BOUNDEDNESS ESTIMATES FOR NONLINEAR NONLOCAL KINETIC KOLMOGOROV-FOKKER-PLANCK EQUATIONS

FRANCESCA ANCESCHI AND MIRCO PICCININI

Abstract. We investigate local properties of weak solutions to the following wide class of kinetic equations,

$$(\partial_t + v \cdot \nabla_x) f = \mathcal{L}_v f.$$

Above, the diffusion term $\mathcal{L}_v$ is an integro-differential operator whose nonnegative kernel is of differentiability order $s \in (0,1)$ and integrability oredr $p \in (2, \infty)$, having merely measurable coefficients. In particular, we provide explicit interpolative L^∞-L^2 estimates for weak subsolutions.

1. Introduction

In this paper we deal with a wide class of kinetic equations, whose diffusion part is an integro-differential operator of differentiability order $s \in (0,1)$ and summability order $p \in (2, \infty)$. In particular, we investigate local properties of weak solutions $f \equiv f(t,x,v)$ to the following class of equations

$$(\partial_t + v \cdot \nabla_x) f = \mathcal{L}_v f \qquad \text{for } (t,x,v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n\,, \tag{1.1}$$

where the diffusion term $\mathcal{L}_v$ is given by

$$\mathcal{L}_v f(t,x,v) := \text{p. v.} \int_{\mathbb{R}^n} |f(v) - f(w)|^{p-2} (f(w) - f(v)) K(t,x,v,w)\, \mathrm{d}w. \tag{1.2}$$

Here, the symbol p. v. stands for "in the principal value sense" and K is a symmetric measurable kernel such that, for a. e. $(t,x) \in \mathbb{R}^{1+n}$ and for $\Lambda > 0$ it satisfies

$$\Lambda^{-1} |v-w|^{-n-sp} \le K(v,w) \le \Lambda |v-w|^{-n-sp} \qquad \text{for a. e. } v,w \in \mathbb{R}^n. \tag{1.3}$$

As a prototype for Equation (1.1), even though in this scenario the difficulties arising when dealing with only measurable coefficients vanishes, one can consider the simpler case when the involved kernel K does coincide with the classical Gagliardo one, i. e. $K \equiv |v-w|^{-n-sp}$. In this setting, equation (1.1) does reduce to

$$(\partial_t + v \cdot \nabla_x) f + (-\Delta_v)^s_p f = 0, \tag{1.4}$$

where $(-\Delta_v)^s_p$ is the classical (s,p)-Laplacian with respect to the v-variable.

In recent years, great attention has been focused on the study of nonlocal operators and their related fractional Sobolev spaces. For this reason, the literature is really too wide to attempt any precise treatment. However, we still mention [9, 10], where

2020 *Mathematics Subject Classification.* 35Q84, 35B45, 35B65, 47G20, 35R11, 35R05.

Key words and phrases. Kolmogorov-Fokker-Planck equations, kinetic equations, fractional Sobolev spaces, fractional Laplacian, nonlinear operators.

Aknowledgements. The first author is partially supported by the INdAM - GNAMPA project "Variational problems for Kolmogorov equations: long-time analysis and regularity estimates", CUP_E55F22000270001. The second author is partially supported by the INdAM - GNAMPA project "Fenomeni non locali in problemi locali", CUP_E55F22000270001.
Both authors are partially supported by the INdAM-GNAMPA Project "Problemi non locali: teoria cinetica e non uniforme ellitticità", CUP_E53C22001930001.

the authors proved various regularity results in the same spirit as the De Giorgi-Nash-Moser theory, and, among other things, they introduced a new quantity to measure the long-range interactions naturally arising when dealing with nonlocal problems, i. e. *the nonlocal tail of a function*

$$\mathrm{Tail}(f;B_r(v_0)) := r^{sp}\int_{\mathbb{R}^n\setminus B_r(v_0)} \frac{|f(t,x,v)|^{p-1}}{|v_0-v|^{n+sp}}\,\mathrm{d}v\,. \tag{1.5}$$

Such quantity has been subsequently proven to be decisive in the analysis of many nonlocal problems when a fine quantitative control of the naturally arising long-range interactions is needed. Indeed, after its introduction, a quite comprehensive nonlocal De Giorgi-Nash-Moser theory has been successfully developed in even more general integro-differential elliptic frameworks; see for example [9, 10, 8, 25, 6], the survey paper [27] and the references therein as well as the recent monograph [12].

The definition of nonlocal tail was later on extended to the parabolic framework and used to prove very fine quantitative estimates; see for example the breakthrough results by Kassmann and Weidner [17] on the Harnack inequality for fractional (linear) parabolic equations with very general measurable (possibly nonsymmetric) kernels. Moreover, in [30] a L^∞-version of (1.5) was used to prove local L^∞-L^2 estimate via Moser iteration technique for weak subsolutions to the nonlinear heat equation in the superquadratic case, when $p \geq 2$. Furthermore, such L^∞-tail was very recently employed to establish classical $C^{0,\alpha}$-regularity for any value of the integrability exponent $p \in (1,\infty)$; see for example [1, 20, 31].

For what regards the fractional panorama for Kolmogorov-Fokker-Planck equations, as in (1.4), the weak regularity theory is far from being complete and, to the best of our knowledge, the known results are only available in the linear case when $p = 2$; see for instance the survey paper [3] and the references therein. In particular, we refer the reader to the Hölder regularity results in [29], possibly including unbounded source terms, as well as the one in [21] covering more general, possibly nonsymmetric diffusion operators. Furthermore, regarding classical estimates, we mention the very recent breakthrough counterexample to the classical Harnack inequality ([18]) as well as its related new formulation in [2], where a strong Harnack inequality is proved provided that solutions have q-summable nonlocal tail along the transport variables for some $q > q^\star(n,s)$, which is in fact naturally implied in literature, e. g., from the usual mass density boundedness (as for the Boltzmann equation without cut-off), and in clear accordance with the aforementioned counterexample in [18]. Still in the flavor of Harnack-type inequalities, it is worth mentioning the very recent paper [22], in which amongst other results, the author proves a strong Harnack inequality for *global solutions*, a priori bounded, periodic in the space variable, and under an integral monotonicity-in-time assumption (see Definition 2.2 there). Finally, we mention [32] for the existence of weak solutions, and [14] for existence, uniqueness and regularity of solutions in the viscosity sense. Always regarding these existence and uniqueness issues, we also recall the very recent works [4, 5].

For what concerns the more general case when a p-growth exponent is involved, the scenario is basically empty. Then, to the best of our knowledge, our contribution would be a veritable first. In this respect, the forthcoming Theorem 1.1 serves as a first step in the direction of proving that solutions to (1.1) enjoy classical qualitative properties. Aside from the novelty of the result, interpolative estimates are very useful when one deals with local regularity or qualitative properties of solutions to (1.1). However, proving an L^∞-L^2 estimate for kinetic equations is not a simple task. Indeed, even in the linear case when $p = 2$ – as proven in the aforementioned work [18] – it is not in general possible to bound the L^∞-norm of a solution in terms of only local quantities *even starting from globally bounded solutions*. Moreover, a deeper analysis of the counterexample in [18] shows that such supremum estimate remains false also when an error term is added on its right-hand side − basically a tail-type contribution as in (1.5) − if the tail belongs to L^q, for $q < (n(1+2s))/(2s)$. Nevertheless, a L^∞-L^2 estimate plus a nonlocal tail remainder can be derived by assuming higher integrability on the tail function along the transport variables; see in particular Theorem 1.1 in the aforementioned [2].

However, despite the already mentioned recent achievement, the difficulties arising when dealing with a p-growth exponent are a concrete stumbling block. Indeed, the nonlinear growth setting precludes the free generalization of tools and techniques that had already proven to be very useful in providing quantitative properties of weak solutions to kinetic linear equations, as e. g., velocity averaging techniques ([7]) or potential estimates via the fundamental solution ([15]). Indeed, in [2], the backbone of the proof of the desired L^∞-L^2 estimate is an hypoelliptic gain of integrability, which is proven by making use of the fundamental solution of the fractional Kolmogorov equation. More specifically, as done in the classical framework for kinetic equations ([26]), the transfer of regularity is based on treating as source term the difference between the constant coefficients diffusion operator and the one with measurable entries, and then estimating its L^2-norm tracking down the long-range interactions appearing as L^q-norm of the tail quantity (1.5) on the right-hand side; see in particular [2, Lemma 3.1]. However, as well as for velocity averaging lemmas, such a procedure can not be pursued in the nonlinear setting we are dealing with. Hence, our analysis is carried out starting form locally bounded solutions, in accordance with what is classically done as e. g. in the case of the Boltzmann equation ([15, 21]), or even for kinetic equations in divergence form ([22]). In particular, in the superlinear case when $p \in (2,\infty)$, we prove that these solutions satisfy interpolative estimates in terms of their local and nonlocal contributions, the latter encoded in their L^σ-tail. The interpolative nature of the estimates below lies specifically in the arbitrariness in the choice of the parameter δ, which plays the role of an interpolation coefficient between the local and nonlocal part of the estimate.

