Partial regularity for variational integrals with Morrey-Hölder zero-order terms, and the limit exponent in Massari's regularity theorem

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Abstract

We revisit the partial $\mathrm{C}^{1,\alpha}$ regularity theory for minimizers of non-parametric integrals with emphasis on sharp dependence of the Hölder exponent α on structural assumptions for general zero-order terms. A particular case of our conclusions carries over to the parametric setting of Massari's regularity theorem for prescribed-mean-curvature hypersurfaces and there confirms optimal regularity up to the limit exponent.

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Contents

1   Introduction

In this paper, for $n, N \in \mathbb{N}$ and bounded open $\Omega \subseteq \mathbb{R}^n$, we further develop the partial $\mathrm{C}^{1,\alpha}$ regularity theory for minimizers of variational integrals

$$\mathcal{F}[w] := \int_\Omega \left[f(\mathrm{D}w) + g(\,\cdot\,, w) \right] \mathrm{d}x \tag{1.1}$$

which depend on $\mathrm{D}w$ and $(\,\cdot\,, w)$ through separate integrands $f \colon \mathbb{R}^{N \times n} \to \mathbb{R}$ and $g \colon \Omega \times \mathbb{R}^N \to \mathbb{R}$, respectively.

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We focus on cases with standard assumptions on f, but very general hypotheses on g and on the sharp dependence of the exponent α in the $\mathrm{C}^{1,\alpha}$ regularity on the latter hypotheses. More precisely, we impose on g a Morrey-Hölder condition of type

$$|g(x,y) - g(x,\widehat{y})| \le \Gamma(x)(1 + |y| + |\widehat{y}|)^{q-\beta}|y - \widehat{y}|^\beta \qquad \text{for all } x \in \Omega \text{ and } y, \widehat{y} \in \mathbb{R}^N \tag{1.2}$$

with Hölder and growth exponents $0 < \beta \le q < \infty$ for the y-variable and with x-dependence controlled by a non-negative function Γ whose precise regularity will be measured on the scale of Morrey spaces. In particular, in the general tradition of [35] we admit non-differentiability of $g(x,y)$ in y and consequently non-availability of the Euler equation of $\mathcal{F}$. Moreover, we directly foreshadow that the formulation of assumption (1.2) is tailored out, to some extent, for an application to the somewhat different setting of Massari's $\mathrm{C}^{1,\alpha}$ regularity theorem [58, 59] for sets with variational mean curvature in L^p. Indeed, as we explain in more detail later on, in the Massari setting we here complement the results of our predecessor paper [74] and achieve the final sharpening of the regularity conclusion.

However, for the moment we return to the setting of (1.1) and aim at giving precise statements of our results, which concern interior regularity and can thus be conveniently formulated for local minimizers in the following sense:

Definition 1.1 (local minimizers). *Assume that f is a Borel function with $\limsup_{|z|\to\infty} \frac{|f(z)|}{|z|^2} < \infty$ and that g is a Carathéodory integrand. Then we call $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ with $g(\,\cdot\,, u) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ a local minimizer of $\mathcal{F}$ from (1.1) if, for every $x \in \Omega$, there exists some $r > 0$ with $\mathrm{B}_r(x) \Subset \Omega$ such that*

$$\int_{\mathrm{B}_r(x)} \left[f(\mathrm{D}u) + g(\,\cdot\,, u)\right] \mathrm{d}x \le \int_{\mathrm{B}_r(x)} \left[f(\mathrm{D}w) + g(\,\cdot\,, w)\right] \mathrm{d}x$$

holds for all $w \in \mathrm{W}^{1,2}_u(\mathrm{B}_r(x), \mathbb{R}^N)$ with $g(\,\cdot\,, w) \in \mathrm{L}^1(\mathrm{B}_r(x))$.

We remark that (1.2) together with below-mentioned assumptions on β, q, Γ ensures that $g(\,\cdot\,, w) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ holds either for all $w \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ or none at all. This said, it becomes apparent that the integrability requirements on the zero-order term in Definition 1.1 only serve to exclude trivial cases which do not allow for a meaningful notion of local minimality, and indeed when additionally imposing a very mild assumption such as $g(\,\cdot\,, 0) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, for instance, we could also drop these requirements completely.

Our main result now reads as follows:

Theorem 1.2 (partial regularity for variational integrals with Morrey-Hölder zero-order integrand). *We consider a strictly quasiconvex (in the sense of the later Definition 2.3) C^2 integrand $f \colon \mathbb{R}^{N \times n} \to \mathbb{R}$ which has at most quadratic growth in the sense of $\limsup_{|z|\to\infty} \frac{|f(z)|}{|z|^2} < \infty$. Moreover, in case $n \ge 3$, we abbreviate $2^* := \frac{2n}{n-2}$ and assume that a Carathéodory integrand $g \colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies (1.2) with*

$$\beta \in (0,1], \qquad q \in [\beta, 2^*), \tag{1.3}$$

and with a non-negative function Γ which in turn satisfies, for the exponents

$$s_\beta := \left(\frac{2^*}{\beta}\right)' = \frac{2^*}{2^* - \beta} \qquad \text{and} \qquad s_q := \left(\frac{2^*}{q}\right)' = \frac{2^*}{2^* - q},$$

the two Morrey conditions (see Section 2 for definitions of the spaces)

$$\Gamma \in \mathrm{L}^{s_\beta, n + s_\beta((2-\beta)\alpha - \beta)}_{\mathrm{loc}}(\Omega) \qquad \text{for some } \alpha \in (0,1) \tag{1.4}$$

and

$$\begin{cases} \Gamma \in \mathrm{L}^{s_q}_{\mathrm{loc}}(\Omega), & \text{if } q > 2, \\ \Gamma \in \mathrm{L}^{s_q, t}_{\mathrm{loc}}(\Omega) \quad \text{with} \quad t > n - s_q q, & \text{if } q \le 2. \end{cases} \tag{1.5}$$

Then, for every local minimizer u of $\mathcal{F}$ from (1.1), there exists an open set $\Omega_{\mathrm{reg}} \subseteq \Omega$ such that

$$|\Omega \setminus \Omega_{\mathrm{reg}}| = 0 \qquad \text{and} \qquad u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega_{\mathrm{reg}}, \mathbb{R}^N).$$

In case $n \in \{1, 2\}$ the result remains valid with the range for q in (1.3) replaced by $q \in [\beta, \infty)$, with s_β in (1.4) replaced by any $s > 1$, and with (1.5) replaced by the requirement $\Gamma \in \mathrm{L}^{\eta, \max\{n-q, 0\}}_{\mathrm{loc}}(\Omega)$ for some $\eta > 1$.

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In Theorem 1.2, our main point is in determining the gradient Hölder exponent α of the minimizers and more precisely its sharp dependence on the parameters β, q, and Γ from the Morrey-Hölder condition (1.2) for the zero-order integrand g. Somewhat surprisingly, this dependence has rarely been addressed, though even with a focus on partial $\mathrm{C}^{1,\alpha}$ regularity in general vectorial frameworks we are aware of an abundance of positive results in [63, 41, 37, 12, 5, 32, 30, 38, 1, 11, 31, 34, 6, 2, 33, 49, 67, 47, 13, 22, 20, 15, 9, 46, 19, 50, 21, 23, 24, 55, 53, 60, 54, 16, 28, 72, 73, 14, 27, 25, 26, 8, 52, 56, 43] and of counterexamples in [18, 42, 66, 75, 76, 65, 77, 62, 61]. However, at least the works [36, 48, 70] consider conditions of types similar to (1.2) with $q = \beta$. Here, [36] establishes in fact full regularity for scalar and specific vectorial cases. In particular, for $\Gamma \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$, this work reaches $\mathrm{C}^{1,\alpha}$ regularity with the optimal exponent $\alpha = \frac{\beta}{2-\beta}$, which is reproduced by our result; cf. situation (A) below. Moreover, a remark in [36] predicts — still for $q = \beta$, the specific vectorial case mentioned, and in the context of full regularity — even the general Hölder exponent of Theorem 1.2 (though in quite different terminology) and the one of the subsequent situation (B) with $\Gamma \in \mathrm{L}^p_{\mathrm{loc}}(\Omega)$. In contrast, the works [48, 70] deal with partial regularity in frameworks closer to ours and reach once more the exponent $\alpha = \frac{\beta}{2-\beta}$ in case $q = \beta$, $\Gamma \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$, but not the finer ones for Γ in Morrey and L^p spaces. Concerning the exponent $\frac{\beta}{2-\beta}$, it should also be mentioned that it has in fact been discovered even before and occurs also in [17, 68, 69, 3, 51], partially in different form and for more specific model cases. Finally, a Morrey-Hölder condition similar to (1.2) has been treated also in [71], where specifically [71, Corollary 3] asserts partial $\mathrm{C}^{1,\alpha}$ regularity of a minimizer u under a condition of type (1.2), possibly with $\beta < q$, plus an a priori Morrey hypothesis on u itself. However, as these results are derived from a general theory of almost-minimizers, the accessible exponents α stay beyond $\frac{\beta}{2}$ and already in case $\Gamma, |u| \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ cannot reach the previously mentioned optimal exponent $\frac{\beta}{2-\beta}$.

At this stage, with the sharp dependence of α still being our main concern, we turn to several technical observations on the governing conditions (1.3), (1.4), (1.5) for the parameters and in particular on the Morrey conditions (1.4) and (1.5) for Γ.

First, we put on record that only (1.4) but not (1.5) influences the Hölder exponent α. Therefore, we have conveniently formulated (1.4) with the exponent $n + s_\beta((2-\beta)\alpha - \beta)$ such that $\alpha \in (0,1)$ eventually turns out to be exactly the exponent of the resulting $\mathrm{C}^{1,\alpha}$ regularity. In addition, we point out that the case of interest is only $\alpha \le \frac{\beta}{2-\beta}$, since in case of larger α we have $\mathrm{L}^{s_\beta, n + s_\beta((2-\beta)\alpha - \beta)}_{\mathrm{loc}}(\Omega) = \{0\}$, and this enforces via (1.4) and (1.2) that $g(x,y)$ is independent of y and that the zero-order term of $\mathcal{F}$ trivializes. Turning to the complementary condition (1.5), we remark that this hypothesis — though not relevant for determining α — seems technically inevitable in making the first steps towards (C^1) regularity in the above generality, while it can be dropped in slightly restricted frameworks to be discussed below. Let us also observe that the case distinction within (1.5) seems formally reasonable, since $q \le 2$ is equivalent with $n - s_q q \ge 0$ and thus the distinction prevents us from considering negative t.

We remark that, similar to the modification mentioned for $n \in \{1,2\}$, also in dimension $n \ge 3$ we could require (1.4) in the form $\Gamma \in \mathrm{L}^{s, n + s((2-\beta)\alpha - \beta)}_{\mathrm{loc}}(\Omega)$ for any $s \ge s_\beta$. However, in view of the standard Morrey space embedding $\mathrm{L}^{s, n + s((2-\beta)\alpha - \beta)}_{\mathrm{loc}}(\Omega) \subset \mathrm{L}^{s_\beta, n + s_\beta((2-\beta)\alpha - \beta)}_{\mathrm{loc}}(\Omega)$ for all $s \ge s_\beta$ (cf. Section 2), this does not win any generality, and thus we have indeed given the above statement for $s = s_\beta$.

Further, we record that if one reads Theorem 1.2 for merely obtaining $\mathrm{C}^{1,\alpha}$ regularity with *some* $\alpha > 0$, then (1.4) reduces to $\Gamma \in \mathrm{L}^{s_\beta, t}_{\mathrm{loc}}(\Omega)$ for some $t > n - s_\beta \beta$. However, even this reduced version of (1.4) remains incomparable with (1.5) (except in the basic case $q = \beta$), essentially since $\mathrm{L}^{s_q, n - s_q q}_{\mathrm{loc}}(\Omega)$ embeds only into $\mathrm{L}^{s_\beta, n - s_\beta q}_{\mathrm{loc}}(\Omega)$ but not into $\mathrm{L}^{s_\beta, n - s_\beta \beta}_{\mathrm{loc}}(\Omega)$.

Let us also briefly point out that the conditions recorded in Theorem 1.2 for the cases $n \in \{1,2\}$ can be formally derived* by replacing 2^* with an arbitrarily large finite exponent and accordingly the occurrences of s_β and s_q in (1.4) and (1.5) with some $s > 1$ and $\eta > 1$ arbitrarily close to 1. In this light, we consider the case $n \in \{1,2\}$ of Theorem 1.2 as a usual and plausible complement to the main case $n \ge 3$.

Finally, we mention that the proof of Theorem 1.2 is based on the *A*-harmonic approximation method introduced in [29] and further developed in [22, 20, 19, 23, 24, 28, 72, 73], for instance. As usual in partial

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*Indeed, the formal replacement transforms (1.5) into either $q > 2$, $\Gamma \in \mathrm{L}^\eta_{\mathrm{loc}}(\Omega)$ for some $\eta > 1$ or $q \le 2$, $\Gamma \in \mathrm{L}^{\eta,t}_{\mathrm{loc}}(\Omega)$ for some $\eta > 1$ and some positive $t > n - \eta q$. This is reasonable for $n = 2$ and with modified case distinction between $q > 1$ and $q \le 1$ also for $n = 1$. Finally, it reduces to the adapted requirement of the theorem, since in case $q \le n$ having $\Gamma \in \mathrm{L}^{\eta,t}_{\mathrm{loc}}(\Omega)$ for some $\eta > 1$ and some positive $t > n - \eta q$ turns out to be equivalent with having $\Gamma \in \mathrm{L}^{\eta, n - q}_{\mathrm{loc}}(\Omega)$ for some $\eta > 1$.

regularity theory, this method also provides an explicit characterization of the regular set Ω_{reg}, which will be stated in the later Section 7 and may prove very handy. In particular, whenever everywhere C^1 regularity or partial C^1 regularity with better information on the size of $\Omega \setminus \Omega_{\mathrm{reg}}$ than just $|\Omega \setminus \Omega_{\mathrm{reg}}| = 0$ is already available, then Theorem 1.2 can boost also these types of regularity to the level of $\mathrm{C}^{1,\alpha}$ with optimal α. The same is true for the subsequent Theorems 1.3, 1.4, and indeed our intended application to the situation of Massari's regularity theorem is one instance which exploits this observation.

We find it reasonable to complement Theorem 1.2 with a variant for minimizers which are known to be in $\mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$, since such local boundedness may be obtained a priori from maximum principles or C^0 regularity methods at least in scalar cases or cases with specific structure. Anyway, under such an assumption we can fully drop the complementary Morrey condition (1.5) and impose on Γ only the sole assumption (1.4):

Theorem 1.3 (partial regularity for a priori $\mathrm{L}^\infty_{\mathrm{loc}}$ minimizers)**.** *We again consider a strictly quasiconvex C^2 integrand $f \colon \mathbb{R}^{N \times n} \to \mathbb{R}$ which has at most quadratic growth. Moreover, we impose on g the same assumptions as in Theorem 1.2 with $q \in [\beta, 2^*)$ replaced by† $q \in [\beta, \infty)$ and with (1.5) dropped. Then, if a local minimizer u of $\mathcal{F}$ satisfies $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, the conclusion of Theorem 1.2 remains valid.*

Another standard observation in C^1 regularity theory is that in the presence of even a priori *gradient* bounds one merely needs locally uniform convexity/ellipticity and can in fact dispense with any global uniformity. This results in the following further variant of our results, suitable for the announced application to Massari's regularity theorem:

Theorem 1.4 (partial regularity for a priori $\mathrm{W}^{1,\infty}_{\mathrm{loc}}$ minimizers in non-uniformly elliptic cases)**.** *We consider a C^2 integrand $f \colon \mathbb{R}^{N \times n} \to \mathbb{R}$ which is strictly convex in the sense of $\mathrm{D}^2 f(z)\xi \cdot \xi > 0$ for all $z, \xi \in \mathbb{R}^{N \times n}$ and has at most quadratic growth. Moreover, we impose on g the same assumptions as in Theorem 1.2 with $q \in [\beta, 2^*)$ replaced by $q \in [\beta, \infty)$ and with (1.5) dropped. Then, if a local minimizer u of $\mathcal{F}$ satisfies $u \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, the conclusion of Theorem 1.2 still holds.*

In order to better illustrate the significance of our assumptions (1.2), (1.3), (1.4), (1.5) for the zero-order integrand $g \colon \Omega \times \mathbb{R}^N \to \mathbb{R}$, we next touch briefly upon specific situations, to which we will return in full detail in the later Section 7.4. For the moment, we tacitly understand that in case $n \in \{1, 2\}$ the previously outlined modifications are in force and that the first-order integrand f is suitably well-behaved. Then two model situations are as follows:

(A) **(uniformly) $\mathrm{C}^{0,\beta}$ integrands g:** In this basic situation, assume (1.2), (1.3) and $\Gamma \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$. Then, the embedding $\mathrm{L}^\infty_{\mathrm{loc}}(\Omega) \subset \mathrm{L}^{s,n}_{\mathrm{loc}}(\Omega)$ implies (1.4) and (1.5), and our results reproduce partial $\mathrm{C}^{1,\alpha}$ regularity with the known exponent $\alpha = \frac{\beta}{2-\beta}$ if $\beta < 1$ and clearly with every $\alpha < 1$ if $\beta = 1$.

