ON DE GIORGI'S CONJECTURE OF NONLOCAL APPROXIMATIONS FOR FREE-DISCONTINUITY PROBLEMS: THE SYMMETRIC GRADIENT CASE

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Abstract. We prove that E. De Giorgi's conjecture for the nonlocal approximation of free-discontinuity problems extends to the case of functionals defined in terms of the symmetric gradient of the admissible field. After introducing a suitable class of continuous finite-difference approximants, we show the compactness of deformations with equibounded energies, as well as their Gamma-convergence. The compactness analysis relies on a generalization of a Fréchet-Kolmogorov approach previously introduced by two of the authors. An essential difficulty is the identification of the limiting space of admissible deformations. We show that if the approximants involve superlinear contributions, a limiting GSBD representation can be ensured, whereas a further integral geometric term appears in the limiting functional in the linear case. We eventually discuss the connection between this latter setting and an open problem in the theory of integral geometric measures.

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1. Introduction

Free-discontinuity problems and their approximation by means of Sobolev formulations, discrete descriptions, or nonlocal functionals are a thriving research area owing to their broad scope of applications, ranging from image reconstruction, to the modeling of failure phenomena in continuum mechanics. A milestone in this direction was a conjecture formulated by E. De Giorgi, and dealing with the approximation via Gamma-convergence of the Mumford-Shah functional by a sequence of nonlocal counterparts. Starting from the seminal proof of this result by M. Gobbino in [26], such analysis has paved the way for further generalizations in [17, 27], eventually leading from [25] to variational studies on point clouds (see, e.g., [6, 11, 19]), as well as to machine-learning applications [10]. All the aforementioned contributions deal with free-discontinuity problems involving the full distributional gradient of the admissible maps. A question that, to the authors' knowledge, was so far left open, was whether similar nonlocal approximation techniques would also prove successful for the study of free-boundary problems involving more general differential operators.

In this paper we show that an analogue approximation result to that conjectured by E. De Giorgi and proved by M. Gobbino holds in the case of free-boundary problems defined in terms of the symmetric gradient of the admissible deformations. To be precise, we introduce a new class of *continuous finite-difference approximations* of linearized Griffith functionals, and we rigorously derive their Γ-convergence as well as the compactness of their minimizing sequences.

In order to describe our findings, we first recall the results [26], where the sequence of functionals

$$\mathscr{F}_\varepsilon(u,\Omega) := \frac{1}{\varepsilon^{n+1}}\int_{\Omega\times\Omega} \arctan\left(\frac{(u(x')-u(x))^2}{\varepsilon}\right) e^{-|\frac{x'-x}{\varepsilon}|^2}\mathrm{d}x'\mathrm{d}x, \quad u\in L^1(\Omega) \tag{1.1}$$

is shown to Γ-converge, as ε tends to zero, to the Mumford-Shah energy

$$\sqrt{\pi}\mathcal{MS}_{1/\sqrt{\pi}}(u,\Omega) := \int_\Omega |\nabla u(x)|^2 \mathrm{d}x + \sqrt{\pi}\mathcal{H}^{n-1}(J_u), \quad u \in \mathrm{GSBV}(\Omega). \tag{1.2}$$

In (1.2), the acronym GSBV stands for the space of functions with generalised special bounded variation (roughly speaking, maps whose truncations are functions of bounded variations with distributional gradients exhibiting null Cantor part), cf. [4]. Note that the multiplicative constants in (1.2) depend only on the choice of the integrand $\arctan(x)$ and of the weight function $e^{-|x|^2}$.

It is thus natural, in a first stage, to ask whether the functional

$$\mathcal{F}_\varepsilon(u,\Omega) := \frac{1}{\varepsilon^{n+1}} \int_{\Omega\times\Omega} \arctan\left(\frac{((u(x')-u(x))\cdot(x'-x))^2}{\varepsilon^3}\right) e^{-|\frac{x'-x}{\varepsilon}|^2} \mathrm{d}x'\mathrm{d}x \tag{1.3}$$

defined for measurable functions $u \in L^0(\Omega;\mathbb{R}^n)$ provides, analogously, a nonlocal approximation, in the sense of Γ-convergence, of linearized free discontinuity problems of the form

$$\mathcal{F}(u,\Omega) := \int_\Omega \varphi(e(u))\,\mathrm{d}x + C\mathcal{H}^{n-1}(J_u), \quad u \in \mathrm{GSBD}(\Omega)$$

for suitable choices of the density φ and of the constant $C > 0$. In the expression above, $\mathrm{GSBD}(\Omega)$ denotes the space of functions with generalized special bounded deformation (see [21]), and $e(u)$ the absolutely continuous part of the symmetric distributional gradient $\mathcal{E}(u) := \frac{Du+Du^\top}{2}$.

The structure of (1.3) aligns with the principles of linearized theories in continuum mechanics, where only the symmetric part of the gradient contributes to the deformation energy. This consideration naturally motivates the assumption that only the component of $u(x') - u(x)$ projected along the direction $x - x'$ should provide energy contributions. By setting $\varepsilon\xi := x' - x$, when the differences $(u(x+\varepsilon\xi) - u(x))\cdot\xi$ are relatively small, the functionals in (1.3) behave like pure bulk energies. For large values of $(u(x+\varepsilon\xi) - u(x))\cdot\xi$, instead, the energies in (1.3) saturate, effectively detecting and penalizing the size of the discontinuities of u.

An important remark concerns the fact that the energies in (1.1) only approximate the Mumford-Shah functional under suitable additional assumptions on the geometry of Ω, such as Lipschitz regularity (see [26, Remark 7.1]). Such requirements may not be fully consistent when handling free-boundary problems where discontinuities are already present within the material. Consider, for example, the case of a crack-initiated domain, where the presence of an $(n-1)$-dimensional set $\Gamma \subset \Omega$, representing an initial crack makes the set $\Omega \setminus \Gamma$ not Lipschitz.

This possible lack of regularity of Ω calls for a more refined approach, which we address by modifying the energies in (1.3) as follows. First, for every $\xi \in \mathbb{S}^{n-1}$ and every Borel set $E \subset \Omega$ we introduce the functionals $F_{\varepsilon,\xi}(u,E)$, defined as

$$F_{\varepsilon,\xi}(u,E) := \frac{1}{\varepsilon}\int_{E\cap(E-\varepsilon\xi)} \arctan\left(\frac{((u(x+\varepsilon\xi)-u(x))\cdot\xi)^2}{\varepsilon}\right)\mathrm{d}x, \quad u \in L^0(\Omega;\mathbb{R}^n).$$

The specific structure of (1.3) allows one to make use of Fubini's theorem and rewrite it as an average of $F_{\varepsilon,\xi}$ with respect to all possible directions ξ. Specifically, we have

$$\mathcal{F}_\varepsilon(u,E) = \int_{\frac{E-E}{\varepsilon}} F_{\varepsilon,\xi}(u,E)e^{-|\xi|^2}\mathrm{d}\xi,$$

where $E - E := \{y \in \mathbb{R}^n : y = x' - x, \text{ for some } x, x' \in E\}$. The relevant functionals for our analysis, denoted by $\mathcal{F}^p_\varepsilon(u,\Omega)$, depend on a further degree of freedom, encoded by a parameter $p \in [1, \infty)$, and take the form

$$\mathcal{F}^p_\varepsilon(u,\Omega) := \sup_{\mathscr{B}} \sum_{B\in\mathscr{B}} \left(\int_{\frac{\Omega-\Omega}{\varepsilon}} F_{\varepsilon,\xi}(u,B)^p e^{-|\xi|^2} \mathrm{d}\xi \right)^{\frac{1}{p}}, \quad u \in L^0(\Omega;\mathbb{R}^n). \tag{1.4}$$

In the expression above, $\mathscr{B}$ represents the class of all finite families of pairwise disjoint open balls contained in Ω. This supremum procedure over $\mathscr{B}$ allows the functional $\mathcal{F}^p_\varepsilon$ to bypass irregularities within the domain Ω. In fact, by choosing $p = 1$, the strict superaddiditity of the set function $F_{\varepsilon,\xi}(u,\cdot)$ leads to the strict inequality $\mathcal{F}^1_\varepsilon < \mathcal{F}_\varepsilon$. For $p > 1$, the presence of the L^p-norm in the definition (1.4) should be viewed as a *regularizing effect*, and its precise role will be explained in more detail below.

Our main results are provided by the following two theorems.

Theorem 1.1 (Closure and compactness). *Let $\Omega \subset \mathbb{R}^n$ be open, let $p > 1$, and let $\{u_\varepsilon\}_{\varepsilon>0} \subset L^0(\Omega;\mathbb{R}^n)$ be such that*

$$\sup_{\varepsilon>0} \mathcal{F}^p_\varepsilon(u_\varepsilon,\Omega) < \infty. \tag{1.5}$$

Then, there exists a subsequence $\varepsilon_k \to 0$ as $k \to \infty$ such that the set

$$A := \{x \in \Omega : |u_{\varepsilon_k}(x)| \to \infty \text{ as } k \to \infty\} \tag{1.6}$$

has finite perimeter, and $u_{\varepsilon_k} \to u$ pointwise almost everywhere in $\Omega \setminus A$ for some measurable function $u\colon \Omega\setminus A \to \mathbb{R}^n$. In addition, for almost every $c \in \mathbb{R}$, the extension of u to the whole of Ω defined as $u = c$ on A, satisfies $u \in \mathrm{GSBD}(\Omega)$ and

$$\liminf_{k\to\infty} \mathcal{F}^p_{\varepsilon_k}(u_k,\Omega) \geq \int_\Omega \varphi_p(e(u))\mathrm{d}x + \beta_p\mathcal{H}^{n-1}(J_u \cup \partial^* A), \tag{1.7}$$

where $\varphi_p\colon \mathbb{M}^{n\times n}_{sym} \to [0,\infty)$ is 2-homogeneous and β_p is a positive constant.

Theorem 1.2 (Γ-convergence). *Let $\Omega \subset \mathbb{R}^n$ be open. For every $p > 1$ the family of functionals $\{\mathcal{F}^p_\varepsilon\}_{\varepsilon>0}$ Γ-converges in $L^0(\Omega;\mathbb{R}^n)$ to the Griffith-type functional $\mathcal{F}^p$ defined as*

$$\mathcal{F}^p(u,\Omega) := \begin{cases} \displaystyle\int_\Omega \varphi_p(e(u))\mathrm{d}x + \beta_p\mathcal{H}^{n-1}(J_u) & \text{for } u \in \mathrm{GSBD}(\Omega), \\ \infty & \text{otherwise in } L^0(\Omega;\mathbb{R}^n), \end{cases} \tag{1.8}$$

where φ_p and β_p are given as in Theorem 1.1.

For the precise definition of the bulk energy density φ_p and of the surface energy density β_p we refer to Lemma 4.6.

In proving Theorems 1.1 and 1.2 we addressed two main hurdles. The first challenge, common to both theorems, involves the characterization of the domain of the Γ-limit. The second difficulty concerns the compactness result in Theorem 1.1, where no integrability assumptions are imposed on the family $\{u_\varepsilon\}_\varepsilon$.

The assumption $p > 1$ plays a key role in tackling the first problem. Broadly speaking, the non-local nature of the approximating functional $\mathcal{F}^p_\varepsilon$ results in a limiting function space consisting of measurable vector fields that exhibit generalized bounded deformation in a weaker sense. To explain this phenomenon more precisely, recall that $u \in \mathrm{GBD}(\Omega)$ if and only if u is a measurable vector field and there exists a finite Radon measure λ on Ω such that, for every $\xi \in \mathbb{S}^{n-1}$, the generalized directional variation $\hat{\mu}^\xi_u$ of $u\cdot\xi$, as introduced in [21, Definition 4.10], satisfies:

$$\hat{\mu}^\xi_u(B) \leq \lambda(B), \quad \text{for every Borel set } B \subset \Omega. \tag{1.9}$$

By a slicing argument, a similar approach to that proposed by M. Gobbino in [26] gives that a measurable vector field $u\colon \Omega \to \mathbb{R}^n$ belongs to the domain of $\mathcal{F}^1$ if and only if the condition in (1.9) holds in a much weaker form, namely only after averaging with respect to ξ,

$$\int_{\mathbb{S}^{n-1}} \hat{\mu}_u^\xi(B)\,\mathrm{d}\mathcal{H}^{n-1}(\xi) \le \lambda(B), \quad \text{for every Borel set } B \subset \Omega. \tag{1.10}$$

The fact that (1.9) is replaced in our setting by (1.10) is a huge obstacle to establish the structural properties of the limiting function space. In particular, the technique presented in [21] for deriving the GBD-structural properties fails, and we need to rely on the more recent integral geometric approach introduced in [2]. The main difference from the approach in [2] lies in the role of the flatness of $\mathbb{R}^n$, where rescaling and translating the domain preserve straight lines as geodesics, maintaining invariance in the associated GBD-space. This property does not generally hold on a Riemannian manifold. In Proposition 3.13, we leverage this feature to avoid relying on a Korn-Poincaré-type inequality to estimate the dimensionality of the set on which the jump of all one-dimensional slices of u concentrate. In this direction, the exponent p in $(1,\infty)$ in (1.4) acts as a regularization, for it provides in the limit an interpolation between the two conditions in (1.9) and (1.10). The assumption that $p > 1$ in Theorems 3.6 and 3.7 is thus instrumental to infer essential structural properties for the maps in the domain of the limiting energy, particularly those related to the slicing of the jump set. Ultimately, we can conclude in Theorem 3.1 that the domain of $\mathcal{F}^p$ coincides precisely with $\mathrm{GSBD}(\Omega)$. For the detailed definitions of the relevant quantities and the proof of the structural properties, we refer the reader to Sections 2 and 3, respectively. It is important to observe that the significant non-linearity in the definition of $\hat{\mu}_u^\xi$ prevents us from controlling the averaged measure in (1.10) through finitely many $\hat{\mu}_u^{\xi_i}$ for $i = 1, \dots, N$, regardless of the chosen finite set of directions $\{\xi_1, \dots, \xi_N\}$. Such a bound can only be established *after* deriving the structural properties of the limiting maps. This contrasts sharply with the GSBV setting analyzed in [26, Lemma 4.1]: in that case, although truncations were also employed to define the space, its structure remained deeply rooted in the theory of distributions.

To clarify why the case $p = 1$ remains unresolved, denote by $\mathrm{GBV}^{\mathcal{E}}(\mathbb{R}^n)$ the space of measurable vector fields $u\colon \mathbb{R}^n \to \mathbb{R}^n$ such that, for $\mathcal{H}^{n-1}$-almost every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-almost every $y \in \Pi^\xi$ (where Π^ξ is the orthogonal hyperplane to ξ passing through the origin), the one-dimensional slice $\hat{u}_y^\xi$ belongs to $BV_{\mathrm{loc}}(\mathbb{R})$, together with the condition in (1.10). For precise definitions of the relevant quantities, we refer to Section 2. For every Borel set $B \in \mathbb{R}^n$ and every $y \in \mathbb{R}^n$, denote further $B_y^\xi := \{t \in \mathbb{R} : y + t\xi \in B\}$.

The setting $p = 1$ is related to the following question.

Open Problem. *Let $u\colon \mathbb{R}^n \to \mathbb{R}^n$ be a measurable vector field, and define the Borel regular measure $\mathscr{I}_{u,1}$ on $\mathbb{R}^n$ as*

$$\mathscr{I}_{u,1}(B) := \int_{\mathbb{S}^{n-1}} \left(\int_{\Pi^\xi} \sum_{t \in B_y^\xi} |[\hat{u}_y^\xi](t)| \wedge 1 \,\mathrm{d}\mathcal{H}^{n-1}(y) \right) \mathrm{d}\mathcal{H}^{n-1}(\xi), \quad B \subset \mathbb{R}^n \text{ Borel.}$$

Is it true that if $u \in \mathrm{GBV}^{\mathcal{E}}(\mathbb{R}^n)$, then $\mathscr{I}_{u,1}$ is an $(n-1)$-rectifiable measure?

This problem is quite delicate, as it hinges on the ability to deduce the rectifiability of a measure expressed in an integral geometric form. As P. Mattila showed in [29], for some explicit counterexamples the answer is a negative one. The question whether similar constructions would apply to the measure described above currently remains open. For a more detailed discussion on this topic, we refer the interested reader to Section 6. Without knowing the answer to the open problem raised above, our Γ-convergence result for $p = 1$ reads as follows.

Theorem 1.3 (Γ-convergence for $p = 1$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open. The family of functionals $\{\mathcal{F}^1_\varepsilon\}_\varepsilon$ Γ-converges in $L^0(\Omega;\mathbb{R}^n)$ to the Griffith-type functional $\mathcal{F}^1$ defined as*

$$\mathcal{F}^1(u,\Omega) := \begin{cases} \displaystyle\int_\Omega \varphi(e(u))\mathrm{d}x + \beta\mathcal{H}^{n-1}(J_u) + \mathcal{I}^s_u(\Omega) & \text{for } u \in \mathrm{GSBV}^{\mathcal{E}}(\Omega), \\ \infty & \text{otherwise in } L^0(\Omega;\mathbb{R}^n), \end{cases},$$

where $\varphi\colon \mathbb{M}^{n\times n}_{sym} \to [0,\infty)$ is 2-homogeneous and β is a positive constant, while the measure $\mathcal{I}^s_u$ is absolutely continuous with respect to the purely $(n-1)$-unrectifiable part of $\mathscr{I}_{u,1}$, and, in particular, gives zero mass to every σ-finite set with respect to $\mathcal{H}^{n-1}$.

Once the identification of the domain of the Γ-limit was obtained, the second main issue which we had to face was the compactness result stated in Theorem 1.1. In this regard, we recall some related works [23, 28, 32], inspired by the corresponding results on the Mumford-Shah functional [7, 30], which provide a nonlocal approximation of a class of Griffith-type functionals in terms of a bulk energy of the form

$$\frac{1}{\varepsilon}\int_\Omega f\big(\varepsilon W(e(u)) * \rho_\varepsilon\big)\,\mathrm{d}x$$

featuring the convolution of a volume density $W(e(u))$ for a Sobolev map u and a potential f whose behavior is similar to that of arctan in (1.3). We further mention [1, 18], which instead proposed a discrete finite-difference approximation in the spirit of [5, 14, 13, 31], with a finite range of interaction.

Comparing our setting with those in the above literature, the main difficulty to overcome in the study of (1.3) was the lack of information on the gradient (or symmetric gradient) of u, which is, a priori, only a measurable function. This was a major obstacle to establish the compactness result in Theorem 1.1. Indeed, we could not directly apply the well-known compactness results of [15, 16] nor modify their technique, which builds upon a slicing argument and the existence of a generalised symmetric gradient/generalised variation. Surprisingly enough, we could still rely on a Fréchet-Kolmogorov approach, which amounts to prove equi-continuity of translations. This technical step is provided in the proof of Theorem 1.1 and shares common ideas with [3], where the proof of the compactness theorem of [15] was first revisited, avoiding the use of Korn and Korn-Poincaré inequalities. Once again, the lack of a symmetric gradient and the structure of $\mathcal{F}_\varepsilon$, which features an extra integration over the directions $\xi \in \mathbb{R}^n$, do not allow us to select a preferred basis to control translations, as it was the case in [3]. Nevertheless, the slicing properties of the functionals $\mathcal{F}_\varepsilon$ make it possible to control the translations of the maps $\arctan(u_\varepsilon(x)\cdot\xi)$ both with respect to x and ξ. Compactness is thus recovered by Fréchet-Kolmogorov Theorem. Eventually, combining Theorem 1.1 with the Γ-convergence of Theorem 1.2, we further deduce convergence of minimisers of $\mathcal{F}^p_\varepsilon$ to minimisers of $\mathcal{F}^p$ under suitable Dirichlet boundary conditions (cf. Theorem 5.2).

Organization of the paper. The paper is organized as follows. In Section 2, we collect a few preliminary definitions and results. Section 3 is devoted to establish the main properties of the spaces of limiting deformations. Sections 4 and 5 are devoted to the proofs of our compactness, and Γ-convergence result, respectively. Eventually, the case $p = 1$ is the subject of Section 6.

Outlook. The analysis provided in this work offers the opportunity to explore several further research lines. On the one hand, the slicing technique has been successfully employed in other finite difference approximations, such as [8, 12, 25] for the Total Variation, the Mumford-Shah, and the perimeter functionals. On the other hand, the use of a slicing argument for the study of the variational limit restricts the choice of the approximating sequence $\mathcal{F}_\varepsilon$, which in turn determines the class of admissible densities φ_p. The study of more general approximations and of integral representation formulas in the spirit of [17] will be the subject of future investigation.