Theorem 1.1. *Let $p \in (2,\infty)$, $s \in (0,1)$ and let $f \in \mathcal{W}$ be a weak subsolution to* (1.1) *in Ω according to* Definition 2.5 *such that $f \in L^\infty_{\rm loc}(Q_1)$, for $Q_1 \equiv Q_1(\mathbf{0}) \subset \Omega$. If for some $\sigma > \max(n/(sp), 2)$, $\mathrm{Tail}(f_+; B) \in L^\sigma_{\rm loc}((-1,0)\times B_1)$ for any $B \Subset B_1$, then it holds*

$$\sup_{Q_{\frac{r}{2}}} f \quad \leq \quad \frac{c}{\delta^\beta}\left(\fint_{Q_r} (1+f_+^p)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} + \delta\left(\fint_{(-r^{sp},0]\times B_{r^{1+sp}}} \mathrm{Tail}(f_+; B_{r/2})^\sigma\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\sigma}},$$

for any $0<r<1$ and any $\delta \in (0,1]$ and where $\beta \equiv \beta(n,s,p,\sigma)$ and $c \equiv c(n,s,p,\sigma,\Lambda)$.

We remark that the proof of our statement strongly relies on the boundedness assumption on the weak solution coupled with the higher integrability requirement on their nonlocal tail. These requirements may seem very strong, but as previously mentioned they are in accordance with the existing literature for Boltzmann equations, see for example [29, 15, 21, 22]. Lastly, we point out that in the "trickier" singular case when $p \in (1,2)$, the adaptation of the available techniques for the proof of a boundedness estimate does not seem straightforward and hence it will be the subject of further studies.

As in the classical theory, a fractional Caccioppoli-type inequality is needed in order to built the proper iteration scheme. Hence, we conclude by explicitly stating it for any $p \in (1,\infty)$, since it has an interest on its own for future developments in the weak regularity theory for solutions to (1.1) and related equations.

Theorem 1.2. *Let* $p \in (1,\infty)$*,* $s \in (0,1)$ *and* $Q_1 \equiv Q_1(\mathbf{0}) \subset \Omega$*. Let* f *be a weak subsolution to* (1.1) *in* Ω *according to* Definition 2.5*. For any* $Q_r \equiv Q_r(\mathbf{0}) \subset Q_1$ *the following estimate holds true*

$$
\begin{aligned}
\sup_{t\in[-r^{sp},0]} \int_{B_{r^{1+sp}}\times B_r} \omega^2\varphi^p \,\mathrm{d}x\,\mathrm{d}v + \int_{(-r^{sp},0]\times B_{r^{1+sp}}} [\omega\varphi]^p_{W^{s,p}(B_r)} \,\mathrm{d}x\,\mathrm{d}t \\
\leq c \int_{(-r^{sp},0]\times B_{r^{1+sp}}} \iint_{B_r\times B_r} \frac{\max\{\omega(v),\omega(w)\}^p|\varphi(v)-\varphi(w)|^p}{|v-w|^{n+sp}} \,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\
+ c\int_{Q_r} \omega\varphi^p \left(\sup_{v\in \operatorname{supp}\varphi} \int_{\mathbb{R}^n\setminus B_r} \frac{\omega^{p-1}(w)\,\mathrm{d}w}{|v-w|^{n+sp}} \right) \mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
+ c\int_{Q_r} |v\cdot\nabla_x\varphi|\varphi^{p-1}\omega^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + c\,r^{sp}\int_{Q_r} \omega^2\varphi^p \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned}
$$

where the constant $c>0$ *depends only on* p *and on the kernel constant* Λ*,* $\omega := (f-k)_+$*, for any* $k\in\mathbb{R}$*, and* $\varphi \equiv \varphi(x,v) \in C_0^\infty(B_{r^{1+sp}}\times B_r)$*.*

Outline of the paper. In Section 2 we introduce preliminary notions about the functional and geometrical setting of this work. Section 3 is devoted to the proof of Theorem 1.2. The proof of Theorem 1.1 is contained in Section 4.

2. Preliminaries

In this section, we recall some well known results about our underlying geometrical and functional setting. After fixing the notation, we introduce an appropriate geometric framework to study integral kinetic equation; then, we recall some properties of fractional Sobolev spaces and the functional setting required to deal with equation (1.1).

2.1. **Notation.** We denote with c a positive universal constant greater than one, which may change from line to line. For the sake of readability, dependencies of the constants will be often omitted within the chains of estimates, therefore stated after the estimate. Relevant dependencies on parameters will be emphasized by using parentheses.

As customary, for any $r>0$ and any $y_0\in\mathbb{R}^m$, $m\in\mathbb{N}$, we denote by

$$
B_r(y_0) \equiv B(y_0;r) := \big\{ y\in\mathbb{R}^m \,:\, |y-y_0|<r \big\},
$$

the open ball with radius r and center y_0. For any $\beta>0$ we will denote with $\beta B_r(y_0)$ the rescaled ball by a factor of β, i. e. $\beta B_r(y_0) = B_{\beta r}(y_0)$.

For any set $E\subset\mathbb{R}^m$ we will denote the Lebesgue measure of E with $|E|$. Moreover, for any $f\in L^1(E)$, we let

$$
(f)_E := \fint_E f \,\mathrm{d}y := \frac{1}{|E|}\int_E f\,\mathrm{d}y.
$$

For any $k\in\mathbb{R}$, we denote the positive and negative part of f, respectively, as

$$
(f(y)-k)_+ := \max\{f(y)-k,0\} \quad \text{and} \quad (f(y)-k)_- := \max\{k-f(y),0\}.
$$

Clearly $(f(y)-k)_+\neq 0$ on the super-level set $\{y\in\mathbb{R}^m \,:\, f(y)>k\}$, whereas $(f(y)-k)_-\neq 0$ on $\{y\in\mathbb{R}^m \,:\, f(y)<k\}$.

2.2. **The underlying geometry.** In a similar fashion as for the geometrical setting of the Boltzmann kernel [16] or as in [23], we start by endowing $\mathbb{R}^{1+2n} = \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ with the following Galilean transformation

$$(t_0, x_0, v_0) \circ (t, x, v) := (t + t_0,\, x + x_0 + tv_0,\, v + v_0),$$

where $(t_0, x_0, v_0), (t, x, v) \in \mathbb{R}^{1+2n}$. With respect to the group law $\circ$ the couple $(\mathbb{R}^{1+2n}, \circ)$ is a Lie group with identity element $\mathbf{0} := (0, 0, 0)$ and inverse element, for any $(t, x, v) \in \mathbb{R}^{1+2n}$, given by $(-t,\, -x + tv,\, -v)$.

For any $r > 0$, we consider the usual fractional nonlinear kinetic scaling $\delta_r : \mathbb{R}^{1+2n} \mapsto \mathbb{R}^{1+2n}$ defined by

$$\delta_r(t, x, v) := (r^{sp} t,\, r^{1+sp} x,\, rv).$$

As customary, we introduce a family of fractional kinetic cylinders respecting the invariant transformations defined above. For any $r > 0$, we denote by Q_r a cylinder centred in $(0, 0, 0)$ of radius r; that is,

$$Q_r \equiv Q_r(\mathbf{0}) := U_r(0, 0) \times B_r(0) = (-r^{sp}, 0] \times B_{r^{1+sp}}(0) \times B_r(0)\,.$$

For every $(t_0, x_0, v_0) \in \mathbb{R}^{1+2n}$ and for every $r > 0$, the slanted cylinder $Q_r(t_0, x_0, v_0)$ is defined as follows,

$$\begin{aligned} Q_r(t_0, x_0, v_0) \; := \; \{(t, x, v) \in \mathbb{R}^{1+2n} :\; & t_0 - r^{sp} < t \le t_0, \\ & |x - x_0 - (t - t_0)v_0| < r^{1+sp},\; |v - v_0| < r\}. \end{aligned}$$

Roughly speaking the integro-differential equation (1.1) is invariant under the kinetic scaling δ_r and left-invariant with respect to the Galilean transform. Namely, for any $(t_0, x_0, v_0) \in \mathbb{R}^{1+2n}$ and any $r > 0$, if f is a solution to (1.1) in $Q_r(t_0, x_0, v_0)$, then $f((t_0, x_0, v_0) \circ \delta_r(\cdot))$ solves an equation of the same ellipticity class as (1.1) in Q_1.

2.3. **Functional setting.** We introduce the needed fractional functional setting; for a more comprehensive treatment we refer the reader to [11].