(B) **L^p-$\mathrm{C}^{0,\beta}$ integrands g:** Now assume that (1.2), (1.3) hold with $\Gamma \in \mathrm{L}^p_{\mathrm{loc}}(\Omega)$ for some $p \in \left(\frac{n}{\beta}, \infty \right)$. Then, via $p \geq s_\beta$ and $\mathrm{L}^p_{\mathrm{loc}}(\Omega) \subset \mathrm{L}^{s_\beta, n - s_\beta n/p}_{\mathrm{loc}}(\Omega)$, we deduce (1.4) with largest admissible exponent $\alpha = \frac{\beta - n/p}{2-\beta}$, and in case of‡ $p \geq s_q$ we also infer (1.5). Thus, our results yield partial $\mathrm{C}^{1,\alpha}$ regularity with $\alpha = \frac{\beta - n/p}{2 - \beta}$ (specifically $\alpha = 1 - \frac{n}{p}$ in case $\beta = 1$), where the additional condition $p \geq s_q$ is required for the general framework of Theorem 1.2, but can be dropped in the a priori bounded cases of Theorems 1.3 and 1.4.

In fact, a decisive motivation for the present work arises in the scalar case $N = 1$ from integrands g of the integral form

$$g(x, y) = -\int_0^y H(x, t)\, \mathrm{d}t \qquad \text{for } x \in \Omega,\, y \in \mathbb{R} \tag{1.6}$$

with suitable $H \in \mathrm{L}^1_{\mathrm{loc}}(\Omega \times \mathbb{R})$. By observing that the indefinite integral of an $\mathrm{L}^r(\mathbb{R})$ function has an $\mathrm{L}^r(\mathbb{R})$ derivative and is $(1 - \frac{1}{r})$-Hölder continuous on $\mathbb{R}$ (or a corresponding explicit estimate with Hölder's inequality), the previous situations with $\beta = q = 1 - \frac{1}{r}$ then give rise to the following ones:

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†In fact, we could also replace the term $(1 + |y| + |\widehat{y}|)^{q - \beta}$ in (1.2) with an arbitrary locally bounded function of y and $\widehat{y}$, but for simplicity we stick to the above form of (1.2).

‡In case $n \geq 3$ one can check by computation that this condition is an extra requirement only for $q > 2^*\left(1 - \frac{\beta}{n}\right) = 2\frac{n - \beta}{n - 2}$.

(A') **(uniformly) $\mathbf{W^{1,r}}$ integrands $\mathbf{g}$:** Assume that g takes the form (1.6) with $H \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega, \mathrm{L}^r(\mathbb{R}))$, i.e. $\sup_{x \in K} \|H(x, \cdot)\|_{\mathrm{L}^r(\mathbb{R})} < \infty$ for all $K \Subset \Omega$, with some $r \in (1, \infty]$. We then conclude partial $\mathrm{C}^{1,\alpha}$ regularity with $\alpha = \frac{r-1}{r+1}$ if $r < \infty$ and clearly with every $\alpha < 1$ if $r = \infty$.

(B') **$\mathbf{L^p}$-$\mathbf{W^{1,r}}$ integrands $\mathbf{g}$:** Now assume that g takes the form (1.6) with $H \in \mathrm{L}^p_{\mathrm{loc}}(\Omega, \mathrm{L}^r(\mathbb{R}))$, i.e. $\int_K \|H(x, \cdot)\|^p_{\mathrm{L}^r(\mathbb{R})} \, \mathrm{d}x < \infty$ for all $K \Subset \Omega$, with some $r \in (1, \infty]$ and $p \in (nr', \infty)$. We then conclude partial $\mathrm{C}^{1,\alpha}$ regularity with $\alpha = \frac{r-1-nr/p}{r+1}$ (specifically $\alpha = 1 - \frac{n}{p}$ in case $r = \infty$).

Our final application exploits Theorem 1.4 in the case (B') with $p = r \in (n+1, \infty)$ and the corresponding optimal Hölder exponent $\alpha = \alpha_{\mathrm{opt}} := \frac{p-(n+1)}{p+1}$. It concerns minimizers of Massari's functional

$$\mathcal{F}^U_H(F) := \mathrm{P}(F, U) - \int_{U \cap F} H \, \mathrm{d}x \,,$$

defined for fixed open $U \subseteq \mathbb{R}^{n+1}$ and $H \in \mathrm{L}^1(U)$ on sets $F \subseteq \mathbb{R}^{n+1}$ of finite perimeter $\mathrm{P}(F, U)$ in U. In this setting, our previous work [74] came up with the explicit exponent α_{opt} and established partial $\mathrm{C}^{1,\alpha}$ regularity of minimizers of $\mathcal{F}^U_H$ with $H \in \mathrm{L}^p_{\mathrm{loc}}(U)$ for all $\alpha < \alpha_{\mathrm{opt}}$, while examples of irregularity were given for all $\alpha > \alpha_{\mathrm{opt}}$. Here, indeed, we finally sharpen these results by pushing regularity to the limit case $\alpha = \alpha_{\mathrm{opt}}$:

Theorem 1.5 (optimal Massari-type regularity). *Consider an open set $U \subseteq \mathbb{R}^{n+1}$ and $H \in \mathrm{L}^1(U) \cap \mathrm{L}^p_{\mathrm{loc}}(U)$ with $n + 1 < p < \infty$. If a set $E \subseteq \mathbb{R}^{n+1}$ of finite perimeter in U minimizes $\mathcal{F}^U_H$ among all sets $F \subseteq \mathbb{R}^{n+1}$ of finite perimeter in U such that $F \triangle E \Subset U$, then $\partial^* E \cap U$ is relatively open in $\partial E \cap U$ and is an n-dimensional $\mathrm{C}^{1,\alpha_{\mathrm{opt}}}$-submanifold with $\alpha_{\mathrm{opt}} := \frac{p-(n+1)}{p+1}$. Moreover, the singular set $(\partial E \setminus \partial^* E) \cap U$ is empty in case $n \leq 6$ and has Hausdorff dimension at most $n - 7$ in case $n \geq 7$.*

The plan of the paper is now as follows. We collect preliminaries and organize our assumptions in Sections 2 and 3, respectively. Then, on the basic level, we follow the known A-harmonic approximation strategy: We deal with a Caccioppoli inequality in Section 4, with approximate A-harmonicity in Section 5, and with excess estimates in Section 6. Finally, we prove Theorems 1.2, 1.3, 1.4 and return to the model situations (A), (B), (A'), (B') in Section 7, while in Section 8 we finally establish Theorem 1.5.

2 Preliminaries

Generalities

By $\mathrm{B}_r(x)$ we denote the ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $r > 0$. We abbreviate $\mathrm{B}_r := \mathrm{B}_r(0)$. By $\mathrm{c}(t_1, \ldots, t_k)$, $k \in \mathbb{N}$, we denote a constant which only depends on the numbers, vectors, matrices or functions $t_1, \ldots, t_k$. For $p \in [1, \infty)$, we define the Sobolev exponent as $p^* := \frac{np}{n-p}$ if $p < n$ and as (arbitrarily large) number[§] $p^* \in (p, \infty)$ if $p \geq n$. Moreover, we denote the conjugate exponent $\frac{p}{p-1}$ of $p \in (1, \infty)$ by p'.
For functions $u \colon \mathbb{R}^n \to \mathbb{R}^N$ and constants $\zeta \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{N \times n}$, we set

$$u_{\xi,\zeta}(x) := u(x) - \zeta - \xi x \qquad \text{for all } x \in \mathbb{R}^n \,. \tag{2.1}$$

If u is even in $\mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, we write $(u)_\Omega$ for the mean value $\fint_\Omega v \, \mathrm{d}x := \frac{1}{|\Omega|} \int_\Omega u \, \mathrm{d}x$ and abbreviate $(u)_{x,r}$ if $\Omega = \mathrm{B}_r(x)$ with $x \in \mathbb{R}^n$ and $r > 0$. For $u \in \mathrm{L}^p_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ with $p \in [1, \infty)$, the Lebesgue differentiation theorem gives that a.e. $x \in \Omega$ is an L^p-Lebesgue point of u in the sense that $\lim_{r \searrow 0} \frac{1}{|\mathrm{B}_r|} \int_{\mathrm{B}_r(x) \cap \Omega} |u - u^*(x)|^p \, \mathrm{d}y = 0$ holds for some corresponding Lebesgue value $u^*(x) \in \mathbb{R}^N$. Moreover, u^* equals u a.e. and is sometimes called the Lebesgue representative of u.

Morrey spaces

In this paper, we use Morrey spaces in the sense of the following definition.

[§]It may seem intuitive to set $p^* = \infty$ for $p > n$. However, in our context, the convention above is more useful.

Definition 2.1 (Morrey spaces). *Let $p \in [1, \infty)$ and $r \in [0, \infty)$. We define the **Morrey space***

$$\mathrm{L}^{p,r}(\Omega) := \left\{ h \in \mathrm{L}^p(\Omega) : \|h\|_{\mathrm{L}^{p,r}(\Omega)} < \infty \right\},$$

where the value

$$\|h\|_{\mathrm{L}^{p,r}(\Omega)} := \sup_{\rho > 0, x \in \Omega} \left(\frac{1}{\rho^r} \int_{\Omega \cap \mathrm{B}_\rho(x)} |h|^p \, \mathrm{d}x \right)^{\frac{1}{p}}$$

is called the $\mathrm{L}^{p,r}$ Morrey norm of h.

Remark 2.2. Hölder's inequality yields the optimal embedding

$$\mathrm{L}^{p,n-\alpha}(\Omega) \subseteq \mathrm{L}^{r,n-a}(\Omega) \qquad \text{for all } r \leq \min\left\{1, \frac{a}{\alpha}\right\} p, \text{ and } p \in [1,\infty], \, \alpha, \, a \in [0,n].$$

In particular, there holds $\mathrm{L}^p(\Omega) = \mathrm{L}^{p,0}(\Omega) \subseteq \mathrm{L}^{r,n-\frac{n}{p}r}(\Omega)$ for all $r \in [1,p]$. Moreover, Lebesgue's differentiation theorem implies $\mathrm{L}^{p,n}(\Omega) = \mathrm{L}^\infty(\Omega)$ and $\mathrm{L}^{p,r}(\Omega) = \{0\}$ for all $r > n$.

Quasiconvexity and growth conditions

Quasiconvexity was introduced by Morrey in [64] as a generalization of convexity. In several cases, quasiconvexity of the integrand turned out to be equivalent to lower semicontinuity of variational integrals. Thus, quasiconvexity is decisive in the existence theory of minimizers. In addition, it is a crucial hypothesis for partial regularity of (local) minimizers as well. We recast the definition as follows.

Definition 2.3 ((strict) quasiconvexity). *A function $h \in \mathrm{C}^0(\mathbb{R}^{N \times n})$ is called **quasiconvex** if*

$$\fint_{\mathrm{B}_1} h(\xi + \mathrm{D}\varphi) \, \mathrm{d}x \geq h(\xi)$$

*is satisfied for all $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathrm{B}_1, \mathbb{R}^N)$ and all $\xi \in \mathbb{R}^{N \times n}$. It is called **strictly quasiconvex** if, for each bound $M > 0$, there exists a positive constant Q_M such that*

$$\fint_{\mathrm{B}_1} h(\xi + \mathrm{D}\varphi) \, \mathrm{d}x \geq h(\xi) + \mathrm{Q}_M \fint_{\mathrm{B}_1} |\mathrm{D}\varphi|^2 \, \mathrm{d}x \tag{2.2}$$

holds for all $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathrm{B}_1, \mathbb{R}^N)$ and all $\xi \in \mathbb{R}^{N \times n}$ with $|\xi| < M$.

Remark 2.4. Whenever a function h is quasiconvex and has at most quadratic growth, (2.2) stays valid for all $\varphi \in \mathrm{W}^{1,2}_0(\mathrm{B}_1, \mathbb{R}^N)$ by approximation.

Next we record that quasiconvexity together with the quadratic growth condition of this paper naturally implies a linear bound for the derivative of the integrand, suitable rescaled growth conditions, and the Legendre-Hadamard condition for the second derivatives of the integrand.

Lemma 2.5. *Let $h \in \mathrm{C}^2(\mathbb{R}^n)$ be a quasiconvex function such that $|h(z)| \leq \lambda(1 + |z|^2)$ for some $\lambda > 0$ and all $z \in \mathbb{R}^n$. Then there exists $\Lambda = \Lambda(n, \lambda) > 0$ such that $|\mathrm{D}h(z)| \leq \Lambda(1 + |z|)$ for all $z \in \mathbb{R}^n$.*

The proof of Lemma 2.5 can be found, for instance, in [39, Lemma 5.3, Proposition 5.2].

Lemma 2.6. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function satisfying the growth condition*

$$|h(x)| \leq \mathrm{c}_1(1 + |x|^2) \qquad \text{and} \qquad |\mathrm{D}h(x)| \leq \mathrm{c}_2(1 + |x|)$$

for some $\mathrm{c}_1, \mathrm{c}_2 > 0$ and all $x \in \mathbb{R}^n$. Then, for each $R > 0$, there exists a positive constant c depending only on $\mathrm{c}_1, \mathrm{c}_2, R$ and $\sup_{\mathrm{B}_{R+1}} |\mathrm{D}^2 h|$ such that, for all $x \in \mathrm{B}_R$ and all $t > 0$, it holds

$$\frac{|h(x+ty) - h(x) - t\mathrm{D}h(x) \cdot y|}{t^2} \leq \mathrm{c}|y|^2 \qquad \text{and} \qquad \frac{|\mathrm{D}h(x+ty) - \mathrm{D}h(x)|}{t} \leq \mathrm{c}|y|,$$

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Lemma 2.6 originates from [1, Lemma II.3]. The proof given there relies on the mean value theorem combined with a case distinction in terms of $|ty|$.

Lemma 2.7 (Legendre-Hadamard condition)**.** *Consider a strictly quasiconvex function $h \in \mathrm{C}^2(\mathbb{R}^{N \times n})$ and $\xi \in \mathbb{R}^{N \times n}$ with $|\xi| < M$ for some $M > 0$. Then $\mathrm{D}^2 h(\xi)$ satisfies the Legendre-Hadamard condition with constant $2\mathrm{Q}_M$, that is, $\mathrm{D}^2 h(\xi)\zeta x^T \cdot \zeta x^T \geq 2\mathrm{Q}_M |\zeta|^2 |x|^2$ for all $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$.*

The proof is analogous to [39, Proposition 5.2]. Indeed, the second-order criterion for the minimum at $t = 0$ of the single-variable function $t \mapsto \int_\Omega h(\xi + t\mathrm{D}\varphi) - \mathrm{Q}_M |t\mathrm{D}\varphi|^2 \, \mathrm{d}x$ gives $\int_\Omega \mathrm{D}^2 h(\xi)\mathrm{D}\varphi \cdot \mathrm{D}\varphi \, \mathrm{d}x \geq 2\mathrm{Q}_M \int_\Omega |\mathrm{D}\varphi|^2 \, \mathrm{d}x$ for all $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)$. This inequality is tested with $\varphi(y) = \eta(y)\cos(\tau x \cdot y)\zeta$ and $\varphi(y) = \eta(y)\sin(\tau x \cdot y)\zeta$, where $\eta \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega)$ and $\tau \in \mathbb{R}$ are a cut-off function and a parameter, respectively, and in the limit $\tau \to \infty$ one obtains the claim. For further details we refer once more to the proof of [39, Proposition 5.2].

A-harmonic approximation

We use the following notion of *A*-harmonic functions or in other words of weak solutions to second-order constant-coefficient linear PDE systems.

Definition 2.8 (*A*-harmonic function, [70, Section 3])**.** *Let A be a bilinear form on $\mathbb{R}^{N \times n}$ satisfying*

$$A(\zeta x^T, \zeta x^T) \geq \lambda |\zeta|^2 |x|^2 \tag{2.3}$$

$$|A| \leq \Lambda \tag{2.4}$$

*for some $\lambda, \Lambda > 0$ and all $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$. A function $h \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$ is called an **A-harmonic function** on Ω if*

$$\int_\Omega A(\mathrm{D}h, \mathrm{D}\varphi) \, \mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{W}^{1,2}_0(\Omega, \mathbb{R}^N).$$

The next lemma can be found in a slightly more general version in [72, Lemma 6.8]. It will be crucial in deriving excess estimates for local minimizers by comparison with *A*-harmonic functions and ultimately by exploiting the good estimates of linear regularity theory.