2. Definitions and notation

Recall the functionals $\mathcal{F}^p_\varepsilon$ and $F_{\varepsilon,\xi}$ defined in the introduction. Given $A \subset \mathbb{R}$, we further set

$$F_\varepsilon(u, A) := \frac{1}{\varepsilon}\int_A \arctan\left(\frac{(u(t+\varepsilon)-u(t))^2}{\varepsilon}\right) \mathrm{d}t,$$

for every measurable function $u\colon \mathrm{Dom}(u) \to \mathbb{R}$ such that $A \subset \mathrm{Dom}(u) \cap (\mathrm{Dom}(u) - \varepsilon)$, where $\mathrm{Dom}(u)$ denotes the domain of u. For the sequel it is convenient to introduce the following class of Mumford-Shah and Griffith functionals. For $\gamma > 0$, the former reads as

$$\mathcal{MS}_\gamma(u,\Omega) = \begin{cases} \gamma \int_\Omega |\nabla u(x)|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_u), & u \in \mathrm{GSBV}(\Omega), \\ +\infty, & u \in L^1_{loc}(\Omega) \setminus \mathrm{GSBV}(\Omega). \end{cases}$$

For $\lambda > 0$, we define the Griffith functional by

$$\mathcal{G}_\lambda(u,\Omega) = \begin{cases} \lambda \int_\Omega |e(u)(x)|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_u), & u \in \mathrm{GSBD}(\Omega), \\ \infty, & u \in L^0(\Omega;\mathbb{R}^n) \setminus \mathrm{GSBD}(\Omega). \end{cases}$$

For every $\xi \in \mathbb{R}^n \setminus \{0\}$, every $x \in \mathbb{R}^n$, and every $E \subset \mathbb{R}^n$, we define

$$E^\xi_x := \{t \in \mathbb{R} : x + t\xi \in E\}.$$

For $u\colon E \to \mathbb{R}^n$ we define $\hat{u}^\xi_x\colon E^\xi_x \to \mathbb{R}$ as

$$\hat{u}^\xi_x(t) := u(x+t\xi)\cdot\xi.$$

For $v\colon E \to \mathbb{R}$, we denote by $v^\xi_x\colon E^\xi_x \to \mathbb{R}$ the function $v^\xi_x(t) := v(x+t\xi)$. For $A \subset \mathbb{R}$ open and $v\colon A \to \mathbb{R}$ measurable, we set $J^1_v := \{t \in J_u \cap A : |v^+(t) - v^-(t)| > 1\}$.

The proof of the following measurability property is postponed to Section 3.

Lemma 2.1. *Let $u\colon \Omega \to \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. Assume that for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, the function $\hat{u}^\xi_y$ belongs to $\mathrm{BV}_{loc}(\Omega^\xi_y)$. Then, for every open set $U \subset \Omega$ the map*

$$(x,\xi) \mapsto |D\hat{u}^\xi_x|(U^\xi_x \setminus J^1_{\hat{u}^\xi_x}) + \mathcal{H}^0(U^\xi_x \cap J^1_{\hat{u}^\xi_x}) \tag{2.1}$$

is $(\mathcal{L}^n \otimes \mathcal{H}^{n-1})$-measurable on $\Omega \times \mathbb{S}^{n-1}$. In particular, for every open set $U \subset \Omega$ we have that the map

$$\xi \mapsto \int_{\Pi^\xi} |D\hat{u}^\xi_y|(U^\xi_y \setminus J^1_{\hat{u}^\xi_y}) + \mathcal{H}^0(U^\xi_y \cap J^1_{\hat{u}^\xi_y}) \mathrm{d}\mathcal{H}^{n-1}(y)$$

is $\mathcal{H}^{n-1}$-measurable on $\mathbb{S}^{n-1}$.

Next, we recall a measurability property originally stated in [21, Lemma 3.6].

Lemma 2.2. *Let $v\colon \Omega \to \mathbb{R}$ be $\mathcal{L}^n$-measurable and let $\xi \in \mathbb{S}^{n-1}$. Assume that for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, the function v^ξ_y belongs to $\mathrm{BV}_{loc}(\Omega^\xi_y)$. Then, for every Borel set $B \subset \Omega$ the map*

$$y \mapsto |Dv^\xi_y|(B^\xi_y \setminus J^1_{v^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{v^\xi_y})$$

is $\mathcal{H}^{n-1}$*-measurable on* Π^ξ*.*

We are now in position to define the space $\mathrm{GBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$.

Definition 2.3. We say that a $\mathcal{L}^n$-measurable function $u\colon\Omega\to\mathbb{R}^n$ belongs to the space $\mathrm{GBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$ if there exists a positive Radon measure $\lambda\in\mathcal{M}_b^+(\Omega)$ such that for $\mathcal{H}^{n-1}$-a.e. $\xi\in\mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ the function $\hat{u}^\xi_y\in\mathrm{BV}_{loc}(\Omega^\xi_y)$ and

$$\int_{\mathbb{S}^{n-1}}\bigg(\int_{\Pi^\xi}|D\hat{u}^\xi_y|(U^\xi_y\setminus J^1_{\hat{u}^\xi_y})+\mathcal{H}^0(U^\xi_y\cap J^1_{\hat{u}^\xi_y})\mathrm{d}\mathcal{H}^{n-1}(y)\bigg)\mathrm{d}\mathcal{H}^{n-1}(\xi)\le\lambda(U),\qquad(2.2)$$

for every open subset U of Ω. Moreover, we say that $u\in\mathrm{GSBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$ if $u\in\mathrm{GBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$ and $\hat{u}^\xi_y\in\mathrm{SBV}_{loc}(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $\xi\in\mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$.

Remark 2.4. Observe that the space $\mathrm{GBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$ introduced in Definition 2.3 does not coincide a priori with $\mathrm{GBD}(\Omega)$ introduced in [21, Definition 4.1]. This is because the control in (2.2) is not pointwise in ξ but only in average with respect to the $\mathcal{H}^{n-1}\llcorner\mathbb{S}^{n-1}$-measure. Nevertheless, we will show in Section 3.2 that, under an additional assumption, actually $\mathrm{GSBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)=\mathrm{GSBD}(\Omega)$.

In view of the measurability property contained in Lemma 2.2 we provide the following definition.

Definition 2.5. Let $\Omega\subset\mathbb{R}^n$ open and let $u\colon\Omega\to\mathbb{R}^n$ be $\mathcal{L}^n$-measurable. Then for every $\xi\in\mathbb{S}^{n-1}$ we define the Borel regular measure μ^ξ_u in Ω as follows. We define μ^ξ_u on Ω as the unique Borel regular measure on Ω that satisfies for every open set $U\subset\Omega$

$$\mu^\xi_u(U):=\int_{\Pi^\xi}|D\hat{u}^\xi_y|(U^\xi_y\setminus J^1_{\hat{u}^\xi_y})+\mathcal{H}^0(U^\xi_y\cap J^1_{\hat{u}^\xi_y})\,\mathrm{d}\mathcal{H}^{n-1}(y),$$

whenever for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ the function $\hat{u}^\xi_y$ belongs to $\mathrm{BV}_{loc}(U^\xi_y)$, while $\mu^\xi_u(U)=\infty$ otherwise.

Remark 2.6. Since the hypothesis of De Giorgi-Letta's Theorem [4] are fulfilled, the extension of the set function $U\mapsto\mu^\xi_u(U)$ to a Borel regular measure on Ω is uniquely guaranteed by the formula

$$\mu^\xi_u(B):=\inf\{\mu^\xi_u(U):B\subset U,\ U\subset\Omega\text{ open}\}.$$

Remark 2.7. In view of (2.2), if $u\in\mathrm{GBV}^{\mathcal{E}}(\Omega)$ we have that $\mu^\xi_u\in\mathcal{M}_b^+(\Omega)$ for $\mathcal{H}^{n-1}$-a.e. $\xi\in\mathbb{S}^{n-1}$. Moreover, for such ξ, we can make use of the measurability property contained in Lemma 2.2 to infer

$$\mu^\xi_u(B)=\int_{\Pi^\xi}|D\hat{u}^\xi_y|(B^\xi_y\setminus J^1_{\hat{u}^\xi_y})+\mathcal{H}^0(B^\xi_y\cap J^1_{\hat{u}^\xi_y})\,\mathrm{d}\mathcal{H}^{n-1}(y),$$

for every Borel set $B\subset\Omega$.

Definition 2.8. For $p\ge1$, $u\in\mathrm{GBV}^{\mathcal{E}}(\Omega)$, and $\xi\in\mathbb{S}^{n-1}$ such that $\mu^\xi_u\in\mathcal{M}_b^+(\Omega)$, we define $\hat{\mu}^p_u$ on Ω as the unique Borel regular measure on Ω that satisfies

$$\hat{\mu}^p_u(U):=\sup_{\mathscr{B}}\sum_{B\in\mathscr{B}}\bigg(\int_{\mathbb{S}^{n-1}}\mu^\xi_u(B)^p\,\mathrm{d}\mathcal{H}^{n-1}(\xi)\bigg)^{\frac{1}{p}}\qquad\text{for }U\subset\Omega\text{ open},\qquad(2.3)$$

where $\mathscr{B}$ denotes any finite family of pairwise disjoints open balls contained in U.

Observe that for $p=1$ we have

$$\hat{\mu}^1_u(U)=\int_{\mathbb{S}^{n-1}}\mu^\xi_u(U)\,\mathrm{d}\mathcal{H}^{n-1}(\xi)\qquad\text{for }U\subset\Omega\text{ open}.\qquad(2.4)$$

Remark 2.9. Also in this case, by De Giorgi-Letta's Theorem [4] the extension of the set function $U \mapsto \hat{\mu}^p_u(U)$ to a Borel regular measure on Ω is uniquely guaranteed by

$$\hat{\mu}^p_u(B) := \inf\{\hat{\mu}^p_u(U) : B \subset U,\ U \subset \Omega \text{ open}\}.$$

The above formula immediately gives also the uniqueness of the extension to a Borel regular measure on Ω.

We conclude this section with a remark concerning the definition of the functionals $\mathcal{F}^p_\varepsilon$ as well as the definition of the measures $\hat{\mu}^p_u$.

Remark 2.10. We observe that in (1.4) and (2.3) we can equivalently take the supremum over any family of countable pairwise disjoint open balls and the result would be the same. In addition, all the results as well as their proofs remain unchanged if we replace $\mathscr{B}$ with any class made of finitely many pairwise disjoint convex and open sets.

We also remark that the functional (1.4) could be replaced by

$$\mathcal{F}'^p_\varepsilon(u,\Omega) := \sup_{\mathscr{B}} \sum_{B\in\mathscr{B}} \left(\int_{\frac{B-B}{\varepsilon}} F_{\varepsilon,\xi}(u,B)^p e^{-|\xi|^2}\, d\xi \right)^{\frac{1}{p}}$$

without any substantial change in the proofs that follow. In particular, with this choice we would have exactly $\mathcal{F}'^1_\varepsilon(u,B) = \mathcal{F}_\varepsilon(u,B)$ for every open ball $B \subset \mathbb{R}^n$.

3. Structure of the space $\mathrm{GBV}^{\mathcal{E}}$

The present section addresses the study of the space $\mathrm{GBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$ introduced in Definition 2.3. In particular, we focus on the structure of functions $u \in \mathrm{GSBV}^{\mathcal{E}}(\Omega,\mathbb{R}^n)$ under the additional assumption $\hat{\mu}^p_u(\Omega) < \infty$. As a byproduct, we prove the following identification Theorem.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ open. A measurable function $u\colon \Omega \to \mathbb{R}^n$ belongs to $\mathrm{GSBD}(\Omega)$ if and only if $u \in \mathrm{GSBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$ with $\hat{\mu}^p_u(\Omega) < \infty$ for some $p \in (1,+\infty)$.*

In the next subsection, we discuss some measurability issues which justify formula (2.2) in Definition 2.3.

3.1. Preliminaries.

We start with a technical proposition for one-dimensional functions. We introduce the class $\mathcal{T}$ of all functions $\tau \in C^1(\mathbb{R})$ such that $-\frac{1}{2} \le \tau \le \frac{1}{2}$ and $0 \le \tau' \le 1$. We recall that, given an open set $I \subset \mathbb{R}$ and given $(\tau_j) \subset C^1(I)$, $\tau \in C^1(I)$, we have $\tau_j \to \tau$ in $C^1_{loc}(I)$ if and only if for every open set $U \Subset I$ it holds $\|\tau_j - \tau\|_{L^\infty(U)} + \|\tau'_j - \tau'\|_{L^\infty(U)} \to 0$ as $j \to \infty$.

Proposition 3.2. *Let $I \subset \mathbb{R}$ be open and let $u \in \mathrm{BV}_{loc}(I)$. Let furthermore $\hat{\mathcal{T}} \subset \mathcal{T}$ be any countable set which is dense with respect to the C^1_{loc}-topology. For every U open subset of I, it holds*

$$|Du|(U \setminus J^1_u) + \mathcal{H}^0(U \cap J^1_u) = \sup_{k\in\mathbb{N}} \sup \sum_{i=1}^k |D(\tau_i(u))|(U_i)\,, \tag{3.1}$$

where the second supremum is taken over all the families $\tau_1,\dots\tau_k \in \hat{\mathcal{T}}$ and all the families of pairwise disjoint open subsets $U_1,\dots,U_k$ of U.

Proof. Since $u \in \mathrm{BV}_{loc}(I)$, for every $U \Subset I$ open and compactly contained in I, the condition (a) of [21, Theorem 3.5] is satisfied. Therefore, we can apply [21, Theorem 3.8] to the one-dimensional function $u \llcorner U \in \mathrm{BV}(U)$ to deduce that (3.1) holds true for every open set $U' \subset U$ if we replace $\hat{\mathcal{T}}$ with the entire family $\mathcal{T}$. In order to pass from a compactly contained open set U to the whole of I, it is enough to consider a sequence $U_k \Subset I$ with $U_k \nearrow I$ and notice that

$$|Du|(U_k \setminus J_u^1) + \mathcal{H}^0(U_k \cap J_u^1) \nearrow |Du|(I \setminus J_u^1) + \mathcal{H}^0(I \cap J_u^1), \quad \text{as } k \to \infty.$$

In order to conclude the proof of formula (3.1) we need to show that it is enough to consider the supremum on the smaller family $\hat{\mathcal{T}}$. To this purpose we simply notice that for every $U \Subset I$, being $u \in \mathrm{BV}(U)$, we have $\|u\|_{L^\infty(U)} \leq m$. Hence, by applying the chain rule for BV-function (see for instance [4]), if $(\tau_j) \subset \hat{\mathcal{T}}$ is such that $\tau_j \to \tau$ in $C^1_{loc}(\mathbb{R})$, then

$$|D(\tau_j(u) - \tau(u))|(U) \leq \|\tau_j - \tau\|_{C^1((-m,m))} |Du|(U) \to 0, \quad \text{as } j \to \infty.$$

With this information at hand we infer that the double supremum in the right hand-side of (3.1) does not change when restricted to $\hat{\mathcal{T}}$. □

We are now in position to prove Lemma 2.1.

Proof of Lemma 2.1. The assumptions of the lemma are equivalent to: for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$, for $\mathcal{L}^n$-a.e. $x \in \Omega$, the function $\hat{u}^\xi_x$ belongs to $\mathrm{BV}_{loc}(\Omega^\xi_x)$.

We claim that for every open set $U \subset \Omega$ and every $\tau \in \mathcal{T}$ the map $(x,\xi) \mapsto |D\tau(\hat{u}^\xi_x)|(U^\xi_x)$ is $(\mathcal{L}^n \otimes \mathcal{H}^{n-1})$-measurable on $\Omega \times \mathbb{S}^{n-1}$. To show the claim, consider a countable dense subset $D \subset C^1_c(\mathbb{R})$ in the C^1_{loc}-topology. Notice that, for every $f \in L^1_{loc}(\mathbb{R})$ and every $I \subset \mathbb{R}$ open, we have

$$|Df|(I) = \sup_{\substack{\varphi \in C^1_c(I) \\ \|\varphi\|_{L^\infty} \leq 1}} \int_{\mathbb{R}} f\, D\varphi \,\mathrm{d}t = \sup_{\substack{\varphi \in C^1_c(I) \cap D \\ \|\varphi\|_{L^\infty} \leq 1}} \int_{\mathbb{R}} f\, D\varphi \,\mathrm{d}t. \tag{3.2}$$

Now consider a sequence of open sets $U'_k \Subset U$ with $U'_k \nearrow U$, and consider a sequence of functions $v_k \in C^1_c(U)$ with $0 \leq v_k \leq 1$ while $v_k = 1$ on U'_k. Notice that, for every $\varphi \in C^1_c(\mathbb{R})$, the map $(x,\xi) \mapsto \int_{\mathbb{R}} \tau(\hat{u}^\xi_x)\, D((v_k)^\xi_x \varphi)\,\mathrm{d}t$ is $(\mathcal{L}^n \otimes \mathcal{H}^{n-1})$-measurable (for instance by virtue of Fubini's Theorem). By formula (3.2), we have for every $(x,\xi) \in \Omega \times \mathbb{S}^{n-1}$ that

$$|D\tau(\hat{u}^\xi_x)|((U'_k)^\xi_x) \leq h_k(x,\xi) := \sup_{\substack{\varphi \in C^1_c(\mathbb{R}) \cap D \\ \|\varphi\|_{L^\infty} \leq 1}} \int_{\mathbb{R}} \tau(\hat{u}^\xi_x)\, D((v_k)^\xi_x \varphi)\,\mathrm{d}t \leq |D\tau(\hat{u}^\xi_x)|(U^\xi_x).$$

Notice that, since the countable supremum of measurable functions is a mesurable function, $h_k(x,\xi)$ is $(\mathcal{L}^n \otimes \mathcal{H}^{n-1})$-measurable. By the monotone property of measures, we know that $|D\tau(\hat{u}^\xi_x)|((U'_k)^\xi_x) \nearrow |D\tau(\hat{u}^\xi_x)|(U^\xi_x)$ for every $(x,\xi) \in \Omega \times \mathbb{S}^{n-1}$ as $k \to \infty$, meaning that $h_k(x,\xi) \to |D\tau(\hat{u}^\xi_x)|(U^\xi_x)$ for every $(x,\xi) \in \Omega \times \mathbb{S}^{n-1}$ as $k \to \infty$. Thanks to the fact that pointwise limits of sequences of measurable functions are measurable functions, the claim follows.

By combining the previous claim with Proposition 3.2, we deduce that the map in (2.1) is the countable supremum of measurable maps, therefore, it is measurable. The proof is thus concluded. □

We discuss here the measurability of $\xi \mapsto \mu^\xi_u(B)$ for B Borel subset of Ω.

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^n$ open and let $u\colon \Omega \to \mathbb{R}^n$ be $\mathcal{L}^n$-measurable and such that for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ and $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the function $\hat{u}^\xi_y$ belongs to $\mathrm{BV}_{loc}(\Omega^\xi_y)$. For every $\xi \in \mathbb{S}^{n-1}$ and for every U open subset of Ω, it holds*

$$\mu^\xi_u(U) = \sup_{k \in \mathbb{N}} \sup \sum_{i=1}^k |D_\xi(\tau_i(u \cdot \xi))|(U_i)\,, \tag{3.3}$$

where the second supremum is taken over all the families $\tau_1, \dots \tau_k \in \mathcal{T}$ and all the families of pairwise disjoint open subsets $U_1, \dots, U_k$ of U.

Proof. Whenever $\xi \in \mathbb{S}^{n-1}$ and $U \subset \Omega$ are such that $\mu_u^\xi(U) < \infty$, equality (3.3) holds in view of [21, Theorem 3.8]. It remains to consider the case $\mu_u^\xi(U) = \infty$. For simplicity of notation, we denote by η the set function given by the right-hand side of (3.3). By contradiction, assume that $\eta(U) < \infty$. In particular, for every $\tau \in \mathcal{T}$ we have

$$|D_\xi \tau(u \cdot \xi)|(U) \leq \eta(U) < \infty \, . \tag{3.4}$$

Hence, $D_\xi \tau(u \cdot \xi) \in \mathcal{M}_b(U)$. By a Carathéodory construction using η as Gauge function and the family of all open balls contained in U as set of generators, we obtain a Borel regular measure λ on U, with λ_δ as approximating measure for $\delta > 0$ (see, e.g., [24, Section 2.10.1]). For every $V \Subset U$, for every $\delta > 0$ small enough, and for every covering $\mathcal{G}$ of V made of open balls in $\mathbb{R}^n$ with diameter smaller than δ, we may assume that $\bigcup_{A \in \mathcal{G}} A \subset U$. By Besicovitch covering theorem, there exists a dimensional constant $c(n) > 0$ and $\mathcal{G}_1, \dots, \mathcal{G}_{c(n)}$ disjoint countable subfamilies of $\mathcal{G}$ such that

$$V \subset \bigcup_{i=1}^{c(n)} \bigcup_{A \in \mathcal{G}_i} A \, .$$

In particular, we have that $\lambda_\delta(V) \leq c(n)\eta(U)$, which yields the inequality $\lambda(V) \leq c(n)\eta(U)$ for every $V \Subset U$. Taking the limit $V \nearrow U$ we conclude that λ is a positive bounded Radon measure on U. By using (3.4) it is not difficult to show that actually for every Borel set $B \subset U$ we have that $|D_\xi \tau(u \cdot \xi)|(B) \leq \lambda(B)$ whenever $\tau \in \mathcal{T}$. Thus, we are in a position to apply [21, Theorem 3.5] to deduce that $\mu_u^\xi(U) \leq \lambda(U) < \infty$, which is a contradiction. Thus, it must be $\eta(U) = \infty$ and equality (3.3) is satisfied. □

Corollary 3.4. *Under the assumptions of Proposition 3.3, for every open set $U \subset \Omega$ the map $\xi \mapsto \mu_u^\xi(U)$ is lower-semicontinuous on $\mathbb{S}^{n-1}$.*

Proof. We notice that equality (3.3) holds for every $U \subset \Omega$ open, for every $\xi \in \mathbb{S}^{n-1}$. In particular, the right-hand side of (3.3) is lower-semicontinuous w.r.t. $\xi \in \mathbb{S}^{n-1}$ for fixed $U \subset \Omega$ open, hence $\xi \mapsto \mu_u^\xi(U)$ is also lower-semicontinuous. □

Thanks to Corollary 3.4, the integral (2.2) in Definition 2.3 is now justified. We further notice that $\hat{\mu}_u^1$ can be extended to a finite Radon measure on Ω, that we still denote by $\hat{\mu}_u^1$.