For $p \in (1, \infty)$, $s \in (0, 1)$ and any $E \subseteq \mathbb{R}^n$, we denote with $W^{s,p}(E)$ the fractional Sobolev space

$$W^{s,p}(E) := \big\{ f \in L^p(E) : [f]_{W^{s,p}(E)} < +\infty \big\},$$

where the fractional seminorm $[f]_{W^{s,p}(E)}$ is the usual one via Gagliardo kernels,

$$[f]_{W^{s,p}(E)} := \left(\iint_{E \times E} \frac{|f(v) - f(w)|^p}{|v - w|^{n+sp}} \,\mathrm{d}v \,\mathrm{d}w \right)^{\frac{1}{p}}.$$

We endow $W^{s,p}(E)$ with the following norm

$$\|f\|_{W^{s,p}(E)} := \|f\|_{L^p(E)} + [f]_{W^{s,p}(E)}.$$

A function f belongs to $W^{s,p}_{\rm loc}(E)$ if $f \in W^{s,p}(E')$ whenever $E' \Subset E$. In a similar fashion, we denote with $W^{s,p}_0(E)$ the closure of $C^\infty_0(E)$ with respect to $\|\cdot\|_{W^{s,p}(E)}$.

Finally, the following fractional Sobolev embedding holds true; see [11].

Theorem 2.1. *Let $p \ge 1$ and $sp < n$, then for any $\tau \in [1, n/(n-sp)]$ and $f \in W^{s,p}(\mathbb{R}^n)$ we have*

$$\|f\|_{L^{\tau p}(\mathbb{R}^n)} \le c\, [f]_{W^{s,p}(\mathbb{R}^n)},$$

for $c \equiv c(n, p, s) > 0$. Moreover, if E is a bounded extension domain for $W^{s,p}$, we have that

$$\|f\|_{L^{\tau p}(E)} \le c\, [f]_{W^{s,p}(E)},$$

for any $f \in W^{s,p}(E)$ and for $c \equiv c(n, s, p, E) > 0$. If $sp = n$, then the statement holds true for any $\tau \in [1, +\infty)$. If $sp > n$, then the second inequality holds true for any $\tau \in [1, +\infty]$.

Since any bounded Lipschitz domain is an extension domain for $W^{s,p}$, the following holds.

Lemma 2.2. *Let $v_0 \in \mathbb{R}^n$, $r > 0$ and $B_r := B_r(v_0)$. Suppose $f \in W_0^{s,p}(B_r(v_0))$. Then for any $\tau \in [1, n/(n-sp)]$*

$$\left(\fint_{B_r} |f|^{\tau p}\,\mathrm{d}v\right)^{\frac{1}{\tau}} \leq c\, r^{sp-n} \iint_{B_r \times B_r} \frac{|f(v)-f(w)|^p}{|v-w|^{n+sp}}\,\mathrm{d}v\,\mathrm{d}w + \; c \fint_{B_r} |f|^p\,\mathrm{d}v,$$

for $c \equiv c(n,p,s) > 0$.

Proof. The proof immediately follows from Theorem 2.1 above, as pointed out in [30, Lemma 2.1]. Indeed, since $f \in W_0^{s,p}(B_r)$, then $f \in W_0^{s,p}(\mathbb{R}^n)$ and $f = 0$ in $\mathbb{R}^n \setminus B_r(v_0)$. Hence, $f \in W^{s,p}(B_r(v_0))$ and the statement is a direct consequence of Theorem 2.1. □

We conclude this section by proving a new Sobolev embedding suitable for the underlying functional setting of the problem we are dealing with.

Proposition 2.3. *Let $s \in (0,1)$ and $p \in [1,\infty)$ be such that $sp < n$ and let $(t_1,t_2) \times B_R \times B_r \subset \mathbb{R}^{1+2n}$, with $R, r > 0$. If $f \in L^p((t_1,t_2) \times B_R; W_0^{s,p}(B_r))$ is such that $f \geq 0$, then, for any $\tau \in [1, n/(n-sp)]$, we have*

$$\begin{aligned}
&\fint_{B_r} \left(\fint_{B_R \times (t_1,t_2)} f\,\mathrm{d}x\,\mathrm{d}t\right)^{\tau p} \mathrm{d}v \\
&\quad \leq c \left(r^{sp-n} \fint_{B_R \times (t_1,t_2)} [f]^p_{W^{s,p}(B_r)}\,\mathrm{d}x\,\mathrm{d}t + \fint_{B_r \times B_R \times (t_1,t_2)} f^p\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\tau},
\end{aligned}$$

where $c \equiv c(n,p,s) > 0$.

Proof. Let $f \in L^p((t_1,t_2) \times B_R; W_0^{s,p}(B_r))$. Then, $(f)_{B_R \times (t_1,t_2)} \in W_0^{s,p}(B_r)$, and by Jensen's Inequality, we have

$$\fint_{B_r} (f)^p_{B_R \times (t_1,t_2)}(v)\,\mathrm{d}v \leq \fint_{B_r \times B_R \times (t_1,t_2)} f^p\;\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t,$$

and

$$[(f)_{B_R \times (t_1,t_2)}]^p_{W^{s,p}(B_r)} \leq \fint_{B_R \times (t_1,t_2)} [f]^p_{W^{s,p}(B_r)}\,\mathrm{d}x\,\mathrm{d}t.$$

Finally, applying Lemma 2.2 to $(f)_{B_R \times (t_1,t_2)}$ we get the desired estimate. □

We give a precise definition of the tail quantity briefly introduced in (1.5), which plays a fundamental role in order to detect qualitative properties of solutions to kinetic nonlocal equations as in [2].

Definition 2.4. *Let f be a measurable function on $(t_1,t_2) \times \Omega_x \times \mathbb{R}^n \subset \mathbb{R}^{1+2n}$. The* "(kinetic) nonlocal tail of f centered in $v_0 \in \Omega_v \subset \mathbb{R}^n$ of diffusion radius r" *is the quantity* $\mathrm{Tail}(f; B_r(v_0))$ *given by*

$$\mathrm{Tail}(f; B_r(v_0)) := r^{sp} \int_{\mathbb{R}^n \setminus B_r(v_0)} \frac{|f(\cdot,\cdot,v)|^{p-1}}{|v_0 - v|^{n+sp}}\,\mathrm{d}v.$$

Now, we consider the following *tail space*

$$L^{p-1}_{sp}(\mathbb{R}^n) := \left\{ g \in L^{p-1}_{\mathrm{loc}}(\mathbb{R}^n) \,:\, \|g\|_{L^{p-1}_{sp}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|g(v)|^{p-1}}{(1+|v|)^{n+sp}}\,\mathrm{d}v < \infty \right\},$$

as firstly defined in [19]; see Section 2 there for related properties.

Given $\Omega := (t_1, t_2) \times \Omega_x \times \Omega_v \subset \mathbb{R}^{1+2n}$ we denote by $\mathcal{W}$ the natural functions space where weak solutions to (1.1) are taken. If $p \in (1, +\infty)$ and $p' := p/(p-1)$ is its conjugate exponent, then we have

$$\mathcal{W} := \Big\{ f \in L^p_{\rm loc}((t_1, t_2) \times \Omega_x; W^{s,p}_{\rm loc}(\Omega_v)) \cap L^{p-1}_{\rm loc}((t_1, t_2) \times \Omega_x; L^{p-1}_{sp}(\mathbb{R}^n)) \\ : (\partial_t + v \cdot \nabla_x) f \in L^{p'}_{\rm loc}((t_1, t_2) \times \Omega_x; (W^{s,p}(\mathbb{R}^n))^*) \Big\},$$

where $(W^{s,p}(\mathbb{R}^n))^*$ is the dual space of $W^{s,p}(\mathbb{R}^n)$.

Furthermore, we denote by $\mathcal{E}$ the nonlocal energy associated with our diffusion term $\mathcal{L}_v$ in (1.2); that is

$$\mathcal{E}(f, \phi) := \iint_{\mathbb{R}^n \times \mathbb{R}^n} |f(v) - f(w)|^{p-2}(f(v) - f(w))(\phi(v) - \phi(w)) K(v, w)\, {\rm d}v\, {\rm d}w\,,$$

for any test function ϕ smooth enough.

We are now in a position to recall the definition of weak sub- and supersolution.

Definition 2.5. *A function $f \in \mathcal{W}$ is a* weak subsolution (*resp.,* supersolution) *to* (1.1) *in Ω if*

$$\int_{t_1}^{t_2} \int_{\Omega_x} \mathcal{E}(f, \phi)\, {\rm d}x\, {\rm d}t + \int_{t_1}^{t_2} \int_{\Omega_x} \langle (\partial_t + v \cdot \nabla_x) f \,|\, \phi \rangle\, {\rm d}x\, {\rm d}t \,\leq\, 0 \quad \big(\geq\, 0,\ \textit{resp.}\big),$$

for any nonnegative $\phi \in L^p((t_1, t_2) \times \Omega_x; W^{s,p}(\mathbb{R}^n))$ such that $\operatorname{supp} \phi(t, x, \cdot) \Subset \Omega_v$; in the display above we denote by $\langle \cdot \,|\, \cdot \rangle$ the usual duality paring between $W^{s,p}(\mathbb{R}^n)$ and its dual.

