as in (1.6), and quantitative stability and Alexandrov-type ridigity were established. Here we will only need the existence of positive minimizers for (5.1). The proof is in Appendix C and follows [MNR23a, Theorem A.1].

Theorem 5.1 (Existence of radial isoperimetric functions on $\mathbb{R}^{n+1}$)**.** *If $v_0 > 0$ and F and V are continuous, non-negative functions such that $F(0) = 0 = V(0)$ and*

$$\lim_{t \to 0^+} \frac{V(t)}{F(t)} = 0, \tag{5.2}$$

then there exists a strictly positive, radial, decreasing minimizer $x \mapsto w(|x|)$ for $\Psi(v_0)$.

We now reformulate (1.1)-(1.2). First, in order to write the decay at infinity condition in a suitable weak sense, when $n \geq 2$ we introduce the space

$$D_n^{1,2}(\Omega;[0,1]) := \{v : v \in L^{2(n+1)/(n-1)}(\Omega;[0,1]),\, \nabla v \in L^2(\Omega)\}\,. \tag{5.3}$$

By the Gagliardo-Nirenberg-Sobolev inequality and an extension argument (to account for the compact $\mathbf{W}$), $D_n^{1,2}(\Omega;[0,1])$ is closed under the topology induced by the norm $\|\cdot\|_{L^{2(n+1)/(n-1)}(\Omega)} + \|\nabla\cdot\|_{L^2(\Omega)}$. If $n = 1$, we cannot use this space since 2 is the critical Sobolev exponent, so we set

$$D_1^{1,2}(\Omega;[0,1]) := \{v : 0 \leq v \leq 1,\, \nabla v \in L^2,\, \mathcal{L}^2(\{v > t\}) < \infty\ \forall t \in (0,1)\}\,. \tag{5.4}$$

Unlike the case when $n \geq 2$, this space is not closed under the norm induced by the L^2 norm of the gradient; however, due to our assumption (1.11) on F when $n = 1$, it will be closed under the convergence one obtains for a minimizing sequence for the generalized formulation of (1.1), which we now state. We may thus reformulate the minimization problems (1.1) and (1.2) as

$$\inf\left\{ \int_\Omega |\nabla u|^2 + F(u) \,:\, u \in D_n^{1,2}(\Omega;[0,1]),\ \{u^* \geq t\} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \text{ for all } t \in (1/2,1) \right\} \tag{5.5}$$

and

$$\inf\left\{ \int_\Omega |\nabla u|^2 + F(u) : \begin{array}{c} u \in W^{1,2}_{\rm loc}(\Omega;[0,1]),\ \int_\Omega V(u) = 1, \\ \{u^* \geq t\} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \text{ for all } t \in (1/2,1) \end{array} \right\}. \tag{5.6}$$

Here, "$\mathcal{C}$-spanning" is interpreted in the sense of Definition A.2. When u is continuous in the above two problems, the condition is equivalent to "$\{u = 1\} \cap \gamma \neq \varnothing$ for all $\gamma \in \mathcal{C}$." To prove Theorem 1.2, we will obtain minimizers for (5.5) and (5.6) and then show that these minimizers are admissible and minimizing for (1.1) and (1.2) respectively.

Remark 5.2 (Euler-Lagrange equations for minimizers in the expanded formulation)**.** The same proof given in Theorem 2.1 for deriving the Euler-Lagrange equations (1.8)-(1.10) for minimizers of (1.1)-(1.2) applies for minimizers of (5.5)-(5.6): one may repeat verbatim the argument of [MNR23a, Theorem 1.3] to deduce this. The only difference to [MNR23a, Proof of Theorem 1.3] is that the paragraphs containing Equations 7.8-7.9 therein can be ignored, since they deal with an alternative spanning condition to the one in (5.5)-(5.6).

Proof of Theorem 1.2. The proof is divided into steps. First we obtain limits of minimizing sequences for (5.5) and (5.6). Then in steps two through four, we verify that these limits are admissible and minimizing for the either (5.5) or (5.6) (using crucially (1.11)) and also (1.1) or (1.2) respectively (by applying the regularity theory in Section 3). Note that we must distinguish between the cases $n = 1$ and $n \geq 2$ when verifying the admissibility for (5.5) and (1.1).

Step one (limits of minimizing sequences): Let $\{u_j\}$ be a minimizing sequence for (5.5) or (5.6). By Lemma C.2, which asserts that the infimums are indeed finite, there exists $u \in W^{1,2}_{\rm loc}(\Omega;[0,1])$ such that (up to a subsequence) $u_j \to u$ strongly in $L^1_{\rm loc}$ and, by the lower-semicontinuity of the Dirichlet energy and Fatou's lemma,

$$\int_\Omega |\nabla u|^2 + F(u)\,dx \leq \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^2 + F(u_j)\,dx\,.$$

By Theorem C.1, $\{u^* \geq t\}$ is $\mathcal{C}$-spanning $\mathbf{W}$ for every $t \in (1/2,1)$.

Step two (admissibility/minimality of u in (5.5) and (1.1) if $n \geq 2$): In this case, by the lower-semicontinuity of the Dirichlet energy and Fatou's lemma, $u \in D_n^{1,2}(\Omega;[0,1])$ and so is admissible for (5.5).