Remark 3.5. We point out that whenever $u \in GBV^{\mathcal{E}}(\Omega; \mathbb{R}^n)$, the integral formula (2.4) can be extended also for $K \subset \Omega$ compact. Indeed, it is enough to take a sequence U_k of open subsets of Ω such that $K = \bigcap_{k \in \mathbb{N}} U_k$. Then, for every $k \in \mathbb{N}$ we have that

$$\hat{\mu}_u^1\Big(\bigcap_{j=1}^k U_j\Big) = \int_{\mathbb{S}^{n-1}} \mu_u^\xi\Big(\bigcap_{j=1}^k U_j\Big) \, d\mathcal{H}^{n-1}(\xi) \, .$$

Passing to the limit as $k \to \infty$, by dominated convergence (recall that $\mu_u^\xi(\Omega) \in L^1(\mathbb{S}^{n-1})$) we deduce that

$$\hat{\mu}_u^1(K) := \int_{\mathbb{S}^{n-1}} \mu_u^\xi(K) \, d\mathcal{H}^{n-1}(\xi) \, .$$

3.2. **Proof of Theorem 3.1.** In this section we prove that if $u \in \mathrm{GSBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$ then $u \in \mathrm{GSBD}(\Omega)$ under the additional assumption that $\hat{\mu}_u^p$ is a finite measure for some $p > 1$ (the other implication being trivially true). Such implication is a consequence of the following two theorems, concerning the structure properties of functions in $\mathrm{GSBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$ with $\hat{\mu}_u^p$ finite.

Theorem 3.6. *Let $u \in \mathrm{GBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$ and assume that $\hat{\mu}_u^p$ is a finite measure for some $p > 1$. Then for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ we have*

$$(J_u)^\xi_y = J_{\hat{u}^\xi_y}, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi \tag{3.5}$$

Theorem 3.7. *Let $u \in \mathrm{GBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$. Then there exists $e(u) \in L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ such that*

$$\operatorname*{ap-lim}_{y\to x} \frac{(u(y)-u(x)-e(u)(x)(y-x))\cdot(y-x)}{|y-x|^2} = 0$$

holds for a.e. $x \in \Omega$. Moreover, for a.e. $\xi \in \mathbb{R}^n \setminus \{0\}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have

$$e(u)^\xi_y \xi \cdot \xi = \nabla \hat{u}^\xi_y \quad \mathcal{L}^1\text{-a.e. on } \Omega^\xi_y. \tag{3.6}$$

Postponing the proofs of Theorems 3.6 and 3.7 to Subsections 3.2.1 and 3.2.2, respectively, we conclude this section showing how to exploit them to prove Theorem 3.1.

Proof of Theorem 3.1. The only non trivial implication is $u \in \mathrm{GSBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$ implies $u \in \mathrm{GSBD}(\Omega)$. From the very definition of $\mathrm{GSBD}(\Omega)$ we need to show the existence of a finite Borel measure λ on Ω such that for every $\xi \in \mathbb{S}^{n-1}$

$$\hat{u}^\xi_y \in \mathrm{SBV}_{loc}(\Omega^\xi_y), \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi \tag{3.7}$$

$$\mu^\xi_u(B) \le \lambda(B), \quad \text{for every } B \subset \Omega \text{ Borel.} \tag{3.8}$$

By combining Theorem 3.6 together with the countably $(n-1)$-rectifiability of J_u (see [22]), we make use of the Area Formula to infer for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ that for every Borel set $B \subset \Omega$

$$\int_{\Pi^\xi} |D^j \hat{u}^\xi_y|(B^\xi_y \setminus J^1_{\hat{u}^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{\hat{u}^\xi_y})\,\mathrm{d}\mathcal{H}^{n-1}(y) = \int_{B\cap J_u} (|[u\cdot\xi]| \wedge 1)\,|\nu_u\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1},$$

where D^j denotes the jump part of the distributional derivative. In particular, there exists a dimensional constant $0 < c_n < 1$ such that for every $v \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$

$$c_n(|v| \wedge 1) \le \int_{\mathbb{S}^{n-1}} (|v\cdot\xi| \wedge 1)\,|\nu\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1}(\xi) \le |v| \wedge 1.$$

In addition, by Theorem 3.7 we have also for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ and for every Borel set $B \subset \Omega$

$$\begin{aligned}\int_{\Pi^\xi} |\nabla \hat{u}^\xi_y|(B^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y) &= \int_{\Pi^\xi}\Big(\int_{B^\xi_y} |e(u)^\xi_y\xi\cdot\xi|\,\mathrm{d}t\Big)\mathrm{d}\mathcal{H}^{n-1}(y)\\ &= \int_B |e(u)\xi\cdot\xi|\,\mathrm{d}x.\end{aligned}$$

Therefore, since $\hat{\mu}^1_u$ is a finite measure (recall also $|e(u)| \in L^1(\Omega)$), the Borel measure $\lambda := |e(u)|\mathcal{L}^n + (|[u]| \wedge 1)\mathcal{H}^{n-1} \llcorner J_u$ is finite and for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ we have

$$\mu^\xi_u(B) \le \lambda(B), \quad \text{for every } B \subset \Omega \text{ Borel.} \tag{3.9}$$

In order to get the full conditions (3.7) and (3.8) for every $\xi \in \mathbb{S}^{n-1}$, given $\xi \in \mathbb{S}^{n-1}$, we consider $\xi_k \to \xi$ such that each ξ_k satisfies (3.7), (3.8) and by virtue of Corollary 3.4 we infer that

$$\hat{\mu}_u^{\xi}(U) \leq \liminf_k \hat{\mu}_u^{\xi_k}(U) \leq \lambda(U), \quad \text{for every } U \subset \Omega \text{ open.}$$

Since λ is a finite Borel measure, the above inequality can be further extended to every Borel set $B \subset \Omega$ by outer regularity. This proves (3.8). In order to verify the validity of (3.7) we notice that, from (3.9) and by the definition of $GSBV^{\mathcal{E}}(\Omega;\mathbb{R}^n)$, we know that for a dense set of $\xi \in \mathbb{S}^{n-1}$ conditions (3.7) and (3.8) hold true. From [21, Theorem 3.5], for every such ξ we have

$$|D\tau(u\cdot\xi)|(B) \leq \lambda(B), \quad \text{for every } B \subset \Omega \text{ Borel,} \tag{3.10}$$

whenever $\tau \in \mathcal{T}$. By exploiting (3.10) and the lower semi-continuity of $\xi \mapsto |D\tau(u\cdot\xi)|(U)$ for every open set $U \subset \Omega$, we argue as before and show that actually (3.10) holds for every $\xi \in \mathbb{S}^{n-1}$ and every Borel set $B \subset \Omega$. By using again [21, Theorem 3.5], we infer that for every $\xi \in \mathbb{S}^{n-1}$ property (3.8) holds true, and also $\hat{u}^{\xi}_y \in BV_{loc}(\Omega^{\xi}_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^{\xi}$. Eventually, condition (3.8) and the specific form of λ, together with a simple disintegration argument with respect to the projection $\pi_\xi \colon \mathbb{R}^n \to \Pi^{\xi}$, give exactly the validity of (3.7). □

3.2.1. *Proof of Theorem 3.6.* In slicing the jump set we will follow the line developed in [2]. As already explained in the introduction, the main difference is that, in contrast to the case of a generic Riemannian manifold, the flatness of $\mathbb{R}^n$ allows to avoid the use of a Korn-Poincaré type of inequality. We start by introducing a class of relevant measures. Before doing this, we recall a measurability lemma.

Lemma 3.8. *Let $u\colon \Omega \to \mathbb{R}^m$ be $\mathcal{L}^n$-measurable. Then, for every Borel set $B \subset \Omega$ we have that*

$$y \mapsto \sum_{t \in B^{\xi}_y} (|[\hat{u}^{\xi}_y(t)]| \wedge 1) \qquad \textit{is } \mathcal{H}^{n-1}\textit{-measurable} \tag{3.11}$$

$$\xi \mapsto \int_{\Pi^{\xi}} \sum_{t \in B^{\xi}_y} \big(|[\hat{u}^{\xi}_y(t)]| \wedge 1\big)\, d\mathcal{H}^{n-1}(y) \qquad \textit{is } \mathcal{H}^{n-1}\textit{-measurable.} \tag{3.12}$$

Proof. See [2, Lemma 4.5]. □

Given an $\mathcal{L}^n$-measurable function $u\colon \Omega \to \mathbb{R}^n$, by virtue of (3.11) we consider for every $\xi \in \mathbb{S}^{n-1}$ the (outer) Borel regular measure η_ξ of $\mathbb{R}^n$ given by

$$\eta_\xi(B) := \int_{\Pi^{\xi}} \sum_{t \in B^{\xi}_y} \big(|[\hat{u}^{\xi}_y(t)]| \wedge 1\big)\, d\mathcal{H}^{n-1}(y) \qquad B \subset \Omega \text{ Borel}\,, \tag{3.13}$$

$$\eta_\xi(E) := \inf\{\eta_\xi(B) : \; E \subset B,\; B \subset \Omega \text{ Borel}\}\,. \tag{3.14}$$

Definition 3.9 (The outer measure $\mathscr{I}_{u,p}$)**.** Let $u\colon \Omega \to \mathbb{R}^n$ be measurable and let $\{\eta_\xi\}_{\xi\in\mathbb{S}^{n-1}}$ be the family of measures in (3.13)–(3.14). Let moreover $f_B(\xi) := \eta_\xi(B)$ for every $\xi \in \mathbb{S}^{n-1}$ and $B \subset \Omega$ Borel. For $p \in [1,\infty]$, by virtue of (3.12) we define the set function

$$\zeta_p(B) := \|f_B\|_{L^p(\mathbb{S}^{n-1})}.$$

Via the classical Caratheodory's construction we define the (outer) Borel regular measure $\mathscr{I}_{u,p}$ on Ω as

$$\mathscr{I}_{u,p}(E) := \sup_{\delta>0} \inf_{G_\delta} \sum_{B \in G_\delta} \zeta_p(B),$$

whenever $E \subset \Omega$ and where G_δ is the family of all countable Borel covers of E made of sets having diameter less than or equal to δ.

Remark 3.10. For every $u \in \mathrm{GBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$ it holds $\mathscr{I}_{u,1}(B) = \zeta_1(B)$ for every Borel set $B \subset \Omega$.

Next we prove a key technical proposition.

Proposition 3.11. *Let $u\colon \Omega \to \mathbb{R}^n$ be measurable and assume that there exists $p \in (1,\infty]$ such that $\mathscr{I}_{u,p}$ is finite. Then,*

$$\mathscr{I}_{u,1}\big(\{x \in \Omega\colon 0 \notin J_{\hat{u}^\xi_x} \text{ for } \mathcal{H}^{n-1}\text{-a.e. } \xi \in \mathbb{S}^{n-1}\}\big) = 0\,. \tag{3.15}$$

Proof. Notice that $\hat{u}^\xi_x(t) = \hat{u}^\xi_{\pi_\xi(x)}(x\cdot\xi + t)$ for every $t \in \Omega^\xi_x$, whenever $\pi_\xi\colon \mathbb{R}^n \to \Pi^\xi$ denotes the orthogonal projection. By letting $t_x := x\cdot\xi$, then the set in (3.15) coincides with

$$\{x \in \Omega\colon t_x \notin J_{\hat{u}^\xi_{\pi_\xi(x)}} \text{ for } \mathcal{H}^{n-1}\text{-a.e. } \xi \in \mathbb{S}^{n-1}\}. \tag{3.16}$$

The proof follows exactly as in [2, Proposition 4.7]. □

Before proving the dimensional estimate on the set where the measure $\mathscr{I}_{u,1}$ is concentrated, we need the following proposition.

Proposition 3.12. *Let $A \subset \mathbb{R}^n$ be measurable. Consider the real vector space $L^0(A)$ made of all Lebesgue equivalence classes of measurable functions $v\colon A \to \mathbb{R}$ endowed with the metric $\mathrm{d}(\cdot,\cdot)$ defined as*

$$\mathrm{d}(v_1,v_2) := \int_A |v_1 - v_2| \wedge 1 \,\mathrm{d}x, \quad v_1, v_2 \in L^0(A),$$

which induces the convergence in measure. Then, any finite dimensional vector subspace $V \subset L^0(A)$ is a complete metric space with respect to the distance $\mathrm{d}(\cdot,\cdot)$.

The proposition above is a direct consequence of Riesz Theorem. To keep the presentation self contained, we include a proof below.

Proof. For every measurable set $B \subset A$ we can consider the linear map $i_B\colon L^0(A) \to L^0(B)$ defined as $i(v) := v \llcorner B$. Clearly, being $V \subset L^0(A)$, the map i_B is a well defined linear map between the vector spaces V and $V(B)$, where we have set $V(B) := \{u \in L^0(B) : u = v \llcorner B, \text{ for some } v \in V\}$. We claim that there exists $\varepsilon_c > 0$ such that for every measurable set $K \subset A$ with $\mathcal{L}^n(A \backslash K) \le \varepsilon_c$ we have that the linear map $i_K\colon V \to V(K)$ is injective. Indeed, fix a basis $\{v_1,\dots,v_k\}$ for V and assume by contradiction that there exists $\varepsilon_j \searrow 0$ and measurable sets $K_j \subset A$ with $\mathcal{L}^n(A \setminus K_j) \le \varepsilon_j$ such that there exists real coefficients, not all identically zero, $\{\alpha^j_1,\dots,\alpha^j_k\}$, satisfying

$$\sum_{i=1}^k \alpha^j_i (v_i \llcorner K_j) = 0. \tag{3.17}$$

By renormalization, with no loss of generality, we may assume that for every $i = 1,\dots,k$ and every $j = 1,2,\dots$ it holds true $|\alpha^j_i| \le 1$ and there exists at least one $i(j) = 1,\dots,k$ such that $|\alpha^j_{i(j)}| = 1$. Up to pass to a not relabelled subsequence in j, we may suppose that the index $i(j)$ does not depend on j, and that $\alpha^j_i \to \alpha_i$ for every $i = 1,\dots,n$, for some real coefficients $\{\alpha_1,\dots,\alpha_k\}$ which are not all identically zero. Therefore, since $K_j \nearrow A$ in measure we deduce that $\sum_{i=1}^k \alpha^j_i i_{K_j}(v_i)$ (extended to zero out of K_j) converges to $\sum_{i=1}^k \alpha_i v_i$ pointwise a.e. in A as $j \to \infty$. From (3.17) we immediately deduce that $\sum_{i=1}^k \alpha_i v_i = 0$ in $L^0(A)$ which gives a contradiction. The claim is thus proved.

Now assume that $\{w_j\}_{j\in\mathbb{N}} \subset V$ is a Cauchy sequence. Since $L^0(A)$ is complete we have $\mathrm{d}(w_j, w) \to 0$ as $j \to \infty$ for some $w \in L^0(A)$. We want to prove that $w \in V$. Up to passing to a subsequence in j, the d-convergence implies $w_j \to w$ pointwise a.e. in A as $j \to \infty$. Therefore, for every $0 < \varepsilon \le \varepsilon_c$, we apply Egorov's Theorem to find a measurable set $K \subset A$ such that $\mathcal{L}^n(A \setminus K) \le \varepsilon$ and $w_j \to w$ uniformly on K as $j \to \infty$. But this means that the sequence $\{i_K(w_j)\}_{j\in\mathbb{N}}$ is a sequence belonging to the finite-dimensional vector space $V(K)$ and converging with respect to the L^1-norm. Hence, we find real coefficients $\{\alpha_1^K, \dots, \alpha_k^K\}$ such that $i_K(w) = \sum_{i=1}^k \alpha_i^K i_K(v_i)$, where $\{v_1, \dots, v_k\}$ is a basis of V. In order to conclude we need to show that, given a sequence of measurable sets $K_j \nearrow A$ the coefficients $\{\alpha_1^{K_j}, \dots, \alpha_k^{K_j}\}$ are definitely constant as $j \to \infty$. So assume by contradiction that for every $j \ge 1$ we find $j < j_1 < j_2$ such that $\{\alpha_1^{K_{j_1}}, \dots, \alpha_k^{K_{j_1}}\} \neq \{\alpha_1^{K_{j_2}}, \dots, \alpha_k^{K_{j_2}}\}$. Now choose j_1 so large such that $\mathcal{L}^n(A \setminus (K_{j_1} \cap K_{j_2})) \le \varepsilon_c$. By our assumption on the non-coincidence between the K_{j_1}-coefficients and the K_{j_2}-coefficients, we deduce that the linear combination $\sum_{i=1}^k (\alpha_i^{K_{j_1}} - \alpha_i^{K_{j_2}}) i_{K_{j_1}\cap K_{j_2}}(v_i)$ is identically null in $L^0(K_{j_1} \cap K_{j_2})$ while its coefficients are not all identically equal to zero. But this gives a contradiction with the fact that, thanks to our previous claim, the linear map $i_{K_{j_1}\cap K_{j_2}} \colon V \to V(K_{j_1} \cap K_{j_2}) \subset L^0(K_{j_1} \cap K_{j_2})$ is injective. □

Recall the definition of Θ^{*n-1} from [24].

Proposition 3.13. *Let $u \in \mathrm{GBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$. Then, for $\hat\mu_u^1$-a.e. $x \in \Omega$, the condition $\Theta^{*n-1}(\hat\mu_u^1, x) = 0$ implies that the set $\{\xi \in \mathbb{S}^{n-1} : 0 \in J_{\hat u_x^\xi}\}$ is $\mathcal{H}^{n-1}$-negligible.*

Proof. In order to simplify the notation we set for every $x \in \Omega$ and $\xi \in \mathbb{S}^{n-1}$

$$O_x^\xi(u) := |D\hat u_x^\xi|(\Omega_x^\xi \setminus J^1_{\hat u_x^\xi}) + \mathcal{H}^0(\Omega_x^\xi \cap J^1_{\hat u_x^\xi})$$
$$O_x(u) := \int_{\mathbb{S}^{n-1}} |D\hat u_x^\xi|(\Omega_x^\xi \setminus J^1_{\hat u_x^\xi}) + \mathcal{H}^0(\Omega_x^\xi \cap J^1_{\hat u_x^\xi})\,\mathrm{d}\mathcal{H}^{n-1}(\xi).$$

Step 1. We claim that

$$\hat u_x^\xi \in \mathrm{BV}_{loc}(\Omega_x^\xi), \quad \text{for } (\mathcal{L}^n \otimes \mathcal{H}^{n-1})\text{-a.e. } (x,\xi) \in \Omega \times \mathbb{S}^{n-1} \tag{3.18}$$
$$(x,\xi) \mapsto O_x^\xi(u) \quad \text{is } (\mathcal{L}^n \otimes \mathcal{H}^{n-1})\text{-measurable} \tag{3.19}$$
$$\int_\Omega O_x(u)\,\mathrm{d}x \le c_n(\Omega)\hat\mu_u^1(\Omega). \tag{3.20}$$

Indeed, from the definition of $\mathrm{GBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$, we know that for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ we have $\hat u_y^\xi \in \mathrm{BV}_{loc}(\Omega_y^\xi)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. Equivalently, for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ we have $\hat u^\xi_{\pi_\xi(x)} \in \mathrm{BV}_{loc}(\Omega^\xi_{\pi_\xi(x)})$ for $\mathcal{L}^n$-a.e. $x \in \Omega$, where $\pi_\xi \colon \mathbb{R}^n \to \Pi^\xi$ denotes the orthogonal projection. Therefore, by using Fubini's Theorem on the product space $\Omega \times \mathbb{S}^{n-1}$ with measure $\mathcal{L}^n \otimes \mathcal{H}^{n-1}$, and the fact that $\hat u_x^\xi = \hat u^\xi_{\pi_\xi(x)}$, we immediately infer the validity of (3.18). To prove (3.20), in view of the measurability (3.19) (see Lemma 2.1), it is enough to make use of Fubini's Theorem and exchange the order of integration in x and ξ.

Step 2. Let us assume by contradiction that the measure $\hat\mu_u^1$ has null $(n-1)$-density at x while the set $\Sigma \subset \mathbb{S}^{n-1}$ defined as

$$\Sigma := \{\xi \in \mathbb{S}^{n-1} : 0 \in J_{\hat u_x^\xi}\}$$

has strictly positive $\mathcal{H}^{n-1}$-measure. Notice also that, thanks to the fact that the set appearing in (3.15) and the set defined in (3.16) coincide, by [2, Lemma 4.4] there exists a Borel set $A \in \Omega \times \mathbb{S}^{n-1}$ (hence $(\hat{\mu}^1_u \otimes \mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1})$-measurable) such that $\{\xi \in \mathbb{S}^{n-1} : (x,\xi) \in A\} = \{\xi \in \mathbb{S}^{n-1} : 0 \in J_{\hat{u}^\xi_x}\}$ for every $x \in \Omega$. In particular, we deduce from Fubini's Theorem that the set $\{\xi \in \mathbb{S}^{n-1} : 0 \in J_{\hat{u}^\xi_x}\}$ is $\mathcal{H}^{n-1}$-measurable for $\hat{\mu}^1_u$-a.e. $x \in \Omega$. With no loss of generality, we may thus assume that $x \in \Omega$ is such that Σ is a $\mathcal{H}^{n-1}$-measurable set.