A function $f \in \mathcal{W}$ is a weak solution *to* (1.1) *if it is both a weak sub- and supersolution.*

3. Energy estimates

This section is devoted to the proof of a fractional Caccioppoli-type estimate for weak subsolutions to (1.1), in which we extend to the nonlinear setting the approach seen in [2, Lemma 3.1]; see in particular Step 1 there. In the upcoming proof we will make use of the following inequality, whose proof is obtained via convexity and a standard iteration process; see e. g. Lemma 3.1 in [10].

Lemma 3.1. *Let $p \geq 1$ and $\varepsilon \in (0, 1]$. Then*

$$|a|^p \leq |b|^p + c_p \varepsilon |b|^p + (1 + c_p \varepsilon) \varepsilon^{1-p} |a - b|^p, \quad c_p := (p-1)\Gamma(\max\{1,\, p-2\})\,,$$

holds for every $a, b \in \mathbb{R}^m$, $m \geq 1$. Here, Γ stands for the standard Gamma function.

Proof of Theorem 1.2. Let $Q_r \equiv Q_r(\mathbf{0}) \Subset \Omega$ and let f be a weak subsolution to (1.1) according to Definition 2.5. Moreover, for any given $k \in \mathbb{R}$ define $\omega := (f-k)_+$. Consider a non-negative cut-off function $\varphi \equiv \varphi(x, v) \in C^\infty_0(B_{r^{1+sp}} \times B_r)$. For a. e. $t \in (-r^{sp}, 0]$, testing Definition 2.5 with $\phi := \omega \varphi^p$ yields

$$\begin{aligned} 0 \quad \geq & \quad \int_{B_{r^{1+sp}} \times B_r} (f_t + v \cdot \nabla_x f) \omega \varphi^p\, {\rm d}x\, {\rm d}v \\ & + \int_{B_{r^{1+sp}}} \mathcal{E}(f, \omega \varphi^p)\, {\rm d}x \\ =: & \quad J_1 + J_2. \end{aligned} \tag{3.1}$$

We separately consider the integrals above.

Let us begin estimating J_1. Using the fact that $\partial_t \varphi = 0$, we have that

$$\begin{aligned} J_1 &\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{r^{1+sp}}\times B_r} \omega^2\varphi^p \,\mathrm{d}x\,\mathrm{d}v - \frac{1}{2}\int_{B_{r^{1+sp}}\times B_r} |v\cdot\nabla_x(\varphi^p)|\omega^2 \,\mathrm{d}x\,\mathrm{d}v \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{r^{1+sp}}\times B_r} \omega^2\varphi^p \,\mathrm{d}x\,\mathrm{d}v - \frac{p}{2}\int_{B_{r^{1+sp}}\times B_r} |v\cdot\nabla_x\varphi|\varphi^{p-1}\omega^2 \,\mathrm{d}x\,\mathrm{d}v. \end{aligned} \tag{3.2}$$

Let us estimate the integral J_2 now. We begin with the splitting

$$\begin{aligned} J_2 &= \int_{B_{r^{1+sp}}\times B_r}\int_{B_r} |f(v)-f(w)|^{p-2}(f(w)-f(v))(\omega\varphi^p(v)-\omega\varphi^p(w))K(v,w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x \\ &\quad + 2\int_{B_{r^{1+sp}}\times B_r}\int_{\mathbb{R}^n\setminus B_r} |f(v)-f(w)|^{p-2}(f(w)-f(v))\omega\varphi^p(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x \\ &=: J_{2,1}+J_{2,2}, \end{aligned}$$

We begin by estimating the term $J_{2,1}$. First, let us assume that $f(v)\geq f(w)$, then

$$\begin{aligned} &|f(v)-f(w)|^{p-2}(f(v)-f(w))(\omega\varphi^p(v)-\omega\varphi^p(w)) \\ &= \big(f(v)-f(w)\big)^{p-1}\big(\omega\varphi^p(v)-\omega\varphi^p(w)\big) \\ &\geq \begin{cases} \big(\omega(v)-\omega(w)\big)^{p-1}\big(\omega\varphi^p(v)-\omega\varphi^p(w)\big) & \text{if } \ f(v),\ f(w)>k, \\ \big(\omega\varphi(v)\big)^p & \text{if } \ f(v)>k\geq f(w), \\ 0 & \text{otherwise} \end{cases} \\ &\geq \ \big(\omega(v)-\omega(w)\big)^{p-1}\big(\omega\varphi^p(v)-\omega\varphi^p(w)\big)\,, \end{aligned}$$

which yields

$$\begin{aligned} &|f(v)-f(w)|^{p-2}(f(v)-f(w))(\omega\varphi^p(v)-\omega\varphi^p(w))K(v,w) \\ &\qquad\qquad \geq \big(\omega(v)-\omega(w)\big)^{p-1}\big(\omega\varphi^p(v)-\omega\varphi^p(w)\big)K(v,w). \end{aligned}$$

If the opposite holds true, i. e. $f(v)\leq f(w)$, then we exchange the roles of v and w and repeat the computations above.

Now, applying Lemma 3.1 and considering that $\omega(v)\geq\omega(w)$ and $\varphi(w)\geq\varphi(v)$, we obtain

$$(1-c_p\varepsilon)\varphi^p(w)-(1+c_p\varepsilon)\varepsilon^{1-p}|\varphi(v)-\varphi(w)|^p\leq\varphi^p(v).$$

Moreover, by choosing

$$\varepsilon := \frac{1}{\max(1,2c_p)}\frac{\omega(v)-\omega(w)}{\omega(v)}\in(0,1],$$

we get

$$\begin{aligned} \big(\omega(v)-\omega(w)\big)^{p-1}\omega\varphi^p(v) \ &\geq \ \big(\omega(v)-\omega(w)\big)^{p-1}\omega(v)\max\{\varphi(v),\varphi(w)\}^p \\ &\quad -\frac{1}{2}\big(\omega(v)-\omega(w)\big)^p\max\{\varphi(v),\varphi(w)\}^p \\ &\quad -c\max\{\omega(v),\omega(w)\}^p|\varphi(v)-\varphi(w)|^p. \end{aligned}$$

The estimates above are trivial when $0=\omega(v)=\omega(w)$, or $\omega(v)\geq\omega(w)$ and $\varphi(v)\geq\varphi(w)$. On the other hand, if $\omega(v)\leq\omega(w)$, then we exchange the roles of v and w. Thus, we have that

$$\begin{aligned} &\big(\omega(v)-\omega(w)\big)^{p-1}\big(\omega\varphi^p(v)-\omega\varphi^p(w)\big) \\ &\qquad \geq \frac{1}{2}\big(\omega(v)-\omega(w)\big)^p\max\{\varphi(v),\varphi(w)\}^p - c\max\{\omega(v),\omega(w)\}^p|\varphi(v)-\varphi(w)|^p. \end{aligned}$$

Finally, observing that

$$
\begin{aligned}
|\omega\varphi(v)-\omega\varphi(w)|^p \quad &\leq \quad 2^{p-1}|\omega(v)-\omega(w)|\max\{\varphi(v),\varphi(w)\}^p \\
&\quad + 2^{p-1}|\varphi(v)-\varphi(w)|^p\max\{\omega(v),\omega(w)\}^p,
\end{aligned}
$$

we conclude

$$
\begin{aligned}
J_{2,1} \quad \geq \quad & c\int_{B_{r^{1+sp}}} [\omega\varphi]^p_{W^{s,p}(B_r)}\,\mathrm{d}x \\
& -c\int_{B_{r^{1+sp}}}\iint_{B_r\times B_r}\frac{\max\{\omega(v),\omega(w)\}^p|\varphi(v)-\varphi(w)|^p}{|v-w|^{n+sp}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x. \qquad (3.3)
\end{aligned}
$$

Now, we deal with the nonlocal term in $J_{2,2}$. Note that

$$
\begin{aligned}
&|f(v)-f(w)|^{p-2}(f(v)-f(w))\omega(v) \\
&\quad\geq -\big(f(w)-f(v)\big)_+^{p-1}\big(f(v)-k\big)_+ \\
&\quad\geq -\big(f(w)-k\big)_+^{p-1}\big(f(v)-k\big)_+ = -\omega^{p-1}(w)\omega(v),
\end{aligned}
$$

From this, we conclude $J_{2,2}$ can be treated as follows:

$$
\begin{aligned}
J_{2,2} \quad \geq \quad & -c\int_{B_{r^{1+sp}}\times B_r}\int_{\mathbb{R}^n\setminus B_r}\frac{\omega^{p-1}(w)\omega(v)\varphi^p(v)}{|v-w|^{n+sp}}\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x \\
\geq \quad & -c\int_{B_{r^{1+sp}}\times B_r}\omega\varphi^p\left(\sup_{v\in\,\mathrm{supp}\,\varphi}\int_{\mathbb{R}^n\setminus B_r}\frac{\omega^{p-1}(w)\,\mathrm{d}w}{|v-w|^{n+sp}}\right)\mathrm{d}v\,\mathrm{d}x. \qquad (3.4)
\end{aligned}
$$