To see that u is in fact admissible for (1.1), we first observe that by Remark 5.2, u satisfies (1.8)-(1.10). Furthermore, since $u \in L^{2(n+1)/(n-1)}(\Omega)$, the function $v = 1-u$ satisfies

$1 - v \in L^{2(n+1)/(n-1)}(\Omega)$, and thus the hypotheses of Theorem 3.1.(ii) are satisfied, by Chebyshev's inequality. Therefore, by Theorem 3.1.(ii), there is $M(v,d)$ such that $N_{v,x}(r) \leq M$ for all $x \in \Omega$ with $\mathrm{dist}(x, \partial\Omega) \geq d$ and $r < \min\{d, r_{**}\}$. Thus, according to Theorem 3.1.(i), u is locally Lipschitz, and the Lipschitz estimate (3.8) holds for all small balls sufficiently far from $\partial\Omega$ with Lipschitz constant C independent of the center. Since $\nabla v = -\nabla u \in L^2(\Omega)$, the uniformity (at infinity) of the Lipschitz estimate (3.8) and the $L^{2(n+1)/(n-1)}$-integrability of u imply that u decays uniformly to 0 at infinity. Finally, by Lemma A.3, $\{u = 1\}$ is $\mathcal{C}$-spanning in the sense of (1.3).

So u is admissible for (1.1), and it remains to show that it is a minimizer. This requires proving that any admissible w for (1.1) with finite energy belongs to the space $D^{1,2}_n(\Omega; [0,1])$. This follows from (5.3) and the fact that $w \in L^{2(n+1)/(n-1)}(B_R^c)$, which is a consequence of $\nabla w \in L^2(B_R^c; [0,1])$ (for R such that $\mathbf{W} \subset\subset B_R$) and the pointwise decay to zero at infinity of w (see for example [Gal11, Theorem II.6.1] for a proof of the integrability of u under these two assumptions).

Step three (admissibility/minimality of u in (5.5) and (1.1) if $n = 1$): If $n = 1$, then by (1.11), there exists $t_k \searrow 0$ such that $F(t_k) > 0$. In order to obtain the decay of u at infinity, we will use this to show that, for all $t \in (0,1)$

$$\sup_j \mathcal{L}^2(\{u_j > t\}) < \infty\,. \tag{5.7}$$

Assuming the validity of the uniform bound (5.7), which depends on t but not j, combined with the L^1_{loc} convergence of u_j to u, we deduce that $u \in D^{1,2}_1(\Omega; [0,1])$ and thus is admissible for (5.5).

To prove (5.7), for R such that $\mathbf{W} \subset\subset B_R$, we let E denote a continuous linear extension operator from $W^{1,2}(B_{2R} \setminus B_R; [0,1])$ to $W^{1,2}(B_{2R}; [0,1])$. In a slight abuse of notation, for u_j we will let Eu_j denote the function on $\mathbb{R}^{n+1}$ which agrees with u_j outside B_R. It thus suffices to prove (5.7) for Eu_j, and in fact for $(Eu_j)^*$, which is the radially symmetric decreasing rearrangment (see e.g. [Gra14, Section 1.4.1]) of Eu_j. Note that the uniform energy bound for u_j implies that

$$\sup_j \int_{\mathbb{R}^{n+1}} |\nabla (Eu_j)^*|^2 + F((Eu_j)^*)\, dx < \infty\,. \tag{5.8}$$

Let us assume for contradiction that the uniform bound (5.7) for $(Eu_j)^*$ does not hold with some $t_0 \in (0,1)$. Then, letting $r_j \to \infty$ be such that $\mathcal{L}^2(\{(Eu_j)^* > t_0\}) = \pi r_j^2$ (up to extracting a subsequence if necessary), we set $\mathcal{F}(t) = \int_0^t \sqrt{F(s)}\, ds$ and use the identity $2ab \leq a^2 + b^2$ to estimate

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}} |\nabla (Eu_j)^*|^2 + F((Eu_j)^*)\, dx &\geq 2 \int_{B_{r_j}^c} |\nabla_x \mathcal{F}((Eu_j)^*(x))|\, dx \\
&\geq 4\pi r_j \int_{r_j}^\infty |\partial_r \mathcal{F}((Eu_j)^*(r))|\, dr \\
&= 4\pi r_j \mathcal{F}((Eu_j)^*(r_j)) \\
&= 4\pi r_j \mathcal{F}(t_0)\,,
\end{aligned}$$

where in the next to last line we have used the fundamental theorem of calculus and the assumption that $(Eu_j)^*$ decays to 0 at infinity (which follows from $u_j \in D^{1,2}_1(\Omega; [0,1])$), while in the last line we have used the fact that $(Eu_j)^*(r_j) = t_0$, by construction. Note that above we have abused notation slightly, interchanging between $(Eu_j)^*$ as a function of x and as a function of $r = |x|$. Now since F is not the zero function on $(0, t_0)$ by (1.11), $\mathcal{F}(t_0) > 0$ and thus the quantity $4\pi r_j \mathcal{F}(t_0)$ diverges to ∞ if $r_j \to \infty$, contradicting (5.8). Therefore, we have shown (5.7), and so $u \in D^{1,2}_1(\Omega; [0,1])$ and is minimizing for (5.5). By Remark 5.2, u satisfies the Euler-Lagrange equations, and again by Theorem 3.1 it is locally Lipschitz continuous. Furthermore, the fact that $\nabla u \in L^2$ combined with (5.7) allow us to invoke Theorem 3.1.(ii) to deduce that $v = 1 - u$ satisfies a uniform frequency bound for $N_{v,x}(r)$ for all large enough x and small enough r, and so by (3.8), u is globally Lipschitz on B_R if $\mathbf{W} \subset B_R$. Combined with the decay in measure of u to 0 at infinity (which is a consequence of (5.7)), this implies that u decays uniformly to 0 at infinity and is thus admissible for (1.1) (using Lemma A.3 to show that $\{u = 1\}$ is $\mathcal{C}$-spanning in the sense of (1.3)). In addition, u is minimizing for the latter problem, since every admissible competitor function in (1.1) is admissible for (5.5) (again by Lemma A.3).