Now define for every $0 < r < \mathrm{dist}(x, \partial\Omega)$ the rescaled function $u_r \colon B_1(0) \to \mathbb{R}^n$ as $u_r(z) := u(x + rz)$ and define the truncated cone

$$\mathrm{co}(\Sigma \cup \{0\}) := \{z = t\xi : \xi \in \Sigma,\ t \in (-1,1) \setminus \{0\}\}.$$

From formula (3.3), we deduce that $u_r \in \mathrm{GBV}^{\mathcal{E}}(B_1(0);\mathbb{R}^n)$ and that $\mu^\xi_{u_r}(B_1(0)) = r^{1-n}\mu^\xi_u(B_r(x))$ for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$. Let now $\{e_1, \dots, e_n\}$ denote the orthonormal basis of $\mathbb{R}^n$, and consider the sets $\Lambda^\pm := \{\xi \in \mathbb{S}^{n-1} : \pm(e_1 \cdot \xi) > 0\}$. We notice that $\mathbb{S}^{n-1} \setminus (\Lambda^+ \cup \Lambda^-) = \mathbb{S}^{n-1} \cap \Pi^{e_1}$, $\Lambda^+ \cap \Lambda^- = \emptyset$, and $\Lambda^- = -\Lambda^+$. Let us further denote the two maps $\phi^\pm \colon \mathbb{R}^n \setminus \Pi^{e_1} \to \Lambda^\pm$ defined as

$$\phi^\pm(z) := \begin{cases} \frac{z}{|z|}, & \text{if } \frac{z}{|z|} \in \Lambda^\pm \\ -\frac{z}{|z|}, & \text{if } \frac{z}{|z|} \in \Lambda^\mp. \end{cases}$$

Notice that, by construction, the functions $t \mapsto \phi^+(t\xi)$ and $t \mapsto \phi^-(t\xi)$ are constants where they are defined, namely, for $\xi \in \mathbb{S}^{n-1} \setminus \Pi^{e_1}$ and $t \in \mathbb{R} \setminus \{0\}$, and hence they can be continuously extended to the whole of $\mathbb{R}$.

Since $\mathcal{H}^{n-1}(\Sigma) > 0$ and since $\mathbb{S}^{n-1} \cap \Pi^{e_1}$ is a negligible set, then $\mathcal{H}^{n-1}(\Sigma \cap \Lambda^+) > 0$ or $\mathcal{H}^{n-1}(\Sigma \cap \Lambda^-) > 0$. Hence, we may assume with no loss of generality that Σ is all contained for instance in Λ^+. The case in which Σ is all contained in Λ^- can be treated exactly in the same way. From our assumption, we have $\mathcal{L}^n(\mathrm{co}(\Sigma \cup \{0\})) > 0$ and for $\xi \in \Sigma$ we obtain

$$(\hat{u}_r)^\xi_0 \to f_\xi \ \text{ in } \ L^1((-1,1)) \text{ as } r \to 0^+, \ \text{ where } \ f_\xi(t) := a_\xi \,\mathrm{sign}(t) + b_\xi \tag{3.21}$$

for some $a_\xi \in \mathbb{R}\setminus\{0\}$ and $b_\xi \in \mathbb{R}$. Condition (3.21) tells us in particular that $u_r \cdot \phi^+ \to f$ in measure on $\mathrm{co}(\Sigma \cup \{0\})$ as $r \to 0^+$, where $f \colon B_1(0) \to \mathbb{R}$ is defined as

$$f(t\xi) := \begin{cases} f_\xi, & \text{if } t\xi \in \mathrm{co}(\Sigma \cup \{0\}) \text{ for } (\xi, t) \in \Lambda^+ \times \mathbb{R}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.22}$$

Step 3. We claim that there exist a subsequence of radii $r_k \searrow 0$ and linearly independent vectors $\{z_1, \dots, z_n\} \subset B_1(0) \setminus \{0\}$, such that z_i are points of approximate continuity for u_{r_k}, while (3.18) holds true for $x = z_i$ and $u = u_{r_k}$, and $O_{z_i}(u_{r_k}) \to 0^+$ for every $k = 1, 2, \dots$ as $k \to \infty$. Notice that, once the claim is proved, by the Fundamental Theorem of Calculus together with a measure theoretic argument we find that for $\mathcal{L}^n$-a.e. $z \in B_1(0)$ and for every $1 = 1, \dots, n$

$$|(u_{r_k}(z_i) - u_{r_k}(z)) \cdot \phi^+(z - z_i)| \le |D(\hat{u}_r)^{\phi^+(z-z_i)}_{z_i}|(B_1(0)^{\phi^+(z-z_i)}_{z_i}),\ k = 1, 2, \dots \tag{3.23}$$

We notice that the right-hand side of (3.23) is $\mathcal{L}^n$-measurable as a function of z. Indeed, we could have chosen the vectors $\{z_1, \dots, z_n\}$ so that $\xi \mapsto |D(\hat{u}_r)^\xi_{z_i}|(B_1(0)^\xi_{z_i})$ are measurable maps between the σ-algebra of $\mathcal{H}^{n-1}$-measurable subsets of $\mathbb{S}^{n-1}$ and the Borel σ-algebra of $\mathbb{R}$ (this follows similarly as in the proof of Lemma 2.1). Then, $z \mapsto |D(\hat{u}_r)^{\phi^+(z-z_i)}_{z_i}|(B_1(0)^{\phi^+(z-z_i)}_{z_i})$ is simply the composition of $\xi \mapsto |D(\hat{u}_r)^\xi_{z_i}|(B_1(0)^\xi_{z_i})$ with $z \mapsto \phi^+(z-z_i)$, which is Borel measurable and whose preimages of $\mathcal{H}^{n-1}$-negligible sets are $\mathcal{L}^n$-negligible.

To prove the claim we first notice that the assumption $\Theta^{n-1}(\hat\mu^1_u, x) = 0$ implies that $\hat\mu^1_{u_r}(B_1(0)) \to 0$ as $r \to 0^+$. By letting $\lambda_r := \hat\mu^1_{u_r}(B_1(0))$ we see from (3.20) applied to $u = u_r$ and $\Omega = B_1(0)$ that

$$\mathcal{L}^n(\{z \in B_1(0) : O_z(u_r) > c_n(\Omega)\sqrt{\lambda_r}\}) \le \sqrt{\lambda_r}, \quad \text{for every } 0 < r < \mathrm{dist}(x, \partial\Omega).$$

Therefore, by passing to a subsequence $r_k \searrow 0$ such that $\sum_k \sqrt{\lambda_{r_k}} \le \mathcal{L}^n(B_1(0))/2$, we have that the set $A := \bigcap_k \{z \in B_1(0) : O_z(u_{r_k}) \le c_n(\Omega)\sqrt{\lambda_{r_k}}\}$ satisfies $\mathcal{L}^n(A) \ge \mathcal{L}^n(B_1(0))/2$. In particular, since $\mathcal{L}^n(A) > 0$, we can find $\{z_1, \dots, z_n\} \subset A$ in generic position such that z_i is an approximate continuity point for u_{r_k} and (3.18) holds true with $u = u_{r_k}$ and $x = z_i$ for every $k = 1, 2, \dots$ and $i = 1, \dots, n$. The fact that $O_{z_i}(u_{r_k}) \to 0^+$ for every $i = 1, \dots, n$ as $k \to \infty$ is a direct consequence of the fact that $\{z_1, \dots, z_n\} \subset A$. The claim is thus proved.

Step 4. Condition $O_{z_i}(u_{r_k}) \to 0^+$ as $k \to \infty$ allows us to say that, up to considering a not relabelled subsequence,

$$g_{i,k}(\xi) \to 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \xi \in \mathbb{S}^{n-1} \quad \text{as} \quad k \to \infty, \tag{3.24}$$

where we have set

$$g_{i,k}(\xi) := |D(\hat u_{r_k})^\xi_{z_i}|(B_1(0)^\xi_{z_i} \setminus J^1_{(\hat u_{r_k})^\xi_{z_i}}) + \mathcal{H}^0(B_1(0)^\xi_{z_i} \cap J^1_{(\hat u_{r_k})^\xi_{z_i}}).$$

In particular, for every sufficiently large k (depending on ξ) we have that for every $i = 1, \dots, n$

$$g_{i,k}(\xi) = |D(\hat u_{r_k})^\xi_{z_i}|(B_1(0)^\xi_{z_i}). \tag{3.25}$$

Now since $\mathcal{L}^n(\mathrm{co}(\Sigma \cup \{0\})) > 0$ and since $\{z_1, \dots, z_n\}$ are linearly independent vectors, we obtain

$$\{\phi^+(z - z_1), \dots, \phi^+(z - z_n)\} \text{ form a basis of } \mathbb{R}^n \text{ for } \mathcal{L}^n\text{-a.e. } z \in \mathrm{co}(\Sigma \cup \{0\}). \tag{3.26}$$

Therefore, for $\mathcal{L}^n$-a.e. $z \in \mathrm{co}(\Sigma \cup \{0\})$ we find real smooth coefficients $\{c_1(z), \dots, c_n(z)\}$ such that $\phi^+(z) = \sum_i c_i(z)\phi^+(z - z_i)$. In addition, combining (3.23) and (3.26) we infer for every $i = 1, \dots, n$ that

$$|(u_{r_k}(z_i) - u_{r_k}(z)) \cdot \phi^+(z - z_i)| \to 0, \text{ pointwise a.e. } z \in \mathrm{co}(\Sigma \cup \{0\}) \text{ as } k \to \infty.$$

Therefore, for every sufficiently large k (depending on z)

$$\begin{aligned} |u_{r_k}(z) \cdot \phi^+(z) - \sum_i c_i(z) u_{r_k}(z_i) \cdot \phi^+(z - z_i)| &\le \sum_i |c_i(z)||(u_{r_k}(z) - u_{r_k}(z_i)) \cdot \phi^+(z - z_i)| \\ &\le \sum_i |c_i(z)||D(\hat u_{r_k})^{\phi^+(z-z_i)}_{z_i}|(B_1(0)^{\phi^+(z-z_i)}_{z_i}) \\ &\le \sum_i |c_i(z)| g_{i,k}(\phi^+(z - z_i)). \end{aligned}$$

By combining (3.24) and (3.25), the above inequality yields that for a.e. $z \in \mathrm{co}(\Sigma \cup \{0\})$

$$|u_{r_k}(z) \cdot \phi^+(z) - \sum_i c_i(z) u_{r_k}(z_i) \cdot \phi^+(z - z_i)| \to 0, \quad \text{as } k \to \infty,$$

thus, by recalling (3.22), we have for a.e. $z \in \mathrm{co}(\Sigma \cup \{0\})$

$$|f(z) - \sum_i c_i(z) u_{r_k}(z_i) \cdot \phi^+(z - z_i)| \to 0, \quad \text{as } k \to \infty.$$

By performing the change of variables $z = t\xi$ for $(\xi, t) \in \Lambda^+ \times \mathbb{R} \setminus \{0\}$, we rewrite the above convergence as: for $\mathcal{H}^{n-1}$-a.e. $\xi \in \Sigma$ it holds

$$|a_\xi \operatorname{sign}(t) + b_\xi - \sum_i c_i(t\xi) u_{r_k}(z_i) \cdot \phi^+(t\xi - z_i)| \to 0, \text{ a.e. } t \in (-1,1) \text{ as } k \to \infty. \quad (3.27)$$

We conclude by showing that (3.27) gives a contradiction. Indeed, fix a $\xi \in \Sigma$ for which (3.27) holds true. Consider the finite-dimensional real vector subspace V of $L^0((-1,1))$ generated by the elements $\{v_{ij} \colon (-1,1) \to \mathbb{R} : i, j = 1, \dots, n\}$, where $v_{ij}(t) := c_i(t\xi)\phi_j^+(t\xi - z_i)$ $(\phi_j^+ := \phi^+ \cdot e_j)$. Condition (3.27) combined with Proposition 3.12 (recall that pointwise convergence implies convergence in measure) imply that the function $v \colon (-1,1) \to \mathbb{R}$, defined as $v(t) := a_\xi \operatorname{sign}(t) + b_\xi$, belongs to V. Since the generators v_{ij} are all smooth functions, any linear combination of them still belongs to $C^\infty((-1,1))$, forcing $a_\xi = 0$. This is not possible because we assumed at the beginning $a_\xi \neq 0$ and we reach a contradiction. The proof is thus concluded. □

Before proving Theorem 3.6, we need to clarify the relation between the set appearing in formula (3.15) and the set where certain oscillations of u do not vanish. In [2] the following notion of *one-dimensional radial oscillation* for a function $u \colon \mathbb{R}^n \to \mathbb{R}^n$ around $x \in \mathbb{R}^n$ was considered.

Definition 3.14 (One dimensional radial oscillation around x)**.** Let $f \colon \mathbb{R} \to \mathbb{R}$ be measurable. We introduce the oscillation of f at scale $r > 0$ around the origin as

$$\operatorname{Osc}_r(f) := \inf_{\operatorname{Lip}(\theta) \le 1} \int_{-1/4}^{1/4} (|f(rt) - \theta| \wedge 1)\, t^{n-1}\, \mathrm{d}t\,.$$

For $\Omega \subseteq \mathbb{R}^n$ open and $u \colon \Omega \to \mathbb{R}^n$ measurable we define the one-dimensional radial oscillation of u around $x \in \Omega$ as

$$\operatorname{Osc}(u, x) := \limsup_{r \searrow 0} \int_{\mathbb{S}^{n-1}} \operatorname{Osc}_r(\hat{u}_x^\xi)\, \mathrm{d}\mathcal{H}^{n-1}(\xi)\,.$$

The set of all points $x \in \Omega$ for which the one-dimensional radial oscillation of u around x do not vanish is given by

$$\operatorname{Osc}_u := \{x \in \Omega : \operatorname{Osc}(u, x) > 0\}.$$

We notice that, for every measurable function $u \colon \Omega \to \mathbb{R}^n$, the following inclusion holds true

$$\{x \in \Omega : 0 \notin J_{\hat{u}_x^\xi} \text{ for } \mathcal{H}^{n-1}\text{-a.e. } \xi \in \mathbb{S}^{n-1}\} \subset \operatorname{Osc}_u.$$

Therefore, proving that the set $\{x \in \Omega : 0 \notin J_{\hat{u}_x^\xi} \text{ for } \mathcal{H}^{n-1}\text{-a.e. } \xi \in \mathbb{S}^{n-1}\}$ is σ-finite with respect to $\mathcal{H}^{n-1}$ does not a priori imply the same for the set Osc_u, which corresponds to assumption (2) of [2, Theorem 4.12] with $\rho = 1$. Nevertheless, in proving that the measure $\mathscr{I}_{u,1}$ is $(n-1)$-rectifiable, by virtue of Proposition 3.11 (which corresponds to [2, Proposition 4.7]), we can replace condition (2) of [2, Theorem 4.12] with

(2') The set $\{x \in \Omega : 0 \notin J_{\hat{u}_x^\xi} \text{ for } \mathcal{H}^{n-1}\text{-a.e. } \xi \in \mathbb{S}^{n-1}\}$ is σ-finite w.r.t. $\mathcal{I}^{n-1}$,

where $\mathcal{I}^{n-1}$ denotes the $(n-1)$-dimensional integral geometric measure in $\mathbb{R}^n$.

We are now in position to prove Theorem 3.6.

Proof of Theorem 3.6. We start by showing that $\hat{\mu}_u^p(\Omega) < \infty$ implies $\mathscr{I}_{u,p}(\Omega) < \infty$. Indeed, given $\delta > 0$, if we denote by $\mathscr{B}_{\delta,c}$ the set consisting of all countable covers of

Ω made of open balls, we clearly have

$$\begin{aligned}\mathscr{I}_{u,p}(\Omega) = \sup_{\delta>0}\inf_{G_\delta}\sum_{B\in G_\delta}\zeta_p(B) &\le \sup_{\delta>0}\inf_{\mathscr{B}_{\delta,c}}\sum_{B\in\mathscr{B}_{\delta,c}}\Big(\int_{\mathbb{S}^{n-1}}\mu_u^\xi(B)^p\,\mathrm{d}\mathcal{H}^{n-1}(\xi)\Big)^{\frac{1}{p}}\\ &\le \sup_{\delta>0}\inf_{\mathscr{B}_{\delta,c}}\sum_{B\in\mathscr{B}_{\delta,c}}\hat{\mu}_u^p(B), \qquad (3.28)\end{aligned}$$

where we used also that $\eta_\xi \le \mu_u^\xi$ as measures for a.e. $\xi\in\mathbb{S}^{n-1}$, where η_ξ is the measure in (3.13)–(3.14) associated to u, and that by construction

$$\Big(\int_{\mathbb{S}^{n-1}}\mu_u^\xi(B)^p\,\mathrm{d}\mathcal{H}^{n-1}(\xi)\Big)^{\frac{1}{p}} \le \hat{\mu}_u^p(B), \quad \text{for every open ball } B\subset\Omega.$$

Finally, since $\hat{\mu}_u^p$ is assumed to be a finite (Borel) measure, then the outcome of (3.28) is exactly $\hat{\mu}_u^p(\Omega)$. This proves in particular the desired implication.

Now recall by [24] that given a Radon measure μ, then the set $\{x : \Theta^{*n-1}(\mu,x)>0\}$ is σ-finite with respect to $\mathcal{H}^{n-1}$. Therefore, we can combine Propositions 3.11 and 3.13 to infer that the assumption $\hat{\mu}_u^p(\Omega)<\infty$ implies that $\mathscr{I}_{u,1}$ is concentrated on a set which is σ-finite with respect to $\mathcal{H}^{n-1}$. We are thus in position to apply [2, Theorem 4.12] with the condition (2) replaced by (2') (see the discussion above) to infer exactly (3.5). The proof is thus concluded. □

3.2.2. *Proof of Theorem 3.7.* The proof follows the line of [21, Theorem 9.1], we report here only the main differences.

We set

$$\Xi := \{\xi\in\mathbb{R}^n\setminus\{0\} : \hat{u}_y^\xi \in \mathrm{BV}(\Omega_y^\xi) \text{ for a.e. } y\in\Pi^\xi\}. \qquad (3.29)$$

Observe that $\mathcal{L}^n(\mathbb{R}^n\setminus\Xi)=0$. Without loss of generality, we may assume that u is a Borel function with compact support in Ω and that $\hat{u}_y^\xi\in\mathrm{BV}(\Omega_y^\xi)$ for every $\xi\in\Xi$ and for every $y\in\Pi^\xi$. For every $x\in\Omega$ we define

$$\hat{u}^\xi(x) := \limsup_{\rho\to0^+}\frac{1}{2\rho}\int_{-\rho}^{\rho}u(x+s\xi)\cdot\xi\,\mathrm{d}s, \qquad (3.30)$$

$$e^\xi(x) := \limsup_{\rho\to0^+}\frac{1}{2\rho}\int_0^\rho\frac{\hat{u}^\xi(x+s\xi)-\hat{u}^\xi(x)}{s}\,\mathrm{d}s.$$

We observe that u^ξ and e^ξ are Borel functions and have compact support on Ω. In particular,

$$e^{\rho\xi}(x) = \rho^2 e^\xi(x) \quad \text{for every } \rho>0 \text{ and every } x\in\Omega. \qquad (3.31)$$

By the Lebesgue Differentation Theorem, for every $y\in\Pi^\xi$ we have

$$(\hat{u}^\xi)_y^\xi = \hat{u}_y^\xi \quad \mathcal{L}^1\text{-a.e. in } \Omega_y^\xi.$$

Since $\hat{u}_y^\xi\in\mathrm{BV}(\Omega_y^\xi)$ and $(\hat{u}^\xi)_y^\xi$ is a good representative of $\hat{u}_y^\xi$ by (3.30), we obtain that

$$\nabla\hat{u}_y^\xi(t) = \lim_{s\to0}\frac{(\hat{u}^\xi)_y^\xi(t+s)-(\hat{u}^\xi)_y^\xi(t)}{s} = (e^\xi)_y^\xi(t) \qquad (3.32)$$

for every $y\in\Pi^\xi$ and for $\mathcal{L}^1$-a.e. $t\in\Omega_y^\xi$.

As in [21, Theorem 9.1], the following parallelogram identity holds

$$e^{\xi+\eta}(x)+e^{\xi-\eta}(x) = 2e^\xi(x)+2e^\xi(x) \quad \text{for a.e. } x\in\Omega, \qquad (3.33)$$

for every $\xi,\eta\in\mathbb{R}^n$ such that $\xi,\eta,\xi+\eta,\xi-\eta\in\Xi$.

Recall (3.29). Let $\xi_1 \in \Xi$. By induction, we consider

$$\xi_k \in \Xi_k := \Xi \cap \left(\bigcap_{1 \le i \le k-1} \bigcap_{q \in \mathbb{Q}} \Xi + q\xi_i \right)$$

and we remark that $\mathcal{L}^n(\mathbb{R}^n \setminus \Xi_k) = 0$. Define X as the vector space over $\mathbb{Q}$ generated by $\{\xi_k\}_{k\in\mathbb{N}}$. Since $\mathcal{L}^n(\mathbb{R}^n \setminus \Xi_k) = 0$ for every $k \in \mathbb{N}$, the sequence $\{\xi_k\}_{k\in\mathbb{N}}$ can be chosen to be dense in $\mathbb{R}^n$. We remark that, since Ξ is closed by multiplication with scalars, then by construction $X \subset \Xi$. Since X is countable and owing to (3.33), there exists a Borel set $N \subset \Omega$ such that $\mathcal{L}^n(N) = 0$ and the parallelogram identity

$$e^{\xi+\eta}(x) + e^{\xi-\eta}(x) = 2e^{\xi}(x) + 2e^{\xi}(x) \tag{3.34}$$

holds for every $x \in \Omega \setminus N$ and for every $\xi, \eta \in X$.

Since $e^{\xi}(x)$ is also positively homogeneous of degree 2 by (3.31), we deduce by [20, Proposition 11.9] that for every $x \in \Omega \setminus N$ there exists a symmetric bilinear form $B_x : X \times X \to \mathbb{R}$ such that

$$e^{\xi}(x) = B_x(\xi, \xi)$$

for every $\xi \in X$. This implies that for every $x \in \Omega \setminus N$ there exists a symmetric matrix $e(u)(x) \in \mathbb{M}^{n\times n}_{sym}$ such that

$$e^{\xi}(x) = e(u)(x)\xi \cdot \xi \tag{3.35}$$

for every $\xi \in X$.