Hence, combining (3.3) and (3.4), it yields that

$$
\begin{aligned}
J_2 \quad \geq \quad & c\int_{B_{r^{1+sp}}}[\omega\varphi]^p_{W^{s,p}(B_r)}\,\mathrm{d}x \\
& -c\int_{B_{r^{1+sp}}}\iint_{B_r\times B_r}\frac{\max\{\omega(v),\omega(w)\}^p|\varphi(v)-\varphi(w)|^p}{|v-w|^{n+sp}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x \\
& -\int_{B_{r^{1+sp}}\times B_r}\omega\varphi^p\left(\sup_{v\in\,\mathrm{supp}\,\varphi}\int_{\mathbb{R}^n\setminus B_r}\frac{\omega^{p-1}(w)\,\mathrm{d}w}{|v-w|^{n+sp}}\right)\mathrm{d}v\,\mathrm{d}x. \qquad (3.5)
\end{aligned}
$$

All in all, by (3.1), (3.2) and (3.5) we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{r^{1+sp}}\times B_r}\omega^2\phi^p\,\mathrm{d}x\,\mathrm{d}v + c\int_{B_{r^{1+sp}}}[\omega\varphi]^p_{W^{s,p}(B_r)}\,\mathrm{d}x \\
&\qquad \leq c\int_{B_{r^{1+sp}}}\iint_{B_r\times B_r}\frac{\max\{\omega(v),\omega(w)\}^p|\varphi(v)-\varphi(w)|^p}{|v-w|^{n+sp}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x \\
&\qquad\quad +c\int_{B_{r^{1+sp}}\times B_r}\omega\varphi^p\left(\sup_{v\in\,\mathrm{supp}\,\varphi}\int_{\mathbb{R}^n\setminus B_r}\frac{\omega^{p-1}(w)\,\mathrm{d}w}{|v-w|^{n+sp}}\right)\mathrm{d}v\,\mathrm{d}x \\
&\qquad\quad +c\int_{B_{r^{1+sp}}\times B_r}|v\cdot\nabla_x\varphi|\varphi^{p-1}\omega^2\,\mathrm{d}x\,\mathrm{d}v\,,
\end{aligned}
$$

for some constant c depending only on p and on the kernel constant Λ.

Thus, by integrating the proceeding estimate over $[\tau_1, \tau_2]$, for $-r^{sp} \leq \tau_1 < \tau_2 \leq 0$, we get

$$\begin{aligned}
&\int_{B_{r^{1+sp}}\times B_r} \omega^2 \phi^p(\tau_2, x, v)\,\mathrm{d}x\,\mathrm{d}v + \int_{\tau_1}^{\tau_2}\int_{B_{r^{1+sp}}} [\omega\varphi]^p_{W^{s,p}(B_r)}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq c\int_{U_r}\iint_{B_r\times B_r} \frac{\max\{\omega(v),\omega(w)\}^p|\varphi(v)-\varphi(w)|^p}{|v-w|^{n+sp}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +c\int_{Q_r}\omega\varphi^p\left(\sup_{v\in\operatorname{supp}\varphi}\int_{\mathbb{R}^n\setminus B_r}\frac{\omega^{p-1}(w)\,\mathrm{d}w}{|v-w|^{n+sp}}\right)\mathrm{d}v\,\mathrm{d}x \\
&\quad +c\int_{Q_r}|v\cdot\nabla_x\varphi|\varphi^{p-1}\omega^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + c\int_{B_{r^{1+sp}}\times B_r}\omega^2\varphi^p(\tau_1,x,v)\,\mathrm{d}v\,\mathrm{d}x\,.
\end{aligned} \tag{3.6}$$

Taking the supremum over τ_2 on the left-hand side and the average integral over $\tau_1 \in [-r^{sp}, 0]$ on both sides of the inequality, we get

$$\begin{aligned}
&\sup_{t\in[-r^{sp},0]}\int_{B_{r^{1+sp}}\times B_r}\omega^2\varphi^p\,\mathrm{d}x\,\mathrm{d}v \\
&\leq c\int_{U_r}\iint_{B_r\times B_r} \frac{\max\{\omega(v),\omega(w)\}^p|\varphi(v)-\varphi(w)|^p}{|v-w|^{n+sp}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +c\int_{Q_r}\omega\varphi^p\left(\sup_{v\in\operatorname{supp}\varphi}\int_{\mathbb{R}^n\setminus B_r}\frac{\omega^{p-1}(w)\,\mathrm{d}w}{|v-w|^{n+sp}}\right)\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +c\int_{Q_r}|v\cdot\nabla_x\varphi|\varphi^{p-1}\omega^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + c\,r^{sp}\int_{Q_r}\omega^2\varphi^p\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned} \tag{3.7}$$

whereas choosing $\tau_1 = -r^{sp}$ in (3.6) and $\tau_2 = 0$ yields (using also (3.7))

$$\begin{aligned}
&\int_{U_r}[\omega\varphi]^p_{W^{s,p}(B_r)}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq c\int_{U_r}\iint_{B_r\times B_r} \frac{\max\{\omega(v),\omega(w)\}^p|\varphi(v)-\varphi(w)|^p}{|v-w|^{n+sp}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +c\int_{Q_r}\omega\varphi^p\left(\sup_{v\in\operatorname{supp}\varphi}\int_{\mathbb{R}^n\setminus B_r}\frac{\omega^{p-1}(w)\,\mathrm{d}w}{|v-w|^{n+sp}}\right)\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +c\int_{Q_r}|v\cdot\nabla_x\varphi|\varphi^{p-1}\omega^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + c\,r^{sp}\int_{Q_r}\omega^2\varphi^p\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}$$

Thus, combining the display above with (3.7) yields the desired Caccioppoli-type estimate. □

4. Interpolative L^∞-L^p-type estimate

This section is devoted to the proof of Theorem 1.1, our main result. Before starting we recall a classical iteration lemma; see for example [13, Lemma 4.3].

Lemma 4.1. *Let $\beta_1, \beta_2 > 0$ and let $\{A_j\}_{j\in\mathbb{N}}$ be a sequence of positive real numbers such that*

$$A_{j+1} \leq c_1\, b^j\,(A_j^{1+\beta_1} + A_j^{1+\beta_2}), \qquad j = 0, 1, 2, \dots$$

with $c_1 > 0$, $b > 1$ and $\beta_1 \geq \beta_2 > 0$. If

$$A_0 \leq \min\left(1, c_1^{-\frac{1}{\beta_2}} b^{-\frac{1}{\beta_2^2}}\right),$$

then $\lim_{j\to\infty} A_j = 0$.

Proof of Theorem 1.1. Fix $r > 0$ such that $Q_r \equiv Q_r(\mathbf{0}) \subset Q_1$. For any fixed $i \in \mathbb{N}$, we define the sequences

$$\rho_i := \sum_{\alpha=0}^{i} 2^{-\alpha-1} r, \quad Q^i \equiv \delta_{\rho_i}(Q_1), \quad \text{and} \quad m_i := \sup_{Q^i} f.$$

Notice that the sequence of radii $\{\rho_i\}_i$ is increasing, and hence $Q^i \subset Q^{i+1}$ for every $i \in \mathbb{N}$. In particular, $\rho_0 \equiv r/2$ whereas $\rho_\infty := \lim_{i\to\infty} = r$. Thus, by definition of supremum, it follows $m_i \leq m_{i+1}$ for every $i \in \mathbb{N}$. Define the following family of radii

$$r_j := \rho_i + 2^{-i-j-2} r \quad \text{and} \quad Q^j \equiv \delta_{r_j}(Q_1).$$

Notice that the sequence of radii $\{r_j\}_j$ is decreasing, and hence $Q^{j+1} \subset Q^j$ for every $j \in \mathbb{N}$. In particular, $r_0 \equiv \rho_{i+1}$ whereas $r_\infty := \lim_{j\to\infty} = \rho_i$.