Step four (admissibility/minimality of u in (5.6) *and* (1.2)*)*: Once again, by Lemma A.3, every admissible w for (1.2) is in fact admissible for (5.6). Then since a minimizer to (5.6) is locally Lipschitz (by Remark 5.2 and Theorem 3.1), to conclude the existence proof for (1.2) under the additional volume constraint, it remains to show that

$$\int_\Omega V(u)\,dx = 1\,. \tag{5.9}$$

Assume for contradiction that $\int_\Omega V(u)$ is strictly less than one and set

$$\varepsilon := \int_\Omega V(u)\,dx \in (0,1)\,.$$

In order to prove (5.9) by contradiction, we make three preliminary claims: first, that

$$\liminf_{j\to\infty} \int_\Omega |\nabla u_j|^2 + F(u_j)\,dx \geq \int_\Omega |\nabla u|^2 + F(u)\,dx + \Psi(1-\varepsilon) \tag{5.10}$$

(recall the definition of Ψ in (5.1)); second, that u is a minimizer for the problem

$$\Psi_{\mathbf{W}}(\varepsilon) := \inf\left\{ \int_\Omega |\nabla v|^2 + F(v) : \begin{array}{c} v \in W^{1,2}(\Omega;[0,1]),\ \int_\Omega V(v) = \varepsilon, \\ \{v^* \geq t\} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \text{ for all } t \in (1/2,1) \end{array} \right\}; \tag{5.11}$$

and third, that

$$\text{the infimum in (5.6) is equal to } \int |\nabla u|^2 + F(u)\,dx + \Psi(1-\varepsilon) = \Psi_{\mathbf{W}}(\varepsilon) + \Psi(1-\varepsilon)\,. \tag{5.12}$$

The lower bound (5.10) follows by a standard localization argument which we omit. For the second two claims (5.11)-(5.12), firstly for any v which is admissible for (5.11) (in particular u itself), we may consider the functions

$$v_j(x) = \max\{v(x), w(x - je_1)\}\,,$$

where w is a radial, decreasing minimizer for the isoperimetric problem (5.1) with $v_0 = 1-\varepsilon$; see Theorem 5.1. Observe that v_j satisfy the spanning condition and also $\int_\Omega V(v_j) \nearrow 1$, since w is decreasing to zero at infinity. Thus by Lemma B.1, we may fix volumes so that v_j are admissible for (5.6). Combining this with the fact that $\{u_j\}$ is a minimizing sequence for the latter, we obtain the upper bound

$$\liminf_{j\to\infty} \int_\Omega |\nabla u_j|^2 + F(u_j)\,dx \leq \lim_{j\to\infty} \int_\Omega |\nabla v_j|^2 + F(v_j)\,dx = \int_\Omega |\nabla v|^2 + F(v)\,dx + \Psi(1-\varepsilon)\,. \tag{5.13}$$

By (5.10) and the fact that (5.13) holds for every admissible v in (5.11), we deduce that

$$\begin{aligned} \Psi_{\mathbf{W}}(\varepsilon) + \Psi(1-\varepsilon) &\leq \int_\Omega |\nabla u|^2 + F(u)\,dx + \Psi(1-\varepsilon) \\ &\leq \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^2 + F(u_j)\,dx \\ &\leq \Psi_{\mathbf{W}}(\varepsilon) + \Psi(1-\varepsilon)\,. \end{aligned}$$

This concludes the arguments for the minimality of u in (5.11) and (5.12).

To prove (5.9) *by contradiction*: Now that we have demonstrated (5.10)-(5.12), we are in a position to prove (5.9). We introduce the notation

$$\mathcal{E}(u;U) = \int_U |\nabla u|^2 + F(u)\,dx\,, \qquad \mathcal{V}(u;U) = \int_U V(u)\,dx\,, \tag{5.14}$$

for $U \subset \Omega$. We simply write $\mathcal{E}(u)$ and $\mathcal{V}(u)$ respectively in the case when $U = \Omega$. We introduce the functions

$$v_k(x) = \max\{u(x),\, w(x - ke_1)\} : \Omega \to [0,1]$$

which have volume strictly less than 1 due to the fact that $w > 0$ (see Theorem 5.1); more precisely, denoting

$$A_k := \{x \in \Omega : 0 < u(x) < w(x - ke_1)\}\,, \quad B_k := \{x \in \Omega : 0 < w(x - ke_1) \le u(x)\} \cup \mathbf{W}\,,$$

which satisfy $A_k \cap B_k = \emptyset$, $|A_k| + |B_k| > 0$ since $w > 0$, we have

$$\mathcal{V}(v_k) = 1 - \mathcal{V}(u; A_k) - \mathcal{V}(w(\cdot - ke_1); B_k) < 1\,.$$

Since u is minimal for (5.6) with potential $\varepsilon^{-1}V$, Corollary 3.3.(iii) applies to u, yielding the Lipschitz bound (3.10) uniformly on small balls away from $\partial\Omega$, so that u decays uniformly to 0 as $|x| \to \infty$. Combined with the fact that w also decays uniformly to 0 (it is radially decreasing), we find that

$$0 < \max\{\sup\{u(x) : x \in A_k\},\, \sup\{w(x - ke_1) : x \in B_k\}\} \le \beta_k$$

for some $\beta_k \to 0$. Therefore, by the assumption (H4) that $\lim_{t\to 0} V(t)/F(t) = 0$,

$$\frac{\mathcal{E}(u;A_k) + \mathcal{E}(w(\cdot - ke_1);B_k)}{\mathcal{V}(u;A_k) + \mathcal{V}(w(\cdot - ke_1);B_k)} \ge \frac{\int_{A_k} F(u) + \int_{B_k} F(w(x - ke_1))}{\mathcal{V}(u;A_k) + \mathcal{V}(w(\cdot - ke_1);B_k)} \ge \inf_{0<t\le\beta_k} \frac{F(t)}{V(t)} \to \infty\,. \tag{5.15}$$