Let us fix $\xi_0 \in \Xi$. We want to prove that (3.35) holds for $\xi = \xi_0$ and for a.e. $x \in \Omega$. Let X_0 be the vector subspace over $\mathbb{Q}$ generated by $X \cup \{\xi_0\}$. Since X_0 is countable, there exists a Borel set $N_0 \subset \Omega$, with $N \subset N_0$ and $\mathcal{L}^n(N_0) = 0$, such that (3.34) holds for every $x \in \Omega \setminus N_0$ and for every $\xi, \eta \in X_0$. Arguing as before, we prove that for every $x \in \Omega \setminus N_0$ there exists a symmetric matrix $A(x) \in \mathbb{M}^{n\times n}_{sym}$ such that

$$e^{\xi}(x) = A(x)\xi \cdot \xi \tag{3.36}$$

for every $\xi \in X_0$. Since $X \subset X_0$ and $N \subset N_0$, equalities (3.35) and (3.36) hold for every $x \in \Omega \setminus N_0$ and for every $\xi \in X$. This implies that $A(x) = e(u)(x)$ for every $x \in \Omega \setminus N_0$. Since (3.36) holds for every $x \in \Omega \setminus N_0$ and for every $\xi \in X_0$, we deduce that the same is true for (3.35). Since $\xi_0 \in X_0$, we conclude that (3.35) holds for $\xi = \xi_0$ for every $x \in \Omega \setminus N_0$.

By the arbitrariness of ξ_0 we have shown that for a.e. $\xi \in \mathbb{R}^n$ we have

$$e^{\xi}(x) = e(u)(x)\xi \cdot \xi \quad \text{a.e. in } \Omega.$$

Lastly, (3.6) follows by combining the above equality and (3.32).

Finally, the fact that $e(u) \in L^1(\Omega; \mathbb{M}^{n\times n}_{sym})$ follows from

$$\begin{aligned} \int_{\Omega} |e(u)(x)| \, \mathrm{d}x &\le c(n) \int_{\Omega} \int_{\mathbb{S}^{n-1}} |e(u)(x)\xi \cdot \xi| \, \mathrm{d}\xi \, \mathrm{d}x \\ &= \int_{\mathbb{S}^{n-1}} \int_{\Pi^{\xi}} \int_{\Omega^{\xi}_y} |e(u)^{\xi}_y(t)\xi \cdot \xi)| \, \mathrm{d}t \, \mathrm{d}y \, \mathrm{d}\xi \\ &\le \hat{\mu}^1_u(\Omega) < \infty. \end{aligned}$$

4. Proof of Theorem 1.1

We divide this section into two subsection. The first one contains a number of preliminary results that will be exploited in the second subsection, where the proof of Theorem 1.1 is carried out.

4.1. **Preliminary results.** We start with a general estimate.

Lemma 4.1. *For every positive integer n it holds true*

$$\sup_{e\in\mathbb{R}^n}\int_{B_4(0)\setminus B_{1/4}(0)}\frac{|e|}{1+|e\cdot\eta|^2}\,\mathrm{d}\mathcal{H}^n(\eta)<\infty. \tag{4.1}$$

Proof. An application of Coarea Formula with the map $f\colon\mathbb{R}^n\to\mathbb{R}$ defined as $f(\eta):=|\eta|$ allows us to write

$$\begin{aligned}
&\sup_{e\in\mathbb{R}^n}\int_{B_4(0)\setminus B_{1/4}(0)}\frac{|e|}{1+|e\cdot\eta|^2}\,\mathrm{d}\mathcal{H}^n(\eta)\\
&\quad=\sup_{e\in\mathbb{R}^n}\int_{1/4}^4\left(\int_{\partial B_\rho(0)}\frac{|e|}{1+|e\cdot\eta|^2}\,\mathrm{d}\mathcal{H}^{n-1}(\eta)\right)\mathrm{d}\rho\\
&\quad=\sup_{e\in\mathbb{R}^n}\int_{1/4}^4\left(\int_{\partial B_1(0)}\frac{\rho^{n-1}|e|}{1+\rho^2|e\cdot\eta|^2}\,\mathrm{d}\mathcal{H}^{n-1}(\eta)\right)\mathrm{d}\rho\\
&\quad\le 4^n\sup_{e\in\mathbb{R}^n}\int_{\mathbb{S}^{n-1}}\frac{|e|}{1+|e\cdot\eta|^2}\,\mathrm{d}\mathcal{H}^{n-1}(\eta).
\end{aligned}$$

Hence, it is enough to prove

$$\sup_{e\in\mathbb{R}^n}\int_{\mathbb{S}^{n-1}}\frac{|e|}{1+|e\cdot\eta|^2}\,\mathrm{d}\mathcal{H}^{n-1}(\eta)<\infty. \tag{4.2}$$

We remark that, since the integrand in (4.2) is rotation invariant, we can assume $e=\lambda e_1$, where $\lambda\in\mathbb{R}$ and e_1 is the first element of the canonical basis. Moreover, we observe that by setting $\eta_1:=\eta\cdot e_1$ it is enough to show

$$\sup_{\lambda>0}\int_{\mathbb{S}^{n-1}_\delta}\frac{\lambda}{1+\lambda^2\eta_1^2}\,\mathrm{d}\mathcal{H}^{n-1}(\eta)<\infty$$

for some $\delta\in(0,1)$, where $\mathbb{S}^{n-1}_\delta=\mathbb{S}^{n-1}\cap\{0\le\eta_1\le\delta\}$. Indeed, the integral

$$\int_{\mathbb{S}^{n-1}\cap\{\eta_1>\delta\}}\frac{\lambda}{1+\lambda^2\eta_1^2}\,\mathrm{d}\mathcal{H}^{n-1}(\eta)$$

is uniformly bounded by a constant only depending on n and δ.

By applying the Coarea Formula with the map $f\colon\mathbb{S}^{n-1}\to\mathbb{R}$ defined as $f(\eta):=\eta_1$ we have

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}_\delta}\frac{\lambda}{1+\lambda^2\eta_1^2}\frac{|\mathrm{J}_{\mathbb{S}^{n-1}}f(\eta)|}{|\mathrm{J}_{\mathbb{S}^{n-1}}f(\eta)|}\,\mathrm{d}\mathcal{H}^{n-1}(\eta)\\
&\qquad\le\sup_{\eta\in\mathbb{S}^{n-1}_\delta}\frac{1}{|\mathrm{J}_{\mathbb{S}^{n-1}}f(\eta)|}\int_0^1\frac{\lambda}{1+\lambda^2t^2}\mathcal{H}^{n-2}(\mathbb{S}^{n-1}_\delta\cap\{\eta_1=t\})\,\mathrm{d}t\\
&\qquad\le C(n,\delta)\int_0^1\frac{\lambda}{1+\lambda^2t^2}\,\mathrm{d}t=C(n,\delta)\arctan(\lambda)\le C(n,\delta)\frac{\pi}{2}.
\end{aligned}$$

This concludes the proof of (4.1). □

We now recall some useful results shown in [26] for the study of nonlocal approximations of the Mumford-Shah functional. For the reader's convenience, we report the proof of Lemma 4.5, highlighting the ambient space Ω in the functional. The proofs of Lemmas 4.2-4.4 coincide with those of [26].

Lemma 4.2 ([26, Lemma 3.2])**.** *Let $I=[a,b]$ be an interval, let $\{u_\varepsilon\}_{\varepsilon>0}\subset L^1_{loc}(\mathbb{R})$, and let $u\in L^1_{loc}(\mathbb{R})$. Let us assume the following:*

(i) $u_\varepsilon \to u$ *in* $L^1_{loc}(\mathbb{R})$;
(ii) a *and* b *are Lebesgue points of* u.

Then

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, I) \geq \min\left\{\frac{\pi}{2}, \frac{(u(b)-u(a))^2}{b-a}\right\}.$$

Lemma 4.3 ([26, Lemma 3.3]). *Let* $u \in L^\infty(\mathbb{R})$. *Then there exists* $a \in \mathbb{R}$ *such that*

(i) $a+q$ *is a Lebesgue point of* u *for every* $q \in \mathbb{Q}$;
(ii) every sequence $\{u_k\}_{k\in\mathbb{N}} \subset L^\infty(\mathbb{R})$ *that satisfies the following conditions:*
 – $u_k(a+\frac{z}{k}) = u(a+\frac{z}{k})$ *for all* $z \in \mathbb{Z}$;
 – *if* $x \in [a+\frac{z}{k}, a+\frac{z+1}{k}]$, *then* $u_k(x)$ *belongs to the interval with endpoints* $u(a+\frac{z}{k})$ *and* $u(a+\frac{z+1}{k})$;

 has a subsequence converging to u *in* $L^1_{loc}(\mathbb{R})$.

Lemma 4.4 ([26, Lemma 5.1]). *Let* $\Omega \subset \mathbb{R}^n$ *be an open set. For every* $u \in L^0(\Omega, \mathbb{R}^m)$, *every* $E \Subset \Omega$, *every* $\delta > 0$, *and every* $\xi \in \mathbb{R}^n$ *such that* $E + \delta\xi \subset \Omega$, *we have that*

$$\int_E |\arctan(u(x+\delta\xi)\cdot\xi) - \arctan(u(x)\cdot\xi)|\mathrm{d}x \leq C_E\delta(1 + F_{\delta,\xi}(u,E)),$$

for some $C_E > 0$ *only depending on* E.

Lemma 4.5 ([26, Lemma 5.3]). *Let* $\Omega \subset \mathbb{R}^n$ *be an open set,* $E \Subset \Omega$, $R \subset \mathbb{R}^n$, *and* $\eta > 0$ *be such that* $R \subset \frac{\Omega-\Omega}{\eta}$ *and*

$$\mathrm{dist}(E, \partial\Omega) \geq \eta \sup_{\xi\in R}|\xi|\,.$$

Then, for every $u \in L^0(\Omega;\mathbb{R}^n)$, $\varepsilon > 0$, *and* $m \in \mathbb{N}$ *such that* $m\varepsilon < \eta$ *we have that*

$$\int_R F_{m\varepsilon,\xi}(u,E)\,\mathrm{d}\xi \leq \int_R F_{\varepsilon,\xi}(u,\Omega)\,\mathrm{d}\xi\,. \tag{4.3}$$

Proof. Given $\varepsilon > 0$, we prove the statement by induction over m. If $m = 1$ there is nothing to prove. Let us assume that (4.3) holds for m and prove it for $m+1$, assuming that $(m+1)\varepsilon < \eta$. In particular, this implies that $E \subset \Omega - (m+1)\varepsilon\xi$ for every $\xi \in R$. For $A, B \in \mathbb{R}$ it holds

$$\arctan\left(\frac{(A+B)^2}{m+1}\right) \leq \arctan(A^2) + \arctan\left(\frac{B^2}{m}\right).$$

Applying such inequality to

$$A = \frac{(u(x+(m+1)\varepsilon\xi) - u(x+m\varepsilon\xi))\cdot\xi}{\sqrt{\varepsilon}} \qquad B = \frac{(u(x+m\varepsilon\xi)-u(x))\cdot\xi}{\sqrt{\varepsilon}}\,,$$

with $x \in E$ and $\xi \in R$, we get that

$$\begin{aligned}
&\frac{1}{(m+1)\varepsilon}\arctan\left(\frac{\big((u(x+(m+1)\varepsilon\xi)-u(x))\cdot\xi\big)^2}{(m+1)\varepsilon}\right)\\
&\quad\leq \frac{1}{(m+1)\varepsilon}\arctan\left(\frac{\big((u(x+(m+1)\varepsilon\xi)-u(x+m\varepsilon\xi))\cdot\xi\big)^2}{\varepsilon}\right)\\
&\qquad + \frac{m}{(m+1)}\frac{1}{m\varepsilon}\arctan\left(\frac{\big((u(x+m\varepsilon\xi)-u(x))\cdot\xi\big)^2}{m\varepsilon}\right)
\end{aligned}$$

Integrating over $x \in E$ and $\xi \in R$ and performing a change of variable in the first integral on the right-hand side we obtain

$$\begin{aligned}\int_R F_{(m+1)\varepsilon,\xi}(u,E)\,\mathrm{d}\xi &\le \frac{1}{(m+1)\varepsilon}\int_R\int_{E+m\varepsilon\xi}\arctan\left(\frac{\big((u(x+\varepsilon\xi)-u(x))\cdot\xi\big)^2}{\varepsilon}\right)\mathrm{d}x\,\mathrm{d}\xi\\ &\quad+\frac{m}{(m+1)}\int_R F_{m\varepsilon,\xi}(u,E)\,\mathrm{d}\xi\\ &\le \frac{1}{(m+1)}\int_R F_{\varepsilon,\xi}(u,\Omega)\,\mathrm{d}\xi+\frac{m}{(m+1)}\int_R F_{m\varepsilon,\xi}(u,E)\,\mathrm{d}\xi\,,\end{aligned}$$

where, in the last inequality, we have used that $E+m\varepsilon\xi\subset\Omega\cap\Omega-\varepsilon\xi$. Hence, we conclude for (4.3) by the induction hypothesis. □

We conclude this section with the following lemma, which characterizes the density φ_p and the constant β_p appearing in (1.7) and in the definition of the functional $\mathcal{F}^p$ in (1.8).

Lemma 4.6. *Let $p>1$, let $\Omega\subset\mathbb{R}^n$ be an open set, and let $\varphi_p\colon\mathbb{M}^{n\times n}\to[0,\infty)$ be the map with quadratic growth defined as*

$$\varphi_p(A):=\left(\int_{\mathbb{R}^n}\left(|A\xi\cdot\xi|^2\right)^p|\xi|^pe^{-|\xi|^2}\mathrm{d}\xi\right)^{\frac1p}\qquad \text{for } A\in\mathbb{M}^{n\times n} \text{ and } p<\infty.$$

Then, there exists a positive constant $\beta_p>0$ such that for every $u\in\mathrm{GSBD}(\Omega;\mathbb{R}^n)$ it holds

$$\begin{aligned}\sup_{\mathscr{B}}\sum_{B\in\mathscr{B}}\frac{\pi}{2}&\left(\int_{\mathbb{R}^n}\left(\int_{\Pi^\xi}\mathcal{MS}_{\frac{2}{\pi}}(\hat u^\xi_y,B^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\right)^p|\xi|^pe^{-|\xi|^2}\mathrm{d}\xi\right)^{\frac1p}\\ &=\int_\Omega\varphi_p(e(u))\,\mathrm{d}x+\beta_p\mathcal{H}^{n-1}(J_u),\end{aligned}$$

where $\mathscr{B}$ is the set of all possible finite families of pairwise disjoint balls in Ω.

Proof. We recall that

$$\frac{\pi}{2}\mathcal{MS}_{\frac{2}{\pi}}(\hat u^\xi_y,B^\xi_y)=\int_{B^\xi_y}|\nabla\hat u^\xi_y|^2\mathrm{d}\tau+\frac{\pi}{2}\mathcal{H}^0(J_{\hat u^\xi_y}\cap B^\xi_y).$$

Integrating the expression above in $y\in\Pi^\xi$, by Theorem 3.7 we obtain for the first term

$$\begin{aligned}\int_{\Pi^\xi}\int_{B^\xi_y}|\xi||\nabla\hat u^\xi_y|^2\,\mathrm{d}\tau\,\mathrm{d}\mathcal{H}^{n-1}(y)&=\int_{\Pi^\xi}\int_{B^\xi_y}|\xi||\nabla u(y+\tau\xi)\cdot\xi|^2\,\mathrm{d}\tau\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &=\int_{\Pi^\xi}\int_{B^\xi_y}|\xi||e(u)(y+\tau\xi)\xi\cdot\xi|^2\,\mathrm{d}\tau\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &=\int_B|\xi||e(u)(x)\xi\cdot\xi|^2\,\mathrm{d}x.\end{aligned}$$

On the other hand, by the area formula and Theorem 3.6, for the second term we get

$$\int_{\Pi^\xi}|\xi|\mathcal{H}^0(J_{\hat u^\xi_y}\cap B)\,\mathrm{d}\mathcal{H}^{n-1}(y)=\int_{J_u\cap B}|\xi||\nu_u(x)\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1}(x).$$

We define for every ball $B\subset\Omega$ the function

$$\zeta(u,B):=\left(\int_{\mathbb{R}^n}\left(\int_B|e(u)(x)\xi\cdot\xi|^2\mathrm{d}x+\frac{\pi}{2}\int_{J_u\cap B}|\nu_u(x)\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1}(x)\right)^p|\xi|^pe^{-|\xi|^2}\,\mathrm{d}\xi\right)^{\frac1p},$$

and for every open $A \subset \Omega$ we set

$$\mu(u, A) := \sup_{\mathscr{B}_A} \sum_{B \in \mathscr{B}_A} \zeta(u, B),$$

where $\mathscr{B}_A$ is any finite family of disjoint balls contained in A. Then, μ can be extended to a Borel measure (which we still denote by μ with a slight abuse of notation). We also observe that there exists a positive constant C only depending on n and p such that for every open set $A \subset \Omega$ we have

$$\mu(u, A) \leq C\mathcal{G}_{\frac{2}{\pi}}(u, A), \tag{4.4}$$

where the inequality follows by the definition of μ. In particular, (4.4) holds for every Borel set. Hence, we decompose μ in the following way

$$\mu(u, B) = \int_B f(x)\,\mathrm{d}x + \int_{J_u \cap B} g(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)$$

for every Borel set $B \subset \Omega$, for some densities f and g.

We start by calculating the density f, which is given by

$$f(x) = \lim_{r \to 0} \frac{\mu(u, B_r(x))}{\omega_n r^n}$$

for $\mathcal{L}^n$-a.e. $x \in \Omega$. Let $x \in \Omega$ be such that

$$\lim_{r \to 0} \frac{1}{r^n} \int_{B_r(x)} |e(u)(y) - e(u)(x)|^2 \mathrm{d}y = 0, \tag{4.5}$$

$$\lim_{r \to 0} \frac{\mathcal{H}^{n-1}(J_u \cap B_r(x))}{r^n} = 0\,. \tag{4.6}$$

We recall that $\mathcal{L}^n$-a.e. $x \in \Omega$ satisfy (4.5) and (4.6). Consider $\mathscr{B}_{B_r(x)}$ a family of covers of $B_r(x)$ such that

$$\lim_{r \to 0} \frac{\mu(u, B_r(x)) - \sum_{B \in \mathscr{B}_{B_r(x)}} \zeta(u, B)}{r^n} = 0\,, \tag{4.7}$$

$$\lim_{r \to 0} \frac{\mathcal{L}^n(B_r(x)) - \sum_{B \in \mathscr{B}_{B_r(x)}} \mathcal{L}^n(B)}{r^n} = 0\,. \tag{4.8}$$

In view of (4.7), for every $\delta > 0$ we have

$$\begin{aligned}
f(x) &= \lim_{r \to 0} \frac{\mu(u, B_r(x))}{\omega_n r^n} = \lim_{r \to 0} \frac{\sum_{B \in \mathscr{B}_{B_r(x)}} \zeta(u, B)}{\omega_n r^n} \\
&= \lim_{r \to 0} \frac{1}{\omega_n r^n} \sum_{B \in \mathscr{B}_{B_r(x)}} \Big(\int_{\mathbb{R}^n} \Big(\int_B (|(e(u)(y) - e(u)(x) + e(u)(x))\xi \cdot \xi|^2 \mathrm{d}y \\
&\quad + \frac{\pi}{2} \int_{J_u \cap B} |\nu_u(y) \cdot \xi|\, \mathrm{d}\mathcal{H}^{n-1}(y) \Big)^p |\xi|^p e^{-|\xi|^2} \mathrm{d}\xi \Big)^{\frac{1}{p}} \\
&\leq \lim_{r \to 0} \frac{1}{\omega_n r^n} \sum_{B \in \mathscr{B}_{B_r(x)}} \Big(\int_{\mathbb{R}^n} (1 + \delta)^p |e(u)(x)\xi \cdot \xi|^{2p} \mathcal{L}^n(B)^p |\xi|^p e^{-|\xi|^2} \mathrm{d}\xi \Big)^{\frac{1}{p}} \qquad (4.9) \\
&\quad + \lim_{r \to 0} \frac{1}{\omega_n r^n} \sum_{B \in \mathscr{B}_{B_r(x)}} \Big(\int_{\mathbb{R}^n} \Big(\int_B (1 + \frac{1}{\delta}) |e(u)(y) - e(u)(x)|^2 \mathrm{d}y \Big)^p |\xi|^{5p} e^{-|\xi|^2} \mathrm{d}\xi \Big)^{\frac{1}{p}} \qquad (4.10)
\end{aligned}$$

$$+\lim_{r\to 0}\frac{1}{\omega_n r^n}\sum_{B\in\mathscr{B}_{B_r(x)}}\bigg(\int_{\mathbb{R}^n}\Big(\frac{\pi}{2}\int_{J_u\cap B}|\nu_u(y)\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1}(y)\Big)^p|\xi|^p e^{-|\xi|^2}\mathrm{d}\xi\bigg)^{\frac{1}{p}}. \tag{4.11}$$

The limit in (4.10) rewrites as

$$(1+\frac{1}{\delta})\Big(\int_{\mathbb{R}^n}|\xi|^{5p}e^{-|\xi|^2}\mathrm{d}\xi\Big)^{\frac{1}{p}}\lim_{r\to 0}\frac{1}{\omega_n r^n}\sum_{B\in\mathscr{B}_{B_r(x)}}\int_B|e(u)(y)-e(u)(x)|^2\mathrm{d}y$$

which is equal to zero by (4.5). The limit in (4.11) is bounded by

$$\frac{\pi}{2}\Big(\int_{\mathbb{R}^n}|\xi|^{2p}e^{-|\xi|^2}\mathrm{d}\xi\Big)^{\frac{1}{p}}\lim_{r\to 0}\frac{1}{\omega_n r^n}\sum_{B\in\mathscr{B}_{B_r(x)}}\mathcal{H}^{n-1}(J_u\cap B)$$

which is also equal to zero by (4.6). Finally, by (4.8) the limit in (4.9) is equal to

$$(1+\delta)\left(\int_{\mathbb{R}^n}\left(|e(u)(x)\xi\cdot\xi|^2\right)^p|\xi|^p e^{-|\xi|^2}\mathrm{d}\xi\right)^{\frac{1}{p}}=(1+\delta)\varphi_p\big(e(u)(x)\big)$$

for every $\delta>0$, and thus $f(x)\le\varphi_p\big(e(u)(x)\big)$. The other inequality follows from similar arguments.