Define a family $\{\varphi_j\}_{j\in\mathbb{N}}$ of test functions $\varphi_j \equiv \varphi_j(x,v) \in C_0^\infty(B_{r_j^{1+sp}} \times B_{r_j})$, such that

$$0 \leq \varphi_j \leq 1, \quad \varphi_j \equiv 1 \,\text{on}\; B_{r_{j+1}^{1+sp}} \times B_{r_{j+1}}, \quad \varphi_j(x,\cdot) = 0 \text{ outside } B_{(r_j+r_{j+1})/2}$$
$$|\nabla_v \varphi_j| \leq c 2^{j+i}/r, \quad \text{and} \quad |v\cdot\nabla_x \varphi_j| \leq c 2^{(j+i)(1+sp)}/r^{sp}.$$

Then, consider two increasing families of parameters

$$k_j := (1-2^{-j})k \quad \text{and} \quad \tilde{k}_j := (k_j + k_{j+1})/2,$$

where $k > 0$ is a positive quantity which will be fixed later on. ù

Define $\tilde{\omega}_j := (f - \tilde{k}_j)_+$ and $\omega_j := (f-k_j)_+$. Note that

$$\begin{aligned}
\tilde{\omega}_j &\geq (f-\tilde{k}_j)\mathbb{1}_{\{f>k_{j+1}\}} \\
&\geq (k_{j+1}-\tilde{k}_j)\mathbb{1}_{\{f>k_{j+1}\}} = \frac{k_{j+1}-k_j}{2}\mathbb{1}_{\{f>k_{j+1}\}} \geq \frac{k}{2^{j+2}}\mathbb{1}_{\{f>k_{j+1}\}},
\end{aligned} \tag{4.1}$$

and, for any $0 \leq \tau \leq q$, it holds

$$\tilde{\omega}_j^\tau \leq c\frac{2^{j(q-\tau)}}{k^{q-\tau}}\omega_j^q. \tag{4.2}$$

Indeed, by the following chain of estimates based on the definition of the involved quantities, we have

$$\begin{aligned}
\omega_j^q &= (f-k_j)_+^q \mathbb{1}_{\{f>k_j\}} \\
&\geq (f-k_j)_+^q \mathbb{1}_{\{f>\widetilde{k}_j\}} = (f-k_j)_+^{q-\tau}(f-k_j)_+^\tau \mathbb{1}_{\{f>\widetilde{k}_j\}} \\
&\geq (\widetilde{k}_j - k_j)^{q-\tau}(f-k_j)_+^\tau \mathbb{1}_{\{f>\widetilde{k}_j\}} = (2^{-j-2}k)^{q-\tau}(f-k_j)_+^\tau \mathbb{1}_{\{f>\widetilde{k}_j\}} \\
&\geq c\left(\frac{k}{2^j}\right)^{q-\tau}\widetilde{\omega}_j^\tau.
\end{aligned}$$

Now, consider the following quantities, which are the construction blocks of the right-hand side of the Caccioppoli inequality in Theorem 1.2,

$$J_1 := r_j^{sp} \fint_{Q_j} \int_{B_j} \frac{\max\{\tilde{\omega}_j(v), \tilde{\omega}_j(w)\}^p |\varphi_j(v)-\varphi_j(w)|^p}{|v-w|^{n+sp}} \,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \tag{4.3}$$

$$J_2 := r_j^{sp} \fint_{Q_j} \tilde{\omega}_j \varphi_j^p \left(\sup_{v\in\operatorname{supp}\varphi_j} \int_{\mathbb{R}^n\setminus B_j} \frac{\tilde{\omega}_j^{p-1}(w)\,\mathrm{d}w}{|v-w|^{n+sp}} \right) \mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \tag{4.4}$$

$$J_3 := r_j^{sp} \fint_{Q_j} |v\cdot\nabla_x\varphi_j|\varphi_j^{p-1}\tilde{\omega}_j^2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \tag{4.5}$$

$$J_4 := r^{2sp} \fint_{Q_j} \tilde{\omega}_j^2 \varphi_j^p \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t. \tag{4.6}$$

We separately estimate the previous terms.

We begin with J_1. Up to exchange the roles of v and w we assume that $\tilde{\omega}_j(v) \geq \tilde{\omega}_j(w)$. Hence, it is estimated as follows (recalling that $|\nabla_v \varphi_j| \leq c2^{j+i}/r$ and φ_j is a Lipschitz function)

$$\begin{aligned}
J_1 &\overset{(4.3)}{=} r_j^{sp} \fint_{Q_j} \tilde{\omega}_j^p \left(\int_{B_j} \frac{|\varphi_j(v) - \varphi_j(w)|^p \,\mathrm{d}w}{|v-w|^{n+sp}} \right) \mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\leq c\, 2^{(j+i)p} r^{sp-p} \fint_{Q_j} \tilde{\omega}_j^p \left(\int_{B_{2r}(w)} \frac{\mathrm{d}w}{|v-w|^{n-p(1-s)}} \right) \mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\leq c\, 2^{(j+i)p} \fint_{Q_j} \tilde{\omega}_j^p \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\overset{(4.2)}{\leq} c 2^{ip+jq} k^p \fint_{Q_j} \frac{\omega_j^q}{k^q} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t. \qquad (4.7)
\end{aligned}$$

As for J_2, by applying Hölder's Inequality twice, the first time with $\tau = \tau' = 2$ and the second one with $\tau = \sigma/2$ and $\tau' = (\sigma-2)/2$ for some $\sigma > 2$, we have

$$\begin{aligned}
J_2 &\overset{(4.4)}{=} \frac{r_j^{sp}}{|Q_j|} \int_{Q_j \cap \operatorname{supp} \varphi_j} \tilde{\omega}_j \varphi_j^p \left(\sup_{v \in \operatorname{supp} \varphi_j} \int_{\mathbb{R}^n \setminus B_j} \frac{\tilde{\omega}_j^{p-1}(w)\,\mathrm{d}w}{|v-w|^{n+sp}} \right) \mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \left(\fint_{Q_j} \tilde{\omega}_j^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2}} \\
&\quad \times \left(\frac{1}{|Q_j|} \int_{Q_j \cap \operatorname{supp}\ \varphi_j} \left(r_j^{sp} \int_{\mathbb{R}^n \setminus B_j} \frac{\tilde{\omega}_j^{p-1}(w)}{|v-w|^{n+sp}} \,\mathrm{d}w \right)^2 \mathbb{1}_{\{f(v) > \tilde{k}_j\}} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2}} \\
&\leq \left(\fint_{Q_j} \tilde{\omega}_j^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2}} \left(\frac{1}{|Q_j|} \int_{Q_j \cap \operatorname{supp}\ \varphi_j} \left(r_j^{sp} \int_{\mathbb{R}^n \setminus B_j} \frac{\tilde{\omega}_j^{p-1}(w)}{|v-w|^{n+sp}} \,\mathrm{d}w \right)^{\sigma} \mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{\sigma}} \\
&\quad \times \left(\frac{|Q_j \cap \{f > \tilde{k}_j\}}{|Q_j|} \right)^{\frac{1}{2}(1-\frac{2}{\sigma})} \\
&\leq c\, 2^{(i+j)(n+sp)} \left(\fint_{Q_j} \tilde{\omega}_j^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2}} \\
&\quad \times \left(\fint_{U_j} \left(r_j^{sp} \int_{\mathbb{R}^n \setminus B_j} \frac{\tilde{\omega}_j^{p-1}(w)}{|w|^{n+sp}} \,\mathrm{d}w \right)^{\sigma} \mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{\sigma}} \left(\frac{|Q_j \cap \{f > \tilde{k}_j\}}{|Q_j|} \right)^{\frac{1}{2}-\frac{1}{\sigma}} \\
&\leq c\, 2^{(i+j)(n+sp)} \left(\fint_{Q_j} \tilde{\omega}_j^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2}} \left(\fint_{U_r} \mathrm{Tail}(f; B_{r/2})^{\sigma} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{\sigma}} \left(\frac{|Q_j \cap \{f > \tilde{k}_j\}}{|Q_j|} \right)^{\frac{1}{2}-\frac{1}{\sigma}}
\end{aligned}$$

where we have used that, for $w \in \mathbb{R}^n \setminus B_j$ and $v \in \operatorname{supp}\ \varphi_j$ (recalling that the support in the v-variable of φ_j is contained in $B_{(r_j + r_{j+1})/2}$)

$$\frac{|w|}{|v-w|} \leq 1 + \frac{|v|}{|w| - |v|}$$

$$\leq \quad 1 + \frac{r_j + r_{j+1}}{r_j - r_{j+1}} \leq c\,2^{i+j}\,.$$

Now, note that by Chebyschev's Inequality it holds for $q > 1$

$$\begin{aligned} |Q_j \cap \{f > \tilde{k}_j\}| &= |Q_j \cap \{f > \tfrac{k_{j+1}+k_j}{2}\}| = |Q_j \cap \{f - k_j > \tfrac{k_{j+1}-k_j}{2}\}| \\ &\leq |Q_j \cap \{f - k_j > 2^{-j-2}k\}| \\ &\leq c2^{q(j+2)} \int_{Q_j} \frac{\omega_j^q}{k^q}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.8}$$

Hence, by applying Hölder's inequality on the integral term to equalize the exponents and up to choosing

$$k \geq \delta \left(\fint_{U_r} \mathrm{Tail}(f; B_{r/2})^\sigma \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{\sigma}} \qquad \text{for } \delta \in (0,1]\,, \tag{4.9}$$

we can estimate further, for $q \geq 2$,

$$\begin{aligned} J_2 \quad &\overset{(4.2),(4.9)}{\leq} \quad c\,2^{i(n+sp)+j(n+sp+\frac{q-2}{2})} \frac{k^2}{\delta} \left(\fint_{Q_j} \frac{\omega_j^q}{k^q}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2}} \\ &\qquad \times \left(\frac{|Q_j \cap \{f > \tilde{k}_j\}|}{|Q_j|} \right)^{\frac{1}{2}-\frac{1}{\sigma}} \\ &\overset{(4.8)}{\leq} \quad 2^{j(n+sp+q-1-\frac{q}{\sigma})} \left(\frac{c\,k^2 2^{i(n+sp)}}{\delta} \right) \left(\fint_{Q_j} \frac{\omega_j^q}{k^q}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{\sigma-1}{\sigma}}. \end{aligned} \tag{4.10}$$