By applying a volume fixing variation to v_k as given by Lemma B.1.(ii) that increases the volume to 1, there is a constant $C_2 > 0$ (independent of k) such that for large k, there is $\tilde{v}_k$ with

$$\mathcal{V}(\tilde{v}_k) = 1,\ \mathcal{E}(\tilde{v}_k)) \le C_2\big(1 - \mathcal{V}(v_k)\big) + \mathcal{E}(v_k) = C_2\big(\mathcal{V}(u;A_k) + \mathcal{V}(w(\cdot - ke_1);B_k)\big) + \mathcal{E}(v_k)\,. \tag{5.16}$$

By (5.15), we may choose some k' large enough so that

$$\frac{\mathcal{E}(u;A_{k'}) + \mathcal{E}(w(\cdot - k'e_1);B_{k'})}{\mathcal{V}(u;A_{k'}) + \mathcal{V}(w(\cdot - k'e_1);B_{k'})} > C_2\,, \tag{5.17}$$

Since $\{u = 1\}$ is $\mathcal{C}$-spanning, $\{\tilde{v}_{k'} = 1\}$ is as well, so it is admissible for (5.6). Furthermore, by (5.16)-(5.17) and the minimality of u for $\Psi_{\mathbf{W}}(\varepsilon)$ we have

$$\begin{aligned}
\mathcal{E}(\tilde{v}_{k'}) &\le C_2\big(\mathcal{V}(u;A_{k'}) + \mathcal{V}(w(\cdot - k'e_1);B_{k'})\big) + \mathcal{E}(v_{k'}) \\
&= C_2\big(\mathcal{V}(u;A_{k'}) + \mathcal{V}(w(\cdot - k'e_1);B_{k'})\big) + \mathcal{E}(u;\Omega\setminus A_{k'}) + \mathcal{E}(w;\Omega\setminus B_{k'}) \\
&< \mathcal{E}(u;A_{k'}) + \mathcal{E}(w(\cdot - k'e_1);B_{k'}) + \mathcal{E}(u;\Omega\setminus A_{k'}) + \mathcal{E}(w;\Omega\setminus B_{k'}) \\
&= \Psi_{\mathbf{W}}(\varepsilon) + \Psi(1-\varepsilon)\,.
\end{aligned}$$

But by the admissibility of $\tilde{v}_k$ for (5.6), this contradicts (5.12). So it must be the case that $\mathcal{V}(u) = 1$, which is (5.9). □

Remark 5.3 (Optimality of the assumptions in $\mathbb{R}^2$ for (1.1))**.** If $n = 1$ and there exists $t_0 > 0$ such that $F(t) = 0$ for $t \in [0, t_0]$, then there do not exist any minimizers for (1.1), and any minimizing sequence converges to a function which is bounded from below by t_0. To see this, take a minimizing sequence $\{u_j\}$ and, for R large enough such that $\mathbf{W} \subset B_R$, consider the functions

$$w_j(x) = \begin{cases} \max\{u_j(x), t_0\} & x \in B_R \cap \Omega \\ \max\{2t_0 - t_0\log(|x|)/\log(R), u_j(x)\} & x \in B_{R^2}\setminus B_R \\ u_j(x) & \text{otherwise}\,. \end{cases} \tag{5.18}$$

Direct computation shows that their energy approaches that of $\max\{u_j, t_0\}$ on Ω as $R \to \infty$, which, since $F = 0$ and u_j decays at infinity, has strictly less energy than u_j. If u_j converged to a function which took values below t_0, this strict inequality would persist in the limit and contradict the minimality of our sequence. So there is no minimizer if $F = 0$ on $[0, t_0]$. Note that when $n \geq 2$, there is no way to truncate in a way that simultaneously ensures decay at infinity and that the Dirichlet energy of the tails decay to zero. Indeed, this phenomenon of lack of existence of global solutions to (1.1) with F vanishing only occurs when $n = 1$.

Appendix A. Homotopic spanning via measure theory

In this appendix we precisely reformulate (1.1)-(1.2) using the generalized spanning condition from [MNR23b, MNR23a]. The following definition is from [DLGM17, Definition 3] and is a slight generalization of the one from [HP16b, pg 359], which has stimulated much recent progress on the Plateau problem; see e.g. [DLGM17, DPDRG16, HP16a, DLDRG19, HP17, FK18, DR18, DPDRG20]. In what follows $\mathbf{W} \subset \mathbb{R}^{n+1}$ is closed and $\Omega = \mathbb{R}^{n+1} \setminus \mathbf{W}$.

Definition A.1 (Homotopic spanning for closed sets)**.** A **spanning class** $\mathcal{C}$ is a family of smooth embeddings of $\mathbb{S}^1$ into Ω which is closed by homotopy relative to Ω, and a relatively closed set $K \subset \Omega$ is said to be **$\mathcal{C}$-spanning W** if $K \cap \gamma \neq \varnothing$ for every $\gamma \in \mathcal{C}$.

