

MEASURE AND DIMENSION THEORY OF PERMEABLE SETS AND ITS APPLICATIONS TO FRACTALS

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ABSTRACT. We study *permeable* sets. These are sets $\Theta \subset \mathbb{R}^d$ which have the property that each two points $x, y \in \mathbb{R}^d$ can be connected by a short path γ which has small (or even empty, apart from the end points of γ) intersection with Θ . We investigate relations between permeability and Lebesgue measure and establish theorems on the relation of permeability with several notions of dimension. It turns out that for most notions of dimension each subset of \mathbb{R}^d of dimension less than $d-1$ is permeable. We use our permeability result on the Nagata dimension to characterize permeability properties of self-similar sets with certain finiteness properties.

INTRODUCTION

Context of the paper. In this article we study the geometric concept of permeability and its variants as introduced in [42]. The intuition behind these concepts is that a line segment in the Euclidean plane, being “infinitely thin”, poses much less of a barrier than a strip of positive thickness. Indeed, if we are willing to take small detours, we may connect each pair of points of

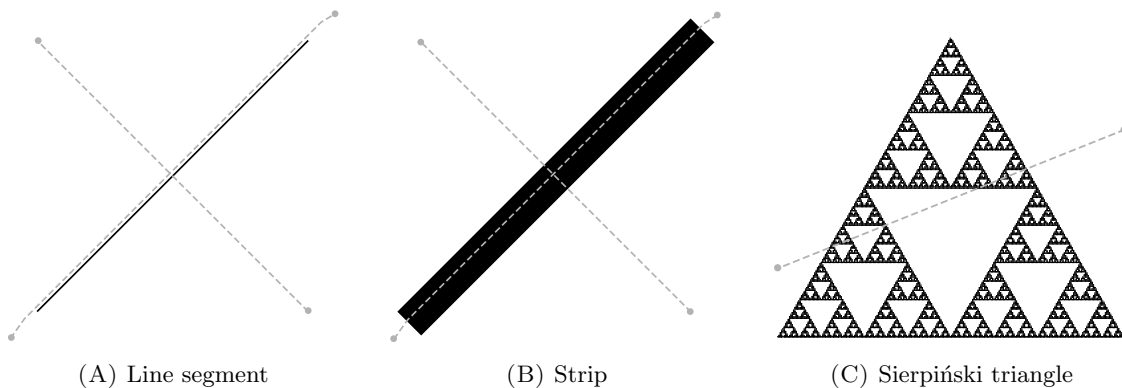


FIGURE 1. Connecting points by short paths. (A) If we allow for small detours, we can connect each pair of points by intersecting the line in at most one point. (B) For the strip we get an uncountable intersection even with small detours. (C) Can we always get countable intersections with the Sierpiński triangle with small detours?

the plane by a path that intersects the line segment in at most one point. On the other hand, for the strip of positive thickness, the intersection with a path connecting two points of the plane may remain uncountable, even if we allow for small detours. A countable subset of the plane or even the middle third Cantor set (when isometrically embedded in the plane) can always be avoided entirely when one is willing to accept small detours from straight lines. According to Marstrand’s intersection theorem [46, Theorem III], straight lines intersecting the Sierpiński triangle often have

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uncountable intersection (see Figure 1). However, as we shall see later, if we choose two points in the plane, there always exist small detours connecting these points, having only countable intersection with the Sierpiński triangle.

The notions of permeability studied in the present paper provide a framework for classifying sets with respect to their intersections with “short” paths that “pass through” them. The different “sizes” of the intersections considered in the present paper are classified as “empty” (for *null permeability*), “finite” (for *finite permeability*), or “having countable closure” (for *permeability*). We refer to Definition 1.2 for the formal definition of these notions.

The name “permeability” appears as a physical quantity in models of flow in porous media, as the *permeability coefficient* in *Darcy’s law*, cf. [10, Chapter 5]. In this context, the permeability coefficient describes the ability of a material to conduct a liquid or a gas. Our notion of permeability can be regarded as an abstract version of this physical concept.

A concept that is closely related to null permeability is *tortuosity*, which describes the ability of a particle of a given size to move through a porous medium (see e.g. [27] for different variants and applications of tortuosity). The notion of null permeability considered in our article is a limit case of tortuosity for infinitesimal particle size.

Another related concept is *percolation* (cf. [28]), in particular *Mandelbrot percolation* (also called *fractal percolation*, see e.g. [12, 13, 58]), which pursues a similar goal as permeability. Indeed, in Mandelbrot percolation one iteratively constructs a random *retaining set* $\mathcal{C} \subset [0, 1]^d$ and studies typical properties of \mathcal{C} . In particular, \mathcal{C} *percolates* if there exists a connected subset of \mathcal{C} that has nonempty intersection with the “bottom” $[0, 1]^{d-1} \times \{0\}$ and the “top” $[0, 1]^{d-1} \times \{1\}$ of the unit cube $[0, 1]^d$. While permeability also aims at connecting points, it investigates the existence of *short paths between arbitrary points* that do not hit any point of $[0, 1]^d \setminus \mathcal{C}$ (or at most finitely or countably many), which is a somewhat different goal. Null permeability can thus be viewed as “fast percolation everywhere”. We provide more details on this relation in Remark 1.4.

The notion of null permeability is also connected to the classical notion of negligibility for *extremal length* (see e.g. [1] for the definition of extremal length and extremal distance). Recall that a set $E \subset \mathbb{R}^d$ is *negligible* for extremal lengths, if the extremal length of the collection of curves between any two disjoint continua is the same as the extremal length of the subset of the above curves that do not intersect E , except possibly at the endpoints. It was shown in [1], that a closed set in the complex plane is negligible for extremal distances if and only if it is *removable* for conformal embeddings. Many results on the topic appeared in the meantime, in particular the study of negligibility for extremal distance in higher dimensions (see [66]) and its implications to removability (cf. [4, 67]) are studied. We also refer to [11] for a related concept that is called “metric removability”. Very recently a modification of this condition, called negligibility for extremal distances with countable intersections, was considered in [51, 52]. The modification is similar to passing from null permeability to permeability: the extremal length of the curve family joining the continua is required to equal the extremal length of the subcollection of curves where the curves are allowed to have countable intersection with E . According to [52], this condition for closed sets implies removability for quasiconformal homeomorphisms.

Our notion of permeability has applications in the analysis of Lipschitz maps. Given a continuous function on \mathbb{R}^d , which has bounded gradient on the complement of a small set $\Theta \subset \mathbb{R}^d$, one can ask whether it is Lipschitz continuous on the whole space. It is obvious that this is the case when Θ is finite or a line in the plane, and it is also obvious that it is not the case if Θ has inner points, for example, if it is a strip of positive thickness. It is much less obvious what happens when Θ is a Lipschitz or Hölder manifold, or when it is a Sierpiński carpet, a Sierpiński triangle, or some other complicated set. It turns out that permeability is the proper framework for answering these more demanding questions, see [42]. For example, we will see in Example 1.3 (9) that the Sierpiński carpet is impermeable. This implies that the Devil’s staircase function (also known as Cantor function), a nonconstant continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is constant in each complementary component of the middle third Cantor set, can be generalized to the Sierpiński carpet in the sense that there is a non-constant continuous function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ that is constant on each complementary component of the Sierpinski carpet. Because the Sierpiński triangle is permeable according to Example 4.13, such a function does *not* exist for the Sierpiński triangle.

The first application of finitely permeable sets — without being called “finitely permeable” there — was to prove Lipschitz continuity of certain functions, and can be found in the context of numerical methods for stochastic differential equations (SDEs) with discontinuous coefficients, see for instance [43, 48, 56, 57]. In each of these papers, classes of SDEs with discontinuous drift coefficient are studied, where the set Θ of discontinuity is a submanifold. The idea is to apply a certain transformation to such an SDE, in order to transform it to another SDE with a Lipschitz continuous drift coefficient. To this transformed SDE classical results for existence, uniqueness, and convergence of numerical methods assuming Lipschitz continuous drift coefficient of the SDE can be applied. The crucial ingredient that ensures the existence of a suitable transformation is provided in [43, Lemma 3.6]. Indeed, this result allows to conclude Lipschitz continuity of the drift coefficient on the whole of \mathbb{R}^d , from Lipschitz continuity (w.r.t. the *intrinsic metric*) on $\mathbb{R}^d \setminus \Theta$, provided that Θ is finitely permeable. This “transformation method” of [43] for studying SDEs with discontinuous drift coefficient stirred some interest in the community, as it made possible a number of results on tractability of a range of numerical methods [17, 48, 49, 56, 57].

Contributions of the present paper. The main contribution of this article is to develop a theory of permeable subsets of \mathbb{R}^d , relating permeability to topological, dimension theoretic, and measure theoretic properties of sets. We present a variety of conditions for checking null permeability, permeability, or impermeability of sets. Our theory is applicable to a variety of natural and well-known examples, for instance, permeability can be characterized for large classes of self-similar sets, fractal curves, and related subsets in \mathbb{R}^d .

In Section 1 we provide fundamental definitions including null permeability, finite permeability, and permeability, the central concepts of this paper. We give a list of simple examples that illustrate these concepts, and we establish basic results on permeability that will be used later on. Moreover, we show that, under mild conditions, our notions of permeability do not depend on the norm we use in order to determine the length of a path in \mathbb{R}^d (see Theorem 1.7). Theorem 1.8 shows that the Euclidean norm furnishes the strongest notion of permeability. This result provides the justification for restricting our attention to the Euclidean norm in the sequel. Section 2 deals with permeability and impermeability criteria for sets having zero (Section 2.1) and positive (Section 2.2) Lebesgue measure, respectively. We present several examples that illustrate and sharpen the results of this section. In particular, we discuss permeable and impermeable versions of so-called *Osgood curves*, *i.e.*, injective paths with positive Lebesgue measure (Examples 2.12 – 2.14). In Section 3 we investigate the relation between permeability and various notions of dimension. It turns out that for most notions of dimension a subset of \mathbb{R}^d with dimension less than $d-1$ is null permeable. Special emphasis is put on the Hausdorff and Nagata dimension. The corresponding result for the Nagata dimension (*cf.* Theorem 3.16) turns out to be useful later to establish criteria for null permeability of self-similar sets. Section 3.7 provides an overview of our dimension results. Section 4 is devoted to permeability properties of self-similar sets in \mathbb{R}^d under certain finiteness properties. Here the classical *finite type condition* plays a crucial role. After proving that this condition is equivalent to a seemingly stronger condition (see Theorem 4.3) we provide quite general permeability criteria for self-similar sets. In particular, Theorem 4.6 shows that self-similar sets in \mathbb{R}^2 are permeable under certain finiteness conditions, while Theorem 4.14 shows that such sets are even null permeable in \mathbb{R}^d for $d \geq 3$. We end Section 4 with a discussion of permeability properties of Bedford-McMullen carpets. This discussion exhibits impermeable subsets of \mathbb{R}^d that are compact, topologically 0-dimensional, and have Lebesgue measure 0.

1. PRELIMINARIES

In this section we define *permeability*, the basic concept of our paper, and some of its variants. We provide simple examples to illustrate permeability and establish auxiliary results that will be needed later on. The section ends with a theorem that shows to what extent our notions of permeability depend on the norm we use to define a metric on \mathbb{R}^d .

1.1. Basic definitions. We recall general definitions and terminology that will be used throughout the paper. Assume that the Euclidean space \mathbb{R}^d is equipped with some norm¹ $\|\cdot\|$. For $Y \subseteq \mathbb{R}^d$ we denote the closure of Y by \bar{Y} and the interior of Y by Y° . If $x \in \mathbb{R}^d$, we set $d(x, Y) := \inf\{\|y - x\| : y \in Y\}$. Let $\varepsilon > 0$. We use the notation

$$\begin{aligned} [Y]_\varepsilon &:= \{x \in \mathbb{R}^d : \|y - x\| \leq \varepsilon \text{ for some } y \in \bar{Y}\}, \\ (Y)_\varepsilon &:= \{x \in \mathbb{R}^d : \|y - x\| < \varepsilon \text{ for some } y \in Y\} \end{aligned} \tag{1.1}$$

for the *closed and open ε -neighborhood* of Y , respectively. In particular, $B_\varepsilon(x) := (\{x\})_\varepsilon$ is the open ball with center $x \in \mathbb{R}^d$ and radius ε . For $X, Y \subset \mathbb{R}^d$, we set $d(X, Y) := \inf\{\|y - x\| : x \in X, y \in Y\}$. A *continuum* is a compact connected Hausdorff space and a *Peano continuum* is a locally connected metric continuum. A continuum is called *nondegenerate* if it contains at least two elements. A *Cantor set* is a nonempty compact totally disconnected perfect subset of \mathbb{R} . A *region* is a nonempty, open, and connected subset of \mathbb{R}^d . A *path* is a continuous map $\gamma : [a, b] \rightarrow \mathbb{R}^d$, where $a, b \in \mathbb{R}$ with $a < b$. We say that γ connects $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ if $x = \gamma(a)$ and $y = \gamma(b)$. An injective path is called an *arc*. A continuous injection $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^d$, where \mathbb{S}^1 is the unit circle, is called a *loop*. Often, when there is no risk of confusion, we abuse notation and identify a path, an arc, or a loop with its image, respectively. For $x, y \in \mathbb{R}^d$ we denote the *directed line segment* from x to y by \overline{xy} . If $x_0, \dots, x_n \in \mathbb{R}^d$ are given, the path consisting of the concatenation of the line segments $\overline{x_{k-1}x_k}$ for $k \in \{1, \dots, n\}$ is called a *polygonal chain* and is denoted by $\overline{x_0 \cdots x_n}$. We regard the directed line segments of a polygonal chain as its elements. For two directed line segments s_1 and s_2 we denote by $\sphericalangle(s_1, s_2) \in [0, \pi]$ the angle between the direction vectors of s_1 and s_2 .

Definition 1.1 (Length of a path). If $\gamma : [a, b] \rightarrow \mathbb{R}^d$ is a path, then its *length* $\ell(\gamma)$ is defined as

$$\ell(\gamma) := \sup \left\{ \sum_{k=1}^n \|\gamma(t_k) - \gamma(t_{k-1})\| : n \in \mathbb{N}, a = t_0 < \cdots < t_n = b \right\}.$$

Let $\gamma : [a, b] \rightarrow \mathbb{R}^d$ be a path and $a = t_0 < \cdots < t_n = b$. The polygonal chain consisting of the directed line segments $\overline{\gamma(t_{k-1})\gamma(t_k)}$, $k \in \{1, \dots, n\}$, is called a *polygonal approximation* of γ .

1.2. Permeability. We are now ready to provide the central definition of the present paper.

Definition 1.2 (Permeability). Let $\Theta \subset \mathbb{R}^d$.

- (1) We call Θ *null permeable*, if for all $x, y \in \mathbb{R}^d$ and all $\delta > 0$, x and y can be connected by a path γ in \mathbb{R}^d with $\ell(\gamma) \leq \|y - x\| + \delta$ and $\gamma \cap \Theta \subset \{x, y\}$.
- (2) We call Θ *finitely permeable*, if for all $x, y \in \mathbb{R}^d$ and all $\delta > 0$, x and y can be connected by a path γ in \mathbb{R}^d with $\ell(\gamma) \leq \|y - x\| + \delta$ and $\gamma \cap \Theta$ finite.
- (3) We call Θ *permeable*, if for all $x, y \in \mathbb{R}^d$ and all $\delta > 0$, x and y can be connected by a path γ in \mathbb{R}^d with $\ell(\gamma) \leq \|y - x\| + \delta$ and $\overline{\gamma \cap \Theta}$ countable.
- (4) We call Θ *impermeable*, if Θ is not permeable.

Note that every null permeable set is finitely permeable, and every finitely permeable set is permeable. Note also that these notions of permeability depend on the norm $\|\cdot\|$ we are using (see Section 1.4 for more on this dependence). Null permeability is equivalent to a purely geometric version of *metric removability* that has been studied in [35]. However, since we saw in the introduction that the term “metric removability” is also used in the context of conformal mappings, we will not use it in the sequel.

Examples 1.3. We start with a number of elementary examples to illustrate the permeability concepts of Definition 1.2. In these examples, we equip \mathbb{R}^d with the Euclidean metric.

- (1) For $d > 1$ every finite subset of \mathbb{R}^d is null permeable.
- (2) Every finite subset of \mathbb{R} is finitely permeable.

¹A metric space would be sufficient to define permeability. However, to avoid pathologies, we restrict ourselves to this setting.

- (3) Every closed and countably infinite (*i.e.*, scattered) subset of \mathbb{R} is permeable (but not finitely permeable).
- (4) The set $\mathbb{Q} \subset \mathbb{R}$ is impermeable.
- (5) The sets $\{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{Q}\}$ and $\{(\cos x, \sin x) \in \mathbb{R}^2 : x \in \mathbb{Q} \cap [0, 2\pi]\}$ are null permeable, their closures in \mathbb{R}^2 are finitely permeable.
- (6) The *Warsaw sine curve* $\{(x, \sin(x^{-1})) : x \in (0, \infty)\} \cup (\{0\} \times [-1, 1]) \subset \mathbb{R}^2$ is permeable, but not finitely permeable.
- (7) If $\Theta \subset \mathbb{R}^d$ has an interior point, then Θ is impermeable ([42, Proposition 12]).
- (8) If $C \subset [0, 1]$ denotes the *middle third Cantor set* (see [18, Chapter 2] for its first appearances in the literature), then $C \times [0, 1]^{d-1} \subset \mathbb{R}^d$ is impermeable.
- (9) $([0, 1] \setminus \mathbb{Q})^2$ is impermeable [42, Proposition 26].
- (10) Supersets of impermeable sets are impermeable. Thus as a consequence of item (8), the classical *Sierpiński carpet* (which goes back to [63]) is impermeable.
- (11) Subsets of null permeable sets are null permeable, and analogous statements hold for finite permeability and permeability.
- (12) A topological submanifold of \mathbb{R}^d of dimension smaller than d , which has bi-Lipschitz charts and is closed as a subset of \mathbb{R}^d , is permeable (*cf.* [42, Theorem 31]). In particular, if $D \subset \mathbb{R}^{d-1}$ is closed and convex, and $\vartheta : D \rightarrow \mathbb{R}$ is a Lipschitz function, then its graph $\{(x, \vartheta(x)) : x \in D\} \subset \mathbb{R}^d$ is permeable.
- (13) In [14, Theorem 3] an example of a Hölder-continuous function $\vartheta : [0, 1] \rightarrow \mathbb{R}$ with impermeable graph (considered as a subset of \mathbb{R}^2) is constructed.

Remark 1.4 (Relation to percolation). Let $N > 1$ and $p \in (0, 1)$. Partition the d -dimensional unit cube $[0, 1]^d$ into N^d subcubes of side length $\frac{1}{N}$. Each of these subcubes is independently retained with probability p or deleted with probability $1 - p$. Repeat this process for each of the previously retained subcubes, *ad infinitum*, to obtain a random set \mathcal{C} , which is called *Mandelbrot fractal percolation set*. The investigation of permeability properties of probabilistic constructions like Mandelbrot percolation offers interesting facts and questions.

- (1) If the inequality $p \leq \frac{1}{N}$ holds, then for any $\delta > 0$ any two points $x, y \in [0, 1]^d$ can be connected by a path γ of length $\|x - y\| + \delta$ that does not intersect $\mathcal{C} \setminus \{x, y\}$ (*i.e.*, \mathcal{C} is null permeable). This is a consequence of \mathcal{C} being totally disconnected (see [22]), having finite $(d - 1)$ -dimensional Hausdorff measure (see *e.g.* [15, Section 1.a], where this is shown for $d = 2$), and [35, Theorem 1.3].
- (2) There exists $p_c \in (0, 1)$ such that for $p \in [p_c, 1)$ there are nondegenerate continua contained in \mathcal{C} (see *e.g.* [22]), so for $d = 2$, \mathcal{C} is not null permeable. We do not know whether \mathcal{C} can be permeable.
- (3) It is unknown what kind of behavior w.r.t. permeability can occur for $p \in (\frac{1}{N}, p_c)$.

1.3. Auxiliary results on permeability. The next two lemmata state that for a set $\Theta \subset \mathbb{R}^d$ with nonempty interior null permeability as well as permeability already follows if one only considers short paths connecting points in the complement of Θ .

Lemma 1.5. *Let $\Theta \subset \mathbb{R}^d$ have empty interior. Assume that for all $x, y \in \mathbb{R}^d \setminus \Theta$ and all $\delta > 0$ there exists a path γ in \mathbb{R}^d connecting x and y with $\ell(\gamma) < \|y - x\| + \delta$ and $\gamma \cap \Theta = \emptyset$. Then Θ is null permeable.*

Proof. Let $x, y \in \mathbb{R}^d$ and let $\delta > 0$. Since Θ has no interior points, there exist sequences $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ of points in $\mathbb{R}^d \setminus \Theta$ such that $\|x_n - x\| < \frac{\delta}{24 \cdot 2^n}$ and $\|y_n - y\| < \frac{\delta}{24 \cdot 2^n}$ for all n . By assumption, there exist paths γ_n from x_{n+1} to x_n with

$$\ell(\gamma_n) < \|x_{n+1} - x_n\| + \frac{\delta}{6 \cdot 2^n} \leq \|x_{n+1} - x\| + \|x_n - x\| + \frac{\delta}{6 \cdot 2^n} < \frac{\delta}{4 \cdot 2^n}$$

and $\gamma_n \cap \Theta = \emptyset$, as well as paths η_n from y_n to y_{n+1} with $\ell(\eta_n) < \|y_{n+1} - y_n\| + \frac{\delta}{6 \cdot 2^n} < \frac{\delta}{4 \cdot 2^n}$ and $\eta_n \cap \Theta = \emptyset$. Moreover, there exists a path γ_0 from x_1 to y_1 with $\ell(\gamma_0) < \|y_1 - x_1\| + \frac{\delta}{6}$ and $\gamma_0 \cap \Theta = \emptyset$. Now let γ be the closure of the concatenation of the paths $\dots, \gamma_2, \gamma_1, \gamma_0, \eta_1, \eta_2, \dots$ By

taking the closure of this concatenation, the only points that are added are x and y , thus γ is a path. For its length we gain

$$\begin{aligned} \ell(\gamma) &= \ell(\gamma_0) + \sum_{n=1}^{\infty} (\ell(\gamma_n) + \ell(\eta_n)) \leq \|y_1 - x_1\| + \frac{\delta}{6} + \sum_{n=1}^{\infty} \left(\frac{\delta}{4 \cdot 2^n} + \frac{\delta}{4 \cdot 2^n} \right) \\ &\leq \|x - x_1\| + \|y - x\| + \|y_1 - y\| + \frac{2\delta}{3} \leq \frac{\delta}{48} + \|y - x\| + \frac{\delta}{48} + \frac{2\delta}{3} \leq \|y - x\| + \delta. \quad \square \end{aligned}$$

Lemma 1.6. *Let $\Theta \subset \mathbb{R}^d$ have empty interior. Assume that for all $x, y \in \mathbb{R}^d \setminus \Theta$ and all $\delta > 0$ there exists a path γ connecting x and y with $\ell(\gamma) < \|y - x\| + \delta$ and $\overline{\gamma \cap \Theta}$ countable. Then Θ is permeable.*

Proof. Let $x, y \in \mathbb{R}^d$ and let $\delta > 0$. We choose sequences $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ as in the proof of Lemma 1.5. Again there are paths γ_n from x_{n+1} to x_n with $\ell(\gamma_n) < \|x_{n+1} - x_n\| + \frac{\delta}{6 \cdot 2^n} < \frac{\delta}{4 \cdot 2^n}$ and $\overline{\gamma_n \cap \Theta}$ countable, and paths η_n from y_n to y_{n+1} with $\ell(\eta_n) < \|y_{n+1} - y_n\| + \frac{\delta}{6 \cdot 2^n} < \frac{\delta}{4 \cdot 2^n}$ and $\overline{\eta_n \cap \Theta}$ countable. Moreover, there exists a path γ_0 from x_1 to y_1 with $\ell(\gamma_0) < \|y_1 - x_1\| + \frac{\delta}{6}$ and $\overline{\gamma_0 \cap \Theta}$ countable. Now let γ be the closure of the concatenation of the paths $\dots, \gamma_2, \gamma_1, \gamma_0, \eta_1, \eta_2, \dots$, and, as in the proof of Lemma 1.5, we get that the closure adds only x and y to the concatenation. Hence, γ is a path with $\ell(\gamma) < \|y - x\| + \delta$. We need to show that, by taking the closure of the concatenation, countability of $\overline{\gamma \cap \Theta}$ is maintained. However, this follows because

$$\overline{\gamma \cap \Theta} = \overline{(\gamma_0 \cap \Theta)} \cup \bigcup_{n=1}^{\infty} \overline{(\gamma_n \cup \eta_n) \cap \Theta} \cup (\{x, y\} \cap \Theta), \quad (1.2)$$

and the right hand side is a union of countably many countable sets. \square

1.4. Invariance under change of norms. In this section, for $1 \leq p \leq \infty$ we will denote the p -norm by $\|\cdot\|_p$. If we concatenate axis-parallel line segments to connect points and apply Lemma 1.5 it is easy to see that the set $(\mathbb{R} \setminus \mathbb{Q})^2$ is permeable in $(\mathbb{R}^2, \|\cdot\|_1)$. However, $(\mathbb{R} \setminus \mathbb{Q})^2$ is impermeable in $(\mathbb{R}^2, \|\cdot\|_2)$ by Example 1.3 (9). This raises the question to what extent permeability depends on the norm used on \mathbb{R}^d . It turns out that norms admitting many geodesics (like $\|\cdot\|_{\infty}$ and $\|\cdot\|_1$) correspond to weaker forms of permeability while norms having a strictly convex closed unit ball all lead to the same class of permeable sets. Analogous assertions hold for finite permeability and null permeability. Indeed, we get the following results.