As for the integral J_3 and J_4, we start noticing that $|v \cdot \nabla_x \varphi_j| \varphi_j^{p-1} \leq c2^{(j+i)(1+sp)}/r^{sp}$ and $\varphi_j \leq 1$, which, since $r < 1$, yields that

$$\begin{aligned} J_3 + J_4 \quad &\overset{(4.5),(4.6)}{=} \quad r_j^{sp} \fint_{Q_j} |v \cdot \nabla_x \varphi_j| \varphi_j^{p-1} \tilde{\omega}_j^2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\ &\qquad + c\,r^{2sp} \fint_{Q_j} \tilde{\omega}_j^2 \varphi_j^p \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\ &\leq \quad c\,2^{(j+i)(1+sp)} \fint_{Q_j} \tilde{\omega}_j^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\overset{(4.2)}{\leq} \quad c\,2^{i(1+sp)+j(q+sp-1)} k^2 \fint_{Q_j} \frac{\omega_j^q}{k^q}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.11}$$

Hence, collecting all previous estimates (4.7), (4.10) and (4.11) yields

$$\begin{aligned} J_1 + J_2 + J_3 + J_4 \leq \left(\frac{c\,2^{i(n+sp)}(k^2 + k^p)}{\delta} \right) 2^{j(n+sp+q-1-\frac{q}{\sigma})} \\ \times \left\{ \fint_{Q_j} \frac{\omega_j^q}{k^q}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \; + \left(\fint_{Q_j} \frac{\omega_j^q}{k^q}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{\sigma-1}{\sigma}} \right\}. \end{aligned} \tag{4.12}$$

Also, note that since $\sigma > 2$ and $q \geq \max\{2, p\}$ the exponent $n + sp + q - 1 - \frac{q}{\sigma} > 0$.

Now, we use the previous computation when $q = p$. With no loss of generality, let us assume that $sp < n$ to treat the remaining cases it is sufficient to rearrange a bit the exponents; see for example [24, Theorem 1.1]. Making use of the definition of φ_j, recalling that $\omega_{j+1} \equiv 0$ on $\{f \le k_{j+1}\}$ and observing $k_j \le \widetilde{k}_j \le k_{j+1}$, by Fubini-Tonelli's Theorem, for $\tau \ge 1$, we get

$$\fint_{Q_{j+1}} \omega_{j+1}^p \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t$$
$$\le m_i^{p-1} \fint_{B_{j+1}} \left(\fint_{U_{j+1}} \tilde{\omega}_j \mathbb{1}_{\{\omega_{j+1}>0\}} \,\mathrm{d}x\,\mathrm{d}t \right)^{\tau p} \left(\fint_{U_{j+1}} \tilde{\omega}_j \mathbb{1}_{\{\omega_{j+1}>0\}} \,\mathrm{d}x\,\mathrm{d}t \right)^{1-\tau p} \mathrm{d}v. \tag{4.13}$$

Moreover, by (4.1) we have that

$$\left(\fint_{U_{j+1}} \tilde{\omega}_j \mathbb{1}_{\{\omega_{j+1}>0\}} \,\mathrm{d}x\,\mathrm{d}t \right)^{1-\tau p} \le c \left(\frac{k}{2^{j+2}} \right)^{1-\tau p} = \frac{c\,2^{j(\tau p-1)}}{k^{\tau p-1}}.$$

Thus, we can estimate (4.13) further

$$\begin{aligned}
&\fint_{Q_{j+1}} \frac{\omega_{j+1}^p}{k^p} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \frac{c\,2^{j(\tau p-1)} m_i^{p-1}}{k^{p(\tau+1)-1}} \fint_{B_{j+1}} \left(\fint_{U_{j+1}} \tilde{\omega}_j \,\mathrm{d}x\,\mathrm{d}t \right)^{\tau p} \mathrm{d}v \\
&\quad\le \frac{c\,2^{j(\tau p-1)} m_i^{p-1}}{k^{p(\tau+1)-1}} \fint_{B_j} \left(\fint_{U_j} \tilde{\omega}_j \varphi_j \,\mathrm{d}x\,\mathrm{d}t \right)^{\tau p} \mathrm{d}v \\
&\quad\le \frac{c\,2^{j(\tau p-1)} m_i^{p-1}}{k^{p(\tau+1)-1}} \Bigg(r_j^{sp} \fint_{Q_j} \int_{B_j} \frac{|\tilde{\omega}_j(v)\varphi_j(v) - \tilde{\omega}_j(w)\varphi_j(w)|^p}{|v-w|^{n+sp}} \,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \qquad (4.14) \\
&\qquad\qquad + \fint_{Q_j} (\tilde{\omega}_j \varphi_j)^p \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \Bigg)^{\tau},
\end{aligned}$$

where in the last line we applied the Sobolev inequality in Proposition 2.3.

Applying now the Caccioppoli inequality in Theorem 1.2, estimate (4.12) with $q = p$, and choosing $\tau := n/(n-sp)$ and $k > 0$ sufficiently large such that

$$k \ge 1\,, \tag{4.15}$$

we get

$$\begin{aligned}
\fint_{Q_{j+1}} \frac{\omega_{j+1}^p}{k^p} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \;&\le\; c\,2^{j\left(\frac{n(n+sp+p-1-p/\sigma)}{n-sp} + \frac{np}{n-sp} - 1\right)} \left(\frac{m_i^{p-1} 2^{\frac{in(n+sp)}{n-sp}} k^{\frac{np}{n-sp}}}{k^{\frac{np}{n-sp}+p-1} \delta^{\frac{n}{n-sp}}} \right) \\
&\quad\times \left\{ \fint_{Q_j} \frac{\omega_j^p}{k^p} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \;+ \left(\fint_{Q_j} \frac{\omega_j^p}{k^p} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{n}{n-sp}} \\
&\le\; c\,2^{j\left(\frac{n(n+sp+p-1-q/\sigma)}{n-sp} + \frac{np}{n-sp} - 1\right)} \left(\frac{m_{i+1}^{p-1} 2^{\frac{in(n+sp)}{n-sp}}}{k^{p-1} \delta^{\frac{n}{n-sp}}} \right) \\
&\quad\times \left\{ \fint_{Q_j} \frac{\omega_j^p}{k^p} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \;+ \left(\fint_{Q_j} \frac{\omega_j^p}{k^p} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{n}{n-sp}} \\
&\le\; c\,2^{j\left(\frac{n(n+sp+p-1-q/\sigma)}{n-sp} + \frac{np}{n-sp} - 1\right)} \left(\frac{2^{\frac{in(n+sp)}{n-sp}}}{\varepsilon^{p-1} \delta^{\frac{n}{n-sp}}} \right)
\end{aligned}$$

$$\times \left\{ \left(\fint_{Q_j} \frac{\omega_j^p}{k^p} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{n}{n-sp}} + \left(\fint_{Q_j} \frac{\omega_j^p}{k^p} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\left(\frac{\sigma-1}{\sigma}\right)\frac{n}{n-sp}} \right\},$$

where: we used the fact that $m_i \leq m_{i+1}$ by definition since the sequence $\{\rho_i\}_{i\in\mathbb{N}}$ is increasing; in the second line, we considered that $(a+b)^\tau \leq 2^{\tau-1}(a^\tau + b^\tau)$, for $a, b > 0$ $\tau > 1$, and choose

$$k \geq \varepsilon\, m_{i+1}, \tag{4.16}$$

for some $\varepsilon > 0$ which will be fixed later on.

Thus, defining $b := 2^{\frac{n(n+sp+p-1-p/\sigma)}{n-sp} + \frac{np}{n-sp} - 1} > 1$ and

$$A_j := \fint_{Q_j} \frac{\omega_j^p}{k^p} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,,$$

we can rewrite the chain of estimates above as follows

$$A_{j+1} \leq c\, b^j \left(\frac{2^{\frac{in(n+sp)}{n-sp}}}{\varepsilon^{p-1}\delta^{\frac{n}{n-sp}}} \right) \left(A_j^{1+\frac{sp}{n-sp}} + A_j^{1+\frac{\sigma sp-n}{\sigma(n-sp)}} \right),$$

Now, choosing $\sigma > \max\{2, n/(sp)\}$ we have that, once defined $\beta_1 := \frac{sp}{n-sp} > 0$ and $\beta_2 := \frac{\sigma sp - n}{\sigma(n-sp)}$, it holds $\beta_1 > \beta_2 > 0$.