To generalize this, we recall the notion of $\mathcal{C}$-spanning introduced by the first two authors and F. Maggi in [MNR23b, Definition B] and applied to the Allen-Cahn energy in [MNR23a]. It uses the notion of measure theoretic connectedness introduced in [CCDPM17, CCDPM14]. A Borel set K **essentially disconnects** another Borel set G if there exist Borel $G_1, G_2 \subset G$ such that

$$\mathcal{L}^{n+1}\big(G \Delta (G_1 \cup G_2)\big) = 0, \quad \mathcal{L}^{n+1}(G_1)\mathcal{L}^{n+1}(G_2) > 0, \quad \mathcal{H}^n\big((G^{(1)} \cap \partial^e G_1 \cap \partial^e G_2) \setminus K\big) = 0\,. \tag{A.1}$$

Here, for any Borel $B \subset \mathbb{R}^{n+1}$, $B^{(t)}$ is the set of points of Lebesgue density $t \in [0,1]$ and $\partial^e B$ is the essential boundary of B, or $\mathbb{R}^{n+1} \setminus (B^{(0)} \cup B^{(1)})$. We also denote by B_1^n the ball of radius one in $\mathbb{R}^n$.

Definition A.2 (Homotopic spanning for Borel sets)**.** Given a spanning class $\mathcal{C}$, the associated **tubular spanning class** $\mathcal{T}(\mathcal{C})$ is the family of all triples (γ, Ψ, T) where $\gamma \in \mathcal{C}$,

$$\Psi : \mathbb{S}^1 \times \overline{B_1^n} \to \Omega \text{ is a diffeomorphism with } \Psi|_{\mathbb{S}^1 \times \{0\}} = \gamma,$$

and $T = \Psi(\mathbb{S}^1 \times B_1^n)$. A Borel set $K \subset \Omega$ is **$\mathcal{C}$-spanning W** if for every $(\gamma, \Psi, T) \in \mathcal{T}(\mathcal{C})$, $\mathcal{H}^1$-a.e. $s \in \mathbb{S}^1$ has the following property: for $\mathcal{H}^n$-a.e. $x \in \Psi(\{s\} \times B_1^n)$, there exists a partition $\{T_1, T_2\}$ of T with $x \in \partial^e T_1 \cap \partial^e T_2$ and such that $K \cup \Psi(\{s\} \times B_1^n)$ essentially disconnects T into $\{T_1, T_2\}$.

Remark A.1 (Consistency of Definitions A.1-A.2)**.** The previous two definitions are consistent because for any relatively closed $K \subset \Omega$, it is $\mathcal{C}$-spanning according to the former if and only if it is $\mathcal{C}$-spanning according to the latter [MNR23b, Theorem A.1].

The $\mathcal{H}^n$-stability of the class of $\mathcal{C}$-spannings sets [MNR23b, Page 8] is the key property that allows for an acceptable definition of $\mathcal{C}$-spanning for the 1-level set of $u \in W^{1,2}(\Omega; [0,1])$. Specifically, if $u \in W^{1,2}(\Omega)$, then $\mathcal{H}^n$-a.e. $x \in \Omega$ is a Lebesgue point of u, and the **precise representative** u^* is given by

$$u^*(x) = \begin{cases} \lim\limits_{r \to 0} \dfrac{1}{\omega_{n+1} r^{n+1}} \displaystyle\int_{\{|z-x|<r\}} u(z)\, d\mathcal{L}^{n+1}(z) & \text{if the limit exists} \\ 0 & \text{otherwise}\,, \end{cases} \tag{A.2}$$

with the above limit existing for $\mathcal{H}^n$-a.e. $x \in \Omega$ (see e.g. [EG92, Chapter 4.8]).

Definition A.3 (Generalization of (1.3))**.** For $u \in W^{1,2}_{\mathrm{loc}}(\Omega; [0,1])$, the reformulation of (1.3) is

$$\{u^* \geq t\} \text{ is } \mathcal{C}\text{-spanning according to Definition A.2 for all } t \in (1/2, 1)\,. \tag{A.3}$$

Remark A.2. If $u \in (W^{1,2} \cap C^0)(\Omega; [0,1])$, then (A.3) holds if and only if $\{u = 1\}$ is $\mathcal{C}$-spanning (Lemma A.3). Also, (A.3) is preserved under uniform Dirichlet energy bounds, proven in [MNR23a] and recalled in Theorem C.1. Lastly, since supersets of $\mathcal{C}$-spanning sets are $\mathcal{C}$-spanning, choosing some other lower bound than $1/2$ does not change whether the condition holds for some u, as it is only those super-level sets where u takes values arbitrarily close to 1 that matter.

We conclude this section with an important lemma that will allow us to work with the spanning condition of Definition A.1 in place of that of Definition A.2 for continuous functions.

Lemma A.3 (Spanning for continuous functions)**.** *If $\mathbf{W} = \mathbb{R}^{n+1} \setminus \Omega$ is compact, $\mathcal{C}$ is a spanning class for $\mathbf{W}$, $\delta \in (1/2, 1]$, $u \in (W^{1,2} \cap C^0)(\Omega; [0,1])$, and $\{u^* \geq t\} = \{u \geq t\}$ is $\mathcal{C}$-spanning $\mathbf{W}$ for every $t \in (1/2, \delta)$, then $\{u \geq \delta\}$ is $\mathcal{C}$-spanning $\mathbf{W}$.*

Proof. By Remark A.1, it suffices to show that for every $\gamma \in \mathcal{C}$, $\{u \geq \delta\} \cap \gamma \neq \varnothing$. Pick a sequence $\{t_j\} \subset (1/2, \delta)$ such that $t_j \nearrow \delta$. Since $\{u \geq t_j\}$ is closed for every t, Remark A.1 implies that $\{u \geq t_j\}$ is $\mathcal{C}$-spanning in the sense of Definition A.1, that is there exists $x_j \in \gamma_j$ such that $u(x_j) \geq t_j$. By the compactness of γ, there must therefore be $x \in \gamma$ such that $x_j \to x$. Thus by the continuity of u,

$$u(x) = \lim_{j\to\infty} u(x_j) \geq \lim_{j\to\infty} t_j = \delta,$$

and so $\{u \geq \delta\} \cap \gamma \neq \varnothing$. □

Appendix B. Variational estimates

Here we collect some basic variational estimates relating to minimizers of (5.5) and (5.6), mostly contained in [MNR23a]. We begin with the following lemma, quoted from [MNR23a, Lemma 4.5], giving the inner variation formulae for the energy and volume.