Theorem 1.7. *Let $\|\cdot\|$ be any norm on \mathbb{R}^d such that the boundary of its unit ball is strictly convex. Then $\Theta \subset \mathbb{R}^d$ is permeable in $(\mathbb{R}^d, \|\cdot\|)$ if and only if it is permeable in $(\mathbb{R}^d, \|\cdot\|_2)$. The same equivalence is true for finite permeability and null permeability.*

Theorem 1.8. *If $\Theta \subset \mathbb{R}^d$ is permeable in $(\mathbb{R}^d, \|\cdot\|_2)$ then it is permeable in $(\mathbb{R}^d, \|\cdot\|)$ for any norm $\|\cdot\|$. The same statement is true for finite permeability and null permeability.*

The following lemma forms the main part of the proofs of Theorem 1.7 and Theorem 1.8. In its statement recall the definitions from Section 1.1 and the fact that the line segments of a polygonal approximation of a path are regarded as its elements.

Lemma 1.9. *We denote by $\hat{\ell}$ the length of a path in \mathbb{R}^d w.r.t. some norm $\|\cdot\|$. For every $n \in \mathbb{N}$ let γ_n be a path in \mathbb{R}^d connecting 0 and z . Consider the following assertions.*

- (1) *For each $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ and all polygonal approximations Z of γ_n we have*

$$\sum_{s \in Z: \langle (s, \overline{0z}) \rangle > \varepsilon} \hat{\ell}(s) < \varepsilon.$$

- (2) $\lim_{n \rightarrow \infty} \hat{\ell}(\gamma_n) = \|z\|$.

Then (1) \Rightarrow (2) holds for any norm and (2) \Rightarrow (1) holds for each norm with strictly convex closed unit ball.

Proof. Let $\|\cdot\|$ be any norm on \mathbb{R}^d . W.l.o.g. we may assume that $\|z\| = 1$. Let us denote the open unit ball of $\|\cdot\|$ by U . Using the norm axioms one easily checks that \bar{U} is convex. Thus there exists an affine hyperplane E_z with $z \in \partial U \cap E_z$ and $E_z \cap U = \emptyset$ (E_z is a *supporting hyperplane* of \bar{U} in z). Denote by π_z the projection parallel to E_z onto $\mathbb{R}z$, and by v the unit normal vector of E_z satisfying $\langle x, v \rangle > 0$ for each $x \in E_z$. By definition, this projection does not increase the norm.

We first prove that (1) \Rightarrow (2) for arbitrary norms. Let $\alpha := \frac{\pi}{2} - \angle(\bar{0v}, \bar{0z})$ and consider the map

$$f : \left\{ x \in E_z : \angle(\bar{0x}, \bar{0z}) \leq \frac{\alpha}{2} \right\} \setminus \{z\} \rightarrow \mathbb{R}; \quad x \mapsto \frac{\|x\| - 1}{\angle(\bar{0x}, \bar{0z})}.$$

Because \bar{U} is nondegenerate and convex, the singularity of f at z is removable, and we can extend f continuously to the whole filled ellipse $\{x \in E_z : \angle(\bar{0x}, \bar{0z}) \leq \frac{\alpha}{2}\}$. Denoting this extension again by f , we define a constant

$$c := \sup \left\{ f(x) : x \in E_z \text{ with } \angle(\bar{0x}, \bar{0z}) \leq \frac{\alpha}{2} \right\}, \quad (1.3)$$

which is finite. With this definition of c , we have for every $x \in E_z$ with $\angle(\bar{0x}, \bar{0z}) \leq \varepsilon \leq \frac{\alpha}{2}$ that

$$\|x\| \leq 1 + c\varepsilon. \quad (1.4)$$

Then (1.4) implies that for each $x \in \mathbb{R}^d$ with $\angle(\bar{0x}, \bar{0z}) \leq \varepsilon \leq \frac{\alpha}{2}$ we have

$$\hat{\ell}(\bar{0x}) = \|x\| \leq (1 + c\varepsilon)\|\pi_z(x)\| = (1 + c\varepsilon)\hat{\ell}(\pi_z(\bar{0x})). \quad (1.5)$$

For each $\varepsilon > 0$ small enough there is $n \geq N$ such that for each Z that approximates γ_n we have

$$\hat{\ell}(Z) \leq \varepsilon + \sum_{s \in Z : \angle(s, \bar{0z}) \leq \varepsilon} \hat{\ell}(s) \leq \varepsilon + (1 + c\varepsilon) \sum_{s \in Z : \angle(s, \bar{0z}) \leq \varepsilon} \hat{\ell}(\pi_z(s)), \quad (1.6)$$

where we used (1.5) in the second inequality. Because by assumption we have

$$\sum_{s \in Z : \angle(s, \bar{0z}) \leq \varepsilon} \hat{\ell}(\pi_z(s)) \leq \sum_{s \in Z : \angle(s, \bar{0z}) \leq \pi/2} \hat{\ell}(\pi_z(s)) = 1 + \sum_{s \in Z : \angle(s, \bar{0z}) > \pi/2} \hat{\ell}(\pi_z(s)) \leq 1 + \varepsilon,$$

we gain from (1.6) that

$$\hat{\ell}(Z) \leq \varepsilon + (1 + c\varepsilon)(1 + \varepsilon) < 1 + (2c + 2)\varepsilon.$$

Since Z was arbitrary, this yields $\hat{\ell}(\gamma_n) \leq 1 + (2c + 2)\varepsilon$ for $n \geq N$ and, hence, $\lim_{n \rightarrow \infty} \hat{\ell}(\gamma_n) = 1$.

To establish the implication (2) \Rightarrow (1), we may assume that the closed unit ball \bar{U} of $\|\cdot\|$ is strictly convex. We define a function $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}$ as

$$\delta(\varepsilon) := \inf \{ \|x\| - 1 : x \in E_z \text{ with } \angle(\bar{0x}, \bar{0z}) > \varepsilon \}.$$

Because \bar{U} is strictly convex, we have $\delta(\varepsilon) > 0$ for all $\varepsilon \in \mathbb{R}_+$. Suppose that there is $\varepsilon > 0$ such that for arbitrarily large n there is an approximation Z of γ_n satisfying

$$\sum_{s \in Z : \angle(s, \bar{0z}) > \varepsilon} \hat{\ell}(s) = \varepsilon' \geq \varepsilon.$$

The projections $\pi_z(s)$ of the line segments $s \in Z$ with $\angle(s, \bar{0z}) > \varepsilon$ cover at most a $\|\cdot\|$ -length $\frac{\varepsilon'}{1 + \delta(\varepsilon)}$ of $\bar{0z}$. Thus the projections of the line segments $s \in Z$ with $\angle(\bar{0x}, \bar{0z}) \leq \varepsilon$ have to cover a $\|\cdot\|$ -length at least $1 - \frac{\varepsilon'}{1 + \delta(\varepsilon)}$ of $\bar{0z}$. Note that $\ell(s) \geq \ell(\pi_z(s))$. We therefore get

$$\begin{aligned} \hat{\ell}(\gamma_n) &\geq \hat{\ell}(Z) = \sum_{s \in Z : \angle(s, \bar{0z}) > \varepsilon} \hat{\ell}(s) + \sum_{s \in Z : \angle(s, \bar{0z}) \leq \varepsilon} \hat{\ell}(s) \\ &\geq \varepsilon' + \sum_{s \in Z : \angle(s, \bar{0z}) \leq \varepsilon} \hat{\ell}(\pi_z(s)) \geq \varepsilon' + 1 - \frac{\varepsilon'}{1 + \delta(\varepsilon)} \geq 1 + \frac{\varepsilon\delta(\varepsilon)}{1 + \delta(\varepsilon)}. \end{aligned}$$

and, hence, $\hat{\ell}(\gamma_n)$ cannot converge to 1. \square

Lemma 1.9 is sharp in the following sense: If $\|\cdot\|$ is a norm whose closed unit ball is not strictly convex, then there exist points $z_1, z_2 \in \mathbb{R}^d$ with $\|z_1\| = \|z_2\| = 1$, $z_1 \neq z_2$, and $\|\frac{1}{2}(z_1 + z_2)\| = 1$. For $z = \frac{1}{2}(z_1 + z_2)$, it is not hard to find $\varepsilon > 0$ and a polygonal chain Z with $\sum_{s \in Z} \hat{\ell}(s) = \|z\|$ and $\sum_{s \in Z: \langle (s, \overline{0z}) \rangle > \varepsilon} \hat{\ell}(s) \geq \varepsilon$ for all $n \in \mathbb{N}$.

We immediately gain the following corollary, which is of interest in its own right.

Corollary 1.10. *Let $\|\cdot\|$ be any norm on \mathbb{R}^d whose unit ball is strictly convex. Further let $x, y \in \mathbb{R}^d$ and let γ_n be a sequence of paths from x to y . Let $\hat{\ell}$ be the length of a path w.r.t. $\|\cdot\|$ and let ℓ be the length of a path w.r.t. $\|\cdot\|_2$. Then $\lim_{n \rightarrow \infty} \hat{\ell}(\gamma_n) = \|y - x\|$ if and only if $\lim_{n \rightarrow \infty} \ell(\gamma_n) = \|y - x\|_2$.*

Proof. By the equivalence of all norms on \mathbb{R}^d , the condition in Lemma 1.9 does not depend on the norm. This implies the result. \square

Proof of Theorem 1.7. Let $\hat{\ell}$ and ℓ be as in Corollary 1.10. Now Θ is permeable in $(\mathbb{R}^d, \|\cdot\|)$ if and only if for each $x, y \in \Theta$ there is a sequence $(\gamma_n)_{n \in \mathbb{N}}$ of paths with $\overline{\gamma_n \cap \Theta}$ countable and $\lim_{n \rightarrow \infty} \hat{\ell}(\gamma_n) = \|x - y\|$. By Corollary 1.10 this is equivalent to the fact that for each $x, y \in \Theta$ there is a sequence $(\gamma_n)_{n \in \mathbb{N}}$ of paths with $\overline{\gamma_n \cap \Theta}$ countable and $\lim_{n \rightarrow \infty} \ell(\gamma_n) = \|x - y\|_2$. But this is equivalent to permeability of Θ in $(\mathbb{R}^d, \|\cdot\|_2)$.

The same argument goes through for finite permeability and null permeability. \square

Proof of Theorem 1.8. This runs along the same lines as the proof of Theorem 1.7 because the implication (1) \Rightarrow (2) in Lemma 1.9 is true for any norm. \square

Because of Theorems 1.7 and 1.8, from now onwards, we will exclusively work in the space $(\mathbb{R}^d, \|\cdot\|_2)$. For this reason, in the sequel we will just write $\|\cdot\|$ for the 2-norm.

2. PERMEABILITY AND LEBESGUE MEASURE

In this section we investigate how properties of the Lebesgue measure of a set relate to our notions of permeability. As we will see in Section 2.1, the fact that a set $\Theta \subset \mathbb{R}^d$ has Lebesgue measure zero allows for easier ways to infer that Θ is permeable. On the other hand, Section 2.2 provides a criterion for impermeability under the assumption that Θ has positive Lebesgue measure. Nevertheless, we will see from various examples that the Lebesgue measure cannot be used to characterize permeability and its variants.

2.1. Sets of measure zero. In this section we will deal with permeability results for subsets of \mathbb{R}^d with zero Lebesgue measure. We start with an auxiliary result.

Lemma 2.1 ([42, Lemma 30]). *Let $\Theta \subset \mathbb{R}^d$ be a Lebesgue nullset. Then for all $x, y \in \mathbb{R}^d$ and all $\varepsilon > 0$ there exists a polygonal chain $\gamma: [a, b] \rightarrow \mathbb{R}^d$ connecting x and y , parametrized by arc length², such that $\ell(\gamma) < \|y - x\| + \varepsilon$ and $\{t \in [a, b] : \gamma(t) \in \Theta\}$ is a Lebesgue nullset in $[a, b]$.*

The next definition is a special case of [32, 2.15].

Definition 2.2 (Quasi convexity). A subset $A \subset \mathbb{R}^d$ is called *C-quasi convex* if there exists $C > 0$ such that for all $x, y \in A$ and $\varepsilon > 0$ there exists a path $\gamma \subset A$ connecting x and y with $\ell(\gamma) \leq C\|y - x\| + \varepsilon$.

Our first result shows that *C*-quasi convexity of its complement implies null permeability of a Lebesgue nullset.

Proposition 2.3. *Let $\Theta \subset \mathbb{R}^d$ and $C \in [1, \infty)$. If Θ is a Lebesgue nullset and $\mathbb{R}^d \setminus \Theta$ is *C*-quasi convex then Θ is null permeable.*

²The assertion that γ is parametrized by arc length is not present in [42, Lemma 30], but is evident from its proof.

Proof. Let $x, y \in \mathbb{R}^d \setminus \Theta$ with $x \neq y$ and $\delta > 0$. Lemma 2.1 provides a path $\gamma: [0, \ell(\gamma)] \rightarrow \mathbb{R}^d$ from x to y , parametrized by arc length, such that $A = \{t \in [0, \ell(\gamma)]: \gamma(t) \in \Theta\}$ is a Lebesgue nullset in $[0, \ell(\gamma)]$ and such that $\ell(\gamma) < \|x - y\| + \frac{\delta}{3}$. Thus there exists a countable collection of disjoint intervals $((a_n, b_n))_{n \in \mathbb{N}}$ such that $A \subset \bigcup_{n=1}^{\infty} (a_n, b_n)$ and $\sum_{k=1}^{\infty} (b_n - a_n) < \frac{\delta}{3C}$. By our assumption $\mathbb{R}^d \setminus \Theta$ is C -quasi convex. Thus for each $n \in \mathbb{N}$ there exists a path $\kappa_n: [a_n, b_n] \rightarrow \mathbb{R}^d \setminus \Theta$ connecting $\gamma(a_n)$ and $\gamma(b_n)$ with

$$\ell(\kappa_n) \leq C\|\gamma(b_n) - \gamma(a_n)\| + \frac{\delta}{3 \cdot 2^n} \leq C\ell(\gamma|_{[a_n, b_n]}) + \frac{\delta}{3 \cdot 2^n}.$$

Then

$$\kappa(t) := \begin{cases} \gamma(t) & \text{if } t \notin \bigcup_{n=1}^{\infty} [a_n, b_n], \\ \kappa_n(t) & \text{if } t \in [a_n, b_n] \end{cases}$$

defines a path $\kappa: [0, \ell(\gamma)] \rightarrow \mathbb{R}^d$ disjoint from Θ satisfying

$$\begin{aligned} \ell(\kappa) &\leq \ell(\gamma) + \sum_{n=1}^{\infty} \ell(\kappa_n) \leq \ell(\gamma) + \sum_{n=1}^{\infty} \left(C\|\gamma(b_n) - \gamma(a_n)\| + \frac{\delta}{3 \cdot 2^n} \right) \\ &\leq \ell(\gamma) + \frac{\delta}{3} + \sum_{n=1}^{\infty} C\ell(\gamma|_{[a_n, b_n]}) \leq \|y - x\| + \frac{2\delta}{3} + \sum_{n=1}^{\infty} C(b_n - a_n) < \|y - x\| + \delta. \end{aligned}$$

Now the assertion follows from Lemma 1.5. \square

Corollary 2.4. *Let $d \geq 2$ and $1 \leq k \leq d-1$. Let $A \subset \mathbb{R}^k$ be a Lebesgue nullset and let $B \subset \mathbb{R}^{d-k}$ be a set with dense complement in \mathbb{R}^{d-1} . Then $A \times B \subset \mathbb{R}^d$ is null permeable.*

Proof. We claim that the complement of $A \times B$ is $\sqrt{2}$ -quasi convex. To show this, note that, as a Lebesgue nullset, A has dense complement. Recall that $(A \times B)^c = (A^c \times B) \cup (A \times B^c) \cup (A^c \times B^c)$. We first consider the case $(x_1, x_2) \in (A^c \times B)$ and $(y_1, y_2) \in (A \times B^c)$. Let $\varepsilon > 0$. Since A^c and B^c are dense in \mathbb{R}^k and \mathbb{R}^{d-k} , respectively, there exist $\xi_2 \in B^c$ and $\eta_1 \in A^c$ with $\|\xi_2 - x_2\| < \varepsilon$ and $\|\eta_1 - y_1\| < \varepsilon$. Now the polygonal chain $\gamma := (x_1, x_2)(x_1, \xi_2)(\eta_1, \xi_2)(\eta_1, y_2)(y_1, y_2)$ is a path in $(A \times B)^c$ with

$$\ell(\gamma) < \varepsilon + \|\eta_1 - x_1\| + \|y_2 - \xi_2\| + \varepsilon < \|y_1 - x_1\| + \|y_2 - x_2\| + 4\varepsilon \leq \sqrt{2}\|y - x\| + 4\varepsilon.$$

Since the other cases can be shown in a similar way, the claim is proved.

Because A is a Lebesgue nullset, the same is true for $A \times B$. Thus it follows from Proposition 2.3 that $A \times B$ is null permeable. \square

The next corollary is a variant of the above corollary.

Corollary 2.5. *Let $\Theta \subset \mathbb{R}^d$ be a closed Lebesgue nullset such that each projection to a coordinate $(d-1)$ -plane is nowhere dense therein. Then Θ is null permeable.*

Proof. Since all $(d-1)$ -dimensional coordinate projections of Θ are nowhere dense, $\mathbb{R}^d \setminus \Theta$ is C -quasi convex for some $C > 0$ by [29, Theorem A]. Thus the corollary follows from Proposition 2.3. \square

Example 2.6 (The Menger sponge is null permeable). The *Menger sponge* (see [18, Chapter 6] for its origins) is obtained by removing the set

$$\left(\left(\frac{1}{3}, \frac{2}{3} \right) \times \left(\frac{1}{3}, \frac{2}{3} \right) \times [0, 1] \right) \cup \left(\left(\frac{1}{3}, \frac{2}{3} \right) \times [0, 1] \times \left(\frac{1}{3}, \frac{2}{3} \right) \right) \cup \left([0, 1] \times \left(\frac{1}{3}, \frac{2}{3} \right) \times \left(\frac{1}{3}, \frac{2}{3} \right) \right)$$

from $[0, 1]^3$ and repeating this operation on the remaining 20 cubes of side length $\frac{1}{3}$ iteratively. Indeed, the Menger sponge is the attractor of a self-similar iterated function system (see Section 4.1 for a definition of these objects). The projection of the Menger sponge to each coordinate plane is equal to the Sierpiński carpet. Because the Sierpiński carpet is nowhere dense, null permeability of the Menger sponge follows from Corollary 2.5.

Note that this argument does not work for the Sierpiński tetrahedron (if one uses the tetrahedron's base plane the projection thereon equals a filled triangle). For the treatment of this set other methods can be used, see Examples 3.7 and 4.16.

We continue with a version of Proposition 2.3 for permeability.

Proposition 2.7. *Let $\Theta \subset \mathbb{R}^d$ and let $C \in [1, \infty)$. Assume that Θ is C -quasi permeable, i.e., for all $x, y \in \mathbb{R}^d$ and all $\delta > 0$, x and y can be connected by a path γ in \mathbb{R}^d with $\ell(\gamma) \leq C\|y - x\| + \delta$ and $\overline{\gamma} \cap \Theta$ countable. If Θ is contained in a closed Lebesgue nullset, then Θ is permeable.*

Proof. W.l.o.g., we may assume that Θ is a closed Lebesgue nullset. Assume that x and y are points in \mathbb{R}^d and $\delta > 0$. We proceed just as in the proof of Proposition 2.3. Using Lemma 2.1, we obtain a path $\gamma: [0, \ell(\gamma)] \rightarrow \mathbb{R}^d$, connecting x and y , parametrized by arc length and with $\ell(\gamma) < \|x - y\| + \frac{\delta}{3}$ and $A = \gamma^{-1}(\Theta)$. As Θ is closed, $A \subset [0, \ell(\gamma)]$ is compact. Thus for some $K > 1$ there exist disjoint intervals $((a_n, b_n))_{1 \leq n \leq K}$ such that $A \subseteq \bigcup_{n=1}^K (a_n, b_n)$ and $\sum_{n=1}^K (b_n - a_n) < \frac{\delta}{3C}$. By assumption, for each $n \in \{1, \dots, K\}$ there exists a path κ_n connecting $\gamma(a_n)$ and $\gamma(b_n)$ with $\overline{\kappa_n} \cap \Theta$ countable and $\ell(\kappa_n) < C\|\gamma(b_n) - \gamma(a_n)\| + \frac{\delta}{3K}$. Replacing γ on the pieces $[a_n, b_n]$ by κ_n for each $n \in \{1, \dots, K\}$, we obtain a path κ that connects x and y and has $\ell(\kappa) < \|y - x\| + \delta$. Since $\overline{\kappa} \cap \Theta$ is countable for each $n \in \{1, \dots, K\}$, $\overline{\kappa} \cap \Theta$ is countable as well and permeability of Θ follows. \square

Example 2.11 shows that one cannot remove the closedness condition in the statement of Proposition 2.7. In Example 1.3 (10) we showed that the Sierpiński carpet, a Lebesgue nullset, is impermeable. In Section 3 we will see more examples of impermeable Lebesgue nullsets, and Section 4.4 contains examples of particularly “small” impermeable sets.

2.2. Sets of positive measure. At the end of the Section 2.1 we mentioned that Lebesgue nullsets may well be impermeable. We start this section with examples that point in the opposite direction in providing null permeable sets with large Lebesgue measure.

Example 2.8 (Null permeable sets with large Lebesgue measure). Let $d \geq 2$. We provide closed null permeable sets $A_n \subset [0, 1]^d$, whose measures tend to 1 for $n \rightarrow \infty$ and a null permeable set $A \subset [0, 1]^d$ with Lebesgue measure 1 (see also [35, Proposition 3.5]). Let $(q_i)_{i \in \mathbb{N}}$ be a dense sequence in \mathbb{R}^d and define $A = [0, 1]^d \setminus \bigcup_{0 \leq i < j} \overline{q_i q_j}$ (recall that \overline{xy} denotes the line segment from x to y). In the same spirit, for each $n \in \mathbb{N}$ define, using the notation (1.1), the closed set $A_n = [0, 1]^d \setminus \bigcup_{0 \leq i < j} (\overline{q_i q_j})_{2^{-i-j-n}}$. By Lemma 1.5, A and A_n are null permeable. Obviously, the Lebesgue measures of A_n tend to 1 for $n \rightarrow \infty$, and A has Lebesgue measure 1.

See also Example 2.14 for a permeable curve with positive Lebesgue measure.

We continue with an impermeability result for sets with positive Lebesgue measure. Recall that $x \in \mathbb{R}^d$ is a *Lebesgue point* of a measurable set $A \subset \mathbb{R}^d$, if and only if

$$\lim_{r \rightarrow 0} \frac{\lambda(B_r(x) \setminus A)}{\lambda(B_r(x))} = 0.$$

Moreover, for a set $A \subset \mathbb{R}^d$ we denote the s -dimensional Hausdorff measure by $\mathcal{H}^s(A)$.

Proposition 2.9. *Let $d, k \in \mathbb{N}$, let $A_1, \dots, A_k \subset \mathbb{R}^d$ have positive Lebesgue measure, and let f_1, \dots, f_k be isometries of \mathbb{R}^d . If the set $A = \bigcap_{i=1}^k f_i(A_i \times \mathbb{R}^{d-1})$ has positive Lebesgue measure, then it is impermeable in \mathbb{R}^d .*

Proof. The case $d = 1$ is trivial, so we assume $d \geq 2$. By considering subsets of A_i , $i \in \{1, \dots, k\}$, we may assume that all the A_i are compact. Let x be a Lebesgue point of A . Applying a shift by $-x$ to all f_i and noting that neither permeability, impermeability nor the measure of A changes through translations, it suffices to consider $x = 0$. Further, using translates B_i of A_i , one can find isometries \tilde{f}_i such that $\tilde{f}_i(0) = 0$ and $f_i(A_i \times \mathbb{R}^{d-1}) = \tilde{f}_i(B_i \times \mathbb{R}^{d-1})$ for all $i \in \{1, \dots, k\}$. Thus we may assume that $f_i(0) = 0$.

We want to find a point y close to 0 so that 0 and y cannot be joined by a short path whose intersection with A is countable. Let us first identify a suitable direction of the point y from 0.