Lemma 2.9 (*A*-harmonic approximation)**.** *Let A be a bilinear form on $\mathbb{R}^{N \times n}$ which satisfies (2.3) and (2.4) for some $\lambda, \Lambda > 0$. For every $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon, n, N, \lambda, \Lambda) > 0$ and $\mathrm{c} = \mathrm{c}(n, N, \lambda, \Lambda) > 0$ such that, whenever $w \in \mathrm{W}^{1,2}(\mathrm{B}_\rho(x_0), \mathbb{R}^N)$ and $\Psi \in (0, 1]$ satisfy*

$$\fint_{\mathrm{B}_\rho(x_0)} |\mathrm{D}w|^2 \, \mathrm{d}x \leq \Psi^2 \quad and \quad \left| \fint_{\mathrm{B}_\rho(x_0)} A(\mathrm{D}w, \mathrm{D}\varphi) \, \mathrm{d}x \right| \leq \delta \Psi \, \|\mathrm{D}\varphi\|_{\mathrm{L}^\infty(\mathrm{B}_\rho(x_0))}$$

for all $\varphi \in \mathrm{W}^{1,\infty}_0(\mathrm{B}_\rho(x_0), \mathbb{R}^N)$, then there exists an A-harmonic function $h \in \mathrm{C}^\infty(\mathrm{B}_\rho(x_0), \mathbb{R}^N)$ with

$$\|\mathrm{D}h\|_{\mathrm{C}\left(\mathrm{B}_{\frac{\rho}{2}}(x_0)\right)} + \rho \|\mathrm{D}^2 h\|_{\mathrm{C}\left(\mathrm{B}_{\frac{\rho}{2}}(x_0)\right)} \leq \mathrm{c} \quad and \quad \fint_{\mathrm{B}_{\frac{\rho}{2}}(x_0)} \left| \frac{w - \Psi h}{\rho} \right|^2 \, \mathrm{d}x \leq \Psi^2 \varepsilon.$$

In fact, the lemma will eventually be used with the choice $A := \mathrm{D}^2 f(\xi)$, where f is the first-order integrand of our theorems and Lemma 2.7 guarantees the validity of the Legendre-Hadamard condition (2.3).

An iteration lemma

The following lemma, for which we refer to [39, Lemma 6.1], will be decisive for the proof of a later Caccioppoli inequality.

Lemma 2.10 (iteration lemma)**.** *Let $0 \leq r < R$ and let $v : [r, R] \to [0, \infty)$ be a bounded and non-negative function such that for all $r_1 < r_2$ in $[r, R]$ the estimate*

$$v(t) \leq \frac{a}{(r_2 - r_1)^\tau} + \frac{b}{(r_2 - r_1)^t} + C + \mu v(s)$$

holds true with exponents $\tau, t > 0$, some constants $a, b, C \geq 0$ and $\mu \in [0, 1)$. Then there exists a constant $\mathrm{c} = \mathrm{c}(\alpha, \mu) > 0$ such that

$$v(r) \leq \mathrm{c} \left(\frac{a}{(R - r)^\tau} + \frac{b}{(R - r)^t} + C \right).$$

Campanato's integral characterization of Hölder continuity

The following proposition connects integral oscillation controls and Hölder continuity in an optimal way. It can be found, for instance, in [39, Theorem 2.9].

Proposition 2.11 (Campanato's integral characterization of Hölder continuity)**.** *Let $p \in [1, \infty)$, $\alpha \in (0, 1]$ and $v \in \mathrm{L}^p(\Omega)$. The Lebesgue representative v^* of v is in $\mathrm{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$ if and only if there exist a constant $\mathrm{c} > 0$ such that for all $x \in \Omega$, $r > 0$ with $\mathrm{B}_r(x) \subseteq \Omega$, the inequality*

$$\fint_{\mathrm{B}_r(x)} |v - (v)_{x,r}|^p \, \mathrm{d}y \leq \mathrm{c}r^{\alpha p}$$

is satisfied.

The perimeter and variational mean curvatures

In Section 8 we adapt our regularity results to sets of variational mean curvature in L^p. We directly warn the reader that we set these parametric considerations in the ambient space $\mathbb{R}^{n+1}$ and then find the optimal exponent $\frac{p-(n+1)}{p+1}$, while in the context of the ambient space $\mathbb{R}^n$ the same exponent clearly reads $\frac{p-n}{p+1}$. However, our setting is convenient in transferring regularity from the non-parametric to the parametric framework.

We now give a brief introduction to the theory of variational mean curvatures. For an $\mathcal{L}^n$-measurable set $E \subseteq \mathbb{R}^{n+1}$ and an open set $U \subseteq \mathbb{R}^{n+1}$, the perimeter of E in U is defined by

$$\mathrm{P}(E, U) := \sup \left\{ \int_E \mathrm{div}\varphi \, \mathrm{d}x : \varphi \in \mathrm{C}^1_{\mathrm{cpt}}(U, \mathbb{R}^{n+1}), \|\varphi\|_{\mathrm{C}(U)} \leq 1 \right\} .$$

We say that E is a set of finite perimeter in U if $\mathrm{P}(E, U) < \infty$. The perimeter measures the boundary of a set. Indeed, the structure theorem of De Giorgi guarantees $P(E, U) = \mathcal{H}^{n-1}(\partial^* E \cap U)$ for sets E of locally finite perimeter in U, where the reduced boundary $\partial^* E$ is defined as in [4, Definition 3.54] and is invariant under modification of E by null sets. In particular, we record $\partial E = \partial^* E$ whenever E has C^1 boundary.

In connection with variational curvatures, we consider the measure theoretic interior

$$E(1) := \left\{ x \in \mathbb{R}^{n+1} : \lim_{r \searrow 0} \frac{|E \cap \mathrm{B}_r(x)|}{|\mathrm{B}_r(x)|} = 1 \right\}$$

as a representative of the $\mathcal{L}^n$-measurable set E. The main advantage is that, for every other representative E' of E, it holds $\partial E' \supseteq \partial E(1) = \overline{\partial^* E}$. For more background on perimeter and BV theory, we refer to [4, 40, 57].

Variational mean curvatures are motivated by the strong connection between the Massari functional $\mathcal{F}^U_H$ and the mean curvature of C^2-submanifolds and in a sense indeed generalize the notion of mean curvature for (reduced) boundaries of arbitrary $\mathcal{L}^n$-measurable sets. However, the perspective differs from the classical one of differential geometry in that variational mean curvatures are indeed functions defined on the entire ambient space $\mathbb{R}^{n+1}$.

Definition 2.12 ((local) variational mean curvatures, [45, p. 197])**.** *Let $U \subseteq \mathbb{R}^{n+1}$ be an open set. We say that a set $E \subseteq \mathbb{R}^{n+1}$ of finite perimeter in U has **(local) variational mean curvature $H \in \mathrm{L}^1(U)$ in U** if*

$$\mathcal{F}^U_H(E) \leq \mathcal{F}^U_H(F) \qquad \text{holds for all measurable sets } F \in \mathbb{R}^{n+1} \text{ such that } F \triangle E \Subset U \,,$$

where we abbreviated

$$\mathcal{F}^U_H(F) := \mathrm{P}(F, U) - \int_{U \cap F} H \, \mathrm{d}x \,.$$

Remark 2.13 ([7, p. 149], [58, p. 357-358]).

   i) If H is a variational mean curvature H of E in U, then each $\mathrm{L}^1(U)$ function which is larger than H on $E \cap U$ a.e. and smaller than H a.e. on $U \setminus E$ is also a variational mean curvature of E in U. In particular, from one variational mean curvature of E in U one can obtain infinitely many others.

ii) If $E \subseteq \mathbb{R}^{n+1}$ has C^1 boundary, if a part of ∂E is the graph of a C^1 function defined on some open $\Omega \subseteq \mathbb{R}^n$ such that $v(\Omega) \Subset (0, R)$ for some $R > 0$, and if H is a variational mean curvature of E in some open U such that $\overline{\Omega} \times [0, R] \subseteq U$, then v minimizes the functional

$$\mathcal{G}[w] := \int_\Omega \sqrt{1 + |\mathrm{D}w(x)|^2} - \int_0^{w(x)} H(x, t) \,\mathrm{d}t \,\mathrm{d}x$$

among all $w \in \mathrm{C}^1_u(\Omega)$ with $w(\Omega) \Subset (0, R)$. This fact allows for transferring regularity from the non-parametric case of minimizers of (1.1) to the parametric case of sets of variational mean curvature in L^p.

Although Remark 2.13 i) shows that variational curvatures are far from being unique, a good integrability of the curvature still implies a strong regularity result, as originally proved by Massari in [59, Theorem 3.1, Theorem 3.2].

Theorem 2.14 (Massari's regularity theorem). *Consider $p \in (n+1, \infty]$, $\alpha := \frac{1}{4}\left(1 - \frac{n+1}{p}\right)$ (where we understand $\frac{1}{4}\left(1 - \frac{n+1}{p}\right) = \frac{1}{4}$ in case $p = \infty$), an open set $U \subseteq \mathbb{R}^{n+1}$, and a set $E \subseteq \mathbb{R}^{n+1}$ of finite perimeter in U. If there exists a variational mean curvature $H \in \mathrm{L}^p(U)$ of E in U, then the following hold.*

i) *$\partial^* E \cap U$ is an n-dimensional $\mathrm{C}^{1,\alpha}$-manifold relatively open in $\partial E \cap U$.*

ii) *For all $t \in (n-7, n+1]$, it holds $\mathcal{H}^t((\partial E \setminus \partial^* E) \cap U) = 0$, where we understand $\mathcal{H}^t := \mathcal{H}^0$ in case $t < 0$.*

Remark 2.15 (optimality of the parameters in Theorem 2.14).

i) The ranges of the parameters p and t are optimal. In fact, [44, Example 2.2] and [45, Section 2] show that the theorem fails for $p \leq n+1$, and in [10, Theorem A] it is proved that the Simons cone $C = \left\{x \in \mathbb{R}^8 : x_1^2 + x_2^2 + x_3^2 + x_4^2 < x_5^2 + x_6^2 + x_7^2 + x_8^2\right\}$ is an area-minimizing cone (i.e. has vanishing variational mean curvature) with $\partial C \setminus \partial^* C = \{0\}$.

ii) As foreshadowed in the introduction, the Hölder exponent $\alpha = \frac{1}{4}\left(1 - \frac{n+1}{p}\right)$ is not optimal. In fact, in our predecessor paper [74] we applied Tamanini's regularity results [78] for almost-minimizers of perimeter to improve the exponent in Theorem 2.14 to arbitrary $\alpha < \frac{p-(n+1)}{p+1}$ and at the same time showed by counterexamples that $\mathrm{C}^{1,\alpha}$ regularity may fail for $\alpha > \frac{p-(n+1)}{p+1}$. In particular, this confirmed the conjecture of [44, Remark 3.4] that Theorem 2.14 should hold with exponents $\alpha(n, p)$ such that $\lim_{p \to \infty} \alpha(n, p) = 1$. Here, with Theorem 1.5 we close the last gap in this regard by showing that, for $p \in (n+1, \infty)$, Theorem 2.14 extends to the case of the limit exponent $\alpha = \alpha_{\mathrm{opt}} := \frac{p-(n+1)}{p+1}$. (As a side remark we record that in case $p = \infty$ the limit exponent 1 and thus $\mathrm{C}^{1,1}$ regularity can be reached only for $n = 1$, but not for $n \geq 2$, as shown in [74, Proposition 3.6] and [44, Remark 3.4], respectively.)

3 Overall assumptions and settings

We generally work with dimensions $n, N \in \mathbb{N}$ and a bounded open set $\Omega \subseteq \mathbb{R}^n$. As first-order integrand we fix a C^2 function $f \colon \mathbb{R}^{N \times n} \to \mathbb{R}$ of at most quadratic growth

$$\limsup_{|z| \to \infty} \frac{|f(z)|}{|z|^2} < \infty, \tag{3.1}$$

and as zero-order integrand a Carathéodory function $g \colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ which satisfies the Morrey-Hölder condition (1.2) with a non-negative function Γ and exponents $\beta \in (0, 1]$ and $q \in [\beta, 2^*)$ as in (1.3). We continue abbreviating $s_\beta := 2^*/(2^* - \beta) \in (1, \infty)$ and $s_q := 2^*/(2^* - q) \in (1, \infty)$.
Next we set up conventions in order to unify the Morrey conditions (1.4), (1.5) imposed on Γ for $n \geq 3$ with their counterparts for $n \in \{1, 2\}$. In fact, since we can replace Ω with subdomains, we may and do assume that these assumptions hold in global instead of local Morrey spaces. Furthermore, on one hand we can assume $\alpha \leq \frac{\beta}{2-\beta}$ (compare the introduction), on the other hand, for $n \in \{1, 2\}$, we understand that $2^* \in (2, \infty)$ is

essentially arbitrary, but large enough to ensure $s_\beta \leq s$ for the exponent $s > 1$ in Theorem 1.2 (plus a similar requirement added a few lines below). We remark that in principle this leads to a dependence of several constants in later estimates on the choice of 2^*. However, whenever we explicitly list such dependencies in the sequel, we always think of the main case $n \geq 3$ with $2^* = \frac{2n}{n-2}$ fully determined by n, and thus we tacitly disregard any additional dependency on the parameter 2^* in case $n \in \{1, 2\}$. With these conventions in force, we may now recast (1.4) for arbitrary dimension n in the convenient form of

$$\Gamma \in \mathrm{L}^{s_\beta, n - s_\beta \beta + s_\beta(2-\beta)\alpha}(\Omega) \text{ with } \alpha \in \left(0, \tfrac{\beta}{2-\beta}\right] \cap (0, 1), \tag{3.2}$$

and we generally abbreviate

$$\Gamma_1 := \|\Gamma\|_{\mathrm{L}^{s_\beta, n + s_\beta((2-\beta)\alpha - \beta))}(\Omega)}.$$

Furthermore, in case $n \in \{1, 2\}$ we require 2^* to be large enough also for $s_q \leq \eta$ with the exponent $\eta > 1$ of Theorem 1.2. Then, we may recast (1.5) for arbitrary dimension n as

$$\begin{aligned}
&\text{in case } q > \min\{2, n\}: \quad && \Gamma \in \mathrm{L}^{s_q}(\Omega), \\
&\text{in case } q \leq \min\{2, n\}: \quad && \Gamma \in \mathrm{L}^{s_q, n - s_q q + 2\delta}(\Omega) \text{ with } \delta \in (0, 1),
\end{aligned} \tag{3.3}$$

where e.g. in case $n \in \{1, 2\}$, $q \leq n$ the passage to (3.3) works by observing $\Gamma \in \mathrm{L}^{s_q, n-q}(\Omega)$ and taking $\delta \in (0, 1)$ such that $2\delta \leq s_q q - q$. We correspondingly abbreviate

$$\Gamma_2 := \|\Gamma\|_{\mathrm{L}^{s_q}(\Omega)} \qquad \text{and} \qquad \Gamma_2 := \|\Gamma\|_{\mathrm{L}^{s_q, n - s_q q + 2\delta}(\Omega)},$$

respectively. Additionally, in case $q > \min\{2, n\}$ we observe $s_q q > n$ and for later convenience still consider a fixed exponent $\delta \in (0, 1)$ such that

$$\text{in case } q > \min\{2, n\}: \quad 2\delta \leq s_q q - n. \tag{3.4}$$

In handling $\mathrm{D}^2 f$, given $t_1, \ldots, t_k > 0$, $k \in \mathbb{N}$, we apply the convention that $\tilde{\Lambda}_{t_1, \ldots, t_k}$ denotes a constant such that

$$\tilde{\Lambda}_{t_1, \ldots, t_k} \geq \sup\{|\mathrm{D}^2 f(z)| : z \in \mathbb{R}^{N \times n}, |z| < \mathrm{c}(t_1, \ldots, t_k)|\}.$$

Moreover, since f is C^2, for each $M > 0$, there exists a modulus of continuity $\nu_M : [0, \infty) \to [0, \infty)$ for $\mathrm{D}^2 f$ such that $\lim_{t \searrow 0} \nu_M(t) = \nu_M(0) = 0$ and

$$|\mathrm{D}^2 f(A) - \mathrm{D}^2 f(B)| \leq \sqrt{\nu_M(|A - B|^2)} \qquad \text{for all } A, B \in \mathbb{R}^{N \times n} \text{ such that } |A| \leq M, \, |B| \leq M + 1. \tag{3.5}$$

Without loss of generality, we can choose ν_M such that it is bounded by $4\tilde{\Lambda}_M^2$. In fact, in the sequel we mostly work with a concave upper bound $\widehat{\nu}_M$ for ν_M in the sense of the subsequent remark.

Remark 3.1. For each bounded function $\nu : [0, \infty) \to [0, \infty)$ with $\lim_{t \searrow 0} \nu(t) = \nu(0) = 0$, there exists a concave upper bound $\widehat{\nu}$ in the sense that

$$\widehat{\nu} \in \mathcal{A}_\nu := \{\Theta : [0, \infty) \to [0, \infty) \mid \Theta \text{ is concave and } \Theta \geq \nu\}$$

with $\lim_{t \searrow 0} \widehat{\nu}(t) = \widehat{\nu}(0) = 0$ and $\sup_{[0, \infty)} \widehat{\nu} \leq \sup_{[0, \infty)} \nu$. These properties imply that $\widehat{\nu}$ is non-decreasing, and clearly one can choose $\widehat{\nu}$ as the concave hull $\inf_{\Theta \in \mathcal{A}_\nu} \Theta$ of ν.