Lemma B.1. (i): *If F, V are C^1, $A \subset \mathbb{R}^{n+1}$ is open, $X \in C_c^\infty(A; \mathbb{R}^{n+1})$, and $f_t(x) = x + t\,X(x)$, then there are positive constants t_0 and C_0 depending on X only, such that, for every $|t| < t_0$, $f_t : A \to A$ is a diffeomorphism, and for every $w \in W^{1,2}(A; [0,1])$ we have*

$$\begin{aligned}\Big| \int_A |\nabla(w \circ f_t)|^2 + F(w \circ f_t) - \int_A |\nabla w|^2 + F(w) \\ -t \int_A \big[|\nabla w|^2 + F(w)\big] \operatorname{div} X - 2(\nabla w) \cdot \nabla X[\nabla w]\Big| \leq C_0 t^2 \int_A |\nabla w|^2 + F(w)\,, \\ \Big| \int_A V(w \circ f_t) - \int_A V(w) - t \int_A V(w) \operatorname{div} X\Big| \leq C_0 t^2 \int_A V(w)\,, \end{aligned} \tag{B.1}$$

(ii): *If F, V are C^1 and V satisfies (H3), $A \subset \mathbb{R}^{n+1}$ is open, $u \in L^1(A; [0,1])$ and u is not constant on A, then there are positive constants η_0, t_0, β_0, and C_0 and a one parameter family of diffeomorphisms $\{f_t\}_{|t|<t_0}$, all depending on A and u, such that $f_0 = \mathrm{id}$, $\{f_t \neq \mathrm{id}\} \subset\subset A$, and for every $w \in W^{1,2}(A; [0,1])$ with $\|u - w\|_{L^1(A)} \leq \beta_0$ and $|\eta| < \eta_0$, there is $t = t(\eta) \in (-t_0, t_0)$ such that $w_t = w \circ f_t$ satisfies*

$$\int_A V(w_t) = \int_A V(w) + \eta\,, \qquad \Big| \int_A |\nabla w_t|^2 + F(w_t) - |\nabla w|^2 - F(w)\Big| \leq C_0|\eta| \int_A |\nabla w|^2 + F(w)\,.$$

Outline of Proof. The first item follows from the area formula and does not depend on the form of V. The second item is the volume-fixing variations argument for perimeter ([Mag12, Lemma 29.13, Theorem 29.14]) adapted to the Allen-Cahn setting. The only required property of V is that the non-constancy of u implies that $V(u)$ is non-constant also; see [MNR23a, Proof of Lemma 4.5.(ii)]. But this is guaranteed by our assumption (H3) that V is strictly increasing. □

Corollary B.2. *If F, V are C^1, $\mathbf{W} = \mathbb{R}^{n+1} \setminus \Omega$ is compact, $\mathcal{C}$ is a spanning class for Ω, and u is a minimizer for* (5.6)*, then there exists positive $\tilde{r}$ and $\tilde{C}$, both depending on u, such that for all $w \in W^{1,2}(\Omega; [0,1])$ with $\{w \geq t\}$ $\mathcal{C}$-spanning $\mathbf{W}$ for all $t \in (1/2, 1)$ and $\{u \neq w\} \subset B_{\tilde{r}}(x_0) \cap \Omega$ for some $x_0 \in \Omega$,*

$$\int_\Omega |\nabla u|^2 + F(u)\, dx \leq \int_\Omega |\nabla w|^2 + F(w)\, dx + \tilde{C} \left| 1 - \int_\Omega V(w) \right| . \tag{B.2}$$

Proof. First, note that u cannot be constant on Ω since $\int_\Omega V(u) = 1$ and Ω is unbounded. Let $A_i \subset \Omega$ for $i = 1, 2$ be such that $\mathrm{dist}(A_1, A_2) > 0$ and u is non-constant on each A_i. Then Lemma B.1.(ii) applies to u and A_i, so we may choose $\tilde{r}$ small enough so that if $w \in W^{1,2}(\Omega; [0,1])$ and $\{u \neq w\} \subset B_{\tilde{r}}(x_0) \cap \Omega$ for any x_0, then $|1 - \int V(w)| < \eta_0$ and $B_{\tilde{r}}(x_0)$ is disjoint from at least one A_i. By fixing the volume of w on this A_i via $w \circ f_{t(\eta)}$ as in the previous lemma, we may modify it so that the modification has volume 1. In addition, we claim that this modification preserves the spanning constraint in (1.2). Indeed, if B is Borel, then $f_t^{-1}(B^{(1)}) = (f_t^{-1}(B))^{(1)}$ and $f_t^{-1}(B^{(0)}) = (f_t^{-1}(B))^{(0)}$ (both immediate consequences of the area formula) imply that

$$\partial^e(f_t^{-1}(B)) = \left((f_t^{-1}(B))^{(1)} \cup (f_t^{-1}(B))^{(0)}\right)^c = \left(f_t^{-1}(B^{(1)}) \cup f_t^{-1}(B^{(0)})\right)^c = f_t^{-1}(\partial^e B)\,; \tag{B.3}$$

also, due to the closure of $\mathcal{C}$ under homotopy,

$$f_t^{-1} \circ \gamma \in \mathcal{C} \quad \forall |t| < t_0\,. \tag{B.4}$$

By (B.3)-(B.4), the verification of (A.3) for $w \circ f_t$ via Definition A.2 with a triple (γ, Ψ, T) reduces to the validity of the same condition for w on $(f_t^{-1} \circ \gamma, f_t^{-1} \circ \Psi, f_t^{-1} \circ T) \in \mathcal{T}(\mathcal{C})$. Testing the minimality of u against this modification and using the estimates from Lemma B.1.(ii) concludes the argument. □

We can in fact further show that if minimizers exist for (5.6), then the sizes of the Lagrange multipliers and almost-minimality properties are uniform among the set of all minimizers for (5.6).