We do this by writing $v_i := f_i((1, 0, \dots, 0))$ for all i and selecting a direction $v \in \mathbb{S}^1$ satisfying $c := \min\{|\langle v, v_1 \rangle|, \dots, |\langle v, v_k \rangle|\} > 0$. Since each A_i is compact, the complement of $f_i(A_i \times \mathbb{R}^{d-1})$ is the disjoint union of countably many components of the form $f_i(U \times \mathbb{R}^{d-1})$, where U is an open interval. We define $w(f_i(U \times \mathbb{R}^{d-1})) := \text{diam}(U)$. For a given $i \in \{1, \dots, k\}$ and $\delta > 0$ let $\{T_{\delta, j}^i\}_{j \in \mathbb{N}}$ be the collection of all complementary components of $f_i(A_i \times \mathbb{R}^{d-1})$ that intersect $B_\delta(0)$. Since 0 is a Lebesgue point of A , (the scalar) 0 is also a Lebesgue point of A_i , so we have, setting

$$a_\delta := \max \left\{ \sum_j w(T_{\delta, j}^i) : i \in \{1, \dots, k\} \right\},$$

that $\limsup_{\delta \rightarrow 0} \frac{a_\delta}{\delta} = 0$. Thus we may choose $\delta > 0$ small enough so that

$$a_\delta < 3\delta q \quad \text{with} \quad q := \frac{(1 - \sqrt{1 - c^2})}{6k} \in (0, 1) \quad (2.1)$$

and we define $y_\delta := \delta v$. Assume that A is permeable. Then there exists a path $\gamma: [0, 1] \rightarrow \mathbb{R}^d$ connecting 0 and y_δ with $\gamma \cap A$ countable and

$$\delta \leq \ell(\gamma) < \|y_\delta\|(1 + q) = \delta(1 + q). \quad (2.2)$$

We may assume w.l.o.g. that γ is an arc (see *e.g.* [3, Theorem 4.4.7], or [2, Proposition 3.4]). Since [2, Proposition 3.5] implies that $\mathcal{H}^1(\gamma) = \ell(\gamma)$ we gain from (2.2) that

$$\delta \leq \mathcal{H}^1(\gamma) = \ell(\gamma) < \|y_\delta\|(1 + q) = \delta(1 + q). \quad (2.3)$$

Because

$$\delta \leq \mathcal{H}^1(\gamma) = \mathcal{H}^1(\gamma \setminus A) = \mathcal{H}^1\left(\bigcup_{i=1}^k (\gamma \setminus f_i(A_i \times \mathbb{R}^{d-1}))\right) \leq \sum_{i=1}^k \mathcal{H}^1(\gamma \setminus f_i(A_i \times \mathbb{R}^{d-1})),$$

for some $i \in \{1, \dots, k\}$ we have

$$\mathcal{H}^1(\gamma \setminus f_i(A_i \times \mathbb{R}^{d-1})) \geq \frac{\delta}{k}. \quad (2.4)$$

Keep this i fixed from now on. Let us abbreviate $c_i := |\langle v, v_i \rangle| \geq c > 0$. By changing v to $-v$ if necessary, we may assume that $\langle v, v_i \rangle = c_i$. By choosing an absolutely continuous reparametrization of γ (which is always possible since γ is of finite length, see [2, Theorem 4.4]) we gain

$$\begin{aligned} \langle \dot{\gamma}(t), v \rangle &= \langle \dot{\gamma}(t), \langle v, v_i \rangle v_i \rangle + \langle \dot{\gamma}(t), v - \langle v, v_i \rangle v_i \rangle \\ &\leq c_i \langle \dot{\gamma}(t), v_i \rangle + \|\dot{\gamma}(t)\| \sqrt{1 - c_i^2} \\ &\leq \langle \dot{\gamma}(t), v_i \rangle + \|\dot{\gamma}(t)\| \sqrt{1 - c^2} \end{aligned}$$

for a.a. parameter values t , where we used $\|v - \langle v, v_i \rangle v_i\| = \sqrt{1 - c_i^2}$. Hence,

$$\begin{aligned} \|y_\delta\| &= \int_0^1 \langle \dot{\gamma}(t), v \rangle dt = \int_0^1 \langle \dot{\gamma}(t), v \rangle \chi_{\mathbb{R}^d \setminus f_i(A_i \times \mathbb{R}^{d-1})}(\gamma(t)) dt + \int_0^1 \langle \dot{\gamma}(t), v \rangle \chi_{f_i(A_i \times \mathbb{R}^{d-1})}(\gamma(t)) dt \\ &\leq \int_0^1 (\langle \dot{\gamma}(t), v_i \rangle + \|\dot{\gamma}(t)\| \sqrt{1 - c^2}) \chi_{\mathbb{R}^d \setminus f_i(A_i \times \mathbb{R}^{d-1})}(\gamma(t)) dt + \int_0^1 \|\dot{\gamma}(t)\| \chi_{f_i(A_i \times \mathbb{R}^{d-1})}(\gamma(t)) dt \\ &\leq \int_0^1 \langle \dot{\gamma}(t), v_i \rangle \chi_{\mathbb{R}^d \setminus f_i(A_i \times \mathbb{R}^{d-1})}(\gamma(t)) dt + \sqrt{1 - c^2} \mathcal{H}^1(\gamma \setminus f_i(A_i \times \mathbb{R}^{d-1})) \\ &\quad + \mathcal{H}^1(\gamma \cap f_i(A_i \times \mathbb{R}^{d-1})) \\ &\leq \int_0^1 \langle \dot{\gamma}(t), v_i \rangle \chi_{\mathbb{R}^d \setminus f_i(A_i \times \mathbb{R}^{d-1})}(\gamma(t)) \chi_{(-\delta, \delta)}(\langle \dot{\gamma}(t), v_i \rangle) dt + \int_0^1 \|\dot{\gamma}(t)\| \chi_{\mathbb{R} \setminus (-\delta, \delta)}(\langle \dot{\gamma}(t), v_i \rangle) dt \\ &\quad + \sqrt{1 - c^2} \mathcal{H}^1(\gamma \setminus f_i(A_i \times \mathbb{R}^{d-1})) + \mathcal{H}^1(\gamma \cap f_i(A_i \times \mathbb{R}^{d-1})). \end{aligned} \quad (2.5)$$

Now note that for $z \in \mathbb{R}^d$ we may write

$$z = \langle z, v_i \rangle v_i + z - \langle z, v_i \rangle v_i = \langle z, v_i \rangle f_i((1, 0, \dots, 0)) + f_i((0, x_2, \dots, x_d)) = f_i(\langle z, v_i \rangle, x_2, \dots, x_d)$$

for some $x_2, \dots, x_d \in \mathbb{R}$. Thus $z \in f_i(A_i \times \mathbb{R}^{d-1})$ iff $\langle z, v_i \rangle \in A_i$, or, equivalently, $z \in \mathbb{R}^d \setminus f_i(A_i \times \mathbb{R}^{d-1})$ iff $\langle z, v_i \rangle \in \mathbb{R} \setminus A_i$. Defining $g(t) := \langle \gamma(t), v_i \rangle$ and noting that $g(1) = \langle \delta v, v_i \rangle = \delta c_i > 0$ we get, using the generalized change of variable theorem [41, Theorem 3] in the penultimate equality,

$$\begin{aligned} & \int_0^1 \langle \dot{\gamma}(t), v_i \rangle \chi_{\mathbb{R}^d \setminus f_i(A_i \times \mathbb{R}^{d-1})}(\gamma(t)) \chi_{(-\delta, \delta)}(\langle \gamma(t), v_i \rangle) dt \\ &= \int_0^1 \langle \dot{\gamma}(t), v_i \rangle \chi_{\mathbb{R} \setminus A_i}(\langle \gamma(t), v_i \rangle) \chi_{(-\delta, \delta)}(\langle \gamma(t), v_i \rangle) dt \\ &= \int_0^1 g'(t) \chi_{(\mathbb{R} \setminus A_i) \cap (-\delta, \delta)}(g(t)) dt = \int_{g(0)}^{g(1)} \chi_{(\mathbb{R} \setminus A_i) \cap (-\delta, \delta)}(s) ds \leq a_\delta. \end{aligned}$$

Combining this with (2.5) finally yields

$$\|y_\delta\| \leq a_\delta + (\ell(\gamma) - \delta) + \sqrt{1 - c^2} \mathcal{H}^1(\gamma \setminus f_i(A_i \times \mathbb{R}^{d-1})) + \mathcal{H}^1(\gamma \cap f_i(A_i \times \mathbb{R}^{d-1}))$$

We proceed, using (2.1), (2.3), and (2.4), to get

$$\begin{aligned} \|y_\delta\| &\leq a_\delta + \delta q + \sqrt{1 - c^2} \mathcal{H}^1(\gamma \setminus f_i(A_i \times \mathbb{R}^{d-1})) + (\mathcal{H}^1(\gamma) - \mathcal{H}^1(\gamma \setminus f_i(A_i \times \mathbb{R}^{d-1}))) \\ &\leq 3q\delta + \delta q + (\sqrt{1 - c^2} - 1) \frac{\delta}{k} + \mathcal{H}^1(\gamma) = \mathcal{H}^1(\gamma) - 2q\delta \leq \mathcal{H}^1(\gamma) \left(1 - \frac{2q}{1+q}\right) \\ &= \frac{1-q}{1+q} \mathcal{H}^1(\gamma). \end{aligned}$$

Thus $\frac{1+q}{1-q} \|y_\delta\| \leq \mathcal{H}^1(\gamma)$, which contradicts (2.3). This shows that A is impermeable. \square

As an immediate consequence of Proposition 2.9 we have the following result.

Corollary 2.10. *Let $A_1, \dots, A_d \subset \mathbb{R}$ be sets with positive Lebesgue measure. Then $\prod_{i=1}^d A_i$ is impermeable in \mathbb{R}^d .*

Example 2.11. With help of Corollary 2.10 we will exhibit a $\sqrt{2}$ -quasi permeable set (as defined in Proposition 2.7) $\Theta \subset \mathbb{R}^2$ of Lebesgue measure zero that is impermeable. Note that, by Proposition 2.7, Θ cannot be closed. The set Θ will be a countable union of boundaries of squares — and hence of Hausdorff dimension 1 — that are defined by using approximations of the Smith-Volterra-Cantor set. Let us start with its definition. Set $F_0 := [0, 1]$. If for some $n \in \mathbb{N}$ the set $F_{n-1} = \bigcup_{k=1}^{2^{n-1}} [a_k, b_k]$ with $0 = a_1 < b_1 < a_2 < b_2 < \dots < a_{2^{n-1}} < b_{2^{n-1}} = 1$ has already been constructed, we inductively define

$$F_n := \bigcup_{k=1}^{2^{n-1}} \left(\left[a_k, \frac{a_k + b_k}{2} - \frac{1}{2^{2n+1}} \right] \cup \left[\frac{a_k + b_k}{2} + \frac{1}{2^{2n+1}}, b_k \right] \right).$$

(Note that all of the 2^n intervals constituting F_n have the same length, denoted by λ_n .) Then $F := \bigcap_{n \geq 0} F_n$ is the *Smith-Volterra-Cantor set*, a Cantor set of positive Lebesgue measure. Our set Θ is now defined in terms of the approximations F_n by setting

$$\Theta := \bigcup_{n \geq 0} \partial(F_n \times F_n)$$

(see Figure 2).

Let us next prove that Θ is $\sqrt{2}$ -quasi permeable. To that end, fix $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and let $\varepsilon > 0$. By symmetry we may assume w.l.o.g. that $x_2 \leq x_1$ and $y_2 \leq y_1$. Let $(c_0, d_0) \in \mathbb{R}^2$ be such that $\|(c_0, d_0) - (x_i, y_i)\| \leq \frac{1}{2} \|(x_1, y_1) - (x_2, y_2)\| + \varepsilon$ for $i \in \{1, 2\}$ and so that $c_0 \notin F_{k_0} \cup \{x_1, x_2\}$ and $d_0 \notin F_{k_0} \cup \{y_1, y_2\}$ for some $k_0 \in \mathbb{N}$. Now, we construct a path γ_1 connecting (c_0, d_0) and (x_1, y_1) (the path γ_2 connecting (c_0, d_0) and (x_2, y_2) can be constructed analogously, and we omit the details). We assume that $c_0 < x_1$ and $d_0 < y_1$ (the other constellations can be treated analogously). For every $i \geq 1$ we iteratively select real numbers $c_i \in (c_{i-1}, x_1)$ and $d_i \in (d_{i-1}, y_1)$ so that $|c_i - x_1| \leq 2^{-i} \|(x_1, y_1) - (x_2, y_2)\|$, $|d_i - y_1| \leq 2^{-i} \|(x_1, y_1) - (x_2, y_2)\|$ and $c_i, d_i \notin F_{k_i}$ for some $k_i \in \mathbb{N}$. Then we let γ_1 be the path obtained by concatenation of the axis-parallel

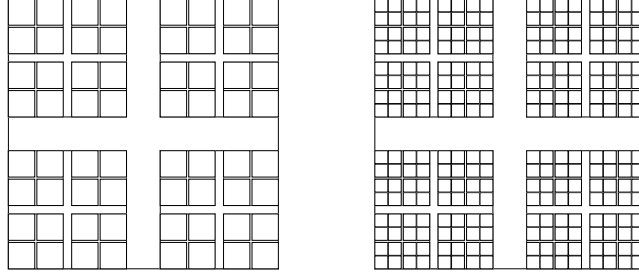


FIGURE 2. Two approximations of the set Θ . The width of the gaps between the squares tends to zero much faster than the width of the squares. We construct a path γ connecting two points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ that consists of axis-parallel line segments inside the gaps. This path γ has countable intersection with Θ .

line segments $\overline{(c_0, d_0)(c_1, d_0)}, \overline{(c_1, d_0)(c_1, d_1)}, \overline{(c_1, d_1)(c_2, d_1)}, \dots$. By construction, each of these line-segments intersects Θ finitely many times. Thus, the only possible accumulation point of $\Theta \cap \gamma_1$ is at (x_1, y_1) . Constructing γ_2 along the same lines, we can concatenate γ_1 and γ_2 to a path γ connecting (x_1, y_1) and (x_2, y_2) with $\ell(\gamma) \leq \sqrt{2}(\|(x_1, y_1) - (x_2, y_2)\| + 2\varepsilon)$. Since $\varepsilon > 0$ was arbitrary, Θ is $\sqrt{2}$ -quasi permeable.

Finally, we show that Θ is impermeable. To this end we consider an arc $\gamma: [0, 1] \rightarrow \mathbb{R}^2$. We claim that

$$\overline{\gamma \cap \Theta} = \gamma \cap \overline{\Theta}. \quad (2.6)$$

Since F has positive measure and $F \times F \subset \overline{\Theta}$, the impermeability of Θ then follows from (2.6) and Corollary 2.10. We have to establish (2.6). Since $\gamma \cap \Theta \subseteq \overline{\gamma \cap \Theta} \subseteq \gamma \cap \overline{\Theta}$ trivially holds, it remains to show that $\gamma \cap (\overline{\Theta} \setminus \Theta) \subseteq \overline{\gamma \cap \Theta}$. To see this, take $x \in \gamma \cap (\overline{\Theta} \setminus \Theta)$ and $\varepsilon > 0$. Let $i \in \mathbb{N}$ be large enough so that $\lambda_i < \min\{\text{diam}(\gamma), \varepsilon\}$. By the construction of Θ there exists a square $Q \subset F_i \times F_i$ of side-length λ_i so that $x \in Q$ and $\partial Q \subset \Theta$. By the choice of i , we have that $\gamma \cap \partial Q \neq \emptyset$. Since such i and Q can be found for all $\varepsilon > 0$, we conclude that $x \in \overline{\gamma \cap \Theta}$. Thus, (2.6) holds.

The remaining examples in this section concern arcs in \mathbb{R}^2 with positive Lebesgue measure, *i.e.*, so-called *Osgood curves*.

Example 2.12. The classical Osgood curve $\gamma: [0, 1] \rightarrow [0, 1]^2$ goes back to [54]. We give a short reminder of its construction and refer to [60, Section 8.2] for more details: Start with the path in Figure 3. Then, iteratively substitute each of the diagonal segments of the path by a scaled-down version of that path, modified in a way that the relative size of the gaps between the segments parallel to a coordinate axis converges to zero exponentially faster than the diameter of the path. Indeed, not only has γ positive measure, it even contains the Cartesian product of a Cantor set of positive measure with itself, which is impermeable by Corollary 2.10. Thus also γ is impermeable.

Example 2.13. Sierpiński's construction (see again [60, Section 8.2]) of an Osgood curve is depicted in Figure 4. It contains the intersection of four sets $f_i(A_i \times \mathbb{R})$, $i \in \{1, \dots, 4\}$, where f_1, f_2, f_3, f_4 are rotations and A_1, A_2, A_3, A_4 are Cantor sets of positive measure and is thus impermeable by Proposition 2.9.

Example 2.14. We next construct an Osgood curve which is permeable. To this end, let C be the Osgood curve from Example 2.12. Let further $(A_m)_{m \geq 1}$ be a sequence of closed permeable subsets of $[0, 1]^2$ with $\lim_{m \rightarrow \infty} \lambda(A_m) = 1$. Such a sequence exists by Example 2.8. Additionally, assume that the dense sequence $(q_i)_{i \in \mathbb{N}}$, chosen in Example 2.8, has the property that for each line segment $\overline{q_i q_j}$ the angle included with the x -axis is not a multiple of $\frac{\pi}{4}$. We then have $\lim_{m \rightarrow \infty} \lambda(C \cap A_m) = \lambda(C)$, so there exists m_0 such that $\lambda(C \cap A_{m_0}) > 0$. We now change Osgood's construction as follows: To every diagonal line segment in the n -th iteration of Osgood's construction there corresponds an axis-parallel square of which the endpoints of the line segments are the square's corners. Let \mathcal{Q}_n denote the collection of those squares and let $\mathcal{Q} := \bigcup \mathcal{Q}_n$. We only refine in those

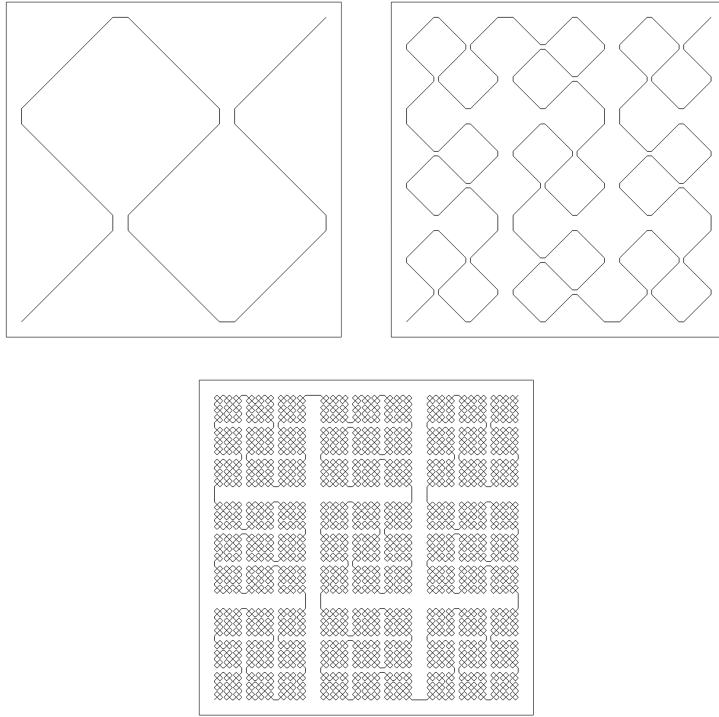


FIGURE 3. One, two, and ten iterations of Osgood's construction

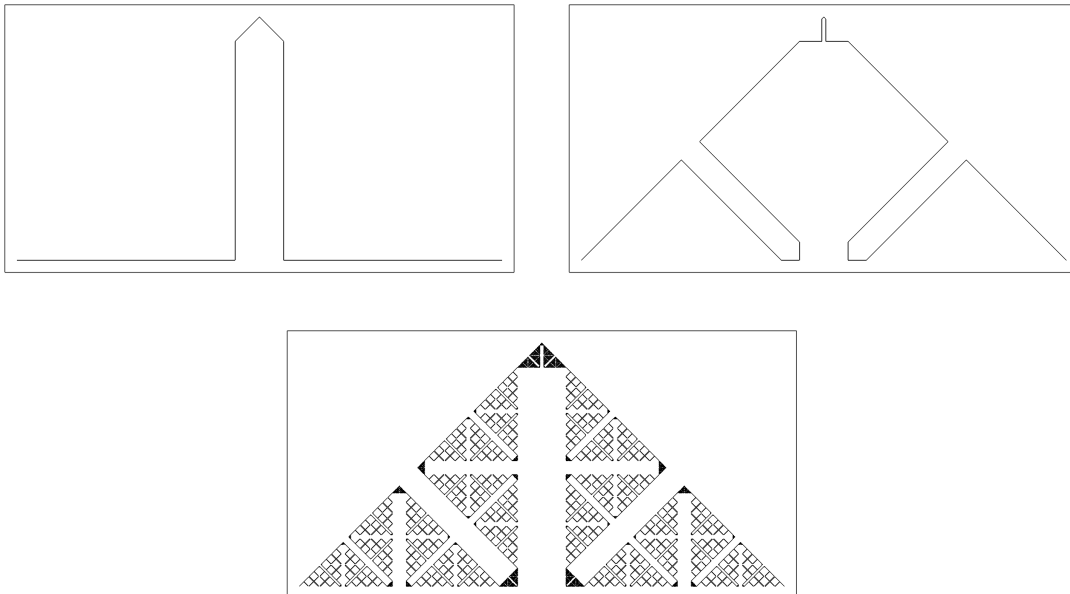


FIGURE 4. One, two, and ten iterations of Sierpiński's construction

squares which contain elements of $C \cap A_{m_0}$ (see Figure 5). We claim that the resulting curve Θ is permeable. Suppose we have two distinct points $x, y \in \{q_i : i \in \mathbb{N}\}$. Then $\overline{xy} \cap A_{m_0} = \emptyset$. Suppose $\overline{xy} \cap \Theta$ is infinite. Then $\overline{xy} \cap \Theta$ has an accumulation point z . Thus there is a sequence $(Q_k)_{k \in \mathbb{N}}$,

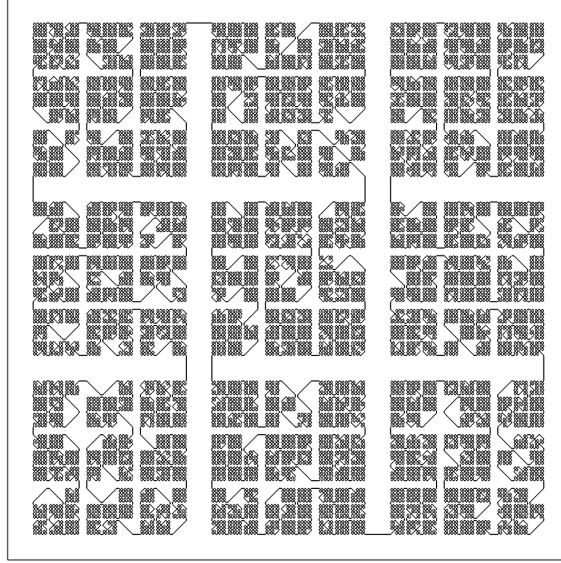


FIGURE 5. Five iterations of a permeable Osgood curve

$Q_k \in \mathcal{Q}$, of axis-parallel squares which contain elements of $C \cap A_{m_0}$ and such that $Q_k \subset B_{\frac{1}{k}}(z)$. But then $z \in C \cap A_{m_0}$, since $C \cap A_{m_0}$ is closed, contradicting $\overline{xy} \cap A_{m_0} = \emptyset$. Now we can conclude permeability of Θ by Lemma 1.6 (because Θ is closed, it suffices to consider points x and y from a dense subset of $\mathbb{R}^2 \setminus \Theta$).

3. PERMEABILITY AND DIMENSION

In this section we investigate the relationship between permeability (and its variants) and various notions of dimension for subsets of \mathbb{R}^d (for the relationship between percolation and dimension we refer to [12]). We will get the same picture for most of the notions of dimension that we study: If the dimension is contained in the interval $[d - 1, d]$, no general statement about permeability can be made. However, sets with dimension less than $d - 1$ are null permeable. Only topological dimension shows a different behavior. Among the dimensions considered in this article, the Nagata dimension is the most challenging one. However, our result on Nagata dimension, namely Theorem 3.16, turns out to be quite useful. Indeed, we can apply this result in Section 4.3 in order to show null permeability for a large class of self-similar sets in \mathbb{R}^d with $d \geq 3$.

3.1. Hausdorff dimension. Let $d \geq 2$. We denote the Hausdorff dimension of a subset $A \subset \mathbb{R}^d$ by $\dim_H(A)$. We start with the following result.