Proof of the claim in Remark 3.1. It holds $\mathcal{A}_\nu \neq \emptyset$ since the constant function with value $\mathrm{C} := \sup_{[0, \infty)} \nu$ is in $\mathcal{A}_\nu$. Setting $\widehat{\nu}(t) := \inf_{\Theta \in \mathcal{A}_\nu} \Theta(t)$ for $t \in [0, \infty)$, we have $\widehat{\nu} \in \mathcal{A}_\nu$ and $\widehat{\nu} \leq \mathrm{C}$. For every $\varepsilon \in (0, \mathrm{C}]$, there exists $\delta > 0$ such that $\nu([0, \delta]) \subseteq [0, \varepsilon]$. Then $\Theta^\varepsilon(t) := \mathbb{1}_{[\delta, \infty)}(t)\mathrm{C} + \mathbb{1}_{[0, \delta)}(t)\left(\varepsilon + \frac{t}{\delta}(\mathrm{C} - \varepsilon)\right)$ is in $\mathcal{A}_\nu$ with $\lim_{t \searrow 0} \Theta^\varepsilon(t) = \Theta^\varepsilon(0) = \varepsilon$. Taking into account $\widehat{\nu} \leq \Theta^\varepsilon$ and the arbitrariness of ε, we arrive at $\lim_{t \searrow 0} \widehat{\nu}(t) = \widehat{\nu}(0) = 0$. Finally, we deduce that $\widehat{\nu}$ is non-decreasing: For arbitrary $t_1 \leq t_2$ in $[0, \infty)$ and $\lambda \in (0, 1]$, we write $t_2 = (1 - \lambda)t_1 + \lambda t_3$ for suitable $t_3 \in [0, \infty)$ and deduce from concavity and non-negativity of $\widehat{\nu}$ that $\widehat{\nu}(t_2) \geq (1 - \lambda)\widehat{\nu}(t_1)$. In the limit $\lambda \searrow 0$ this implies $\widehat{\nu}(t_2) \geq \widehat{\nu}(t_1)$. $\qquad\square$

In all statements to follow, we now **consider a local minimizer** $u \in \mathbf{W}^{1,2}(\Omega, \mathbb{R}^N)$ of the integral (1.1) in the sense of Definition 1.1, and we **fix a ball** $\mathbf{B}_\rho(x_0) \Subset \Omega$ **with** $\rho \in (0, 1]$. Beyond that, we will work in one of the following settings, which reproduce the frameworks of Theorems 1.2, 1.3, 1.4 in the introduction (except again for using global instead of local spaces).

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Setting 1. This setting is essentially the one of Theorem 1.2 and the basic one for our purposes. We impose all assumptions previously discussed in this section and additionally assume that f is strictly quasiconvex.

Setting 2. This setting is essentially the one of Theorem 1.3. We assume that the local minimizer u satisfies $u \in \mathrm{L}^\infty(\Omega, \mathbb{R}^N)$. Among the Morrey conditions discussed we keep only (3.2), while we drop (3.3) and (3.4). Instead of $q \in [\beta, 2^*)$ we allow even arbitrary $q \in [\beta, \infty)$, and we again assume that f is strictly quasiconvex.

Setting 3. This setting is essentially the one of Theorem 1.4. We assume that the local minimizer u satisfies $u \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^N)$, and we define $\mathcal{M} := \|u\|_{\mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^N)}$. Again, we keep only (3.2), but drop (3.3) and (3.4), and we allow arbitrary $q \in [\beta, \infty)$. Finally, we assume that, for each $M > 0$, there exists $\lambda_M > 0$ such that we have

$$\mathrm{D}^2 f(z)\xi \cdot \xi \geq \lambda_M |\xi|^2 \qquad \text{for all } \xi, z \in \mathbb{R}^{N \times n} \text{ such that } |z| \leq M. \tag{3.6}$$

Since (3.6) still implies that f is quasiconvex (though not necessarily strictly quasiconvex in the precise sense of Definition 2.3), Lemma 2.5 and the growth condition (3.1) imply that in each setting there exist constants $\Lambda > 0$ and $\Lambda_{\mathrm{D}} = \Lambda_{\mathrm{D}}(n, N, \Lambda) > 0$ such that

$$|f(z)| \leq \Lambda(1 + |z|^2) \quad \text{and} \quad |\mathrm{D}f(z)| \leq \Lambda_{\mathrm{D}}(1 + |z|) \qquad \text{for all } z \in \mathbb{R}^{N \times n}. \tag{3.7}$$

4 Caccioppoli inequality

Caccioppoli-type inequalities are crucial in the partial regularity theory for minimizers u of quasiconvex integrals. Our subsequent version essentially controls the L^2 norm of $\mathrm{D}u_{\xi,\varsigma}$ via L^2 and L^{2^*} norms of $u_{\xi,\varsigma}$, where it is crucial that the L^{2^*} terms are either superlinear in $u_{\xi,\varsigma}$ or come with an arbitrarily small ε in front.

Lemma 4.1 (Caccioppoli's inequality). *We consider Setting 1. For every $M > 0$ there exist a constant $\mathrm{c} = \mathrm{c}(n, N, \beta, q, \mathrm{Q}_M, \Gamma_1, \Gamma_2, \Lambda, M, \tilde{\Lambda}_M) > 0$ such that*

$$\fint_{\mathrm{B}_{\frac{\rho}{2}}(x_0)} |\mathrm{D}u_{\xi,\varsigma}|^2 \, \mathrm{d}x \leq \mathrm{c} \left[\fint_{\mathrm{B}_\rho(x_0)} \left| \frac{u_{\xi,\varsigma}}{\rho} \right|^2 \mathrm{d}x + \varepsilon \left(\fint_{\mathrm{B}_\rho(x_0)} \left| \frac{u_{\xi,\varsigma}}{\rho} \right|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}} + \fint_{\mathrm{B}_\rho(x_0)} \left| \frac{u_{\xi,\varsigma}}{\rho} \right|^{2^*} \mathrm{d}x + \varepsilon^{-\frac{\beta}{2-\beta}} \rho^{2\min\{\alpha,\delta\}} \right]$$

for all $\varepsilon \in (0,1]$, $\varsigma \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{N \times n}$ with $|\varsigma| + |\xi| \leq M$, and the affine function $u_{\xi,\varsigma}$ defined in (2.1).

Proof. W.l.o.g. we assume $x_0 = 0$. In order to apply Lemma 2.10 for $R = \rho$ and $r = \frac{\rho}{2}$, we fix $r_1 < r_2$ in $\left(\frac{\rho}{2}, \rho \right)$. Moreover, let $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$ be a cut-off function with $\mathrm{spt}(\eta) \subseteq \mathrm{B}_{r_2}$, $\eta = 1$ on B_{r_1} and $|\mathrm{D}\eta| \leq \frac{2}{r_2 - r_1}$. We set $v(x) := u_{\xi,\varsigma}(x) = u(x) - \varsigma - \xi x$, $\varphi := \eta v$ and $\psi := (1-\eta)v$. By the product rule we have $\varphi, \psi \in \mathrm{W}^{1,2}_0(\mathrm{B}_\rho, \mathbb{R}^N)$ with $\varphi = v$ on B_{r_1}. Employing the quasiconvexity of f and the bound $|\xi| \leq M$ we then estimate

$$\mathrm{Q}_M \int_{\mathrm{B}_{r_1}} |\mathrm{D}v|^2 \, \mathrm{d}x \leq \mathrm{Q}_M \int_{\mathrm{B}_{r_2}} |\mathrm{D}\varphi|^2 \, \mathrm{d}x \leq \int_{\mathrm{B}_{r_2}} f(\xi + \mathrm{D}\varphi) - f(\xi) \, \mathrm{d}x. \tag{4.1}$$

Now, the equality $\mathrm{D}u - \xi = \mathrm{D}v = \mathrm{D}\varphi + \mathrm{D}\psi$ on B_ρ gives

$$\mathrm{Q}_M \int_{\mathrm{B}_{r_1}} |\mathrm{D}v|^2 \, \mathrm{d}x \leq \int_{\mathrm{B}_{r_2}} f(\xi + \mathrm{D}\varphi) - f(\xi) \, \mathrm{d}x$$

$$= \mathcal{F}[u] - \mathcal{F}[u - \varphi] + \int_{\mathrm{B}_{r_2}} g(x, u - \varphi) - g(x, u) \, \mathrm{d}x \tag{4.2}$$

$$+ \int_{\mathrm{B}_{r_2}} f(\mathrm{D}u - \mathrm{D}\psi) - f(\mathrm{D}u) \, \mathrm{d}x + \int_{\mathrm{B}_{r_2}} f(\xi + \mathrm{D}\psi) - f(\xi) \, \mathrm{d}x.$$

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The vanishing of $D\psi$ on B_{r_1} leads to

$$Q_M \int_{B_{r_1}} |Dv|^2\, dx \le \left(\mathcal{F}[u] - \mathcal{F}[u-\varphi]\right) + \int_{B_{r_2}} g(x, u-\varphi) - g(x,u)\, dx$$

$$+ \int_{B_{r_2}\setminus B_{r_1}} f(Du - D\psi) - f(Du) + Df(\xi)D\psi\, dx$$

$$+ \int_{B_{r_2}\setminus B_{r_1}} f(\xi + D\psi) - f(\xi) - Df(\xi)D\psi\, dx$$

$$=: I + II + III + IV.$$

Next we suitably estimate the right-hand side terms I, II, III, and IV. Clearly, we can control I by observing $\mathcal{F}[u] - \mathcal{F}[u-\varphi] \le 0$ thanks to the minimizing property of u. Next we handle the term II, which for our purposes is the decisive one. To this end we recall assumption (1.2) which gives

$$II = \int_{B_{r_2}} g(x, u-\varphi) - g(x,u)\, dx \le \int_{B_{r_2}} \Gamma(1 + |u| + |\varphi|)^{q-\beta} |\varphi|^\beta\, dx \le \int_{B_{r_2}} \Gamma(1 + |u| + |v|)^{q-\beta} |v|^\beta\, dx. \quad (4.3)$$

Taking into account $|\zeta| + |\xi| \le M$ we deduce

$$II \le c(\beta, q, M)\left(\int_{B_\rho} |v|^\beta \Gamma\, dx + \int_{B_\rho} |v|^q \Gamma\, dx\right). \quad (4.4)$$

The application of Hölder's inequality with exponents $\frac{2^*}{\beta}$ and $s_\beta = \frac{2^*}{2^* - \beta}$ gives

$$\int_{B_\rho} |v|^\beta \Gamma\, dx \le \left(\int_{B_\rho} |v|^{2^*}\, dx\right)^{\frac{\beta}{2^*}} \left(\int_{B_\rho} \Gamma^{s_\beta}\, dx\right)^{\frac{1}{s_\beta}}.$$

For further estimating the preceding term, we distinguish the cases $n \ge 3$ and $n \in \{1,2\}$. In the case $n \ge 3$, we employ Young's inequality with exponents $\frac{2}{\beta}$ and $\frac{2}{2-\beta}$ to conclude (with the arbitrary $\varepsilon > 0$ from the statement of Lemma 4.1)

$$\int_{B_\rho} |v|^\beta \Gamma\, dx \le \varepsilon \left(\int_{B_\rho} |v|^{2^*}\, dx\right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} \left(\int_{B_\rho} \Gamma^{s_\beta}\, dx\right)^{\frac{2}{s_\beta(2-\beta)}}.$$

Taking into account the Morrey condition $\Gamma \in L^{s_\beta, n - s_\beta \beta + s_\beta(2-\beta)\alpha}(\Omega)$ from (3.2) for Γ and the equality $(n - s_\beta \beta)\frac{2}{s_\beta(2-\beta)} = n$ (where the latter follows in case $n \ge 3$ from the definitions of s_β and 2^*), we deduce

$$\int_{B_\rho} |v|^\beta \Gamma\, dx \le \varepsilon \left(\int_{B_\rho} |v|^{2^*}\, dx\right)^{\frac{2}{2^*}} + \Gamma_1^{\frac{2}{2-\beta}} \varepsilon^{-\frac{\beta}{2-\beta}} \rho^{n+2\alpha}.$$

In view of $\frac{2}{2^*}n + 2 = n$ we further get

$$\int_{B_\rho} |v|^\beta \Gamma\, dx \le c(n, \beta, \Gamma_1)\left[\varepsilon \rho^n \left(\fint_{B_\rho} \left|\frac{v}{\rho}\right|^{2^*}\, dx\right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} \rho^{n+2\alpha}\right]. \quad (4.5)$$

Next we turn to the case $n \in \{1,2\}$ which requires a slight technical modification of the preceding estimates. Indeed, we still apply Young's inequality with exponents $\frac{2}{\beta}$ and $\frac{2}{2-\beta}$, but with the ε in the preceding now replaced by $\varepsilon \rho^{n - \frac{2}{2^*}n - 2}$. This means that we first get

$$\int_{B_\rho} |v|^\beta \Gamma\, dx \le \varepsilon \rho^{n - \frac{2}{2^*}n - 2} \left(\int_{B_\rho} |v|^{2^*}\, dx\right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} \rho^{\frac{\beta}{2-\beta}\left(\frac{2}{2^*}n + 2 - n\right)} \left(\int_{B_\rho} \Gamma^{s_\beta}\, dx\right)^{\frac{2}{s_\beta(2-\beta)}}.$$

Then we arrive at (4.5) also in this case and essentially as before, since in spite of now $n\frac{2}{2^*} + 2 > n$ we still have $\frac{\beta}{2-\beta}(\frac{2}{2^*}n + 2 - n) + (n - s_\beta\beta)\frac{2}{s_\beta(2-\beta)} = n$ even with our convention of an arbitrary $2^* > 2$ in case $n \in \{1, 2\}$.

At this point we continue our reasoning back in arbitrary dimension n, but in estimating the second integral on the right-hand side of (4.4) we distinguish cases for q. We first treat the case $q \le \min\{2, n\}$. By Young's inequality with exponents $\frac{2^*}{q}$ and s_q and by the Morrey condition $\Gamma \in L^{s_q, n - s_q q + 2\delta}(\Omega)$ of (3.3) it follows

$$\int_{B_\rho} |v|^q \Gamma \, dx \le \int_{B_\rho} \left|\frac{v}{\rho}\right|^{2^*} dx + \rho^{s_q q} \int_{B_\rho} \Gamma^{s_q} \, dx \le \int_{B_\rho} \left|\frac{v}{\rho}\right|^{2^*} dx + \Gamma_2^{s_q} \rho^{n + 2\delta} \, . \tag{4.6}$$

In the opposite case $q > \min\{2, n\}$, taking into account that $\Gamma \in L^{s_q}(\Omega)$ and $qs_q \ge n + 2\delta$ by (3.3) and (3.4) and further recalling $\rho \le 1$, we readily observe that the estimate (4.6) is also valid.

Now, back to the general case we collect the estimates (4.4), (4.5), (4.6) and obtain

$$\text{II} \le c(n, \beta, q, M, \Gamma_1, \Gamma_2) \left[\varepsilon\rho^n \left(\fint_{B_\rho} \left|\frac{v}{\rho}\right|^{2^*} dx\right)^{\frac{2}{2^*}} + \int_{B_\rho} \left|\frac{v}{\rho}\right|^{2^*} dx + \varepsilon^{-\frac{\beta}{2-\beta}} \rho^{n + 2\min\{\alpha, \delta\}}\right] \, .$$

Next we turn to the term III. Taking (3.7) into account, we apply Lemma 2.6 to estimate

$$\text{III} = \int_{B_{r_2} \setminus B_{r_1}} f(\xi + Dv - D\psi) - f(\xi) + f(\xi) - f(\xi + Dv) + Df(\xi)D\psi \, dx$$

$$= \int_{B_{r_2} \setminus B_{r_1}} f(\xi + Dv - D\psi) - f(\xi) - Df(\xi)(Dv - D\psi) - \int_0^1 [Df(\xi + tDv) - Df(\xi)] \, Dv \, dt \, dx$$

$$\le c(n, N, \Lambda, M, \tilde{\Lambda}_M) \int_{B_{r_2} \setminus B_{r_1}} |Dv - D\psi|^2 + |Dv|^2 \, dx \, .$$

For the term IV, the same type of estimate implies

$$\text{IV} = \int_{B_{r_2} \setminus B_{r_1}} \int_0^1 [Df(\xi + tD\psi) - Df(\xi)] \, D\psi \, dt \, dx \le c(n, N, \Lambda, \mathcal{M}, \tilde{\Lambda}_M) \int_{B_{r_2} \setminus B_{r_1}} |D\psi|^2 \, dx.$$

The previous estimates can then be combined in order to jointly estimate III and IV by

$$\text{III} + \text{IV} \le c(n, N, \Lambda, M, \tilde{\Lambda}_M) \int_{B_{r_2} \setminus B_{r_1}} |Dv|^2 + |D\psi|^2 \, dx \le c(n, N, \Lambda, M, \tilde{\Lambda}_M) \int_{B_{r_2} \setminus B_{r_1}} |Dv|^2 + \left|\frac{v}{r_2 - r_1}\right|^2 dx \, .$$

Collecting all estimates, we have

$$\int_{B_{r_1}} |Dv|^2 \, dx$$

$$\le c \left(\int_{B_{r_2} \setminus B_{r_1}} |Dv|^2 \, dx + \int_{B_\rho} \left|\frac{v}{r_2 - r_1}\right|^2 dx + \varepsilon\rho^n \left(\fint_{B_\rho} \left|\frac{v}{\rho}\right|^2 dx\right)^{\frac{2}{2^*}} + \int_{B_\rho} \left|\frac{v}{\rho}\right|^{2^*} dx + \varepsilon^{-\frac{\beta}{2-\beta}} \rho^{n + 2\min\{\alpha, \delta\}}\right) \, .$$

In order to apply Lemma 2.10, we need a constant less than 1 in front of the first term on the right-hand side. Therefore, using Widman's hole filling trick, we first add $c \int_{B_{r_1}} |Dv|^2 \, dx$ and then divide by $1 + c$ to get

$$\int_{B_{r_1}} |Dv|^2 \, dx$$

$$\le \frac{c}{1 + c} \int_{B_{r_2}} |Dv|^2 \, dx + \int_{B_\rho} \left|\frac{v}{r_2 - r_1}\right|^2 dx + \varepsilon\rho^n \left(\fint_{B_\rho} \left|\frac{v}{\rho}\right|^2 dx\right)^{\frac{2}{2^*}} + \int_{B_\rho} \left|\frac{v}{\rho}\right|^{2^*} dx + \varepsilon^{-\frac{\beta}{2-\beta}} \rho^{n + 2\min\{\alpha, \delta\}} \, .$$

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Finally, by Lemma 2.10 we conclude

$$\fint_{B_{\frac{\rho}{2}}} |Dv|^2 \, dx \le c \left(\fint_{B_\rho} \left|\frac{v}{\rho}\right|^2 dx + \varepsilon \left(\fint_{B_\rho} \left|\frac{v}{\rho}\right|^2 dx \right)^{\frac{2}{2^*}} + \fint_{B_\rho} \left|\frac{v}{\rho}\right|^{2^*} dx + \varepsilon^{-\frac{\beta}{2-\beta}} \rho^{n+2\min\{\alpha,\delta\}} \right).$$

This completes the proof of the lemma. $\qquad\square$

5 Approximate harmonicity

The following estimate prepares the ground for eventually applying the A-harmonic approximation of Lemma 2.9 with $A = D^2 f(\xi)$.