Corollary B.3 (Uniform almost-minimality and bounds on Lagrange multipliers)**.** *If F, V are C^2, $\mathbf{W} = \mathbb{R}^{n+1} \setminus \Omega$ is compact, $\mathcal{C}$ is a spanning class for $\mathbf{W}$, and there exist minimizers for* (5.6)*, then*

$$\sup\{\lambda : u \textit{ is a minimizer for } (5.6) \textit{ and satisfies } (1.8)\text{-}(1.9) \textit{ with } \Phi = F - \lambda V\} < \infty \tag{B.5}$$

and there exists $C_0 > 0$ and $r_0 > 0$, depending only on F, V, $\mathbf{W}$, and n, such that for all $w \in W^{1,2}(\Omega; [0,1])$ with $\{w \geq t\}$ $\mathcal{C}$-spanning $\mathbf{W}$ for all $t \in (1/2, 1)$ and $\{u \neq w\} \subset B_{r_0}(x_0) \cap \Omega$ for some $x_0 \in \Omega$,

$$\int_\Omega |\nabla u|^2 + F(u)\, dx \leq \int_\Omega |\nabla w|^2 + F(w)\, dx + C_0 \left| 1 - \int_\Omega V(w) \right| . \tag{B.6}$$

Proof. Let u_j be a sequence of minimizers such that the limit $\lim \lambda_j$ of the corresponding Lagrange multipliers achieves the supremum in (B.5). By Lemma B.1, we may obtain a uniform set of volume fixing variations for the tail of the sequence $\{u_j\}$. By the proof of Corollary B.2, there exist $\tilde{C}$ and $\tilde{r}$ such that the almost-minimality property (B.2) holds along the tail of the sequence $\{u_j\}$ with $\tilde{C}$ and $\tilde{r}$. The Lagrange multipliers may now be bounded uniformly in j by adapting the corresponding classical argument for bounding the Lagrange multipliers of volume-constrained perimeter minimizers in terms of the constant in the almost-minimality inequality. The proof of (B.6) is analogous. □

Appendix C. Preliminaries for Theorem 1.2

Theorem C.1 (Compactness)**.** *If $\mathbf{W} \subset \mathbb{R}^{n+1}$ is compact, $\mathcal{C}$ is a spanning class for $\mathbf{W}$, $\{\delta_j\}_j \subset (1/2, 1]$, $\delta_j \to \delta_0 \in (1/2, 1]$, and $\{u_j\} \subset W^{1,2}_{\rm loc}(\Omega; [0, 1])$ are such that $u_j \to u$ in $L^1_{\rm loc}(\Omega)$ for some u, $\{u_j^* \geq t\}$ is $\mathcal{C}$-spanning for each j and $t \in [1/2, \delta_j)$, and*

$$\sup_j \int_\Omega |\nabla u_j|^2 \, dx < \infty \,, \tag{C.1}$$

then $\{u^* \geq t\}$ is $\mathcal{C}$-spanning for every $t \in [1/2, \delta_0)$.

Outline of Proof. Fix a triple $(\gamma, \Phi, T) \in \mathcal{T}(\mathcal{C})$ for which we must verify that Definition A.2 holds for every $\{u^* \geq t\}$ with $t \in [1/2, \delta_0)$. We modify our function so as to allow for the application of [MNR23a, Theorem 3.2]. Let $w \in W^{1,2}_{\rm loc}(\Omega; [0, 1])$ be such that

$$w = 0 \text{ on } \operatorname{cl} T, \; w = 1 \text{ on } \Omega \setminus B_R(0) \text{ for some large } R \tag{C.2}$$

and consider the functions

$$v_j = \max\{u_j, w\} \,, \qquad v = \max\{u, w\} \,. \tag{C.3}$$

Note that

$$\sup_j \int_\Omega |\nabla v_j|^2 \, dx < \infty \,, \tag{C.4}$$

$$v_j \xrightarrow{L^2_{\rm loc}} v, \text{ and } \{v_j^* \geq t\} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \text{ for every } t \in [1/2, \delta_j) \;. \tag{C.5}$$

Since $v = u$ on $\operatorname{cl} T$, the super-level sets of v satisfy the spanning condition on this fixed tube T if and only if those of u do as well. So it suffices to explain why $\{v^* \geq t\}$ satisfies Definition A.2 for this (γ, Φ, T). This can be done by following the compactness result [MNR23a, Theorem 3.2], which gives conditions under which the spanning condition is preserved under limits of functions. Our assumptions (C.4)-(C.5) on v_j the same as in [MNR23a, Theorem 3.2] up to the facts that there, the uniform bound

$$\sup_j \int_\Omega \varepsilon |\nabla v_j|^2 + \frac{W(v_j)}{\varepsilon} \, dx < \infty \,, \qquad \varepsilon > 0 \,, \tag{C.6}$$

where W is a double-well potential with $W(1) = 0 = W(0)$ is assumed instead of (C.4), and the functions v_j are assumed to belong to L^2 rather than the $L^2_{\rm loc}$. By (C.2)-(C.3), the functions v_j satisfy (C.6), and the class $L^2_{\rm loc}$ is enough to repeat [MNR23a, Proof of Theorem 3.2] verbatim. (The spanning condition on a single tube (γ, Φ, T) is local in nature, in that it does not depend on the values of v outside T, so this last claim should be heuristically clear without referencing the details of [MNR23a, Proof of Theorem 3.2].) ☐