Proposition 3.1. *Let $d \geq 2$.*

- (1) *For each $s \in [0, d]$ there exists a null permeable set $\Theta \subset \mathbb{R}^d$ with $\dim_H(\Theta) = s$.*
- (2) *For each $s \in [d - 1, d]$ there exists an impermeable set $\Theta \subset \mathbb{R}^d$ with $\dim_H(\Theta) = s$.*

Note that the situation is different for $d = 1$. Any set $\Theta \subset \mathbb{R}$ with $\dim_H(\Theta) > 0$ is impermeable.

Proof. To prove (1) first observe that by Example 2.8 there are null permeable subsets of \mathbb{R}^d with positive Lebesgue measure. Thus a null permeable set can have Hausdorff dimension $s = d$. Let now $s \in (0, d)$ be arbitrary. For $a := \frac{s}{d} \in (0, 1)$ let C_a be a self-similar Cantor set with $\dim_H(C_a) = a$. Then $\dim_H((C_a)^d) = ad = s$ (here we use that, if the box counting dimension of Y coincides with its Hausdorff dimension, then $\dim_H(X \times Y) = \dim_H(X) + \dim_H(Y)$, see [21, Corollary 7.4]). The set $(C_a)^d$ is null permeable for each $a \in (0, 1)$ by Proposition 2.3. Because for $d \geq 2$ the singleton $\{0\}$ is null permeable with $\dim_H(\{0\}) = 0$, assertion (1) follows also for $s = 0$.

To show (2), observe that for an uncountable set $P \subset [0, 1]$ satisfying $\dim_H(P) = 0$, we obtain the (obviously) impermeable set $[0, 1]^{d-1} \times P$ with $\dim_H([0, 1]^{d-1} \times P) = d - 1$ (see again [21, Corollary 7.4]). Let $a \in (0, 1]$ be arbitrary. With a self-similar Cantor set $P = C_a$ satisfying $\dim_H(C_a) = a \in (0, 1]$ we get the impermeable set $[0, 1]^{d-1} \times C_a$ satisfying $\dim_H([0, 1]^{d-1} \times C_a) = a + d - 1 \in (d - 1, d]$. Since $a \in (0, 1]$ was arbitrary, this proves (2). \square

Proposition 3.1 indicates that permeability and Hausdorff dimension are quite “independent”. However, we are able to give some results showing that low Hausdorff dimension implies permeability and even null permeability. We refer to Kalmykov *et al.* [35, Theorem 1.3] for a related (but different) result. In the sequel we use U^\perp to denote the orthogonal complement of a linear subspace U of \mathbb{R}^d .

Theorem 3.2. *Let $d \geq 1$. If $\Theta \subset \mathbb{R}^d$ satisfies $\mathcal{H}^{d-1}(\Theta) = 0$, then Θ is null permeable.*

Proof. Since the case $d = 1$ is trivial, we may assume that $d \geq 2$. Let $x, y \in \mathbb{R}^d$ and $\delta > 0$. Let B be the $(d - 1)$ -dimensional ball in $\frac{x+y}{2} + (\mathbb{R}(y-x))^\perp$ with center $\frac{x+y}{2}$ and radius δ . Consider the map

$$c: [-1, 1] \times B \rightarrow \mathbb{R}^d, (s, z) \mapsto \frac{1}{2}(x+y) + s\frac{y-x}{2} + (1-|s|)z. \quad (3.1)$$

This parametrizes the double cone $\mathcal{C} := c([-1, 1] \times B)$ over $B + \frac{1}{2}(x+y)$ with apices x and y (see Figure 6). Note that the restriction of c to $(-1, 1) \times B$, which takes away the apices $\{x, y\}$, is bijective. For $s \in (-1, 1)$ let p_s be the map

$$p_s: c([\min(s, 0), \max(s, 0)] \times B) \rightarrow c(\{0\} \times B), c(t, z) \mapsto c(0, z).$$

Note that for each $s \in (-1, 1)$ the map p_s is Lipschitz. Take a strictly increasing sequence

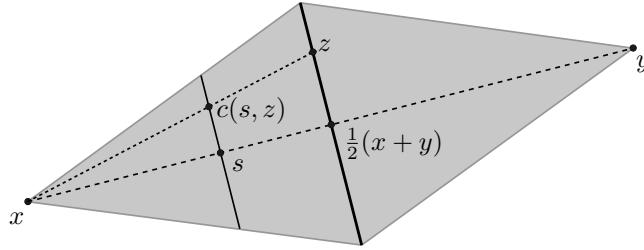


FIGURE 6. Illustration of the double cone \mathcal{C} and its parametrization c .

$(s_n)_{n \geq 0}$ with $s_n \in (0, 1)$ and $\lim_{n \rightarrow \infty} s_n = 1$. Then, because $p_{\pm s_n}$ is Lipschitz for each $n \in \mathbb{N}$ and $\mathcal{H}^{d-1}(\Theta) = 0$,

$$\Theta_\infty := \bigcup_{n=1}^{\infty} p_{-s_n}(\Theta \cap c([-s_n, 0] \times B)) \cup \bigcup_{n=1}^{\infty} p_{s_n}(\Theta \cap c([0, s_n] \times B))$$

is a countable union of subsets of $c(\{0\} \times B) = \frac{1}{2}(x+y) + B$, each of them having zero \mathcal{H}^{d-1} -measure by [21, Proposition 2.2 and (2.9)]. Therefore, $\mathcal{H}^{d-1}(\Theta_\infty) = 0$ and, hence, Θ_∞ cannot contain a $(d-1)$ -dimensional ball in $\frac{1}{2}(x+y) + B$. Thus there is $z^* \in (\frac{1}{2}(x+y) + B) \setminus \Theta_\infty$, such that $\gamma := \bigcup_{n \geq 1} p_{-s_n}^{-1}(z^*) \cup \bigcup_{n \geq 1} p_{s_n}^{-1}(z^*)$ satisfies $\gamma \cap \Theta = \emptyset$. By construction, $\bar{\gamma} := \gamma \cup \{x, y\}$ is a path connecting x and y that satisfies $\bar{\gamma} \cap \Theta \subseteq \{x, y\}$, whose length is bounded by $(\|x-y\|^2 + 4\delta^2)^{\frac{1}{2}}$. Since $x, y \in \mathbb{R}^d$ and $\delta > 0$ were arbitrary, this establishes null permeability of Θ . \square

From Theorem 3.2 we obtain the following corollary.

Corollary 3.3. *For $d \geq 2$, a countable subset of \mathbb{R}^d is null permeable.*

Example 3.4 (Brownian motion). With probability one, a path B of a d -dimensional *Brownian motion* in \mathbb{R}^d for $d \geq 2$ satisfies $\dim_H(B) = 2$ and $\mathcal{H}^2(B) = 0$ (this is the content of [65], for $d = 2$ it goes back to [44]; for a modern survey and more detailed information about Hausdorff measures of random fractals and graphs of Markov processes see [68]). Therefore, if $d \geq 3$, a path of a d -dimensional Brownian motion is almost surely null permeable in \mathbb{R}^d .

In the proof of the next result we need the following special case of a result from Mattila [47] (recall that the 0-dimensional Hausdorff measure is the counting measure).

Lemma 3.5 (see [47, Theorem 10.10]). *Let $\Theta \subset \mathbb{R}^d$ be a Borel set with $\mathcal{H}^{d-1}(\Theta) < \infty$. Then for each line $g \subset \mathbb{R}^d$ passing through the origin we have $\#((g+a) \cap \Theta) < \infty$ for almost all $a \in g^\perp$.*

Note that [47, Theorem 10.10] contains the assumption $\mathcal{H}^{d-1}(\Theta) > 0$. However, it is obvious (and mentioned in [47, beginning of Chapter 10]) that this assumption can be dropped for the part of the result that is formulated in our lemma.

Theorem 3.6. *Let $d \geq 1$. If $\Theta \subset \mathbb{R}^d$ is a Borel set with $\mathcal{H}^{d-1}(\Theta) < \infty$, then Θ is permeable.*

Proof. Since the case $d = 1$ is trivial, we may assume that $d \geq 2$. Let $x, y \in \mathbb{R}^d$ and $\delta > 0$. We use the notation from the proof of Theorem 3.2. Consider the cone $c([-1, 1] \times B)$. For each $s \in [0, 1)$, the restriction $c_s: [-s, s] \times B \rightarrow c([-s, s] \times B)$ of the map c from (3.1) is a bi-Lipschitz map, so $\mathcal{H}^{d-1}(\Theta \cap c([-s, s] \times B)) < \mathcal{H}^{d-1}(\Theta) < \infty$ implies that $\mathcal{H}^{d-1}(c_s^{-1}(\Theta \cap c([-s, s] \times B))) < \infty$ (see again [21, Proposition 2.2 and (2.9)]). By Lemma 3.5, for each such s there is a subset $D_s \subset B$ with $\mathcal{H}^{d-1}(D_s) = \mathcal{H}^{d-1}(B)$ such that for each $z \in D_s$ we have that $(-s, s) \times \{z\}$ has finite intersection with $c_s^{-1}(\Theta \cap c([-s, s] \times B))$. Taking the image under c_s , we conclude that $c([-s, s] \times \{z\}) \cap \Theta$ is finite for each $z \in D_s$. Take a strictly increasing sequence $(s_n)_{n \geq 1}$ with $s_n \in (0, 1)$ and $\lim_{n \rightarrow \infty} s_n = 1$. Then, $D_\infty := \bigcap_{n \geq 1} D_{s_n}$ has the same $(d-1)$ -dimensional Hausdorff measure as B and for each $z \in D_\infty$, $c([-1, 1] \times \{z\})$ is a polygonal chain of length at most $(\|x - y\|^2 + 4\delta^2)^{\frac{1}{2}}$ connecting x and y , such that $c([-s_n, s_n] \times \{z\}) \cap \Theta$ is finite for each $n \in \mathbb{N}$. Thus $c([-1, 1] \times \{z\}) \cap \Theta$ can have accumulation points only in $\{x, y\}$ and, hence, $\overline{c([-1, 1] \times \{z\}) \cap \Theta}$ is countable. Since $x, y \in \mathbb{R}^d$ and $\delta > 0$ were arbitrary, this implies that Θ is permeable. \square

Example 3.7. The *Sierpiński tetrahedron* $T \subset \mathbb{R}^3$ has Hausdorff dimension 2 with positive and finite Hausdorff measure. This can easily be derived from the observations in [34, p. 335]. Just note that the one-to-one mapping mentioned there is Lipschitz to get positivity of \mathcal{H}^2 ; finiteness of \mathcal{H}^2 — which is important for us — is obtained by choosing a suitable sequence of s -covers (see Definition 3.10) with $s \rightarrow 0$. Thus T is permeable by Theorem 3.6.

In view of Theorem 3.6, one may ask whether a totally disconnected set $\Theta \subset \mathbb{R}^d$ with $\mathcal{H}^{d-1}(\Theta) < \infty$ is always null permeable. This has been answered affirmatively for closed sets in [35, Theorem 1.3]. We mention that the assumption of total disconnectedness cannot be dropped as a line segment in \mathbb{R}^2 is obviously not null permeable.

3.2. Box counting and packing dimension. Denote by $\dim_B(\Theta)$ the box counting dimension of a subset $\Theta \subset \mathbb{R}^d$.

Proposition 3.8. *Let $d \geq 2$.*

- (1) *For each $s \in [0, d]$ there exists a null permeable set $\Theta \subset \mathbb{R}^d$ with $\dim_B(\Theta) = s$.*
- (2) *For each $s \in [d-1, d]$ there exists an impermeable set $\Theta \subset \mathbb{R}^d$ with $\dim_B(\Theta) = s$.*
- (3) *Each $\Theta \subset \mathbb{R}^d$ with $\dim_B(\Theta) < d-1$ is null permeable.*

The same assertions hold for the packing dimension.

Proof. We start with (1). The case $s = 0$ is trivial. For the case $s \in (0, d)$, we may choose a self-similar Cantor set $C_a \subset \mathbb{R}$ with $\dim_B(C_a) = a < 1$ and, hence, $\dim_B((C_a)^d) = ad = s$. By Proposition 2.3, $(C_a)^d$ is null permeable. For the case $s = d$ we use again the set A from Example 2.8 and the inequality $\dim_B(A) \geq \dim_H(A)$.

Assertion (2) follows for $s \in (d-1, d]$ by using the same sets as the ones we used in the case of Hausdorff dimension in the proof of Proposition 3.1. For $s = d-1$ we construct a Cantor set

C of box counting dimension 0 (take a Cantor middle set with proportion of the “middle parts” increasing sufficiently fast, [20, Section 3.2]) and obtain the impermeable set $\Theta = C \times [0, 1]^{d-1}$.

It remains to show (3). Since $\dim_B(\Theta) = \dim_B(\Theta) < d - 1$ we may assume that Θ is closed. Because $\dim_B(\Theta) \geq \dim_H(\Theta)$ (see *e.g.* [21, Inequality (3.17)]), the result follows from Theorem 3.2.

By [21, Inequality (3.29)], the same assertions are true for the packing dimension by combining our results on the Hausdorff dimension from Section 3.1 with the above assertions (1), (2), and (3). \square

3.3. Assouad dimension. Recently, also the Assouad dimension \dim_A has been studied extensively (see *e.g.* Fraser [25] or Robinson [59, Chapter 9]). In the following we show that w.r.t. permeability it behaves similarly as the dimensions treated above. Indeed, we have the following analog of Proposition 3.8.

Proposition 3.9. *Let $d \geq 2$.*

- (1) *For each $s \in [0, d]$ there exists a null permeable set $\Theta \subset \mathbb{R}^d$ with $\dim_A(\Theta) = s$.*
- (2) *For each $s \in [d - 1, d]$ there exists an impermeable set $\Theta \subset \mathbb{R}^d$ with $\dim_A(\Theta) = s$.*
- (3) *Each $\Theta \subset \mathbb{R}^d$ with $\dim_A(\Theta) < d - 1$ is null permeable.*

Proof. We start with (1). Since for $d \geq 2$ a singleton is null permeable with Assouad dimension zero, the assertion holds for $s = 0$. We know from Proposition 3.1 (1) that there are null permeable sets with Hausdorff dimension $s = d$ in \mathbb{R}^d . Since $\dim_H(\Theta) \leq \dim_A(\Theta)$ (*cf.* [25, Lemma 2.4.3]) these sets also have Assouad dimension d . Moreover, let C_a be a self-similar Cantor set of Hausdorff dimension $a \in (0, 1)$. Because $(C_a)^d$ is a null-permeable self-similar IFS attractor satisfying the open set condition, we have $\dim_H((C_a)^d) = \dim_A((C_a)^d)$ (see [53]). Thus the case $0 < s < d$ follows from Proposition 3.1 (1) as well.

To prove (2) observe that there exists an uncountable set B of Assouad dimension zero (*e.g.* a Cantor set as constructed in [20, Section 3.2]). Hence, $\dim_A([0, 1]^{d-1} \times B) = d - 1$ by [59, Lemma 9.7]. With an analogous construction with a self-similar Cantor set $P = C_a$ of Assouad dimension $a \in (0, 1]$ we can find impermeable subsets of \mathbb{R}^d of any Assouad dimension in $s \in [d - 1, d]$.

Assertion (3) follows from Theorem 3.2, because $\dim_H(\Theta) \leq \dim_A(\Theta)$. \square

3.4. Nagata dimension. The integer-valued Nagata dimension (see *e.g.* [5]; this kind of dimension is also called Assouad-Nagata dimension) provides conditions for extensions of Lipschitz functions (see for instance [39]) and is therefore of particular interest. Since we will study it in some detail, we recall its definition for the case where the ambient space is \mathbb{R}^d . This requires some preparations.

Definition 3.10 (*s-cover and r-multiplicity*). An *s-cover* \mathcal{U} of a set $\Theta \subset \mathbb{R}^d$ is a collection of subsets of \mathbb{R}^d such that $\Theta \subset \bigcup_{U \in \mathcal{U}} U$ and $\text{diam}(U) \leq s$ for all $U \in \mathcal{U}$. The *r-multiplicity* of \mathcal{U} is given by the smallest $k \in \mathbb{N}$ such that each subset of Θ with diameter at most r has a nonempty intersection with at most k elements of \mathcal{U} .

Definition 3.11 (*s-separated sets*). We say that the sets $A, B \subset \mathbb{R}^d$ are *s-separated*, for some constant $s \geq 0$, if $d(A, B) \geq s$. A collection \mathcal{U} of subsets of \mathbb{R}^d is called *s-separated* if its elements are pairwise *s-separated*.

Using this terminology, according to [39, Proposition 2.5], the Nagata dimension of a set $\Theta \subseteq \mathbb{R}^d$ can be defined in two equivalent ways as follows.

Definition 3.12 (*Nagata dimension; cf. [39, Proposition 2.5]*). Let $\Theta \subset \mathbb{R}^d$. The following are equivalent definitions of the *Nagata dimension*³.

- (1) The Nagata dimension $\dim_N(\Theta)$ is defined as the smallest $n \in \mathbb{N}$ for which there exists $C > 0$ such that for all $r > 0$ the set Θ has a Cr -cover with *r-multiplicity* at most $n + 1$.

³We only need item (2). We included item (1) just because it is the classical definition of Nagata dimension.

- (2) The Nagata dimension $\dim_N(\Theta)$ is the minimum of all $n \in \mathbb{N}$ with the following property: There exists $c > 0$ such that for all $s > 0$ the set Θ admits an s -cover $\mathcal{U} = \bigcup_{i=0}^n \mathcal{U}_i$, where \mathcal{U}_i is a collection of cs -separated sets for each $i \in \{0, \dots, n\}$. In this case we say that Θ has Nagata dimension $\dim_N(\Theta)$ *with constant* c .

For the relation between the constants C and c as well as for other equivalent characterizations of Nagata dimension we refer to [39, Proposition 2.5] and its proof. It is evident from the definition that each $\Theta \subset \mathbb{R}^d$ with Nagata dimension $\dim_N(\Theta)$ has Nagata dimension $\dim_N(\Theta)$ with constant c for some $c \in (0, 1]$.

Proposition 3.13. *Let $\Theta \subset \mathbb{R}^d$ be given. Then $\dim_N(\Theta) = \dim_N(\overline{\Theta})$.*

Proof. Assume that for some $c > 0$, $s > 0$ and $n \in \mathbb{N}$ the set Θ admits an s -cover $\mathcal{U} = \bigcup_{i=0}^n \mathcal{U}_i$, where \mathcal{U}_i is cs -separated. As \mathcal{U}_i is cs -separated for each $i \in \{0, \dots, n\}$, also $\overline{\mathcal{U}_i} := \{\overline{U}_i : U_i \in \mathcal{U}_i\}$ is cs -separated for each $i \in \{0, \dots, n\}$. Since \mathcal{U} covers Θ , and every \mathcal{U}_i is cs -separated (and thus locally finite), we have

$$\overline{\Theta} \subset \overline{\bigcup_{U \in \mathcal{U}} U} = \bigcup_{i=0}^n \overline{\bigcup_{U \in \mathcal{U}_i} U} = \bigcup_{i=0}^n \bigcup_{U \in \mathcal{U}_i} \overline{U} = \bigcup_{U \in \mathcal{U}} \overline{U}.$$

Thus $\overline{\Theta}$ admits the s -cover $\overline{\mathcal{U}} := \bigcup_{i=0}^n \overline{\mathcal{U}_i}$, where $\overline{\mathcal{U}_i}$ is cs -separated for each $i \in \{0, \dots, n\}$ and, hence, Definition 3.12 (2) yields $\dim_N(\overline{\Theta}) \leq \dim_N(\Theta)$. Since the reverse inequality follows from the monotonicity of the Nagata dimension, the result is proved. \square

The following result is an immediate consequence of the definition of the Nagata dimension.

Lemma 3.14. *Assume that $\Theta \subset \mathbb{R}^d$ satisfies $\dim_N(\Theta) = k$ with constant c . If $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a similarity transformation then $\dim_N(f(\Theta)) = \dim_N(\Theta)$ with constant c .*

Also for the Nagata dimension we get an analog of Proposition 3.1.

Proposition 3.15. *Let $d \geq 2$.*

- (1) *For each $s \in \{0, \dots, d\}$ there exists a null permeable set $\Theta \subset \mathbb{R}^d$ with $\dim_N(\Theta) = s$.*
- (2) *For each $s \in \{d-1, d\}$ there exists an impermeable set $\Theta \subset \mathbb{R}^d$ with $\dim_N(\Theta) = s$.*

Proof. To prove (1) we note that $\mathbb{Z}^s \times \{0\}^{d-s}$ is null permeable and has Nagata dimension s for each $s \in \{0, \dots, d\}$.

To prove (2) take the uncountable set B with Assouad dimension 0 mentioned in the proof of Proposition 3.9 and consider the impermeable set $\Theta = B \times [0, 1]^{d-1}$. This set has $\dim_N(\Theta) = d-1$ because $\dim_N(B) \leq \dim_A(B) = 0$ (the inequality is due to [40, Theorem 1.1]) and

$$d-1 = \dim_N(\{0\} \times [0, 1]^{d-1}) \leq \dim_N(B \times [0, 1]^{d-1}) = \dim_N(B) + \dim_N([0, 1]^{d-1}) \leq d-1$$

(see [39, Theorem 2.6] for the last inequality). This proves the case $s = d-1$. Since \mathbb{R}^d is an impermeable set with $\dim_N(\mathbb{R}^d) = d$, (2) follows. \square

The main goal of the present section is to establish the following result.

Theorem 3.16. *Let $d \geq 2$. If $\Theta \subset \mathbb{R}^d$ satisfies $\dim_N(\Theta) \leq d-2$ then Θ is null permeable.*

Theorem 3.16 can be viewed as a quantified version of a classical result by Mazurkiewicz on the topological dimension of cuts of the Euclidean space (see [38, §59, II, Theorem 1]).

The proofs of the analogs of Theorem 3.16 for the Hausdorff dimension in Section 3.1 are relatively easy. This is due to the fact that the Hausdorff dimension (and the Hausdorff measures) behaves well under Lipschitz maps and, *a fortiori*, under projections. In particular, the Hausdorff dimension cannot increase under a Lipschitz map. Since this is no longer true for the Nagata dimension, the proof of Theorem 3.16 is much harder and relies on several auxiliary results.

Proposition 3.17. *If a measurable set $\Theta \subset \mathbb{R}^d$ has a Lebesgue point then $\dim_N(\Theta) = d$.*

Before we turn to the proof of this result we recall the definition of a porous set. Recall that $B_r(x)$ denotes the open ball with radius r around x .

Definition 3.18 (Porosity; see [69, Section 2]). Let Θ be a subset of \mathbb{R}^d and $q > 0$.

- (1) A point x is called a *q-porosity point* of Θ , if for each $r > 0$, there exists $y \in \mathbb{R}^d$ with $B_{qr}(y) \subset B_r(x) \setminus \Theta$.
- (2) Θ is called *q-porous* if every point of Θ is a *q-porosity point* of Θ .

From this definition it is immediate that a *q-porous* set $\Theta \subset \mathbb{R}^d$ does not have a Lebesgue point, neither does its closure $\overline{\Theta}$. Thus the d -dimensional Lebesgue measure of $\overline{\Theta}$ is zero. Proposition 3.17 now follows by combining [45, Theorem 5.2] with [40, Theorem 1.1] (see also [16, Remark 6.14]). However, this proof runs via the Assouad dimension. For the sake of self-containment, we therefore provide a short and direct proof of Proposition 3.17. Indeed, we will prove Proposition 3.17 by showing that each $\Theta \subset \mathbb{R}^d$ with $\dim_N(\Theta) \leq d - 1$ is *q-porous* for some $q > 0$ (see Lemma 3.20). To establish this lemma, we need the following classical result from dimension theory.

Lemma 3.19 ([37, §28, II, Theorem 6]). *Let $S = [p_0, \dots, p_d]$ be a nondegenerate closed simplex with closed faces $S_i = [p_0, \dots, p_{i-1}, p_{i+1}, \dots, p_d]$, $i \in \{0, \dots, d\}$. If A_0, \dots, A_d are $d+1$ closed sets with $S = A_0 \cup \dots \cup A_d$ and $A_i \cap S_i = \emptyset$ for each $i \in \{0, \dots, d\}$, then $A_0 \cap \dots \cap A_d \neq \emptyset$.*

Proposition 3.17 is an immediate consequence of the following lemma. The uniformity of the porosity constant q asserted in this lemma will be needed later (see Lemma 3.23).

Lemma 3.20. *Let $d \in \mathbb{N}$ and $c > 0$. Then there exists $q > 0$ such that every set $\Theta \subset \mathbb{R}^d$ with $\dim_N(\Theta) \leq d - 1$ with constant c is *q-porous*.*

Proof. Let $e_0 := 0$ and let e_i be the i -th standard basis vector, $i \in \{1, \dots, d\}$. Consider the closed d -simplex $S = \frac{1}{2}[e_0, e_1, \dots, e_d] \subset B_1(0)$ with faces $S_j = \frac{1}{2}[e_0, \dots, e_{j-1}, e_{j+1}, \dots, e_d]$ ($0 \leq j \leq d$). We fix $\varepsilon > 0$ so small that $[S]_{c\varepsilon} \subset B_1(0)$ and that each set $X \subset \mathbb{R}^d$ with $\text{diam}(X) \leq \varepsilon$ satisfies $d(X, S_j) \geq c\varepsilon$ for at least one $j \in \{0, \dots, d\}$. Now set $q = \frac{c\varepsilon}{3}$.