Then, up to choosing k such that

$$\begin{aligned} k \quad &:= \quad \varepsilon\, m_{i+1} + c^{\frac{1}{p\beta_2}} b^{\frac{1}{p\beta_2^2}} \left(\delta^{-\frac{n\sigma}{\sigma sp-n}} \fint_{Q^{(i)}} (1+f_+^p) \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{p}} \left(\frac{2^{\frac{in\sigma(n+sp)}{p(\sigma sp-n)}}}{\varepsilon^{\frac{p-1}{p\beta_2}}} \right) \\ &\quad + \delta \left(\fint_{U_r} \mathrm{Tail}(f_+; B_{r/2})^\sigma \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{\sigma}}, \end{aligned}$$

which is in clear accordance with (4.9) and (4.15) and (4.16), Lemma 4.1 yields that $\lim_{j\to\infty} A_j = 0$, and thus $f \leq k$ for a. e. $(v, x, t) \in Q^{(i)}$. Thus,

$$\begin{aligned} m_i \quad &\leq \quad \varepsilon\, m_{i+1} + c(\varepsilon)\, 2^{\frac{in\sigma(n+sp)}{p(\sigma sp-n)}} \left(\delta^{-\frac{n\sigma}{\sigma sp-n}} \fint_{Q_r} (1+f_+^p) \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{p}} \\ &\quad + \delta \left(\fint_{U_r} \mathrm{Tail}(f_+; B_{r/2})^\sigma \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{\sigma}}. \end{aligned}$$

Iterating the previous estimate we get

$$\begin{aligned} m_0 \quad &\leq \quad \varepsilon^{i+1} m_{i+1} + \delta \left(\fint_{U_r} \mathrm{Tail}(f_+; B_{r/2})^\sigma \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{\sigma}} \sum_{j=0}^{i} \varepsilon^j \\ &\quad + c(\varepsilon) \left(\delta^{-\frac{n\sigma}{\sigma sp-n}} \fint_{Q_r} (1+f_+^p) \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{p}} \sum_{j=0}^{i} \left(\varepsilon 2^{\frac{n\sigma(n+sp)}{p(\sigma sp-n)}} \right)^j. \end{aligned} \tag{4.17}$$

Hence, choosing now $\varepsilon \equiv \varepsilon(n, s, p, \sigma) > 0$ so that the series on the right-hand side of (4.17) converges i.e. $\varepsilon < 2^{-\frac{n\sigma(n+sp)}{p(\sigma sp-n)}}$, and then sending $i \to \infty$ we get

$$\sup_{Q_{\frac{r}{2}}} f \leq \frac{c}{\delta^{\frac{n\sigma}{p(\sigma sp-n)}}} \left(\fint_{Q_r} (1+f_+^p) \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{p}} + \delta \left(\fint_{U_r} \mathrm{Tail}(f_+; B_{r/2})^\sigma \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{\sigma}}.$$

□

References

[1] K. Adimurthi, H. Prasad, V. Tewary: Local Hölder regularity for nonlocal parabolic p-Laplace equation. https://arxiv.org/abs/2205.09695 (2022).

[2] F. Anceschi, G. Palatucci, M. Piccinini: Harnack inequalities for kinetic integral equations. https://arxiv.org/abs/2401.14182 (2024)

[3] F. Anceschi, M. Piccinini, A. Rebucci: New perspectives on recent trends for Kolmogorov operators. Springer INdAM Series, 56, pp. 57–92 (2024).

[4] P. Auscher, C. Imbert, L. Niebel: Weak solutions to Kolmogorov-Fokker-Planck equations: regularity, existence and uniqueness (2024). https://arxiv.org/abs/2403.17464

[5] P. Auscher, C. Imbert, L. Niebel: Fundamental solutions to Kolmogorov-Fokker-Planck equations with rough coefficients: existence, uniqueness, upper estimates (2024). https://arxiv.org/abs/2403.17468

[6] V. Bögelein, F. Duzaar, N. Liao, G. Molica Bisci, R. Servadei: Regularity for the fractional p-Laplace equation. *Preprint* (2024). arXiv:2406.01568

[7] F. Bouchut: Hypoelliptic regularity in kinetic equations. *J. Math. Pures Appl.* (9) 81 (2002), no. 11, 1135–1159.

[8] L. Brasco, E. Lindgren, A, Schikorra: Higher Hölder regularity for the fractional p-Laplacian in the superquadratic case. *Adv. Math.* **338** (2018), 782–846.

[9] A. Di Castro, T. Kuusi, G. Palatucci: Nonlocal Harnack inequalities. *J. Funct. Anal.* **267** (2014), no. 6, 1807–1836.

[10] A. Di Castro, T. Kuusi, G. Palatucci: Local behavior of fractional p-minimizers. *Ann. Inst. H. Poincaré Anal. Non Linéaire.* **33** (2016), 1279–1299.

[11] E. Di Nezza, G. Palatucci, E. Valdinoci: Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), 521–573.

[12] X. Fernández-Real, X. Ros-Oton: *Integro-Differential Elliptic Equations*. Progress in Mathematics, Birkhauser, 2024.

[13] K. Ho, I. Sim: Corrigendum to "Existence and some properties of solutions for degenerate elliptic equations with exponent variable". [*Nonlinear Anal.* **98**, 146–164 (2014)]. *Nonlinear Anal.* **128**, 423–426 (2015)

[14] C. Imbert: A non-local regularization of first order Hamilton-Jacobi equations. *J. Differential Equations*, **211** (2005), no. 1, 218–246.

[15] C. Imbert, L. Silvestre: The weak Harnack inequality for the Boltzmann equation without cut-off. *J. Eur. Math. Soc. (JEMS)*, 22(2) (2020): 507–592.

[16] C. Imbert, L. Silvestre: Global regularity estimates for the Boltzmann equation without cut-off. *J. Amer. Math. Soc.* no. 3 (2022), 625–703.

[17] M. Kassmann, M. Weidner: The parabolic Harnack inequality for nonlocal equations. To appear in *Duke Math. J.* (2024). arXiv:2303.05975

[18] M. Kassmann, M. Weidner: The Harnack inequality for nonlocal kinetic equations (2024). https://arxiv.org/abs/2405.05223

[19] J. Korvenpää, T. Kuusi, G. Palatucci: The obstacle problem for nonlinear integro-differential operators. *Calc. Var. Partial Differential Equations* **55** (2016), no. 3, Art. 63.

[20] N. Liao: Hölder regularity for parabolic fractional p-Laplacian. *Calc. Var. Partial Differential Equations* **63**, no. 22,(2024)

[21] A. Loher: Quantitative De Giorgi methods in kinetic theory for non-local operators. To appear in *J. Funct. Anal.* (2024).

[22] A. Loher: Semi-local behaviour of non-local hypoelliptic equations: divergence form. https://arxiv.org/abs/2404.05612v3 (2024).

[23] M. Manfredini, S. Pagliarani, S. Polidoro: Intrinsic Hölder spaces for fractional kinetic operators. (2024) https://arxiv.org/abs/2309.16350

[24] M. Manfredini, G. Palatucci, M. Piccinini, S. Polidoro: Hölder continuity and boundedness estimates for nonlinear fractional equations in the Heisenberg group. *J. Geom. Anal.* **33** (2023), no. 3, Art. 77.

[25] S. Nowak: Higher Hölder regularity for nonlocal equations with irregular kernel. *Calc. Var. Partial Differential Equations* **60** (2021), no. 1, Art. 24.

[26] A. Pascucci, S. Polidoro: The Moser's iterative method for a class of ultraparabolic equations. *Comm. Cont. Math.* **6** (2004), no. 3, 395–417.

[27] G. Palatucci: The Dirichlet problem for the p-fractional Laplace equation. *Nonlinear Anal.* 177 (2018), 699–732.

[28] H. Prasad, V. Tewary: Local boundedness of variational solutions to nonlocal double phase parabolic equations. *J. Differential Equations* **351** (2023).

[29] L. F. Stokols: Hölder continuity for a family of nonlocal hypoelliptic kinetic equations. *SIAM J. Math. Anal.* **51** (2019), no. 6, 4815–4847.

[30] M. Strömqvist: Local boundedness of solutions to non-local parabolic equations modeled on the fractional p-Laplacian. *J. Differential Equations* **266** no. 12 (2019), 7948–7979.

[31] A. Tavakoli: A perturbative approach to Hölder continuity of solutions to a nonlocal p-parabolic equation. *J. Evol. Equ.* **24**, 27 (2024).

[32] M. Wang, J. Duan: Existence and regularity of a linear nonlocal Fokker-Planck equation with growing drift. *J. Math. Anal. Appl.* **449** (2017), no. 1, 228–243.

Francesca Anceschi, Dipartimento di Ingegneria Industriale e Scienze Matematiche, Universitá Politecnica delle Marche, Via Brecce Bianche, 12, 60131 Ancona, Italy
Email address: f.anceschi@staff.univpm.it

Mirco Piccinini, Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Università di Parma, Parco Area delle Scienze 53/a, Campus, 43124 Parma, Italy, and Dipartimento di Matematica, Università di Pisa, L.go B. Pontecorvo 5, 56127, Pisa, Italy
Email address, Mirco Piccinini: mirco.piccinini@dm.unipi.it