Lemma 5.1 (approximate A-harmonicity)**.** *We consider Setting 1. For each bound $M > 0$, there exists a constant* $c = c(n, N, \beta, q, \Gamma_1, \Gamma_2, \Lambda, M, \tilde\Lambda_M) > 0$ *such that*

$$\left| \fint_{B_\rho(x_0)} D^2 f(\xi) Du_{\xi,\zeta} \cdot D\varphi \, dx \right| \le c \left(\fint_{B_\rho(x_0)} \left|\frac{u_{\xi,\zeta}}{\rho}\right|^{2^*} dx + \sqrt{\widehat\nu_M(\Phi)\Phi} + \Phi + \rho^\alpha \right) \|D\varphi\|_{L^\infty(B_\rho(x_0))}$$

for all $\varphi \in W_0^{1,\infty}(B_\rho(x_0), \mathbb{R}^N)$, $\zeta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{N\times n}$ with $|\zeta| + |\xi| \le M$, where we use

$$\Phi := \fint_{B_\rho(x_0)} \left|Du_{\xi,\zeta}\right|^2 dx,$$

$\widehat\nu_M$ *from Remark 3.1, and $u_{\xi,\zeta}$ from (2.1).*

Proof. W.l.o.g. we assume $x_0 = 0$ and $\|D\varphi\|_{L^\infty(B_\rho)} = 1$, which in particular implies $\|\varphi\|_{L^\infty(B_\rho)} \le \rho$. We abbreviate once more $v := u_{\xi,\zeta}$ and — in order to achieve convenient balance between the terms $\sigma^{\beta-1} \rho^{(2-\beta)\alpha}$ and σ in the subsequent estimates (5.4) and (5.5), respectively — choose

$$\sigma := \rho^\alpha \in (0, 1].$$

We then split integrals as follows (where in particular we use $\int_{B_\rho} D\varphi \, dx = 0$):

$$
\begin{aligned}
\fint_{B_\rho} D^2 f(\xi) Dv \cdot D\varphi \, dx = {} & \fint_{B_\rho} D^2 f(\xi) Dv \cdot D\varphi - Df(Du) \cdot D\varphi + Df(\xi) \cdot D\varphi \, dx \\
& + \fint_{B_\rho} Df(Du) \cdot D\varphi + \frac{1}{\sigma} \left[f(Du - \sigma D\varphi) - f(Du) \right] dx \\
& + \frac{1}{\sigma} \fint_{B_\rho} f(Du) - f(Du - \sigma D\varphi) \, dx \\
=: {} & \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}
\tag{5.1}
$$

We proceed by estimating the term III via minimality. Indeed, since u is a local minimizer of $\mathcal{F}$, we have $\mathcal{F}[u] \le \mathcal{F}[u - \sigma\varphi]$ and may then rearrange terms and divide by $\sigma\omega_n\rho^n$ to find

$$\mathrm{III} = \frac{1}{\sigma} \fint_{B_\rho} f(Du) - f(Du - \sigma D\varphi) \, dx \le \frac{1}{\sigma} \fint_{B_\rho} g(x, u - \sigma\varphi) - g(x, u) \, dx.
\tag{5.2}$$

Assumption (1.2) on g and the bound $\|\varphi\|_{L^\infty(B_\rho)} \le \rho$ then imply

$$
\begin{aligned}
\mathrm{III} \le {} & \frac{1}{\sigma} \fint_{B_\rho} \Gamma(1 + 2|u| + \sigma\rho)^{q-\beta} (\sigma\rho)^\beta \, dx \le \frac{c(\beta, q, M)}{\sigma} \fint_{B_\rho} \Gamma(1 + |v|)^{q-\beta} (\sigma\rho)^\beta \, dx \\
& \le c(\beta, q, M) \sigma^{\beta-1} \left(\rho^\beta \fint_{B_\rho} \Gamma \, dx + \rho^\beta \fint_{B_\rho} \Gamma|v|^q \, dx \right).
\end{aligned}
\tag{5.3}
$$

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In further estimating the terms on the right-hand side of (5.3), on one hand we exploit that (3.2) gives $\Gamma \in \mathrm{L}^{s_\beta, n + s_\beta(2-\beta)\alpha - s_\beta\beta}(\Omega) \subseteq \mathrm{L}^{1, n + (2-\beta)\alpha - \beta}(\Omega)$ and thus

$$\rho^\beta \fint_{\mathrm{B}_\rho} \Gamma \, \mathrm{d}x \leq \mathrm{c}(n, \beta, \Gamma_1) \, \rho^{(2-\beta)\alpha} \,.$$

On the other hand, we re-use the estimate (4.6) in form

$$\rho^\beta \fint_{\mathrm{B}_\rho} \Gamma |v|^q \, \mathrm{d}x \leq \rho^\beta \fint_{\mathrm{B}_\rho} \left| \frac{v}{\rho} \right|^{2^*} \mathrm{d}x + \mathrm{c}(n, q, \Gamma_2) \, \rho^{\beta + 2\delta} \leq \rho^{(1-\beta)\alpha} \fint_{\mathrm{B}_\rho} \left| \frac{v}{\rho} \right|^{2^*} \mathrm{d}x + \mathrm{c}(n, q, \Gamma_2) \, \rho^{(2-\beta)\alpha} \,,$$

where in the second step we exploited $\alpha \leq \frac{\beta}{2-\beta}$ in discarding (at this stage insignificant) factors $\rho^{\beta - (1-\beta)\alpha} \leq 1$ and $\rho^{\beta + 2\delta - (2-\beta)\alpha} \leq 1$. Collecting the estimates and inserting the choice $\sigma = \rho^\alpha$, we arrive at

$$\mathrm{III} \leq \mathrm{c} \, \sigma^{\beta - 1} \left(\rho^{(1-\beta)\alpha} \fint_{\mathrm{B}_\rho} \left| \frac{v}{\rho} \right|^{2^*} \mathrm{d}x + \rho^{(2-\beta)\alpha} \right) = \mathrm{c} \left(\fint_{\mathrm{B}_\rho} \left| \frac{v}{\rho} \right|^{2^*} \mathrm{d}x + \rho^\alpha \right) \tag{5.4}$$

with $\mathrm{c} = \mathrm{c}(n, \beta, q, \Gamma_1, \Gamma_2, M)$. Next, we return to the term I from the right-hand side of (5.1). We partially rewrite the term by integration, then in view of $|\xi| \leq M$ use (3.5) on the set $U := \{|\mathrm{D}v| < 1\}$ and (3.7) on its complement U^c, and finally apply the Cauchy-Schwarz and Jensen inequalities. In this way we find

$$
\begin{aligned}
\mathrm{I} &= \frac{1}{|\mathrm{B}_\rho|} \int_{\mathrm{B}_\rho \cap U} \int_0^1 \left[\mathrm{D}^2 f(\xi) - \mathrm{D}^2 f(\xi + \tau \mathrm{D}v) \right] \mathrm{D}v \cdot \mathrm{D}\varphi \, \mathrm{d}\tau \, \mathrm{d}x \\
&\quad + \frac{1}{|\mathrm{B}_\rho|} \int_{\mathrm{B}_\rho \cap U^c} \mathrm{D}^2 f(\xi) \mathrm{D}v \cdot \mathrm{D}\varphi - \mathrm{D}f(\mathrm{D}u) \cdot \mathrm{D}\varphi + \mathrm{D}f(\xi) \cdot \mathrm{D}\varphi \, \mathrm{d}x \\
&\leq \fint_{\mathrm{B}_\rho} \sqrt{\widehat{\nu}_M(|\mathrm{D}v|^2)} \, |\mathrm{D}v| \, \mathrm{d}x + \frac{1}{|\mathrm{B}_\rho|} \int_{\mathrm{B}_\rho \cap U^c} \tilde{\Lambda}_M |\mathrm{D}v| + \Lambda_{\mathrm{D}}(2 + |\xi| + |\mathrm{D}u|) \, \mathrm{d}x \\
&\leq \sqrt{\fint_{\mathrm{B}_\rho} \widehat{\nu}_M(|\mathrm{D}v|^2) \, \mathrm{d}x} \sqrt{\fint_{\mathrm{B}_\rho} |\mathrm{D}v|^2 \, \mathrm{d}x} + \mathrm{c}(n, N, M, \Lambda, \tilde{\Lambda}_M) \fint_{\mathrm{B}_\rho} |\mathrm{D}v|^2 \, \mathrm{d}x \\
&\leq \sqrt{\widehat{\nu}_M(\Phi)} \sqrt{\Phi} + \mathrm{c}(n, N, M, \Lambda, \tilde{\Lambda}_M) \, \Phi \,.
\end{aligned}
$$

Similarly, in order to control the term II, we work on the sets $\widetilde{U} := \{|\mathrm{D}u| < 1\}$ and $\widetilde{U}^c$ and deduce

$$
\begin{aligned}
\mathrm{II} &= \fint_{\mathrm{B}_\rho} \int_0^\sigma \left[\mathrm{D}f(\mathrm{D}u) - \mathrm{D}f(\mathrm{D}u - t\mathrm{D}\varphi) \right] \cdot \mathrm{D}\varphi \, \mathrm{d}t \, \mathrm{d}x \\
&= \frac{1}{|\mathrm{B}_\rho|} \int_{\mathrm{B}_\rho \cap \widetilde{U}} \int_0^\sigma \int_0^1 \mathrm{D}^2 f(\mathrm{D}u - \tau t \mathrm{D}\varphi) \, t \mathrm{D}\varphi \cdot \mathrm{D}\varphi \, \mathrm{d}\tau \, \mathrm{d}t \, \mathrm{d}x \\
&\quad + \frac{1}{|\mathrm{B}_\rho|} \int_{\mathrm{B}_\rho \cap \widetilde{U}^c} \int_0^\sigma \left[\mathrm{D}f(\mathrm{D}u) - \mathrm{D}f(\mathrm{D}u - t\mathrm{D}\varphi) \right] \cdot \mathrm{D}\varphi \, \mathrm{d}t \, \mathrm{d}x \\
&\leq \tilde{\Lambda} \fint_{\mathrm{B}_\rho} \int_0^\sigma t |\mathrm{D}\varphi|^2 \, \mathrm{d}t \, \mathrm{d}x + \frac{\Lambda_{\mathrm{D}}}{|\mathrm{B}_\rho|} \int_{\mathrm{B}_\rho \cap \widetilde{U}^c} \int_0^\sigma 2 + |\mathrm{D}u| + |\mathrm{D}u - t\mathrm{D}\varphi| \, \mathrm{d}t \, \mathrm{d}x \\
&\leq \tilde{\Lambda} \sigma + \mathrm{c}(n, N, M, \Lambda) \fint_{\mathrm{B}_\rho} |\mathrm{D}v|^2 \, \mathrm{d}x \\
&= \tilde{\Lambda} \rho^\alpha + \mathrm{c}(n, N, M, \Lambda) \, \Phi \,.
\end{aligned}
\tag{5.5}
$$

The combination of the estimates for I, II, and III gives

$$\fint_{\mathrm{B}_\rho(x_0)} \mathrm{D}^2 f(\xi) \mathrm{D}v \cdot \mathrm{D}\varphi \, \mathrm{d}x \leq \mathrm{c} \left(\fint_{\mathrm{B}_\rho(x_0)} \left| \frac{v}{\rho} \right|^{2^*} \mathrm{d}x + \sqrt{\widehat{\nu}_M(\Phi)\Phi} + \Phi + \rho^\alpha \right) \,.$$

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The corresponding lower bound can be derived similarly: The term I in (5.1) is kept unchanged, the terms II and III in (5.1) are replaced by

$$\fint_{B_\rho} Df(Du) \cdot D\varphi - \frac{1}{\sigma} \left[f(Du + \sigma D\varphi) - f(Du) \right] dx \qquad \text{and} \qquad \frac{1}{\sigma} \fint_{B_\rho} f(Du + \sigma D\varphi) - f(Du) \, dx \, ,$$

respectively, and all three single terms are now estimated from below instead of above. □

6 Excess estimates

We have now collected the main auxiliary results in order to derive estimates for the quadratic excess, defined by

$$\Phi(x_0, \rho) := \fint_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \, dx \qquad \text{and} \qquad \Phi(\rho) := \Phi(0, \rho) \, .$$

Before turning to these estimates, however, we put on record one basic lemma on properties of the excess.

Lemma 6.1. *For all $r \in (0, \rho]$ and all $\xi \in \mathbb{R}^{N \times n}$, we have the inequalities*

$$\Phi(x_0, \rho) \leq \fint_{B_\rho(x_0)} |Du - \xi|^2 \, dx \qquad \text{and} \qquad \Phi(x_0, r) \leq \left(\frac{\rho}{r} \right)^n \Phi(x_0, \rho) \, .$$

Proof. The first estimate follows by checking that the quadratic function $\mathbb{R}^{N \times n} \to \mathbb{R}$, $\xi \mapsto \fint_{B_\rho(x_0)} |Du - \xi|^2 \, dx$ has its minimum point at $(Du)_{x_0,\rho}$. For $r \in (0, \rho]$, an application of this estimate then gives

$$\Phi(x_0, r) \leq \fint_{B_r(x_0)} |Du - (Du)_{x_0,\rho}|^2 \, dx \leq \frac{|B_\rho(x_0)|}{|B_r(x_0)|} \Phi(x_0, \rho) = \left(\frac{\rho}{r} \right)^n \Phi(x_0, \rho)$$

and thus confirms the second claim. □

Lemma 6.2 (excess improvement). *We consider Setting 1. For every $\kappa \in \left(\min\{\alpha, \delta\}, 1 \right)$ and every $M > 0$, there exist constants $\theta \in (0, 1)$, $\varepsilon > 0$, $c > 0$ such that*

$$\rho + \Phi(x_0, \rho) \leq \varepsilon \qquad \text{and} \qquad |(Du)_{x_0,\rho}| + |(u)_{x_0,\rho}| \leq M \qquad (6.1)$$

together imply

$$\Phi(x_0, \theta\rho) \leq \theta^{2\kappa} \Phi(x_0, \rho) + c\rho^{2\min\{\alpha, \delta\}} \, .$$

The constants c and θ all depend only on n, N, β, q, Q_M, Γ_1, Γ_2, Λ, M, $\tilde{\Lambda}_{n,N,Q_M,\Lambda,M}$, κ, and ε additionally depends on α and ν_M.