Consider $\Theta \subset \mathbb{R}^d$ with $\dim_N(\Theta) \leq d - 1$ with constant c , $x \in \mathbb{R}^d$, and $r > 0$. We need to find $y \in \mathbb{R}^d$ with $B_{qr}(y) \subset B_r(x) \setminus \Theta$. In view of Lemma 3.14, by applying a suitable similarity transformation to Θ , we may assume w.l.o.g. that $x = 0$ and $r = 1$, *i.e.*, we need to find $y \in \mathbb{R}^d$ with $B_q(y) \subset B_1(0) \setminus \Theta$. By Definition 3.12, the set Θ admits an ε -cover $\mathcal{U} = \bigcup_{i=0}^{d-1} \mathcal{U}_i$, where \mathcal{U}_i is $c\varepsilon$ -separated for each $i \in \{0, \dots, d-1\}$. Set

$$\mathcal{V}_j := \{U \in \mathcal{U} : U \text{ is a distance at least } c\varepsilon \text{ away from } S_j\} \quad (j \in \{0, \dots, d\}),$$

$$A_0 := \bigcup_{U \in \mathcal{V}_0} [U]_{c\varepsilon/3}, \text{ and for } j \in \{1, \dots, d\}$$

$$A_j := \bigcup_{U \in \mathcal{V}_j \setminus (\mathcal{V}_0 \cup \dots \cup \mathcal{V}_{j-1})} [U]_{c\varepsilon/3}.$$

Note that each \mathcal{V}_j is locally finite and, hence, each A_j is closed. By the choice of ε , $\mathcal{U} = \bigcup_{j=0}^d \mathcal{V}_j$. Thus $A_0 \cup \dots \cup A_d \supseteq [\Theta]_{c\varepsilon/3}$. Suppose that $[\Theta]_{c\varepsilon/3} \supseteq S$. Then $A_0 \cap S, \dots, A_d \cap S$ satisfy the conditions of Lemma 3.19. Thus $A_0 \cap \dots \cap A_d \neq \emptyset$. This implies that there exist pairwise distinct $U_0, \dots, U_d \in \mathcal{U}$ such that $[U_0]_{c\varepsilon/3} \cap \dots \cap [U_d]_{c\varepsilon/3} \neq \emptyset$. By the pigeonhole principle at least two of the sets U_0, \dots, U_d belong to the same \mathcal{U}_i , and this \mathcal{U}_i therefore cannot be $c\varepsilon$ -separated. This contradiction yields $[\Theta]_{c\varepsilon/3} \not\supseteq S$. This implies that there is $y \in S$ such that the ball $B_q(y)$ of radius $q = \frac{c\varepsilon}{3}$ satisfies $B_q(y) \subset [S]_q \subset [S]_{c\varepsilon} \subset B_1(0)$ and $B_q(y) \cap \Theta = \emptyset$. This proves the lemma. \square

We recall one more result from dimension theory. In its statement, we use the following terminology. For sets $X, Y, W, Z \subset \mathbb{R}^d$ with $X \subset W$ and $Y \subset W$ we say that Z *separates* W between X and Y , if there exist M and N such that $W \setminus Z = M \cup N$ with $(\overline{M} \cap N) \cup (M \cap \overline{N}) = \emptyset$, $X \subset M$, and $Y \subset N$ (sloppily speaking, X and Y lie in different connected components of $W \setminus Z$).

Lemma 3.21 ([37, §28, II, Theorem 8]). *Let $S = [p_0, \dots, p_d]$ be a nondegenerate closed simplex with closed faces $S_i = [p_0, \dots, p_{i-1}, p_{i+1}, \dots, p_d]$, $i \in \{0, \dots, d\}$. If A_1, \dots, A_d are d open sets in S with $A_i \cap S_i = \emptyset$ for each $i \in \{1, \dots, d\}$, and such that $A_1 \cup \dots \cup A_d$ separates S between $\{p_0\}$ and S_0 , then $A_1 \cap \dots \cap A_d \neq \emptyset$.*

Let $d \in \mathbb{N}$, $c > 0$, and let q be the according constant from Lemma 3.20. Choose a nondegenerate closed $(d-1)$ -simplex $E_0 = [p_1, \dots, p_d] \subset \mathbb{R}^{d-1} \times \{0\}$ with barycenter $b_0 = 0$ and diameter bounded by q . Set $p_0 = (0, \dots, 0, 4)$ and define a nondegenerate closed simplex $E := [p_0, \dots, p_d]$. For $i \in \{0, \dots, d\}$ let $E_i = [p_0, \dots, p_{i-1}, p_{i+1}, \dots, p_d]$ be a closed face of E (note that for $i = 0$ this coincides with the definition of E_0 above). Let

$$T := E \setminus (B_q(p_0) \cup B_q(b_0))$$

(see Figure 7). Throughout this section fix $\varrho > 0$ in a way that each set B of diameter bounded by

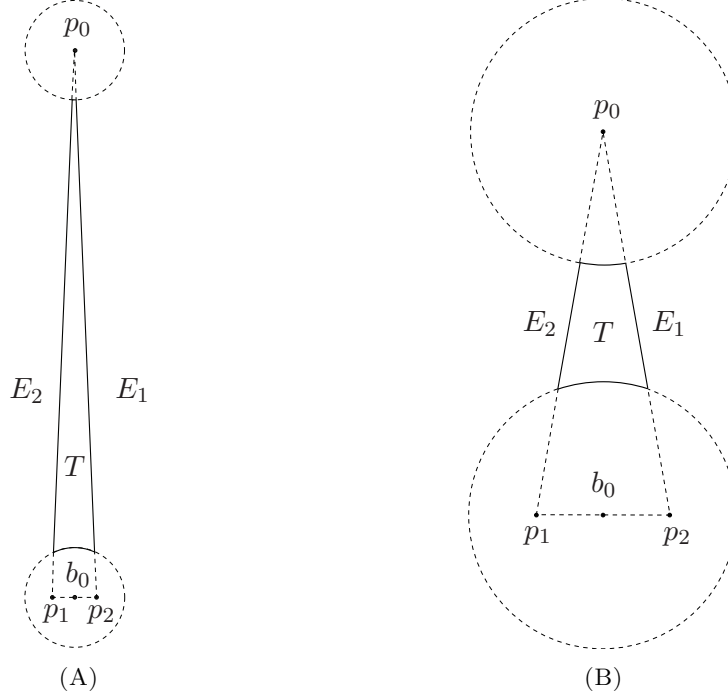


FIGURE 7. Two examples of T . In (A) the radius q of the dotted circles is small, in (B) it is larger, which is not scaled properly, but better suited for the visualization of the cover \mathcal{U} in Figure 8.

$(1+c)\varrho$ intersecting T satisfies $B \cap (\{p_0\} \cup E_0) = \emptyset$ and $B \cap E_i = \emptyset$ for some $i \in \{1, \dots, d\}$. Then, also throughout this section, fix a finite cover \mathcal{U} of T by open balls of diameter $\frac{c\varrho}{4}$. An example of a cover \mathcal{U} is shown in Figure 8. There we illustrate that ϱ is chosen small enough.

Lemma 3.22. *Let X_1, \dots, X_d be unions of elements of \mathcal{U} and set $\mathcal{P} = \{X_1, \dots, X_d\}$. If there is $\delta > 0$ with $[X_i]_\delta \cap E_i = \emptyset$ for each $i \in \{1, \dots, d\}$ and $\bigcap_{i=1}^d [X_i]_\delta = \emptyset$, then there is a path $\alpha_{\mathcal{P}}$ of finite length $\ell(\alpha_{\mathcal{P}})$ in $E \setminus \overline{X_1 \cup \dots \cup X_d}$ connecting p_0 and b_0 .*

Proof. Because each set X_i , $i \in \{1, \dots, d\}$, is a union of elements of \mathcal{U} , by the conditions on \mathcal{U} we have $\overline{X_1 \cup \dots \cup X_d} \cap (\{p_0\} \cup E_0) = \emptyset$. Thus we may choose $\delta > 0$ small enough so that $[X_1 \cup \dots \cup X_d]_\delta \cap (\{p_0\} \cup E_0) = \emptyset$. Then there is a nondegenerate simplex $E' = [p'_0, \dots, p'_d] \subset E^\circ$ with faces E'_i , satisfying $\|p'_i - p_i\| < \frac{\delta}{2}$ ($i \in \{0, \dots, d\}$), such that $E'_i \cap [X_i]_{\delta/2} = \emptyset$ for each $i \in \{1, \dots, d\}$, $[X_1 \cup \dots \cup X_d]_{\delta/2} \cap (\{p'_0\} \cup E'_0) = \emptyset$, and such that there exists a line segment $\beta_1 \subset E \setminus (X_1 \cup \dots \cup X_d)$ connecting p_0 and p'_0 and a line segment $\beta_2 \subset E \setminus (X_1 \cup \dots \cup X_d)$ connecting the barycenter b'_0 of E'_0 and b_0 . Thus it suffices to construct a path $\alpha'_{\mathcal{P}}$ of finite length $\ell(\alpha'_{\mathcal{P}})$ in $E \setminus \overline{X_1 \cup \dots \cup X_d}$ connecting p'_0 and b'_0 .

Since $(X_1)_{\delta/2} \cap \dots \cap (X_d)_{\delta/2} = \emptyset$, Lemma 3.21 implies that $E' \cap ((X_1)_{\delta/2} \cup \dots \cup (X_d)_{\delta/2})$ does not separate E' between $\{p'_0\}$ and E'_0 . Thus $E' \cap \overline{X_1 \cup \dots \cup X_d}$ does not separate E' between $\{p'_0\}$ and E'_0 . Because $E'_0 \cap \overline{X_1 \cup \dots \cup X_d} = \emptyset$, the set $E' \cap \overline{X_1 \cup \dots \cup X_d}$ does not separate E' between

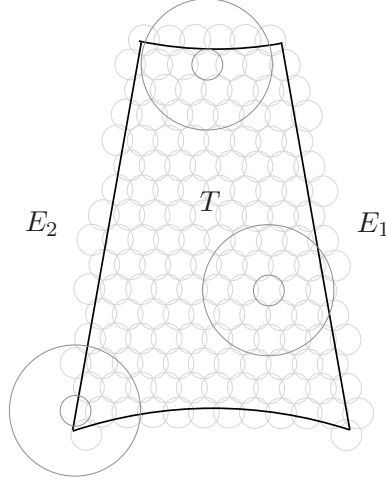


FIGURE 8. The set T from Figure 7 (B) enlarged and drawn together with the elements of a cover \mathcal{U} . The radius ϱ is chosen so that no enlarged ball intersects both E_1 and E_2 . Three of the enlarged balls are also drawn in the figure.

$\{p'_0\}$ and $\{b'_0\}$ and, *a fortiori*, it does not separate E° between $\{p'_0\}$ and $\{b'_0\}$. Now [38, §49, IV, Theorem 1] implies that there is a region $R \subset E^\circ \setminus \overline{X_1 \cup \dots \cup X_d}$ (open in E° and, hence, open in \mathbb{R}^d) with $\{p'_0, b'_0\} \in R$. Thus p'_0 and b'_0 can be connected by a path $\alpha'_\mathcal{P} \subset R$ of finite length (even by a polygonal chain, see [38, §59, I, Theorem 1]). Obviously, $\alpha'_\mathcal{P}$ is contained in $E \setminus \overline{X_1 \cup \dots \cup X_d}$. Thus $\alpha_\mathcal{P} = \beta_1 \cup \alpha'_\mathcal{P} \cup \beta_2$ is the desired path. \square

Note that because \mathcal{U} is finite, there are only finitely many choices of collections $\mathcal{P} = \{X_1, \dots, X_d\}$ that satisfy the assumptions of Lemma 3.22. Let \mathcal{K} be the finite set of all these collections. Note that $\mathcal{K} \neq \emptyset$ because we may choose $X_1 = \dots = X_d = \emptyset$. For all that follows we fix a path $\alpha_\mathcal{P}$ as in Lemma 3.22 for each $\mathcal{P} \in \mathcal{K}$. Then

$$\ell_{\max} := \max\{\ell(\alpha_\mathcal{P}) : \mathcal{P} \in \mathcal{K}\}$$

is finite. The finiteness of this maximum is important for bounding the lengths of the paths α uniformly in the following lemma.

Lemma 3.23. *Suppose $\Theta \subset E \subset \mathbb{R}^d$ has $\dim_N(\Theta) \leq d - 2$ with constant c , and let q be as in Lemma 3.20. Then there exists a path $\alpha \subset E$ with $\ell(\alpha) \leq \ell_{\max}$ connecting p_0 with b_0 such that $\Theta \cap \alpha \subset B_q(p_0) \cup B_q(b_0)$.*

The assertion of Lemma 3.23, is illustrated in Figure 9.

Proof. Let \mathcal{U} and ϱ be as described before Lemma 3.22. Because Θ satisfies $\dim_N(\Theta) \leq d - 2$ with constant c , the set $\Theta \setminus (B_q(p_0) \cup B_q(b_0))$ admits a ϱ -cover $\mathcal{C} = \bigcup_{i=0}^{d-2} \mathcal{C}_i$, where \mathcal{C}_i is $c\varrho$ -separated for each $i \in \{0, \dots, d - 2\}$. For each $C \in \mathcal{C}$ write

$$U_C := \bigcup_{\substack{U \in \mathcal{U} \\ U \cap C \neq \emptyset}} U.$$

For $i \in \{0, \dots, d - 2\}$ define $\mathcal{V}'_i := \{U_C : C \in \mathcal{C}_i\}$, which is $\frac{c\varrho}{2}$ -separated for each $i \in \{0, \dots, d - 2\}$ and set $\mathcal{V} := \bigcup_{i=0}^{d-2} \mathcal{V}'_i$. Let

$$\mathcal{V}_1 := \{V \in \mathcal{V} : [V]_{c\varrho/4} \cap E_1 = \emptyset\}$$

and for $j \in \{2, \dots, d\}$ let

$$\mathcal{V}_j := \{V \in \mathcal{V} : [V]_{c\varrho/4} \cap E_j = \emptyset\} \setminus (\mathcal{V}_1 \cup \dots \cup \mathcal{V}_{j-1}).$$

By construction, each $V \in \mathcal{V}$ has diameter bounded by $(1 + \frac{\epsilon}{2})\varrho$ and, hence, satisfies $[V]_{c\varrho/8} \cap E_j = \emptyset$ for some $j \in \{1, \dots, d\}$. Thus $\mathcal{V} = \mathcal{V}_1 \cup \dots \cup \mathcal{V}_d$ and, hence, $\mathcal{V}_1 \cup \dots \cup \mathcal{V}_d$ is a cover of $\Theta \setminus (B_q(p_0) \cup B_q(b_0))$. Define

$$X_j := \bigcup_{V \in \mathcal{V}_j} V, \quad j \in \{1, \dots, d\},$$

and set $\delta = \frac{c\varrho}{8}$. Assume that there is $x \in \bigcap_{j=1}^d [X_j]_\delta$. Then there exist pairwise distinct $V_1, \dots, V_d \in \mathcal{V}$ so that $x \in [V_j]_\delta$ for all $j \in \{1, \dots, d\}$. Since there are d of these sets, two of them are from the same collection \mathcal{V}'_i for some $i \in \{0, \dots, d-2\}$, a contradiction, because \mathcal{V}'_i is 4δ -separated. Thus, $\bigcap_{j=1}^d [X_j]_\delta = \emptyset$ and so the sets X_i satisfy the conditions of Lemma 3.22. This lemma exhibits a path $\alpha_{\mathcal{P}}$, $\mathcal{P} = \{X_1, \dots, X_d\}$, with $\ell(\alpha_{\mathcal{P}}) \leq \ell_{\max}$ that is contained in $E \setminus \overline{X_1 \cup \dots \cup X_d}$ and connects p_0 and b_0 . Because $\overline{X_1 \cup \dots \cup X_d} \supset \Theta \setminus (B_q(p_0) \cup B_q(b_0))$, the path $\alpha := \alpha_{\mathcal{P}}$ has the properties asserted in the statement of the lemma. \square

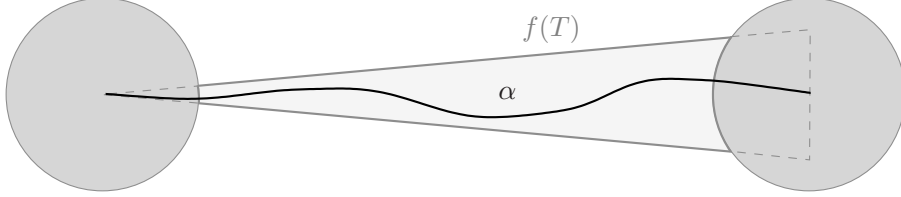


FIGURE 9. Together with Lemma 3.14, Lemma 3.23 implies that for each similarity transformation $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ we can find a path $\alpha \subset f(E)$ that does not intersect Θ outside the shaded disks $f(B_q(p_0) \cup B_q(b_0))$, *i.e.*, for α we have $(\alpha \cap f(T)) \cap \Theta = \emptyset$.

Set $a_1 = (-1, 0, \dots, 0)$ and $a_2 = (1, 0, \dots, 0)$. Let $\Theta \subset \mathbb{R}^d$ have $\dim_N(\Theta) \leq d-2$ with constant c . By Lemma 3.20, Θ is q -porous with q depending only on d and c , so for every $i \in \{1, 2\}$ there exists a sequence $(s_{i,n})_{n \geq 0}$ with $\|s_{i,n} - a_i\| \leq 2^{-n}$ such that

$$B_{2^{-n}q}(s_{i,n}) \cap \Theta = \emptyset \quad (n \geq 0). \quad (3.2)$$

This implies that

$$\|s_{1,0} - s_{2,0}\| \leq 4 \quad \text{and} \quad \|s_{i,n} - s_{i,n+1}\| \leq 2^{1-n} \quad (i \in \{1, 2\}, n \geq 0). \quad (3.3)$$

We illustrate this sequence in Figure 10.⁴

For a similarity transformation $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ we call the unique number $r > 0$ satisfying $\|f(x) - f(y)\| = r\|x - y\|$ for all $x, y \in \mathbb{R}^d$ the *similarity ratio* of f and denote it by $\text{sr}(f)$.

Lemma 3.24. *Let $\Theta \subset \mathbb{R}^d$ with $\dim_N(\Theta) \leq d-2$ with constant c and for $i \in \{1, 2\}$ define the sequences $(s_{i,n})_{n \geq 0}$ as above. Then the following assertions hold.*

- (1) *For each $i \in \{1, 2\}$ and each $n \geq 0$ there is a path $\alpha_{i,n} \subset \mathbb{R}^d \setminus \Theta$ connecting $s_{i,n}$ and $s_{i,n+1}$ with $\ell(\alpha_{i,n}) \leq 2^{-n-1}\ell_{\max}$.*
- (2) *There exists a path $\beta \subset \mathbb{R}^d \setminus \Theta$ connecting $s_{1,0}$ and $s_{2,0}$ with $\ell(\beta) \leq \ell_{\max}$.*

Proof. We start with the proof of (1). Fix $i \in \{1, 2\}$ and $n \geq 0$. We may assume that $s_{i,n} \neq s_{i,n+1}$ because otherwise there is nothing to prove. Fix a similarity transformation $f_{i,n}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ that maps p_0 to $s_{i,n}$ and b_0 to $s_{i,n+1}$. By Lemma 3.14, $f_{i,n}^{-1}(\Theta)$ satisfies $\dim_N(f_{i,n}^{-1}(\Theta)) \leq d-2$ with constant c . Moreover, (3.2) and (3.3) imply that $\text{sr}(f_{i,n}) \leq 2^{-n-1}$ and

$$f_{i,n}^{-1}(\Theta) \cap B_q(p_0) = \emptyset \quad \text{and} \quad f_{i,n}^{-1}(\Theta) \cap B_q(b_0) = \emptyset. \quad (3.4)$$

⁴If Θ is disjoint from $\{a_1, a_2\}$ (which we will assume later) and closed, we could define $(s_{i,n})_{n \geq 0}$ in a way that there exists n_0 such that $s_{i,n} = a_i$ holds for $n \geq n_0 > 0$ as in the figure ($i \in \{1, 2\}$). However, this observation is not crucial for the current proof.

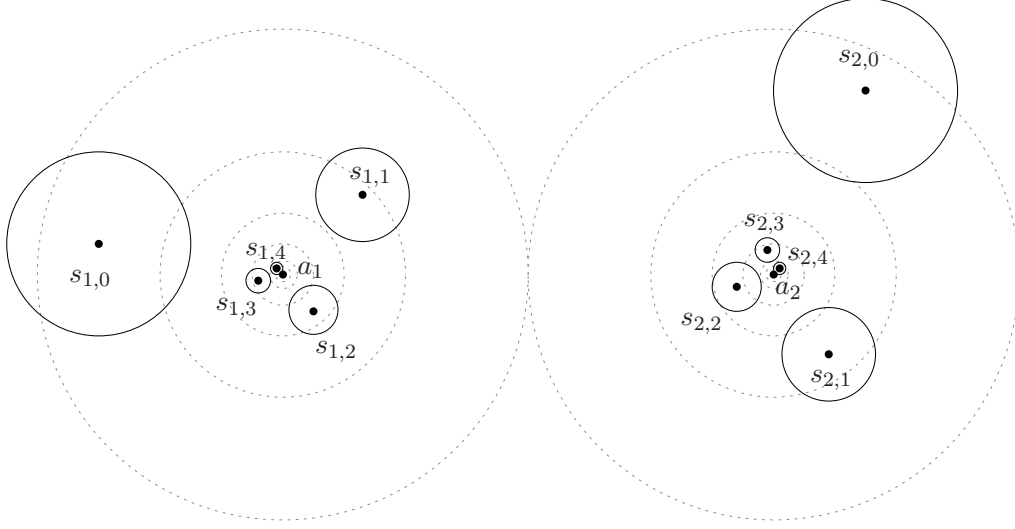


FIGURE 10. The sequences $(s_{i,n})_{n \geq 1}$ together with the circles $B_{2^{-n}q}(s_{i,n})$ for $i \in \{1, 2\}$ and small n .

Thus the set $f_{i,n}^{-1}(\Theta) \cap E$ satisfies the conditions of Lemma 3.23 and we obtain a path $\alpha \subset E$ connecting p_0 and b_0 of length $\ell(\alpha) \leq \ell_{\max}$ with $f_{i,n}^{-1}(\Theta) \cap \alpha \subset B_q(p_0) \cup B_q(b_0)$. From (3.4) we even obtain that $f_{i,n}^{-1}(\Theta) \cap \alpha = \emptyset$. Thus $\alpha_{i,n} = f_{i,n}(\alpha)$ is a path with the desired properties.

To prove (2) we may assume that $s_{1,0} \neq s_{2,0}$ because otherwise there is nothing to prove. Choose a similarity transformation $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ mapping p_0 to $s_{1,0}$ and b_0 to $s_{2,0}$ and observe that the similarity ratio of f is bounded by 1 because of (3.3). Then proceed as in the proof of (1). \square

Lemma 3.25. *Let $\Theta \subset \mathbb{R}^d \setminus \{(\pm 1, 0, \dots, 0)\}$ with $\dim_N(\Theta) \leq d - 2$ with constant c . Then there exists a path $\alpha \subset \mathbb{R}^d \setminus \Theta$, connecting $a_1 = (-1, 0, \dots, 0)$ and $a_2 = (1, 0, \dots, 0)$, with length $\ell(\alpha) \leq 3\ell_{\max}$.*

The construction of the path α is illustrated in Figure 11.

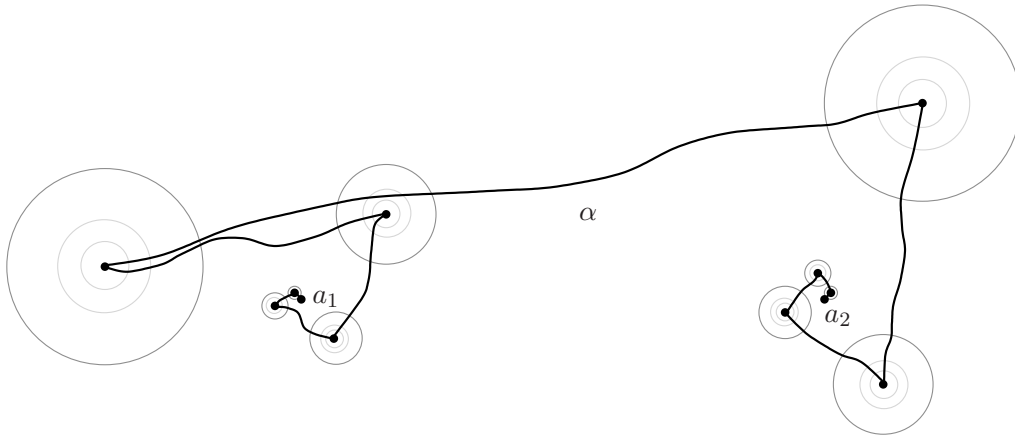


FIGURE 11. An illustration of the path α constructed in Lemma 3.25. The sub-paths are given by Lemma 3.24. They avoid the set Θ near the endpoints due to porosity, while further away from the endpoints the avoidance follows from Lemma 3.23.