In order to calculate the density g we consider $x\in J_u$ such that

$$\lim_{r\to 0}\frac{\int_{B_r(x)}|e(u)(y)\xi\cdot\xi|^2\mathrm{d}y}{r^{n-1}}=0, \tag{4.12}$$

$$\lim_{r\to 0}\frac{\mathcal{H}^{n-1}(J_u\cap B_r(x))}{\omega_{n-1}r^{n-1}}=1, \tag{4.13}$$

$$\lim_{r\to 0}\frac{1}{\omega_{n-1}r^{n-1}}\int_{B_r(x)\cap J_u}|\nu_u(y)-\nu_u(x)|\,\mathrm{d}\mathcal{H}^{n-1}(y)=0. \tag{4.14}$$

We remark that $\mathcal{H}^{n-1}$-a.e. $x\in J_u$ satisfy (4.12)–(4.14). We consider a family of covers $\mathscr{B}_{B_r(x)}$ of $B_r(x)$ such that

$$\lim_{r\to 0}\frac{\mu(u,B_r(x))-\sum_{B\in\mathscr{B}_{B_r(x)}}\zeta(u,B)}{r^{n-1}}=0\,. \tag{4.15}$$

Thanks to (4.15) we obtain

$$\begin{aligned}
g(x)&=\lim_{r\to 0}\frac{\mu(u,B_r(x))}{\omega_{n-1}r^{n-1}}=\lim_{r\to 0}\frac{\sum_{B\in\mathscr{B}_{B_r(x)}}\zeta(u,B)}{\omega_{n-1}r^{n-1}}\\
&=\lim_{r\to 0}\frac{1}{\omega_{n-1}r^{n-1}}\sum_{B\in\mathscr{B}_{B_r(x)}}\bigg(\int_{\mathbb{R}^n}\Big(\frac{\pi}{2}\int_{J_u\cap B}|(\nu_u(y)-\nu_u(x)+\nu_u(x))\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1}(y)\\
&\quad+\int_B(|e(u)(y)\xi\cdot\xi|^2\mathrm{d}y\Big)^p|\xi|^p e^{-|\xi|^2}\mathrm{d}\xi\bigg)^{\frac{1}{p}}\\
&\le\lim_{r\to 0}\frac{1}{\omega_{n-1}r^{n-1}}\sum_{B\in\mathscr{B}_{B_r(x)}}\bigg(\int_{\mathbb{R}^n}\Big(\frac{\pi}{2}\int_{J_u\cap B}|\nu_u(x)\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1}(y)\Big)^p|\xi|^p e^{-|\xi|^2}\mathrm{d}\xi\bigg)^{\frac{1}{p}}
\end{aligned} \tag{4.16}$$

$$+\lim_{r\to 0}\frac{1}{\omega_{n-1}r^{n-1}}\sum_{B\in\mathscr{B}_{B_r(x)}}\left(\int_{\mathbb{R}^n}\left(\frac{\pi}{2}\int_{J_u\cap B}|\nu_u(y)-\nu_u(x)|\,\mathrm{d}\mathcal{H}^{n-1}(y)\right)^p|\xi|^{2p}e^{-|\xi|^2}\mathrm{d}\xi\right)^{\frac{1}{p}} \tag{4.17}$$

$$+\lim_{r\to 0}\frac{1}{\omega_{n-1}r^{n-1}}\sum_{B\in\mathscr{B}_{B_r(x)}}\left(\int_{\mathbb{R}^n}\left(\int_B|e(u)(y)|^2\mathrm{d}y\right)^p|\xi|^{5p}e^{-|\xi|^2}\mathrm{d}\xi\right)^{\frac{1}{p}}. \tag{4.18}$$

Similarly as before, the limit in (4.17) is equal to zero by (4.13) and (4.14). From (4.12) it follows that also (4.18) is zero. Finally, we observe that (4.16) is equal to

$$\frac{\pi}{2}\left(\int_{\mathbb{R}^n}|\nu_u(x)\cdot\xi|^p|\xi|^pe^{-|\xi|^2}\mathrm{d}\xi\right)^{\frac{1}{p}}=:\beta_p,$$

hence $g(x)\le\beta_p$. The reverse inequality follows from similar computations.

□

4.2. **Proof of Theorem 1.1.** Consider any subsequence $\varepsilon_k\to 0$ and set $u_k:=u_{\varepsilon_k}$. By Hölder's inequality, we have

$$M:=\sup_{k\in\mathbb{N}}\mathcal{F}^1_{\varepsilon_k}(u_k,\Omega)\le C(n,p)\sup_{k\in\mathbb{N}}\mathcal{F}^p_{\varepsilon_k}(u_k,\Omega)<\infty.$$

We divide the proof into 6 steps. In Steps 1 and 2 we prove that u_k converges pointwise, up to an exceptional set A, to some limit function u by a Fréchet-Kolmogorov argument. Steps 3–6 are devoted to show that u is indeed a $\mathrm{GSBV}^{\mathcal{E}}$-function and that A is of finite perimeter.

Step 1: Fix two open sets $F\Subset E\Subset\Omega$ and define $f_k\colon\Omega\times\mathbb{R}^n\to\mathbb{R}$ by

$$f_k(x,\xi):=\tau(u_k(x)\cdot\xi):=\arctan(u_k(x)\cdot\xi).$$

Let $k\in\mathbb{N}$ be sufficiently large so that $B_4(0)\setminus B_{1/4}(0)\subset\frac{\Omega-\Omega}{\varepsilon_k}$ (notice that $\Omega-\Omega$ is an open set containing the origin) and $4\varepsilon_k\le\mathrm{dist}(E,\partial\Omega)$. In particular, this implies that $F\subset E\subset\Omega-\varepsilon_kt\xi$ for every $\xi\in B_4(0)\setminus B_{1/4}(0)$ and every $t\in[0,1]$. For simplicity of notation, let us set $R:=B_2(0)\setminus B_{1/2}(0)$. In order to apply Fréchet-Kolmogorov Theorem on $F\times R$, we start by showing that for every $\alpha>0$ there exist $\bar t\in(0,1)$ such that

$$\int_{F\times R}|f_k(x+\eta,\xi)-f_k(x,\xi)|\,\mathrm{d}x\,\mathrm{d}\xi\le\alpha\qquad\text{for }\eta\in B_{\bar t}(0). \tag{4.19}$$

Instead of proving (4.19), we prove a slightly different equivalent statement: for every $\alpha>0$ there exist $\bar t\in(0,1)$ and $\bar k\in\mathbb{N}$ such that

$$\int_{F\times R}|f_k(x+\eta,\xi)-f_k(x,\xi)|\,\mathrm{d}x\,\mathrm{d}\xi\le\alpha\qquad\text{for }\eta\in B_{\bar t}(0)\text{ and }k\ge\bar k. \tag{4.20}$$

For every $\eta\in B_1(0)$, $\xi\in\mathbb{R}^n$ and $j=1,2,\dots$, we define $\xi^j_\eta\in\mathbb{R}^n$ as

$$\xi^j_\eta:=\xi+\frac{1}{j}\eta.$$

We remark that by Lemma 4.1 for every $\eta\in B_1(0)$ and $j\in\mathbb{N}$ we have that

$$\sup_{e\in\mathbb{R}^n}\int_R\frac{|e|}{1+|e\cdot\xi^j_\eta|^2}\,\mathrm{d}\xi\le\sup_{e\in\mathbb{R}^n}\int_{B_4(0)\setminus B_{1/4}(0)}\frac{|e|}{1+|e\cdot\xi|^2}\,\mathrm{d}\xi\le C \tag{4.21}$$

for some positive constant C only depending on the dimension n.

Let us now fix $\bar j\ge 4$ such that $\bar j\ge 4|F|C/\alpha$. We repeatedly apply the triangle inequality to obtain for $t\in[0,1]$

$$|f_k(x+t\eta,\xi)-f_k(x,\xi)| \tag{4.22}$$

$$\leq |f_k(x+t\eta,\xi)-f_k(x+t\eta,\xi^{\bar{j}}_\eta)|+|f_k(x+t\eta,\xi^{\bar{j}}_\eta)-f_k(x-\bar{j}t\xi,\xi^{\bar{j}}_\eta)|$$
$$+|f_k(x-\bar{j}t\xi,\xi^{\bar{j}}_\eta)-f_k(x-\bar{j}t\xi,\xi)|+|f_k(x-\bar{j}t\xi,\xi)-f_k(x,\xi)|.$$

The first term in (4.22) is bounded as follows

$$\begin{aligned}
&|f_k(x+t\eta,\xi)-f_k(x+t\eta,\xi^{\bar{j}}_\eta)| \\
&\quad=|\tau(u_k(x+t\eta)\cdot\xi)-\tau(u_k(x+t\eta)\cdot\xi^{\bar{j}}_\eta)|=\left|\int_{u_k(x+t\eta)\cdot\xi}^{u_k(x+t\eta)\cdot\xi^{\bar{j}}_\eta}\frac{\mathrm{d}s}{1+s^2}\right| \\
&\quad\leq\max\left\{\frac{1}{1+|u_k(x+t\eta)\cdot\xi|^2},\frac{1}{1+|u_k(x+t\eta)\cdot\xi^{\bar{j}}_\eta|^2}\right\}\left|u_k(x+t\eta)\cdot(\xi^{\bar{j}}_\eta-\xi)\right| \\
&\quad\leq\frac{1}{\bar{j}}\max\left\{\frac{1}{1+|u_k(x+t\eta)\cdot\xi|^2},\frac{1}{1+|u_k(x+t\eta)\cdot\xi^{\bar{j}}_\eta|^2}\right\}|u_k(x+t\eta)|.
\end{aligned}\tag{4.23}$$

Thanks to (4.21), we deduce from (4.23) that

$$\int_R|f_k(x+t\eta,\xi)-f_k(x+t\eta,\xi^{\bar{j}}_\eta)|\,\mathrm{d}\xi\leq\frac{1}{\bar{j}}C\,. \tag{4.24}$$

for the same constant C defined in (4.21). Hence, we infer from (4.24) and from the choice of $\bar{j}$ that for every $x\in F$, $t\in[0,1]$, $k\in\mathbb{N}$, and $\eta\in B_1(0)$, it holds

$$\int_R|f_k(x+t\eta,\xi)-f_k(x+t\eta,\xi^{\bar{j}}_\eta)|\,\mathrm{d}\xi\leq\frac{\alpha}{4|F|}. \tag{4.25}$$

The very same argument allows us to bound the third term on the right-hand side of (4.22) for every $x\in F$, $t\in[0,1]$, $k\in\mathbb{N}$, and $\eta\in B_1(0)$ as

$$\int_R|f_k(x-\bar{j}t\xi,\xi^{\bar{j}}_\eta)-f_k(x-\bar{j}t\xi,\xi)|\,\mathrm{d}\xi\leq\frac{\alpha}{4|F|}. \tag{4.26}$$

For the second term on the right-hand side of (4.22) we have

$$\begin{aligned}
&\int_R\int_F|f_k(x+t\eta,\xi^{\bar{j}}_\eta)-f_k(x-\bar{j}t\xi,\xi^{\bar{j}}_\eta)|\mathrm{d}x\,\mathrm{d}\xi \\
&=\int_R\int_{F-\bar{j}t\xi}|f_k(x+t\bar{j}\xi^{\bar{j}}_\eta,\xi^{\bar{j}}_\eta)-f_k(x,\xi^{\bar{j}}_\eta)|\mathrm{d}x\,\mathrm{d}\xi \\
&\leq\int_{E\times R}|\tau(u_k(x+t\bar{j}\xi^{\bar{j}}_\eta)\cdot\xi^{\bar{j}}_\eta)-\tau(u_k(x)\cdot\xi^{\bar{j}}_\eta)|\mathrm{d}x\,\mathrm{d}\xi,
\end{aligned}$$

if $\bar{j}t\ll 1$, since $F\Subset E$. We apply the change of variable $\xi=\xi^{\bar{j}}_\eta$ to the last integral above to obtain

$$\begin{aligned}
&\int_R\int_F|f_k(x+t\eta,\xi^{\bar{j}}_\eta)-f_k(x-\bar{j}t\xi,\xi^{\bar{j}}_\eta)|\mathrm{d}x\,\mathrm{d}\xi \\
&\qquad\leq\int_{E\times(B_4(0)\setminus B_{1/4}(0))}|\tau(u_k(x+t\bar{j}\xi)\cdot\xi)-\tau(u_k(x)\cdot\xi)|\mathrm{d}x\,\mathrm{d}\xi\,.
\end{aligned}$$

Recall that, by Lemma 4.4, for $t\in[0,1)$ and $\xi\in B_4(0)\backslash B_{1/4}(0)$ such that $E+t\bar{j}\xi\subset\Omega$ we have

$$\int_E|\tau(u_k(x+t\bar{j}\xi)\cdot\xi)-\tau(u_k(x)\cdot\xi)|\mathrm{d}x\leq C_Et\bar{j}(1+F_{t\bar{j},\xi}(u_k,E)). \tag{4.27}$$

Choose $t_\alpha\in(0,1)$ sufficiently small so that

$$F\subset E+\bar{j}t\xi\subset\Omega\qquad\text{for every }\xi\in(B_4(0)\setminus B_{1/4}(0))\text{ and every }t\in[0,t_\alpha], \tag{4.28}$$

$$C_E \bar{j} t_\alpha (1+M) \leq \frac{\alpha}{4}.$$

Let $\bar{k} \in \mathbb{N}$ sufficiently large and let us set $t_k := i_k \varepsilon_k \in (\frac{t_\alpha}{2}, t_\alpha)$ for $k \geq \bar{k}$ and for some $i_k \in \mathbb{N}$. Thanks to (4.28), we apply (4.27) for $t = t_k$ and Lemma 4.5 with the choice $m = \bar{j} i_k$, obtaining

$$\begin{aligned}
&\int_{E\times(B_4(0)\setminus B_{1/4}(0))} |\tau(u_k(x+\bar{j}t_k\xi)\cdot\xi) - \tau(u_k(x)\cdot\xi)| \mathrm{d}x\,\mathrm{d}\xi \\
&\qquad = \int_{E\times(B_4(0)\setminus B_{1/4}(0))} |\tau(u_k(x+\bar{j}i_k\varepsilon_k\xi)\cdot\xi) - \tau(u_k(x)\cdot\xi)| \mathrm{d}x\,\mathrm{d}\xi \\
&\qquad \leq C_E \bar{j} i_k \varepsilon_k \int_{(B_4(0)\setminus B_{1/4}(0))} (1 + F_{\bar{j}i_k\varepsilon_k,\xi}(u_k, E))\mathrm{d}\xi \\
&\qquad \leq C_E \bar{j} i_k \varepsilon_k (1 + \mathcal{F}^1_{\varepsilon_k}(u_k, \Omega)).
\end{aligned}$$

Summarizing we have thus obtained for every $k \geq \bar{k}$ and every $\eta \in B_1(0)$

$$\begin{aligned}
\int_{F\times R} |f_k(x+t_k\eta, \xi^{\bar{j}}_\eta) - f_k(x - \bar{j}t_k\xi, \xi^{\bar{j}}_\eta)| \mathrm{d}x\,\mathrm{d}\xi &\leq C_E \bar{j} i_k \varepsilon_k (1 + \mathcal{F}^1_{\varepsilon_k}(u_k, \Omega)) \\
&\leq C_E \bar{j} t_\alpha (1+M) \leq \frac{\alpha}{4}.
\end{aligned} \tag{4.29}$$

From the very same argument, we infer that for every $k \geq \bar{k}$ and every $\eta \in B_1(0)$

$$\int_{F\times R} |f_k(x - \bar{j}t_k\xi, \xi) - f_k(x,\xi)| \mathrm{d}x\,\mathrm{d}\xi \leq C_E \bar{j} i_k \varepsilon_k (1 + \mathcal{F}^1_{\varepsilon_k}(u_k, \Omega)) \leq \frac{\alpha}{4}\,. \tag{4.30}$$

Combining (4.25), (4.26), (4.29), (4.30), we have shown that for every $k \geq \bar{k}$ and every $\eta \in B_1(0)$ it holds

$$\int_{F\times R} f_k(x+t_k\eta, \xi) - f(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \leq \alpha\,, \tag{4.31}$$

where we recall that $t_k \in (\frac{t_\alpha}{2}, t_\alpha)$. Finally, setting $\bar{t} := t_\alpha/2$, (4.31) yields

$$\int_R \int_F |f_k(x+\eta,\xi) - f_k(x,\xi)| \mathrm{d}x\,\mathrm{d}\xi \leq \alpha,$$

for every $k \geq \bar{k}$ and every $\eta \in B_{\bar{t}}(0)$, which is precisely (4.20).

Step 2: Now we prove that for every $\alpha > 0$ there exists $\bar{t} \in (0,1)$ such that

$$\int_{F\times R} |f_k(x, \xi+\eta) - f_k(x,\xi)|\,\mathrm{d}x\,\mathrm{d}\xi \leq \alpha \qquad \text{for } \eta \in B_{\bar{t}}(0) \text{ and } k \in \mathbb{N}. \tag{4.32}$$

Fix $\eta \in \mathbb{S}^{n-1}$ and let $t \in (0,1)$. By arguing as in (4.23) we have that for $x \in F$

$$\begin{aligned}
&|\tau(u_k(x)\cdot(\xi+t\eta)) - \tau(u_k(x)\cdot\xi)| \\
&\leq t\,\max\Big\{\frac{1}{1+|u_k(x)\cdot\xi|^2}, \frac{1}{1+|u_k(x)\cdot(\xi+t\eta)|^2}\Big\}|u_k(x)\cdot\eta|.
\end{aligned}$$

From this last inequality we deduce from Lemma 4.1 that

$$\int_R |\tau(u_k(x)\cdot(\xi+t\eta)) - \tau(u_k(x)\cdot\xi)|\,\mathrm{d}\xi \leq Ct\,, \tag{4.33}$$

for a positive constant C only depending on the dimension n. By (4.33) there exists $\bar{t} \in (0,1)$ such that for every $x \in F$, $k \in \mathbb{N}$, $\eta \in \mathbb{S}^{n-1}$, and every $t \in [0, \bar{t}]$

$$\int_R |f_k(x, \xi+t\eta) - f_k(x,\xi)|\,\mathrm{d}\xi \leq \alpha. \tag{4.34}$$

Eventually, (4.34) leads to (4.32).

Thanks to (4.19) and (4.32), it is not difficult to see that we are in position to apply Fréchet-Kolmogorov Theorem (see, e.g., [9, Theorem 4.26]) and obtain that the sequence $\{f_k\}_{k\in\mathbb{N}}$ is relatively compact in $L^1(F\times R)$ for every open set F compactly contained in Ω. Therefore, up to passing to a subsequence, we have $f_k \to f_\infty$ as $k\to\infty$ strongly in $L^1_{loc}(\Omega\times R)$. By a diagonal argument we can possibly pass to another subsequence such that

$$f_k \to f_\infty \qquad \text{pointwise a.e. in } \Omega\times R. \tag{4.35}$$

From (4.35), we deduce the existence of an orthonormal basis $\{\eta_1,\dots,\eta_n\}$ of $\mathbb{R}^n$ such that

$$\tau(u_k\cdot\eta_i)\to f^i_\infty \qquad \text{pointwise a.e. in } \Omega \text{ for every } i=1,\dots,n \text{ as } k\to\infty,$$

for some measurable function $f^i_\infty\colon\Omega\to[-\pi/2,\pi/2]$. Since arctan is a diffeomorphism with its image, we deduce the existence of a measurable function $u\colon\Omega\to\mathbb{R}^n$ such that $u=0$ on A and

$$u_k\to u \qquad \text{a.e. in } \Omega\setminus A \text{ as } k\to\infty \tag{4.36}$$

$$|u_k|\to\infty \qquad \text{a.e. in } A \text{ as } k\to\infty, \tag{4.37}$$

where A is the set defined in (1.6). Eventually, up to passing to a subsequence, we also assume that $u_k\to u$ in measure in $\Omega\setminus A$. We further remark that

$$x\in A \iff \lim_{k\to\infty}|u_k(x)\cdot\xi|=\infty \quad \text{for a.e. } \xi\in\mathbb{R}^n. \tag{4.38}$$

Step 3: Let $\lambda>0$ and let $\tau_\lambda\colon\mathbb{R}\to[-\lambda,\lambda]$ be a strictly increasing function such that $\tau'_\lambda\le 1$, $\lim_{t\to\pm\infty}\tau_\lambda(t)=\pm\lambda$, and $\tau_\lambda(t)=t$ for $t\in[-\lambda/2,\lambda/2]$. Fix $\xi\in\mathbb{R}^n$, and let $y\in\Pi^\xi$. For every $\lambda>0$, we want to construct suitable modifications $\{v_{j,\lambda}\}_{j\in\mathbb{N}}\subset \mathrm{SBV}(\Omega^\xi_y)$ of $(\hat{u}_k)^\xi_{y,\lambda}(t):=\tau_\lambda(u_k(y+t\xi)\cdot\xi)$, as in [26, Theorem 3.4, Step 2], such that

$$v_{j,\lambda}\to\hat{u}^\xi_{y,\lambda} \text{ in } L^1_{loc}(\Omega^\xi_y) \text{ as } j\to\infty, \tag{4.39}$$

and

$$\liminf_{k\to\infty} F_{\varepsilon_k}\big((\hat{u}_k)^\xi_{y,\lambda}, I\cap(I-\varepsilon_k)\big)\ge\frac{\pi}{2}\mathcal{MS}_{\frac{2}{\pi}}(v_{j,\lambda},I) \tag{4.40}$$

for every j sufficiently large and every interval $I\Subset\Omega^\xi_y$, where $\hat{u}^\xi_{y,\lambda}(t):=\tau_\lambda(u(y+t\xi)\cdot\xi)$ for $y+t\xi\in\Omega\setminus A$ and $\hat{u}^\xi_{y,\lambda}(t)=\pm\lambda$ for $y+t\xi\in A$.