Proof. We assume $x_0 = 0$ and $\Phi := \Phi(\rho) > 0$ and abbreviate

$$\xi := (Du)_{0,\rho} \, , \qquad \zeta := (u)_{0,\rho} \, , \qquad v(x) := u_{\xi,\zeta}(x) = u(x) - \zeta - \xi x \, , \qquad A := D^2 f(\xi) \, .$$

Our first aim is now applying the A-harmonic approximation lemma with this choice of A. To this end we first deduce from the quasiconvexity assumption on f, Lemma 2.7, and the bound $|\xi| \leq M$ that the conditions (2.3), (2.4) are satisfied. Therefore, Lemma 5.1 guarantees

$$\left| \fint_{B_\rho} A(Dv, D\varphi) \, dx \right| \leq c_1 \left(\fint_{B_\rho} \left| \frac{v}{\rho} \right|^{2^*} dx + \sqrt{\widehat{\nu_M(\Phi)}} \sqrt{\Phi} + \Phi + \rho^\alpha \right) \|D\varphi\|_{L^\infty(B_\rho)}$$

for all $\varphi \in W_0^{1,\infty}(B_\rho, \mathbb{R}^N)$. Taking into account $(v)_{0,\rho} = 0$ and the Sobolev-Poincaré inequality, for a slightly modified constant $c_1 = c_1(n, N, \beta, q, \Gamma_1, \Gamma_2, M, \tilde{\Lambda}_M) > 0$, we also get the estimate in form

$$\left| \fint_{B_\rho} A(Dv, D\varphi) \, dx \right| \leq c_1 \left(\Phi^{\frac{2^*}{2}} + \sqrt{\widehat{\nu_M(\Phi)}} \sqrt{\Phi} + \Phi + \rho^\alpha \right) \|D\varphi\|_{L^\infty(B_\rho)} \, .$$

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Now, for $\varepsilon_{\mathrm{HA}} \in (0,1]$ to be fixed below, we consider the corresponding $\delta_{\mathrm{HA}} = \delta_{\mathrm{HA}}(\varepsilon_{\mathrm{HA}}, n, N, Q_M, \Lambda) > 0$ of Lemma 2.9, and we set

$$\Psi := \sqrt{\Phi} + \frac{2c_1}{\delta_{\mathrm{HA}}} \rho^\alpha \,.$$

Then we have

$$\left| \fint_{\mathrm{B}_\rho} A(\mathrm{D}v, \mathrm{D}\varphi) \, \mathrm{d}x \right| \leq \left(c_1 \left(\sqrt{\widehat{\nu}_M(\Phi)} + \sqrt{\Phi} + \Phi^{\frac{2^*-1}{2}} \right) + \frac{1}{2} \delta_{\mathrm{HA}} \right) \Psi \, \|\mathrm{D}\varphi\|_{\mathrm{L}^\infty(\mathrm{B}_\rho)}$$

for all $\varphi \in W_0^{1,\infty}(\mathrm{B}_\rho, \mathbb{R}^N)$. Since we can choose $\varepsilon = \varepsilon(n, N, \beta, q, Q_M, \tilde{\Lambda}_M, \Lambda, \Gamma_1, \Gamma_2, M, \alpha, \nu_M, \varepsilon_{\mathrm{HA}}) \in (0,1)$ small enough for deducing from (6.1) that $c_1 \left(\sqrt{\widehat{\nu}_M(\Phi)} + \sqrt{\Phi} + \Phi^{\frac{2^*-1}{2}} \right) \leq \frac{1}{2} \delta_{\mathrm{HA}}$ and $\Psi \leq 1$, we can ensure

$$\left| \fint_{\mathrm{B}_\rho} A(\mathrm{D}v, \mathrm{D}\varphi) \, \mathrm{d}x \right| \leq \delta_{\mathrm{HA}} \Psi \, \|\mathrm{D}\varphi\|_{\mathrm{L}^\infty(\mathrm{B}_\rho)}$$

for all $\varphi \in W_0^{1,\infty}(\mathrm{B}_\rho, \mathbb{R}^N)$ in order to then apply Lemma 2.9. Thus, there exist an A-harmonic function $h \in \mathrm{C}^\infty(\mathrm{B}_\rho, \mathbb{R}^N)$ and a constant $c_2 = c_2(n, N, Q_M, \Lambda) > 0$ such that we have

$$\|\mathrm{D}h\|_{\mathrm{C}(\mathrm{B}_{\frac{\rho}{2}})} + \rho \|\mathrm{D}^2 h\|_{\mathrm{C}(\mathrm{B}_{\frac{\rho}{2}})} \leq c_2 \tag{6.2}$$

and

$$\fint_{\mathrm{B}_{\frac{\rho}{2}}} \left| \frac{v - \Psi h}{\rho} \right|^2 \mathrm{d}x \leq \varepsilon_{\mathrm{HA}} \Psi^2 \,. \tag{6.3}$$

In order to take proper advantage of (6.2) and (6.3) we first observe that (6.2) and Taylor expansion yield the bound

$$\|h(x) - h(0) - \mathrm{D}h(0)x\|_{\mathrm{C}(\mathrm{B}_{2\theta\rho}(0))} \leq 4c_2 \theta^2 \rho \,.$$

In the sequel we abbreviate

$$\tilde{\xi} := \xi + \Psi \, \mathrm{D}h(0) \qquad \text{and} \qquad \tilde{\zeta} := \zeta + \Psi \, h(0) \,,$$

and, for $\theta \in \left(0, \frac{1}{4}\right]$ to be determined at the end of the proof, we now fix $\varepsilon_{\mathrm{HA}} := \theta^{n+4}$. Then, from (6.3) and the previous bound we obtain

$$\begin{aligned}
\fint_{\mathrm{B}_{2\theta\rho}} \left| \frac{u_{\tilde{\xi}, \tilde{\zeta}}}{\theta\rho} \right|^2 \mathrm{d}x &\leq \frac{1}{4^n \theta^{n+2}} \fint_{\mathrm{B}_{\frac{\rho}{2}}} \left| \frac{v - \Psi h}{\rho} \right|^2 \mathrm{d}x + \Psi^2 \sup_{x \in \mathrm{B}_{2\theta\rho}} \left| \frac{h(x) - h(0) - \mathrm{D}h(0)x}{2\theta\rho} \right|^2 \\
&\leq c \left(\frac{\varepsilon_{\mathrm{HA}}}{\theta^{n+2}} + \theta^2 \right) \Psi^2 \\
&\leq c\theta^2 \Psi^2
\end{aligned} \tag{6.4}$$

with $c = c(n, N, Q_M, \Lambda) > 0$. Furthermore, we claim that the auxiliary estimates

$$|\tilde{\xi} - \xi| = \Psi \, |\mathrm{D}h(0)| \leq c \, \Psi \qquad \text{and} \qquad |\tilde{\zeta} - \zeta| = \Psi \, |h(0)| \leq c \, \Psi \rho \tag{6.5}$$

hold with $c = c(n, N, Q_M, \Lambda, M)$. Indeed, the first estimate in (6.5) is evident from (6.2), while the second one is now derived as follows. We first observe that (6.2) implies $|h - h(0)| \leq c\rho$ on $\mathrm{B}_{\frac{\rho}{2}}$. Then, via (6.3), $\varepsilon_{\mathrm{HA}} < 1$, and the Poincaré inequality we conclude

$$\begin{aligned}
\Psi^2 |h(0)|^2 &\leq c \left(\fint_{\mathrm{B}_{\frac{\rho}{2}}} |\Psi h|^2 \, \mathrm{d}x + \Psi^2 \rho^2 \right) \\
&\leq c \left(\fint_{\mathrm{B}_{\frac{\rho}{2}}} |v - \Psi^2 h|^2 \, \mathrm{d}x + \fint_{\mathrm{B}_{\frac{\rho}{2}}} |v|^2 \, \mathrm{d}x + \Psi^2 \rho^2 \right) \leq c(\varepsilon_{\mathrm{HA}} \Psi^2 \rho^2 + \rho^2 \Phi + \Psi^2 \rho^2) \leq c \, \Psi^2 \rho^2
\end{aligned}$$

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This completes the verification of (6.5) and in particular ensures $|\tilde{\xi}| + |\tilde{\zeta}| \leq c(n, N, Q_M, \Lambda, M)$. At this stage, we are ready for the main estimations, which draw on Lemma 6.1, on the Caccioppoli inequality of Lemma 4.1 with arbitrary $\varepsilon_{\mathrm{CI}} \in (0,1]$ (to be determined at the end of this proof), on the Sobolev-Poincaré inequality, and on the estimates (6.4) and (6.5). In fact, with a constant $c = c(n, N, \beta, q, Q_M, \Gamma_1, \Gamma_2, \Lambda, M, \tilde{\Lambda}_{n,N,Q_M,\Lambda,M})$, which may change from line to line, we find

$$\Phi(\theta\rho) \leq \fint_{B_{\theta\rho}} |\mathrm{D}u - \tilde{\xi}|^2 \,\mathrm{d}x$$

$$\leq c\left(\fint_{B_{2\theta\rho}} \left|\frac{u_{\xi,\tilde{\zeta}}}{\theta\rho}\right|^2 \mathrm{d}x + \varepsilon_{\mathrm{CI}}\left(\fint_{B_{2\theta\rho}} \left|\frac{u_{\xi,\tilde{\zeta}}}{\theta\rho}\right|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}} + \fint_{B_{2\theta\rho}} \left|\frac{u_{\xi,\tilde{\zeta}}}{\theta\rho}\right|^{2^*} \mathrm{d}x + \varepsilon_{\mathrm{CI}}^{-\frac{\beta}{2-\beta}}(\theta\rho)^{2\min\{\alpha,\delta\}} \right)$$

$$\leq c\left(\fint_{B_{2\theta\rho}} \left|\frac{u_{\xi,\tilde{\zeta}}}{\theta\rho}\right|^2 \mathrm{d}x + \varepsilon_{\mathrm{CI}}\left(\theta^{-n-2^*} \fint_{B_\rho} \left|\frac{v}{\rho}\right|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}} + \varepsilon_{\mathrm{CI}}\left(\frac{|\tilde{\zeta} - \zeta|}{\theta\rho} + |\tilde{\xi} - \xi| \right)^2 \right.$$

$$\left. + \theta^{-n-2^*} \fint_{B_\rho} \left|\frac{v}{\rho}\right|^{2^*} \mathrm{d}x + \left(\frac{|\tilde{\zeta} - \zeta|}{\theta\rho} + |\tilde{\xi} - \xi| \right)^{2^*} + \varepsilon_{\mathrm{CI}}^{-\frac{\beta}{2-\beta}}(\theta\rho)^{2\min\{\alpha,\delta\}} \right)$$

$$\leq c\left(\theta^2 \Psi^2 + \varepsilon_{\mathrm{CI}}\theta^{-\frac{2}{2^*}n-2}\Phi + \varepsilon_{\mathrm{CI}}\theta^{-2}\Psi^2 + \theta^{-n-2^*}\Phi^{\frac{2^*}{2}} + \theta^{-2^*}\Psi^{2^*} + \varepsilon_{\mathrm{CI}}^{-\frac{\beta}{2-\beta}}(\theta\rho)^{2\min\{\alpha,\delta\}} \right).$$

By using $\Phi \leq \Psi^2$ and by reducing to the worst powers of θ, we simplify the result of these estimations and arrive at

$$\Phi(\theta\rho) \leq c_3 \left[\left(\theta^2 + \varepsilon_{\mathrm{CI}}\theta^{-\frac{2}{2^*}n-2} + \Psi^{2^*-2}\theta^{-n-2^*} \right)\Psi^2 + \varepsilon_{\mathrm{CI}}^{-\frac{\beta}{2-\beta}}(\theta\rho)^{2\min\{\alpha,\delta\}} \right].$$

for $c_3 = c_3(n, N, \beta, q, Q_M, \Gamma_1, \Gamma_2, \Lambda, M, \tilde{\Lambda}_{n,N,Q_M,\Lambda,M})$. We now finalize the proof by determining the remaining parameters. Taking into account $\kappa < 1$, we first take $\theta = \theta(n, N, \beta, q, Q_M, \Gamma_1, \Gamma_2, \Lambda, M, \kappa, \tilde{\Lambda}_{n,N,Q_M,\Lambda,M})$ small enough for having $c_3\theta^2 \leq \frac{1}{6}\theta^{2\kappa}$. Then we make $\varepsilon_{\mathrm{CI}}$ (which depends on the same parameters as θ) small enough to ensure $c_3\varepsilon_{\mathrm{CI}}\theta^{-\frac{2}{2^*}n-2} \leq \frac{1}{6}\theta^{2\kappa}$, and we decrease ε (which after all depends additionally on α and ν_M and now also on κ) such that, in addition to the earlier smallness requirement, (6.1) implies also $c_3\Psi^{2^*-2}\theta^{-n-2^*} \leq \frac{1}{6}\theta^{2\kappa}$. Altogether, we finally end up with

$$\Phi(\theta\rho) \leq \frac{1}{2}\theta^{2\kappa}\Psi^2 + c_3\varepsilon_{\mathrm{CI}}^{-\frac{\beta}{2-\beta}}(\theta\rho)^{2\min\{\alpha,\delta\}} \leq \theta^{2\kappa}\Phi + c\rho^{2\min\{\alpha,\delta\}},$$

where we exploited that $\frac{1}{2}\Psi^2 \leq \Phi + \frac{2c_1}{\delta_{\mathrm{HA}}}\rho^{2\alpha} \leq \Phi + \frac{2c_1}{\delta_{\mathrm{HA}}}\rho^{2\min\{\alpha,\delta\}}$ by choice of Ψ. The constant c in this final estimate once more depends on the same parameters as θ and $\varepsilon_{\mathrm{CI}}$, and the proof is complete. $\square$

Lemma 6.3 (excess decay). *We consider Setting 1. For every $\kappa \in \big(\min\{\alpha, \delta\}, 1\big)$ and every $M > 0$, there exist constants $\varepsilon \in (0,1]$ and $c > 0$ such that*

$$\rho + \Phi(x_0, \rho) \leq \varepsilon \qquad and \qquad |(\mathrm{D}u)_{x_0,\rho}| + |(u)_{x_0,\rho}| \leq M \tag{6.6}$$

together imply

$$\Phi(x_0, r) \leq c\left(\left(\frac{r}{\rho}\right)^{2\kappa}\Phi(x_0, \rho) + r^{2\min\{\alpha,\delta\}} \right) \qquad for\ all\ r \in (0, \rho].$$

Moreover, both ε and c only depend on the parameters n, N, β, q, Q_M, Γ_1, Γ_2, Λ, M, κ, and $\tilde{\Lambda}_{n,N,Q_M,\Lambda,M}$, and ε additionally depends on α and ν_M.

Proof. We assume $x_0 = 0$ and for the moment abbreviate

$$\gamma := \min\{\alpha, \delta\}.$$

We denote by θ, $\tilde{\varepsilon}$, and $\tilde{c}$ the constants from Lemma 6.2 (for the given κ and M). Moreover, we fix a new quantity $\varepsilon = \varepsilon(n, N, \beta, q, Q_M, \Gamma_1, \Gamma_2, \Lambda, M, \kappa, \tilde{\Lambda}_{n,N,Q_M,\Lambda,M}, \alpha, \nu_M) > 0$ small enough that (6.6) implies

$$\rho + C\rho^{2\gamma} + \Phi(\rho) \leq \tilde{\varepsilon}, \qquad \frac{\theta^{-\frac{n}{2}}}{1-\theta^\kappa}\sqrt{\Phi(\rho)} + \frac{\theta^{-\frac{n}{2}}}{1-\theta^\gamma}\sqrt{C}\rho^\gamma \leq M \quad and \quad \frac{c_1\theta^{-n}}{1-\theta}\rho\left(\sqrt{\tilde{\varepsilon}} + 2M\right) \leq M \tag{6.7}$$

with $C := \frac{\tilde{c}}{\theta^{2\gamma} - \theta^{2\kappa}}$ and the constant $c_1 = c_1(n)$ of the Poincaré inequality relevant below. We claim that, for all integers $i \geq 0$, there hold

$$\Phi(\theta^i \rho) \leq \theta^{2\kappa i} \Phi(\rho) + C(\theta^i \rho)^{2\gamma}, \tag{6.8}$$

$$\theta^i \rho + \Phi(\theta^i \rho) \leq \tilde{\varepsilon}, \tag{6.9}$$

$$|(Du)_{\theta^i \rho}| \leq 2M, \tag{6.10}$$

$$|(u)_{\theta^i \rho}| \leq 2M. \tag{6.11}$$

In fact, our main aim is proving (6.8), but it will be convenient to establish the above set of inequalities by the following induction argument.