Proof. This is an immediate consequence of Lemma 3.24. Just concatenate the paths

$$\dots, \alpha_{1,3}, \alpha_{1,2}, \alpha_{1,1}, \alpha_{1,0}, \beta, \alpha_{2,0}, \alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}, \dots$$

(with the orientations appropriately adapted) and take the closure to obtain the desired path α . \square

We can now finish the proof of our main theorem.

Proof of Theorem 3.16. Let $x, y \in \mathbb{R}^d \setminus \Theta$. If $x = y$ there is nothing to prove. Thus assume $x \neq y$. There exists a similarity transform f which maps x to $(1, 0, \dots, 0)$ and y to $(-1, 0, \dots, 0)$. Hence, $\text{sr}(f) = \frac{2}{\|y-x\|}$. By Lemma 3.14 and Lemma 3.25 there exists a path $\alpha \subset \mathbb{R}^d \setminus f(\Theta)$ connecting $(1, 0, \dots, 0)$ and $(-1, 0, \dots, 0)$ with $\ell(\alpha) \leq 3\ell_{\max}$. Therefore $f^{-1} \circ \alpha$ is a path in $\mathbb{R}^d \setminus \Theta$ connecting x and y with $\ell(f^{-1} \circ \alpha) = \text{sr}(f^{-1})\ell(\alpha) = \frac{\|y-x\|}{2}3\ell_{\max}$. Thus $\mathbb{R}^d \setminus \Theta$ is $\frac{3}{2}\ell_{\max}$ -quasi convex. Since Θ is a Lebesgue nullset by Proposition 3.17, the result now follows from Proposition 2.3. \square

3.5. Lipschitz dimension. Let $d \geq 2$. In [16], the integer valued *Lipschitz dimension* \dim_{Lip} of Cheeger-Kleiner is treated. In [16, Corollary 3.5] it is shown that $\dim_N(\Theta) \leq \dim_{\text{Lip}}(\Theta)$ holds for each $\Theta \subset \mathbb{R}^d$. This immediately yields that if for $\Theta \subset \mathbb{R}^d$ we have $\dim_{\text{Lip}}(\Theta) < d - 1$, then Θ is null permeable. An example for an impermeable set with $\dim_{\text{Lip}}(\Theta) = d - 1$ is furnished by $\Theta = C \times [0, 1]^{d-1}$ with C being the middle third Cantor set. Indeed, from [16, Proposition 3.6] it follows that $\dim_{\text{Lip}}(C) = 0$ and it is easy to see by definition ([16, Definition 1.3]) that $\dim_{\text{Lip}}([0, 1]^{d-1}) = d - 1$. By [16, Proposition 4.2], we have that $\dim_{\text{Lip}}(C \times [0, 1]^{d-1}) \leq d - 1$ and since $\dim_N(C \times [0, 1]^{d-1}) = d - 1$, we conclude that $\dim_{\text{Lip}}(C \times [0, 1]^{d-1}) = d - 1$. An example for a permeable set with Lipschitz dimension d is given by \mathbb{Z}^d .

3.6. Topological dimension. The topological dimension of a subset of \mathbb{R}^d is defined for instance in [37, §25, I], and we denote it by \dim_T . Concerning permeability, it behaves differently from the other notions of dimension. Sets of dimension 0 may be impermeable, take for example the product $F^d \subset \mathbb{R}^d$, where F is the Smith-Volterra-Cantor set of Example 2.11. On the other hand, a hyperplane H in \mathbb{R}^d is a finitely permeable set with $\dim_T(H) = d - 1$, and any linear subspace of topological dimension $d - 2$ is null permeable. However, a set $\Theta \subset \mathbb{R}^d$ with $\dim_T(\Theta) = d$ has nonempty interior by the Menger-Urysohn Theorem (see *e.g.* [19, Theorem 1.8.10]) and, hence, it is impermeable. Summing up, it is easy to get the following result.

Proposition 3.26. *Let $d \geq 1$.*

- (1) *For each $s \in \{0, \dots, d\}$ there exists an impermeable set $\Theta \subset \mathbb{R}^d$ with $\dim_T(\Theta) = s$.*
- (2) *For each $s \in \{0, \dots, d - 1\}$ there exists a finitely permeable set $\Theta \subset \mathbb{R}^d$ with $\dim_T(\Theta) = s$.*
- (3) *For each $s \in \{0, \dots, d - 2\}$ there exists a null permeable set $\Theta \subset \mathbb{R}^d$ with $\dim_T(\Theta) = s$.*
- (4) *Each $\Theta \subset \mathbb{R}^d$ with $\dim_T(\Theta) = d$ is impermeable.*

For $d = 2$, item (3) is best possible, *i.e.*, there is no null permeable set of topological dimension $d - 1 = 1$. Indeed, let $\Theta \subset \mathbb{R}^2$ be null permeable, $x \in \Theta$, and $r > 0$. There exist points $w_1, w_2, w_3 \in \mathbb{R}^2$ such that $\overline{w_1 w_2 w_3 w_1} \subset B_r(x)$, and such that $\text{wind}(\overline{w_1 w_2 w_3 w_1}, x) = 1$, where $\text{wind}(\kappa, z)$ denotes the winding number of a closed path $\kappa \subset \mathbb{R}^2$ relative to $z \in \mathbb{R}^2$. Since Θ has no interior points, for every $i \in \{1, \dots, 4\}$ we can shift the corner point w_i by a sufficiently small distance, to find points v_i in the complement of Θ , such that still $\overline{v_1 v_2 v_3 v_1} \subset B_r(x)$ and $\text{wind}(\overline{v_1 v_2 v_3 v_1}, x) = 1$. Since Θ is null permeable we can find paths $\gamma_1, \gamma_2, \gamma_3$ of finite length connecting v_1 and v_2 , v_2 and v_3 , and v_3 and v_1 , respectively. If we require the paths $\gamma_1, \gamma_2, \gamma_3$ to be sufficiently short, as we may by null permeability of Θ , the concatenation γ of $\gamma_1, \gamma_2, \gamma_3$ satisfies $\gamma \subset B_r(x)$, and $\text{wind}(\gamma, x) = 1$. By [38, §61, II, Theorem 5], we may assume that γ is a loop. By the Jordan Curve Theorem there exist disjoint open sets A_1, A_2 with $\mathbb{R}^2 = A_1 \cup A_2 \cup \gamma$ and $x \in A_1$. By construction, $x \in A_1 \subset B_r(x)$. Since $\gamma \cap \Theta = \emptyset$, A_1 is a neighborhood of x whose boundary has empty intersection with Θ .

We have shown that, for every $x \in \Theta$ and $r > 0$, there exists an open set $A_1 \subset B_r(x)$ such that $A_1 \cap \Theta$ is a neighborhood of x w.r.t. to the subspace topology on Θ having empty boundary relative to this topology. Since r — and thus also $A_1 \cap \Theta$ — can be made arbitrarily small, $\dim_T(\Theta) = 0$ by definition.

If $d \geq 3$ and $\Theta \subset \mathbb{R}^d$ is closed, then also the optimality of (3) still holds, *i.e.*, Θ cannot be null permeable if $\dim_T(\Theta) = d - 1$: Assume $\dim_T(\Theta) = d - 1$. Then by [24] (for $d = 3$) and [23] (for $d > 3$), Θ does not contain a region, but separates a region. Let G be such a region separated by Θ and let $x \in \Theta \cap G$. Then there exists $r > 0$ such that Θ separates $B_r(x)$, *i.e.*, $B_r(x)$ is the disjoint union of $A_1, A_2, \Theta \cap B_r(x)$, where A_1, A_2 are open sets. Choose $x_1 \in A_1, x_2 \in A_2$. Then every path $\gamma \in B_r(x)$ connecting x_1 and x_2 has nonempty intersection with Θ . Every path γ connecting x_1 and x_2 and with $\gamma \cap (\mathbb{R}^d \setminus B_r(x)) \neq \emptyset$ satisfies $\ell(\gamma) > \|x_2 - x_1\| + c$ for some constant c . Thus Θ is not null permeable.

For $d \geq 3$ it is unclear if there exists a non-closed null permeable set $\Theta \subset \mathbb{R}^d$ with $\dim_T(\Theta) = d - 1$. The example in Sitnikov [64, Section 3.3] indicates that this might be a difficult question.

3.7. Synopsis. The interplay between the various notions of dimension we treated in this section and their relation to permeability is summarized concisely in Figure 12.

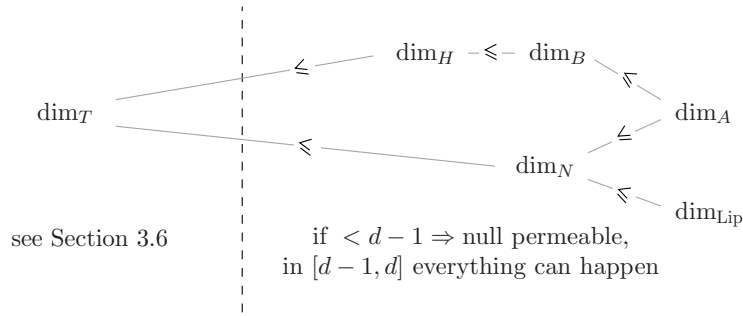


FIGURE 12. Relations between different notions of dimension and permeability

4. PERMEABILITY AND SELF-SIMILAR SETS

We now turn to permeability results for self-similar (and, in the last part of this section, self-affine) sets. It turns out that the well-known *finite type condition* (*cf.* [8, Definition 8] and see [50] for a predecessor of this condition) is crucial for permeability of self-similar sets. This section is organized as follows. In Section 4.1 we recall the terminology on iterated function systems that is relevant for us. Moreover, we prove that in the case of self-similar sets, the finite type condition is equivalent to the seemingly stronger condition of *finite types of neighborhoods*. This result is needed in the subsequent sections. Section 4.2 gives a criterion for a two-dimensional self-similar set to be permeable. In dimensions greater than 2, we can use Theorem 3.16 in order to give a criterion for a self-similar set to be null permeable. This will be established in Section 4.3. Finally, Section 4.4 is devoted to the impermeability of certain self-affine Bedford-McMullen carpets. Self-affine sets show a completely different behavior *w.r.t.* their permeability properties. This becomes apparent from the examples exhibited in [14].

4.1. Iterated function systems and self-similar sets. Recall that $\{f_1, \dots, f_m\}$ is an *iterated function system* (IFS, for short) in \mathbb{R}^d , if, for each $j \in \{1, \dots, m\}$, $f_j: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a contraction having *contraction ratio* $r_j \in (0, 1)$. It is known from Hutchinson [33] that there exists a unique nonempty compact set K satisfying

$$K = \bigcup_{j=1}^m f_j(K).$$

This set K is called the *attractor* of the iterated function system $\{f_1, \dots, f_m\}$. If the mappings f_1, \dots, f_m are similarity transformations, we call the iterated function system $\{f_1, \dots, f_m\}$ *self-similar*. In this case K is called a *self-similar set*. The definition of a *self-affine set* is analogous if the mappings f_1, \dots, f_m are affine transformations.

We introduce some notation. For a nonempty set M we denote by M^* and $M^{\mathbb{N}}$ the set of finite and infinite sequences of elements of M , respectively. For $k \in \mathbb{N}$ and $\mathbf{i} = i_1 \cdots i_k \in M^*$ we write

$|\mathbf{i}| := k$ for the *length* of \mathbf{i} and we write $M^{\leq k} = \{\mathbf{i} \in M^* : |\mathbf{i}| \leq k\}$. Let $\mathcal{F} = \{f_1, \dots, f_m\}$ be a self-similar IFS in \mathbb{R}^d with similarity ratios $r_i := \text{sr}(f_i)$ for $i \in \{1, \dots, m\}$. Set $r_{\min} := \min\{r_1, \dots, r_m\}$ and $r_{\max} := \max\{r_1, \dots, r_m\}$. For $\mathbf{i} = i_1 \cdots i_k \in \{1, \dots, m\}^*$ let $f_{\mathbf{i}} = f_{i_1} \circ \cdots \circ f_{i_k}$. For $\mathbf{i} = \epsilon$, the empty word, this means that $f_{\epsilon} = \text{id}$. In the sequel, we will deal with attractors K satisfying $\#f_i(K) \cap f_j(K) < \infty$ for $1 \leq i < j \leq m$.

Definition 4.1 (Open set condition). We say that an iterated function system $\mathcal{F} := \{f_1, \dots, f_m\}$ in \mathbb{R}^d satisfies the *open set condition* if there exists a nonempty open set $U \subset \mathbb{R}^d$ such that $\bigcup_{i=1}^m f_i(U) \subset U$ and $f_i(U) \cap f_j(U) = \emptyset$ for $i \neq j$.

We now recall the *finite type condition* of [8, Definition 8] together with a stronger version of it. We use the notation $[Y]_{\epsilon}$ from (1.1).

Definition 4.2 (Neighbor maps, finite types of neighborhoods, and finite type condition). Let K be the attractor of a self-similar IFS $\mathcal{F} = \{f_1, \dots, f_m\}$.

- (1) A *neighbor map* for \mathcal{F} is a map $h: \mathbb{R}^d \rightarrow \mathbb{R}^d$ of the form $h = f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}}$ with $h \neq \text{id}$. For every $\epsilon \in [0, \infty)$ denote by $N_{\epsilon}(\mathcal{F})$ the set of neighbor maps h with $[K]_{\epsilon} \cap h([K]_{\epsilon}) \neq \emptyset$ and $\text{sr}(h) \in [r_{\min}, r_{\min}^{-1}]$;
- (2) \mathcal{F} has *finite types of neighborhoods* for $\epsilon > 0$ if $N_{\epsilon}(\mathcal{F})$ is finite;
- (3) \mathcal{F} has *finite types of neighborhoods* if there exists $\epsilon > 0$ such that \mathcal{F} has finite types of neighborhoods for ϵ .
- (4) \mathcal{F} satisfies the *finite type condition* if $N_0(\mathcal{F})$ is finite (cf. [8, Definition 8]).

Set

$$H(\mathcal{F}) := \{x \in \mathbb{R}^d : x \in K \cap h(K) \text{ for some } h \in N_0(\mathcal{F})\} = \bigcup_{h \in N_0(\mathcal{F})} K \cap h(K),$$

which we call the set of *intersection points* of \mathcal{F} . It is immediate that $H(\mathcal{F}) = \emptyset$ if and only if $N_0(\mathcal{F}) = \emptyset$. Note that if $0 < \epsilon_1 < \epsilon_2$, then $N_{\epsilon_1}(\mathcal{F}) \subset N_{\epsilon_2}(\mathcal{F})$, so it makes sense to define

$$N(\mathcal{F}) := \lim_{\epsilon \rightarrow 0^+} N_{\epsilon}(\mathcal{F}) = \bigcap_{\epsilon \in (0, \infty)} N_{\epsilon}(\mathcal{F}).$$

Theorem 4.3. *Let $\mathcal{F} := \{f_1, \dots, f_m\}$ be a self-similar IFS. Then \mathcal{F} has finite types of neighborhoods if and only if it satisfies the finite type condition.*

Proof. For $\#K = 1$, the assertion is immediate, so we assume $\#K > 1$. We only prove necessity, sufficiency being trivial. Thus we assume that the finite type condition holds, i.e., $N_0(\mathcal{F})$ is finite. For convenience, let $\mathcal{M} := \{1, \dots, m\}$ and $C := \left\lceil \frac{\log(r_{\min})}{\log(r_{\max})} \right\rceil$. Define

$$R := \{g = f_{\mathbf{i}}^{-1} \circ h \circ f_{\mathbf{j}} : h \in N_0(\mathcal{F}) \cup \{\text{id}\}, \mathbf{i} \in \mathcal{M}, \mathbf{j} \in \mathcal{M}^{\leq C}, \text{sr}(g) \in [r_{\min}, r_{\min}^{-1}], K \cap g(K) = \emptyset\},$$

$$R(\mathbf{i}) := \{g = f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}} : \mathbf{j} \in \mathcal{M}^*, \text{sr}(g) \in [r_{\min}, r_{\min}^{-1}], K \cap g(K) = \emptyset\} \quad (\mathbf{i} \in \mathcal{M}^*).$$

These sets are finite, since $N_0(\mathcal{F})$ is finite and since there are only finitely many contractions in \mathcal{F} . Because $\#K > 1$, the minimum

$$n_1 := \min\{n \in \mathbb{N} : \text{there exists } \mathbf{i} \in \mathcal{M}^{\leq n} \text{ with } R(\mathbf{i}) \neq \emptyset\}$$

is finite. Thus

$$e_n := \sup\{\epsilon > 0 : [K]_{\epsilon} \cap h([K]_{\epsilon}) = \emptyset \text{ for each } \mathbf{i} \in \mathcal{M}^{\leq n} \text{ and each } h \in R(\mathbf{i})\} \quad (n \geq n_1)$$

are all finite. Moreover, let

$$e := \sup\{\epsilon > 0 : [K]_{\epsilon} \cap g([K]_{\epsilon}) = \emptyset, \text{ for all } g \in R\},$$

and note that $e = \infty$ if $R = \emptyset$. Since K is compact and R as well as $R(\mathbf{i})$ are finite, we have and $e_n > 0$ for $n \geq n_1$ and $e \in (0, \infty]$. Moreover, the sequence $(e_n)_{n \geq n_1}$ is nonincreasing.

The result will follow if we show that each element of the sequence $(e_n)_{n \geq n_1}$ is greater than or equal to $\min\{e, e_{n_1}\}$. To establish this claim it suffices to show that $e_{n+1} < e_n$ implies $e_{n+1} \geq e$. To prove this, let $n \geq n_1$ with $e_{n+1} < e_n$. Since K is compact and $\bigcup_{|\mathbf{i}| \leq n+1} R(\mathbf{i})$ is finite, there exist $\mathbf{i} \in \mathcal{M}^{\leq n+1}$ and $\mathbf{j} \in \mathcal{M}^*$ with $\text{sr}(f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}}) \in [r_{\min}, r_{\min}^{-1}]$ and $[K]_{e_{n+1}} \cap (f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}})([K]_{e_{n+1}}) \neq \emptyset$

but $[K]_\varepsilon \cap (f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}})([K]_\varepsilon) = \emptyset$ for all $0 \leq \varepsilon < e_{n+1}$. Since $e_{n+1} < e_n$, we need to have $|\mathbf{i}| = n + 1$ and $|\mathbf{j}| \geq n + 1$ (if $|\mathbf{j}| \leq n$, one can swap \mathbf{i} and \mathbf{j} to get a contradiction). Let \mathbf{i}' be obtained by cutting off the last index from \mathbf{i} . There exists $k \in \{0, \dots, C\}$ such that if \mathbf{j}' is obtained from \mathbf{j} by cutting off the last k indices, then $\text{sr}(f_{\mathbf{i}'}^{-1} \circ f_{\mathbf{j}'}) \in [r_{\min}, r_{\min}^{-1}]$. We show that $K \cap (f_{\mathbf{i}'}^{-1} \circ f_{\mathbf{j}'})(K) \neq \emptyset$: Otherwise $f_{\mathbf{i}'}^{-1} \circ f_{\mathbf{j}'} \in R(\mathbf{i}')$, and if we let $e_{n+1} < \varepsilon < e_n$, then

$$f_{\mathbf{i}}([K]_\varepsilon) \subset f_{\mathbf{i}'}([K]_\varepsilon) \quad \text{and} \quad f_{\mathbf{j}}([K]_\varepsilon) \subset f_{\mathbf{j}'}([K]_\varepsilon), \quad (4.1)$$

since all occurring maps are similarity transformations. Hence,

$$\emptyset \neq f_{\mathbf{i}}([K]_\varepsilon) \cap f_{\mathbf{j}}([K]_\varepsilon) \subset f_{\mathbf{i}'}([K]_\varepsilon) \cap f_{\mathbf{j}'}([K]_\varepsilon).$$

But this is impossible by the definition of e_n . Now $K \cap (f_{\mathbf{i}'}^{-1} \circ f_{\mathbf{j}'})(K) \neq \emptyset$ with $\text{sr}(f_{\mathbf{i}'}^{-1} \circ f_{\mathbf{j}'}) \in [r_{\min}, r_{\min}^{-1}]$ means that $f_{\mathbf{i}'}^{-1} \circ f_{\mathbf{j}'} \in N_0(\mathcal{F}) \cup \{\text{id}\}$. Thus $f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}} = f_{i_{n+1}}^{-1} \circ h \circ f_{\mathbf{j}}$ for $h \in N_0(\mathcal{F}) \cup \{\text{id}\}$, $i_{n+1} \in \mathcal{M}$ and $\mathbf{j} \in \mathcal{M}^{\leq C}$, in other words, $f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}} \in R$. From $f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}} \in R$ we gain $e_{n+1} \geq e$ and the claim is proved. Thus \mathcal{F} has finite types of neighborhoods for every $\varepsilon \in (0, \min\{e, e_{n_1}\})$. \square

Remark 4.4.

- (1) In the preceding proof we obtained, as an intermediate result, that $e_n > e_{n+1}$ implies $R \neq \emptyset$. Conversely, if $R = \emptyset$, then $e_n = e_{n_1}$ for every $n \geq n_1$.
- (2) The inclusions in (4.1) only hold for self-similar IFS. We believe that the conclusion of Theorem 4.3 does not hold if “self-similar” is replaced by “self-affine”.
- (3) Note that Theorem 4.3 implies $N_0(\mathcal{F}) = N(\mathcal{F})$.

Lemma 4.5. *Let $K \subset \mathbb{R}^d$ be the attractor of a self-similar IFS $\mathcal{F} := \{f_1, \dots, f_m\}$ satisfying*

- (1) $f_i(K) \cap f_j(K) < \infty$ for $1 \leq i < j \leq m$;
- (2) \mathcal{F} is of finite type.

Then $N(\mathcal{F})$ and $H(\mathcal{F})$ are both finite sets and there exists $\varepsilon \in (0, \infty)$ with $N(\mathcal{F}) = N_\varepsilon(\mathcal{F})$.

Proof. By Theorem 4.3, \mathcal{F} has finite types of neighborhoods for some $\varepsilon_0 > 0$. $H(\mathcal{F})$ is a finite set because $\#N_0(\mathcal{F}) < \infty$ and $\#K \cap h(K) < \infty$ for each $h \in N_0(\mathcal{F})$.

Note further that, since $N_\varepsilon(\mathcal{F})$ is a finite set for every $\varepsilon \in (0, \varepsilon_0)$, there exists $\varepsilon \in (0, \infty)$ such that for all $\varepsilon' \in (0, \varepsilon]$ we have $N_{\varepsilon'}(\mathcal{F}) = N_\varepsilon(\mathcal{F})$, and thus $N(\mathcal{F}) = \lim_{\varepsilon' \rightarrow 0^+} N_{\varepsilon'}(\mathcal{F}) = N_\varepsilon(\mathcal{F})$. \square

4.2. Two-dimensional self-similar sets. In this section we prove the following theorem, which provides a criterion of permeability of planar self-similar sets in terms of the finite type condition from Definition 4.2.

Theorem 4.6. *Let $K \subset \mathbb{R}^2$ be the attractor of a self-similar IFS $\mathcal{F} := \{f_1, \dots, f_m\}$ satisfying $\#f_i(K) \cap f_j(K) < \infty$ for $1 \leq i < j \leq m$. If K is connected and \mathcal{F} satisfies the finite type condition, then K is permeable.*

The proof of this result will be subdivided into lemmata.