Lemma C.2 (Non-triviality of (5.5)-(5.6))**.** *If F and V are continuous with $F(0) = 0 = V(0)$ and $V(t) > 0$ for $t \in (0, 1]$, $\mathbf{W} = \mathbb{R}^{n+1} \setminus \Omega$ is compact, and $\mathcal{C}$ is a spanning class for $\mathbf{W}$ satisfying (1.7), then (5.5) and (5.6) have finite infimums.*

Proof. We will proceed to explicitly construct an admissible function u for both (5.5) and (5.6) with finite energy. For some $\delta > 0$ fixed, let us consider the $\mathbf{W}_\delta$ to be those points in Ω such that $\operatorname{dist}(x, \mathbf{W}) \leq \delta$. We claim that:

$$\text{if } \gamma \in \mathcal{C} \text{ and (the image of) } \gamma \text{ is contained in } \mathbf{W}_\delta^c, \text{ then } \operatorname{diam} \gamma > \delta/2. \tag{C.7}$$

Indeed, if this were not the case, and there were some γ with image contained in $\mathbf{W}_\delta^c$ with diameter no more than $\delta/2$, then, choosing some ball $\mathbf{W}_\delta^c \supset B_{\delta/2}(x) \supset \gamma$, we would have $\operatorname{dist}(B_{\delta/2}(x), \mathbf{W}) \geq \delta/2$. But then $\gamma \subset B_{\delta/2}(x) \subset \Omega$, and thus γ is homotopic to a point, contradicting (1.7). Let $R > 0$

be such that $\mathbf{W} \subset B_R$. Now, defining the grid $G_{\delta/2} = \cup_{z\in\mathbb{Z}^{n+1}} \frac{\delta}{2\sqrt{n+1}}\big(z + \partial([0,1]^{n+1})\big)$ of diameter $\frac{\delta}{2}$, we claim that

$$\big[(\Omega \cap \mathbf{W}_\delta) \cup \partial B_R \cup (B_R \cap G_{\delta/2})\big] \cap \gamma \neq \varnothing \text{ for all } \gamma \in \mathcal{C}\,. \tag{C.8}$$

To see that this is the case, first notice that if $\mathrm{dist}(\gamma, \mathbf{W}) \le \delta$, then clearly the intersection is non-empty. On the other hand, if $\gamma \subset \mathbf{W}_\delta^c$, then it must intersect $\partial B_R \cup (B_R \cap G_{\delta/2})$, because otherwise it would be contained in a single cube and contradict (C.7) or be contained in $\overline{B}_R^c$ and again be homotopic to a point, contradicting (1.7). Finally, for $\varepsilon > 0$ to be determined, we define

$$u(x) = \max\{1 - \mathrm{dist}(x, \mathbf{W}_\delta \cup \partial B_R \cup (B_R \cap G_{\delta/2}))/\varepsilon, 0\}\,.$$

Since u is Lipschitz with compact support and $\{u = 1\}$ contains the $\mathcal{C}$-spanning set from (C.8), u is admissible in (5.5). Furthermore, for (5.6), note that $\int_\Omega V(u)$ is continuous and increasing in ε, so the intermediate value theorem yields some ε such that u satisfies the volume constraint. Finally, clearly u has finite energy, since it has compact support and is Lipschitz. □

Proof of Theorem 5.1. The argument is the same as the one in [MNR23a, Theorem A.1] and depends on (5.2). Let $\{w_j\}_j$ be a minimizing sequence for $\Psi(v)$. By the Pólya-Szegö inequality, we may as well assume that $w_j(x) = g_j(|x|)$ are radially decreasing. Due to the uniform Dirichlet and L^∞ bounds on w_j, there exists $w \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}; [0,1])$ with finite Dirichlet energy such that $w_j \to w$ in L^1_{loc}, $w(x) = g(|x|)$, and

$$\int_{\mathbb{R}^{n+1}} |\nabla w|^2 + F(w)\, dx \le \liminf_{j\to\infty} \int_{\mathbb{R}^{n+1}} |\nabla w_j|^2 + F(w_j)\, dx\,.$$

To show that w is a minimizer for (5.1), we only need to show that $\int_{\mathbb{R}^{n+1}} V(w)\, dx = v$, which would follow from showing that

$$\lim_{R\to\infty} \sup_j \int_{B_R^c} V(w_j)\, dx = 0\,. \tag{C.9}$$

Since $g_j \to g$ a.e. on $(0,\infty)$ and g is radially decreasing, it follows that $\lim_{R\to\infty} \sup_j g_j(R) = 0$. Now by (5.2), we estimate

$$0 \le \lim_{R\to\infty} \sup_j \int_{B_R^c} V(g_j(|x|))\, dx \le \lim_{R\to\infty} \begin{cases} \sup_j \dfrac{V(g_j(R))}{F(g_j(R))} \displaystyle\int_{B_R^c} F(g_j(|x|))\, dx & g_j(R) \neq 0 \\ 0 & g_j(R) = 0\,; \end{cases} \tag{C.10}$$

note that $g_j(R) \neq 0$ implies that $F(g_j(R)) \neq 0$ by (5.2) and (H3). By using $\lim_{R\to\infty} \sup_j g_j(R) = 0$ in (C.10), we find (C.9). The fact $w > 0$ follows from the Euler-Lagrange equations as in (2.1). □

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