We notice that by applying Fubini's Theorem for a.e. ξ we have

$$\mathcal{L}^n(\{x\in A:\liminf_{k\to\infty}|u_k(x)\cdot\xi|<\infty\})=0.$$

Then thanks to (4.36)–(4.37), up to a subsequence which does not depend on ξ and y we have (for every $\lambda>0$)

$$(\hat{u}_k)^\xi_{y,\lambda}\to\hat{u}^\xi_{y,\lambda} \qquad \text{in } L^1_{loc}((\Omega\setminus A)^\xi_y) \text{ for every } \xi \text{ and for } \mathcal{H}^n\text{-a.e. } y\in\Pi^\xi, \tag{4.41}$$

$$|(\hat{u}_k)^\xi_{y,\lambda}|\to|\hat{u}^\xi_{y,\lambda}|=\lambda \qquad \text{in } L^1_{loc}(A^\xi_y) \text{ for a.e. } \xi, \text{ for } \mathcal{H}^{n-1}\text{-a.e. } y\in\Pi^\xi, \tag{4.42}$$

as $k\to\infty$. We further observe that, since $0\le\tau'_\lambda\le 1$, then $|\tau_\lambda(t)-\tau_\lambda(t')|\le|t-t'|$ for every $t,t'\in\mathbb{R}$, leading to

$$F_\varepsilon(f,B)\ge F_\varepsilon(\tau_\lambda(f),B), \tag{4.43}$$

for every $\varepsilon>0$, every measurable set $B\subset\mathbb{R}$, and every measurable function $f\colon B\to\mathbb{R}$.

We extend the function $\hat{u}^\xi_{y,\lambda}$ to $\mathbb{R}$ in such a way that $\hat{u}^\xi_{y,\lambda}(t)=0$ for $t\in\mathbb{R}\setminus\Omega^\xi_y$. Let $a\in\mathbb{R}$ satisfy conditions (i) and (ii) of Lemma 4.3 for $\hat{u}^\xi_{y,\lambda}$, and let $I^z_j:=[a+\frac{z}{j},a+\frac{z+1}{j}]\Subset\mathbb{R}$. We define $v_{j,\lambda}$ in every interval I^z_j in the following way:

- If $j(\hat{u}^\xi_{y,\lambda}(a+\frac{z+1}{j})-\hat{u}^\xi_{y,\lambda}(a+\frac{z}{j}))^2 \le \frac{\pi}{2}$, then $v_{j,\lambda}$ is the affine function that coincides with $\hat{u}^\xi_{y,\lambda}$ at the endpoints of I^z_j;
- if $j(\hat{u}^\xi_{y,\lambda}(a+\frac{z+1}{j})-\hat{u}^\xi_{y,\lambda}(a+\frac{z}{j}))^2 > \frac{\pi}{2}$, then $v_{j,\lambda}$ is the piecewise constant function that coincides with $\hat{u}^\xi_{y,\lambda}$ at the endpoints of I^z_j and has a unique discontinuity in the middle point of the interval.

Then by Lemma 4.3, up to subsequences, (4.39) holds. Moreover, $v_{j,\lambda} \in \mathrm{SBV}(\Omega^\xi_y)$ and

$$\frac{\pi}{2}\mathcal{MS}_{\frac{2}{\pi}}(v_{j;\lambda}, I^z_j) = \min\left\{\frac{\pi}{2}, j\left(\hat{u}^\xi_{y,\lambda}\left(a+\frac{z+1}{j}\right)-\hat{u}^\xi_{y,\lambda}\left(a+\frac{z}{j}\right)\right)^2\right\}$$

for every $j \in \mathbb{N}$ and $z \in \mathbb{Z}$ such that $I^z_j \Subset \Omega^\xi_y$.

Let $I \subset \Omega^\xi_y$ be an interval and $F \Subset I \cap (I-\varepsilon_k)$ for k sufficiently large, note that such F exists since $I \cap (I-\varepsilon_k) \to I$ as $k \to \infty$. We notice that for every j sufficiently large, depending on F, every intervals I^z_j such that $I^z_j \subset I \cap (I-\varepsilon_k)$ also satisfy $F \subset \cup_z I^z_j$. By virtue of (4.41) and (4.42) we can make use of Lemma 4.2 in I^z_j and infer, for a.e. $\xi \in \mathbb{R}^n$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, that

$$\liminf_{k\to\infty} F_{\varepsilon_k}((\hat{u}_k)^\xi_{y,\lambda}, I^z_j) \ge \frac{\pi}{2}\mathcal{MS}_{\frac{2}{\pi}}(v_{j,\lambda}, I^z_j).$$

To prove (4.40) we sum over all z such that $I^z_j \subset I \cap (I-\varepsilon_k)$ for every j sufficiently large

$$\liminf_{k\to\infty} F_{\varepsilon_k}((\hat{u}_k)^\xi_{y,\lambda}, I \cap (I-\varepsilon_k)) \ge \frac{\pi}{2}\mathcal{MS}_{\frac{2}{\pi}}(v_{j,\lambda}, F).$$

From the semicontinuity of the Mumford-Shah functional with respect to the L^1_{loc}-convergence, taking the limit as $j \to \infty$ and then as $F \nearrow I$, we obtain

$$\liminf_{k\to\infty} F_{\varepsilon_k}\big((\hat{u}_k)^\xi_{y,\lambda}, I \cap (I-\varepsilon_k)\big) \ge \frac{\pi}{2}\mathcal{MS}_{\frac{2}{\pi}}(\hat{u}^\xi_{y,\lambda}, I). \tag{4.44}$$

We recall that

$$\begin{aligned}\mathcal{F}^1_\varepsilon(u,\Omega) &= \sup_{\mathscr{B}} \sum_{B\in\mathscr{B}} \int_{\frac{\Omega-\Omega}{\varepsilon}} F_{\varepsilon,\xi}(u,B)e^{-|\xi|^2}\mathrm{d}\xi \\ &= \sup_{\mathscr{B}} \sum_{B\in\mathscr{B}} \int_{\frac{\Omega-\Omega}{\varepsilon}} \left(\int_{\Pi^\xi} F_\varepsilon(\hat{u}^\xi_y, B^\xi_y \cap (B-\varepsilon\xi)^\xi_y)\mathrm{d}\mathcal{H}^{n-1}(y)\right)|\xi|e^{-|\xi|^2}\mathrm{d}\xi.\end{aligned}$$

From (4.43), Fatou's Lemma, and (1.5), we get

$$\begin{aligned}&\sup_{\mathscr{B}} \sum_{B\in\mathscr{B}} \int_{\mathbb{R}^n}\int_{\Pi^\xi} \liminf_{k\to\infty} F_{\varepsilon_k}((\hat{u}_k)^\xi_{y,\lambda}, B^\xi_y \cap (B-\varepsilon_k\xi)^\xi_y)|\xi|e^{-|\xi|^2}\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}\xi \\ &\le \sup_{\mathscr{B}} \sum_{B\in\mathscr{B}} \int_{\mathbb{R}^n}\int_{\Pi^\xi} \liminf_{k\to\infty} F_{\varepsilon_k}((\hat{u}_k)^\xi_{y}, B^\xi_y \cap (B-\varepsilon_k\xi)^\xi_y)|\xi|e^{-|\xi|^2}\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}\xi \\ &\le \liminf_{k\to\infty} \sup_{\mathscr{B}} \sum_{B\in\mathscr{B}} \int_{\frac{\Omega-\Omega}{\varepsilon_k}}\int_{\Pi^\xi} F_{\varepsilon_k}((\hat{u}_k)^\xi_{y}, B^\xi_y \cap (B-\varepsilon_k\xi)^\xi_y)|\xi|e^{-|\xi|^2}\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}\xi \\ &\le \liminf_{k\to\infty} \mathcal{F}^1_{\varepsilon_k}(u_k,\Omega) \le M,\end{aligned}$$

which yields for every open ball $B \subset \Omega$

$$\liminf_{k\to\infty} F_{\varepsilon_k}((\hat{u}_k)^\xi_{y,\lambda}, B^\xi_y \cap (B-\varepsilon_k\xi)^\xi_y) \le C \quad \text{for a.e. } \xi \in \mathbb{R}^n, \text{ for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi \tag{4.45}$$

where the constant $C > 0$ may depend on ξ and y but not on λ and B. The above inequality combined with (4.44) implies that $\hat{u}^\xi_{y,\lambda} \in \mathrm{SBV}_{loc}(\Omega^\xi_y)$ for a.e. $\xi \in \mathbb{R}^n$ and $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$.

Step 4: We claim that for a.e. $\xi \in \mathbb{R}^n$, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, and for every bounded interval $I \subset \Omega^\xi_y$ we have

$$\mathcal{H}^0(A^\xi_y \cap I) \neq 0 \implies \mathcal{H}^0(J_{\hat{u}^\xi_{y,\lambda}} \cap I) \neq 0 \text{ or } I \subset A^\xi_y. \tag{4.46}$$

If $I \subset A^\xi_y$ we have nothing to prove. Let us assume that there exists $s \in I \setminus A^\xi_y$. By contradiction, assume $\mathcal{H}^0(J_{\hat{u}^\xi_{y,\lambda}} \cap I) = 0$. For a.e. $\xi \in \mathbb{R}^n$, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, by (4.44) and (4.45) we get

$$\int_I |\nabla \hat{u}^\xi_{y,\lambda}(t)|^2 \,\mathrm{d}t \leq C. \tag{4.47}$$

Since $\hat{u}^\xi_{y,\lambda}$ belongs to $\mathrm{SBV}(I)$ and since $\mathcal{H}^0(J_{\hat{u}^\xi_{y,\lambda}} \cap I) = 0$, it is absolutely continuous in I. Hence, for $\bar{t} \in I$ we have

$$\begin{aligned}
|\hat{u}^\xi_{y,\lambda}(\bar{t})| &= \left| \hat{u}^\xi_{y,\lambda}(s) + \int_s^t \nabla \hat{u}^\xi_{y,\lambda}(t)\,\mathrm{d}t \right| \\
&\leq |\hat{u}^\xi_{y,\lambda}(s)| + \left| \int_s^t |\nabla \hat{u}^\xi_{y,\lambda}(t)|\,\mathrm{d}t \right| \\
&\leq |\hat{u}^\xi_{y,\lambda}(s)| + \mathrm{lenght}(I)^{1/2} \left| \int_s^t |\nabla \hat{u}^\xi_{y,\lambda}(t)|^2\,\mathrm{d}t \right|^{1/2} \leq C,
\end{aligned}$$

for every $\bar{t} \in I$, where the last inequality follows from (4.47) and the fact that $s \in I \setminus A^\xi_y$. Since $\mathcal{H}^0(A^\xi_y \cap I) \neq 0$ and (4.38) holds, there exists $\bar{t} \in A^\xi_y \cap I$ such that $|\hat{u}^\xi_{y,\lambda}(\bar{t})| > C$ for every $\lambda > 0$ sufficiently large, which is a contradiction. This concludes the proof of (4.46).

Step 5: Let $I \subset \Omega^\xi_y$ be a bounded interval. We claim that, for a.e. $\xi \in \mathbb{R}^n$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, the set A^ξ_y is a finite union of intervals and

$$\frac{\pi}{2}\mathcal{H}^0(\partial^* A^\xi_y \cap I) \leq \liminf_{k\to\infty} F_{\varepsilon_k}((\hat{u}_k)^\xi_{y,\lambda}, I \cap (I - \varepsilon_k)). \tag{4.48}$$

Moreover, A is a set of finite perimeter.

Denote by $(A^\xi_y)^d$ the points of density d of A^ξ_y. We want to prove that $(A^\xi_y)^1$ is an open set. By contradiction, suppose that there exist $t \in (A^\xi_y)^1$ and a sequence $\{s_j\}_{j\in\mathbb{N}} \subset \Omega^\xi_y \setminus (A^\xi_y)^1$ such that $s_j \to t$. Since almost every points of $\Omega^\xi_y \setminus (A^\xi_y)^1$ belong to $(A^\xi_y)^0$, we find another sequence $\{t_j\}_{j\in\mathbb{N}} \subset (A^\xi_y)^0$ such that $t_j \to t$. Without loss of generality we assume $t^0_j < t^0_{j+1}$ for every $j \in \mathbb{N}$. By definition of points of density zero, for every j, there exists a point $t^1_j \in [t^0_j, t^0_{j+1})$ of density one. By the previous step, there exists another increasing sequence $t_j \in [t^0_j, t^0_{j+1}) \cap J_{\hat{u}^\xi_{y,\lambda}}$. Since the sequence $\{t_j\}_{j\in\mathbb{N}}$ is strictly increasing, there exist disjoint open intervals I_j such that $t_j \in I_j$. Now, by applying (4.44) and (4.45) on every I_j we obtain $\sum_j 1 \leq \sum_j \mathcal{H}^0(J_{\hat{u}^\xi_{y,\lambda}} \cap I_j) \leq C$, which is a contradiction. Hence, $(A^\xi_y)^1$ is open.

From now on we assume that A^ξ_y coincides with $(A^\xi_y)^1$. Suppose that $|I \setminus A^\xi_y| > 0$, otherwise there is nothing to prove. Let K be a connected component of $A^\xi_y \cap I$, and let $t_1 \in K$, $t_2 \in I \setminus A^\xi_y$. Then, by Step 4, there exists $t \in J_{\hat{u}^\xi_{y,\lambda}} \cap I$ between t_1 and t_2. Using a similar argument as above, we prove that $\mathcal{H}^0(J_{\hat{u}^\xi_{y,\lambda}} \cap I)$ is greater than or equal to the number of connected components of $A^\xi_y \cap I$ and is uniformly bounded, from which it follows that A^ξ_y is the union of a finite number of intervals. Finally, (4.48) is a consequence of $\mathcal{H}^0(\partial^* A^\xi_y \cap I) \le \mathcal{H}^0(J_{\hat{u}^\xi_{y,\lambda}} \cap I)$ and of (4.44).

To conclude we need to prove that A is of finite perimeter. Multiplying (4.48) by $|\xi|e^{-|\xi|^2}$ and then integrating in $\xi \in \mathbb{R}^n$ and $y \in \Pi^\xi$ we obtain for every ball $B \subset \Omega$

$$
\begin{aligned}
C(n)\mathcal{H}^{n-1}(\partial^* A \cap B) &\le \int_{\mathbb{R}^n} \Big(\int_{\Pi^\xi} \mathcal{H}^0(\partial^* A^\xi_y \cap B^\xi_y) \mathrm{d}\mathcal{H}^{n-1}(y) \Big) |\xi| e^{-|\xi|^2} \mathrm{d}\xi \\
&\le \int_{\mathbb{R}^n} \Big(\int_{\Pi^\xi} \liminf_{k\to\infty} F_{\varepsilon_k}((\hat{u}_k)^\xi_{y,\lambda}, B^\xi_y \cap B^\xi_y - \varepsilon_k) \mathrm{d}\mathcal{H}^{n-1}(y) \Big) |\xi| e^{-|\xi|^2} \mathrm{d}\xi \\
&\le \liminf_{k\to\infty} \mathcal{F}^1_{\varepsilon_k}(u_k, B),
\end{aligned}
$$

where the last inequality follows from Fatou's Lemma. This proves in particular that A has locally finite perimeter in Ω. By Besicovitch covering Theorem there exist a dimensional constant $j(n)$ and $\mathcal{B}_1, \dots, \mathcal{B}_{j(n)}$ countable families of pairwise disjoint open balls contained in Ω, such that $\Omega \subset \cup_{i=1}^{j(n)} \cup_{B\in\mathcal{B}_i} B$.

$$
\begin{aligned}
C(n)\mathcal{H}^{n-1}(\partial^* A) &\le \sum_{i=1}^{j(n)} \sum_{B\in\mathcal{B}_i} C(n)\mathcal{H}^{n-1}(\partial^* A \cap B) \\
&\le \sum_{i=1}^{j(n)} \sum_{B\in\mathcal{B}_i} \liminf_{k\to\infty} \mathcal{F}^1_{\varepsilon_k}(u_k, B) \\
&\le j(n) \liminf_{k\to\infty} \sum_{B\in\mathcal{B}_i} \mathcal{F}^1_{\varepsilon_k}(u_k, B) \\
&\le j(n) \liminf_{k\to\infty} \mathcal{F}^1_{\varepsilon_k}(u_k, \Omega) < \infty,
\end{aligned}
$$

where, in the last but one inequality above, we used Remark 2.10.

Step 6: We conclude by proving that $u \in \mathrm{GSBD}(\Omega)$ and that (1.7) holds.

Fix $\mathscr{B}$ be any finite family of disjoint balls in Ω and let $B \in \mathscr{B}$. For a.e. $\xi \in \mathbb{R}^n$, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, for every interval $I \subset B^\xi_y$ of the form $I = (t-\delta, t+\delta)$ for $t \in \partial^* A^\xi_y$ and $\delta > 0$, by (4.43), (4.44) and (4.48), we have

$$
\begin{aligned}
&\liminf_{k\to\infty} F_{\varepsilon_k}((\hat{u}_k)^\xi_y, B^\xi_y \cap (B - \varepsilon_k\xi)^\xi_y) \\
&\qquad \ge \liminf_{k\to\infty} F_{\varepsilon_k}((\hat{u}_k)^\xi_y, I \cap (B - \varepsilon_k\xi)^\xi_y) + \liminf_{k\to\infty} F_{\varepsilon_k}((\hat{u}_k)^\xi_y, B^\xi_y \setminus I \cap (B - \varepsilon_k\xi)^\xi_y) \\
&\qquad \ge \liminf_{k\to\infty} F_{\varepsilon_k}((\hat{u}_k)^\xi_{y,\lambda}, I \cap (B - \varepsilon_k\xi)^\xi_y) + \liminf_{k\to\infty} F_{\varepsilon_k}((\hat{u}_k)^\xi_{y,\lambda}, B^\xi_y \setminus I \cap (B - \varepsilon_k\xi)^\xi_y) \\
&\qquad \ge \frac{\pi}{2}\mathcal{H}^0(\partial^* A^\xi_y \cap I) + \frac{\pi}{2}\mathcal{MS}_{\frac{2}{\pi}}(\hat{u}^\xi_{y,\lambda}, B^\xi_y \setminus I) \\
&\qquad = \frac{\pi}{2}\mathcal{H}^0(\partial^* A^\xi_y \cap I) + \frac{\pi}{2}\mathcal{H}^0(J_{\hat{u}^\xi_{y,\lambda}} \setminus I) + \int_{B^\xi_y \setminus I} |\nabla \hat{u}^\xi_{y,\lambda}(t)|^2 \,\mathrm{d}t.
\end{aligned}
$$

Since the above inequality holds for every $\delta > 0$ and $t \in \partial^* A^\xi_y$, we get

$$
\liminf_{k\to\infty} F_{\varepsilon_k}((\hat{u}_k)^\xi_y, B^\xi_y \cap (B - \varepsilon_k\xi)^\xi_y) \ge \frac{\pi}{2}\mathcal{H}^0(\partial^* A^\xi_y \cup J_{\hat{u}^\xi_{y,\lambda}}) + \int_{B^\xi_y} |\nabla \hat{u}^\xi_{y,\lambda}(t)|^2 \,\mathrm{d}t.
$$

Since $\tau_\lambda(t) = t$ for $t \in [-\lambda/2, \lambda/2]$ and $J_{\hat{u}^\xi_{y,\lambda}} = J_{\hat{u}^\xi_y}$, we infer that

$$
\liminf_{k\to\infty} F_{\varepsilon_k}((\hat{u}_k)^\xi_y, B^\xi_y \cap (B - \varepsilon_k\xi)^\xi_y) \ge \frac{\pi}{2}\mathcal{H}^0(\partial^* A^\xi_y \cup J_{\hat{u}^\xi_{y,\lambda}}) + \int_{B^\xi_y} |\nabla \hat{u}^\xi_{y,\lambda}(t)|^2 \,\mathrm{d}t
$$

$$\geq \frac{\pi}{2}\mathcal{H}^0(\partial^* A^\xi_y \cup J_{\hat{u}^\xi_y}) + \int_{\left\{t\in B^\xi_y : |\hat{u}^\xi_{y,\lambda}(t)|\leq \lambda/2\right\}} |\nabla \hat{u}^\xi_y(t)|^2 \, \mathrm{d}t.$$

Now, by sending $\lambda \to \infty$ we get for every $c \in \mathbb{R}$

$$\begin{aligned}\liminf_{k\to\infty} F_{\varepsilon_k}((\hat{u}_k)^\xi_y, B^\xi_y \cap (B-\varepsilon_k\xi)^\xi_y) &\geq \frac{\pi}{2}\mathcal{H}^0(\partial^* A^\xi_y \cup J_{\hat{u}^\xi_y}) + \int_{B^\xi_y} |\nabla \hat{u}^\xi_y(t)|^2 \, \mathrm{d}t \\ &\geq \frac{\pi}{2}\mathcal{MS}_{\frac{2}{\pi}}(\hat{u}^\xi_y(1-\chi_{A^\xi_y}) + c\chi_{A^\xi_y}, B^\xi_y).\end{aligned}$$

By combining the above inequality with Fatou's Lemma we deduce

$$\begin{aligned}&\sup_{\mathscr{B}} \sum_{B\in\mathscr{B}} \frac{\pi}{2}\left(\int_{\mathbb{R}^n}\left(\int_{\Pi^\xi} \mathcal{MS}_{\frac{2}{\pi}}(\hat{u}^\xi_y(1-\chi_{A^\xi_y}) + c\chi_{A^\xi_y}, B^\xi_y)\mathrm{d}\mathcal{H}^{n-1}(y)\right)^p |\xi|^p e^{-|\xi|^2}\mathrm{d}\xi\right)^{\frac{1}{p}} \\ &\leq \liminf_{k\to\infty}\sup_{\mathscr{B}} \sum_{B\in\mathscr{B}} \left(\int_{\frac{\Omega-\Omega}{\varepsilon_k}}\left(\int_{\Pi^\xi} F_{\varepsilon_k}((\hat{u}_k)^\xi_y, B^\xi_y\cap(B-\varepsilon_k\xi)^\xi_y)\mathrm{d}\mathcal{H}^{n-1}(y)\right)^p |\xi|^p e^{-|\xi|^2}\mathrm{d}\xi\right)^{\frac{1}{p}} \\ &= \liminf_{k\to\infty} \mathcal{F}^p_{\varepsilon_k}(u_k,\Omega) < \infty,\end{aligned}$$

which implies that $u \in \mathrm{GSBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$ and $\hat{\mu}^p_u(\Omega) < \infty$. Thus by Theorem 3.1 we conclude that $u \in \mathrm{GSBD}(\Omega)$. In addition, recall that, by Lemma 4.6, we have for every $c \in \mathbb{R}$

$$\begin{aligned}\sup_{\mathscr{B}} \sum_{B\in\mathscr{B}} \frac{\pi}{2}&\left(\int_{\mathbb{R}^n}\left(\int_{\Pi^\xi} \mathcal{MS}_{\frac{2}{\pi}}(\hat{u}^\xi_y(1-\chi_{A^\xi_y}) + c\chi_{A^\xi_y}, B^\xi_y)\mathrm{d}\mathcal{H}^{n-1}(y)\right)^p |\xi|^p e^{-|\xi|^2}\mathrm{d}\xi\right)^{\frac{1}{p}} \\ &= \int_\Omega \varphi_p(e(u_c(1-\chi_A)))\mathrm{d}x + \beta_p\mathcal{H}^{n-1}(J_{u_c}),\end{aligned}$$

where $u_c\colon \Omega \to \mathbb{R}^n$ is defined as $u_c := u(1-\chi_A) + c\chi_A$. Since a measure theoretic argument yields that $\partial^* A \subset J_{u_c}$ for a.e. $c \in \mathbb{R}$, inequality (1.7) follows.