Lemma 4.7. *For $d \geq 2$ let $K \subset \mathbb{R}^d$ be the attractor of a self-similar IFS $\mathcal{F} := \{f_1, \dots, f_m\}$ satisfying $\#f_i(K) \cap f_j(K) < \infty$ for all $1 \leq i < j \leq m$. Then K has no interior points.*

Proof. The conclusion trivially holds if K is a finite set. Thus assume that K is infinite. Suppose there exists $x \in K$ and $\varepsilon > 0$ such that $B_\varepsilon(x) \subset K$. There exists $\mathbf{i} \in \{1, \dots, m\}^*$ with $f_{\mathbf{i}}(K) \subsetneq B_\varepsilon(x)$. Let $J = \{1, \dots, m\}^{|\mathbf{i}|} \setminus \{\mathbf{i}\}$. By assumption, for every $\mathbf{j} \in J$ we have that $f_{\mathbf{i}}(K) \cap f_{\mathbf{j}}(K)$ is finite. Then

$$(f_{\mathbf{i}}(K) \cap B_\varepsilon(x)) \cup \left(\bigcup_{\mathbf{j} \in J} f_{\mathbf{j}}(K) \cap B_\varepsilon(x) \right) = B_\varepsilon(x),$$

but $F := (f_{\mathbf{i}}(K) \cap B_\varepsilon(x)) \cap \bigcup_{\mathbf{j} \in J} f_{\mathbf{j}}(K) \cap B_\varepsilon(x)$ is finite, thus $B_\varepsilon(x) \setminus F = (B_\varepsilon(x) \setminus f_{\mathbf{i}}(K)) \cup (B_\varepsilon(x) \setminus \bigcup_{\mathbf{j} \in J} f_{\mathbf{j}}(K))$, that is, we have written $B_\varepsilon(x) \setminus F$ as the disjoint union of two nonempty open sets. But if $d \geq 2$, $B_\varepsilon(x) \setminus F$ is clearly path connected and therefore connected, yielding a contradiction. \square

Lemma 4.8. *Let $K \subset \mathbb{R}^d$ be a nondegenerate continuum which is the attractor of a self-similar IFS $\mathcal{F} := \{f_1, \dots, f_m\}$. If $\#f_i(K) \cap f_j(K) < \infty$ for $1 \leq i < j \leq m$, then for any two distinct $\mathbf{i}, \mathbf{j} \in \{1, \dots, m\}^*$ we have $f_{\mathbf{i}} \neq f_{\mathbf{j}}$.*

Proof. Assume, on the contrary, that $f_{\mathbf{i}} = f_{\mathbf{j}}$. We may assume, w.l.o.g., that $\mathbf{i} = i_1 \cdots i_k$ and $\mathbf{j} = j_1 \cdots j_n$ with $i_1 \neq j_1$. Then,

$$f_{i_1}(K) \cap f_{j_1}(K) \supseteq f_{\mathbf{i}}(K) \cap f_{\mathbf{j}}(K) = f_{\mathbf{i}}(K)$$

is uncountable, because K is uncountable, and the similarity transformation f_{i_1} is a bijection. This contradicts the finiteness of the intersections. \square

Now the following lemma is an easy consequence of Lemma 4.8 and [8, Theorem 9].

Lemma 4.9. *Let $K \subset \mathbb{R}^d$ be a nondegenerate continuum which is the attractor of a self-similar IFS $\mathcal{F} := \{f_1, \dots, f_m\}$. If $\#f_i(K) \cap f_j(K) < \infty$ for $1 \leq i < j \leq m$ and \mathcal{F} satisfies the finite type condition, then \mathcal{F} satisfies the open set condition.*

Let $\varrho \in (0, 1)$ be given. We curtail each infinite word $i_1 i_2 \cdots \in \{1, \dots, m\}^{\mathbb{N}}$ after the first term i_k satisfying

$$r_{\min} \varrho < \text{sr}(f_{i_1 \cdots i_k}) = r_{i_1} r_{i_2} \cdots r_{i_k} \leq \varrho$$

and let \mathcal{Q}_ϱ be the set of all finite sequences in $\{1, \dots, m\}^*$ obtained in this way. Then it is easy to see that (recall that $\varrho < 1$)

$$K = \bigcup_{\mathbf{i} \in \mathcal{Q}_\varrho} f_{\mathbf{i}}(K).$$

Lemma 4.10. *Let $K \subset \mathbb{R}^d$ be the attractor of a self-similar IFS $\mathcal{F} := \{f_1, \dots, f_m\}$ satisfying the open set condition. Then there exists a constant c such that for all $\varrho \in (0, \infty)$ and all balls $B \subset \mathbb{R}^d$ with diameter ϱ there exists $\mathcal{D} \subset \mathcal{Q}_\varrho$ such that $\#\mathcal{D} \leq c$ and*

$$\bigcup_{\mathbf{i} \in \mathcal{D}} f_{\mathbf{i}}(K) \supseteq K \cap \bar{B}.$$

Proof. See, e.g., the proof of [21, Theorem 9.3]. \square

Next, we consider the case of plane IFS: Let $K \subset \mathbb{R}^2$ be a nondegenerate continuum which is the attractor of a self-similar IFS $\mathcal{F} := \{f_1, \dots, f_m\}$ satisfying $\#f_i(K) \cap f_j(K) < \infty$ for $1 \leq i < j \leq m$. Suppose that \mathcal{F} has finite type and hence, by Theorem 4.3 satisfies the finite types of neighborhoods condition for some $\varepsilon > 0$. W.l.o.g., ε is such that $N_\varepsilon(\mathcal{F}) = N(\mathcal{F})$.

For $\eta > 0$ let

$$C(\eta) := \left\{ \left(-\frac{\eta}{2}, \frac{\eta}{2} \right)^2 + z : \left(\left(-\frac{\eta}{2}, \frac{\eta}{2} \right)^2 + z \right) \cap K \neq \emptyset, z \in \frac{\eta}{2} \mathbb{Z}^2 \right\}.$$

The following rather technical choice of δ will be needed in the sequel, particularly in the proof of Lemma 4.11. Since $H(\mathcal{F})$ is finite by Lemma 4.5 (and $\#H(\mathcal{F}) > 2$ because K is a nondegenerate continuum) we may choose $\delta > 0$ with

$$\delta < \frac{1}{4} \min \{1, \varepsilon, \min \{\|x - y\| : x, y \in H(\mathcal{F}), x \neq y\}\} \quad (4.2)$$

and such that for each $x \in H(\mathcal{F})$ and each $h \in N(\mathcal{F})$

$$h(K) \cap B_{2\delta}(x) \neq \emptyset \implies x \in h(K). \quad (4.3)$$

The latter implication says that a “neighbor” $h(K)$ of K that is 2δ -close to an intersection point of K (i.e., to an element of the finite set $H(\mathcal{F})$) already contains this intersection point.

The set $K \setminus (H(\mathcal{F}))_\delta$ is nonempty (by the choice of δ and by connectedness of K) and compact. Since $N(\mathcal{F})$ is finite, and $h(K)$ is compact for each $h \in N(\mathcal{F})$, there is some $k \geq 1$ such that

$$\min \left\{ \|x - y\| : x \in K \setminus (H(\mathcal{F}))_\delta, y \in \bigcup_{h \in N(\mathcal{F})} h(K) \right\} > \frac{\delta}{2^{k-1}}. \quad (4.4)$$

Note that (4.4) guarantees that, for every open set V with diameter less than $\frac{\delta}{2^{k-1}}$ and every $h \in N(\mathcal{F})$,

$$\overline{V} \cap K \neq \emptyset \text{ and } \overline{V} \cap h(K) \neq \emptyset \implies \overline{V} \cap (H(\mathcal{F}))_\delta \neq \emptyset. \quad (4.5)$$

Set

$$\mathcal{U}_1 := \{V \in C(\delta 2^{-k}): V \cap (K \setminus (H(\mathcal{F}))_\delta) \neq \emptyset\}. \quad (4.6)$$

In Figure 13 we illustrate this cover for the *Hata tree*, a self-similar set that will appear in the examples later (see Examples 4.13). The shaded trees are (parts of) $h(K)$ for $h \in N(\mathcal{F})$. The set $(H(\mathcal{F}))_\delta$ is indicated by shaded disks; the estimate in (4.4) states that outside these disks the attractor K is separated from its “neighbors” $h(K)$ by a distance of at least $\frac{\delta}{2^{k-1}}$. Thus no element of \mathcal{U}_1 can intersect such a neighbor. We now iteratively define a sequence $(\mathcal{U}_n)_{n \geq 1}$ of covers of K

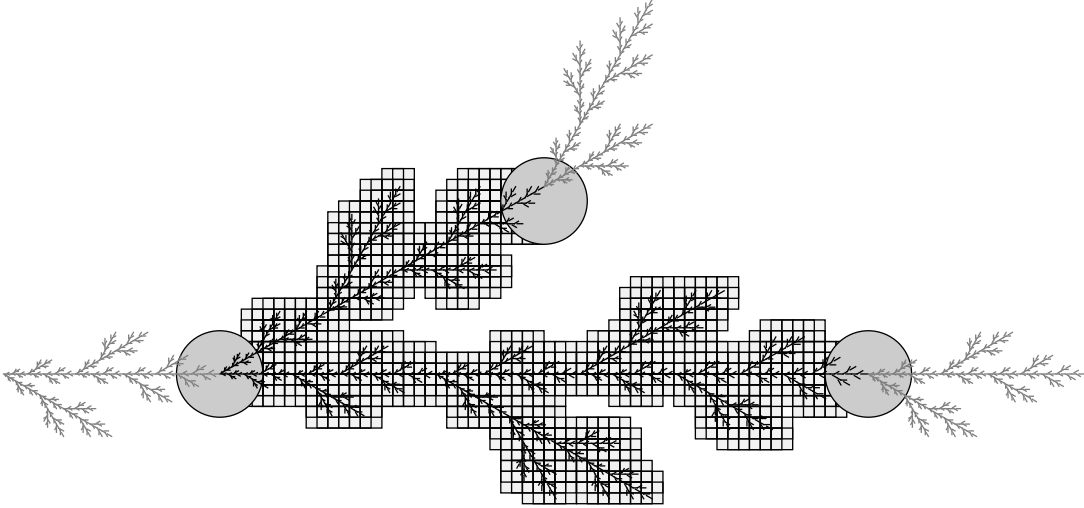


FIGURE 13. The cover $\mathcal{U}_1 \cup \{(H(\mathcal{F}))_\delta\}$ of the Hata tree K .

as follows. Suppose that $\mathcal{U}_1, \dots, \mathcal{U}_{n-1}$ have already been defined. Let

$$\mathcal{P}_n = \{\mathbf{i} \in \mathcal{Q}_{\delta^n} : f_{\mathbf{i}}(K) \cap [H(\mathcal{F})]_{\delta^{n-1}} \neq \emptyset\}. \quad (4.7)$$

For each $\mathbf{i} \in \mathcal{P}_n$ we set

$$\mathcal{V}_{\mathbf{i}} = \{V \in f_{\mathbf{i}}(C(\delta 2^{-k})): V \cap (K \setminus (H(\mathcal{F}))_{\delta^n}) \neq \emptyset\}. \quad (4.8)$$

We set

$$\mathcal{U}_n = \mathcal{U}_{n-1} \cup \bigcup_{\mathbf{i} \in \mathcal{P}_n} \mathcal{V}_{\mathbf{i}}. \quad (4.9)$$

Note that, by Lemma 4.10, there exists a constant c such that $\#\mathcal{P}_n \leq c$ for all $n \in \mathbb{N}$. This uniform boundedness will be essential later.

Lemma 4.11. *Let $(\mathcal{U}_n)_{n \geq 1}$ be defined as above and let $U_n := \bigcup_{U \in \mathcal{U}_n} U$. Then the following assertions hold.*

- (1) *For each $n \geq 1$, the collection \mathcal{U}_n is an open cover of $K \setminus (H(\mathcal{F}))_{\delta^n}$.*
- (2) *For each $n \geq 1$, each $\varrho > 0$, and each $\mathbf{j} \in \mathcal{Q}_{\varrho}$, we have*

$$f_{\mathbf{j}}(\overline{U}_n) \cap \bigcup_{\substack{\mathbf{k} \in \mathcal{Q}_{\varrho} \\ \mathbf{k} \neq \mathbf{j}}} f_{\mathbf{k}}(K) = \emptyset.$$

Proof. In this proof the choice of δ satisfying (4.2) and (4.3) will be of great importance.

Assertion (1) is proved by induction. For \mathcal{U}_1 this assertion follows from its definition in (4.6). Now assume that (1) holds for \mathcal{U}_{n-1} . By the definition of \mathcal{P}_n in (4.7) the union $\bigcup_{\mathbf{i} \in \mathcal{P}_n} \mathcal{V}_{\mathbf{i}}$ covers $[H(\mathcal{F})]_{\delta^{n-1}} \setminus (H(\mathcal{F}))_{\delta^n}$ and, hence, by (4.9), (1) also holds for \mathcal{U}_n .

To prove assertion (2), we use induction as well. Choose two distinct elements $\mathbf{j}, \mathbf{k} \in \mathcal{Q}_\varrho$ arbitrary and set $h := f_{\mathbf{j}}^{-1} \circ f_{\mathbf{k}}$. By the definition of \mathcal{Q}_ϱ we have $\text{sr}(h) \in (r_{\min}, r_{\min}^{-1})$, and by Lemma 4.8, $h \neq \text{id}$.

To show (2) for $n = 1$ it suffices to prove that

$$\overline{U}_1 \cap h(K) = \emptyset. \quad (4.10)$$

If $h \notin N(\mathcal{F})$ then $[K]_\varepsilon \cap h([K]_\varepsilon) = \emptyset$ by our choice of ε . Thus, because $\delta < \frac{\varepsilon}{4}$ and $U_1 \subset [K]_\delta$ by (4.6), we get (4.10). It remains to deal with $h \in N(\mathcal{F})$. In this case (4.10) follows immediately from (4.4) and (4.6) because the diameter of the elements of $C(\delta 2^{-k})$ is $\delta 2^{-k+1/2} < \delta 2^{-k+1}$.

Now assume that assertion (2) holds for $n - 1$. In view of (4.9) it is sufficient to show that for each $\mathbf{i} \in \mathcal{P}_n$ and each $V \in \mathcal{V}_{\mathbf{i}}$ we have

$$\overline{V} \cap h(K) = \emptyset. \quad (4.11)$$

If $h \notin N(\mathcal{F})$ then $[K]_\varepsilon \cap h([K]_\varepsilon) = \emptyset$. By (4.8) we have $V \subset [K]_{\delta 2^{-k+1/2}} \subset [K]_\varepsilon$ from which we immediately gain (4.11). Next we assume that $h \in N(\mathcal{F})$. We argue by contradiction. Suppose that (4.11) does not hold. Then there exists $\mathbf{l} \in \{1, \dots, m\}^*$ such that

$$\overline{V} \cap h \circ f_{\mathbf{l}}(K) \neq \emptyset, \quad (4.12)$$

and such that $g := f_{\mathbf{i}}^{-1} \circ h \circ f_{\mathbf{l}} = f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}}^{-1} \circ f_{\mathbf{k}} \circ f_{\mathbf{l}}$ has a similarity ratio $\text{sr}(g) \in [r_{\min}, 1)$. Applying $f_{\mathbf{i}}^{-1}$ to (4.12) we gain

$$f_{\mathbf{i}}^{-1}(\overline{V}) \cap g(K) \neq \emptyset. \quad (4.13)$$

Since $f_{\mathbf{i}}^{-1}(\overline{V}) \in C(\delta 2^{-k})$ has diameter bounded by $\delta 2^{-k+1/2} < \frac{\delta}{2^{k-1}} < \delta < \frac{\varepsilon}{4}$ and satisfies $f_{\mathbf{i}}^{-1}(\overline{V}) \cap K \neq \emptyset$, we conclude that $g \in N(\mathcal{F})$. Thus (4.5) implies that

$$f_{\mathbf{i}}^{-1}(\overline{V}) \cap (H(\mathcal{F}))_\delta \neq \emptyset. \quad (4.14)$$

Observe that, by (4.2), δ is so small that $f_{\mathbf{i}}^{-1}(\overline{V})$ can intersect the δ -neighborhood of at most one point $x \in H(\mathcal{F})$. By (4.13), for this point x we have $f_{\mathbf{i}}^{-1}(\overline{V}) \subset B_{2\delta}(x)$. Thus (4.14) yields $B_{2\delta}(x) \cap g(K) \neq \emptyset$, and by the definition of δ in (4.3) we gain that $x \in K \cap g(K)$. Combining this with (4.14) again, yields $f_{\mathbf{i}}^{-1}(\overline{V}) \cap (K \cap g(K))_\delta \neq \emptyset$. Applying the similarity transformation $f_{\mathbf{i}}$ and observing that $\text{sr}(f_{\mathbf{i}}) \leq \delta^n$ we derive

$$\emptyset \neq \overline{V} \cap (f_{\mathbf{i}}(K) \cap h \circ f_{\mathbf{l}}(K))_{\delta^{n+1}} \subset \overline{V} \cap (K \cap h(K))_{\delta^{n+1}} \subset \overline{V} \cap (H(\mathcal{F}))_{\delta^{n+1}}. \quad (4.15)$$

On the other hand, by the definition of $\mathcal{V}_{\mathbf{i}}$ we have that $\overline{V} \setminus (H(\mathcal{F}))_{\delta^n} \neq \emptyset$. And because $\text{diam}(\overline{V}) < \delta^{n+1} < \delta^n - \delta^{n+1}$, since $\delta < \frac{1}{4}$, we have $\overline{V} \cap (H(\mathcal{F}))_{\delta^{n+1}} = \emptyset$, a contradiction to (4.15). This concludes the proof. \square

We are now in a position to exhibit a path α that surrounds small copies of K by intersecting K in finitely many points.

Lemma 4.12. *Let $(U_n)_{n \geq 1}$ be as in Lemma 4.11, and $O := \bigcup_{n \geq 1} U_n$. The boundary of the unbounded component⁵ of $\mathbb{R}^2 \setminus O$ is a path α of finite length satisfying $f_{\mathbf{i}}(\alpha) \cap K = f_{\mathbf{i}}(H(\mathcal{F}))$ for each $\mathbf{i} \in \{1, \dots, m\}^*$.*

Proof. Each element of \mathcal{U}_n is a square. We want to calculate the sum ℓ_n of the lengths $\ell(\partial U)$ of all loops⁶ in the set $L_n := \{\partial U : U \in \mathcal{U}_n\}$. Because $\#\mathcal{P}_n \leq c$ uniformly in n , we gain from (4.8),

⁵Note that in general, $\mathbb{R}^2 \setminus O$ is not connected.

⁶Recall that a loop in \mathbb{R}^d is a continuous injection $\alpha : \mathbb{S}^1 \rightarrow \mathbb{R}^d$. Since α can be considered as a path in \mathbb{R}^d , its length is well defined.

(4.9), and (4.6)

$$\begin{aligned}
\ell_n &:= \sum_{U \in \mathcal{U}_n} \ell(\partial U) = \sum_{U \in \mathcal{U}_1} \ell(\partial U) + \sum_{j=2}^n \sum_{\mathbf{i} \in \mathcal{P}_j} \sum_{U \in \mathcal{V}_j} \ell(\partial U) \\
&\leq 4\delta 2^{-k} \#C(\delta 2^{-k}) + \sum_{j=2}^n c4\delta^j 2^{-k} \#C(\delta 2^{-k}) \\
&\leq 2^{-k+2} c \#C(\delta 2^{-k}) \sum_{j=1}^n \delta^j < 2^{-k+2} c \#C(\delta 2^{-k})
\end{aligned} \tag{4.16}$$

(in the last inequality we used that $\delta < \frac{1}{4}$). Because the last term in (4.16) does not depend on n , the sequence $(\ell_n)_{n \geq 1}$ is bounded.

Set $V_n := U_n \cup (H(\mathcal{F}))_{\delta^{n-1}}$. Let α_n be the boundary of the unbounded complementary component of V_n . By Lemma 4.11 (1), $\mathcal{U}_n \cup \{B_{\delta^{n-1}}(x) : x \in H(\mathcal{F})\}$ is an open cover of the connected set K each of whose elements has nonempty intersection with K . Thus its closure \bar{V}_n is a non-degenerate Peano continuum without cut points and, hence, Torhorst's Theorem (see [38, §61, II, Theorem 4, (ii)]) shows that α_n is a loop. Because this loop is contained in $L_n \cup \partial[H(\mathcal{F})]_{\delta^{n-1}}$, also the length of α_n , $n \in \mathbb{N}$, is bounded by some constant ℓ_{\max} (note that $\partial[H(\mathcal{F})]_{\delta^{n-1}}$ is the union of finitely many circles of radius δ^{n-1}).

By the definition of V_n , for each $\eta > 0$ there is $N \in \mathbb{N}$ such that $V_n \Delta V_m \subset [H(\mathcal{F})]_\eta$ for all $n, m > N$. Thus the sequence $(\alpha_n)_{n \in \mathbb{N}}$ converges uniformly to a loop α , whereby α coincides with α_n on the complement of $[H(\mathcal{F})]_{\delta^n}$ for every n . Hence the length of α is bounded by ℓ_{\max} . By construction, α is the boundary of the unbounded complementary component of O and $\alpha \cap K = H(\mathcal{F})$, which is finite. From this we get $f_{\mathbf{i}}(\alpha) \cap K = f_{\mathbf{i}}(H(\mathcal{F}))$. Because we can regard α as a path, the result follows. \square

Proof of Theorem 4.6. Because of the finiteness condition $\#f_{\mathbf{i}}(K) \cap f_{\mathbf{j}}(K) < \infty$, it follows from Lemma 4.7 that K has empty interior. Moreover, by Lemma 4.9, K satisfies the open set condition. However, according to [61, (2.3) Corollary], a self-similar set satisfying the open set condition has empty interior if and only if it has Lebesgue measure zero. Thus K has Lebesgue measure zero. Moreover, Theorem 4.3 implies that K has finite types of neighborhoods. Let O be as in Lemma 4.12. By Lemma 4.12, the boundary of the unbounded complementary component of O is a path α of finite length satisfying $f_{\mathbf{i}}(\alpha) \cap K = f_{\mathbf{i}}(H(\mathcal{F}))$ for each $\mathbf{i} \in \{1, \dots, m\}^*$.

Let x, y be two points in $x, y \in \mathbb{R}^2 \setminus K$. Since K is compact, we may choose $\delta > 0$ such that $\min\{d(x, K), d(y, K)\} > \frac{\delta}{4}$. By Lemma 1.6, to prove permeability of K , we have to construct a path $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ connecting x and y with $\ell(\gamma) < \|x - y\| + \delta$, and with $K \cap \gamma$ countable.

Let g be the line passing through x and y . Because K is a Lebesgue nullset, Fubini's theorem implies that there is a line $g' \subset [g]_{\delta/4}$, parallel to g , such that $\lambda_{g'}(g' \cap K) = 0$ (here, $\lambda_{g'}$ is the one-dimensional Lebesgue measure on g'). Let x' and y' be the orthogonal projections of x and y to g' , respectively. Since $\min\{d(x, K), d(y, K)\} > \frac{\delta}{4}$, the line segments $\overline{x'x'}$ and $\overline{y'y'}$ have empty intersection with K . Let c be the constant from Lemma 4.10 and note that c only depends on f_1, \dots, f_m . Since $\lambda_{g'}(g' \cap K) = 0$, and $g' \cap K$ is closed, we can cover $g' \cap K$, regarded as a subset of g' , with finitely many open intervals with total measure less than $\frac{\delta}{4c\ell(\alpha)}$. That means that $g' \cap K$, regarded as a subset of \mathbb{R}^2 , can be covered by finitely many disks $B_{r_1}(x_1), \dots, B_{r_k}(x_k)$ with $\sum_{j=1}^k 2r_j < \frac{\delta}{4c\ell(\alpha)}$. By Lemma 4.10, for every $j \in \{1, \dots, k\}$ there exists $\mathcal{D}_j \subset \mathcal{Q}_{2r_j}$ with $\#\mathcal{D}_j \leq c$ and

$$\bigcup_{\mathbf{i} \in \mathcal{D}_j} f_{\mathbf{i}}(K) \supseteq K \cap \overline{B_{r_j}(x_j)}.$$

Set $I := \mathcal{D}_1 \cup \dots \cup \mathcal{D}_k$. Then

$$\sum_{\mathbf{i} \in I} \text{sr}(f_{\mathbf{i}}) \leq \sum_{j=1}^k 2r_j c < 2c \frac{\delta}{4c\ell(\alpha)} = \frac{\delta}{2\ell(\alpha)}.$$

Consider the unbounded component of $\mathbb{R}^2 \setminus \left(g' \cup \bigcup_{i \in I} f_i(O)\right)$. Its boundary is contained in $g' \cup \bigcup_{i \in I} f_i(\alpha)$, which is path-connected and contains x' and y' , and so there exists a path $\beta \subset g' \cup \bigcup_{i \in I} f_i(\alpha)$ which connects x' and y' with

$$|\ell(\beta) - \|x' - y'\|| \leq \sum_{i \in I} \ell(f_i(\alpha)) = \ell(\alpha) \sum_{i \in I} \text{sr}(f_i) < \frac{\delta}{2},$$

and $\beta \cap f_i(K)$ is finite. Concatenating $\overline{xx'}$, β , $\overline{y'y}$ yields the desired path γ . \square

Examples 4.13. Theorem 4.6 applies to a wealth of examples. For instance, it implies permeability of the *von Koch Curve* (see [18, Chapter 3]), the *Sierpiński triangle* (see [62]), and the *Hata tree* (see [30]) (for standard references to these famous self-similar sets we refer for instance to [9, 21, 31]). Besides that, the examples of self-similar sets K in Barnsley [9, Chapter 7, Example 1.14], satisfying $\#f_i(K) \cap f_j(K) < \infty$ (which can be verified easily), are permeable by Theorem 4.6. In each of these examples the finite type condition is obviously satisfied.

We also refer to [6, 7], where the computer system `ifstile`⁷ is used to create interesting carpets. This system can be used to create further examples to which Theorem 4.6 can be applied.

The *Sierpiński carpet*⁸ K mentioned in Example 1.3 (10) is an impermeable self-similar set in the plane. It is a nondegenerate Peano continuum (this follows immediately, see [30, Theorem 4.6]) with $\dim_H(K) = \dim_B(K) = \frac{\log 8}{\log 3} \approx 1.8928$ (by [21, Theorem 9.3]). Moreover, it is clearly porous in the sense of Definition 3.18. This example shows that one cannot get rid of the condition $\#f_i(K) \cap f_j(K) < \infty$ in Theorem 4.6.