For $i = 0$, all claims follow readily from (6.6) and the preceding choice of ε. So, we assume (6.8)–(6.11) for $i = 0, 1, 2, \ldots, k$ and deduce the validity of these claims for $i = k + 1$ as well. To this end, since we have (6.9)–(6.11) for $i = k$, we may use Lemma 6.2 with $\theta^k \rho$ instead of ρ. We find

$$\Phi(\theta^{k+1}\rho) \leq \theta^{2\kappa} \Phi(\theta^k \rho) + \tilde{c}(\theta^k \rho)^{2\gamma}.$$

Then, taking into account (6.8) for $i = k$, we conclude

$$\Phi(\theta^{k+1}\rho) \leq \theta^{2\kappa(k+1)} \Phi(\rho) + C(\theta^k \rho)^{2\gamma} \theta^{2\kappa} + \tilde{c}(\theta^k \rho)^{2\gamma}$$

$$\leq \theta^{2\kappa(k+1)} \Phi(\rho) + \tilde{c}(\theta^k \rho)^{2\gamma} \frac{\theta^{2\kappa}}{\theta^{2\gamma} - \theta^{2\kappa}} + \tilde{c}(\theta^k \rho)^{2\gamma}$$

$$= \theta^{2\kappa(k+1)} \Phi(\rho) + \tilde{c}(\theta^k \rho)^{2\gamma} \left(\frac{\theta^{2\kappa}}{\theta^{2\gamma} - \theta^{2\kappa}} + 1 \right)$$

$$= \theta^{2\kappa(k+1)} \Phi(\rho) + C(\theta^{k+1} \rho)^{2\gamma}.$$

Thus, we arrive at (6.8) for $i = k + 1$, and this implies (6.9) for $i = k + 1$ via $\theta < 1$ and the smallness assumption (6.7). Next we turn to (6.10). By the Cauchy-Schwarz inequality and (6.8) for $i = 0, 1, 2, \ldots, k$, we estimate

$$\left| (Du)_{\theta^{k+1}\rho} - (Du)_\rho \right| \leq \sum_{i=0}^{k} \left| (Du)_{\theta^{i+1}\rho} - (Du)_{\theta^i \rho} \right|$$

$$\leq \sum_{i=0}^{k} \left(\fint_{B_{\theta^{i+1}\rho}} |Du - (Du)_{\theta^i \rho}|^2 \, dx \right)^{\frac{1}{2}}$$

$$= \theta^{-\frac{n}{2}} \sum_{i=0}^{k} \sqrt{\Phi(\theta^i \rho)}$$

$$\leq \theta^{-\frac{n}{2}} \sum_{i=0}^{\infty} \left(\theta^{\kappa i} \sqrt{\Phi(\rho)} + \sqrt{C} \theta^{\gamma i} \rho^\gamma \right)$$

$$= \frac{\theta^{-\frac{n}{2}}}{1 - \theta^\kappa} \sqrt{\Phi(\rho)} + \frac{\theta^{-\frac{n}{2}}}{1 - \theta^\gamma} \sqrt{C} \rho^\gamma.$$

Then, (6.10) for $i = k+1$ follows via (6.7) and (6.6). For (6.11), we argue similarly, but also involve Poincaré's

inequality. In fact, we have

$$
\begin{aligned}
\left|(u)_{\theta^{k+1}\rho} - (u)_\rho\right| &\leq \theta^{-n} \sum_{i=0}^{k} \fint_{B_{\theta^i\rho}} |u - (u)_{\theta^i\rho}| \, dx \\
&\leq c_1 \theta^{-n} \sum_{i=0}^{k} \theta^i \rho \fint_{B_{\theta^i\rho}} |Du| \, dx \\
&\leq c_1 \theta^{-n} \sum_{i=0}^{k} \theta^i \rho \left(\fint_{B_{\theta^i\rho}} |Du - (Du)_{\theta^i\rho}| \, dx + |(Du)_{\theta^i\rho}| \right) \\
&\leq c_1 \theta^{-n} \sum_{i=0}^{k} \theta^i \rho \left(\sqrt{\Phi(\theta^i\rho)} + 2M \right) \\
&\leq \frac{c_1 \theta^{-n}}{1 - \theta} \rho \left(\sqrt{\varepsilon} + 2M \right).
\end{aligned}
$$

Then, also (6.11) for $i = k+1$ follows via (6.7) and (6.6), and the induction argument is complete.

Now, consider $r \in (0, \rho]$. There exists an integer $i \geq 0$ such that $\theta^{i+1}\rho < r \leq \theta^i\rho$. By (6.8) and Lemma 6.1, for a constant $c = c(n, N, \beta, q, Q_M, \Gamma_1, \Gamma_2, \Lambda, M, \kappa, \tilde\Lambda_{n,N,Q_M,\Lambda,M})$, we may estimate

$$
\begin{aligned}
\Phi(r) &\leq \theta^{-n} \Phi(\theta^i\rho) \\
&\leq \theta^{-n} \left(\theta^{2\kappa i} \Phi(\rho) + C(\theta^i\rho)^{2\gamma} \right) \\
&\leq \theta^{-n-2\kappa} \left(\theta^{2\kappa(i+1)} \Phi(\rho) + C(\theta^{i+1}\rho)^{2\gamma} \right) \\
&\leq c \left(\theta^{2\kappa(i+1)} \Phi(\rho) + (\theta^{i+1}\rho)^{2\gamma} \right) \\
&\leq c \left(\left(\frac{r}{\rho} \right)^{2\kappa} \Phi(\rho) + r^{2\gamma} \right),
\end{aligned}
$$

and the proof is complete. $\qquad\square$

7 Proofs of the partial regularity theorems

The excess estimates prepare the ground for deducing regularity of Du on the regular set Ω_{reg}, defined as

$$
\Omega_{\mathrm{reg}} := \left\{ x \in \Omega \, : \, \liminf_{\rho \searrow 0} \fint_{B_\rho(x)} |Du - (Du)_{x,\rho}|^2 \, dy = 0 \text{ and } \limsup_{\rho \searrow 0} |(Du)_{x,\rho}| + |(u)_{x,\rho}| < \infty \right\}. \tag{7.1}
$$

Specifically for $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ as in Setting 3, the set Ω_{reg} is nothing but the set of L^2-Lebesgue points of Du. However, even for arbitrary $u \in W^{1,2}(\Omega, \mathbb{R}^N)$, it is a standard consequence of the Lebesgue differentiation theorem that a.e. point belongs to Ω_{reg} and in other words we have $|\Omega \setminus \Omega_{\mathrm{reg}}| = 0$. Thus, this part of the conclusions in our main results is not addressed in the sequel anymore.

7.1 Basic regularity conclusion

As announced above, we now apply the excess estimates of Proposition 2.11 to establish regularity on Ω_{reg}. However, in the situation of Setting 1 we will initially reach $C^{1,\min\{\alpha,\delta\}}$ regularity with exponent $\min\{\alpha, \delta\}$ only, while the full claim of Theorem 1.2 on $C^{1,\alpha}$ regularity with the optimal exponent α is obtained only a posteriori in a further step. We now work out the last details of the initial step, while the final sharpening of the exponent is postponed to the subsequent Section 7.2

Proof of partial $C^{1,\min\{\alpha,\delta\}}$ regularity in the situation of Setting 1. We consider an arbitrary $x_0 \in \Omega_{\mathrm{reg}}$ and the exponents $\alpha, \delta \in (0, 1)$. Then we fix the $\varepsilon > 0$ from Lemma 6.3 which corresponds to $\kappa := \frac{1+\min\{\alpha,\delta\}}{2}$ and $M := 1 + \limsup_{\rho \searrow 0} \left(|(Du)_{x_0,\rho}| + |(u)_{x_0,\rho}| \right)$. By the choice of Ω_{reg}, we have $\lim_{\rho \searrow 0} \Phi(x_0, \rho) = 0$. Hence, there

exists $\rho_0 \in \left(0, \frac{1}{2}\right]$ with $B_{2\rho_0}(x_0) \Subset \Omega$ and $\frac{\rho_0}{2^n} + \Phi(x_0, 2\rho_0) \leq \frac{\varepsilon}{2^n}$ and $|(Du)_{x_0,\rho_0}| + |(u)_{x_0,\rho_0}| < M$. Further, by continuity of $x \mapsto |(Du)_{x,\rho_0}| + |(u)_{x,\rho_0}|$, there exists $r_0 \in (0, \rho_0]$ such that we have $|(Du)_{x,\rho_0}| + |(u)_{x,\rho_0}| \leq M$ for all $x \in B_{r_0}(x_0)$. Lemma 6.1 gives

$$\Phi(x, \rho_0) \leq \fint_{B_{\rho_0}(x)} |Du - (Du)_{x_0, 2\rho_0}|^2 \, dx \leq 2^n \Phi(x_0, 2\rho_0) \,,$$

and thus $\rho_0 + \Phi(x, \rho_0) \leq \varepsilon$ holds for all $x \in B_{r_0}(x_0)$. We may then apply Lemma 6.3 to infer

$$\Phi(x, r) \leq c \left(\left(\frac{r}{\rho_0} \right)^{1+\min\{\alpha,\delta\}} \Phi(x, \rho_0) + r^{2\min\{\alpha,\delta\}} \right)$$

for all $x \in B_{r_0}(x_0)$ and all $r \in (0, \rho_0]$, and we end up with

$$\fint_{B_r(x)} |Du - (Du)_{x,r}|^2 \, dx \leq c \left(\frac{\rho_0^{1-\min\{\alpha,\delta\}}}{\rho_0^{1+\min\{\alpha,\delta\}}} \varepsilon + 1 \right) r^{2\min\{\alpha,\delta\}} \leq c \left(\varepsilon \rho_0^{-2\min\{\alpha,\delta\}} + 1 \right) r^{2\min\{\alpha,\delta\}}$$

for all $x \in B_{r_0}(x_0)$ and $r \in (0, \rho_0]$. By Proposition 2.11 with $p = 2$ we find that (the Lebesgue representative of) Du is in $C^{0,\min\{\alpha,\delta\}}_{\text{loc}}(B_{r_0}(x_0), \mathbb{R}^{N \times n})$ and therefore (the Lebesgue representative of) u is in $C^{1,\min\{\alpha,\delta\}}_{\text{loc}}(B_{r_0}(x_0), \mathbb{R}^N)$. In particular, we find $B_{r_0}(x_0) \subseteq \Omega_{\text{reg}}$, and we read off that Ω_{reg} is open and that $u \in C^{1,\min\{\alpha,\delta\}}_{\text{loc}}(\Omega_{\text{reg}}, \mathbb{R}^N)$ holds. $\qquad\square$

7.2 The refined Hölder exponent and a priori L^∞ minimizers

Once we know a minimizer is $L^\infty_{(\text{loc})}$ (either by previous reasoning or by assumption), we may improve the Hölder exponent from $\min\{\alpha, \delta\} = \min\{\alpha, \delta\}$ to α and at the same time may drop the additional Morrey assumption of (3.3). This follows from the subsequent adaptations of Lemma 4.1 and Lemma 5.1, respectively.

Lemma 7.1 (Caccioppoli inequality for a priori L^∞ minimizers)**.** *We consider Setting 2. For every $M > 0$, there exists a constant $c = c\left(n, N, \beta, q, Q_M, \Gamma_1, \Lambda, M, \tilde{\Lambda}_M, \|u\|_{L^\infty(B_\rho(x_0))}\right)$ such that we have*

$$\fint_{B_{\frac{\rho}{2}}(x_0)} |Du_{\xi,\zeta}|^2 \, dx \leq c \left(\fint_{B_\rho(x_0)} \left| \frac{u_{\xi,\zeta}}{\rho} \right|^2 dx + \varepsilon \left(\fint_{B_\rho(x_0)} \left| \frac{u_{\xi,\zeta}}{\rho} \right|^{2^*} dx \right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} \rho^{2\alpha} \right)$$

for all $\varepsilon > 0$, $\zeta \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{N \times n}$ with $|\zeta| + |\xi| \leq M$.

Proof. The proof is analogous to the proof of Lemma 4.1 with the slight difference that the a priori bound ensures $|\varphi| + |u| \leq c\left(M, \|u\|_{L^\infty(B_\rho(x_0))}\right)$ on $B_\rho(x_0)$. Therefore, we can improve the estimate (4.3) in order to control the term II from the proof of Lemma 4.1 by

$$\text{II} = \int_{B_{r_2}} g(x, u - \varphi) - g(x, u) \, dx \leq c\left(\beta, q, M, \|u\|_{L^\infty(B_\rho)}\right) \int_{B_\rho} \Gamma |v|^\beta \, dx \,.$$

We combine this with the estimate (4.5), whose derivation is not changed, to find

$$\text{II} \leq c\left(n, \beta, q, M, \Gamma_1, \|u\|_{L^\infty(B_\rho)}\right) \left(\varepsilon \rho^n \left(\fint_{B_\rho} \left| \frac{v}{\rho} \right|^{2^*} dx \right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} \rho^{n+2\alpha} \right).$$

Estimating the other in exactly the same way as in the proof of Lemma 4.1, we come out with the claimed form of the inequality. $\qquad\square$

Lemma 7.2 (approximate A-harmonicity for a priori L^∞ minimizers)**.** *We consider Setting 2. For each bound $M > 0$, there exists a constant $c = c\left(n, N, \beta, q, \Gamma_1, \Lambda, M, \tilde{\Lambda}_M, \|u\|_{L^\infty(B_\rho(x_0))}\right)$ such that we have*

$$\left| \fint_{B_\rho(x_0)} D^2 f(\xi) Du_{\xi,\zeta} \cdot D\varphi \, dx \right| \leq c \left(\sqrt{\widehat{\nu}_M(\Phi)\Phi} + \Phi + \rho^\alpha \right) \|D\varphi\|_{L^\infty(B_\rho(x_0))}$$

for all $\varphi \in W^{1,\infty}_0(B_\rho(x_0), \mathbb{R}^N)$, $\zeta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{N \times n}$ with $|\xi| < M$, where Φ is defined as in Lemma 5.1.

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Proof. The proof is analogous to the proof of Lemma 5.1 with the estimates from (5.3) to (5.4) replaced by simply

$$\mathrm{III} \le c\sigma^{\beta-1}\rho^\beta \fint_{\mathrm{B}_\rho} \Gamma \, \mathrm{d}x \le c\sigma^{\beta-1}\rho^{(2-\beta)\alpha} = c\rho^\alpha$$

for $c = c\big(n, \beta, \Gamma_1, q, \|u\|_{\mathrm{L}^\infty(\mathrm{B}_\rho)}\big)$. The estimates for the other terms remain the same as in the proof of Lemma 5.1. $\qquad\square$

At this stage, we finalize the proof of the first two results from the introduction.

Proof of Theorem 1.2 and Theorem 1.3. We first work in the situation of Setting 2 and thus prove Theorem 1.3. Indeed, with Lemmas 7.1 and 7.2 at hand we may go over the proofs of Lemma 6.2 and Lemma 6.3 in order to reach the same conclusions with $\min\{\alpha, \delta\}$ replaced by α (and with constants which now depend on $\|u\|_{\mathrm{L}^\infty(\mathrm{B}_\rho(x_0))}$, but no longer on Γ_2). Then, in analogy with the reasoning of Section 7.1 we arrive at the regularity claim $u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega_{\mathrm{reg}}, \mathbb{R}^N)$.

Now we turn to Setting 1 and finalize the proof of Theorem 1.2. To this end, we observe that the regularity proved in Section 7.1 implies in particular $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega_{\mathrm{reg}}, \mathbb{R}^N)$. Hence, on every open $\widetilde{\Omega} \Subset \Omega_{\mathrm{reg}}$ we are back precisely to the situation of Setting 2, and $u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega_{\mathrm{reg}}, \mathbb{R}^N)$ is available from the previous reasoning. $\qquad\square$

7.3   Non-uniform ellipticity and a priori $\mathrm{W}^{1,\infty}$ minimizers

Once we know a minimizer is $\mathrm{W}^{1,\infty}_{(\mathrm{loc})}$, we can even deal with *locally* uniform ellipticity in the sense of merely (3.6). This standard observation leads to Theorem 1.4 and eventually also helps in proving Theorem 1.5. On the technical side, it rests on yet another slightly adapted Caccioppoli inequality, as stated next.

Lemma 7.3 (Caccioppoli inequality for a priori $\mathrm{W}^{1,\infty}$ minimizers in non-uniformly elliptic cases)**.** *We consider Setting 3. For every $M > 0$, there exists a constant $c = c(n, N, \beta, q, \Gamma_1, \Lambda, M, \mathcal{M}, \tilde{\Lambda}_M, \lambda_{M+\mathcal{M}})$ such that we have*

$$\fint_{\mathrm{B}_{\frac{\rho}{2}}(x_0)} |\mathrm{D}u_{\xi,\zeta}|^2 \, \mathrm{d}x \le c\left(\fint_{\mathrm{B}_\rho(x_0)} \left|\frac{u_{\xi,\zeta}}{\rho}\right|^2 \mathrm{d}x + \varepsilon\left(\fint_{\mathrm{B}_\rho(x_0)} \left|\frac{u_{\xi,\zeta}}{\rho}\right|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}}\rho^{2\alpha}\right)$$

for all $\varepsilon > 0$, $\zeta \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{N \times n}$ with $|\zeta| + |\xi| \le M$ and all $x_0 \in \Omega$, $\rho \in (0,1]$ such that $\mathrm{B}_\rho(x_0) \Subset \Omega$ and $\mathcal{M} := \|u\|_{\mathrm{W}^{1,\infty}(\mathrm{B}_\rho(x_0))}$.