5. Γ-convergence and convergence of quasi-minimisers

This section is devoted to the proof of Theorem 1.2 as well as to the convergence, up to subsequences, of quasi-minimizers of $\mathcal{F}^p_\varepsilon$ under Dirichlet boundary conditions. The former is performed in Section 5.1, the latter in Section 5.2.

5.1. Γ-convergence.

We start with the following pointwise convergence result following the strategy in [26, Theorem 3.4].

Proposition 5.1. *Let $I \subset \mathbb{R}$ be an interval and let $u \in \mathrm{GSBV}(I)$. For every $\varepsilon > 0$, it holds*

$$F_\varepsilon(u, I\cap(I-\varepsilon)) \leq \frac{\pi}{2}\mathcal{MS}_{\frac{2}{\pi}}(u, I). \tag{5.1}$$

Proof. Since I is an interval, the set $I\cap(I-\varepsilon)$ is also an interval or it is empty. Assume that it is not empty otherwise there is nothing to prove. Define $A^\varepsilon := \{t \in I\cap(I-\varepsilon) : [t,t+\varepsilon]\cap J_u \neq \emptyset\}$. Notice that $t \in I\cap(I-\varepsilon)$ implies $[t,t+\varepsilon] \subset I$. We have

$$\frac{1}{\varepsilon}\int_{A_\varepsilon} \arctan\left(\frac{(u(t+\varepsilon)-u(t))^2}{\varepsilon}\right)\mathrm{d}t \leq \frac{\pi}{2\varepsilon}|A_\varepsilon| \leq \frac{\pi}{2}\mathcal{H}^0(J_u)\,.$$

Since $\arctan x \leq x$ for $x \geq 0$, we have

$$\frac{1}{\varepsilon}\int_{(I\cap(I-\varepsilon))\setminus A_\varepsilon} \arctan\left(\frac{(u(t+\varepsilon)-u(t))^2}{\varepsilon}\right)\mathrm{d}t$$

$$\leq \frac{1}{\varepsilon}\int_{(I\cap(I-\varepsilon))\setminus A_\varepsilon} \arctan\left(\int_t^{t+\varepsilon} |\nabla u(\tau)|^2 \, d\tau\right) dt$$

$$\leq \frac{1}{\varepsilon}\int_{(I\cap(I-\varepsilon))\setminus A_\varepsilon} \int_t^{t+\varepsilon} |\nabla u(\tau)|^2 \, d\tau \, dt \leq \int_I |\nabla u(t)|^2 dt.$$

By combining the above two inequalities we conclude the proof. □

We are now in a position to conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. We notice that the Γ-liminf inequality is a direct consequence of Theorem 1.1 (cf. (1.7)). To conclude for the Γ-convergence, it is enough to show that for every $u \in L^0(\Omega;\mathbb{R}^n)$ it holds true

$$\lim_{\varepsilon\to 0} \mathcal{F}^p_\varepsilon(u,\Omega) \leq \mathcal{F}^p(u,\Omega)\,. \tag{5.2}$$

We begin by recalling that

$$\mathcal{F}^p_\varepsilon(u,\Omega) = \sup_{\mathscr{B}} \sum_{B\in\mathscr{B}} \left(\int_{\frac{\Omega-\Omega}{\varepsilon}} F_{\varepsilon,\xi}(u,B)^p e^{-|\xi|^2} d\xi\right)^{\frac{1}{p}}$$

$$= \sup_{\mathscr{B}} \sum_{B\in\mathscr{B}} \left(\int_{\frac{\Omega-\Omega}{\varepsilon}} \left(\int_{\Pi^\xi} F_\varepsilon(\hat{u}^\xi_y, B^\xi_y \cap (B-\varepsilon\xi)^\xi_y) dy\right)^p |\xi|^p e^{-|\xi|^2} d\xi\right)^{\frac{1}{p}},$$

where the supremum is taken over the set of finite families of disjoint open balls contained in Ω. Then, since $(B-\varepsilon\xi)^\xi_y = B^\xi_y - \varepsilon$ and since $B^\xi_y \subset \mathbb{R}$ is an interval, we combine the above equation with (5.1) and we deduce

$$\limsup_{\varepsilon\to 0} \mathcal{F}^p_\varepsilon(u,\Omega) \leq \limsup_{\varepsilon\to 0} \sup_{\mathscr{B}} \sum_{B\in\mathscr{B}} \frac{\pi}{2} \left(\int_{\frac{\Omega-\Omega}{\varepsilon}} \left(\int_{\Pi^\xi} \mathcal{MS}_{\frac{2}{\pi}}(\hat{u}^\xi_y, B^\xi_y) dy\right)^p |\xi|^p e^{-|\xi|^2} d\xi\right)^{\frac{1}{p}}$$

$$\leq \sup_{\mathscr{B}} \sum_{B\in\mathscr{B}} \frac{\pi}{2} \left(\int_{\mathbb{R}^n} \left(\int_{\Pi^\xi} \mathcal{MS}_{\frac{2}{\pi}}(\hat{u}^\xi_y, B^\xi_y) dy\right)^p |\xi|^p e^{-|\xi|^2} d\xi\right)^{\frac{1}{p}}.$$

Thanks to Lemma 4.6 we infer (5.2). This concludes the proof of the theorem. □

5.2. **Convergence of quasi-minimisers.** For the purposes of this subsection, we fix two open sets $\Omega \subset \Omega' \subset \mathbb{R}^n$. We further assume that $\partial_D\Omega$, namely, the Dirichlet part of the boundary, satisfies $\partial_D\Omega = \partial\Omega \cap \Omega'$. As it is customary in free discontinuity problems, we consider a relaxed boundary condition on $\partial_D\Omega$. To this purpose, our Dirichlet datum is given by a function $f\colon \partial_D\Omega \to \mathbb{R}^n$ which we identify, with a slight abuse of notation, with the trace on $\partial_D\Omega$ of a function $f \in H^1(\Omega';\mathbb{R}^n)$. The domain of our functionals is thus denoted by $L^0_f(\Omega';\mathbb{R}^n)$ and defined as

$$L^0_f(\Omega';\mathbb{R}^m) := \big\{u \in L^0(\Omega';\mathbb{R}^n) : u = f \text{ a.e. in } \Omega'\setminus\Omega\big\}.$$

In addition we define for every $\varepsilon > 0$ the functionals $\mathcal{F}^f_\varepsilon : L^0(\Omega';\mathbb{R}^n) \to [0,\infty]$ as

$$\mathcal{F}^{p,f}_\varepsilon(u,\Omega') := \begin{cases} \mathcal{F}^p_\varepsilon(u,\Omega') & \text{if } u \in L^0_f(\Omega';\mathbb{R}^n) \\ +\infty & \text{otherwise in } L^0(\Omega';\mathbb{R}^n) \end{cases}$$

and the limit functional $\mathcal{F}^{p,f} : L^0(\Omega';\mathbb{R}^n) \to [0,\infty]$ as

$$\mathcal{F}^{p,f}(u,\Omega') := \begin{cases} \mathcal{F}^p(u,\Omega') & \text{if } u \in \mathrm{GSBD}(\Omega') \cap L^0_f(\Omega';\mathbb{R}^n) \\ +\infty & \text{otherwise in } L^0(\Omega';\mathbb{R}^n). \end{cases}$$

We further observe that the functional $\mathcal{F}^{p,f}$ takes the form

$$\mathcal{F}^{p,f}(u,\Omega') = \int_{\Omega'} \varphi_p(e(u))\mathrm{d}x + \beta_p\Big(\mathcal{H}^{n-1}(J_u\cap\Omega) + \mathcal{H}^{n-1}(J_u\cap\partial_D\Omega)\Big),$$

and that the term $\mathcal{H}^{n-1}(J_u\cap\partial_D\Omega)$ penalizes the part of $\partial_D\Omega$ on which the Dirichlet condition is not attained, namely, the set $\{x\in\partial_D\Omega : u^-(x)\neq f(x)\}$ where $u^-(x)$ denotes the trace with respect to the inner normal to $\partial\Omega$ of u at x .

We have the following theorem.

Theorem 5.2. *Let $\Omega,\Omega'\subset\mathbb{R}^n$ be two open sets, let $\partial_D\Omega=\partial\Omega\cap\Omega'$ be the Dirichlet part of the boundary, and let $f\in H^1(\Omega';\mathbb{R}^n)$ be an admissible Dirichlet datum as above. Assume that $\{u_\varepsilon\}_{\varepsilon>0}$ is a family of quasi-minimisers for $\{\mathcal{F}^{p,f}_\varepsilon\}_{\varepsilon>0}$, namely,*

$$\lim_{\varepsilon\to0}\Big(\mathcal{F}^{p,f}_\varepsilon(u_\varepsilon,\Omega') - \inf_{w\in L^0(\Omega';\mathbb{R}^n)}\mathcal{F}^{p,f}_\varepsilon(w,\Omega')\Big)=0\,. \tag{5.3}$$

Then, there exist a minimizer $u\in \mathrm{GSBD}(\Omega')\cap L^0_f(\Omega';\mathbb{R}^n)$ of $\mathcal{F}^{p,f}$ and a subsequence $\varepsilon_k\to0$ as $k\to\infty$ such that $u_{\varepsilon_k}\to u$ almost everywhere in Ω'.

Proof. By (5.2) we have that $\sup_{\varepsilon>0}\mathcal{F}^{p,f}_\varepsilon(f,\Omega')<\infty$. Hence, condition (5.3) yields $\sup_{\varepsilon>0}\mathcal{F}^{p,f}_\varepsilon(u_\varepsilon,\Omega')<\infty$. We are thus in position to apply Theorem 1.1 to infer the existence of a subsequence $\varepsilon_k\to0$ as $k\to\infty$ and a finite perimeter set $A\subset\Omega'$ such that $A=\{x\in\Omega' : |u_{\varepsilon_k}|\to\infty \text{ as } k\to\infty\}$, $u_{\varepsilon_k}\to u$ pointwise a.e. in $\Omega'\setminus A$ for $u\in\mathrm{GSBD}(\Omega')$ with $u=0$ in A. In addition, (1.7) is fulfilled. Since $u_\varepsilon=f$ a.e. in $\Omega'\backslash\Omega$ for every $\varepsilon>0$, we deduce $A\subset\Omega$.

It remains to prove that u is actually a minimiser of $\mathcal{F}^{p,f}(\cdot,\Omega')$. We observe that we cannot immediately infer that u is a minimiser from the Γ-convergence given by Theorem 1.2, since the sequence u_{ε_k} does not convergence in measure to u in the whole of Ω'. Nevertheless, the proof can be achieved with the very same argument as in the standard case. Indeed, take $w\in L^0(\Omega';\mathbb{R}^n)$. Since we want to prove that $\mathcal{F}^{p,f}(w,\Omega')\geq\mathcal{F}^{p,f}(u,\Omega')$, we assume with no loss of generality that $w\in\mathrm{GSBD}(\Omega')\cap L^0_f(\Omega';\mathbb{R}^n)$. By virtue of (1.7), (5.2), and (5.3), we estimate

$$\begin{aligned}
\mathcal{F}^{p,f}(w,\Omega') &= \lim_{k\to\infty}\mathcal{F}^{p,f}_{\varepsilon_k}(w,\Omega')\geq\limsup_{k\to\infty}\inf_{v\in L^0(\Omega';\mathbb{R}^m)}\mathcal{F}^{p,f}_\varepsilon(v,\Omega')\\
&=\limsup_{k\to\infty}\mathcal{F}^{p,f}_\varepsilon(u_{\varepsilon_k},\Omega')\geq\int_{\Omega'}\varphi_p(e(u))\,\mathrm{d}x+\beta_p\mathcal{H}^{n-1}(J_u\cup\partial^*A)\\
&\geq\int_{\Omega'}\varphi_p(e(u))\,\mathrm{d}x+\beta_p\mathcal{H}^{n-1}(J_u)\\
&=\int_{\Omega'}\varphi_p(e(u))\,\mathrm{d}x+\beta_p\Big(\mathcal{H}^{n-1}(J_u\cap\Omega)+\mathcal{H}^{n-1}(J_u\cap\partial_D\Omega)\Big)=\mathcal{F}^{p,f}(u,\Omega').
\end{aligned}$$

Thanks to the arbitrariness of w we deduce that $\partial^*A\subset J_u$ and that u is a minimizer of $\mathcal{F}^{p,f}$. □

6. The case $p=1$

In this final section we recall and discuss the following problem which has already been formulated in the introduction.

Open Problem. *Let $u\colon\mathbb{R}^n\to\mathbb{R}^n$ be a measurable vector field, and define the Borel regular measure $\mathscr{I}_{u,1}$ in $\mathbb{R}^n$ as*

$$\mathscr{I}_{u,1}(B):=\int_{\mathbb{S}^{n-1}}\Big(\int_{\Pi^\xi}\sum_{t\in B^\xi_y}|[\hat{u}^\xi_y](t)|\wedge1\,d\mathcal{H}^{n-1}(y)\Big)d\mathcal{H}^{n-1}(\xi),\qquad B\subset\mathbb{R}^n \textit{ Borel.}$$

If $u \in \mathrm{GBV}^{\mathcal{E}}(\mathbb{R}^n)$, is it true that $\mathscr{I}_{u,1}$ is an $(n-1)$-rectifiable measure?

The consequence of an affirmative answer to the above problem is that for every $u \in \mathrm{GBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$ and for $\mathcal{H}^{n-1}$-a.e $\xi \in \mathbb{S}^{n-1}$

$$(J_u)^\xi_y = J_{\hat{u}^\xi_y}, \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi. \tag{6.1}$$

The key tool giving (6.1) from the above open problem, is a trace property for functions having bounded directional variation contained in [21, Theorem 5.1]. Loosely speaking, for a function $v\colon \Omega \to \mathbb{R}$ which is L^1-integrable and such that $D_\xi v$ is a measure, there exists a notion of traces $v^\pm$ a.e. on both sides of graphs which are given by Lipschitz functions of the form $g\colon \Pi^\xi \to \mathbb{R}$. Moreover, $v^\pm(y+t\xi)$ coincides with $v^{\xi\pm}_y(t)$ (up to a change of sign) for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ and for every $t \in \mathrm{graph}(g)^\xi_y$. This can be rigorously proved by following [21, Section 5].

Condition (6.1) is the key missing component needed to establish the desired Γ-convergence for $p=1$. In the language of Theorem 1.3, this is equivalent to proving that $\mathcal{I}^s_u = 0$. This measure is finite and absolutely continuous with respect to the purely $(n-1)$-unrectifiable part of $\mathscr{I}_{u,1}$. Moreover, it assigns zero mass to any set that is σ-finite with respect to the $(n-1)$-dimensional Hausdorff measure. At the same time, it holds that $\mathcal{I}^s_u \perp \mathcal{L}^n$. While we believe the above open problem to be true, as suggested in [29], we cannot a priori exclude that $\mathscr{I}_{u,1}$ might concentrate on a purely $(n-1)$-unrectifiable set. Unfortunately, the techniques presented in this paper are insufficient to rule out this last scenario.

We dedicate this last part of the section to explain how Theorem 1.3 can be derived plus a few further comments. We observe that the results provided by Theorems 1.1 and 1.2 still hold for $p=1$ by replacing the space GSBD with $\mathrm{GSBV}^{\mathcal{E}}$ and the limit functional $\mathcal{F}^p$ with $\mathcal{F}$, this latter being defined as

$$\mathcal{F}(u,\Omega) := \begin{cases} \int_{\mathbb{R}^n} \left(\int_{\Pi^\xi} \mathcal{MS}_{\frac{2}{\pi}}(\hat{u}^\xi_y, \Omega^\xi_y) \mathrm{d}\mathcal{H}^{n-1}(y) \right) |\xi| e^{-|\xi|^2} \mathrm{d}\xi & u \in \mathrm{GSBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n) \\ \infty & \text{otherwise in } L^0(\Omega;\mathbb{R}^n). \end{cases}$$

By virtue of Theorem 3.7, a similar argument as in Lemma 4.6 gives for every $u \in \mathrm{GSBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \left(\int_{\Pi^\xi} \int_{\Omega^\xi_y} |\nabla \hat{u}^\xi_y|^2 \,\mathrm{d}t \,\mathrm{d}\mathcal{H}^{n-1}(y) \right) |\xi| e^{-|\xi|^2} \mathrm{d}\xi = \frac{\pi^{\frac{n}{2}}}{2} \int_\Omega \left(|e(u)|^2 + \frac{1}{2} \mathrm{div}(u)^2 \right) \mathrm{d}x.$$

Therefore, Theorem 1.3 follows by defining the measure $\mathcal{I}^s_u$ for every Borel set $B \subset \Omega$ as

$$\mathcal{I}^s_u(B) := \int_{\mathbb{R}^n} \left(\int_{\Pi^\xi} \mathcal{H}^0(J_{\hat{u}^\xi_y} \cap B^\xi_y) \mathrm{d}\mathcal{H}^{n-1}(y) \right) |\xi| e^{-|\xi|^2} \mathrm{d}\xi - \pi^{\frac{n-1}{2}} \mathcal{H}^{n-1}(B \cap J_u).$$

In the computation of the Γ-liminf one can get rid of the measure $\mathcal{I}^s_u$. Indeed, by applying [21, Theorem 5.1] and since the jump set of a measurable function is always countably $(n-1)$-rectifiable (see [22]), the same aforementioned trace property for functions with directional bounded variation can be applied to show that every $u \in \mathrm{GSBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$ satisfies for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ that

$$(J_u)^\xi_y \subset J_{\hat{u}^\xi_y}, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi.$$

Hence, we infer the following inequality for every $u \in \mathrm{GSBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \left(\int_{\Pi^\xi} \mathcal{H}^0(J_{\hat{u}^\xi_y}) \mathrm{d}\mathcal{H}^{n-1}(y) \right) |\xi| e^{-|\xi|^2} \mathrm{d}\xi \geq \pi^{\frac{n-1}{2}} \mathcal{H}^{n-1}(J_u) \tag{6.2}$$

This means that the Γ-lim inf of the family $\{\mathcal{F}^1_\varepsilon\}_\varepsilon$ can be always bounded from below by the Griffith-type functional

$$\frac{\pi^{\frac{n}{2}}}{2}\int_\Omega\Big(|e(u)|^2+\frac{1}{2}\mathrm{div}(u)^2\Big)\,\mathrm{d}x+\frac{\pi^{\frac{n+1}{2}}}{2}\mathcal{H}^{n-1}(J_u).$$

The delicate part, which does not allow to conclude that $\mathcal{F}(u;\Omega)$ coincides with the above functional, is the construction of a recovery sequence. Here is exactly where the condition (6.1) becomes essential. Indeed, by following the proof of Theorem 1.2, the construction of a recovery sequence is immediately obtained by showing that (6.2) holds as an equality for every $u \in \mathrm{GSBV}^{\mathcal{E}}(\Omega;\mathbb{R}^n)$. As discussed earlier, this condition ultimately reduces to proving the rectifiability of $\mathscr{I}_{u,1}$.

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