4.3. Null permeability of self-similar sets in higher dimensions. We now turn to self-similar sets in dimension $d \geq 3$. It turns out that Theorem 3.16 on the null permeability of sets with small Nagata dimension is useful here. The following theorem, which gives a criterion for null permeability of self-similar sets of dimension $d \geq 3$ is a consequence of this result. Indeed, we will show that the finite type condition from Definition 4.2 provides a criterion for a self-similar set to have Nagata dimension less than or equal to 1.

Theorem 4.14. *For $d \geq 3$ let $K \subset \mathbb{R}^d$ be the attractor of a self-similar IFS $\mathcal{F} := \{f_1, \dots, f_m\}$ satisfying $\#f_i(K) \cap f_j(K) < \infty$ for $1 \leq i < j \leq m$. Suppose further that \mathcal{F} is of finite type. Then K is null permeable.*

To prove this result, we show that the conditions of the theorem imply that the Nagata dimension of K is less than or equal to 1. This result is of interest in its own right.

Proposition 4.15. *For $d \geq 2$ let $K \subset \mathbb{R}^d$ be the attractor of a self-similar IFS $\mathcal{F} := \{f_1, \dots, f_m\}$ satisfying the finite type condition and $\#f_i(K) \cap f_j(K) < \infty$ for $1 \leq i < j \leq m$. Then $\dim_N(K) \leq 1$.*

Proof. Since the case where $\#K = 1$ is trivial, we assume that $\#K > 1$, so in particular, $\text{diam}(K) > 0$. W.l.o.g., we may assume that $\text{diam}(K) = 1$. By Theorem 4.3, K has finite types of neighborhoods, so, by Lemma 4.5, $H(\mathcal{F})$ is finite and there exists $\varepsilon \in (0, 1)$ such that $N_\varepsilon(\mathcal{F}) = N(\mathcal{F})$, and $N(\mathcal{F})$ is finite. Thus $Z := H(\mathcal{F}) \cup \{h(H(\mathcal{F})) : h \in N(\mathcal{F})\}$ is finite as well. W.l.o.g., we assume⁹

$$\varepsilon < \min\{\|x - y\| : x, y \in Z, x \neq y\}. \quad (4.17)$$

As before, let r_{\min} denote the minimum of the similarity ratios of f_1, \dots, f_m . Set

$$c_1 := \frac{\varepsilon r_{\min}}{8}, \quad (4.18)$$

$$c_2 := \frac{r_{\min}}{2} \min\{\|x - y\| : x \in K \setminus (H(\mathcal{F}))_{c_1}, y \in h(K \setminus (H(\mathcal{F}))_{c_1}), h \in N(\mathcal{F})\}, \quad (4.19)$$

⁷See <https://ifstile.com>.

⁸See e.g. [36, Example 1.3.17] for the definition of the Sierpiński carpet.

⁹As usual, a minimum over an empty set is infinite.

which are strictly positive by the finiteness of $N(\mathcal{F})$ and the compactness of $K \setminus (H(\mathcal{F}))_{c_1}$. Further define $c := \min\{c_1, c_2\}$ and let $s > 0$. We define a collection

$$\mathcal{U}_0 := \{f_{\mathbf{i}}(B_{2c_1}(x)) : x \in H(\mathcal{F}), \mathbf{i} \in \mathcal{Q}_{s/2}\}.$$

Note that \mathcal{U}_0 is not cs -separated, because it may contain different balls with the same center. Let \mathcal{U}_1 be the collection of sets obtained from \mathcal{U}_0 , if for every such center only the ball with the largest radius is chosen, *i.e.*,

$$\mathcal{U}_1 := \{B \in \mathcal{U}_0 : \text{there is no } B' \in \mathcal{U}_0 \text{ with } B \subsetneq B'\}.$$

Further set

$$\mathcal{U}_2 := \{f_{\mathbf{i}}(K \setminus [H(\mathcal{F})]_{c_1}) : \mathbf{i} \in \mathcal{Q}_{s/2}\}.$$

and consider the union $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$. From our choice (4.18) of c_1 , \mathcal{U} is an s -cover of K . Indeed, this is true for each element of \mathcal{U}_1 because $c_1 < \frac{s}{8}$ and, hence, $\text{diam}(f_{\mathbf{i}}(B_{2c_1}(x))) \leq \frac{s}{2} \cdot 4c_1 < s$ for every $\mathbf{i} \in \mathcal{Q}_{s/2}$. Moreover, $V = f_{\mathbf{i}}(K \setminus [H(\mathcal{F})]_{c_1}) \in \mathcal{U}_2$ has diameter less than or equal to s since $\text{sr}(f_{\mathbf{i}}) \leq \frac{s}{2}$ and $\text{diam}(K) = 1$.

It remains to show that both, \mathcal{U}_1 and \mathcal{U}_2 , are cs -separated. First consider two distinct elements $B_1, B_2 \in \mathcal{U}_1$ with centers x_1 and x_2 , respectively. Note that each of their radii is less than or equal to $c_1 s$. There exist $\mathbf{i}_1, \mathbf{i}_2 \in \mathcal{Q}_{s/2}$ and $z_1, z_2 \in H(\mathcal{F})$ such that $x_1 = f_{\mathbf{i}_1}(z_1)$ and $x_2 = f_{\mathbf{i}_2}(z_2)$. If $f_{\mathbf{i}_1}^{-1}(x_2) \notin Z$, *i.e.*, $f_{\mathbf{i}_1}^{-1} \circ f_{\mathbf{i}_2}(z_2) \notin Z$ then $f_{\mathbf{i}_1}^{-1} \circ f_{\mathbf{i}_2} \notin N(\mathcal{F}) \cup \{\text{id}\}$. Thus $[H(\mathcal{F})]_{\varepsilon} \cap (f_{\mathbf{i}_1}^{-1} \circ f_{\mathbf{i}_2})([H(\mathcal{F})]_{\varepsilon}) = \emptyset$, and, hence,

$$\begin{aligned} \|x_1 - x_2\| &= \|f_{\mathbf{i}_1}(z_1) - f_{\mathbf{i}_2}(z_2)\| \geq \frac{s}{2} r_{\min} \|z_1 - f_{\mathbf{i}_1}^{-1} \circ f_{\mathbf{i}_2}(z_2)\| \\ &\geq \frac{s}{2} r_{\min} \min\{\|z_1 - y\| : y \in f_{\mathbf{i}_1}^{-1} \circ f_{\mathbf{i}_2}(H(\mathcal{F}))\} \geq \frac{s}{2} r_{\min} \varepsilon = 4c_1 s, \end{aligned} \quad (4.20)$$

where we used (4.18) in the last equality. Now suppose $f_{\mathbf{i}_1}^{-1}(x_2) \in Z$. Since $f_{\mathbf{i}_1}$ is bijective and $x_1 \neq x_2$, this implies $f_{\mathbf{i}_1}^{-1}(x_2) \in Z \setminus \{z_1\}$. Hence, $\|z_1 - f_{\mathbf{i}_1}^{-1}(x_2)\| \geq \frac{8}{r_{\min}} c_1$ by (4.17) and (4.18), and because $\text{sr}(f_{\mathbf{i}_1}) \geq \frac{s}{2} r_{\min}$, we gain

$$\|x_1 - x_2\| \geq 4c_1 s. \quad (4.21)$$

Now, since B_1, B_2 are balls with radius less than or equal to $c_1 s$, we get from (4.20), (4.21), and the inequality $c_1 \geq c$ that

$$\min\{\|x - y\| : x \in B_1, y \in B_2\} \geq \|x_1 - x_2\| - 2c_1 s \geq 4c_1 s - 2c_1 s > cs,$$

so \mathcal{U}_1 is cs -separated.

Finally, let $\mathbf{i}, \mathbf{j} \in \mathcal{Q}_{s/2}$, with $\mathbf{i} \neq \mathbf{j}$. If $f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}} \notin N(\mathcal{F})$, then $f_{\mathbf{i}}([K]_{\varepsilon}) \cap f_{\mathbf{j}}([K]_{\varepsilon}) = \emptyset$, and therefore

$$\begin{aligned} &\min\{\|x - y\| : x \in f_{\mathbf{i}}(K \setminus (H(\mathcal{F}))_{c_1}), y \in f_{\mathbf{j}}(K \setminus (H(\mathcal{F}))_{c_1})\} \\ &\geq \min\{\|x - y\| : x \in f_{\mathbf{i}}(K), y \in f_{\mathbf{j}}(K)\} > 2\varepsilon \frac{s}{2} r_{\min} = \varepsilon s r_{\min} \geq cs, \end{aligned}$$

where we used (4.18) and the inequality $c_1 \geq c$ in the last estimate. If $f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}} \in N(\mathcal{F})$, then by (4.19) and the inequality $c_2 \geq c$ we have

$$\begin{aligned} &\min\{\|x - y\| : x \in f_{\mathbf{i}}(K \setminus (H(\mathcal{F}))_{c_1}), y \in f_{\mathbf{j}}(K \setminus (H(\mathcal{F}))_{c_1})\} \\ &= \text{sr}(f_{\mathbf{i}}) \min\{\|x - y\| : x \in K \setminus (H(\mathcal{F}))_{c_1}, y \in f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}}(K \setminus (H(\mathcal{F}))_{c_1})\} \\ &\geq \text{sr}(f_{\mathbf{i}}) \frac{2c_2}{r_{\min}} \geq \frac{s}{2} r_{\min} \frac{2c_2}{r_{\min}} \geq cs. \end{aligned}$$

We have shown that \mathcal{U}_2 is cs -separated.

Thus, according to Definition 3.12 (2), K has Nagata dimension less than or equal to 1. \square

On close examination of the previous proof one sees that, if $H(\mathcal{F}) = \emptyset$, then already \mathcal{U}_2 is a cs separated set, and therefore $\dim_N(K) = 0$. The proof remains correct also in this case.

Proof of Theorem 4.14. From Proposition 4.15 we gain $\dim_N(K) \leq 1$. Since $d \geq 3 \Leftrightarrow 1 \leq d - 2$ the result follows from Theorem 3.16. \square

It is reasonable to conjecture that, with suitable assumptions on \mathcal{F} , we have $\dim_N(K) \leq k+1$ if $\dim_N(f_i(K) \cap f_j(K)) \leq k$ for $i \neq j$. Although this would lead to a generalization of Theorem 4.14, we do not pursue this here because in general, for a given self-similar set K , the inequalities $\dim_N(f_i(K) \cap f_j(K)) \leq k$ ($i \neq j$) do not seem to be easier to check than $\dim_N(K) \leq k+1$.

Example 4.16 (Sierpiński tetrahedron). In Example 3.7 we have already shown that the Sierpiński tetrahedron in \mathbb{R}^3 is permeable. Theorem 4.14 gives the stronger result that the Sierpiński tetrahedron is null permeable.

4.4. Final example: A Bedford-McMullen carpet. The classical Sierpiński carpet is the attractor of a self-similar IFS, it can be represented explicitly as

$$\left\{ \sum_{k=1}^{\infty} 3^{-k} x_k : x_1, x_2, \dots \in \{0, 1, 2\}^2 \setminus \{(1, 1)\} \right\} \subset [0, 1]^2.$$

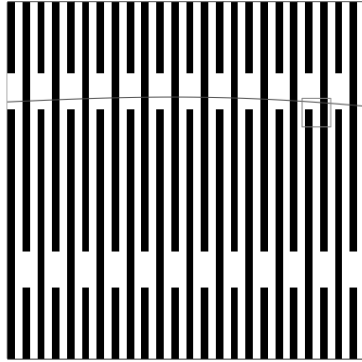
As already stated in Examples 1.3(10), the classical Sierpiński carpet is impermeable. It has Lebesgue measure 0 and topological dimension 1. We want to construct an impermeable subset of \mathbb{R}^2 of measure 0 and topological dimension 0. The construction is similar to that of [29, Section 4.C.1] and can be generalized to subsets of \mathbb{R}^d for $d \geq 1$, *cf.* Theorem 4.19.

A *Bedford-McMullen carpet* (BMC, for short; see, *e.g.*, [26] for a recent survey on these objects) is a self-affine subset of $[0, 1]^2$ that is defined as follows: Fix $n, m \in \mathbb{N}$ and consider a fixed subset $R \subset \{0, \dots, n-1\} \times \{0, \dots, m-1\}$, which we will call $n \times m$ -*pattern*. Set

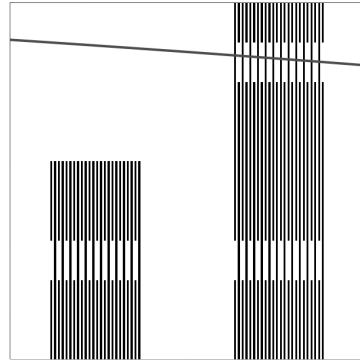
$$K_R := \left\{ \sum_{k=1}^{\infty} T^k x_k : x_1, x_2, \dots \in R \right\} \subset [0, 1]^2,$$

where $T(x, y) := \left(\frac{x}{n}, \frac{y}{m}\right)$, $(x, y) \in \mathbb{R}^2$. The closed set K_R is the attractor of the self-affine IFS $\{f_{(i,j)} : (i, j) \in R\}$ consisting of the functions

$$f_{(i,j)}(x, y) := \left(\frac{x}{n}, \frac{y}{m}\right) + \left(\frac{i}{n}, \frac{j}{m}\right), \quad (i, j) \in R. \quad (4.22)$$



(A) A path crossing the BMC



(B) A magnified section of (A)

FIGURE 14. A Bedford-McMullen carpet

For our purposes we consider the 48×10 -pattern R depicted in Figure 14 (A), *i.e.*, we set

$$R := \{(2i, j) : 0 \leq i < 24, 0 \leq j < 10, j \not\equiv 5i + 7 \pmod{10}\}.$$

Clearly, $K := K_R$ is closed and has Lebesgue measure 0. We show that K is impermeable. For this purpose we formulate a lemma. In its statement recall that the total variation $V_a^b(f)$ of a

function $f: [a, b] \rightarrow \mathbb{R}$ is defined as

$$V_a^b(f) := \sup \left\{ \sum_{k=1}^N |f(t_k) - f(t_{k-1})| : N \in \mathbb{N}, a = t_0 < \dots < t_N = b \right\}.$$

For a path γ in \mathbb{R}^2 we denote the coordinate functions of γ by $(\gamma)_1$ and $(\gamma)_2$, respectively.

Having $n = 48$ and $m = 10$, let $R_2 := R \cup (R + \{(0, 10)\})$ and let

$$f_{(i,j)}(x, y) := \left(\frac{x}{n}, \frac{y}{m} \right) + \left(\frac{i}{n}, \frac{j}{m} \right), \quad (i, j) \in R_2. \quad (4.23)$$

In the next lemma and its proof, we say that $\gamma: [a, b] \rightarrow \mathbb{R}^2$ is a path from $X \subset \mathbb{R}^2$ to $Y \subset \mathbb{R}^2$, if $\gamma(a) \in X$ and $\gamma(b) \in Y$.

Lemma 4.17. *Let $\gamma: [a, b] \rightarrow \mathbb{R}^2$ be a path from $\{0\} \times [0, 2]$ to $\{1\} \times [0, 2]$, satisfying $\gamma([a, b]) \subset [0, 1] \times [0, 2]$ and $V_a^b((\gamma)_2) < 1$. Then there exist elements $(i_1, j_1), (i_2, j_2) \in R_2$ with $i_1 < i_2$ and $(i_1, j_1 + 1), (i_2, j_2 + 1) \in R_2$ with the following property: There exist $a_1 < b_1 < a_2 < b_2$ such that, for $k \in \{1, 2\}$, $\gamma_k := \gamma|_{[a_k, b_k]}$ is a path from $f_{(i_k, j_k)}(\{0\} \times [0, 2])$ to $f_{(i_k, j_k)}(\{1\} \times [0, 2])$ satisfying $\gamma_k([a_k, b_k]) \subset f_{(i_k, j_k)}([0, 1] \times [0, 2])$ and $V_{a_k}^{b_k}((\gamma_k)_2) < \frac{1}{10}$.*

Proof. For every $i \in \{0, \dots, 47\}$ we define $s_i := \sup\{t \in [a, b] : (\gamma)_1(t) = \frac{i}{48}\}$ and $t_i := \inf\{t \in [s_i, b] : (\gamma)_1(t) = \frac{i+1}{48}\}$. Since γ is a path from $\{0\} \times [0, 2]$ to $\{1\} \times [0, 2]$, s_i and t_i are well-defined and $s_i, t_i \in [a, b]$ with $s_i < t_i$.

Since $a \leq s_i < t_i \leq s_{i+1} \leq b$ for all $i \in \{0, \dots, 47\}$, the intervals from the collection $\{[s_{4\nu}, t_{4\nu+3}] : \nu \in \{0, \dots, 11\}\}$ have mutually disjoint interiors. Moreover,

$$\sum_{\nu=0}^{11} V_{s_{4\nu}}^{t_{4\nu+3}}((\gamma)_2) \leq V_a^b((\gamma)_2) < 1,$$

so there are at least 2 indices $\nu_1, \nu_2 \in \{0, \dots, 11\}$, $\nu_1 < \nu_2$, with $V_{s_{4\nu_k}}^{t_{4\nu_k+3}}((\gamma)_2) < 10^{-1}$ for $k \in \{1, 2\}$. Thus for each $k \in \{1, 2\}$ there exist $j_k \in \{0, \dots, 18\}$ with

$$\gamma([s_{4\nu_k}, t_{4\nu_k+3}]) \subset \left[\frac{4\nu_k}{48}, \frac{4\nu_k+4}{48} \right] \times \left[\frac{j_k}{10}, \frac{j_k+2}{10} \right].$$

By the definition of R (resp. R_2) for every $k \in \{1, 2\}$ there is at least one index $i_k \in \{4\nu_k, 4\nu_k+2\}$ with $\{(i_k, j_k), (i_k, j_k+1)\} \subseteq R_2$ (see Figure 15 for an illustration). With this choice we have

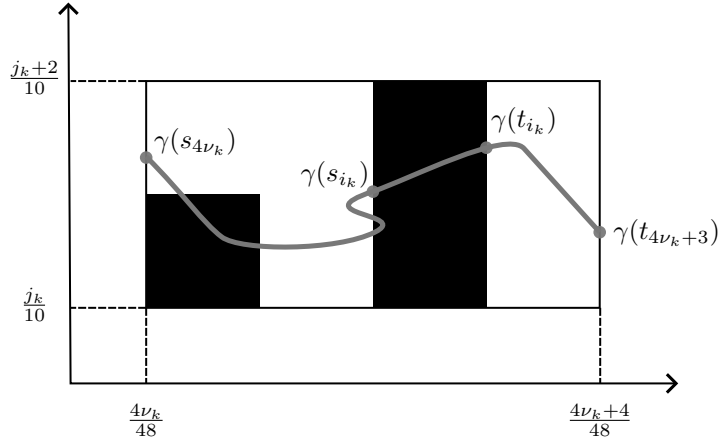


FIGURE 15. An illustration of a 4×2 subblock of the pattern in Figure 14. The black squares correspond to elements $(i, j) \in R$. Each such subblock contains a vertical column of two black squares.

$\gamma|_{[s_{i_k}, t_{i_k}]} \subseteq f_{(i_k, j_k)}([0, 1] \times [0, 2])$. Thus, setting $a_k = s_{i_k}$ and $b_k = t_{i_k}$, the path $\gamma_k := \gamma|_{[a_k, b_k]}$ satisfies the assertions of the lemma (note that $i_1 < i_2$ by construction). \square

Now let $\gamma: [0, 1] \rightarrow \mathbb{R}^2$ be a path connecting the points $(0, \frac{1}{2})$ and $(1, \frac{1}{2})$ with $V_0^1((\gamma)_2) < \frac{1}{2}$. Let $a = a_{0,1} := \sup\{t \in [0, 1]: (\gamma)_1(t) = 0\}$ and $b = b_{0,1} := \inf\{t \in [a, 1]: (\gamma)_1(t) = 1\}$. Then $\gamma|_{[a,b]}$ satisfies the assumptions of Lemma 4.17 and thus, by iteratively applying this lemma, there exist double sequences $((i_{l,k}, j_{l,k}))_{l \geq 0, 1 \leq k \leq 2^l}$ in R_2 and $([a_{l,k}, b_{l,k}])_{l \in \mathbb{N}, 1 \leq k \leq 2^l}$ of intervals, such that $([a_{l,k}, b_{l,k}])_{1 \leq k \leq 2^l}$ are disjoint for every l , $\bigcup_{k=1}^{2^l} [a_{l,k}, b_{l,k}] \subset \bigcup_{k=1}^{2^{l-1}} [a_{l-1,k}, b_{l-1,k}]$ and

$$\gamma\left(\bigcup_{k=1}^{2^l} [a_{l,k}, b_{l,k}]\right) \subset \bigcup_{k=1}^{2^l} f_{(i_{1,k}, j_{1,k})} \circ \cdots \circ f_{(i_{l,k}, j_{l,k})}([0, 1] \times [0, 2]) \subset \bigcup_{j \in R^l} f_j([0, 1]^2),$$

where the $f_{(i_{l,k}, j_{l,k})}$ map into $[0, 1]^2$ since γ is contained in $[0, 1]^2$ (otherwise its variation would exceed $\frac{1}{2}$). Note that $(\gamma)_1\left(\bigcup_{k=1}^{2^l} [a_{l,k}, b_{l,k}]\right)$ is the disjoint union of 2^l closed intervals of length $\frac{1}{48^l}$. Thus

$$\bigcap_{l \geq 0} (\gamma)_1\left(\bigcup_{k=1}^{2^l} [a_{l,k}, b_{l,k}]\right)$$

is a Cantor set, and, hence,

$$\bigcap_{l \geq 0} \gamma\left(\bigcup_{k=1}^{2^l} [a_{l,k}, b_{l,k}]\right) \subset \bigcap_{l \geq 0} \bigcup_{j \in R^l} f_j([0, 1]^2) = K.$$

is uncountable. Note that the $\|\cdot\|_1$ -length of γ is $V_0^1((\gamma)_1) + V_0^1((\gamma)_2)$. We have shown that if $\gamma: [0, 1] \rightarrow \mathbb{R}^2$ is a path from $(0, \frac{1}{2})$ to $(1, \frac{1}{2})$ with $V_0^1(\gamma_2) < \frac{1}{2}$, then $\gamma \cap K$ is uncountable, and thus K is impermeable with respect to the 1-norm on \mathbb{R}^2 . From impermeability of K with respect to the 1-norm follows impermeability with respect to the Euclidean norm, cf. Theorem 1.8.

It is not hard to see that K is totally disconnected. Since it is also compact, by [55, II. Section 3.2, Theorem 5] we have $\dim_T(K) = 0$. We thus have shown the following proposition.

Proposition 4.18. *There exists an impermeable set in \mathbb{R}^2 which is compact, has Lebesgue measure 0 and topological dimension 0.*

The example is easily generalized to several dimensions: using the same $n \times m$ -pattern R as above, consider

$$\tilde{K}_d := K \times [0, 1]^{d-2} = \left\{ \sum_{k=1}^{\infty} \{T^k x_k\} \times [0, 1]^{d-2} : x \in R^{\mathbb{N}} \right\} \subset [0, 1]^d,$$

which provides us with an example of a set of topological dimension $d - 2$ and measure 0, which is impermeable. Alternatively, we may write

$$\tilde{K}_d = \left\{ \sum_{k=1}^{\infty} \tilde{T}^k x_k : x \in (R \times \{0, \dots, 9\}^{d-2})^{\mathbb{N}} \right\} \subset [0, 1]^d,$$

where $\tilde{T}y = (48^{-1}y_1, 10^{-1}y_2, \dots, 10^{-1}y_d)$ for $y \in \mathbb{R}^d$. Now define

$$P_k: \mathbb{R}^d \rightarrow \mathbb{R}^d: (y_1, \dots, y_d) \mapsto (y_1, y_{2+(k \pmod{d-1})}, \dots, y_{2+(k+d-2 \pmod{d-1})}).$$

The purpose of P_k is to cyclically shift the last $d - 1$ coordinates of an element of \mathbb{R}^d by k places modulo $d - 1$. Define

$$K_d := \left\{ \sum_{k=1}^{\infty} \tilde{T}^k P_k x_k : x \in (R \times \{0, \dots, 9\}^{d-2})^{\mathbb{N}} \right\} \subset [0, 1]^d.$$

Then K_d is compact and totally disconnected, therefore has topological dimension 0 (by [55, II. Section 3.2, Theorem 5]), has measure 0 and, along the same lines as for the 2-dimensional case, one can show that it is impermeable. We end by stating this result as a theorem, adding the trivial case $d = 1$, where the middle third Cantor set is a suitable example.

Theorem 4.19. *For each $d \geq 1$ there exists an impermeable set in \mathbb{R}^d which is compact, has Lebesgue measure 0 and topological dimension 0.*

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