Proof. Since u is in particular in $\mathrm{L}^\infty(\Omega, \mathbb{R}^N)$, we can basically repeat the proof of Lemma 4.1 with the modifications of Lemma 7.1. However, in this reasoning we replace the estimate (4.1) based on quasiconvexity of f with the following computation. Indeed, we first recall $\mathrm{D}v = \mathrm{D}u_{\xi,\zeta} = \mathrm{D}u - \xi$ and record $|\xi + \tau \mathrm{D}v| \le |\xi| + |\mathrm{D}u| \le M + \mathcal{M}$ for all $\tau \in [0,1]$ in the present situation. On the basis of this observation we then use the possibly non-uniform ellipticity (3.6) to estimate and rewrite

$$\begin{aligned}
\frac{\lambda_{M+\mathcal{M}}}{2} \int_{\mathrm{B}_{r_1}} |\mathrm{D}v|^2 \, \mathrm{d}x &\le \int_{\mathrm{B}_{r_1}} \int_0^1 \int_0^1 \mathrm{D}^2 f(\xi + \tau t \mathrm{D}v)\mathrm{D}v \cdot \mathrm{D}v \, t \, \mathrm{d}\tau \, \mathrm{d}t \, \mathrm{d}x \\
&\le \int_{\mathrm{B}_{r_2}} \int_0^1 \int_0^1 \mathrm{D}^2 f(\xi + \tau t \mathrm{D}\varphi)\mathrm{D}\varphi \cdot \mathrm{D}\varphi \, t \, \mathrm{d}\tau \, \mathrm{d}t \, \mathrm{d}x \\
&= \int_{\mathrm{B}_{r_2}} f(\xi + \mathrm{D}\varphi) - f(\xi) - \mathrm{D}f(\xi) \cdot \mathrm{D}\varphi \, \mathrm{d}x \\
&= \int_{\mathrm{B}_{r_2}} f(\xi + \mathrm{D}\varphi) - f(\xi) \, \mathrm{d}x \, .
\end{aligned}$$

As foreshadowed above, the resulting inequality is then used as a one-to-one substitute for (4.1) in the arguments already used for Lemma 4.1 and Lemma 7.1. $\qquad\square$

Proof of Theorem 1.4. We rely on Lemma 7.3 and on Lemma 7.2 and otherwise on the same arguments explicated in Sections 6, 7.1, and 7.2 (where several constants now depend on $\mathcal{M} := \|u\|_{\mathrm{W}^{1,\infty}(\mathrm{B}_\rho(x_0), \mathbb{R}^N)}$ and $\lambda_{M+\mathcal{M}}$, but no longer on Q_M, and where in the proof of Lemma 6.2 the requirement (2.3) comes directly from (3.6)). By these arguments we then arrive at $u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega_{\mathrm{reg}}, \mathbb{R}^N)$ as before. $\qquad\square$

7.4 L^p-Hölder zero-order terms and L^p-$W^{1,r}$ zero-order terms

In view of the embeddings of L^p spaces into Morrey spaces (cf. Remark 2.2), our main results have straightforward corollaries for the case of an L^p-Hölder zero-order term. Basically, this issue has already been touched upon in situations (A) and (B) of the introduction, but still we prefer to explicate it here:

Corollary 7.4 (partial regularity for variational integrals with L^p-Hölder zero-order integrand). *We suppose that f is as Setting 1, Setting 2, or Setting 3. For the Carathéodory integrand $g\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ we assume* (1.2) *with merely*

$$\beta \in (0,1]\,, \qquad q \in [\beta, 2^*)\,, \qquad \Gamma \in L^p_{\mathrm{loc}}(\Omega) \text{ for some } p \in \left(\frac{n}{\beta}, \infty\right]$$

and in case of Setting 1 additionally with $p \geq s_q$ (where, for $n \in \{1,2\}$, the range for q should be read as $q \in [\beta, \infty)$ and the additional requirement as void). Then, for every local minimizer u of $\mathcal{F}$ from (1.1)*, we have*

$$u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_{\mathrm{reg}}, \mathbb{R}^N) \qquad with \qquad
\begin{cases}
\alpha = \dfrac{\beta - n/p}{2 - \beta} & in\ case\ p < \infty\,, \\[2mm]
\alpha = \dfrac{\beta}{2 - \beta} & in\ case\ p = \infty\,, \beta < 1\,, \\[2mm]
any\ \alpha < 1 & in\ case\ p = \infty\,, \beta = 1\,.
\end{cases}$$

Proof. We first treat the case $p < \infty$. Since in case $n \geq 3$ we have $p > \frac{n}{\beta} > s_\beta$ and in case $n \in \{1,2\}$ can deduce $p \geq s_\beta$ from $p > \frac{n}{\beta} \geq 1$ by taking 2^* large enough, in any dimension we infer $\Gamma \in L^p(\Omega) \subseteq L^{s_\beta, n - s_\beta \frac{n}{p}}(\Omega) = L^{s_\beta, n + s_\beta((2-\beta)\alpha - \beta)}(\Omega)$ for $\alpha = \frac{\beta - n/p}{2-\beta}$. In case of Setting 1, by assumption or choice of 2^* we additionally get $p \geq s_q$ and $\Gamma \in L^p(\Omega) \subseteq L^{s_q, n - s_q \frac{n}{p}}(\Omega)$ with $n - s_q \frac{n}{p} > n - s_q \beta \geq n - s_q q$. Thus, we may apply Theorem 1.2, Theorem 1.3, or Theorem 1.4, respectively, to deduce the claimed regularity.

The case $p = \infty$ is similar. Since we have $\Gamma \in L^\infty(\Omega) \subset L^{s,n}(\Omega)$ for all $s \in [1, \infty)$, the claimed regularity comes from Theorem 1.2, Theorem 1.3, or Theorem 1.4, respectively, with the choice $\alpha = \frac{\beta}{2-\beta}$ in case $\beta < 1$ and with arbitrary $\alpha \in (0,1)$ in case $\beta = 1$. $\qquad\square$

We also explicate a corresponding result for an L^p-$W^{1,r}$ zero-order term or, in other words, for a zero-order term of certain integral form. This essentially reproduces situations (A') and (B') of the introduction.

Corollary 7.5 (partial regularity for scalar variational integrals with L^p-$W^{1,r}$ zero-order integrands). *We suppose that f is as Setting 1, Setting 2, or Setting 3 in the scalar case $N = 1$. Moreover, we consider $H \in L^1_{\mathrm{loc}}(\Omega \times \mathbb{R})$ which satisfies*

$$H \in L^p(\Omega, L^r(\mathbb{R})) \qquad with\ r \in (1, \infty]\ and\ p \in (nr', \infty]$$

(with integrability understood in the sense of $\int_\Omega \|H(x, \cdot)\|^p_{L^r(\mathbb{R})}\, \mathrm{d}x < \infty$). Then, for every local minimizer $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ of the variational integral

$$\mathcal{F}[w] := \int_\Omega \left[f(\mathrm{D}w(x)) - \int_0^{w(x)} H(x,t)\, \mathrm{d}t \right] \mathrm{d}x\,, \tag{7.2}$$

we have

$$u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_{\mathrm{reg}}) \qquad with \qquad
\begin{cases}
\alpha = \dfrac{r - 1 - nr/p}{r + 1} & in\ case\ p < \infty\,, r < \infty\,, \\[2mm]
\alpha = 1 - n/p & in\ case\ p < \infty\,, r = \infty\,, \\[2mm]
\alpha = \dfrac{r - 1}{r + 1} & in\ case\ p = \infty\,, r < \infty\,, \\[2mm]
any\ \alpha < 1 & in\ case\ p = \infty\,, r = \infty\,.
\end{cases}$$

Proof. We set

$$g(x,y) := -\int_0^y H(x,t)\, \mathrm{d}t \qquad for\ x \in \Omega\ and\ y \in \mathbb{R}$$

and then observe that Hölder's inequality yields the L^p-Hölder condition

$$|g(x,y) - g(x,\widehat{y})| \leq \Gamma(x)|y - \widehat{y}|^{\beta} \qquad \text{for all } x \in \Omega \text{ and } y, \widehat{y} \in \mathbb{R}$$

with $\Gamma \in \mathrm{L}^p(\Omega)$ given by $\Gamma(x) := \|H(x,\,\cdot\,)\|_{\mathrm{L}^r(\mathbb{R})}$ and with $\beta := \frac{1}{r'} \in (0,1]$. Since we assume $p > nr' = \frac{n}{\beta}$, this brings us in position to apply Corollary 7.4 with $q = \beta$ (where now in view of $s_q = s_\beta$ and the initial reasoning in the previous proof the additional requirement $p \geq s_q$ is always valid). By inserting $\beta = \frac{1}{r'} = \frac{r-1}{r}$ into the exponents α of Corollary 7.4 the regularity outcome then takes the form of the current claim. $\qquad\square$

Remark 7.6. At least in case $p = r < \infty$, the Hölder exponent α reached in Corollary 7.5 is optimal. This can be confirmed by transferring the counterexamples of [74, Section 4] from the parametric to the non-parametric setting (compare also Section 8 for transferring regularity the opposite way round).

8 The optimal Hölder exponent in Massari's regularity theorem

At this stage we recall that our final aim is improving Massari's regularity theorem up to the limit Hölder exponent α_{opt}. This will be achieved by considering the non-parametric Massari-type functional (7.2) with $H \in \mathrm{L}^p(\Omega \times \mathbb{R}) = \mathrm{L}^p(\Omega, \mathrm{L}^p(\mathbb{R}))$ and on applying Corollary 7.5 in the particular case $p = r$. We already record that in this case the requirement $p > nr'$ means nothing but $p > n + 1$ and that, for $p = r < \infty$, the Hölder exponent reached in Corollary 7.5 actually boils down to our target exponent $\alpha_{\mathrm{opt}} = \frac{p-(n+1)}{p+1}$. In fact, with Corollary 7.4 at hand, the proof of Theorem 1.5 essentially reduces to deducing a suitable non-parametric minimality property from the parametric one assumed:

Proof of Theorem 1.5. We consider a set $E \subseteq \mathbb{R}^{n+1}$ of finite perimeter in an open set $U \subseteq \mathbb{R}^{n+1}$ and a variational mean curvature $H \in \mathrm{L}^p(U)$ of E in U. Since we may replace H by $H^+\mathbb{1}_E - H^-\mathbb{1}_{E^c}$ according to Remark 2.13 i), we can assume $H \geq 0$ on E and $H \leq 0$ on E^c. We now derive the optimal regularity near an arbitrary fixed $x_0 \in \partial^* E \cap U$.

Step 1: *Non-parametric rewriting with smooth variations.* By isometry invariance of perimeter and variational mean curvatures and by Theorem 2.14, we may directly assume that $\partial E \cap \Omega = \partial^* E \cap \Omega$ is C^1 and $E \cap (\overline{\Omega} \times [0, R])$ is the subgraph of $u \in \mathrm{C}^1(\overline{\Omega})$ for a bounded open set $\Omega \subseteq \mathbb{R}^n$ and a constant $R > 0$ such that $u(\overline{\Omega}) \subseteq (0, R)$ and $x_0 \in \Omega \times (0, R) \Subset U$. In this situation, Remark 2.13 ii) asserts that u minimizes

$$\mathcal{G}[w] := \int_\Omega \left[\sqrt{1 + |\mathrm{D}w(x)|^2} - \int_0^{w(x)} H(x,t)\,\mathrm{d}t \right] \mathrm{d}x$$

among all functions $w = u + \varphi$ with $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega)$ such that $w(\overline{\Omega}) \subseteq (0, R)$.
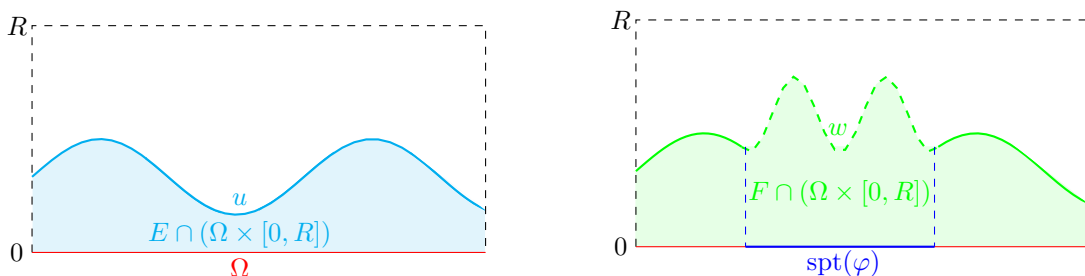

Figure 1: Local parametrization of E via u and of test sets F via w

Step 2: *Extension to $\mathrm{W}^{1,2}$ variations with image in $[\varepsilon, R - \varepsilon]$.* We first argue briefly that we can approximate $\psi \in \mathrm{W}^{1,2}_0(\Omega)$ with $\varepsilon \leq u + \psi \leq R - \varepsilon$ a.e. on Ω for some fixed $\varepsilon > 0$ by $\varphi_k \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega)$ such that $(u + \varphi_k)(\overline{\Omega}) \subseteq (0, R)$ with respect to the $\mathrm{W}^{1,2}$ norm and a.e. on Ω. Indeed, possibly decreasing ε we may assume $u(\overline{\Omega}) \subseteq (\varepsilon, R - \varepsilon)$, and we approximate ψ by $\psi_k \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega)$ in $\mathrm{W}^{1,2}(\Omega)$ and a.e. on Ω. In order to truncate suitably, we choose $T_\varepsilon \in \mathrm{C}^1\left(\mathbb{R}, \left[\frac{1}{3}\varepsilon, R - \frac{1}{3}\varepsilon\right]\right)$ with bounded derivative $(T_\varepsilon)'$ such that $T_\varepsilon(y) = y$ for $y \in \left[\frac{2}{3}\varepsilon, R - \frac{2}{3}\varepsilon\right]$. It is then a standard matter to check that, for $k \to \infty$, we have convergence of $T_\varepsilon(u + \psi_k)$ to $u + \psi$ in $\mathrm{W}^{1,2}(\Omega)$

and a.e. on Ω and consequently also of $\varphi_k := T_\varepsilon(u + \psi_k) - u \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega)$ to ψ in the same sense. Taking into account that $(u + \varphi_k)(\overline{\Omega}) \subseteq \left[\frac{1}{3}\varepsilon, R - \frac{1}{3}\varepsilon\right]$ by choice of T_ε, we have established the approximation claim.

Now, the convergence in $\mathrm{W}^{1,2}(\Omega)$ and a.e. on Ω implies convergence of the first-order and zero-order terms of the functional $\mathcal{G}$, respectively (where, for the latter one, we make use of the dominated convergence theorem). All in all, we infer convergence of $\mathcal{G}[u + \varphi_k]$ to $\mathcal{G}[u + \psi]$. Hence, we can conclude that u minimizes $\mathcal{G}$ among all functions in $u + \psi \in \mathrm{W}^{1,2}_u(\Omega)$ with $\varepsilon \leq u + \psi \leq R - \varepsilon$ a.e. on Ω for some fixed $\varepsilon > 0$.

Step 3: *Extension to arbitrary* $\mathrm{W}^{1,2}$ *variations.* Finally, we pass on to fully arbitrary competitors without need for the $(\varepsilon, R - \varepsilon)$ requirement of Step 2. We consider an arbitrary $\Phi \in \mathrm{W}^{1,2}_u(\Omega)$ and, for $\varepsilon > 0$ small enough that $u(\overline{\Omega}) \subseteq (\varepsilon, R - \varepsilon)$, record that the sharp truncation $\Psi := \max\{\min\{\Phi, R - \epsilon\}, \epsilon\}$ is in $\mathrm{W}^{1,2}_u(\Omega)$ with $\mathrm{D}\Psi = \mathbb{1}_{(\epsilon, R - \epsilon)}(\Phi)\, \mathrm{D}\Phi$ a.e. on Ω. We further define $\tilde{H}(x,t) := \mathbb{1}_{[0,R]}(t)H(x,t)$. Then, since $\tilde{H} \in \mathrm{L}^1(\Omega \times \mathbb{R}) \cap \mathrm{L}^p(\Omega \times \mathbb{R})$ satisfies $H \geq 0$ on $\Omega \times [0, \varepsilon] \subseteq E$ and $H \leq 0$ on $\Omega \times [R - \varepsilon, R] \subseteq E^{\mathrm{c}}$, it follows

$$
\begin{aligned}
\tilde{\mathcal{G}}[\Phi] &:= \int_\Omega \left[\sqrt{1 + |\mathrm{D}\Phi(x)|^2} - \int_0^{\Phi(x)} \tilde{H}(x,t)\,\mathrm{d}t \right] \mathrm{d}x \\
&\geq \int_\Omega \left[\sqrt{1 + |\mathrm{D}\Psi(x)|^2} - \int_0^{\Psi(x)} \tilde{H}(x,t)\,\mathrm{d}t \right] \mathrm{d}x = \mathcal{G}[\Psi] \geq \mathcal{G}[u] = \tilde{\mathcal{G}}[u]\,,
\end{aligned}
$$

where the last rewritings are based on the minimality established in Step 2 and on $u(\overline{\Omega}) \subseteq (0, R)$, respectively.

Step 4: *Application of the non-parametric regularity result.* The first-order integrand $f(z) := \sqrt{1 + |z|^2}$ of the functional $\widetilde{\mathcal{G}}$ is smooth and of at most quadratic growth (in fact, linear growth) with

$$
\mathrm{D}^2 f(z)\xi \cdot \xi \geq \frac{|\xi|^2}{(1 + M^2)^{\frac{3}{2}}} \qquad \text{for all } z, \xi \in \mathbb{R}^n \text{ with } |z| \leq M\,.
$$

Thus, f is as in Setting 3 with $N = 1$, while $\tilde{H}$ satisfies the conditions of Corollary 7.5 with $r = p \in (n + 1, \infty)$, and moreover we already know $u \in \mathrm{C}^1(\overline{\Omega})$ and thus read off $\Omega_{\mathrm{reg}} = \Omega$ from (7.1). All in all, as indicated at the beginning of this section, we may then apply Corollary 7.5 to deduce that the minimizer u of $\tilde{\mathcal{G}}$ is in $\mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega)$ with exactly the claimed exponent $\alpha = \alpha_{\mathrm{opt}} = \frac{p - (n+1)}{p+1}$. Since we worked near an arbitrary $x_0 \in \partial^* E \cap U$, we have verified that $\partial^* E \cap U$ is an n-dimensional $\mathrm{C}^{1,\alpha_{\mathrm{opt}}}$-submanifold (and relatively open in $\partial E \cap U$). Finally, the assertions on the size of the singular set $(\partial E \setminus \partial^* E) \cap U$ are already contained in Theorem 2.14. $\qquad\square$

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